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MOMENT PROPERTIES OF ESTIMATORS FOR AN EXTREME VALUE REGRESSION MODEL WITH TYPE 2 CENSORING

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SUMMARY

An extreme value regression model for grouped data with type 2 censoring is considered. The response variable is taken to have a type 1 extreme value distribution for smallest values and a standard linear regression model is assumed for the means. Large sample approximations to the variances of the maximum likelihood estimators are derived. The small sample moment properties of the maximum likelihood estimators are evaluated by simulation for the case of simple linear regression. The results show that the estimator of the scale parameter has a strong bias in small samples, particularly when there is a heavy degree of censoring. Finally, small sample variance and mean square error efficiences of the best linear unbiased estimators relative to the maximum likelihood estimators are assessed.

CONTENTS

1. INTRODUCTION

We consider a regression model for grouped data in which there are g groups of individuals, the ith group containing n_i individuals. Let Y_{ij} be a random variable representing the response for the jth individual in the ith group. It is assumed that the $\{Y_{ij}\}$ are independently distributed and that $Y_{i_1},....,Y_{in_i}$ have the same type 1 extreme value (EV) distribution for smallest values with density

$$f_i(y)=\frac{1}{\theta}\exp\left\{\frac{y-\mu_i}{\theta}-\gamma-\exp\left(\frac{y-\mu_i}{\theta}-\gamma\right)\right\},\quad -\infty<y<\infty. \tag{1.1}$$

We have

$$E(Y_{ij})=\mu_i,\qquad \mathrm{var}(Y_{ij})=\frac{1}{6}\pi^2\theta^2 \tag{1.2}$$

for i = 1 , . . . ,g, j = 1, . . . , n_i. The individuals in the ith group have the same values $X_{i_1},....,X_{ik}$ for k repressor variates and a linear model for the mean μ_i is assumed with

$$\mu_i = \underset{\sim}{X}_i'\underset{\sim}{\beta}\qquad\qquad i=1,....,g \tag{1.3}$$

where $X_i' = (1,X_{i_1},s....,X_{ik})$ and $\beta' = (\beta_0,\beta_1,...,\beta_k)$ is a vector of regression coefficients with unknown values.

A common application for the extreme value regression model occurs in life-testing when the response variable represents the logarithm of the time to failure. Right censoring of the observations is common in such cases because of the need for early termination of the investigation. Several forms of censoring are possible. Here we shall consider type 2 censoring within groups. For the ith group, we suppose that the r_i. smallest observations denoted by $Y_{i(1)}<Y_{i(2)}<...<Y_{i(r_i)}$ are observed, the remaining n_i-r_i observations being right censored at the value $Y_{i(r_i)}$. The $\{r_i\}$ are fixed integers satisfying $1\le r_i \le n_i$. We let $R = \sum_i r_i$ denote the total number of uncensored observations.

The most commonly used method of estimation for the regression coefficient vector $\underset{\sim}{\beta}$ and scale parameter θ is maximum likelihood, which is described in section 2. The large sample variances of the maximum likelihood estimators are given in section 3, together with the small sample moment properties as estimated by simulation for the case of simple linear regression. The results show that the bias of the estimator of the scale parameter θ is large when there is a heavy degree of censoring within the groups, indicating that bias correction will be necessary in statistical inference procedures for $\underset{\sim}{\beta}$ and θ. With type 2 censoring, best linear unbiased estimation based on the ordered observations available within the groups provides an alternative to maximum likelihood. The variances of the best linear unbiased estimators are given in section 5. Finally in section 6 the small sample variance and mean square error efficiencies of the best linear unbiased estimators relative to the maximum likelihood estimators are evaluated for the case of simple linear regression.

2. MAXIMUM LIKELIHOOD ESTIMATION

The likelihood for the ith group is

$$\ell_i = \{n_i!/(n_i - r_i)!\} \prod_{j=1}^{r_i} f_i(y_{i(j)} \quad \{1 - F_i(y_{i(r_i)}\}^{n_i - r_i} \tag{2.1}$$

where

$$F_i(Y) = 1 - \exp\left[-\exp\left\{\theta^{-1}(y - \underset{\sim}{X}_i' \underset{\sim}{\beta}) - \gamma\right\}\right], \quad -\infty < y < \infty \tag{2.2}$$

is the c.d.f. for the ith group. Set

$$Z_{i(j)} = \theta^{-1}(y_{i(j)} - \underset{\sim}{X}_i' \underset{\sim}{\beta}) - \gamma, \quad i = 1, \ldots, g, \quad j = 1, \ldots, r_i \tag{2.3}$$

$$V_i^{(a)} = \sum_{j=1}^{r_i} (z_{i(j)} + \gamma)^a e^{Z_{i(j)}} + (n_i - r)(z_{i(r_i)} + \gamma)^a e^{Z_i(r_i)}, \quad i = 1, \ldots, g. \tag{2.4}$$

The log-likelihood $L = \sum_i \log \ell_i$ over all groups is

$$L = c - R\log\theta + \sum_i \left\{ \sum_{j=1}^{r_i} Z_{i(j)} - V_i^{(0)} \right\} \tag{2.5}$$

where $C = \sum_i \log\{n_i!/(n_i - r_i)!\}$. using

$$\frac{\partial z_{i(j)}}{\partial \beta_s} = -\frac{X_{is}}{\theta}, \qquad \frac{\partial z_{i(j)}}{\partial \theta} = -\frac{(z_{i(j)} + \gamma)}{\theta} \tag{2.6}$$

we obtain

$$\frac{\partial L}{\partial \beta_s} = \theta^{-1} \sum_i X_{is}(V_i^{(0)} - r_i), \quad s = 0, 1, \ldots, k \tag{2.7}$$

$$\frac{\partial L}{\partial \theta} = \theta^{-1}\left\{\sum_i V_i^{(1)} - \sum_i \sum_j Z_{i(j)} - R(\gamma+1)\right\}. \tag{2.8}$$

The likelihood equations are

$$\sum_i X_{is}\hat{V}_i^{(0)} = \sum_i r_i x_{is}, \qquad s = 0,1,.....,k \tag{2.9}$$

$$\sum_i \hat{V}_i^{(1)} - \sum_i \sum_j \hat{Z}_{i(j)} = R(\gamma+1) \tag{2.10}$$

where $\hat{V}_i^{(a)}$ and $\hat{Z}_{i(j)}$ denote the values of $V_i^{(a)}$ and $Z_{i(j)}$, respectively, evaluated at $\underset{\sim}{\beta} = \hat{\underset{\sim}{\beta}}$ and $\theta = \hat{\theta}$.

The second order derivatives of the log-likelihood are

$$\frac{\partial^2 L}{\partial \beta_s \partial \beta_t} = -\theta^{-2}\sum_i X_{is}X_{it}V_i^{(0)}, \qquad s,t = 0,1, ...,k \tag{2.11}$$

$$\frac{\partial^2 L}{\partial \beta_s \partial \theta} = -\theta^{-2}\sum_i X_{is}(V_i^{(1)} + V_i^{(0)} - r_i), \;\; s = 0,1,...,k \tag{2.12}$$

$$\frac{\partial^2 L}{\partial \theta^2} = -\theta^{-2}\left\{\sum_i (V_i^{(2)} + 2V_i^{(1)}) - 2\sum_i \sum_j Z_{i(j)} - R(1+2\gamma)\right\}. \tag{2.13}$$

The solution of the likelihood equations (2.9) and (2.10) can be found by the Newton-Raphson method using the (k+2)×(k+2) observed information matrix

$$\underset{\sim}{I}_0 = -\begin{pmatrix} \frac{\partial^2 L}{\partial \beta_0^2} & \frac{\partial^2 L}{\partial \beta_0 \partial \beta_1} & \cdots & \frac{\partial^2 L}{\partial \beta_0 \partial \beta_k} & \frac{\partial^2 L}{\partial \beta_0 \partial \theta} \\ \frac{\partial^2 L}{\partial \beta_1 \partial \beta_0} & \frac{\partial^2 L}{\partial \beta_1^2} & \cdots & \frac{\partial^2 L}{\partial \beta_1 \partial \beta_k} & \frac{\partial^2 L}{\partial \beta_1 \partial \theta} \\ \frac{\partial^2 L}{\partial \theta \partial \beta_0} & \frac{\partial^2 L}{\partial \theta \partial \beta_1} & \cdots & \frac{\partial^2 L}{\partial \theta \partial \beta_k} & \frac{\partial^2 L}{\partial \theta^2} \end{pmatrix}_{\substack{\underset{\sim}{\beta} = \hat{\underset{\sim}{\beta}} \\ \theta = \hat{\theta}}} \tag{2.14}$$

with the appropriate partial derivatives given by (2.11), (2.12), (2.13). $\underset{\sim}{I}_0$ can be simplified slightly using the likelihood equations. For example, consider the important case of a single regressor variate, where $\underset{\sim}{X}_i' \underset{\sim}{\beta} = \beta_0 + \beta_1 X_i$. We then have

$$\underset{\sim}{I}_0 = \hat{\theta}^{-2}\begin{bmatrix} R & \sum_i r_i X_i & \sum_i \hat{V}_i^{(1)} \\ \sum_i r_i X_i & \sum_i X_i^2 \hat{V}_i^{(0)} & \sum_i X_i \hat{V}_i^{(1)} \\ \sum_i \hat{V}_i^{(1)} & \sum_i X_i \hat{V}_i^{(1)} & \sum_i \hat{V}_i^{(2)} + R \end{bmatrix} \tag{2.15}$$

An alternative method to Newton-Raphson to obtain the ML estimates was proposed by Roger and Peacock (1982), for implementaton with GLIM. The justification for their method is as follows. Put

$$Y_{i(j)} = Y_{i(r_i)}, \; Z_{i(j)} = Z_{i(r_i)} \quad \text{for} \quad j = r_i, r_i+1, \ldots.., n_i \tag{2.16}$$

and let

$$\delta_{ij} = \begin{cases} 1 & \text{for} \quad j = 1, \ldots., r_i \\ 0 & \text{for} \quad j = r_i + 1, \ldots., n_i. \end{cases} \tag{2.17}$$

From (2.5) the log-likelihood may be written as

$$L = c - R\log\theta + \sum_{i=1}^{g} \sum_{j=1}^{n_i} \left\{\delta_{ij} Z_{i(j)} - \exp(Z_{i(j)})\right\}. \tag{2.18}$$

Put $\alpha = \theta^{-1}$ and

$$m_{ij} = \exp(Z_{i(j)}) \qquad , \qquad i = 1, \ldots, \; j = 1, \ldots, n_i \, . \tag{2.19}$$

We may write

$$\log m_{ij} = \alpha y_{i(j)} + \underset{\sim}{X}_i' \underset{\sim}{\beta}^* \tag{2.20}$$

Where

$$\underset{\sim}{\beta}^{*\prime} = \{-(\alpha\alpha_0 + \gamma), -\alpha\beta_1, \ldots., -\alpha\beta_k\} \; . \tag{2.21}$$

From (2.18) we have

$$L = c + R \log \alpha + \sum_{i=1}^{g} \sum_{i=1}^{n_i} (\delta_{ij} \log m_{ij} - m_{ij}) \; . \tag{2.22}$$

Ignoring constants, this expression is equivalent to the log-likelihood that would be obtained for realised values $\{\delta_{ij}\}$ for independent Poisson random variables with means m_{ij} satisfying (2,20) and a realised value R for an independent binomial random variable based on R trials with trial success probability α. Roger and Peacock give a computer program using the user defined fitted model facilities in GLIM from which the values of α and $\underset{\sim}{\beta}^*$ which maximise (2.22) can be found. The required ML estimates are then given by

$$\hat{\theta} = \hat{\alpha}^{-1}\ , \quad \hat{\beta}_0 = (\hat{\beta}_0^* + \gamma)\hat{\theta}\ , \quad \hat{\beta}_s = -\hat{\theta}\hat{\beta}_s^*\ , \quad s = 1,\ldots,k\ . \tag{2.23}$$

3. MOMENT PROPERTIES OF THE ML ESTIMATORS

The maximum likelihood estimators are asymptotically unbiased with asymptotic covariance matrix given by the inverse of the expected information matrix. To obtain the elements in this matrix, we require expressions for $E(Z_{j,n})$ and

$$E_{j,n}^{(a)} = E\{Z_{j,n}^a \exp(Z_{j,n})\}, \qquad a = 0,1,2 \tag{3.1}$$

where $Z_{j,n}$ denotes the jth order statistic in a sample of n observations from the standard extreme value distribution with p.d.f. $f(z) = \exp(z-e^z)$, $-\infty < Z < \infty$.

From order statistic theory, the p.d.f. of $Z_{j,n}$ is

$$\begin{aligned} g_{j,n}(z) &= \frac{n!}{(j-1)!(n-j)!}\{F(z)\}^{j-1}\{1-F(z)\}^{n-j}f(z) \\ &= \frac{n!}{(j-1)!(n-j)!}\sum_{u=0}^{j-1}(-1)^u\binom{j-1}{u}\exp\{z-(n-j+u+1)e^z\}\ . \end{aligned} \tag{3.2}$$

The moment generating function of the distribution of $Z_{j,n}$ is

$$\begin{aligned} M_{j,n}(t) &= \int_{-\infty}^{\infty}\exp(tz)g_{j,n}(z)dz \\ &= \frac{n!}{(j-1)!(n-j)!}\sum_{u=0}^{j-1}(-1)^u\binom{j-1}{u}\frac{\Gamma(t+1)}{(n-j+u+1)^{t+1}}\ . \end{aligned} \tag{3.3}$$

Since

$$E_{j,n}^{(a)} = \left\{\frac{d^{(a)}M_{j,n}(t)}{dt}\right\}_{t=1} \tag{3.4}$$

we obtain

$$E_{j,n}^{(0)} = \frac{n!}{(j-1)!(n-j)!}\sum_{u=0}^{j-1}(-1)^u\binom{j-1}{u}\frac{1}{(n-j+u+1)^2} \tag{3.5}$$

$$E_{j,n}^{(2)} = \frac{n!}{(j-1)!(n-j)!}\sum_{u=0}^{j-1}(-1)^u\binom{j-1}{u}\frac{\Gamma'(2)-\log(n-j+u+1)}{(n-j+u+1)^2} \tag{3.6}$$

$$E_{j,n}^{(2)} = \frac{n!}{(j-1)!(n-j)!}\sum_{u=0}^{j-1}(-1)^u\binom{j-1}{u}\left\{\frac{\Gamma''(2)-2\Gamma'(2)\log(n-j+u+1)+\log^2(n-j+u+1)}{(n-j+u+1)^2}\right\} \tag{3.7}$$

Also

$$E(Z_{j,n}) = M'_{j,n}(0) = \Gamma'(1) - \frac{n!}{(j-1)!(n-j)!}\sum_{u=0}^{j-1}(-1)^u\binom{j-1}{u}\frac{\log(n-j+u+1)}{n-j+u+1}\,. \tag{3.8}$$

We have

$$E(V_i^{(0)}) = \sum_{j=1}^{\Gamma_i} E_{j,n_i}^{(0)} + (n_i - r_i)E_{r_i,n_i}^{(0)} \tag{3.9}$$

$$E(V_i^{(1)}) = \sum_{j=1}^{r_i}(E_{j,n_i}^{(1)} + \gamma E_{j,n_i}^{(0)})(n_i - r_i)(E_{r_i,n_i}^{(1)} + \gamma E_{r_i,n_i}^{(0)}) \tag{3.10}$$

$$E(V_i^{(2)}) = \sum_{j=1}^{\Gamma_i}(E_{j,n_i}^{(2)} + 2\gamma E_{j,n_i}^{(1)} + \gamma^2 E_{j,n_i}^{(0)}) + (n_i - r_i)(E_{r_i,n_i}^{(2)} + 2\gamma E_{r_i,n_i}^{(1)} + \gamma^2 E_{r_i,n_i}^{(0)}) \tag{3.11}$$

which may be used to find the expectations of the negative values of the second derivatives of the log-likelihood given by (2.11), (2.12) and (2.13). The value of $E(Z_{i(j)})$ required in (2.13) is given by (3.8) with $n = n_i$.

In order to examine the moment properties of the ML estimators, a Monte Carlo simulation study was made for the case of a single explanatory variable with grouped data, the model without censoring being

$$Y_{ij} = \beta_0 + \beta_1 X_i + \in_{ij} \quad , \quad i = 1,\dots.. \quad , \quad j = 1,\dots.,m_i \tag{3.12}$$

where $E\{\varepsilon_{ij}\} = 0$, $\text{var}\{\varepsilon_{ij}\} = \frac{1}{6}\pi^2\theta^2$ and the $\{Y_{ij}\}$ are independently distributed with p.d.f. for Y_{ij} given by

$$f_{ij}(y) = \frac{1}{\theta}\exp\left\{\frac{y-\beta_0-\beta_1 X_i}{\theta} - \gamma - \exp\left(\frac{y-\beta_0-\beta_1 X_i}{\theta} - \gamma\right)\right\}. \qquad (3.13)$$

Equally spaced values of x were used with $x_i = i - \frac{1}{2}(g+1)$, $i = 1,\dots,g$. Equal sample sizes $m_i = m = 5, 10, 20$ were used with $g = 5, 10$ and equal censoring proportions $p = 0.0, 0.20, 0.40, 0.60$ were applied in each group. Without loss of generality, the y-observations were generated putting $\beta_0 = \beta_1 = 0$ and $\theta = 1$ in the regression model. The ML estimates were obtained using a GLIM program based on the Roger/Peacock method. A run-size of 2000 was used in each case.

Values of the biases, variances and skewness coefficients of the ML estimators are shown in tables 1, 2 and 3 for β_0, β_1 and θ respectively. The approximating variances given by the diagonal elements in the inverse of the expected information matrix are shown in parentheses. The main findings are as follows.

a) For estimation of β_0, the bias of the ML estimator was negligible when no censoring was present. With censoring there was a negative bias which became more pronounced as the degree of censoring increased. The large sample variance approximations obtained from the inverse of the information matrix gave good agreement with the simulation variances, although there was a slight underestimation when there was a heavy degree of censoring. The skewness coefficients were positive but small in all cases showing that the distribution of $\hat{\beta}_0$ was almost symmetrical.

b) For estimation of β_1, the biases of the ML estimators were negligible in all cases. The large sample variance approximations gave slightly smaller variances than obtained by simulation, but the differences were generally very small. The skewness coefficients were all close to zero.

c) For estimation of θ, the negative bias of $\hat{\theta}$ for the uncensored case became more pronounced as the degree of censoring increased. The large sample approximating variances gave higher values than the simulation variances, particularly when there was heavy censoring. The skewness coefficients for $\hat{\theta}$ increased with the degree of censoring, but in general the skewness was small.

Table 1

Biases × $10^2\ \theta^{-1}$, variances × $10^2\ \theta^{-2}$ and skewness coefficients of the ML estimates of the β_0

	p	Bias	Variance	Skewness
	0.0	0.54	6.918(6.728)	0.028
m=5	0.2	-1.13	7.128(6.790)	0.033
g=5	0.4	-4.46	8.015(7.318)	0.048
	0.6	-13.47	11.729(10.021)	0.006
	0.0	0.10	3.186(3.364)	0.031
m=10	0.2	-0.68	3.196(3.371)	0.029
g=5	0.4	-2.48	3.553(3.599)	0.047
	0.6	-7.38	5.409(5.077)	0.052
	0.0	0.53	1.571(1.644)	0.025
m=20	0.2	0.17	1.583(1.565)	0.023
g=5	0.4	-0.72	1.712(1.753)	0.017
	0.6	-3.10	2.571(2.588)	0.034
	0.0	0.04	3.362(3.364)	0.034
m=5	0.2	-0.71	3.439(3.395)	0.038
g=10	0.4	-2.37	3.815(3.659)	0.036
	0.6	-6.40	5.547(5.010)	0.007
	0.0	0.08	1.687(1.682)	0.007
m=10	0.2	-0.27	1.706(1.686)	0.006
g=10	0.4	-1.15	1.855(1.799)	0.009
	0.6	-3.43	2.807(2.539)	0.017
	0.0	0.31	0.756(0.822)	0.027
m=20	0.2	0.15	0.760(0.782)	0.031
g=10	0.4	-0-31	0.834(0.876)	0.032
	0.6	-1.67	1.277(1.294)	0.051

Table 2

Biases × $10^2\ \theta^{-1}$, variances × $10^2\ \theta^{-2}$ and skewness coefficients of the ML estimates of β_1

	P	Bia	Variance	Skewness
	0.0	-0.11	2.151(2.000)	0.060
m=5	0.2	-0.13	2.757(2.500)	0.025
g =5	0.4	0.37	3.746(3.334)	0.030
	0.6	1.00	5.814(5.000)	0.019
	0.0	-0.09	1.061(1.000)	0.001
m=10	0.2	0.03	1.303(1.250)	0.011
g=5	0.4	-0.04	1.799(1.666)	0.003
	0.6	-0.08	2.929(2.500)	0.017
	0.0	0.04	0.521(0.502)	0.003
m=20	0.2	0.01	0.664(0.624)	0.002
g=5	0.4	0.09	0.869(0.821)	0.006
	0.6	0.23	1.316(1.253)	0.002
	0.0	-0.03	0.260(0.242)	0.001
m=5	0.2	0.02	0.336(0.303)	0.000
g=10	0.4	-0.05	0.452(0.404)	0.007
	0.6	-0.08	0.715(0.606)	0.019
	0.0	0.12	0.133(0.121)	0.007
m=10	0.2	0.24	0.158(0.152)	0.015
g=10	0.4	0.19	0.217(0.202)	0.030
	0.6	0.15	0.324(0.303)	0.035
	0.0	0.04	0.063(0.060)	0.010
m=20	0.2	0.02	0.078(0.076)	0.003
g=10	0.4	-0.05	0.102(0.100)	0.005
	0.6	0.00	0.153(0.152)	0.002

Table 3

Biases × $10^2\ \theta^{-1}$, variances × $10^2\ \theta^{-2}$ and skewness coefficients of the ML estimates of θ

	P	Bias	Variance	Skewness
	0.0	-5.69	2.433(2.728)	0.093
m=5	0.2	-8.20	3.389(3.834)	0.140
g =5	0.4	-10.97	4.580(5.335)	0.186
	0.6	-16.49	6.689(7.803)	0.276
	0.0	-2.52	1.208(1.364)	0.005
m=10	0.2	-3.65	1.776(2.008)	0.068
g=5	0.4	-5.27	2.461(2.886)	0.041
	0.6	-8.42	3.498(4.392)	0.069
	0.0	-1.73	0.584(0.651)	0.011
m=20	0.2	-2.17	0.894(0.765)	0.010
g=5	0.4	-2.87	1.335(1.325)	0.024
	0.6	-4.37	2.087(2.523)	0.069
	0.0	-2.83	1.275(1.364)	0.085
m=5	0.2	-3.96	1.772(1.917)	0.079
g=10	0.4	-5.34	2.385(2.668)	0.095
	0.6	-7.70	3.432(3.902)	0.092
	0.0	-1.13	0.612(0.682)	0.018
m=10	0.2	-1.62	0.890(1.004)	0.036
g=10	0.4	-2.36	1.252(1.443)	0.039
	0.6	-3.75	1.862(2.196)	0.049
	0.0	-0.93	0.303(0.326)	0.009
m=20	0.2	-1.16	0.456(0.397)	0.009
g=10	0.4	-1.53	0.647(0.663)	0.013
	0.6	-2.47	1.028(1.261)	0.003

Since the biases of $\hat{\theta}$ and $\hat{\beta}_0$ become increasingly marked as the degree of censoring increases, approximations to the biases, denoted by $b_{\hat{\theta}}(n,p)$ and $b_0(n,p)$ respectively, are required. Preliminary plots for the biases against p suggested the use of the quadratic models

$$b_{\hat{\theta}}(n,p) = b_{\hat{\theta}}(n,0)(1 + a_1 p + a_2 p^2) \qquad (3,14)$$

$$b_0(n,p) = b_0(n,0)(1 + a_1' p + a_2' p^2) \qquad (3.15)$$

where from Young and Haddow (1985)

$$b_{\hat{\theta}}(n,0) = -\frac{1.3794\theta}{g^n}, \quad b_0(n,0) = -\frac{0.1834\theta}{g^n}. \qquad (3.16)$$

Least squares fits of the models using the biases obtained by simulation gave the coefficient estimates

$a_1 = 1.099$, $a_2 = 3.800$, $a_1' = 8.908$, $a_2' = -67.700$.

The approximating formulae for the biases are therefore

$$b_{\hat{\theta}}(n,p) = -(gn)^{-1}\theta(1.3794 + 1.516p + 5.24p^2) \qquad (3.17)$$

$$b_0(n,p) = (gn)^{-1}\theta(0.1834 + 1.634p - 12.42p^2) \qquad (3.18)$$

Values of the approximate biases of $\hat{\theta}$ given by (3.17) are shown in table 4 together with the simulation estimates. The agreement between the values is satisfactory for all values of p and m.

Table 4

Values of $10^2\theta^{-1}b_{\hat{\theta}}$ given by (i) approximation (3,17), (ii) Simulation estimate

g =5

	m = 5		m = 10		m = 20	
p	(i)	(ii)	(i)	(ii)	(i)	(ii)
0.0	-5.52	-5.69	-2.76	-2.52	-1.38	-1.73
0.2	-7.57	-8.20	-3.78	-3.65	-1.89	-2.17
0.4	-11.30	-10.97	-5.65	-5.27	-2.82	-2.87
0.6	-16.70	-16.49	-8.35	-8.42	-4.18	-4.37

g=10

	m = 5		m = 10		m = 20	
P	(i)	(ii)	(i)	(ii)	(i)	(ii)
0.0	-2.76	-2.83	-1.38	-1.13	-0.69	-0.93
0.2	-3.78	-3.96	-1.89	-1.62	-0.95	-1.16
0.4	-5.65	-5.34	-2.82	-2.36	-1.41	-1.53
0.6	-8.35	-7.70	-4.18	-2.47	-2.09	-2.47

4. Moment Properties Of The Best Linear Unbiased Estimators Based On The Within Group Order Statistics

With type II censoring within groups, an alternative to ML estimation is to use best linear unbiased estimation. We outline the procedure. Full details are given by Young and Haddow (1985).

Let $e_{j,m} = E(X_{(j)})$ and $c_{j,j',m} = \text{cov}(X_{(j)}, X_{(j')})$, where $X_{(j)}$ denotes the j th order statistic in a sample of m from the standardised type 1 EV distribution with c.d.f. $F(x) = 1-\exp\{-\exp(x)\}$, $-\infty < X < \infty$. Put

$$\underset{\sim}{A} = \begin{bmatrix} 1 & e_{1,m} \\ 1 & e_{2,m} \\ . & . \\ . & . \\ 1 & e_{r,m} \end{bmatrix}, \qquad \underset{\sim}{C} = \begin{bmatrix} c_{11,m} & c_{12,m} & \cdots & c_{1r,m} \\ c_{21,m} & c_{22,m} & \cdots & c_{2r,m} \\ . & & & \\ . & & & \\ c_{r1,m} & c_{r2,m} & \cdots & c_{rr,m} \end{bmatrix} \tag{4.1}$$

and let a_j (r, m), b_j (r, m) denote the elements in the first and second rows of the 2×r matrix $(\underset{\sim}{A}'\underset{\sim}{C}^{-1}\underset{\sim}{A})^{-1}\underset{\sim}{A}'\underset{\sim}{C}^{-1}$ Write

$$\underset{\sim}{A}'\underset{\sim}{C}^{-1}\underset{\sim}{A} = \begin{bmatrix} V_{r,m}^{(1)} & V_{r,m}^{(2)} \\ V_{r,m}^{(2)} & V_{r,m}^{(3)} \end{bmatrix} \tag{4.2}$$

Values of the expectations $\{e_{j,m}\}$, covariances $\{c_{jj,m}\}$, linear co-efficients $\{a_j(r,m)\}$, $\{b_j(r,m)\}$ and the elements $V_{r,m}^{(1)}, V_{r,m}^{(2)}, V_{r,m}^{(3)}$ are given by White (1964) for sample sizes $m \leq 20$ and for $2 \leq r \leq m$. Putting $\xi_i = \mu_i + \gamma\theta$, the BLUE' s for ξ_i and θ based on the order statistics $Y_{i(1)} < Y_{i(2)} < \ldots < Y_{i(r_i)}$ in the ith groups are

$$\hat{\xi}_i = \sum_{j=1}^{r_i} a_j(r_i, m_i) Y_{i(j)}, \quad \hat{\theta}_i = \sum_{j=1}^{r_i} b_j(r_i, m_i) Y_{i(j)} \tag{4.3}$$

Pooling the estimates for θ over all groups, the minimum variance linear unbiased estimator for θ is

$$\hat{\theta}_* = \frac{\sum_{i=1}^{g} \hat{\theta}_i / V_{r_i,m_i}^{(3)}}{\sum_{i=1}^{g} (1/V_{r_i,m_i}^{(3)})} = \sum_{i=1}^{g} W_i \hat{\theta}_i \quad \text{say.} \tag{4.4}$$

Set $\hat{\mu}_{*i} = \hat{\xi}_i - \gamma\hat{\theta}_*$ and $\hat{\underset{\sim *}{\mu}}' = (\hat{\mu}_{*1}, \ldots. \hat{\mu}_{*g})$. Based on the $\{\hat{\mu}_{*i}\}$ the BLUE of $\underset{\sim}{\beta}$ is

$$\hat{\underset{\sim}{\beta}}_* = (\underset{\sim}{X}_1' \underset{\sim}{W}^{-1} \underset{\sim}{X}_1)^{-1} \underset{\sim}{X}_1' \underset{\sim}{W}^{-1} \hat{\underset{\sim}{\mu}}_* \tag{4.5}$$

where

$$\underset{\sim}{X}_1 = \begin{bmatrix} 1 & X_{11} & \ldots & X_{1k} \\ 1 & X_{21} & \ldots & X_{2k} \\ . & . & & \\ . & . & & \\ 1 & X_{g1} & \ldots & X_{gk} \end{bmatrix}, \quad \underset{\sim}{W} = \begin{bmatrix} W_{11} & W_{11} & \ldots & W_{1g} \\ W_{21} & W_{22} & \ldots & W_{2g} \\ . & & & \\ . & & & \\ W_{g1} & W_{g2} & \ldots & W_{gg} \end{bmatrix} \tag{4.6}$$

and

$$W_{ij} = \frac{\left\{\gamma^2 - \gamma(V^{(2)}_{r_i,m_i}/V^{(3)}_{r_i,m_i} + V^{(2)}_{r_j,m_j}/V^{(3)}_{r_j,m_j}\right\}}{\sum_i \left(1/V^{(3)}_{r_i,m_i}\right)}. \tag{4.7}$$

The variance of $\hat{\theta}_*$ is

$$\mathrm{var}(\hat{\theta}_*) = \theta^2 \left\{ \sum_i \left(1/V^{(3)}_{r_i,m_i}\right) \right\}^{-1} \tag{4.8}$$

and the covariance matrix for $\hat{\underset{\sim}{\beta}}_*$ is

$$\mathrm{cov}(\hat{\underset{\sim}{\beta}}_*) = (\underset{\sim}{X}_1' \underset{\sim}{W}^{-1} \underset{\sim}{X}_1)^{-1} \theta^2. \tag{4.9}$$

Values of the exact variances ($\times 10^2\ \theta^{-2}$)of the BLUE's $\hat{\theta}_*$, $\hat{\beta}_{*0}$ and $\hat{\beta}_{*1}$ have been computed using (4.8) and (4.9) for the simple linear regression model used for the study of the ML estimators (see section 3). The results are shown in table 5.

Table 5

Exact variances ($\times 10^2\ \theta^{-2}$)of the BLUE's $\hat{\beta}_{*0}, \hat{\beta}_{*1}, \hat{\theta}_*$ for simple linear regression with type II censoring.

		g = 5			g = 10		
	P	m = 5	m=10	m=20	m =5	m =10	m = 20
	0.0	6.523	3.244	1.616	3.262	1.622	0.808
$\hat{\beta}_{*0}$	0.2	6.637	3.276	1.628	3.318	1.638	0.814
	0.4	7.931	3.696	1.797	3.966	1.848	0.899
	0.6	18.392	6.563	2.900	9.196	3.281	1.450
	0.0	2.314	1.133	0.559	0.281	0.137	0.068
	0.2	2.918	1.341	0.647	0.354	0.163	0.078
$\hat{\beta}_{*1}$	0.4	5.294	2.144	0.981	0.642	0.260	0.119
	0.6	17.892	5.593	2.320	2.169	0.678	0.281
	0.0	3.333	1.432	0.663	1.667	0.716	0.331
$\hat{\theta}_*$	0.2	5.076	2.150	0.997	2.538	1.075	0.498
	0.4	8.336	3.315	1.502	4.168	1.658	0.751
	0.6	17.901	5.951	2.547	8.951	2.975	1.273

5. Small Sample Efficiency Results

We let

$$E_{1\theta} = \frac{\mathrm{var}(\hat{\theta})}{\mathrm{var}(\hat{\theta}_*)}, \qquad E_{1r} = \frac{\mathrm{var}(\hat{\beta}_r)}{\mathrm{var}(\hat{\beta}_{*r})}, \qquad r = 1,....,k \tag{5.1}$$

denote the variance efficiencies of the BLUE's relative to the ML estimators. The corresponding mean square error efficiencies are denoted by

$$E_{2\theta} = \frac{\mathrm{mse}(\hat{\theta})}{\mathrm{mse}(\hat{\theta}_*)}, \qquad E_{2r} = \frac{\mathrm{mse}(\hat{\beta}_r)}{\mathrm{mse}(\hat{\beta}_{*r})}, \; r = 1,....,k. \tag{5.2}$$

Values of the small sample efficiencies have been estimated for the extreme value simple linear regression model described in section 3, and are shown in table 6. The exact variances and mean square errors of the BLUE's are equal and given in table 5. The small sample variances and mean square errors for the ML estimates were based on the simulation estimates given in tables 1, 2 and 3 for the biases and variances.

The general findings are as follows

a) For estimation of β_0, the efficiency of BLUE relative to ML is high and changes only marginally for $0 \leq p \leq 0.4$. At the highest level of censoring ($p = 0.6$), the efficiency drops markedly.

b) For estimation of β_1, the differences between the variance and mean square error efficiencies are negligible. The efficiencies increase marginally as p changes from 0.00 to 0.20, but higher values of p lead to marked decreases in the efficiencies.

c) For estimation of θ, the mean square error efficiency of BLUE relative to ML is appreciably higher than the variance efficiency, particularly at the higher levels of censoring. Both variance and mean square error efficiencies decrease rapidly when p is large

Table 6

Variance and mean square error efficiencies of BLUE's relative to ML estimators for extreme value simple linear regression with censoring

	P	E_{10}	E_{20}	E_{11}	E_{21}	$E_{1\theta}$	$E_{2\theta}$
	0.0	1.06	1.06	0.93	0.93	0.73	0.83
m=5	0.2	1.07	1.08	0.95	0.95	0.67	0.80
g =5	0.4	1.01	1.04	0.71	0.71	0.55	0.69
	0.6	0.64	0.74	0.33	0.33	0.37	0.53
	0.0	0.98	0.98	0.94	0.94	0.84	0.90
m=10	0.2	0.98	0.98	0.97	0.97	0.83	0.89
g =5	0.4	0.96	0.98	0.84	0.84	0.74	0.83
	0.6	0.82	0.91	0.52	0.52	0.59	0.71
	0.0	0.97	0.97	0.93	0.93	0.88	0.93
m=20	0.2	0.97	0.97	1.03	1.03	0.90	0.94
g =5	0.4	0.95	0.96	0.89	0.89	0.89	0.94
	0.6	0.89	0.92	0.57	0.57	0.82	0.90
	0.0	1.03	1.03	0.92	0.92	0.77	0.82
m=5	0.2	1.04	1.05	0.93	0.93	0.70	0.76
g=10	0.4	0.96	0.98	0.70	0.70	0.57	0.64
	0.6	0.60	0.65	0.33	0.33	0.38	0.45
	0.0	1.04	1.04	0.97	0.98	0.86	0.87
m=10	0.2	1.04	1.04	0.97	0.98	0.83	0.85
g=10	0.4	1.00	1.01	0.83	0.84	0.76	0.79
	0.6	0.86	0.89	0.48	0.48	0.63	0.68
	0.0	0.94	0.94	0.92	0.92	0.91	0.94
m=20	0.2	0.93	0.93	1.00	1.00	0.92	0.94
g =10	0.4	0.93	0.93	0.86	0.86	0.86	0.89
	0.6	0.88	0.90	0.54	0.54	0.81	0.86

References

Haddow, A.A. and Young, D.H. (1985). Moment properties of estimators for a type 1 extreme value regression model. Technical Report TR/15/85. Brunel University.

Lawless, J.F. (1982). Statistical Models and Methods for Lifetime Data. New York, Wiley.

Roger, J.H. and Peacock, S,D. (1982). Fitting the scale as a glim parameter for Weibull Extreme-value, Logistic and Log-logistic regression models with censored data. GLIM Newsletter, 6, 30-37.

White, J.S. (1964). Least squares unbiased censored linear estimation for the log Weibull (Extreme Value) distribution. J. Industrial Math. Soc., 14, 21-60.