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Arnold A. Eniego

Ian June L. Garces

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Characterization of completely k-magic regular graphs

A A Eniego[1] **and I J L Garces**[2]

[1] Science and Mathematics Department, National University, Manila, The Philippines
[2] Department of Mathematics, Ateneo de Manila University, Quezon City, The Philippines

E-mail: `aaeniego@national-u.edu.ph, ijlgarces@ateneo.edu`

Abstract. Let $k \in \mathbb{N}$ and $c \in \mathbb{Z}_k$. A graph G is said to be c-sum k-magic if there is a labeling $\ell : E(G) \to \mathbb{Z}_k \setminus \{0\}$ such that $\sum_{u \in N(v)} \ell(uv) \equiv c \pmod{k}$ for every vertex v of G, where $N(v)$ is the neighborhood of v in G. We say that G is completely k-magic whenever it is c-sum k-magic for every $c \in \mathbb{Z}_k$. In this paper, we characterize all completely k-magic regular graphs.

1. Introduction

Let $G = (V(G), E(G))$ be a finite, simple (unless otherwise stated) graph with vertex set $V(G)$ and edge set $E(G)$. A *factor* of G is a subgraph H with $V(H) = V(G)$. In particular, if a factor H of G is h-regular, then we say that H is an h-*factor* of G. An h-*factorization* of G is a partition of $E(G)$ into disjoint h-factors. If such factorization of G exists, then we say that G is h-*factorable*.

The following theorem is attributed to Petersen [7], which we state using the versions of Akiyama and Kano [2] and Wang and Hu [10].

Theorem 1.1 ([2, Theorem 3.1], [7], [10, Theorem 10])**.** *Let G be a $2r$-regular connected general graph (not necessarily simple), where $r \geq 1$. Then G is 2-factorable, and it has a $2k$-factor for every k, $1 \leq k \leq r$. Moreover, if G is of even order, then it is r-factorable.*

A graph G is λ-*edge connected* if it remains connected whenever fewer than λ edges are removed.

Theorem 1.2. [6] *Let r and k be integers such that $1 \leq k < r$, and G be a λ-edge connected r-regular general graph, where $\lambda \geq 1$. If one of the following conditions holds:*

(1) *r is even, k is odd, $|G|$ is even, and $\frac{r}{\lambda} \leq k \leq r(1 - \frac{1}{\lambda})$,*

(2) *r is odd, k is even, and $2 \leq k \leq r(1 - \frac{1}{\lambda})$, or*

(3) *r and k are both odd and $\frac{r}{\lambda} \leq k$,*

then G has a k-regular factor.

Let k be a positive integer. A finite simple graph $G = (V(G), E(G))$ is said to be k-*magic* if there exists an edge labeling $\ell : E(G) \to \mathbb{Z}_k \setminus \{0\}$, where $\mathbb{Z}_1 = \mathbb{Z}$ the group of integers, and $\mathbb{Z}_k = \{0, 1, 2, \ldots, k-1\}$ the group of integers modulo $k \geq 2$, such that the induced vertex labeling $\ell^+ : V(G) \to \mathbb{Z}_k$, defined by $\ell^+(v) = \sum_{uv \in E(G)} \ell(uv)$, is a constant map. If $c \in \mathbb{Z}_k$

and $\ell^+(v) = c$ for all $v \in V(G)$, then we call c is a *magic sum* of G. In particular, if G is k-magic with magic sum c, then we say that G is *c-sum k-magic.* If G is c-sum k-magic for all $c \in \mathbb{Z}_k$, then it is said to be *completely k-magic.* The set of all magic sums $c \in \mathbb{Z}_k$ of G is the *sum spectrum of G with respect to k* and is denoted by $\Sigma_k(G)$. If $c = 0$, then we say that G is *zero-sum k-magic.* The *null set* of G, denoted by $N(G)$, is the set of all positive integers k such that G is a zero-sum k-magic graph.

Remark 1.3. *If $c \in \mathbb{Z}_k$ and ℓ is a c-sum k-magic labeling of G, then the labeling ℓ', defined by $\ell'(e) = k - \ell(e)$, is a $(k - c)$-sum k-magic labeling of G.*

Remark 1.4. *Any 2-magic graph is not completely 2-magic.*

The concept of A-magic graphs is due to Sedlacek [9]. Over the years, many papers have been published in connection with magic graphs. Akbari, Rahmati, and Zare [1] investigated the zero-sum k-magic labelings and null sets of regular graphs. Dong and Wang [4] solved affirmatively a conjecture posed in [1] on the existence of a zero-sum 3-magic labeling of 5-regular graphs. Salehi [8] determined the integer-magic spectra of certain classes of cycle-related graphs.

Using the term "index set," Wang and Hu [10] initially studied the concept of completely k-magic graphs. They gave a partial list of completely 1-magic regular graphs. Eniego and Garces [5] completely added the remaining cases in this list. They also presented the sum spectra of some regular graphs that are not completely k-magic.

Theorem 1.5 ([1, Theorem 13]). *Let G be an r-regular graph, where $r \geq 3$ and $r \neq 5$. If r is even, then $N(G) = \mathbb{N}$ (the set of positive integers); otherwise, $\mathbb{N} \setminus \{2, 4\} \subseteq N(G)$.*

Theorem 1.6 ([4, Theorem 2.1]). *Every 5-regular graph admits a zero-sum 3-magic labeling.*

Theorem 1.7 ([5, Theorem 3.3]). *Let $n \geq 3$ and $k \geq 3$ be integers, and C_n the cycle with n vertices.*

(1) *If n is even, then C_n is completely k-magic for all k.*

(2) *If n is odd, then C_n is not completely k-magic for any k. Moreover, we have*

$$\Sigma_k(C_n) = \begin{cases} \mathbb{Z}_k \setminus \{0\} & \text{if } k \text{ is odd,} \\ \{0, 2, \ldots, k-2\} & \text{if } k \text{ is even.} \end{cases}$$

Theorem 1.8 ([5, Lemma 3.4]). *Let $k \geq 4$ be an even integer. Then there exists no k-magic graph of odd order that is completely k-magic. In particular, if c is a magic sum of a k-magic graph of odd order, then c must be even.*

Theorem 1.9 ([5, Theorem 3.6]). *Let $k, r \geq 3$ be integers, and G an r-regular graph. If $\gcd(r, k) = 1$, then $\{1, 2, \ldots, k-1\} \subseteq \Sigma_k(G)$.*

Theorem 1.10 ([5, Theorem 3.7]). *Let G be a zero-sum k-magic r-regular graph, where $k \geq 3$ and $r \geq 3$. If G has a 1-factor, then G is completely k-magic.*

Theorem 1.11 ([10, Theorem 13], [5, Theorem 2.1]). *Let G be an r-regular graph of order n. Then*

$$\Sigma_1(G) = \begin{cases} \mathbb{Z} \setminus \{0\} & \text{if } r = 1, \\ \mathbb{Z} & \text{if } r = 2 \text{ and } G \text{ contains even cycles only,} \\ 2\mathbb{Z} \setminus \{0\} & \text{if } r = 2 \text{ and } G \text{ contains an odd cycle,} \\ 2\mathbb{Z} & \text{if } r \geq 3,\ r \text{ even, and } n \text{ odd,} \\ \mathbb{Z} & \text{if } r \geq 3 \text{ and } n \text{ even,} \end{cases}$$

where $2\mathbb{Z}$ is the set of all even integers.

With Remark 1.4 and Theorem 1.11, it remains to characterize all completely k-magic regular graphs for $k \geq 3$. This characterization is the main theorem of this paper, which we state as follows.

Theorem 1.12 (**Main Theorem**). *Let $r \geq 2$ and $k \geq 3$ be integers, and G an r-regular graph of order $n \geq 3$. Then G is completely k-magic if and only if one of the following properties holds:*

(1) $k \geq 3$, $r = 2$, *and G contains even cycles only,*
(2) $k \geq 5$ *and* $r \geq 3$ *odd,*
(3) $k \geq 5$, $r \geq 4$ *even, and* n *even,*
(4) $k \geq 5$ *odd,* $r \geq 4$ *even, and* n *odd,*
(5) $k = 4$, $r \geq 3$, n *even, and G zero-sum 4-magic, or*
(6) $k = 3$ *and any one of the following conditions holds:*
   (i) $r \not\equiv 0 \pmod 3$,
   (ii) $r \equiv 0 \pmod 6$, *or*
   (iii) $r \equiv 0 \pmod 3$, r *odd, and G has a factor H such that* $d_H(v) \equiv 1 \pmod 3$ *for all* $v \in V(H)$.

For convenience, we only consider graphs that are finite and simple (unless otherwise stated). We also write $\mathbb{Z}_k^*$ to mean $\mathbb{Z}_k \setminus \{0\}$. For graph-theoretic terms that are not explicitly defined in this paper, see [3].

2. Proof of the Main Theorem

We divide the proof into several results.

It is not difficult to see that if G is 1-regular, then $\Sigma_k(G) = \mathbb{Z}_k^*$. For 2-regular graphs, the following remark is a consequence of Theorem 1.7.

Remark 2.1. *Let $k \geq 3$ and G a 2-regular graph. If G has an odd cycle, then*

$$\Sigma_k(G) = \begin{cases} \mathbb{Z}_k^* & \text{if } k \text{ is odd} \\ \{0, 2, \ldots, k-2\} & \text{if } k \text{ is even.} \end{cases}$$

Otherwise, we have $\Sigma_k(G) = \mathbb{Z}_k$.

Clearly, if G is 1-factorable, then G is completely k-magic. The following theorem considers regular graphs that has a factor that is completely k- magic.

Theorem 2.2. *Let $r \geq 2$, $2 \leq h \leq r$, $k \neq 2$, and G an r-regular graph. If G has an h-factor that is completely k-magic, then G is completely k-magic.*

Proof. The case when $h = r$ is trivial, so we assume $h < r$. Let H be an h-factor of G that is completely k-magic. Let $\alpha = c - (r - h) \pmod k$ and f_α be an α-sum k-magic labeling of H for each $c \in \mathbb{Z}_k$.

Define $\ell_c : E(G) \to \mathbb{Z}_k^*$ by

$$\ell_c(e) = \begin{cases} f_\alpha(e) & \text{if } e \in E(H) \\ 1 & \text{if } e \in E(G \setminus H). \end{cases}$$

Observe that ℓ_c is a c-sum k-magic labeling of G for each $c \in \mathbb{Z}_k$. Hence, G is completely k-magic. □

The following construction will be useful.

Remark 2.3. *Let G be an r-regular graph with $E(G) = \{e_1, e_2, e_3, \ldots, e_m\}$, where $r \geq 1$. Then we can construct a graph G' (with parallel edges) such that $V(G') = V(G)$ and $E(G') = E(G) \cup \{e'_1, e'_2, e'_3, \ldots, e'_m\}$, where e'_i is a duplicate edge of e_i in G for each i (that is, edges e_i and e'_i have the same end vertices). By Theorem 1.1, G' has a $2h$-factor H' for each h, $1 \leq h \leq r$. Also, $G' \setminus H'$ is a $(2r-2h)$-factor of G' obtained by removing the edges of H' from G'.*

Theorem 2.4. *Let G be a 5-regular graph. Then $\mathbb{N} \setminus \{2,4\} \subseteq N(G)$.*

Proof. We know from Theorem 1.11 and Theorem 1.6 that $1, 3 \in N(G)$. For $k \geq 5$, we consider two cases.

Case 1. Suppose $k \geq 5$ and $k \neq 8$. Using the construction described in Remark 2.3, let H' and $G' \setminus H'$ be a 2-factor and 8-factor of G', respectively.

Define a zero-sum k-magic labeling ℓ' on G' by

$$\ell'(e) = \begin{cases} k-4 & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Note that the labeling ℓ on G defined by $\ell(e_i) = \ell'(e_i) + \ell'(e'_i)$ for $e_i \in E(G)$ is a zero-sum k-magic labeling on G.

Case 2. Suppose $k = 8$. Using again the construction in Remark 2.3, let H' and $G' \setminus H'$ be a 4-factor and 6-factor of G', respectively.

Define a zero-sum labeling ℓ' on G' by

$$\ell'(e) = \begin{cases} 2 & \text{if } e \in E(H') \\ 4 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the labeling ℓ on G defined by $\ell(e_i) = \frac{1}{2}[\ell'(e_i) + \ell'(e'_i)]$ for $e_i \in E(G)$ is a zero-sum 8-magic labeling on G.

Therefore, $\mathbb{N} \setminus \{2,4\} \subseteq N(G)$. □

Note that an odd-regular graph may not be zero-sum 4-magic. It was remarked in [1, Remark 10] that an odd-regular graph G is not zero-sum 4-magic if G has a vertex such that every edge incident to it is a cut-edge.

Theorem 2.5. *Let G be an r-regular graph, where $r \geq 3$ is odd and $k \geq 5$. Then G is completely k-magic.*

Proof. We know from Theorems 1.5 and 2.4 that $0 \in \Sigma_k(G)$. Let $E(G) = \{e_1, e_2, e_3, \ldots, e_m\}$. As constructed in Remark 2.3, let H' and $G' \setminus H'$ be a 2-factor and $(2r-2)$-factor of G', respectively. We consider two cases.

Case 1. Suppose $r \equiv 1 \pmod{k}$. Then $\gcd(r, k) = 1$. By Theorem 1.9, G is completely k-magic.

Case 2. Suppose $r \not\equiv 1 \pmod{k}$. Assume $\gcd(r, k) = d$ so that $r = ad$ and $k = bd$ for some positive integers a and b. Note that, since r is odd, d is also odd. We consider two sub-cases.

Sub-Case 2.1. Suppose $k \geq 5$ is odd. Then b is odd.

For each $c \in \mathbb{Z}_k^* \setminus \{k-b, k-2b\}$, define $\ell'_c : E(G') \to \mathbb{Z}_k^*$ by

$$\ell'_c(e) = \begin{cases} x & \text{if } e \in E(H') \\ \frac{1}{2}(k+b) & \text{if } e \in E(G' \setminus H'), \end{cases}$$

where $x = \frac{1}{2}(b+c)$ if c is odd, and $x = \frac{1}{2}(b+c+k)$ if c is even. Observe that ℓ'_c is a c-sum k-magic labeling of G' for each $c \neq 0$.

For each $c \notin \{0, k-b, k-2b\}$, define $\ell_c : E(G) \to \mathbb{Z}_k^*$ by $\ell_c(e_i) = \ell'_c(e_i) + \ell'_c(e'_i)$ for $1 \leq i \leq m$. Since ℓ'_c is a c-sum k-magic labeling of G', ℓ_c is a c-sum k-magic labeling of G for each $c \in \mathbb{Z}_k^* \setminus \{k-b, k-2b\}$.

If $k \neq 3b$, then, by Remark 1.3, $k-b, k-2b \in \Sigma_k(G)$. If $k = 3b$, it is enough to show that $k - 2b \in \Sigma_k(G)$. To do that, we provide a different labeling using a different set of factors of G'. Let J' and $G' \setminus J'$ be a 4-factor and $(2r-4)$-factor of G' respectively. In addition, we let $J' = J'_1 \cup J'_2$, where J'_1 and J'_2 are 2-factors of J'.

Define $\ell' : E(G') \to \mathbb{Z}_k^*$ by

$$\ell'(e) = \begin{cases} \frac{1}{2}(b+1) & \text{if } e \in E(J'_1) \\ \frac{1}{2}(b-1) & \text{if } e \in E(J'_2) \\ b & \text{if } e \in E(G' \setminus J'). \end{cases}$$

Since $k = 3b$, $d = 3$ and $r = 3a$. Thus, the magic sum in G' is given by $2[\frac{1}{2}(b+1)] + 2[\frac{1}{2}(b-1)] + b(2r-4) \equiv -2b \pmod{k}$. Define $\ell : E(G) \to \mathbb{Z}_k^*$ by $\ell(e_i) = \ell'(e_i) + \ell'(e'_i)$ for $1 \leq i \leq m$. Note that ℓ is also a $(k-2b)$-sum k-magic labeling of G.

Sub-Case 2.2. Suppose $k \geq 6$ is even. Then b is even.

By labeling all the edges of G with $\frac{1}{2}k$, we see that $\frac{1}{2}k \in \Sigma_k(G)$.

Suppose $r - 1 \equiv \frac{1}{2}k \pmod{k}$. For each $c \in \mathbb{Z}_k^* \setminus \{k-1, \frac{1}{2}k\}$, define $\ell'_c : E(G') \to \mathbb{Z}_k^*$ by

$$\ell'_c(e) = \begin{cases} c & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the sum of the labels of the edges incident to each vertex in G' is $2(r-1) + 2c \equiv 2c \pmod{k}$. Using a similar argument as in Sub-Case 2.1, it can be shown that G is also e-sum k-magic for all even $e \neq 0$. Thus, we are left to show that G is c-sum k-magic as well for all odd c.

For each odd $c \neq k-1$, define $\ell_c : E(G) \to \mathbb{Z}_k^*$ by $\ell_c(e_i) = \frac{1}{2}[\ell'_c(e_i) + \ell'_c(e'_i)]$ for each i, $1 \leq i \leq m$. Note that, since ℓ'_c is a $2c$-sum k-magic labeling of G', ℓ_c is a c-sum k-magic labeling of G for each odd $c \neq k-1$. Again, by Remark 1.3, we see that $k - 1 \in \Sigma_k(G)$.

Suppose $r - 1 \equiv r_0 \pmod{k}$, where $r_0 \neq \frac{1}{2}k$. For each $c \in \mathbb{Z}_k^* \setminus \{r_0, r_0 + \frac{1}{2}k, r_0 - 1\}$, define $\ell'_c : E(G') \to \mathbb{Z}_k^*$ by

$$\ell'_c(e) = \begin{cases} c - r_0 & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the sum of the labels of the edges incident to each vertex in G' is $2r_0 + 2c - 2r_0 \equiv 2c \pmod{k}$. As in Sub-Case 2.1, it can be shown that G is also even-sum k-magic. So again, we are left to show that G is odd-sum k-magic.

As what we did earlier, for each odd $c \neq r_0 - 1$ (and, possibly, $r_0 + \frac{1}{2}k$), define $\ell_c : E(G) \to \mathbb{Z}_k^*$ by $\ell_c(e_i) = \frac{1}{2}[\ell'_c(e_i) + \ell'_c(e'_i)]$ for all i, $1 \leq i \leq m$. Since ℓ'_c is a $2c$-sum k-magic labeling of G', ℓ_c is a c-sum k-magic labeling of G for each odd $c \neq r_0 - 1$ (and, possibly, $r_0 + \frac{1}{2}k$). If $r_0 - 1$ and $r_0 + \frac{1}{2}k$ are not inverses, then, by Remark 1.3, $\mathbb{Z}_k^* \subset \Sigma_k(G)$.

If $r_0 - 1$ and $r_0 + \frac{1}{2}k$ are inverses, then it is enough to show that $r_0 - 1 \in \Sigma_k(G)$. Define ℓ' on G' by

$$\ell'(e) = \begin{cases} k-1 & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Note that the magic sum using ℓ' is $2r_0-2$. Define ℓ on G by $\ell(e_i) = \frac{1}{2}[\ell'(e_i)+\ell'(e_i')]$ for $e_i \in E(G)$. Clearly, ℓ is an (r_0-1)-sum k-magic labeling on G. Thus, by Remark 1.3, $r_0 + \frac{1}{2}k \in \Sigma_k(G)$, and so $\mathbb{Z}_k^* \subset \Sigma_k(G)$.

In any case, G is completely k-magic. □

Theorem 2.6. *Let $k \geq 5$ and G a $2r$-regular graph of order $n \geq 3$, where $r \geq 2$.*

(1) *If n is even, then G is completely k-magic.*

(2) *If n is odd, then*

  (i) *G is completely k-magic if k is odd, and*

  (ii) *$\Sigma_k(G) = \{0, 2, 4, \ldots, k-2\}$ if k is even.*

Proof. Let $E(G) = \{e_1, e_2, e_3, \ldots, e_m\}$. By Theorem 1.5, G is zero-sum k-magic.

(1) Suppose $r = 2$. To prove the theorem, we only show that $\mathbb{Z}_k^* \subset \Sigma_k(G)$. We consider two cases.

Case 1. Suppose k is odd. Then $\gcd(4, k) = 1$. By Theorem 1.9, $\mathbb{Z}_k^* \subseteq \Sigma_k(G)$.

Case 2. Suppose k is even. It is not difficult to see that, being 4-regular, G is 2-edge connected. By Remark 2.3, we can construct G' so that G' is a 4-edge-connected 8-regular graph. By Theorem 1.2, G' has a 3-factor, say H'. Let $G' \setminus H'$ be the 5-factor of G' obtained by removing the edges of H' from G'.

Sub-Case 2.1. Let $k = 2d$, d even. For each $c \in \mathbb{Z}_k^* \setminus \{\frac{1}{2}k, \frac{1}{4}k\}$, define $f_c : E(G') \to \mathbb{Z}_k^*$ by

$$f_c(e) = \begin{cases} 2c & \text{if } e \in E(H') \\ k-c & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Observe that the sum of the labels of the edges incident to each of the vertices in G' is equal to $5(k-c) + 3(2c) \equiv c \pmod{k}$. This shows that f_c is a c-sum k-magic labeling of G' for all $c \neq 0, \frac{1}{2}k, \frac{1}{4}k$. By Remark 1.3, $\frac{1}{4}k \in \Sigma_k(G')$.

For each $c \in \mathbb{Z}_k^* \setminus \{\frac{1}{2}k, \frac{1}{4}k\}$, define $\ell_c : E(G) \to \mathbb{Z}_k^*$ by $\ell_c(e_i) = f_c(e_i) + f_c(e_i')$ for all i, $1 \leq i \leq m$. Clearly, ℓ_c is a c-sum k-magic labeling of G for each $c \in \mathbb{Z}_k^* \setminus \{\frac{1}{2}k, \frac{1}{4}k\}$. By Remark 1.3, we see that $\mathbb{Z}_k^* \setminus \{\frac{1}{2}k\} \subset \Sigma_k(G)$.

By Theorem 1.1, G is 2-factorable. Let G_1 and G_2 be the two 2-factors of G. Label the edges in G_1 with d and the edges in G_2 with $\frac{1}{2}(k-d)$. This shows that $d = \frac{1}{2}k \in \Sigma_k(G)$.

Sub-Case 2.2. Let $k = 2d$, $d \geq 3$ odd. Observe that, for $c \neq 0, \frac{1}{2}k$, the labeling ℓ_c in Sub-Case 2.1 is a c-sum k-magic labeling of G. We are left to show that $\frac{1}{2}k \in \Sigma_k(G)$.

Let $d \neq 3$ and 9. We give a labeling for the factors of G' defined above (namely, H' and $G' \setminus H'$) and the 2-factors of G (namely, G_1 and G_2) to show that G is d-sum k-magic.

Let $f : E(G) \to \mathbb{Z}_k^*$ be defined by

$$f(e) = \begin{cases} d+1 & \text{if } e \in E(G_1) \\ \frac{1}{2}(k-d-1) & \text{if } e \in E(G_2). \end{cases}$$

Clearly, f is $(d+1)$-sum k-magic labeling of G.

Let $g' : E(G') \to \mathbb{Z}_k^*$ be defined by

$$g'(e) = \begin{cases} k-2 & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'). \end{cases}$$

Define also $g : E(G) \to \mathbb{Z}_k^*$ by $g(e_i) = g'(e_i) + g'(e_i')$ for all i, $1 \leq i \leq m$. Note that g' is a $(k-1)$-sum k-magic labeling of G', so g is a $(k-1)$-sum k-magic labeling of G.

Finally, define $\ell : E(G) \to \mathbb{Z}_k^*$ by $\ell(e) = f(e) + g(e)$ for all $e \in E(G)$. Since f and g are $(d+1)$-sum and $(k-1)$-sum k-magic labeling of G, respectively, ℓ is a d-sum k-magic labeling of G.

Suppose $d = 3$ or 9. Define $g' : E(G') \to \mathbb{Z}_k^*$ be defined by

$$g'(e) = \begin{cases} 2x & \text{if } e \in E(H') \\ 1 & \text{if } e \in E(G' \setminus H'), \end{cases}$$

where $x = 1$ if $d = 3$, and $x = 3$ if $d = 9$. Note that g' is a 5-sum k-magic labeling of G'. Define a labeling g on G by $g(e_i) = g'(e_i) + g'(e_i') + 1$ for all i, $1 \le i \le m$. Note that g is a d-sum k-magic labeling on G. Thus, $d = \frac{1}{2}k \in \Sigma_k(G)$, and so G is completely k-magic.

Suppose $r \ge 3$ is odd. By Theorem 1.1, G is r-factorable. By Theorem 2.5, the r-factors of G are completely k-magic for all $k \ge 5$. Thus, by Theorem 2.2, G is also completely k-magic.

If $r \ge 4$ is even, then, by Theorem 1.1, G has a 6-factor, say H. Using the case for r is odd, H is completely k-magic. Thus, by Theorem 2.2, G is also completely k-magic.

(2(i)) By Theorem 1.1, G is 2-factorable. Let $G_1, G_2, \ldots, G_r$ be the 2-factors of G. If k is odd, then, by Remark 2.1, $\mathbb{Z}_k^* \subseteq \sum_k(G_i)$ for all i, $1 \le i \le r$. For each i and $c \in \mathbb{Z}_k^*$, let ℓ_c^i be a c-sum k-magic labeling of G_i. We consider two cases.

Case 1. Suppose $r \equiv 1 \pmod k$. For each $c \in \mathbb{Z}_k^*$, define $\ell_c : E(G) \to \mathbb{Z}_k^*$ by

$$\ell_c(e) = \begin{cases} \ell_c^1(e) & \text{if } e \in E(G_1) \\ \ell_1^i(e) & \text{if } e \in E(G_i) \text{ for some } i = 2, 3, \ldots, r. \end{cases}$$

Note that ℓ_c is a c-sum k-magic labeling of G for all $c \ne 0$.

Case 2. Suppose $r \not\equiv 1 \pmod k$. For each $c \in \mathbb{Z}_k^* \setminus \{r - 1 \pmod k\}$, define $l_c : E(G) \to \mathbb{Z}_k^*$ by

$$l_c(e) = \begin{cases} l_{c-x}^1(e) & \text{if } e \in E(G_1) \\ l_1^i(e) & \text{if } e \in E(G_i) \text{ for some } i = 2, 3, \ldots, r, \end{cases}$$

where $x \equiv r - 1 \pmod k$. The sum of the labels of the edges incident to each vertex is $c \pmod k$. Thus, G is c-sum k-magic for each $c \ne x$. By Remark 1.3, G is x-sum k-magic since G is $(k - x)$-sum k-magic. In this case, G is completely k-magic.

(2(ii)) This follows from Remark 2.1, Lemma 1.8, and Theorem 2.2. □

The proof of the following theorems are similar to Theorem 2.5 and Theorem 2.6.

Theorem 2.7. *Let $r \ge 3$, and G a zero-sum 4-magic r-regular graph. Then*

(1) *If the order of G is even, then G is completely 4-magic.*
(2) *If the order of G is odd, then $\Sigma_4(G) = \{0, 2\}$.*

Theorem 2.8. *Let G be an r-regular graph, where $r \ge 3$.*

(1) *If $r \not\equiv 0 \pmod 3$ or $r \equiv 0 \pmod 6$, then G is completely 3-magic.*
(2) *If $r \equiv 0 \pmod 3$ and r odd, then G is completely 3-magic if and only if G has a factor H such that $d_H(v) \equiv 1 \pmod 3$ for all $v \in V(H)$.*

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