THEORETICAL STUDIES ON THE DESIGN OF SCHOOL CHOICE MECHANISMS

Nickesha Ayoade

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By: Nickesha Ayoade

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Signed by the final examining committee:

____Chair
Dr. Kregg Hetherington

____ External Examiner
Dr. Lars Ehlers

____ External to Program
Dr. Joel Bothello

____ Examiner
Dr. Ming Li

____ Examiner
Dr. Dipjyoti Majumdar

____Thesis Supervisor
Dr. Szilvia Papai

Approved by ____
Dr. Christian Sigouin, Graduate Program Director

January 23, 2020 ____
Dr. André Roy, Dean
Faculty of Arts and Science

Abstract

Theoretical Studies On The Design Of School Choice Mechanisms

Nickesha Ayoade, Ph.D.

Concordia University, 2020

This thesis consists of three papers on market design which address broadly applicable questions on the design of school choice mechanisms, refugee placement, assignment in entry-level labour markets and similar matching rules.

In the first paper a new family of rules is introduced for many-to-one matching problems, the Preference Rank Partitioned (PRP) rules. PRP rules are Student-Proposing Deferred Acceptance rules where the schools use a choice function based on the students' preference orderings in addition to the schools' strict priority orderings. Each PRP rule uses a choice function which is a function of a fixed partition of both student preference ranks and school priority ranks: the choice function first seeks to select students based on the priority classes and then based on the preference classes. The strict priorities are only used for tie-breaking. PRP rules include many well-known matching rules and some interesting new rules, and we analyze them in this unified framework.

In the second paper we study a new class of matching rules, called Deferred Acceptance with Improvement Trading Cycles (DA-ITC), which start with the DA, and if the DA outcome is not Pareto-efficient then there is an iterated improvement trading cycle phase which allows for Pareto-improvements until a Pareto-efficient outcome is reached. We first revisit EADAM (Kesten, 2010) and show that a simple algorithm which retraces cycles in the DA procedure in a backward order of the rejections is equivalent to the EADAM rule. The new class of DA-ITC rules contains the EADAM and DA-TTC as its two extreme members and exhibits some of their desirable properties.

In the third paper we focus on matching problems where stability need not be satisfied if the violation of priorities is "small," such as when a small priority difference is considered insignificant or when one is willing to consent but only if the priority reversal is small. Based on the degree of stability which specifies what is considered a small priority gap, we define two families of matching rules, the k-Consent rules and the k-DA rules, and explore their attributes.

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Contribution of Authors

Chapters 3, 4, and 5 of the thesis are joint work with my supervisor, Professor Szilvia Pápai.

Contents

Chapter 1

Introduction

1.1 Matching Theory

As aptly put by Alvin Roth (2002, [48]) "Economist have been lately called upon not only to analyze markets, but to design them." Market design seeks to translate economic theory and analysis into practical solutions to real-world problems. By re-designing both the rules that guide market transactions and the infrastructure that enables those transactions to take place, market designers can address a broad range of market failures. In this context, this thesis consists of three studies on market design, which draw on the fields of matching theory, mechanism design, axiomatic resource allocation and game theory, and address broadly applicable questions in the design of school choice mechanisms, refugee placement, entry-level labour markets and similar assignment problems.

There are two main types of matching models that are closest to our analysis: one-sided and two-sided matching models. In both types of matching models the goal is to assign the agents or objects on one side of the market to the agents on the other side of the market based on some desirable criteria. In one-sided matching one side of the market has indivisible goods (often referred to as *objects*), while the other side has agents. The latter is considered the only active side (hence the name "one-sided"), since only the agents' side matters for the required normative and incentive goals. By contrast, in two-sided matching both sides of the market has agents. The focus of this thesis is on one-sided matching.

There are three key objectives in matching theory: fairness, efficiency and strategyproofness. For one-sided matching, an agent is said to have *justified envy* if there exists an object which he prefers to his assignment and for which he has a higher priority than another agent who is matched to this object. A matching is said to be *fair* if there is no agent who has justified envy. Thus, fairness is achieved by eliminating all justified envy. This concept is also closely related to the well-known criterion of stability for the two-sided matching model. The other two criteria, Pareto-efficiency and strategyproofness are standard axioms on market design and axiomatic resource allocation. A matching is considered *Pareto-efficient* if there is no way to make any agent better off without making at least one agent worse off. Finally, *strategyproofness* ensures that no agent has an opportunity to misrepresent her preferences, regardless of what preferences the other agents report, such that the agent can obtain a better object assigned to her by doing so.

1.2 School Choice and Deferred Acceptance

A school choice model, introduced by Abdulkadiroğlu and Sönmez (2003, [2]), consists of a set of a set of students and a set of schools, where a quota is specified for each school indicating the number of available school seats. Each student has a preference ranking over schools and her outside option. Being matched to the outside option means that the student remains unassigned in the problem and could be interpreted in the school choice context as going to a private school or being home-schooled. Each school has a priority ranking over students. The priority rankings are determined by local (state or city) laws and education policies, and typically do not reflect the school's preferences. In other contexts, such as refugee matching, we can also take the priorities to be mandated. Thus, only student preferences are relevant for normative welfare and fairness requirements, and only students are subject to incentive criteria. The school choice model is closely related to the college admissions problem introduced by Gale and Shapley (1962, [30]). The key difference between the two models is that in school choice schools are objects to be "consumed" by the students, whereas in college admissions schools themselves are agents and have preferences over students, as opposed to being objects that have mandated priorities over students.

The outcome for a school choice problem with given student preferences and school priorities is a matching of schools to students, such that each student is matched to either a school or to the outside option, and no school is matched to more students than its quota. A school choice mechanism, which we will also refer to as a matching rule, is a function that assigns a matching to each school choice problem with fixed preferences and priorities. The benchmark mechanism, which is widely used in school choice programs and in entry-level labour markets, such as in hospital-residence matching, is the Gale-Shapley Deferred Acceptance (DA) algorithm due to Gale and Shapley (1962, [30]).

For each school choice problem the DA rule operates as follows:

Step 1 Each student applies to her highest-ranked school. Each school tentatively accepts, up to its capacity, the best students according to its priority list and rejects the rest.

Step k ($k \geq 2$) Each student rejected in the previous step applies to her next highest-ranked school. Each school tentatively accepts, up to its capacity, the best students among its new applicants and the students who are tentatively already matched to the school according to its priority list and rejects the rest.

The algorithm stops when no more students are rejected.

The DA is strategyproof (Dubins and Freedman, 1981, [18]; Roth, 1982, [47]) and fair (Gale and Shapley, 1962, [30]) in the one-sided matching model. However, the

DA is not Pareto-efficient, despite the fact that the DA allocation Pareto-dominates all other fair allocations at each preference profile (Gale and Shapley (1962, [30]).

The DA algorithm proposed by Gale and Shapley (1962, [30]) has had a substantial impact on market design, as it is the fundamental algorithm for a number of labor market clearinghouses around the world. It has also been recently implemented in school choice systems in Boston and New York City, among many others. New York City replaced its decentralized system of high school admissions with a centralized clearinghouse based on the DA algorithm (Abdulkadiroğlu et al., 2005, [4], [5]). In July 2005 the Boston School Committee voted to replace the existing Boston school choice mechanism with a DA mechanism, which simplifies the strategic choices facing parents. Abdulkadiroğlu et al. (2005, [4], [5]) report details on Boston and New York City.

1.3 Theoretical Background

Abdulkadiroğlu and Sönmez (2003, [2]) was the first paper to apply matching theory to the practical design of public school choice systems. They argued that placement mechanisms used in many cities, such as Boston, were flawed or didn't function well, and proposed alternative centralized mechanisms that would expand the choice of available schools and have good normative and incentive properties. This led to several cities adopting a new school choice mechanism, and this continues to be an active area of research which focuses on designing improved mechanisms and studying the performance of mechanisms in use.

Kesten (2004, [36]) argues that there are three important properties that a good student placement mechanism should ideally satisfy: equity (such as eliminating or minimizing justified envy in some manner); optimality (such as Pareto-efficiency or a constrained version of efficiency if full efficiency is not possible) and immunity to strategic behaviour (ideally strategyproofness). It is well known in the literature that there are trade-offs among strategyproofness, fairness and efficiency. Most notably, Gale and Shapley (1962, [30]) and Balinski and Sönmez (1999, [?]) show that there is an incompatibility between eliminating justified envy and Pareto-efficiency. This incompatibility has inspired a substantial literature with various approaches to reconcile the properties of fairness (or stability) and efficiency. There are quite a few papers on weakening stability in some manner, which may allow for improving efficiency. We mention a few here without intending to provide an exhaustive list: Klijn and Massó (2003, [38]), Kesten (2004, [36]), Alcalde and Romero-Medina (2015, [?]), Cantala and Pápai (2014, [14]), Pápai (2015, [43]), and Morrill (2014, [41]; 2017, [25]).

Kesten (2010, [37]) provides theoretical and computational evidence which shows that the Deferred Acceptance matching outcome may suffer large welfare losses. Kesten (2010, [37]) proposes a mechanism that aims to improve efficiency compared to the Deferred Acceptance mechanism. He introduces the idea of consent in school choice, where a consenting student cannot be made worse off by agreeing to have his priority violated, but may help other students as a consequence. This thesis draws inspiration from Kesten (2010, [37]). In particular, Chapter 4 introduces a new class of rules which are all Pareto-efficient and improve upon the Deferred Acceptance outcome in terms of efficiency. In a similar spirit to consenting, Chapter 5 proposes an alternative relaxation of stability where, unlike Kesten (2010, [37]), a student does not consent unconditionally.

Another strand of the literature focuses on priority structures for which Pareto-efficiency and fairness can be reconciled. For strict priorities, Ergin (2002, [29]) showed that "acyclicity" of the priority structure is a necessary and sufficient condition for Pareto-efficiency of the Deferred Acceptance rule. A priority structure is acyclic if it never gives rise to situations where an agent can block a trade between any other agents without affecting his own allocation. Kesten (2002) showed that the Deferred Acceptance and Top Trading Cycle rules are equivalent if and only if the priority structure is acyclic, but his acyclicity condition is stronger than that of Ergin's when there are multiple copies of objects (i.e., in the school choice model). Furthermore, Kesten's stronger acyclicity condition is a necessary and sufficient condition for the Top Trading Cycle rule to always yield a stable matching. Ehlers and Erdil (2010, [22]) generalize this result to the case where school priorities are coarse, that is, school priorities may have indifferences. Morill (2013, [42]) and Hakimov and Kesten (2014,[31])) introduce alternative matching rules, in the school choice model, to the well-known Top Trading Cycles rule, which are also Pareto-efficient and strategyproof just like the Top Trading Cycles rule, and compare favorably in terms of justified envy, although a direct comparison of justified envy at particular preference profiles is not conclusive.

The Deferred Acceptance algorithm produces the student-optimal matching when we have strict priorities over all students, which is a nice theoretical property. However, in reality schools often do not have strict priorities over all students. For example in the Boston Public School system students are prioritized based on whether they have a sibling enrolled at the same school or whether the student lives within the walk zone of the school. This places many students in the same priority class. When using the Deferred Acceptance algorithm, which is only defined for strict priorities, ties among students are typically broken randomly. Abdulkadiroğlu et al. (2009, [1]) demonstrate that when priorities are coarse there does not exist a unique stable matching in general which is weakly Pareto-efficient for students, and this may harm student welfare.

Abdulkadiroğlu et al. (2009, [1]) showed, using simulations, that the Deferred Acceptance rule with single tie-breaking, albeit preferred to the Deferred Acceptance with multiple tie-breaking, can also lead to a matching that is not a student-optimal stable matching. A single tie-breaking rule uses the same tie-breaker at each school, while a multiple tie-breaking rule may use a different tie-breaker at each school. A theoretical result of Abdulkadiroğlu et al. (2009, [1]) implies that there does not exist a strategyproof mechanism (stable or not) that Pareto-improves on the Deferred Acceptance outcome with single tie-breaking. That is, the potential inefficiency of the student-proposing Deferred Acceptance rule with single tie-breaking is the cost of strategyproofness. Erdil and Ergin (2008, [27] created an algorithm, called stable improvement cycles, to find a student-optimal stable matching that Pareto-dominates the outcome of the student-proposing Deferred Acceptance rule with an arbitrary tie breaking rule.

In the same vein as Erdil and Ergin (2008, [27], in Chapters 4 and 5 we introduce alternative algorithms which are aimed at improving upon the student-proposing Deferred Acceptance rule. Moreover, one of the desirable features of the mechanisms that we propose and analyze in Chapter 3 is that they are compatible with coarse priorities, and therefore add to the existing literature by offering alternative approaches to breaking ties.

1.4 Summary of the Chapters

This thesis consists of three studies on market design, which draw on the fields of matching theory, mechanism design, axiomatic resource allocation and game theory, and address broadly applicable questions in the design of school choice mechanisms, refugee placement, entry-level labour markets and similar assignment problems.

The definitions of all relevant matching rules are collected in Chapter 2. Chapter 3 introduces a unified framework to study the stability and incentive properties of new and existing matching rules that rely on student preferences directly in the schools' choice functions. Chapter 4 studies new Pareto-efficient matching rules by revisiting the EADAM rule of Kesten (2010, [37]). Chapter 5 proposes a relaxed definition of stability which may be applicable in several contexts and may lead to matching rules with improved efficiency compared to stable matching rules. More detail to motivate each of the main chapters is provided below.

1.4.1 Chapter 3: A Unified Analysis of Some School Choice Rules

In Chapter 3 we introduce a new family of rules for many-to-one matching problems, the Preference Rank Partitioned (PRP) rules. PRP rules are Student-Proposing Deferred Acceptance rules where the schools use a choice function to select among applicants. These choice functions are based on the students' strict preference orderings, in addition to the schools' strict priority orderings. Often only the priorities are used by the choice function, such as in the standard DA rule, but there are quite a few matching rules that have been studied in the literature, or have been used in practice, for which this is not the case, and student preferences play a role in the schools' selections among applying students. For instance, the Boston mechanism, used in Boston prior to 2005, is a rule which relies heavily on submitted student preferences in its selection. We wish to unify all these rules by specifying a large class of rules, the PRP rules, which includes all known matching rules of this kind, and also some new rules that have not been studied so far, and analyze this class of mechanisms.

Specifically, to capture all school choice rules which rely on preferences for selection among applicants, we introduce choice functions based on a partition of both student preference ranks and school priority ranks: the choice function first seeks to select students based on the priority classes and then, when needed, based on the preference classes. Finally, if the selection of students is still not determined then the strict priorities are used for tie-breaking. PRP rules include many well-known matching rules, such as the Deferred Acceptance and Boston rules, and the family of Application-Rejection mechanisms of Chen and Kesten (2017, [[16]]), in addition to interesting new matching rules. We study the stability and incentive properties of PRP matching rules, including a special sub-family of PRP rules which treat students' preferences symmetrically.

1.4.2 Chapter 4: Revisiting Kesten's EADAM Rule

In Chapter 4 we study a new class of matching rules, called Deferred Acceptance with Improvement Trading Cycles (DA-ITC), which start with the DA, and if the DA outcome is not Pareto-efficient then there is an iterated improvement trading cycle phase which allows for Pareto-improvements until a Pareto-efficient outcome is reached. We first revisit EADAM and show that a simple algorithm which re-traces cycles in the DA procedure in a backward order of the rejections, following the DA rounds starting from the last round, and restores these cycles in this backward order, is equivalent to the EADAM rule (Kesten, 2010, [37]). This class of rules has the DA-TTC as one of its extreme members and EADAM as the other extreme member. Matching rules between the two extremes are rules that combine features of both the DA-TTC and EADAM.

1.4.3 Chapter 5: Relaxing Stability When Small Priority Violations Are Acceptable

In Chapter 5 we study matching problems where stability need not be satisfied if the violation of priorities is "small," such as when a small priority difference is considered insignificant or when one is willing to consent but only if the priority reversal is small. Based on the degree of stability which specifies what is considered a small priority gap, we define two families of matching rules, the k-Consent rules and the k-DA rules, and analyze their properties. The standard stability notion of a matching requires no justified envy, that is, if student i prefers a school to the school she is assigned at the given matching, then each agent j who is assigned a seat at the preferred school has a higher priority for this object than i does, in addition to individual rationality (no student should be matched to an unacceptable school) and non-wastefulness (no school seat should remain empty that at least one student desires). This stability concept can be relaxed in many ways. One interesting and natural way to relax it is to simply consider "small" priority violations acceptable. For example, the priorities might be based on some criterion that is subject to random variation or inaccuracy of measurement, such as a test score. Or consider the idea of consent introduced by Kesten (2010, [37]). Instead of unconditional consent (which may be ideal to increase efficiency), one can easily imagine that a potentially consenting party may agree to consent only if the priority violation is deemed small. This type of weakening of the standard stability notion is the focus of this paper.

We study this more relaxed stability compared to the usual concept, which depends on the size of the priority violation measured by the distance in the priority ranking, assuming strict priorities to begin with. For example, test scores of 162 and 166 may be considered close enough to deem the difference insignificant and thus allow for a priority violation in the matching, by letting the applicant with 162 points be matched to a university over the applicant with 166 points, while the latter applicant prefers this university to the university he is matched to, but if this applicant had 167 points then a similar violation would not be allowed, since a 5-point difference is no longer considered small. We refer to this as the degree of stability and define the concept of k-stability which specifies the allowed size of the priority violation. In the above example, 5-stability would be violated if an applicant with 167 points had justified envy in the described situation, but 5-stability would be satisfied if this applicant's test score was 166. Note that 1-stability coincides with standard stability, while 2-stability requires a more relaxed degree of stability that allows for a small priority violation, namely, when two students' priority rankings are adjacent for the object in question. Clearly, a larger degree of stability is associated with a smaller k-stability (minimally 1), and a smaller degree of stability requires a larger k-stability. At the extreme of $k = n$ there is no stability implication since any priority violation is acceptable.

Chapter 2

Definitions of Matching Rules

In this chapter we provide definitions of the known matching rules that we use in the thesis. The school choice problem consists of a set of students S, a set of schools C, priority ordering $\succ_c$ and capacity constraint q_c for each school $c \in C$, and preference ordering P_s for each student $s \in S$.

2.1 Deferred Acceptance Rule (Gale and Shapley, 1962)

The Deferred Acceptance (DA) mechanism was proposed by Gale and Shapley (1962) both for the one-to-one matching model (marriage markets) and for the many-to-one college admission model, where the latter is formally identical to our model.

Step 1: Each student applies to his first-ranked acceptable school. Each school rejects the lowest-ranked applicants under $\succ_c$ who are in excess of its capacity, and keeps its remaining applicants tentatively.

Step *k* $(\mathbf{k \geq 2})$: Each student who is rejected at Step $k-1$ applies to his next-ranked school. Each school c considers its new applicants together with its applicants who were kept at Step $k-1$ and rejects the lowest-ranked under $\succ_c$ who are beyond its capacity, and keeps its remaining applicants tentatively.

The algorithm terminates whenever each student is either kept by a school or has applied to all the schools. The outcome is the matching that results by matching students to the schools that kept them tentatively in the last step of the procedure, in while other students (if any) remain unmatched.

2.2 Boston Rule (Abdulkadiroğlu and Sönmez, 2003)

Step 1: Each student applies to his first-ranked school. Each school rejects the lowest-ranked applicants under c who are in excess of its capacity, and immediately accepts the remaining applicants. Each school's capacity is now reduced by the number of students that it has just accepted. Let q_c^k be the number of units that remain of school c after Step 1.

Step $\boldsymbol{k}$ $(\mathbf{k \geq 2})$: Each student who is rejected at Step $k-1$ applies to his kth-ranked school. Each school consider its new applicants, rejects the lowest ranked ones under $\succ_c$ who are in excess of its capacity that remains after Step $k-1$, and immediately accepts the remaining applicants. Each school's current capacity is now reduced by the number of students that it has just accepted after Step k. Let q_c^k be the number of units that remain of school c after the current step.

This procedure terminates when each student is either accepted by an school or has applied to all of the schools. In the latter case the student remains unassigned.

2.3 First Preference First Rules (Pathak and Sönmez, 2013)

The set of students is partitioned into two sets. One set is the *equal preference schools* and the other one is the *first preference first schools*. Given a priority and a preference profile $(\succ, P)$, the first preference first rule is the same as the DA with P and the following strict priorities:

1. Each equal-preference school c uses the strict priority $\succ_c$.

2. Each first-preference-first rule school c uses the following strict priority: any student who ranks school c at a certain rank has higher priority than any student who ranks the same school lower than that rank. Among students who rank school c the same, the priority is determined according to $\succ_c$.

Note that the DA and the Boston rules are special cases of the First Preference First rules. The DA is the special case when each school is an equal-preference school, and the Boston rule is the special case when each school is a first-preference-first school.

2.4 Secure Boston Rule (Dur et al., 2018)

In this variation of the Boston rule no student loses his priority at a guaranteed school. Student s has a guaranteed seat at school c with basic priority order $\succ_c$ students (including student s) who either have higher priority than student s or are in tie with student s under $\succ_c$. Let $G_{\succ_c}$ denote the set of students who have guaranteed seats at school c under $\succ_c$. Note that this set may be empty.

Given a priority and preference profile $(\succ, P)$. For each preference profile P, the outcome that the mechanism selects for it is the application of DA for P and the following modified priority profile $\hat{\succ}$:

For each school c and each $s; \hat{s} \in I$:

1. if $s \in G_{\succ_c}$ and $s \succ_c \hat{s}$, then $s \hat{\succ}_c \hat{s}$,

2. if $s, \hat{s} \notin G_{\hat{\succ}_c}$, then

   (a) if student s has ranked school c higher than student $\hat{s}$ under R, then $s \hat{\succ}_c \hat{s}$,

   (b) if student s and $\hat{s}$ have ranked school c the same and $s \succ_c \hat{s}$, then $s \hat{\succ}_c \hat{s}$.

2.5 General French Rules (Bonkoungou, 2017)

In the French university admissions system, there are selective schools which strictly order students based on their academic performances and non-selective schools which define priority based on the students' educational district. France is divided into 30 educational districts. For each non-selective school, each student who belongs to the same district as the school has higher priority over each student who lives outside this district. But students who live in the same district remain tied for schools in their district, and similarly, students outside a district remain tied for schools in the district in question.

The first step is to use the French tie-breaking procedure to obtain a strict priority ordering. The DA is then applied to the modified priority profile $\hat{\succ}_c$.

The **French tie-breaking** rule can be described as follows. To begin, the set of schools C (i.e., the schools) is partitioned into T categories (i.e., districts). Let $(C_1, ..., C_T)$ be such a partition. A school $c \in C_t$ for some district t is said to be ranked *relatively higher* by student s than by another student $\hat{s}$ if s ranks c higher than $\hat{s}$ among schools in C_t. A school $c \in C$ is said to be ranked *absolutely higher* by student s than by another student $\hat{s}$ if s ranks c higher than $\hat{s}$ among all schools in C. Given a preference profile P, ties are broken based on the preferences as follows.

1. For any tie in the coarse priorities, a student who ranks the school relatively higher than another student has higher priority for the school than the other student.

2. Among students who are tied for a school based on coarse priorities and relative rankings, a student who ranks the school absolutely higher than another student has higher priority for the school than the other student.

3. Among students who are still tied for a school based on coarse priorities, relative rankings and absolute rankings, a student who is ordered higher in the tie-breaker $\succ_c$ of the school has higher priority than another student who is ordered lower in $\succ_c$.

2.6 The French Rule (Bonkoungou, 2019)

Given the priority profile $\succ$, for each preference profile P the French Rule constructs the following strict priority profile $\succ^* (P)$.

For all pairs $i, j \in S$:

1. if $i \succ_c j$ then $i \succ_c^* j$;

2. if $i \sim_c j$ and student i has ranked school c at a higher position than did student j under P then $i \succ_c^* j$;

3. if $i \sim_c j$ and student i and j have ranked school c the same position under P and student i is ranked higher than j under $\succ_c$ then $i \succ_c^* j$.

Run the DA with the constructed priority profile $\succ^* (P)$ at every preference profile P.

Note that the French rule is a General French rule with one school per category or with only one category, both of which result in the same matching rule.

2.7 Application-Rejection Rules (Chen and Kesten, 2017)

Step 0: Each student applies to his first choice. Each school c considers its applicants. Those students with the highest $\succ_c$ are tentatively assigned to school c up to its quota. The rest are rejected.

Each rejected student, who is yet to apply to his e-th choice school, applies to his next choice. If a student has been rejected from all his first e choices, then he remains unassigned in this round and does not make any applications until the next round. Each school c considers its applicants. Those students with highest $\succ_c$ are tentatively assigned to school c up to its quota. The rest of the applicants are rejected.

The round terminates whenever each student is either assigned to a school or is unassigned in this round, i.e., he has been rejected by all his first e choice schools. At this point, all tentative assignments become final and the quota of each school is reduced by the number of students permanently assigned to it.

Step *k* $(\mathbf{k} \geq \mathbf{1})$: Each unassigned student from the previous round applies to his te+1-st choice school. Each school c considers its applicants. Those students with the highest $\succ_c$ are tentatively assigned to school c up to its quota. The rest of the applicants are rejected.

The algorithm terminates when either each student or each school has been assigned. After this step all tentative assignments become final.

2.8 Taiwan Deduction Rules (Dur et al., 2018)

A Taiwan rule can be implemented by deducting points from the student priority scores and then applying DA to the resulting problem. Define a **deduction rule** as $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{|S|+1})$ such that $\lambda_1 = 0$ and $\lambda_k \leq \lambda_{k+1}$ for any $k \in (1, 2, \ldots, |S|)$.

When deduction points are zero, that is, if $\lambda = (0, 0, 0, \ldots, 0)$ the associated Taiwan deduction rule produces the same outcome as the DA. When deduction points are very large, that is if $\lambda = (0, \pi_{max}, 2\pi_{max}, \ldots, |S|\pi_{max})$, the associated Taiwan deduction rule produces the same outcome as the Boston rule.

2.9 Serial Dictatorships (Satterthwaite and Sonnenschein, 1981)

Given an ordering σ of students from 1 to n (i.e., any permutation of the set S), a Serial Dictatorship f^σ assigns the schools to students as follows:

1. The first student is assigned her favorite school among all the schools.

2. The second student is assigned her favorite school with remaining seats.

And so on, each student is assigned her favorite school with remaining seats.

2.10 Gale's Top Trading Cycle Rule (Shapley and Scarf, 1974)

The Top Trading Cycles (TTC) rule is based on an initial "endowment" of schools to students, and has the following steps:

Step 1: Each student points to the school who is endowed with her first-ranked school. A student may point to herself. There exists at least one cycle, which may consist of one or more pointing students. Assign each student in each cycle (a top trading cycle) to the school owned by the student to whom she is pointing. These are final assignments, so we remove all the students with their assigned schools who were in a top trading cycle and have received their assignments.

Step k ($\mathbf{k \geq 2}$) : After the assigned schools and students have been removed in round $k-1$, repeat the same procedure in the reduced market as in round 1, except that now each students points to the student who is endowed with her first-ranked school among the schools that are still in the market.

The algorithm stops when all students who were given an initial endowment have been removed from the market with an assigned school.

2.11 DA-TTC (Cantala and Pápai, 2014)

Step 1 (DA step): Run the DA algorithm.

Step 2 (TTC step): Using the DA outcome as endowments, run the TTC algorithm.

2.12 EADAM (Kesten, 2010)

Student i is an **interrupter** at a particular preference profile if the student is rejected by a school c in the DA algorithm in round t, and there is at least one more student who had been previously rejected by school c prior to round t, while student i was temporarily matched to school c in that round. Then student i is an interrupter in round t and the pair (i, c) is an **interrupting pair** in round t.

Step 1 (DA step): Run the DA algorithm.

Step 2 (Efficiency adjustment step):

Find the last round of the DA algorithm run in Step 1 in which an interrupter is rejected by any school. If there is no interrupter for any school, the algorithm terminates. Otherwise identify all interrupting pairs (i, c) of this particular round and remove school c from the preferences of student i without changing the relative order of the remaining schools in R. Keeping the rest of the preference profile unchanged, re-run the DA algorithm with the new preference profile.

In general, in round k, $k \geq 2$, find the last round of the DA algorithm run in round $k-1$ in which an interrupter is rejected by any school. If there is no interrupter left for any school, the algorithm terminates. Otherwise identify all interrupting pairs (i, c) of this particular round and remove school c from the preferences of student i without changing the relative order of the remaining schools in R. Keeping the rest of the preference profile unchanged, re-run the DA algorithm with the new preference profile.

2.13 Simplified EADAM (Tang and Yu, 2014)

A school c is **underdemanded** at a matching μ if no student prefers c to her assignment under μ.

The steps of the simplied EADAM are as follows:

Step 0: Run DA for the problem $(P, \succ)$

Step $\boldsymbol{k}$ $(\mathbf{k \geq 2})$: This step consists of three substeps:

1. Identify the schools that are underdemanded at the DA matching found in the previous step, settle the matching at these schools, and remove these school seats and the students matched to them.

2. For each removed student i who does not consent, each remaining school c that student i desires and each remaining student j such that $i \succ j$, remove c from j's preference.

3. Re-run DA for the remaining school seats and students.

Stop when all school seats or all students are removed.

Chapter 3

School Choice with Preference Rank Classes

3.1 Introduction

We study a new family of matching rules, which we call Preference Rank Partitioned (PRP) rules, for many-to-one matching problems that assign heterogeneous indivisible objects to agents, where objects have strict priorities over agents and several agents may be assigned to the same object. Since this model is known as the school choice model due to Abdulkadiroğlu and Sönmez (2003, [2]), we call the objects schools and refer to the agents as students. However, the theoretical approach and results pertain to a broad range of applications, not just to school choice, such as centralized university admissions, refugee resettlement, and dormitory room assignments, among others. Balinski and Sönmez (1999, [?]) introduced this model first, which only differs from the college admissions model of Gale and Shapley (1962, [30]) in that school priorities are mandated by policies or by the law and thus schools can be viewed as objects to be allocated, while in college admissions schools have preferences and are considered to be strategic agents. This has implications for the efficiency and incentive axioms used in the two models. In the school choice model, only the students' welfare and incentives are considered. Stability, on the other hand, translates into fairness in the school choice model, since the exogeneously given school priorities are taken into account from the students' perspective.

PRP matching rules are student-proposing Deferred Acceptance (DA) rules (Gale and Shapley, 1962, [30]) in which schools use a choice function to select among applicants that rely not only on the school priorities, but also on the student preferences. Choice-based DA mechanisms are studied and characterized by Kojima and Manea (2010,[39]) and Ehlers and Klaus (2016, [24]). Choice functions are employed in matching with diversity constraints (e.g., Ehlers et al., 2014, [23]), matching with distributional constraints (e.g., Kamada and Kojima, 2018, [35]), or more generally by numerous papers in matching with contracts. None of these papers consider choice functions which depend on preferences.

PRP rules are determined by a partition of each school's priority ordering of students and by a partition of each student's preference ranking over schools, which lead to priority and preference rank classes respectively. Students who are in a higher priority rank class are selected by the school's choice function first, followed by a comparison of the preference rank classes in which applying students place the school in question, in order to make further selections. If ties remain then the school-specific strict priorities over students are used for tie-breaking. Since the given priorities are assumed to be strict, a PRP matching rule specifies, as the first selection criterion, priority rank classes which lead to coarse priorities that are consistent with the given strict priorities. As the second selection criterion, a PRP rule bases the selection of students on their preference rank classes in the instances where some students applying to the school are in the same priority rank class and selecting all of them would result in exceeding the school's capacity. The strict priorities within the priority rank classes specified by the PRP rule are used only for tie-breaking, as a last resort, when neither the priority rank classes nor the preference rank classes can determine the selection of students by a school in a particular round of the iterated Deferred Acceptance procedure.

The set of PRP rules includes many well-known matching rules, such as the Deferred Acceptance (Gale and Shapley, 1962, [30]) and Boston (Immediate Acceptance) rules (Abdulkadiroğlu and Sönmez, 2003, [2]), as well as the family of Application-Rejection mechanisms of Chen and Kesten (2017, [16]), the French Priority mechanisms of Bonkoungou (2019, [12]), the Taiwan Deduction mechanisms of Dur et al. (2018, [20]) and the Secure Boston mechanism introduced by Dur et al. (2019, [21]), among others. We analyze the family of PRP matching rules, and also study a subfamily of PRP rules, the Equitable PRP rules, which treat students symmetrically. For matching rules in this subfamily the preference rank classes are homogeneous across students. All previously studied rules and families of rules that are PRP rules are Equitable PRP rules. We specify new PRP rules which are not Equitable PRP rules, namely, the class of Favored Students rules, which treat students in one of two ways: each student has either the coarsest or the finest preference rank partition. We also identify some further classes of PRP rules which are Equitable PRP rules and include the Deferred Acceptance and Boston rules, but are distinct from such families of rules already studied in the literature, such as the Application-Rejection rules, the Taiwan Deduction rules, and the generalized class of Secure Boston rules, proposed by Dur et al. (2019, [21]). One such class is what we call the Deferred Boston rules, which are PRP rules that have homogeneous priority rank partitions combined with the finest preference rank partitions. We also identify the class of Homogeneous PRP rules, characterized by having both homogeneous priority and preference partitions, which makes Homogeneous PRP rules a superset of almost all previously studied PRP rules (except for some French Priority rules such as First Preference First and Secure Boston rules) and a subset of Equitable PRP rules.

Several PRP rules have been used worldwide in school choice, university admissions, and hospital-intern matching. Apart from the widely used Deferred Acceptance rule, which was adopted (with some variations) by the school boards of several large US cities such as New York City, Boston, and New Orleans, and is being used extensively in hospital-intern matching in North-America, the Boston rule was in use in Boston for school choice until 2005 and is still a popular procedure for student placement.[1] The Boston rule is a special case of the priority matching mechanisms of Roth (1991,[49]), which were used in several UK cities starting in the 1960s for allocating hospital positions to graduating medical students and were subsequently abandoned, since these mechanisms have poor stability and incentive properties. Priority matching mechanisms rely both on the exact preference ranks of hospitals by students and the exact priority ranks of students by hospitals to determine the matching, and use a formula which orders the pairs of ranks starting with $(1, 1)$ to make matches. The only priority matching rule that is a PRP rule is the Boston rule. The First Preference First rule used in England for school choice was banned in 2007 (Pathak and Sönmez, 2013, [46]). The French Priority rules are employed in centralized university admissions in France (Bonkoungou, 2019, [12]). The Parallel Mechanisms (more generally, Application-Rejection rules) are in use in China (Chen and Kesten, 2017, [16]). The Taiwan Deduction rules of Dur et al. (2018, [20]) have been used for high school assignment in Taiwan since 2014. Hence, PRP rules and their properties are not just of theoretical interest, but also have practical relevance.

Our results pertain to the stability, efficiency, and incentive properties of PRP rules. We first demonstrate that PRP rules choose the optimal matching that is consistent with the specific school choice functions at each preference profile (Proposition 1). We also characterize the subclass of PRP rules which treat students symmetrically, the Equitable PRP rules, by applying a natural weak stability property that relies on preferences in addition to the priorities to justify assignments (Proposition 2). This characterization generalizes to the larger class of rules which are not necessarily optimal but share the choice-function-specific stability properties of PRP rules (Theorem 1). We also show that the only Pareto-efficient PRP rules (Pareto efficient for the students' side of the market) are what we call the Near-Boston rules, and this result also holds for the

[1]See Pathak (2017, [44]) for the literature on the practical aspects of school choice design.

larger set of rules mentioned above, which form a superset of PRP rules (Theorem 2). Surprisingly, the set of Near-Boston rules includes some PRP rules other than the Boston rule, which itself is well-known to be efficient (Abdulkadiroğlu and Sönmez, 2003, [2]; Kojima and Ünver, 2014, [40]). However, the class of Pareto-efficient PRP rules is still quite restricted, since for these matching rules only one student's preference rank partition may differ from that of the Boston rule, which itself calls for the finest preference partition for each student. A serious issue with the efficiency of the Near-Boston rules is that these rules are not strategyproof (note that in this context for strategyproofness we only take into account the students' incentives, as the schools are not considered strategic agents). In fact, the Boston rule is well-known to be highly manipulable. Thus, since many students (or their parents, in the case of school choice) misrepresent their true preferences, the matching outcome is unlikely to be Pareto-efficient, and in fact the welfare loss can be significant (Ergin and Sönmez, 2006, [28]; Pathak and Sönmez, 2008, [45]). These concerns also carry over to the other Near-Boston rules.

We also prove that the only strategyproof PRP rule is the Deferred Acceptance rule (Theorem 3). Hirata and Kasuya (2017, [32]) study stability and strategyproofness in a general matching market with contracts, and find that the number of stable and strategy-proof rules is at most one. They also show that if the student-optimal stable rule exists then it is the only candidate for a stable and strategyproof rule. These results, nonetheless, don't imply ours, since in their setup the choice function selects a feasible set of contracts from each set of contracts independently of the preferences. Alva and Manjunath (2019) also deal with related topics.

While PRP rules are not strategyproof except for the DA, they can be shown to be less manipulable than their non-optimal counterparts, using the criterion of Pathak and Sönmez (2013, [46]) for comparing manipulability. This result follows from the general results of Pathak and Sönmez (2013, [46]) and Chen et al. (2016, [15]). Our main result on incentives is that students cannot manipulate PRP rules to obtain a school that was unattainable when reporting their preferences truthfully by placing this school in the same preference rank class or in a lower one when reporting untruthfully (Theorem 4). This theorem sheds new light on the incentive properties of well-known PRP rules and offers insight into their manipulability properties in general.

The closest papers to ours are Bonkoungou (2019, [12]),[2] who studies an important subclass of PRP rules, the French Priority rules, in a somewhat different setup, and Chen and Kesten (2017, [16]) and Dur et al. (2018, [20]), both of which study a class of PRP rules which is distinct from the French Priority rules and only have two members in common: the DA and the Boston rule. Bonkoungou (2019, [12]) has coarse priorities as primitives of his model, thus the French Priority rules, which always have the finest preference partition for students (as in the Boston rule), collapse to one single rule in his paper. He explores the incentive properties of the French Priority rule from both an ex-ante and an ex-post perspective. Bonkoungou also introduces a notion called strategic accessibility, which serves as a basis for further manipulability comparisons (see also Bonkoungou and Nesterov, 2019, [13]). Our analysis is from the ex-post perspective, and our results complement the ex-post perspective results of Bonkoungou (2019, [12]). Specifically, Bonkongou (2019, [12]) makes comparisons based on how fine the given coarse priorities are, while our main theorem on incentives pertains to the preference rank classes. Chen and Kesten (2017, [16]) study a subclass of PRP rules in which the preference rank classes are homogeneous but the priority rank classes are the coarsest. The Taiwan Deduction rules of Dur et al. (2018, [20]) translate into the same class of PRP rules in our framework as the Application-Rejection rules of Chen and Kesten (2017, [16]). We generalize the findings of Chen and Kesten (2017, [16]) about the extreme members of the class of Application-Rejection rules (Theorem 2 and Theorem 3). The results of Chen and Kesten (2017, [16]) and Dur et al. (2018, [20]) on manipulability comparisons of different matching rules are independent of our main result on incentives (Theorem 4).

[2]See also a previous version of this paper, Bonkoungou (2017, [11].)

Although we assume that there are exogenously determined strict priorities, PRP matching rules can also be useful when only coarse (weak) priorities are given, as would be typical for public schools that cannot strictly order the students based on a few criteria only, or for countries which cannot distinguish among all refugee families but wish to prioritize refugees with certain skills and attributes. In Bonkoungou's (2019, [12]) approach the exogeneously given priorities are coarse, following the practice of universities in France, and consequently the use of preference rankings is viewed as breaking the ties in the given coarse priorities. In cases like this, when priorities are weak, the DA and most other prominent matching rules that rely on priorities are not well-defined, and the ties need to be resolved. Abdulkadiroğlu et al. (2009, [1]) propose to use a single tie-breaking procedure in the DA, which is a tie-breaking lottery that applies to each school, and show this tie-breaking method to be superior to the multiple tie-breaking procedure, which means separate tie-breaking lotteries for schools. Erdil and Ergin (2008, [27]) introduce stable improvement cycles, which is essentially a different tie-breaking procedure at different preference profiles with the aim of improving student welfare while preserving stability with respect to the coarse priorities. Further interesting papers on tie-breaking are Ehlers and Erdil (2010, [22]) and Ehlers and Westkamp (2018, [26]).

3.2 Model

Let S be the set of n students and C the set of $m \geq 3$ schools. Each school c has capacity $q_c \geq 1$. In order to simplify the exposition, we assume that $m \geq 3$ and that there exist schools $a, b, c \in M$ such that $q_a + q_b + q_c < n$. The latter assumption ensures that there is scarcity for at least the three schools with the least capacity, so we don't need to take care of special cases where this minimum scarcity doesn't hold.

Each student $s \in S$ has a preference relation P_s, a strict ordering over $C \cup \{0\}$, where assigning 0 to student s represents staying unmatched (or being matched to an outside option). If $0P_sc$ then school c is *unacceptable* to student s, and otherwise the school is *acceptable* to s. For c, $c' \in C$ we write cP_sc' if student s strictly prefers school c to school c', and cR_sc' if either cP_sc' or $c = c'$. Let $P_s \in (c)$ denote that student s has one acceptable school only, namely c, and let $P_s \in (0)$ denote that s has no acceptable schools. More generally, we will write $P_s \in (a, b, c)$, for example, to indicate that student s ranks school a first, b second, and c third, and that these are the only acceptable schools to s. We will also use the notation $r_s(c)$ for student s's ranking of school c for each acceptable school c. For example, $r_s(c) = k$ indicates that c is ranked in the kth position by s. Note that if $r_s(c) < r_{\hat{s}}(c)$ then s ranks school c *higher* than $\hat{s}$, in the sense that a lower rank number indicates higher preference. Let $\mathcal{P}_s$ denote the set of all preference relations for student $s \in S$ and let $\mathcal{P} = \mathcal{P}_{s_1} \times \ldots \times \mathcal{P}_{s_n}$. A preference profile is $P = (P_{s_1}, \ldots, P_{s_n})$, where $P \in \mathcal{P}$.

Each school $c \in C$ has a strict priority ordering $\succ_c$ of students in S. Let π be the set of all priority orderings (i.e., permutations) of students. Then for all $c \in C$, $\succ_c \in \pi$. Let $\Pi = \pi \times \ldots \times \pi$ be the m-fold Cartesian product of π. A priority profile is $\succ = (\succ_{c_1}, \ldots, \succ_{c_m})$, where $\succ \in \Pi$. We assume that a fixed strict priority profile $\succ \in \Pi$ is a primitive of the model. We will discuss in Section 9 how we can relax this assumption and give a different interpretation to Preference Rank Partitioned matching rules when coarse priorities are the primitives instead of strict priorities.

The outcome of a matching problem is an assignment of students to schools, which we refer to as a matching. Formally, a **matching** is a function $\mu : S \to C$, where $\mu(s) \in C$ indicates the school to which student s is matched. If a student s is unassigned in matching μ, we will write $\mu(s) = s$. For ease of notation we let μ_s denote $\mu(s)$ and μ_c denote $\mu^{-1}(c)$, the set of students assigned to c. For all $c \in C$, $|\mu_c| \leq q_c$, that is, the school capacity q_c cannot be exceeded. Let the set of matchings be denoted by M. A **matching rule** φ assigns a matching to each priority and preference profile pair $(\succ, P)$, or a *profile* for short. Thus, $\varphi : \mathcal{P} \times \Pi \to M$.

A matching μ is **blocked by student** $s \in S$ at $P \in \mathcal{P}$ if s prefers being single to being matched to μ_s, that is, $sP_s\mu_s$. A matching is **individually rational** at P if it is not blocked by any student at P. A matching μ is **non-wasteful** at P if no student s prefers a school to μ_s which has empty seats at μ, that is, for all $s \in S$ and $c \in C$, if $cP_s\mu_s$ then $|\mu_c| = q_c$.

Student s has **justified envy** at μ, given a profile $(\succ, P)$, if there exist school c and student $\hat{s}$ such that $cP_s\mu_s$, $s \succ_c \hat{s}$ and $\mu_{\hat{s}} = c$. That is, student s has justified envy for c, given that $\hat{s}$ is matched to c and $\hat{s}$ has lower priority for c than s. A matching μ is **stable** at $(\succ, P)$ if it is individually rational, non-wasteful, and there is no student who has justified envy at μ, given $(\succ, P)$. A matching rule is stable if it assigns a stable matching to each profile $(\succ, P)$.

A matching μ is Pareto-efficient if there is no $\eta \in M$ which Pareto-dominates μ, considering the student's preferences. A matching $\eta \in M$ Pareto-dominates μ if for all $s \in S$, $\eta_s R_s \mu_s$ and, for some $\hat{s} \in S$, $\eta_{\hat{s}} P_{\hat{s}} \mu_{\hat{s}}$. A matching rule is Pareto-efficient if it assigns a Pareto-efficient matching to each preference profile $(\succ, P)$. A matching is **optimal** if it is stable and Pareto-dominates all other stable matchings for the set of students. By Gale and Shapley (1962, [30]), there is a unique optimal stable matching for students at each profile. A matching rule is optimal if it assigns the optimal matching for students to each preference profile.

3.3 Preference Rank Partitioned Matching Rules

We now describe the family of matching rules that we study in this paper, called Preference Rank Partitioned (PRP) matching rules. Each PRP rule is determined by a profile of "partitions" of both the priority rankings of schools and the preference rankings of students. Each school's priority rankings are partitioned by specifying the number of consecutively ranked students in each member of the partition, starting from the top of the rankings.[3] Each student's preference rankings are also partitioned

[3] A partition of priority rankings may arise naturally when priorities are coarse, as is often the case in school choice, but here we treat the partitions as part of the matching rule, given the strict priorities

similarly by specifying the number of consecutively ranked schools in each member of the partition.

PRP rules are choice-based Deferred Acceptance rules, that is, each school uses a choice function to select among applicants in each round of the DA procedure. For each school $c \in C$, we define a choice function Ch_c such that for all $S' \subseteq S$, $Ch_c(S') \subseteq S'$ with the following properties. If $|S'| \leq q_c$ then $Ch_c = S'$ and if $|S'| \geq q_c$ then $|Ch_c(S')| = q_c$. We may also write $Ch_c(S', (\succ, P))$, if we want to indicate explicitly that Ch_c depends not just on S' but also on the profile $(\succ, P)$.

Choice-Based Deferred Acceptance (DA) rules:

- **Round 1:**

  Each student applies to her highest-ranked school (assuming that the highest-ranked school is acceptable to the student). Each school tentatively assigns its seats according to its choice function. Any remaining applicants are rejected.

- **Round k:**

  Each student who was rejected in round $k-1$ applies to her next highest-ranked acceptable school (if any remains). Each school considers the students who are tentatively assigned to the school, if any, together with its new applicants (henceforth the *"applicant pool"*), and tentatively assigns its seats according to its choice function. Any remaining applicants are rejected.

The algorithm terminates when each student is either tentatively assigned to some school or has been rejected by each school that is acceptable to the student, in which case the student remains unassigned.

A PRP rule is a choice-based DA rule with a choice function Ch_c for each school $c \in C$ which selects among students as a function of the given partitions of priorities and preferences. Given a partition of the priority rankings for school c, the **priority (rank)**

for each school. We will discuss how to start from naturally arising coarse priorities in Section 9.

classes, and given a partition of the preferences for each student, the **preference (rank) classes**, school c first selects students from its applicant pool in its highest priority class(es). If this does not determine the selected set of students, where the selected number of students is up to the fixed capacity q_c of the school, given that priority classes are not necessarily singletons, the choice function then considers the partitioned preferences, and selects students who have school c in their highest possible preference class(es) relative to each other. If the preference partitions still do not determine the selected set of students for school c, then the choice is resolved based on the strict priority ordering $\succ_c$. This last round based on the strict priority orderings can be seen as tie-breaking. This defines a choice function for each school $c \in S$, that is, it determines unambiguously the set of selected students up to the school capacity from any given applicant pool $S' \subseteq S$ for each $\succ_c$ and preference profile P. This, in turn, defines a PRP matching rule.

In sum, the school choice functions select students from the applicant pool lexicographically in the following order:

1. based on the **priority classes**;
2. based on the **preference classes**;
3. based on the **tie-breaker** given by the strict priority ordering.

Given their central role in the definition of PRP rules, we now define priority and preference rank classes formally. For all $c \in C$, let the cardinalities of the priority rank classes be denoted by $v_c^1, v_c^2, \ldots$, starting with the top-ranked students, such that $\sum_t v_c^t = n$. The **coarsest priority rank partition** is when $v_c^1 = n$ includes all students and the partition has one member only, and the **finest priority rank partition** is given by $(v_c^1, \ldots, v_c^n) = (1, \ldots, 1)$ with n classes, where each class contains one student. More generally, the rank partition is the finest or coarsest, respectively, whenever the resulting matching rule is outcome equivalent with the above. Given that a school capacity may be greater than one, this implies that a priority rank partition for school c is the finest if the students in priority ranks $q_c + 1, q_c + 2, \ldots, n$ are in their own singleton priority classes, and the priority classes of ranks 1 to q_c are irrelevant (may or may not consist of singletons).

Given a priority profile $\succ$, for all $c \in C$, let $\succ_c^1$ be the set of students ranked by $\succ_c$ between 1 and v_c^1 and, for all $t \geq 2$, let $\succ_c^t$ be the set of students ranked by $\succ_c$ between $\sum_{l=1}^{t-1} v_c^l + 1$ and v_c^t. Note that for all $t \geq 1$, $| \succ_c^t | = v_c^t$. Let $v_c = (v_c^1, \ldots)$ denote the priority rank class list for school $c \in C$, and let $v = (v_c)_{c \in C}$ be the priority rank class profile.

For all $s \in S$, let the cardinalities of the preference rank classes be denoted by $p_s^1, p_s^2, \ldots$, starting with the top-ranked schools, such that $\sum_t p_s^t = m$. The **coarsest preference rank partition** is when $p_s^1 = m$ includes all schools and the partition has one member only, and the **finest preference rank partition** is given by $(p_s^1, \ldots, p_s^m) = (1, \ldots, 1)$ with m classes, where each class contains one school. Given our general remark above, technically the preference rank partition is also coarsest if having more than one preference class always leads to the same matching as having just one. For example, if $n \leq \sum_{c \in C} q_c$ then no student is rejected by her mth-ranked school, and thus $p_i^1 = m - 1$ also yields a coarsest partition.

Given a preference profile P, for all $s \in S$, let P_s^1 be the set of *acceptable* schools ranked by P_s between 1 and p_s^1 and, for all $t \geq 2$, let P_s^t be the set of *acceptable* schools ranked by P_s between $\sum_{l=1}^{t-1} p_s^l + 1$ and p_s^t. Note that for all $t \geq 1$, either $|P_s^t| = p_s^t$, if all schools are acceptable to student s, or if some schools are unacceptable then there exists $\hat{t}$ such that for all $l = 1, \ldots, \hat{t} - 1$, $|P_s^l| = p_s^l$ and $|P_s^{\hat{t}}| < p_s^{\hat{t}}$, and for all $l > \hat{t}$, $P_s^l = \emptyset$. Let $p_s = (p_s^1, \ldots)$ denote the preference rank class list for student $s \in S$, and let $p = (p_s)_{s \in S}$ be the preference rank class profile.

Using the above notation, each PRP matching rule is determined by a priority and a preference rank class profile (v, p). This is not to be confused with the profile $(\succ, P)$, a pair of a strict priority profile and a strict preference profile, which determines a specific problem and is a primitive of our model, while (v, p) is part of the matching rule. We will indicate explicitly the priority and preference rank class profiles for a

PRP matching rule and denote it by $f^{v,p}$. Thus, $f^{v,p}(\succ, P) = \mu$ indicates that the PRP rule $f^{v,p}$ assigns matching μ to profile $(\succ, P)$.

We can now formally define the choice function Ch_c for each school $c \in C$ for a PRP rule $f^{v,p}$. Fix $c \in C$ and let $S' \subseteq S$. As already stated, the set of students $T \subseteq S'$ is selected from applicant pool S' based on priority rank classes first, then based on preference rank classes, and finally based on the tie-breaker given by strict priorities. Formally, if $|S'| > q_c$ then $Ch_c(S') = T$ if the following are satisfied:

- $T \subset S', \ |T| = q_c$
- there exists $k \geq 1$ such that for all $s \in T$, $s \in \bigcup_{t=1}^{k} \succ_c^t$;
- for all $\hat{s} \in S' \setminus T$, if $\hat{s} \in \bigcup_{t=1}^{k} \succ_c^t$ then $\hat{s} \in \succ_c^k$;
- for all $s \in T$ and $\hat{s} \in S' \setminus T$ such that $s, \hat{s} \in \succ_c^k$, if there is no $k' \geq 1$ such that $c \in \bigcup_{t=1}^{k'} P_s^t$ and $c \notin \bigcup_{t=1}^{k'} P_{\hat{s}}^t$, then $s \succ_c \hat{s}$.

Example 1 (**PRP choice functions**)**.** Consider the following matching problem with four students and four schools $(n = 4, m = 4)$. The preferences are given in the table below, in which the bars indicate the preference rank classes for the PRP rule. That is, $p_1 = (1, 2, 1), p_2 = (1, 2)$, $p_3 = (3, 1), p_4 = (1, 3)$.

P_1	P_2	P_3	P_4
b	b	d	b
c	a	b	a
a	d	a	d
d	0	c	c

Let $\succ_a = (1, 2, 3, 4)$ indicate the strict priorities in descending order for school a, and let $v_a = (3, 1)$. Assume that school a has capacity 1. Let the applicant pool for school a be $\{1, 3, 4\}$. Student 4 is eliminated based on the priority rank classes of school a, since $\{1, 3\} \subset \succ_a^1$ and $4 \in \succ_a^2$. This leaves students 1 and 3. Student 3 is selected based on the preference rank classes, since 1 ranks a in the second highest preference class and 3 ranks a in the highest preference class: $a \in P_1^2$ and $a \in P_3^1$.

Now consider the same problem with a slightly different PRP rule, where student 3's preference rank classes are different: $p_3 = (2, 1, 1)$. Given applicant pool $\{1, 3, 4\}$ for school a, student 4 is eliminated based on the priority rank classes of school a, as before, and given the new preference rank classes for student 3, now the selection cannot be made between 1 and 3 based on the preference rank classes, since both rank a in their second preference class: $a \in P_1^2$ and $a \in P_3^2$. Thus, we apply the tie-breaker strict priority order $\succ_a$ and student 1 is selected, since $1 \succ_a 3$. □

PRP choice functions satisfy standard properties of choice functions that are associated with choice-based DA rules, such as Acceptance, Monotonicity, Substitutability and Consistency (Ehlers and Klaus, 2016), but these properties only hold when the preference profile is fixed. This is in contrast to typical choice functions used in conjunction with the DA, such as when quotas are specified for different types of agents, and the choice function is not a function of the preferences. This is the most salient feature of PRP rules, namely, that the choice functions of the schools depend on the students' preferences, and specifically on the students' preference rank classes, which is why we call these matching rules Preference Rank Partitioned rules. Thus, in contrast to choice-based DA rules which use choice functions that are independent of the students' preferences, this feature of PRP rules accounts for the loss of strategyproofness (see Theorem 3 in Section 8).

If both the priority rank partition for each school and the preference rank partition for each student are the coarsest, then the PRP rule relies only on the strict priorities as tie-breakers, and therefore this rule is the standard Deferred Acceptance rule, which simply selects the top priority students from each applicant pool. Equivalently, we can let the priority rank partition be arbitrary. As long as the preference rank partition is the coarsest, the PRP rule is the DA. Hence, as there may be multiple representations (v, p) of the same PRP rule, we will use the convention that the role of the tie-breaker should be minimized as much as possible by making the priority partition finer. As a consequence, given a finer priority partition, the preference partition should be left as coarse as possible, which clarifies the impact of the preferences. This specification of a PRP rule delineates which information is used by the choice function, whether it is the priority partition or the preference partition, and specifically which rank classes may play a role in any particular selection. In the case of the standard DA rule, this means that we let the priority partition be the finest for each school, so as to entirely eliminate tie-breaking, which then clarifies that the preference partitions don't play any role in the choice function, since all selections can be made based on the priority partitions, and therefore we can let each student's preference partition be the coarsest. Consequently, this convention not only lets us have a unique representation of each PRP rule, but it also provides an intuitive representation.

Formally, we can find the unique representation of a PRP rule f based on this convention as follows. Fix the priority profile $\succ \in \Pi$. Let $c \in C$ and $s, \hat{s} \in N$ such that $\hat{s} \succ_c s$. If there exists a preference profile $P \in \mathcal{P}$ such that $f_s(\succ, P) = c$ and $cP_{\hat{s}} f_{\hat{s}}(\succ, P)$ then let s and $\hat{s}$ be in the same priority class, and let all students $\tilde{s}$ such that $s \succ_c \tilde{s} \succ_c \hat{s}$ be also in the same priority class, and otherwise let each student for school c be in a separate priority class. This determines the priority rank partition profile v which satisfies the convention that v_c is as fine as possible for each school c. Note that since f is a PRP rule, we would get the same v using any priority profile $\succ \in \Pi$.

In order to determine the preference rank partition profile p, let $c \in C$ such that there is at least one priority class according to v_c with a minimal size of 2, that is, the priority class in v_c contains at least two ranks: $v_c^k \geq 2$ for some $k \geq 1$. If there is no such a school then all school priority partitions v_c are the finest and all preference partitions p_s are the coarsest, and f is the DA. Let students $s, \hat{s} \in S$ occupy these two ranks in the same priority class of v_c : $s, \hat{s} \in \succ_c^k$. Then, from the construction of v, there exists a priority profile $\succ \in \Pi$ with $\hat{s} \succ_c s$ and a preference profile $P \in \mathcal{P}$ such that

$f_s(\succ, P) = c$ and $cP_{\hat{s}} f_{\hat{s}}(\succ, P)$. Find such a profile $(\succ, P)$ which maximizes $r_s(c)$ and minimizes $r_{\hat{s}}(c)$. Let $r_s(c) \in p_s^t$ and $r_{\hat{s}}(c) \in p_{\hat{s}}^{\hat{t}}$, where $t + 1 = \hat{t}$. If we repeat the same exercise for pairs of students $s, \hat{s}$ with an arbitrary school c for different profiles $(\succ, P)$ that meet the above specifications in terms of v_c then we can trace out the preference rank classes for each student and get a unique preference rank class profile p for the fixed PRP rule f such that p satisfies the convention that p_s is as coarse as possible for each student s, given the uniquely specified priority rank class profile v which is as fine as possible.

3.4 Special Subclasses of PRP Matching Rules

As already noted, the standard DA rule is a PRP rule, which is described by the finest priority partition profile and the coarsest preference partition profile. Another well-studied PRP rule besides the DA rule is the Boston (Immediate Acceptance) rule (Abdulkadiroğlu and Sönmez, 2003, [2]). The Boston rule is a PRP rule with the coarsest priority rank partition and selects among students based on the finest preference rank partition, hence tie-breaking is only necessary when students have the same ranking for a school: if students s and $\hat{s}$ are competing for school c then s is chosen over $\hat{s}$ if s ranks a better than $\hat{s}$ (i.e., $r_s(c) < r_{\hat{s}}(c)$) or if s and $\hat{s}$ rank c equally (i.e., $r_s(c) = r_{\hat{s}}(c)$)and $s \succ_c \hat{s}$.

Previously studied classes of matching rules that belong to the set of PRP rules include the Application-Rejection rules (Chen and Kesten, 2017, [16]) used in China, the First Preference First rules (Pathak and Sönmez, 2013, [46]) that were banned in England, the Secure Boston rules and their generalizations proposed by Dur et al. (2019, [21]) to replace the Boston rule, and a special member of the French Priority rule introduced by Bonkoungou (2019, [12]), which corresponds to a broad class of PRP rules in our setting and includes all the previously mentioned rules (see more on this in Section 9), while in Bonkoungou's setup it is a single rule, based on the given coarse priorities.[4]

We list some further notable subfamilies of PRP rules in Table 1 which have not been studied before. The Deferred Boston rules include both the standard DA and the Boston rules and allow for any homogeneous priority partition profile (i.e., the same priority partition for each school), while the preference partitions are the finest. If both the preference and priority partition profiles are homogeneous then we have a Homogeneous PRP rule. All Deferred Boston rules are Homogeneous PRP rules, but the French Priority rules in general are not Homogeneous PRP rules, and specifically the First Preference First and the Secure Boston rules are not Homogeneous PRP rules. On the other hand, the Application-Rejection rules are Homogeneous PRP rules. The class of Equitable PRP rules, characterized by a homogeneous preference partition profile, is even larger than the class of Homogeneous PRP rules, and contains all of the above mentioned rules and families of PRP rules. Lastly, to give a specific class of PRP rules which does not belong to the class of Equitable PRP rules, we included on the table the family of Favored Students rules, which allow for different treatments of students. Specifically, Favored Students rules have the coarsest priority partition profile, and each student is either favored or not, favored students have the coarsest preference partition, and not favored students have the finest.

[4]A previous version of the paper, Bonkoungou (2017, [11]), introduces a larger set of rules, which has the French Priority rule as one of its distinct members. This larger set of rules, which we will refer to as the class of General French Priority rules, is defined in Appendix A, where we clarify the relationship between PRP rules and the General French Priority rules of Bonkoungou (2017, [11]). We show that although the two families of rules have some important members in common, neither families of rules contains the other.

PRP Matching Rules	**Priority Partition**	**Preference Partition**
Deferred Acceptance (DA)	Finest	Coarsest
Boston	Coarsest	Finest
Deferred Boston	Homogeneous	Finest
First Preference First	Equal-preference schools: finest Preference-first schools: coarsest	Finest
Secure Boston	For each school c: finest for top q_c, then coarsest	Finest
French Priority	Arbitrary	Finest
Application-Rejection	Coarsest	Homogeneous
Taiwan Deduction	Coarsest	Homogeneous
Homogeneous PRP	Homogeneous	Homogeneous
Equitable PRP	Arbitrary	Homogeneous
Favored Students	Coarsest	Favored students: coarsest Non-favored students: finest

Although we defined PRP matching rules by using first the priority rank partition when selecting students, note that for the PRP matching rules that have the coarsest priority partition the priority rankings do not play any role in the selection of students up front (such as the Boston rule and more generally the Application-Rejection rules), and we understand intuitively that these rules make selections based on the preference rank partitions primarily, and the strict priorities are used for tie-breaking only when needed. In general, the PRP rules which don't have the coarsest priority partition profile, such as the Deferred Boston rules (excluding the Boston rule) or the First

Preference First rules, the priority rank partitions play a role in student selection.

3.5 Stability and Optimality of PRP Matching Rules

The dependence of PRP choice functions on student preferences, which is the most notable general feature of PRP rules, accounts for violating typical stability conditions that are independent of the preferences. Given that when preference partitions are the coarsest the preferences play no role in choosing among applying students, and given that the only such PRP rule is the standard DA, this is the only rule which satisfies the standard stability axiom in the class of PRP matching rules.

We will now consider a stability concept inspired by PRP rules, which we call rank-partition stability. Given the rank partition profiles (v, p), for each profile $(\succ, P)$ we construct a strict priority profile $\bar{\succ}$ as follows. For each school c the orderings of students across priority rank classes based on $\succ_c$ and v_c remain the same in $\bar{\succ}_c((\succ, P), (v, p))$, and within priority rank classes we order students according to the preference rank partition of P based on p. If ties remain, then we use the strict priority ordering $\succ_c$ as a tie-breaker. More formally, let $s, \hat{s} \in S$, let $k, k' \geq 1$ such that $s \in \succ_c^k, \hat{s} \in \succ_c^{k'}$, and let $t, t' \geq 1$ such that $c \in P_s^t$ and $c \in P_{\hat{s}}^{t'}$. If $k \neq k'$ then s and $\hat{s}$ are in different priority rank classes and $s \tilde{\succ}_c \hat{s}$ if and only if $s \succ_c \hat{s}$. If $k = k'$ then s and $\hat{s}$ are in the same priority rank class. Then, if $t \neq t'$ then if $t < t'$ then $s \bar{\succ}_c \hat{s}$, and if $t > t'$ then $\hat{s} \bar{\succ}_c s$. Finally, if $k = k'$ and $t = t'$ then $s \bar{\succ}_c \hat{s}$ if and only if $s \succ_c \hat{s}$. Note that the priority profile $\bar{\succ}$ is a function of the preference profile P and thus it can change as preferences vary. From now on we will refer to $\bar{\succ}(\succ, P(v, p))$ as the *constructed priority profile.*

A matching rule φ is **rank-partition stable** if there exists a pair of rank partitions (v, p) such that $\varphi(\succ, P)$ is stable with respect to the constructed priority profile $\bar{\succ}((\succ, P), (v, p))$ at each profile $(\succ, P)$ and, for all $\succ, \succ' \in \Pi$, if $\bar{\succ}((\succ, P), (v, p)) = \bar{\succ}((\succ', P), (v, p))$ then $\varphi(\succ, P) = \varphi(\succ', P)$. We will also say that a matching rule φ is rank-partition stable with respect to (v, p).

If a matching rule assigns a matching to each pair of priority and preference profiles $(\succ, P)$ which is stable with respect to the constructed priority profile $\bar{\succ}((\succ, P),(v,p))$, and if the selection of a stable matching only depends on this constructed priority profile at each preference profile then the matching rule is rank-partition stable.

Rank-partition stability can be seen as a straightforward stability property of PRP rules, based on stability with respect to the appropriately constructed priority profile at each preference profile. A similar stability property is used by Bonkongou (2019, [12]) for French Priority rules.

We provide an example below to illustrate the construction of the preference-profile-specific constructed priority profile $\bar{\succ}(\succ, P(v,p))$.

Example 2 (A constructed priority profile for rank-partition stability). Consider the following matching problem with five students and four schools ($n = 5, m = 4$). The preference profile P and the priority profile $\succ$ are given below, and the rank partitions (v,p) are specified by the bars in the two tables.

Student preferences

P_1	P_2	P_3	P_4	P_5
b	b	b	b	d
c	c	a	a	b
a	a	d	c	a
d	d	c	d	c

School priorities

$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
4	3	4	4
1	1	1	5
2	4	5	3
3	2	2	1
5	5	3	2

Constructed priority profile $\bar{\succ}$

$\bar{\succ}_a$	$\bar{\succ}_b$	$\bar{\succ}_c$	$\bar{\succ}_d$
1	3	1	5
2	1	4	4
4	4	5	3
5	2	2	1
3	5	3	2

□

It is easy to see that each PRP rule is rank-partition stable. Moreover, it is straightforward to verify that each PRP rule is optimal within the set of rules that are rank-partition stable with respect to a given (v, p) due to a classic result by Gale and Shapley (1962, [30]), which implies in our setting that for each pair of priority and preference rank partition profiles (v, p) and for each profile $(\succ, P)$ there exists a rank-partition stable matching which is stable with respect to the priorities in the constructed priority profile $\bar{\succ}(\succ, P(v, p))$ for (v, p) at $(\succ, P)$, and each student weakly prefers this matching to any other matching which is stable with respect to $\bar{\succ}(\succ, P(v, p))$ at preference profile P. We call this matching the **$(\mathbf{v}, \mathbf{p})$-optimal matching** at $(\succ, P)$. On can easily verify that the PRP rule $f^{v,p}$ is rank-partition stable with respect to (v, p) and $f^{v,p}(\succ, P)$ is the (v, p)-optimal matching at each profile $(\succ, P)$. We summarize these findings below.

Proposition 1. *Each PRP rule $f^{v,p}$ is rank-partition stable and selects the unique (v, p)-optimal rank-partition stable matching at each profile $(\succ, P)$.*

This representation of PRP rules and the underlying stability concept serve as a foundation for later results, as they highlight the parallel features between PRP rules and the standard DA rule, and allow us to see the PRP rules as optimal rules within the set of rank-partition stable rules, which are stable with respect to the modified priority profile at each preference profile, where the modifications of the priorities correspond to the selections made by PRP choice functions. Note also that Proposition 1 implies that PRP rules are individually rational and non-wasteful.

3.6 Equitable PRP Matching Rules

Not all PRP matching rules treat students symmetrically with respect to their preferences. We call the subfamily of PRP rules which treat students symmetrically in terms of their preference partitions **Equitable PRP rules**, which can be described in terms of a homogeneous partition of preferences across students (i.e., the preference partition for each student is the same). We call these rules equitable since, for example, if student s has a coarser preference partition than student $\hat{s}$ then s gets a preferential treatment compared to student $\hat{s}$, given a fixed priority profile $\succ$. Although a PRP rule with the same preference partition for each student treats students equitably in terms of their preference rankings, we note that these rules only treat the preferences of students equally, but students may still be treated differently based on the school priority partition profile v.

Formally, a preference rank partition profile p is homogeneous if for all $s, s' \in S$, $p_s = p_{s'}$. Since the priority and preference rank partitions that describe a PRP rule are not always unique, as already discussed, we can follow the specified convention to determine if a PRP rule is an Equitable PRP rule. More generally, the requirement is that *there exists at least one* homogeneous preference rank partition (along with a priority rank partition) which yields the PRP rule. In other words, no matter which homogeneous preference rank partition profile and which arbitrary priority rank partition profile are used, for at least one pair of preference and priority rank partition profile (v, p) the outcome does not correspond to the outcome prescribed by the PRP rule.

To aid our analysis, we propose a general stability property of matching rules, called PP-stability, which weakens the standard stability axiom by basing the matching on a comparison of students' preference ranks of a school that they compete for, in addition to the students' priority rankings by this school. Characterizations of the Boston rule, such as the ones given by Kojima and Ünver (2014, [40]) and Doğan and Klaus (2018, [17]), rely on axioms comparing the preference ranking of alternatives, which are similar to the idea for this stability concept, since the Boston rule makes matches primarily based on the preference rankings. Our concept combines a priority-based and a preference-based justification for students to have justified envy, and this stronger justified envy is ruled out by our new stability concept. We can readily see that this is a stronger justified envy concept, since it is justified on the grounds that neither the priorities nor the preference rankings can explain the selection of one student over another at a school to which both students have applied. Afacan (2013, [6]) uses a similar property combining priority and preference rankings, but his axiom makes explicit use of the school capacity.

Preference and Priority Rank Stability (PP-Stability): Student s has **PP-justified envy** at μ, given $(\succ, P)$, if there exist school c and student $\hat{s}$ such that $cP_s\mu_s$, $s \succ_c \hat{s}$, $r_s(c) \leq r_{\hat{s}}(c)$, and $\mu_{\hat{s}} = c$. Student s has PP-justified envy for c, given that $\hat{s}$ is matched to c and $\hat{s}$ has both lower priority for c than s and ranks c the same or higher than $\hat{s}$. A matching μ is **PP-stable** at $(\succ, P)$ if it is individually rational, non-wasteful, and there is no student who has PP-justified envy at μ, given P. A matching rule is PP-stable if it assigns a PP-stable matching to each profile $(\succ, P)$.

It may appear at first that all PRP rules are PP-stable, but this is not the case. The following example shows a Favored Students PRP rule which is not PP-stable.

Example 3 (**A Favored Students PRP rule which is not PP-stable**)**.** Consider the following matching problem with five students and four schools ($n = 5, m = 4$). Schools a, b, c have a capacity of one, and school d has a capacity of two. The preferences and priorities are below, with bars indicating the preference partitions in the preference profile.

Student preferences

P_1	P_2	P_3	P_4	P_5
b	b	b	b	d
c	c	a	a	b
a	a	d	c	a
d	d	c	d	c

School priorities

$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
4	3	1	4
1	1	2	1
2	4	3	5
3	2	4	2
5	5	5	3

In this example students 1 and 2 are so-called favored students, which means that there is only one preference rank class for these students, which includes all the schools (the coarsest preference rank partition), while the non-favored students, 3, 4, and 5, have the finest preference rank partition: $p_1^1 = p_2^1 = 4$, and $p_s = (1, 1, 1, 1)$ for $s \in \{3, 4, 5\}$.

The rounds of the specified Favored Students rule at the given profile are summarized in the table below, with the selected students in each round underlined.

Round	a	b	c	d
1		$1, 2, \underline{3}, 4$		$\underline{5}$
2	$\underline{4}$	$\underline{3}$	$\underline{1}, 2$	$\underline{5}$
3	$\underline{2}, 4$	$\underline{3}$	$\underline{1}$	$\underline{5}$
4	$\underline{2}$	$\underline{3}$	$\underline{1}, 4$	$\underline{5}$
4	$\underline{2}$	$\underline{3}$	$\underline{1}$	$\underline{4}, \underline{5}$

In this example $r_4(a) = 2 < 3 = r_2(a)$ and $4 \succ_a 2$. Since aP_4d and $\mu_2 = a$, this rule is not PP-stable. □

First we verify that PP-stability is independent of rank-partition stability. This is not surprising, but since both concepts weaken the standard stability axiom, it is in order. The Favored Students rules are rank-partition stable PRP rules by Proposition 1, but are not PP-stable, as seen in Example 3. A matching rule which assigns a non-PRP matching (e.g., the outcome of the Top Trading Cycles rule) at each profile $(\succ, P)$ where this matching is PP-stable and otherwise assigns the DA outcome is PP-stable but not rank-partition stable.

We remark that both the DA and Boston rules are PP-stable. Since stability implies PP-stability, the DA is clearly PP-stable. The Boston rule is also PP-stable, since if either $r_s(c) < r_{\hat{s}}(c)$ or if $r_s(c) = r_{\hat{s}}(c)$ and $s \succ_c \hat{s}$, in both cases $\hat{s}$ cannot be assigned to c unless s is also assigned to school $\tilde{c}$, whenever $\tilde{c}R_sc$. Indeed, the Boston rule is preference-rank stable in the sense that whenever $r_s(c) < r_{\hat{s}}(c)$, s does not envy $\hat{s}$ when $\hat{s}$ is assigned c. PRP rules in general are not preference-rank stable in this stronger sense. In fact, it is easy to check that the Boston rule is the only such preference-rank stable rule within the class of PRP rules. On the other hand, the DA rule is the only priority-rank stable (i.e., stable) rule in the class of PRP rules.

While not all PRP rules satisfy PP-stability, as demonstrated by Example 3, the subfamily of PRP rules which satisfy PP-stability is much larger than just the DA and

Boston rules, and we will show in the next proposition that it exactly corresponds to the Equitable PRP rules. We remark that all previously studied PRP rules are Equitable PRP rules.[5] Proposition 2 below provides an explanation for this, since it demonstrates that all the studied rules are PP-stable, which is an intuitive feature of matching rules and an attractive attribute in school choice.

Proposition 2. *A PRP rule is PP-stable if and only if it is an Equitable PRP rule.*

Proof.

Claim 1: An Equitable PRP matching rule is PP-stable.

Let $f^{v,p}$ be an Equitable PRP rule. That is, $f^{v,p}$ is a PRP rule and p is homogeneous across students. Suppose that there exists a profile $(\succ, P)$ such that $f^{v,p}(\succ, P)$ is not PP-stable. Then there exist $s, \hat{s} \in S$ and $c \in C$ such that $r_s(c) \leq r_{\hat{s}}(c)$ at P, $s \succ_c \hat{s}$, and s envies $\hat{s}$ at $(\succ, P)$ for being assigned to c. That is, $f^{v,p}_{\hat{s}}(\succ, P) = c$ and $cP_s f^{v,p}_s(\succ, P)$.

Let $t, t' \geq 1$ such that $c \in P^t_s$ and $c \in P^{t'}_{\hat{s}}$. Then, given $r_s(c) \leq r_{\hat{s}}(c)$, since p is homogeneous across agents, $t \leq t'$. Let $k, k' > 0$ such that $s \in \succ^k_c$ and $\hat{s} \in \succ^{k'}_c$. Then $s \succ_c \hat{s}$ implies that $k \leq k'$. Given that $f^{v,p}_{\hat{s}}(\succ, P) = c$ and $cP_s f^{v,p}_s(\succ, P)$, $k \leq k'$ implies that $k = k'$. Then $t \leq t'$ implies that $t = t'$. Hence, the tie-breaker $\succ_c$ would be used by $f^{v,p}$ to select between s and $\hat{s}$ at school c, but $s \succ_c \hat{s}$ implies that $f^{v,p}_{\hat{s}}(\succ, P) \neq c$ when $cP_s f^{v,p}_s(\succ, P)$, which is a contradiction.

Claim 2: A PP-stable PRP rule is an Equitable PRP rule.

Let $f^{v,p}$ be a PP-stable PRP rule. Suppose that $f^{v,p}$ is not an Equitable PRP rule, that is, p is not homogeneous across students. Then there exist $(\succ, P) \in \mathcal{P}$, $c \in C$, $s, \hat{s} \in S$ and $t, t' \geq 1$ such that

1. $f^{v,p}_{\hat{s}} = c$ and $cP_s f^{v,p}_s(\succ, P)$
2. $r_s(c) \leq r_{\hat{s}}(c)$ at P

[5] The General French Priority rules of Bonkoungou (2017), which also use preference rankings in the schools' choice functions are not all PP-stable, but these are also not PRP rules in general, as we show in Appendix A. Bonkoungou (2017) focuses on the Simplified French rule which corresponds to a subclass of PRP rules that we call the French rules, and these rules are PP-stable.

3. $t > \hat{t}$, where $c \in P_s^t$ and $c \in P_{\hat{s}}^{\hat{t}}$

4. s and $\hat{s}$ are in the same priority rank class for c, given $\succ_c$ and v_c : there exists $k \geq 1$ such that $s, \hat{s} \in \succ_c^k$.

Since $f^{v,p}$ is PP-stable , $\hat{s} \succ_c s$. Note that $f^{v,p}$ selects $\hat{s}$ over s for school c based on the preference rank classes, since $t > \hat{t}$, not based on priority rank classes, given that s and $\hat{s}$ are in the same priority rank class for c. The fact that $\hat{s} \succ_c s$ is irrelevant, since the choice is not based in the tie-breaking. Let $\bar{\succ}_c$ be the same as $\succ_c$ except for the position of student $\hat{s}$: place student s directly above $\hat{s}$ in $\bar{\succ}_c$, and leave all other orderings in $\bar{\succ}_c$ the same as in $\succ_c$.

Let $\bar{\succ}' \equiv (\bar{\succ}_c, \succ_{-c})$. Then $f^{v,p}(\succ', P) = f^{v,p}(\succ, P)$, as we will show. Note first that only $\succ_c$ is changed to $\bar{\succ}_c$, so the only difference can be in the selections made by school c. Note that since s and $\hat{s}$ were in the same priority class at $\succ_c$ and since v_c remains the same, the students in the priority rank classes of c have not changed. Specifically, s and $\hat{s}$ are still in the same priority class at $\bar{\succ}_c$, and so are all $\bar{s} \in S$ such that $\hat{s} \succ_c \bar{s} \succ_s s$. Thus, $\succ_c^k = \bar{\succ}_c^k$, and for all k', $\succ_c^{k'} = \bar{\succ}_c^{k'}$. Therefore, all selections of school c based on the priority rank classes are the same, and subsequently all selections based on preference rank classes are the same at $(succ', P)$ and at $(\succ, P)$. Thus, Ch_c makes the same selections at $\bar{\succ}_c$ as at $\succ_c$, and $f^{v,p}(\bar{\succ}, P) = f^{v,p}(\succ, P)$, as desired. Therefore, $f^{v,p}(\bar{\succ}, P) = c$ and $cP_s f^{v,p}(\succ', P)$. Since $r_s(c) \leq r_{\hat{s}}(c)$ at P and $s \bar{\succ}_c \hat{s}$, $f^{v,p}$ is not PP-stable. This is a contradiction, which implies that $f^{v,p}$ is an Equitable PRP rule. □

We will say that a matching rule is **equitable-rank-partition stable** if the definition of rank-partition stability for this rule is satisfied based on (v, p), where p is homogeneous. Equivalently, if there is a homogeneous preference rank partition p across students, that can be used in the construction of the priority table $\tilde{\succ}((\succ, P), (v, p))$ at each profile $(\succ, P)$. Then the matching rule equitable-rank-partition stable. Clearly, equitable-rank-partition stability implies rank-partition stability of a matching rule.

The next result clarifies the relationship between rank-partition stability and equitable-rank-partition stability. The proof of this theorem and all subsequent omitted proofs are in Appendix B.

Theorem 1. *A rank-partition stable matching rule is PP-stable if and only if it is equitable-rank-partition stable.*

It is not difficult to see that each Equitable PRP rule is equitable-rank-partition stable. As in the case of PRP rules, we can easily verify that the Equitable PRP rule $f^{v,p}$ is equitable-rank-partition stable, that is, rank-partition stable with respect to (v,p) where p is a homogeneous preference rank partition profile, and $f^{v,p}(\succ, P)$ is the (v,p)-optimal matching at each profile $(\succ, P)$. Thus, we can state a similar result to Proposition 1 for Equitable PRP rules: each Equitable PRP rule $f^{v,p}$ is equitable-rank-partition stable and selects the (v,p)-optimal rank-partition stable matching at each profile $(\succ, P)$. Therefore, Theorem 1 is a generalization of Proposition 2. We provided a separate proof of Proposition 2 to give some intuition for the result which cannot easily be gleaned from the proof of Theorem 1.

3.7 Efficiency of PRP Matching Rules

PRP rules select the optimal rank-partition stable matching at each profile, as seen in Section 5. This implies that if the selected matching is not Pareto-efficient, it can only be Pareto-dominated by a matching that is not rank-partition stable. We know that the DA matching is in general not Pareto-efficient (Gale and Shapley, 1962, [30]; Balinski and Sönmez, 1999, [**?**]) so PRP rules, which choose the DA matching for the constructed priority profile, as stated by Proposition 1, are generally not Pareto-efficient. One notable exception is the Boston rule, which is Pareto-efficient (Abdulkadiroğlu and Sönmez, 2003, [2]), due to the fact that it assigns the school seats based primarily on the student preferences, and uses the school priorities only for tie-breaking, and thus cannot be Pareto-dominated by a matching which is not rank-partition stable. One may therefore conjecture that the only Pareto-efficient PRP rule is the Boston rule, but it turns out that a somewhat larger set of PRP rules is Pareto-efficient. Namely, there may be one student whose preference rank partition is not necessarily the finest, and may be chosen arbitrarily, but all other students' preference partitions have to be the finest. We call this class of rules, including the Boston rule, Near-Boston rules.

Near-Boston rules are PRP rules such that:

1. each school has the coarsest priority partition;
2. there exists $s_j \in S$ such that each student $s \in S \setminus \{s_j\}$ has the finest preference partition (and student s_j has an arbitrary preference partition).

Theorem 2. *A rank-partition stable rule is Pareto-efficient if and only if it is a Near-Boston rule.*

Corollary to Theorem 2. *An equitable-rank-partition stable rule is Pareto-efficient if and only if it is the Boston rule.*

This is an immediate corollary to Theorem 2, since if the rule is equitable-rank-partition stable then the rule uses a homogeneous preference rank partition profile p, and thus Theorem 2 implies that all students have the finest preference rank partition. This corollary generalizes the result of Chen and Kesten (2017, [16]) which shows that only the Boston rule is Pareto-efficient within the class of Application-Rejection rules.

3.8 Incentive Properties of PRP Matching Rules

Since PRP rules are generally not strategyproof, it is important to study their incentive properties.

For matching rule φ, given a profile $(\succ, P)$, if there is a student $s \in S$ and an alternative preference ranking $P'_s \in \mathcal{P}_s$ such that $\varphi_s(\succ, (P'_s, P_{-s})) P_s \; \varphi_s(\succ, P)$ then s can **manipulate** φ at P via P'_s, and rule φ is **manipulable** at P. We will also say that s can manipulate at P to obtain school $\varphi_s(\succ, (P'_s, P_{-s}))$. If a rule is not manipulable at any preference profile then the rule is **strategyproof.**

The standard Deferred Acceptance rule is well-known to be strategyproof when only the students' incentives are taken into account (Dubins and Freedman, 1981, [18]; Roth, 1982, [47]). However, we can find examples of preference profiles where a PRP rule (other than the DA) is manipulable, and we prove a negative result for all PRP rules excluding the DA rule. This makes sense intuitively: the DA is the only strategyproof rule in the class of PRP matching rules, since it is the only PRP rule for which the school choice functions are independent of the preferences. This result extends a similar result by Chen and Kesten (2017, [16]) for Application-Rejection rules.

Theorem 3. *A rank-partition stable rule is strategyproof if and only if it is the Deferred Acceptance rule.*

Although PRP rules are not strategyproof in general, we can compare them based on their manipulability, using a simple comparison criterion that was first put forward by Pathak and Sönmez (2013, [46]) and subsequently studied by Chen et al. (2016, [15]).

Given a matching rule φ, for all students $s \in S$ and all profiles $(\succ, P)$, let $I(s, \varphi, (\succ, P)) = \{\varphi_s(\succ, (P'_s, P_{-s})) : P'_s \in \mathcal{P}_s,\ \varphi_s(\succ, (P'_s, P_{-s})) P_s \varphi_s(\succ, P)\}$. A matching rule φ is **less manipulable** than matching rule ψ if for all $s \in S$ and all profiles $(\succ, P)$, $I(s, \varphi, (\succ, P)) \subseteq I(s, \psi, (\succ, P))$ and there exists a profile $(\bar{\succ}, \bar{P})$ for which $I(s, \varphi, (\bar{\succ}, \bar{P})) \subset I(s, \psi, (\bar{\succ}, \bar{P}))$.

A PRP rule and all other rules which satisfy rank-partition stability with respect to the same (v, p) are comparable in terms of how vulnerable to manipulability they are according to the above definition, and the PRP rule stands out as the least manipulable in this class of rules. Formally, each PRP rule $f^{v,p}$ is less manipulable than any other matching rule which is rank-partition stable with respect to (v, p). A similar result has been obtained by Bonkoungou (2019, [12]) for the French Priority rules. These results follow from a general relationship between weak Pareto-domination and relative manipulability, as shown by Chen et al. (2016, [15]). According to their results, if a rule φ weakly Pareto-dominates another one rule ψ, then φ is less manipulable than ψ. Thus, the result follows from the optimality of PRP rules.

The following theorem is our main theorem on the incentive properties of PRP rules. The theorem says that when a PRP matching rule is used, a student cannot manipulate to obtain a seat at school c by placing c in the same or a lower preference rank class than the preference rank class where c belongs truthfully. This theorem has some interesting and wide-ranging implications for PRP rules, as we will explain below.

Theorem 4. *Let $f^{v,p}$ be a PRP rule and fix a profile $(\succ, P)$. Let $s \in S$ and $c \in C$ such that $cP_s f_s^{v,p}(\succ, P)$. Let $P'_s \in \mathcal{P}_s$ such that c is in the same or lower preference rank class in P'_s than in P_s, given p. Then $f_s^{v,p}(\succ, (P'_s, P_{-s})) \neq c$.*

By this theorem, a seat at a school can only be obtained by manipulation when reporting the school to be in a higher preference class than it truthfully is, regardless of what the reported preferences are otherwise. This gives a good idea about how the PRP rules are manipulable in general. The theorem also offers an intuitive explanation for two well-known results: why the DA rule is not manipulable, and why the Boston rule is so markedly manipulable (see, for example, Troyan and Morrill, 2019, [53]). Notably, given that the standard DA rule is the PRP rule with the coarsest preference partition for each student, this theorem implies that the Deferred Acceptance rule is strategyproof, since there is only one preference class for each student, and thus no school can be obtained by manipulation at any profile. At the other extreme, the Boston rule is the PRP rule with the finest preference partition for each student combined with the coarsest priority partition for each school (so priorities are only used to break ties), and thus the theorem sheds light on why the Boston rule is so manipulable: each change in the reported preferences results in placing at least one school in a higher preference class, and the top-ranked school may be the only one which cannot be obtained by manipulation when the Boston rule is used.

PRP rules between the DA and Boston rules are moderately manipulable, and the extent depends on how coarse their preference partitions are. For example, the Application-Rejection rules have homogeneous preference rank classes, and larger preference rank classes would imply, based on the theorem, that there is generally less room for manipulation. Indeed, this exact result is shown by Chen and Kesten (2017, [16]), although this is not a direct implication of Theorem 4. When the preference rank partition is not homogeneous, that is, the PRP rule is not an Equitable PRP rule, the extent to which the PPR rule is vulnerable to manipulation varies with the student. In the extreme case of Favored Students rules, where each student has either the coarsest or the finest preference rank partition, it follows from the theorem that the favored students with the coarsest preference partitions cannot manipulate at all. However, the non-favored students, having the finest preference partitions, have lots of room to manipulate, according to the theorem.

We obtain the following results immediately from Theorem 4.

Corollary to Theorem 4. *Let $f^{v,p}$ be a PRP rule.*

1. *For all $s \in S$, if some school c is ranked in s's top preference rank class at some preference profile P, given p, then s cannot manipulate to obtain school c at P when using rule $f^{v,p}$.*

2. *For all students $s \in S$, if $p_s = (n)$ then student s cannot manipulate $f^{v,p}$ at any profile $(\succ, P)$.*

The first result in the corollary says that a school which is ranked in the top preference class of a student cannot be obtained as a result of successful manipulation by this student. This implies the result by Bonkoungou (2017, [11]) that a student cannot obtain his first-ranked school by manipulating any French Priority rule, and specifically the Boston rule. Our corollary is more general, since it allows for coarse preference partitions and thus for multiple schools in the top preference class that cannot be profitably obtained by misreporting the preferences. This, in turn, is further generalized by Theorem 4, since it also shows that trying to manipulate by ranking a school in a preference class that is either the same or below the preference class of the school to which a student would be matched otherwise is futile. Note that Theorem 4 is unrelated to the incentive analysis of Bonkoungou (2019, [12]), since he studies the given coarse priorities of French Priority rules, and compares the matching rules using a new criterion based on how fine the priority partitions are, while our results concern the preference partitions.

The second result in the corollary states that if a student has the coarsest preference partition then this student cannot manipulate any PRP rule. This is because in this case the student has every school in her top preference class. This is the case for all students in the DA rule, and this also holds for all favored students in Favored Students rules.

3.9 Coarse Priorities: An Extension

PRP matching rules can be extended naturally to the case where the given priorities are coarse (i.e., given by a weak order), as would be typical for New York City high schools (Abdulkadiroğlu et al., 2009, [1]). Another well-known example is Boston, where school priorities are set up based on older siblings attending the school and walk-zone priorities (Abdulkadiroğlu et al., 2005, [4]). In international refugee assignment, countries may have mandated priorities based on the level of danger faced by refugees, or may prioritize specific vocational or language skills, which leads to coarse priorities, given the number of refugees who are looking for asylum (Jones and Teytelboym, 2017, [34]). If priorities are coarse, a strict priority ordering $\succ_c$ that is consistent with the given coarse priorities would be determined by a lottery for each school c and used for tie-breaking only if the choice function cannot make the required selection based on the coarse priorities and the preference rank partitions. Then the given coarse priorities for schools would be primitives of the model, not part of the specification of the rule, and the randomly selected strict priority profile $\succ$, which is consistent with the coarse priorities, would become part of the matching rule as a tie-breaker. This would imply that each member of the family of PRP rules is associated with the strict tie-breaker priority profile $\succ$ and the preference rank partition profile p. This is the setup of Bonkongou (2019, [12]), since universities in France only have weak priorities over students.

This lends additional applicability to PRP rules and makes the current study relevant to situations where coarse priorities arise naturally. We could further enhance the applicability of PRP rules by combining the primary and secondary interpretations (strict versus coarse priorities as primitives) in a natural manner, which would allow for coarse priorities as primitives but let the matching rule further partition the priority rankings within the priority rank classes given by the fixed coarse priorities, while breaking the ties randomly over the remaining weak priorities to create a tie-breaker. Our results extend to all these "hybrid" cases as well in a natural manner, but in the interest of a more accessible exposition we omit the formal presentation of this more general setup.

3.10 Conclusion

We have explored PRP rules, which are generalized Deferred Acceptance rules that allow students' preference rankings to play a role in the schools' choice functions, that is, the selection among competing applicants for a school is based partially on how high students rank the particular school in their preferences, and not only on their priority rankings by this school. Since we consider a large class of matching rules that includes many previously known rules, in addition to interesting new rules that are studied here for the first time, this paper offers a unified approach to these rules and establishes results that apply to all of them. As a foundation for our analysis that ties together all PRP rules, we show that all these matching rules can be understood as the DA matching selected on the basis of an appropriately modified priority profile at each preference profile, where the modifications of the priorities reflect how PRP school choice functions select students based on preference ranks (Proposition 1). Thus, each PRP rule is optimal in this sense, just like the Deferred Acceptance rule.

We have proved that the only Pareto-efficient PRP rules are the so-called Near-Boston rules (Theorem 2), a class of rules which includes the Boston rule and other similar rules which only differ in the preference rank partition of one student from the Boston rule. We have also shown that the only strategyproof PRP rule is the standard Deferred Acceptance rule (Theorem 3). These two results underline the difficulty of obtaining matching rules with both good incentives and strong efficiency properties, when it is desirable to allow the preferences of students to directly affect their chances of being accepted by the school, but the students' priorities are also taken into account.

We explore the classic tension among stability, efficiency, and incentive properties in our setup, which has been studied by a vast array of papers in various matching models and settings. Our main contribution to this literature is to show the specifics of this tension within the class of PRP rules, that is, when student selection by schools takes into account the preference rankings, and Theorems 2 and 3 can be best understood in terms of trade-offs. Namely, when the school priorities have a small impact (mainly used for tie-breaking), we get more efficiency, since the matching rule relies primarily on the preference rankings, as in the Boston rule and similar PRP rules. When the school priorities have a large impact, preference rankings play only a small role in the schools' selection, and the incentive properties are improved, but it is only in the extreme case of the DA rule, when the preferences have no impact on the school choice functions, that full strategyproofness can be achieved.

The main trade-off is between efficiency and incentives when rank-partition stability is required, a stability condition which allows matchings to be based on student preference rankings in addition to school priorities in a systematic manner: namely, depending on the preference rank classes, students who place the school in question in a higher preference rank class have a higher priority for the school. As we have seen, we get some extreme rules only in the family of PRP rules or, more broadly, among rank-partition stable rules, that satisfy efficiency (the Near-Boston rules) and, similarly, only the DA rule satisfies strategyproofness among PRP or rank-partition stable rules. The Boston and DA rules can be seen as the two extreme members of the family of PRP rules. The intuition is that as we place more emphasis on the preferences, we attain more efficiency and get closer to the Boston rule, and as we place more emphasis on the priorities, we get better incentives and get closer to the DA rule. This already gives a general idea to the designer about how to select among PRP rules, depending on the relative importance of the objectives: is efficiency more desirable, or are correct incentives more preferable? If efficiency comes first, choose coarser priority partitions and finer preference partitions, and if incentives are more important then choose finer priority partitions and coarser preference partitions.

One may argue, however, that incentives should always come first, given that if preferences are not reported accurately then efficiency cannot be enforced, since any normative criterion can only be based on the reported preferences, the only input into the mechanism regarding student welfare. But, especially in light of the substantial trade-offs, this argument also has some limitations, and thus the extent and specifics of manipulation are of considerable interest to the designer. Our main result on incentives (Theorem 4) provides guidance on this, as it sheds light on how students can manipulate PRP rules: we show that this is only possible by placing a desired but unattainable school in a higher preference rank class in the reported preferences than it is truthfully. This theorem also clarifies the different potential extent of manipulation for different PRP rules, both in terms of how and whether a school can be obtained by manipulation, depending on its preference rank classes as reported by different students, and also regarding the scope of manipulation which may differ widely among students, even for Equitable PRP rules.

Theorem 4 therefore provides insight and contributes to a more sophisticated understanding of PRP rules than afforded by the rough comparison of the Boston and DA rules themselves. Between the extremes of the DA and Boston rules, several different dimensions may be considered when searching for an appropriate matching rule, as evidenced by the fact that the family of PRP rules encompasses multiple subfamilies of PRP rules that bridge these same two matching rules, such as the First Preference First rules (or more generally the generalized Secure Boston rules), the Deferred Boston rules, the Application Rejection (or Taiwan Deduction) rules and the Favored Student rules. Given that quite a few members of the class of PRP rules are being used in real-life school choice and student placement, our theoretical analysis provides practically relevant policy insight. Therefore, our findings should be helpful for the design of assignment mechanisms that rely on the preference rankings directly when choosing among competing applicants.

3.11 Appendix A: Comparison of PRP and General French Priority Rules

We show below that PRP rules are not a subset of General French Priority (GFP) rules, and GFP rules are not a subset of PRP rules.

Claim 1. *$PRP \not\subset GFP$*

Proof. Let the priority profile $\succ$ be given as below, and let all priority partitions be the coarsest for a fixed PRP rule f. Each school has a capacity of one.

$\succ_a$	$\succ_b$	$\succ_c$
1	2	3
3	1	4
	3	

Consider preference profile $P = (P_1, P_2, P_3, P_4)$, where the bar indicates the homogeneous preference partitions. The unspecified preferences are arbitrary.

P_1	P_2	P_3	P_4
b	b	a	c
a		b	
		c	

PRP procedure at P			
Round	a	b	c
1	$\underline{3}$	$1, \underline{2}$	$\underline{4}$
2	$\underline{1}, 3$	$\underline{2}$	$\underline{4}$
3	$\underline{1}$	$\underline{2}, 3$	$\underline{4}$
4	$\underline{1}$	$\underline{2}$	$3, \underline{4}$

The rounds of the PRP procedure at P are displayed in the table above. Note that in round 4 the choice function of school c for PRP rule f selects student 4 over 3. If students 3 and 4 were not tied, then any GFP rule would select 3 over 4, given $3 \succ_c 4$. Thus, 3 and 4 are tied for c in this example if the rule is a GFP rule.

Now consider preference profile $P' = (P'_1, P'_2, P'_3, P'_4)$, where the bar indicates the same homogeneous preference partitions as before for PRP rule f.

P_1	P_2	P_3	P_4
a	b	a	c
		c	

PRP procedure at P'			
Round	a	b	c
1	$\underline{1}, 3$	$\underline{2}$	$\underline{4}$
2	$\underline{1}$	$\underline{2}$	$\underline{3}, 4$

The first two rounds of the PRP procedure f at P' are displayed in the table above. Given that students 3 and 4 are tied for school c for any GFP rule, a GFP rule would select student 4 over 3 for school c, since 4 ranks c first, while 3 ranks c second. Thus, if school a is in the same category for the GFP rule as school c, 4 would be selected by c over 3 based on the relative ranking of c, and if school a is not in the same category for the GFP rule as school c, 4 would be selected by c over 3 based on the absolute ranking of c. This proves that the specified PRP rule is not a GFP rule, since the PRP rule chooses 3 over 4. $\square$

More generally, note that matching rules which allow for coarse preference partitions (not the finest) are generally not GFP rules. For example, the class of Application-Rejection rules of Chen and Kesten (2017) is not a subclass of GFP rules. Note that our example is an Application-Rejection rule, with the coarsest priority partition profile and homogeneous preference partitions. This also demonstrates that there are Equitable PRP rules which are not GFP rules.

Claim 2. *$GFP \not\subset PRP$*

Proof. Consider a GFP rule φ with the following two categories: category $\{a, b\}$ and category $\{c, d\}$. Let the priority profile $\succ$ be given as below, and let all priority partitions be the coarsest for GFP rule φ. Each school has a capacity of one.

$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
4	5	4	3
2	2	1	1
1	1	2	

Consider preference profile $P = (P_1, P_2, P_3, P_4, P_5)$. The unspecified preferences are arbitrary.

P_1	P_2	P_3	P_4	P_5
d	b	d	c	b
c	a			
a	c			
b	d			

GFP procedure at P				
Round	a	b	c	d
1		$2, \underline{5}$	$\underline{4}$	$1, \underline{3}$
2	$\underline{2}$	$\underline{5}$	$1, \underline{4}$	$\underline{3}$
3	$\underline{1}, 2$	$\underline{5}$	$\underline{4}$	$\underline{3}$

The first three rounds of the GFP procedure at P are displayed in the table above. Note that in round 3 the choice function of school a for GFP rule φ selects student 1 over 2, based on the relative ranking of a in category $\{a, b\}$. This selection implies for any PRP rule that $p_1^1 \geq 3$ and $p_2^1 = 1$.

Now consider preference profile $P' = (P'_1, P'_2, P'_3, P'_4, P'_5)$.

P'_1	P'_2	P'_3	P'_4	P'_5
d	b	d	a	b
c	a			
a	c			
b	d			

GFP procedure at P				
Round	a	b	c	d
1	$\underline{4}$	$2, \underline{5}$		$1, \underline{3}$
2	$2, \underline{4}$	$\underline{5}$	$\underline{1}$	$\underline{3}$
3	$\underline{4}$	$\underline{5}$	$1, \underline{2}$	$\underline{3}$

The first three rounds of the GFP procedure at P' are displayed in the table above. Note that in round 3 the choice function of school c for GFP rule φ selects student 2 over 1, based on the relative ranking of d in category $\{c, d\}$. Since for any PRP rule $p_1^1 \geq 3$ and $p_2^1 = 1$, as shown above, $c \in P_1^1$ and $c \notin P_2^1$. This implies that any PRP rule would select 1 over 2 for school c. This proves that the specified GFP rule is not a PRP rule. □

However, the class of French Priority (FP) rules, for which there is only a single category (or alternatively, each school is in its own category), is a strict subclass of PRP rules. Indeed, in our example the GFP rule is not an FP rule, since there are two categories with two schools in each category.

3.12 Appendix B: Proofs

Proof of Theorem 1

Theorem 1. *A rank-partition stable matching rule is PP-stable if and only if it is equitable-rank-partition stable.*

Proof.

Claim 1.1: An equitable-rank-partition stable matching rule is PP-stable.

Proof: Let φ be an equitable-rank-partition stable matching rule. Suppose that there exists a profile $(\succ, P)$ such that $\varphi(\succ, P)$ is not PP-stable. Then there exist $s, \hat{s} \in S$ and $c \in C$ such that $r_s(c) \leq r_{\hat{s}}(c)$ at P, $s \succ_c \hat{s}$, and s envies $\hat{s}$ at $(\succ, P)$ for being assigned to c. That is, $\varphi_{\hat{s}}(\succ, P) = c$ and $cP_s\varphi_s(\succ, P)$. Let φ be equitable-rank-partition stable with respect to (v, p). Then p is homogeneous across students. Given p, let $t, t' \geq 1$ such that $c \in P_s^t$ and $c \in P_{\hat{s}}^{t'}$. Given v, let $k, k' \geq 1$ such that $s \in \succ_c^k$ and $\hat{s} \in \succ_c^{k'}$. Let $\tilde{\succ}$ denote the constructed strict priority profile $\tilde{\succ}((\succ, P), (v, p))$. Suppose $k < k'$. Then $s \tilde{\succ}_c \hat{s}$ and s has justified envy at $\varphi(\succ, P)$ based on $\tilde{\succ}_c$. This contradicts the fact that φ is equitable-rank-partition stable. Thus, $k \geq k'$. If $k > k'$ then $\hat{s} \succ_c s$ would follow, which is a contradiction. Therefore $k = k'$.

Now suppose $t < t'$. Then, since $k = k'$, $s \tilde{\succ}_c \hat{s}$, and s has justified envy at $\varphi(\succ, P)$ based on $\tilde{\succ}_c$. This again contradicts the fact that φ is equitable-rank-partition stable, and thus $t \geq t'$. Then, since $r_{s(c)} \leq r_{\hat{s}}(c)$ and p is homogeneous across students, it must be the case that $t = t'$. But then $s \succ_c \hat{s}$ implies that $s \tilde{\succ}_c \hat{s}$, since the tie-breaker is applied in the construction of $\tilde{\succ}_c$ when determining the relative positions of s and $\hat{s}$ in $\tilde{\succ}_c$. This implies that s has justified envy at $\varphi(\succ, P)$, based on $\tilde{\succ}_c$, which contradicts the fact that φ is equitable-rank-partition stable. Therefore, $\varphi(\succ, P)$ is PP-stable, and since this holds for all $(\succ, P)$, φ is PP-stable.

Claim 1.2: If a rank-partition stable matching rule is PP-stable then it is equitable-rank-partition stable.

Proof: Let φ be a rank-partition stable matching rule which is PP-stable. Let φ be rank-partition stable with respect to (v, p). Suppose that p is not homogeneous across students. Then there exist $(\succ, P)$, $c \in C$ and $s, \hat{s} \in S$ such that

1. $\varphi_{\hat{s}}(\succ, P) = c$ and $cP_s\varphi_s(\succ, P)$

2. $r_s(c) \leq r_{\hat{s}}(c)$ at P

3. $t > t'$, where $c \in P_s^t$ and $c \in P_{\hat{s}}^{t'}$

4. s and $\hat{s}$ are in the same priority rank class for c, given $\succ_c$ and v_c: there exists $k \geq 1$ such that $s, \hat{s} \in \succ_c^k$.

Let $\tilde{\succ}$ denote the constructed priority profile $\tilde{\succ}((\succ, P), (v, p))$. Since φ is PP-stable, given $r_{s(c)} \leq r_{\hat{s}}(c)$, we have $\hat{s} \succ_c s$. Since s and $\hat{s}$ are in the same priority rank class for c, given $\succ_c$ and v_c, and since $t > t'$, $\hat{s} \tilde{\succ}_c s$. Let $\succ_c'$ be the same as $\succ_c$, except for the positions of $\hat{s}, s$ and students ranked between $\hat{s}$ and s in $\succ_c$. Let $s \succ_c' \hat{s}$, and for all $\bar{s}$ such that $\hat{s} \succ_c \bar{s} \succ_c s$, if $c \in P_{\bar{s}}^{t'}$, let $\hat{s} \succ_c' \bar{s}$, and if $c \in P_{\bar{s}}^t$, let $\bar{s} \succ_c' s$. Leave all other orderings in $\succ_c'$ as in $\succ_c$ and specifically keep the set of students in the priority class of s and $\hat{s}$ the same.

Let $\succ' \equiv (\succ_c', \succ_{-c})$. Let $\tilde{\succ}'$ denote the constructed priority profile $\tilde{\succ}'((\succ', P), (v, p))$. We will show that $\tilde{\succ}' = \tilde{\succ}$. Given that both are a function of $(\succ, P)$ and (v, p), and that only $\succ_c$ has changed, the only possible difference is between $\tilde{\succ}_c'$ and $\tilde{\succ}_c$, while $\tilde{\succ}'_{-c} = \tilde{\succ}_{-c}$.

We will show that $\tilde{\succ}'_{-c} = \tilde{\succ}_{-c}$. Given the difference between $\tilde{\succ}_c'$ and $\tilde{\succ}_c$, all students are still in the same priority rank class for school c at $\tilde{\succ}_c'$ as at $\tilde{\succ}_c$, and since the preference profile is unchanged, given $t > t'$ and $\hat{s} \tilde{\succ}_c' s$. Note that whenever the selection is based on the preference rank classes among the students in $\succ_c'^k$, the strict priority order $\succ_c'$ of the students within this set is irrelevant. Moreover, if the choice is based on the tie-breaker, the relevant orderings are preserved by construction. Thus, $\tilde{\succ}_c' = \tilde{\succ}_c$, which implies that $\tilde{\succ}' = \tilde{\succ}$. Since $s \succ_c' \hat{s}$ and $r_s(c) \leq r_{\hat{s}}(c)$ at $(\succ, P)$, PP-stability of φ implies that $\varphi(\succ', P) \neq \varphi(\succ, P)$. However, since $\tilde{\succ}' = \tilde{\succ}$, the rank-partition stability of φ implies that $\varphi(\succ', P) = \varphi(\succ, P)$, and we have a contradiction. Thus, p is homogeneous and φ is equitable-rank-partition stable. □

Proof of Theorem 2

Theorem 2. *A rank-partition stable rule is Pareto-efficient if and only if it is a Near-Boston rule.*

Proof.

Claim 2.1: A Near-Boston rule is rank-partition stable and Pareto-efficient.

Proof: A Near-Boston rule is a PRP rule, and thus it is rank-partition stable by Proposition 1. We need to show that Near-Boston rules are Pareto-efficient. Let $f^{v,p}$ be a Near-Boston rule and fix a profile $(\succ, P)$. We will prove that $f^{v,p}(\succ, P)$ is Pareto-efficient.

Let $\mu \equiv f^{v,p}(\succ, P)$. Note that since $f^{v,p}$ is a Near-Boston rule, all priority partitions are the coarsest and there exists $j \in S$ such that for for all $s \in S \setminus \{j\}$, s's preference partition is the finest. Choose a permutation σ of the set of students S which follows the order of when the assignments are made in the $f^{v,p}$ procedure at $(\succ, P)$. Formally, for all $s \in S$, let t_s denote the round in the $f^{v,p}$ procedure at $(\succ, P)$ in which s is accepted by μ_s for the first time, and let σ be such that for all $i, l \in S$, $\sigma(i) \leq \sigma(l)$ if and only if $t_i \leq t_l$. It is well-known (see, for example, Balinski and Sönmez (1999, [9]) and Svensson (1999, [50])) that if σ is a permutation of S such that if students choose their favorite school with remaining seats following the order specified by σ and the resulting matching is μ, that is, if σ is used as in a Serial Dictatorship f^{σ}, then μ is Pareto-efficient. Clearly, σ has this property if $f^{v,p}$ is the Boston rule, that is, if j has the finest preference partition along with all other students, since in this case there are no temporary assignments in the procedure: once a school accepts a student, the assignment is final. If σ does not have this property then there exist $i, l \in S$ such that $\sigma(i) < \sigma(l)$ and $\mu_l P_i \mu_i$. This implies that student i applied for school μ_l in some round

$t < t_i$ and got rejected, which in turn implies that μ_l was already filled to capacity in round t. Since $t < t_i \leq t_l$ and l is assigned to μ_l in round t_l, it must be the case that $l = j$. Let $h \in S$ be the student who is assigned μ_j in round t, but $\mu_h \neq \mu_j$. Then h is selected over i in round t by school μ_j, but since in round t_j student j is assigned to μ_j, h is only temporarily assigned to μ_j and is rejected by μ_j in round t_j, where j is selected by μ_j over h. Note that h is unique since only p_j is a preference partition which is not the finest, and $h \neq i, j$.

Suppose, by contradiction, that μ is not Pareto-efficient. Then, given that whenever $i, l \in S$ are such that $\sigma(i) < \sigma(l)$ and $\mu_l P_i \mu_i$, we have $l = j$, as we have shown, there must be an envy chain from j to such a student i, that is, there exists student $\hat{i}$ who is assigned $\mu_{\hat{i}}$ in round $t_{\hat{i}}$, where $t_i \leq t_{\hat{i}} \leq t_j$ and $\mu_{\hat{i}} P_j \mu_j$. Note that $\hat{i} \neq j, h$ but $\hat{i} = i$ is possible. Thus, $r_j(\mu_{\hat{i}}) < r_j(\mu_j)$. Moreover, $\mu_{\hat{i}} \in P_{\hat{i}}^{t_{\hat{i}}}$, since $\hat{i} \neq h$, given that $t < t_{\hat{i}}$.

Since $p_i = p_h = (1, \ldots, 1)$, for all $c \in C,\ c \in P_i^{r_i(c)}$ and $c \in P_h^{r_h(c)}$. Moreover, in each round k where a student gets rejected by a school prior to round t_j, the student ranks the school in the kth position. Since h is selected over i in round t by μ_j, $r_i(\mu_j) = t$, $\mu_j \in P_i^t$ and thus, given that all school priorities are the coarsest, $\mu_j \in P_h^{t'}$, where $t' \leq t$.

Since $\mu_j \in P_h^{t'}$ and j is selected over h in round t_j by μ_j, we have $\mu_j \in P_j^{t''}$ such that $t'' \leq t'$, given that all the priority partitions are the coarsest. Then, since $r_j(\mu_{\hat{i}}) < r_j(\mu_j)$, $\mu_{\hat{i}} \in P_j^{\bar{t}}$ such that $\bar{t} \leq t''$. Observe that $\bar{t} \leq t'' \leq t' \leq t < t_i \leq t_{\hat{i}}$. Therefore, $\bar{t} < t_{\hat{i}}$. Since $\mu_{\hat{i}} \in P_{\hat{i}}^{t_{\hat{i}}}$ and $\mu_{\hat{i}} \in P_j^{\bar{t}}$, where $\bar{t} < t_{\hat{i}}$, it is not possible that $\hat{i}$ is selected over j by $\mu_{\hat{i}}$, and we have a contradiction. Thus, μ is Pareto-efficient and Claim 2.1 is proved.

Claim 2.2: A rank-partition stable and Pareto-efficient rule is a Near-Boston rule.

Proof: We will show that a Pareto-efficient PRP rule is a Near-Boston rule. Note that this is sufficient to prove the claim since a PRP rule chooses the optimal rank-partition stable matching at each profile $(\succ, P)$, which Pareto-dominates all other rank-partition stable matchings at this profile.

Step A: We show that if $f^{v,p}$ is a Pareto-efficient PRP rule then each school has the *coarsest priority partition.*

Let $f^{v,p}$ be a Pareto-efficient PRP rule. Suppose that there exists $a \in M$ such that v_a is not the coarsest. For now we assume that $q_a = q_b = q_c = 1$ and thus $n \geq 4$. Specify $(\succ, P)$ as follows. Let $i, j, l, h \in N$ such that

- l has the top-priority for a and j has the lowest priority for a in $\succ_a$. Note that this implies that $l \in v_a^1$ and $j \notin v_a^1$, given our assumption that v_a is not the coarsest.
- i has the lowest priority for a in $\succ_a$, except for j. Thus, $i \succ_a j$.
- j has the top priority for b and i has the lowest priority for b in $\succ_b$.
- h has the top priority for c in $\succ_c$.

All other priorities in $\succ$ are arbitrary.

Case A.1 : $v_a^1 = n - 1$

Let P be given as shown in the table below.

P_i	P_j	P_l	P_h
c	a	c	c
b	b	b	
a		0	

Assume that further preferences for $i, j, l, h \in S$ are arbitrary at P, and for all students $h' \in S \setminus \{i, j, l, h\}$, $P_{h'} \in (0)$ (i.e., they don't have any acceptable school). Assume without loss of generality that $p_i^1 \leq p_l^1 \leq p_j^1$. The rounds of the $f^{v,p}$ procedure at $(\succ, P)$ are displayed below.

Round	a	b	c
1	$\underline{j}$		$i, l, \underline{h}$
2	$\underline{j}$	$i, \underline{l}$	$\underline{h}$
3	$\underline{i}, j$	$\underline{l}$	$\underline{h}$
4	$\underline{i}$	$\underline{j}, l$	$\underline{h}$

In round 1, h is selected over i and j by c, since either h is in a higher priority class than i and l for c, or h wins on the tie-breaker, given that $r_i(l) = r_l(c) = r_h(c) = 1$ and $h \succ_c i, l$.

In round 2, l is selected over i by b, since either l is in a higher priority class than i for b, or l is selected based on preference rank classes, given that $p_i^1 \leq p_l^1$ and $r_i(b) = r_l(b) = 2$, or if neither the priority classes nor the preference classes provide a basis for selecting l over i, then l wins on the tie-breaker, given that $l \succ_b i$.

Note that since $v_a^1 = n - 1$, $i \in v_a^1$ and $i \in v_a^2$, given $\succ_a$. Thus, in round 3 student i is selected over j by a based on the priority rank classes.

In round 4, j is selected over l by b, since either j is in a higher priority class than l for b, or j is selected based on the preference rank classes, given that $p_l^1 \leq p_j^1$ and $r_j(b) = r_l(b) = 2$, or if neither the priority classes nor the preference classes provide a basis for selecting j over l, then j wins on the tie-breaker, given that $j \succ_b l$.

Now note that since i is assigned a and j is assigned b, i and j would prefer to trade their assignments and thus we have a contradiction to Pareto-efficiency.

Case A.2 : $v_a^1 < n - 1$

Let P be given as shown in the table below.

P_i	P_j	P_l	P_h
a	a	c	c
b	b	b	
0		a	

Assume that further preferences for $i, j, l, h \in S$ are arbitrary at P and for all students $h' \in N \setminus \{i, j, l, h\}$, $P_{h'} \in (0)$. Assume without loss of generality that $p_l^1 \leq p_i^1 \leq p_j^1$. The rounds of the $f^{v,p}$ procedure at $(\succ, P)$ are displayed below.

Round	a	b	c
1	$\underline{i}, j$		$l, \underline{h}$
2	$\underline{i}$	$\underline{j}, l$	$\underline{h}$
3	$\underline{i}, l$	$\underline{j}$	$\underline{h}$
4	$\underline{l}$	$i, \underline{j}$	$\underline{h}$

In round 1, i is selected over j by a, since either i is in a higher priority class than j for a, or i wins on the tie-breaker, given that $r_i(a) = r_j(a) = 1$ and $i \succ_a j$. Similarly, h is selected by c over l.

In round 2, j is selected over l by b since j has top priority for b, and thus j may be selected based on the priority classes. If j does not get selected based on the priority classes, then $p_l^1 \leq p_j^1$ and $r_j(b) = r_l(b) = 2$ imply that j may be selected based on preference classes. Finally, if j is not selected based on either the priority or preference classes, then j wins on the tie-breaker, since $j \succ_b l$.

In round 3, l is selected over i by a based on the priority classes, since $l \in v_a^1$ and $i \notin v_a^1$, given that $v_a^1 < n - 1$.

In round 4, j is selected over i by b, since either j is in a higher priority class than i for b, or j is selected based on the preference classes, given that $p_i^1 \leq p_j^1$ and $r_i(b) = r_j(b) = 2$, or if neither the priority classes nor the preference classes provide a basis for selecting j over c, then j wins on the tie-breaker, given that $j \succ_b i$.

Now note that since l is assigned a and j is assigned b, l and j would prefer to trade their assignments and thus we have a contradiction to Pareto-efficiency.

In order to relax the assumption that $q_a = q_b = q_c = 1$ and generalize both Case A.1 and Case A.2, since there exist $a, b, c \in C$ such that $q_a + q_b + q_c < n$, we can introduce additional students with the top priorities for the relevant schools such that each student with the top priority for a relevant school ranks the school first, and this would allow for getting the same contradiction in both cases as before. Since this is a straightforward extension, we omit the tedious details. Finally, note that since v_a is not the coarsest, Cases A.1 and A.2 cover all possible cases and thus Step A is completed.

Step B: We show that if $f^{v,p}$ is a Pareto-efficient PRP rule then it is a Near-Boston rule.

Suppose that $f^{v,p}$ is Pareto-efficient but it is not a Near-Boston rule. Then there exist $j, l \in N$ with at least one preference class each which are minimally size 2. Since the preference classes which are larger than size 1 have to be relevant, that is, there must exist a profile where these sizes matter, it means that there are enough students who can have top priorities at relevant schools, and since we can assume that these students will rank their top-priority schools first, to reduce the technical details we can assume without loss of generality that the size 2 preference classes are the top preference classes for students j and l, and by a similar argument we can assume without loss of generality that $q_a = q_b = q_c = 1$ for schools $a, b, c \in C$, where $q_a + q_b + q_c < n$.

We specify $(\succ, P)$ as follows. Let the preferences for $i, j, l \in S$ be given as shown in the table below.

P_i	P_j	P_l
a	a	b
	b	a

Note that student j has both a and b in the top preference class, and so does student l: $a, b \in P_j^1$ and $a, b \in P_l^1$.

Let $l \succ_a i \succ_a j$ and $j \succ_b l$. Note that all priority partitions are coarse by Step A, since $f^{v,p}$ is a Pareto-efficient rule. The rounds of the PRP rule applied to $(\succ, P)$ are displayed below, with the selections underlined in each round.

Round	a	b
1	$\underline{i}, j$	$\underline{l}$
2	$\underline{i}$	$\underline{j}, l$
3	$i, \underline{l}$	$\underline{j}$

Now note that since l is assigned a and j is assigned b, l and j would prefer to trade their assignments and thus we have a contradiction to Pareto-efficiency. □

Proof of Theorem 3

Theorem 3. *A rank-partition stable rule is strategyproof if and only if it is the Deferred Acceptance rule.*

Proof.

Step 1: *The only strategyproof PRP rule is the DA.*

Let $f^{v,p}$ be a PRP rule which is not the DA. Suppose that $f^{v,p}$ is strategyproof. Since $f^{v,p}$ is not the DA, there exists $c_1 \in C$ such that v_{c_1} is not the finest, and there exists $s^* \in S$ such that p^*_s is not the coarsest. Then there exists rank $r \geq q_{c_1}$, such that the students ranked in the rth and $(r+1)$th positions are in the same priority rank class according to v_{c_1}. Thus, note that $q_{c_1} < n$. Let $s_n \equiv s^*$. Fix $\succ_{c_1}$ with ranks $r_{c_1}(s^*) = r$, $r_{c_1}(s_1) = r+1$, and for all $i = 2, \ldots, q_{c_1}$, $r_{c_1}(s_i) < r$. Let $c_2, c_3, \ldots, c_m$ be arranged in ascending order of their capacities, that is, $q_{c_2} \leq q_{c_3} \leq \ldots \leq q_{c_m}$. Let $\succ$ be such that for all $t = 2, \ldots, m$, $\succ_{c_t}$ ranks s^* last: $r_{c_t}(s^*) = n$.

Consider each seat of each school as a separate item, denoted by $\bar{c}_1, \ldots, \bar{c}_{\bar{m}}$, where $\bar{m} \equiv \sum_{c \in C} q_c$. For all $t = 1, 2, \ldots, m$, let $Q^t \equiv \sum_{l=1}^{t} q_{c_l}$. Moreover, let $\bar{c}_1, \ldots, \bar{c}_{q_c}$ denote the seats at school c_1, and for all $t = 2, \ldots, m$, let $\bar{c}_{Q^{t-1}+1}, \ldots, \bar{c}_{Q^t}$ denote the seats at school c_t.

- **Case 1:** $n \leq \bar{m}$

  Let $\tilde{m} \geq 2$ be such that $Q^{\tilde{m}} \geq n$ and $Q^{\tilde{m}-1} < n$. Let $P_{s^*} \in (c_2, c_3, \dots, c_{\tilde{m}-1}, c_1, c_{\tilde{m}})$, and for all $i = 1, \dots, n-1$, let $P_{s_i} \in (\bar{c}_i)$.

  The rounds of the PRP procedure at $(\succ, P)$ are as follows.

  Round 1: The only conflict is for c_2, among students $s_{q_{c_1}+1}, \dots, s_{q_{c_1}+q_{c_2}}$ and s^*. Based on $\succ_{c_2}$, since $\succ_{c_2}$ ranks s^* last, either students $s_{q_{c_1}+1}, \dots, s_{q_{c_1}+q_{c_2}}$ are selected or all students applying to c_2 are selected, depending on v_{c_2}. If it is the latter, then since all students in the applicant pool rank c_2 first, regardless of P the selection is based on the tie-breaker $\succ_{c_2}$. Given that $\succ_{c_2}$ ranks s^* last, students $s_{q_{c_1}+1}, \dots, s_{q_{c_1}+q_{c_2}}$ are selected by c_2 in this case, too. In sum, s^* is rejected by c_2 and all other applications are accepted in round 1.

  Round 2: s^* applies to c_3, and the only conflict is for c_3, among students $s_{q_{c_1}+q_{c_2}+1}, \dots, s_{q_{c_1}+q_{c_2}+q_{c_3}}$ and s^*. Here again s^* is rejected by c_3, and all the previous tentative acceptances remain. The arguments are similar to the round 1 arguments, except that since s^* ranks c_3 second, while all other students in the applicant pool for c_3 rank c_3 first, depending on p, s^* could be rejected based on the preference rank classes, if s^* was not rejected already based on the priority rank classes of c_3, given that $\succ_{c_3}$ ranks s^* last. Again, if s^* is not rejected based on priority or preference rank classes by c_3, then s^* is rejected by c_3 based on the tie-breaker $\succ_{c_3}$.

  And so on, we can repeat similar arguments for Rounds 3 to $\tilde{m} - 2$: s^* applies to schools $c_4, ..., c_{\tilde{m}-1}$ in these rounds, respectively, and gets rejected in each round, while all other students remain tentatively matched to their first-choice school.

  Round $\tilde{m} - 1$: s^* applies to c_1, and the only conflict is for c_1, among students $s_1, \dots, s_{q_c}$ and s^*. Since for all $i = 2, \dots, q_{c_1}$, $r_{c_1}(s_i) < r$, $r_{c_1}(s^*) = r$ and $r_{c_1}(s_1) = r + 1$, and since according to v_{c_1} students s^* and s_1, are in the same priority rank class, each student in the applicant pool of c_1 is selected, given $\succ_{c_1}$. Now note that, given $q_{c_2} \leq q_{c_3} \leq \ldots \leq q_{c_m}$ and $Q^{\tilde{m}} \geq n$, s^* is guaranteed to be assigned at worst the $\tilde{m}$th-ranked school under any reported preferences of s^*. Then, since p_{s^*} is not the coarsest, $c_1 \notin P^1_{s^*}$. Since $s_1, \ldots, s_{q_{c_1}}$ rank c_1 first, s^* is rejected by c_1 based on the preference rank classes. This implies that $f^{v,p}_{s^*}(\succ, P) = c_{\tilde{m}}$ and $c_1 P_{s^*} f^{v,p}_{s^*}(\succ, P)$.

Now let $\tilde{P}_{s^*} \in (c_1)$. In round 1 of the PRP procedure at $(\succ, (\tilde{P}_{s^*}, P_{-s^*}))$ based on the priority rank classes specified by v_c, none of the students in the applicant pool for c_1, namely $s_1, \ldots, s_{q_c}$ and s^* are rejected, as shown before. Since all students rank c_1 first in this applicant pool, none of them are rejected based on the preference rank classes. Therefore, the selection is based on the tie-breaker $\succ_{c_1}$. Given that for all $i = 2, \ldots, q_c$, $r_{c_1}(s_i) < r$, $r_{c_1}(s^*) = r$ and $r_{c_1}(s_1) = r + 1$, s_1 is rejected and all other applying students are tentatively matched to c_1. Since $P_{s_1} \in (c_1)$, individual rationality of $f^{v,p}$ implies that the procedure stops after round 1 at $(\succ, (\tilde{P}_{s^*}, P_{-s^*}))$, and thus $f^{v,p}_{s^*}(\succ, (\tilde{P}_{s^*}, P_{-s^*})) = c_1$. This means that s^* can manipulate at P via $\tilde{P}_{s^*}$, which contradicts the strategyproofness of $f^{v,p}$.

- **Case 2:** $n > \bar{m}$

  Let $s^* \equiv s_{\bar{m}+1}$ and let $P_{s^*} \in (c_2, c_3, \ldots, c_{m-1}, c_m, c_1)$. For all $i = 1, \ldots, \bar{m}$, let $P_{s_i} \in (\bar{c}_i)$, and for all $i = \bar{m} + 2, \ldots, n$, let $P_{s_i} \in (s_i)$. For this case we can use a similar argument as for Case 1, given the above modifications. Since c_1 is ranked last by s^* and P_{s^*} is not the coarsest, $c_1 \notin P^1_{s^*}$. Therefore, we have $f^{v,p}_{s^*}(\succ, P) = 0$ and $c_1 P_s \ f^{v,p}_{s^*}(\succ, P)$. Moreover, we can show similarly to Case 1 that $f^{v,p}_{s^*}(\succ, (\tilde{P}_{s^*}, P_{-s^*})) = c_1$, where $\tilde{P}_{s^*} \in (c_1)$. This means that s^* can manipulate at P via $\tilde{P}_{s^*}$, which contradicts the strategyproofness of $f^{v,p}$.

Step 2: *The only strategyproof rank-partition stable rule is the DA.*[6]

Suppose that φ is a rank-partition stable rule which is strategyproof, and suppose that φ is not the DA. Let φ be rank-partition stable with respect to (v, p). Then, by Step

[6]Given Step 1, this step also follows from Alva and Manjunath (2019, [8]). We provide a direct proof of this step for completeness.

1, φ is not the PRP rule $f^{v,p}$. Thus, there exists $(\succ, P)$ such that $\varphi(\succ, P) \neq f^{v,p}(\succ, P)$. Then Proposition 1 implies that $f^{v,p}(\succ, P)$ Pareto-dominates $\varphi(\succ, P)$.

For all $s \in S$, let $\tilde{P}_s$ be the same as P_s from the first-ranked school to $f_s^{v,p}(P)$, and let all other schools (schools ranked below $f_s^{v,p}(\succ, P)$) be unacceptable to s at preferences $\tilde{P}_s$. In other words, $\tilde{P}_s$ is the truncation of P_s directly below $f_s^{v,p}(\succ, P)$. Let $\bar{P}$ be any preference profile in which for all $s \in S$, $\bar{P}_s \in \{P_s, \tilde{P}_s\}$. Then it easy to see that for all such $\bar{P} \in \mathcal{P}$, $f^{v,p}(\succ, \bar{P}) = f^{v,p}(\succ, P)$.

Let $s_1 \in S$ such that $f_{s_1}^{v,p}(\succ, P) \neq \varphi_{s_1}(\succ, P)$. Then $f_{s_1}^{v,p}(\succ, P) P_{s_1} \varphi_{s_1}(\succ, P)$. Note that $f^{v,p}(\succ, P) = f^{v,p}(\succ, \tilde{P}^1)$, where $\tilde{P}^1 \equiv (\tilde{P}_{s_1}, P_{-s_1})$. Since φ is strategyproof and individually rational, $\varphi_{s_1}(\succ, \tilde{P}^1) = 0$. Now suppose that, for all $s \in S \setminus \{s_1\}$, $f_s^{v,p}(\succ, \tilde{P}^1) = \varphi_s(\succ, \tilde{P}^1)$. Then $\varphi_{s_1}(\succ, \tilde{P}^1) \neq 0$, otherwise φ would be wasteful, which contradicts our assumption that φ is rank-partition stable. Hence, there exists $s_2 \in S \setminus \{s_1\}$ such that $f_{s_2}^{v,p}(\succ, \tilde{P}^1) \neq \varphi_{s_2}(\succ, \tilde{P}^1)$. Then, by Proposition 1, $f_{s_2}^{v,p}(\succ, \tilde{P}^1) P_{s_2} \varphi_{s_2}(\succ, \tilde{P}^2)$. Note that $f^{v,p}(\succ, \tilde{P}^1) = f^{v,p}(\succ, \tilde{P}^2)$, where $\tilde{P}^2 = (\tilde{P}_{s_1}, \tilde{P}_{s_2}, P_{-s_1,s_2})$. Now suppose that for all $s \in S \setminus \{s_1, s_2\}, f_s^{v,p}(\succ, \tilde{P}^2) = \varphi_s(\succ, \tilde{P}^2)$. Then $\varphi_{s_1}(\succ, \tilde{P}^2) \neq 0$ and $\varphi_{s_2}(\succ, \tilde{P}^2) \neq 0$, otherwise φ would be wasteful, which contradicts our assumption that φ is rank-partition stable. Hence, there exists $s_3 \in S \setminus \{s_1, s_2\}$ such that $f_{s_3}^{v,p}(\succ, \tilde{P}^2) \neq \varphi_{s_3}(\succ, \tilde{P}^2)$. And so on, if we keep iterating the same argument then, due to the finiteness of S, we run out of students. This is a contradiction, which shows that if φ is rank-partition stable but not the DA then φ is not strategyproof. □

Proof of Theorem 4

Theorem 4. *Let $f^{v,p}$ be a PRP rule and fix a profile $(\succ, P)$. Let $s \in S$ and $c \in C$ such that $c P_s f_s^{v,p}(\succ, P)$. Let $P'_s \in \mathcal{P}_s$ such that c is in the same or lower preference rank class in P'_s than in P_s, given p. Then $f_s^{v,p}(\succ, (P'_s, P_{-s})) \neq c$.*

Substitutability: If student s is chosen from some applicant pool $S' \subseteq S$ by a school, then student s would still be chosen from the applicant pool T by this school where T is a strict subset of S' and s is a member of T. Formally, a choice function Ch satisfies substitutability if $s \in T \subseteq S' \subseteq S$ and $s \in Ch_c(S')$ then $s \in Ch_c(T)$. It is easy to verify that PRP choice functions satisfy substitutability.

Proof. Let $f^{v,p}$ be a PRP rule and fix a profile $(\succ, P)$. Let $s \in S$, and $c, d \in C$ such that $f_s^{v,p}(\succ, P) = d$ and $cP_s d$. Let $P'_s \in \mathcal{P}_s$. Let $\bar{\succ}$ denote the constructed priority profile for $((\succ, P), (v, p))$ and let $\bar{\succ}'$ denote the constructed priority profile for $((\succ, (P'_s, P_{-s})), (v, p))$.

Case 1: Assume that P'_s is different from P_s only by reshuffling schools *within* s's preference rank classes only. For the constructed priority profile only the set of schools in each preference class is relevant, while the ordering of the schools within a preference class is irrelevant. Thus, $\bar{\succ} = \bar{\succ}'$. By Proposition 1, $f^{v,p}(\succ, P) = f^{DA}(\bar{\succ}, P)$ and $f^{v,p}(\succ, (P'_s, P_{-s})) = f^{DA}(\bar{\succ}', (P'_s, P_{-s}))$. Then $f^{v,p}(\succ, (P'_s, P_{-s})) = f^{DA}(\bar{\succ}, (P'_s, P_{-s}))$. Therefore, given that $cP_s f_s^{v,p}(\succ, P)$ and thus $cP_s f_s^{DA}(\bar{\succ}, P)$, by the strategyproofness of the DA, we have $f^{DA}\ (\bar{\succ},\ (P'_s, P_{-s})) \neq c$. This means that $f^{v,p}(\succ, (P'_s, P_{-s})) \neq c$.

Case 2: Assume that P'_s exchanges two schools (a and b) only when compared to P_s such that a and b are in two *adjacent* preference rank classes. Specifically, assume that a is the bottom-ranked school in its preference rank class, and b is the top-ranked school in the preference rank class just below the one a is in. Otherwise P'_s is the same as P_s, including the preference rankings within each preference rank class.

First note that only the priority rankings of a and b are different in the constructed priority profile $\bar{\succ}'$ compared to $\bar{\succ}$, while the priorities for the remaining schools are the same. Moreover, only the position of student s may change, as follows: s has the same or higher ranking in $\bar{\succ}'_b$ compared to $\bar{\succ}_b$, while s has the same or lower ranking in $\bar{\succ}'_a$ compared to $\bar{\succ}_a$. All other students have the same relative rankings in $\bar{\succ}'_b$ versus $\bar{\succ}_b$, as well as in $\bar{\succ}'_a$ versus $\bar{\succ}_a$.

Claim 4.1. *Let school e be ranked directly above school a by P_s. Then for all $\tilde{e} \in C$ such that $\tilde{e} R_s e$, if $f_s^{v,p}(\succ, P) = \tilde{e}$ then $f_s^{v,p}(\succ, (P'_s, P_{-s})) = \tilde{e}$ and otherwise $f_s^{v,p}(\succ, (P'_s, P_{-s})) \neq \tilde{e}$.*

Proof. Note first that school e is ranked directly above school b by P'_s. Then each round of the PRP procedure in which student s proposes to school $\tilde{e}$ such that $\tilde{e}R_s e$ (and consequently also $\tilde{e}R'_s e$), the PRP rounds at $(\succ, P)$ and at $(\succ, (P'_s, P_{-s}))$ are identical, given that $\tilde{e}P_s a, b$, $\tilde{e}P'_s a, b$ and only Ch_a and Ch_b have changed with respect to the selection of s only, when the two preference profiles are compared. $\square$

Case 2a: $dP_s a$. By Claim 3.12, $f^{v,p}_s(\succ, (P'_s, P_{-s})) = f^{v,p}_s(\succ, P) = d$.

Case 2b: $d = a$. Given Claim 3.12, since $cP_s d$, $f^{v,p}_s(\succ, (P'_s, P_{-s})) \neq c$. Claim 3.12 also implies that we have one of the following scenarios:

scenario (i): $f^{v,p}_s(\succ, (P'_s, P_{-s})) = d = a$

scenario (ii) : $f^{v,p}_s(\succ, (P'_s, P_{-s})) = b$ (note: $b \neq c$, since $cP_s dP_s b$)

scenario (iii): $dP_s f^{v,p}_s(\succ, (P'_s, P_{-s}))$

Case 2c: $d = b$. Given Claim 3.12, for all $\tilde{e} \in C$ such that $\tilde{e}R_s e$, $f^{v,p}_s(P'_s, P_{-s}) \neq \tilde{e}$. Suppose that s is rejected by school $d = b$ in the PRP procedure at profile $(\succ, (P'_s, P_{-s}))$. Then, since the PRP rule is not wasteful, there exists at least one student $\hat{s} \in S$ such that $f^{v,p}_{\hat{s}}(\succ, (P'_s, P_{-s})) = b$ and $f^{v,p}_{\hat{s}}(P) \neq b$. Since $f^{v,p}_{\hat{s}}(\succ, (P'_s, P_{-s})) = b$ and $bP_s f^{v,p}_s(\succ\ (P'_s, P_{-s}))$, given that the PRP rule $f^{v,p}$ is stable with respect to the constructed priority profile at each preference profile by Proposition 1, $\hat{s}\bar{\succ}'_b s$. Since s cannot have a lower position in $\bar{\succ}'_b$ compared to $\bar{\succ}_b$, while all other students' relative positions are unchanged, this implies that $\hat{s}\bar{\succ}_b s$. If $bP_{\hat{s}} f^{v,p}_{\hat{s}}(\succ, P)$, then since $\hat{s}\bar{\succ}_b s$ and $f(\succ, P)$ is stable with respect to $\bar{\succ}$, we have a contradiction. Thus, since $f^{v,p}_{\hat{s}}(\succ, P) \neq b$, we have $f^{v,p}_{\hat{s}}(\succ, P)P_{\hat{s}} b$. Note that this implies that $\hat{s}$ does not apply to school b in the $f^{v,p}$ procedure at P.

Assume that s is rejected by e in round $k-1$ of the $f^{v,p}$ procedure applied to both $(\succ, P)$ and $(\succ, (P'_s, P_{-s}))$. This is without loss of generality due to Claim 3.12. Then s applies to school a in Step k at $(\succ, P)$, and s applies to school b in round k at $(\succ, (P'_s, P_{-s}))$. Let $T \subset S$ be the set of students who apply to b in any round after round $k-1$ in the $f^{v,p}$ procedure at $(\succ, P)$, and let $T' \subset S$ be the set of students who apply to b in any round after round $k-1$ in the $f^{v,p}$ procedure at $(\succ, (P'_s, P_{-s}))$. We will show that $T' \subseteq T$.

It follows from Claim 3.12 that in round k at $(\succ, P)$ each student except s applies to the same school as in round k at $(\succ, (P'_s, P_{-s}))$. If student s gets rejected by school a at $(\succ, P)$ in round k then there is no difference in applicants to b at the two profiles after round k and $T' = T$, due to the substitutability property of the school choice functions. If student s gets accepted and some other student gets rejected by a at $(\succ, P)$ who is not rejected by a at $(\succ, (P'_s, P_{-s}))$ then this student applies to her next most-preferred school, which in turn my cause another student to be rejected by a school at $(\succ, P)$ who is not rejected at $(\succ, (P'_s, P_{-s}))$, etc., creating a rejection chain at $(\succ, P)$. Therefore, all students who applied to b at $(\succ, (P'_s, P_{-s}))$, and potentially more students, apply to b at $(\succ, P)$ after round k, since all students involved in this rejection chain get a lower-ranked school at $(\succ, P)$ than at $(\succ, (P'_s, P_{-s}))$, given the substitutability property of the school choice functions. Thus, if $t \in T'$ then $t \in T$, as claimed. Note, however, that this contradicts the existence of student $\hat{s}$ since, as shown above, $\hat{s} \in T' \setminus T$.

Case 2d: bP_sd**.** Note that $b \neq c$, since c is either in the same or a lower preference rank class in P'_s than in P_s, given the fixed p_s. We will show that one of the following three scenarios holds:

scenario (i): $f^{v,p}_s(\succ, (P'_s, P_{-s})) = d$;

scenario (ii): $f^{v,p}_s(\succ, (P'_s, P_{-s})) = b$;

scenario (iii): $dP_sf^{v,p}_s(\succ, (P'_s, P_{-s}))$.

By Claim 4.1, for all $\tilde{e} \in C$ such that $\tilde{e}R_se$, $f^{v,p}_s(\succ, (P'_s, P_{-s})) \neq \tilde{e}$. If $f^{v,p}_s(\succ, (P'_s, P_{-s})) = b$ then we are done, as scenario (ii) holds in this case. Thus, we can assume that $f^{v,p}_s(\succ, (P'_s, P_{-s})) \neq b$. Then $bP_sf^{v,p}_s(\succ, (P'_s, P_{-s}))$.

First we show that $f^{v,p}_s(\succ, (P'_s, P_{-s})) \neq a$. Suppose that $f^{v,p}_s(\succ, (P'_s, P_{-s})) = a$.

Then we can show that $f_s^{v,p}((\succ, P)) = a$, using a similar argument to the one for Case 2c. For Case 2c we showed that if $f_s^{v,p}(\succ, P) = b$ then $f_s^{v,p}(\succ, (P_s', P_{-s})) = b$, and here we can use a similar argument applied to a instead of b, to show that $f_s^{v,p}(\succ, P) = a$, which contradicts $f_s^{v,p}(\succ, P) = d$, since aP_sbP_sd. Therefore, $f_s^{v,p}(\succ, (P_s', P_{-s})) \neq a$, and thus we have shown so far that $aP_s'f_s^{v,p}(\succ, (P_s', P_{-s}))$.

Claim 4.2. *Assume that P_s' is different from P_s only by the ordering of schools within s's preference rank class which contains the school assigned to s at $(\succ, P)$, say school c, that is, when $f_s^{v,p}(\succ, P) = c$. Assume furthermore that the upper contour set of school c in this preference class is weakly smaller at P_s' than at P_s, that is, if c' is in the same preference class in P_s as c, and if $c'P_s'c$ then $c'P_sc$. Then $f_s^{v,p}(\succ, (P_s', P_{-s})) = c$.*

Proof. As noted in the argument for Case 1, for the constructed priority profile only the set of schools in each preference class is relevant, while the ordering of the schools within a preference class is irrelevant. Thus $\bar{\succ} = \bar{\succ}'$. Then $f^{v,p}(\succ, P) = f^{DA}(\bar{\succ}, P)$ and $f^{v,p}(\succ, (P_s', P_{-s})) = f^{DA}(\bar{\succ}', (\succ, (P_s', P_{-s}))) = f^{DA}(\bar{\succ}, (P_s', P_{-s}))$. Therefore, if $f_s^{v,p}(\succ, P) = c$, or equivalently $f_s^{DA}(\bar{\succ}, P) = c$, by strategyproofness of the DA, $f_s^{DA}(\bar{\succ}, (P_s', P_{-s})) = c$ or, equivalently, $f_s^{v,p}(\succ, (P_s', P_{-s})) = c$. □

Given Claim 4.2, we show next that if $f_s^{v,p}(\succ, (P_s', P_{-s})) \neq b$ then dR_s $f_s^{v,p}(\succ, (P_s', P_{-s}))$. Since $a, bP_sf_s^{v,p}(\succ, P)$, $a, bP_s'f_s^{v,p}(\succ, (P_s', P_{-s}))$, and only the order of a and b has changed between P_s and P_s', where a and b are adjacent in the preference ordering, suppose that there exists $c \in C$ such that bP_scP_sd and $f_s^{v,p}(\succ, (P_s', P_{-s})) = c$. This means that $aP_s'cP_s'd$.

Consider P_s'' which is the same as P_s', except that c is lifted to be ranked directly below a (note: if c was already ranked directly below a by P_s', then $P_s' = P_s''$). Since this change only affects the preference class that contains a, c and d, while all other orderings in P_s' are preserved, Claim 4.2 implies that $f_s^{v,p}(\succ, (P_s'', P_{-s})) = c$. Consider $\hat{P}_s$ which is the same as P_s, except that c is lifted to the top of the preference class that contains b, c and d, while all other orderings in P_s are preserved.

Now consider P''_s versus $\hat{P}_s$. Given that $f^{v,p}_s(\succ, (P''_s, P_{-s})) = c$, Claim 4.1 implies that for all $\tilde{e} R''_s e$, $f^{v,p}_s(\succ, (\hat{P}_s, P_{-s})) \neq \tilde{e}$. Now suppose that $f^{v,p}_s(\succ, (\hat{P}_s, P_{-s})) \neq a$. We will show that then $f^{v,p}_s(\succ, (P''_s, P_{-s})) = c$ implies that $f^{v,p}_s(\succ, (\hat{P}_s, P_{-s})) = c$. Suppose that s is rejected by school c in the $f^{v,p}$ procedure at profile $(\succ, (\hat{P}_s, P_{-s}))$. Then, since the PRP rule is not wasteful, there exists at least one student $\tilde{s} \in S$ such that $f^{v,p}_{\tilde{s}}(\succ, (\hat{P}_s, P_{-s})) = c$ and $f^{v,p}_{\tilde{s}}(\succ, (P''_s, P_-s)) \neq c$. Since $f^{v,p}_{\tilde{s}}(\succ, (\hat{P}_s, P_{-s})) = c$ and $cP_s f^{v,p}_s(\succ, (\hat{P}_s, P_{-s}))$, given that the PRP rule $f^{v,p}$ is stable with respect to the constructed priority profile at each preference profile by Proposition 1, $\tilde{s} \hat{\succ}_c s$, where $\hat{\succ}$ is the constructed priority profile at $(\succ, (\hat{P}_s, P_{-s}))$. Thus, since c is in the same preference class in P''_s and in $\hat{P}_s$, we also have $\tilde{s} \succ''_c s$, where $\succ''$ is the constructed priority profile at $(\succ, (P''_s, P_{-s}))$. If $cP_{\tilde{s}} f^{v,p}_{\tilde{s}}(\succ, (P''_s, P_{-s}))$, then since $\tilde{s} \succ''_c s$ and $f(\succ, (P''_s, P_{-s}))$ is stable with respect to $\succ''$, we have a contradiction. Thus, $f^{v,p}_{\tilde{s}}(\succ, (P''_s, P_-s))P_{\tilde{s}}c$, which implies that $\tilde{s}$ does not apply to school c in the $f^{v,p}$ procedure at $(\succ, (P''_s, P_-s))$.

Assume that s is rejected by e in round $k-1$ of the $f^{v,p}$ procedure applied to $(\succ, (P''_s, P_{-s}))$ and to $(\succ, (\hat{P}_s, P_{-s}))$. Then s applies to school b in round k at $(\succ, (P''_s, P_{-s}))$ and to school a at $(\succ, (\hat{P}_s, P_{-s}))$. Let $V \subset S$ be the set of students who apply to c in any round after round $k-1$ in the $f^{v,p}$ procedure at $(\succ, (P''_s, P_{-s}))$ and let $V' \subset S$ be the set of students who apply to c in any round after round $k-1$ in the $f^{v,p}$ procedure at $(\succ, (\hat{P}_s, P_{-s}))$. We will show that $V' \subseteq V$.

It follows from Claim 4.1 that in round k at $(\succ, (P''_s, P_{-s}))$ each student except s applies to the same school as in round k at $(\succ, (\hat{P}_s, P_{-s}))$. If student s gets rejected by school b at $(\succ, (P''_s, P_{-s}))$ in round k then there is no difference in applicants to a at the two profiles after round k, which implies that $V' = V$,

due to the substitutability property of the school choice functions. If student s gets accepted by b at $(\succ, (P''_s, P_{-s}))$ and some other student gets rejected after round $k-1$ who is not rejected by b at $(\succ, (\hat{P}_s, P_{-s}))$, then this student applies to her next most-preferred school, which in turn my cause another student to be rejected by a school at $(\succ, (P''_s, P_{-s}))$ who is not rejected at $(\succ, (\hat{P}_s, P_{-s}))$, etc., creating a rejection chain at $(\succ, (P''_s, P_{-s}))$. Therefore, all students who applied to c at $(\succ, (\hat{P}_s, P_{-s}))$, and potentially more students, apply to c at $(\succ, (P''_s, P_{-s}))$, since all students involved in this rejection chain get a lower-ranked school at $(\succ, (P''_s, P_{-s}))$ than at $(\succ, (\hat{P}_s, P_{-s}))$, again due to the substitutability property of the school choice functions. Thus, if $v \in V'$ then $v \in V$, as claimed. This, however, contradicts the existence of student $\tilde{s}$ since, as shown above, $\tilde{s} \in V' \setminus V$. Therefore, $f_s^{v,p}(\succ, (\hat{P}_s, P_{-s})) \in \{a, c\}$. Finally note that, given P_s, Claim 4.2 implies that $f_s^{v,p}(\succ, (\hat{P}_s, P_{-s})) = d$. This is a contradiction. Therefore,
$f_s^{v,p}(\succ, (P'_s, P_{-s})) \neq c$ and it follows that either $f_s^{v,p}(\succ, (P'_s, P_{-s})) = d$ or
$d P_s f_s^{v,p}(\succ, (P'_s, P_{-s}))$, corresponding to scenarios (i) and (iii) respectively.

Conclusion of the proof:

By repeatedly applying the transformation of P_s in Case 1, which allows for the reshuffling of schools within the preference rank classes of students, as well as the transformation of P_s in Case 2, which allows for exchanging two adjacent schools in the ordering which switches the preference rank classes in which these two schools are, we can transform P_s into an arbitrary $\check{P}_s$ such that c is never moved to a higher preference rank class in any round of the transformation. Thus, based on the proofs for the individual cases above, we can conclude that one of the following three cases holds for $f_s^{v,p}(\succ, (P'_s, P_{-s}))$ for each intermediate round P'_s when transforming P_s into an arbitrary $\check{P}_s$:

1. $f_s^{v,p}(\succ, (P'_s, P_{-s}))$ is the same as in the previous round, denoted by d, and thus does not equal c in Case 1, Case 2a, Case 2b scenario (i), Case 2c, and Case 2d

scenario (i).

2. $f_s^{v,p}(\succ, (P'_s, P_{-s}))$ becomes the school which moves up to a higher preference rank class, denoted by b, where $b \neq c$, since c never moves to a higher preference rank class in any round of the transformation. This happens in Case 2b scenario (ii), and Case 2d scenario (ii).

3. $f_s^{v,p}(\succ, (P'_s, P_{-s}))$ is less preferred than the school assigned in the previous outcome which is denoted by d: $dP_s f_s^{v,p}(\succ, (P'_s, P_{-s}))$, where $cP_s d$. Thus, $f_s^{v,p}(\succ, P'_s, P_{-s}) \neq c$. This happens in Case 2b scenario (iii) and in Case 2d scenario (iii).

In sum, for arbitrary $\check{P}_s$ such that c is in the same or lower preference rank class in $\check{P}_s$ than in P_s, $f_s^{v,p}(\succ, (\check{P}_s, P_{-s})) \neq c$.

$\square$

Chapter 4

Deferred Acceptance with Efficiency Improvements

4.1 Introduction

We study the school choice problem (Abdulkadiroğlu and Sönmez, 2003, [2]) that assign students to schools based on priorities of the schools over the students. Our main interest is in studying the trade-offs between fairness, efficiency, and incentive properties in the context where the rules are Pareto-improvements over the DA assignment, and we study a class of rules which implement Pareto-improving trades starting from the student-proposing Deferred Acceptance outcome as endowments for the students. A notable rule in this class is EADAM, introduced by Kesten (2010, [37]), which allows for Pareto improvements over the DA outcome by iteratively removing interrupters, students who interfere with the efficiency of the DA algorithm based on their priority ranking for certain schools, but cannot themselves be assigned to these schools. The original algorithm of EADAM provided by Kesten (2010, [37]) has been simplified by Tang and Yu (2014, [51]), who show that if we iteratively remove students with underdemanded DA endowments and re-run the DA then the resulting procedure is outcome-equivalent to the EADAM. Another equivalent alternative description of

EADAM based on a trading graph has been proposed by Dur et al. (2019, [19]).

Alternative justifications and results related to EADAM have also been offered by several recent papers, including Ehlers and Morrill (2019, [25]) and Troyan et al. (2020, [52]). Further papers concerning Pareto-improvement over the DA in various contexts include Abdulkadiroğlu et al. (2009, [1]), Alcalde and Romero-Medina (2015, [7]), Bando (2014, [10]), Cantala and Pápai (2014, [14]), and Erdil and Ergin (2008, [27]). In the school choice model alternative matching rules have been put forward to the TTC (Abdulkadiroglu and Sönmez, 2003, [2]) by Hakimov and Kesten (2014, [31]) and Morrill (2015, [42]), arguing that TTC may not be the best strategyproof and efficient rule in terms of fairness properties when schools have capacities.

Another matching rule in the class of Pareto-efficient rules that weakly Pareto-dominate the DA besides EADAM is the DA-TTC rule, and we study a new class of matching rules, called Deferred Acceptance with Improvement Trading Cycles (DA-ITC), whose extreme members are EADAM and DA-TTC. The DA-ITC rules start with the DA outcome, and if it is not Pareto-efficient then it carries out Pareto-improving trading cycles iteratively until no more such trading cycles can be found and thus a Pareto-efficient outcome is reached.

We first revisit EADAM and show that a simple algorithm which re-traces cycles in the DA procedure in a backward order of the rejections in the DA rounds and restores these cycles in this backward order is equivalent to the EADAM rule. We call this simple trading algorithm the DA-BCR (DA with Backward Cycle Restoration) procedure, which identifies in an intuitive manner the trading cycles that are carried out, starting from the DA endowments, in order to reach the EADAM outcome. This algorithm is simpler than that of Dur et al. (2019, [19]) and follows closely the original EADAM algorithm of Kesten (2010, [37]).

The class of DA-ITC rules are parameterized by k, where k is the number of the rounds in the DA in which cycles are restored in the backward restoration step following the DA, and when k is not large enough to ensure that the resulting matching is Pareto-efficient then the TTC rule is applied to the current endowments to reach Pareto-efficiency. This set of rules includes the DA-TTC as one of its extreme members (when $k = 0$) and EADAM as the other extreme member (when k is allowed to be as large as needed to achieve Pareto-efficiency). Matching rules between these two extreme members combine features of both the DA-TTC and EADAM. All of these rules are Pareto-efficient by construction, and it is well-known that none of them are strategyproof since they all produce Pareto-improving outcomes over the DA outcome (Kesten, 2010, [37]; Abdulkadiroğlu et al. 2009, [1]). Although not strategyproof and hence manipulable, all such rules are shown to be non-obviously manipulable by Troyan and Morrill (2020, [53]).

4.2 Model

Let N be the set of n students, and let M be the set of m schools. Each school $x \in M$ has a capacity (or quota) q_x, which is the number of identical school seats at school x that can be assigned to a student. Each student $i \in N$ is matched to at most one school, and each school $x \in M$ may be matched to at most as many students as q_x. Each student $i \in N$ has a strict preference ordering over the schools in M. For $x, y \in M$ if x is strictly preferred to y then we will write xP_iy, where P_i denotes strict preferences. Weak preferences are written as xR_iy, which holds if either xP_iy or if $x = y$, given that preference orderings are strict. The preferences of student i will be denoted by R_i in general, and a preference profile of the students is $R = (R_i)_{i \in N}$, where $R \in \mathcal{R}$ and $\mathcal{R}$ denotes the set of preference profiles consisting of strict preferences.

Each school x has a strict priority ordering $\succ_x$ of students in N. Let Π be the set of all priority orderings and let $\Pi^m = \Pi \times \ldots \times \Pi$ denote the m-fold Cartesian product of Π. A priority profile of the schools is $\succ = (\succ_x)_{x \in M}$, where $\succ \in \Pi^m$. In the following we will assume that the priority profile $\succ$ is fixed, along with N and M and $(q_x)_{x \in M}$. Thus, a matching market is determined by simply a preference profile $R \in \mathcal{R}$.

A **matching** $\mu : N \to M$ is an assignment of students to schools, where $\mu(i)$ indicates the assignment of student $i \in M$. For ease of notation we let μ_i denote $\mu(i)$ and μ_x denote $\mu^{-1}(x)$ such that $|\mu_i| \leq 1$ for all $i \in N$ and $|\mu_x| \leq q_x$ for all $x \in M$. If a student i is unassigned in matching μ, we will write $\mu_i = 0$, where 0 represents being unassigned. Similarly, if no student is assigned to school x in μ, we will write $\mu_x = 0$. Let the set of matchings be denoted by $\mathcal{M}$.

A matching μ is **non-wasteful** if there is no student i and school x such that $xP_i\mu_i$ and $|\mu_x| < q_x$. Student $i \in N$ has **justified envy** at μ, given R, if there exist school $x \in M$ and student $j \in N$ such that $xP_i\mu_i$, $i \succ_x j$, and $\mu_j = x$. We will say that student i has justified envy for x, given that j is matched to x.

A matching μ is **fair** at preference profile R, if it is non-wasteful and if there is no student $i \in N$ who has justified envy at μ, given R.

A matching μ is **Pareto-efficient** if there is no $\nu \in M$ such that for all $i \in N$, $\nu_i P_i \mu_i$ and, for some $j \in N$, $\nu_j P_j \mu_j$. A matching is **constrained efficient** if it is not Pareto-dominated by another stable matching.

Given a fixed priority profile $\succ$, a **matching rule** assigns a matching to each preference profile: $f : \mathcal{R} \to \mathcal{M}$. A matching rule is **fair** if it assigns a fair matching to each preference profile. A matching rule is **Pareto-efficient** if it assigns a Pareto-efficient matching to each preference profile.

4.3 EADAM Revisited

Kesten (2010, [37]) introduced the EADAM rule (Efficiency Adjusted Deferred Acceptance Mechanism), which starts with the DA algorithm and then it carries out efficiency adjustments based on relevant students consenting to give up their priorities for relevant schools. This leads to an iterative Pareto improvement over the DA outcome whenever it is not Pareto-efficient, while simultaneously relaxing the fairness requirement by allowing for priority violations for the relevant consenting students as they appear sequentially in the efficiency adjustment process.

DA with Backward Cycle Restoration (DA-BCR)

Step 1 - DA step: Run the DA algorithm.

Step 2 - Efficiency adjustment step: Backward Cycle Restoration (BCR)

If the DA outcome in Step 1 is Pareto-efficient, it becomes the final outcome. If the DA outcome in Step 1 is not Pareto-efficient, let the DA assignments be the endowments of the students, and find the graph of trading by moving backward in the DA-table (i.e., from a table which indicates the rounds of the initial DA algorithm) in the order of rejections as they occurred in the rounds of the Deferred Acceptance procedure in Step 1, where each rejected student i points to the students who were accepted by the school in the current round that rejected student i in the same round. Once any cycle is found, carry out the corresponding trade. If multiple cycles are completed in the same round of the DA procedure while moving backwards through the DA rounds, then carry out all such cycles simultaneously. Note that, given the multiple school seats for the same school, cycles may be selected differently when multiple cycles occur, but regardless of how these cycles are selected the resulting assignments are always the same.

Since trading once does not result in a final assignment (unlike in the TTC), keep both the students and schools in the market after trading, but update the endowments of the trading students to reflect the trade that has just been carried out. Continue similarly backwards in the DA-table by having students point to the school that rejected them in the previous round, and keep finding pointing students until a cycle is completed. Then carry out the resulting trading cycles, and keep repeating until no more Pareto-improvement can be made and we get back to round 1 of the DA algorithm in Step 1.

We remark that this is a very simple algorithm: it does not require re-running the

DA multiple times, unlike the original algorithm of Kesten (2010, [37]) or the simplified algorithm of Tang and Yu (2014, [51]). It is also a much simpler trading cycle algorithm than the one proposed in Dur et al. (2019, [19]), since in their algorithm it takes much more preparation to find out where the students are pointing and how to identify trading cycles. In the DA-BCR algorithm one can simply read off the sequence of pointing by students from the Step 1 initial DA-table. This simplicity makes perfect sense and underlines the intuition behind the EADAM procedure: it simply restores the cycles working backwards that were created in the process of DA algorithm. While we could choose trading cycles differently when starting from the DA endowments, and this cycle selection and its consequences is the focus of this paper, the DA-BCR procedure seems to "organically" choose the trading cycles to be carried out, based on the the DA algorithm, which makes a lot of intuitive sense in view of the findings of a growing recent literature that seems to frequently single out and revisit Kesten's EADAM, even when starting from seemingly quite different premises (see, for example, Dur et al. (2016, [19]), Ehlers and Morrill (2018, [25]), and Troyan et al. (2020, [52]).

Theorem 5. *The DA-BCR rule is outcome equivalent to the EADAM rule.*

Proof. Fix the preference profile R, and take the first completed cycle in the DA-BCR rule, as defined, starting from the last round of the DA algorithm. If there are multiple such cycles then select one. As described, if cycles can be selected in different ways (which only occurs if schools have multiple seats) then any such cycle can be selected. Since this is a cycle and students can trade their school assignments along this cycle, it must be the case that the cycle is completed when it returns to a student, for the first time, who has the same school as a school prior to this in the cycle. Let this school be school x and let the round where the cycle returns to x be round t, while the round where the cycle originates is round $t' > t$ in the DA algorithm. Note that the student who points to a student who is assigned to school x, call this student i, is not going to be part of this cycle, since the backward chain is initiated by this rejected student in round t'. Moreover, students in the cycle form a rejection chain in the sense that if we start with the student in this cycle who is rejected by school x in round t, then all subsequent students in the cycle (that is, working backwards in the cycle compared to how we constructed the cycle in the DA-BCR procedure, and hence working forward in the cycle in terms of rounds in the DA) are rejected in a row, subsequently to each other, due to the proposal of the student before them in this cycle. Now note that student i, who is the pointing student in the BA-BCR procdure that initiates this cycle but not part of the cycle, is rejected by school x in round t' of the DA, and for the cycle to be completed, had to be accepted by school x in round t, that is, this student's application to school x has initiated the rejection chain corresponding to this cycle that started in round t. This implies that the interrupter for this cycle is student i, and accordingly (i, x) form an interrupting pair in round t' in the DA. Therefore, i is an interrupter in the DA.

If i is an interrupter in the last round of the DA where there is at least one interrupter, then this exactly corresponds to Kesten's procedure of removing school x from student i's preferences and re-running the DA. Multiple simultaneous cycles can be handled the same way, as it is straightforward to see that the EADAM would handle multiple interrupters in the same round similarly. Finally, if i is not an interrupter in the last round of the DA (but in a prior round), then this interrupter and the corresponding cycle are independent of the cycles associated by the last interrupters, as found by the DA-BCR procedure, and will be taken care of the EADAM procedure in a later step.

For iterations of the DA-BCR procedure, note that each round of the DA-BCR creates a new setup that corresponds to the same DA algorithm but with fewer rounds, compared to the orgininal DA algorithm with all rounds. So the table for the DA in Step 1 is now truncated and stops at the relevant round. The reason this table is the same is because trading along any cycle in the DA-BCR restores earlier temporary assignments of students in the DA algorithm. Therefore, if we run each iteration of the DA-BCR procedure procedure, it corresponds to the EADAM procedure. □

As the proof shows and the following example demonstrates, we can easily identify the interrupting pairs of the original EADAM algorithm by simply noting that the last pointing student from the outside of a trading cycle that will emerge from this procedure, who points to a student in the cycle, the student who points directly to the student who starts a cycle and with whom the cycle ends, forms the interrupting pair with the school that the student is endowed to whom the interrupter is pointing.

We provide an example to illustrate the DA-BCR procedure.

Example 4. Consider the following matching problem with six students and five schools. This is a close variant of Example 3 in Kesten (2010) with a unit capacity for each school. *The EADAM outcome is*

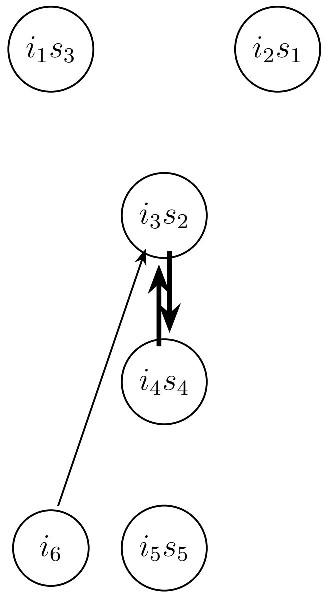
$$\begin{bmatrix} s_1 & s_2 & s_3 & s_4 & s_5 & \mathit{0} \\ i_4 & i_1 & i_2 & i_3 & i_5 & i_6 \end{bmatrix}$$

Student Preferences					
i_1	i_2	i_3	i_4	i_5	i_6
s_2	s_3	s_3	s_1	s_1	s_4
s_1	s_1	s_4	s_2	s_5	s_1
s_3	s_5	s_2	s_4		s_3
					s_2
					s_5

School Priorities				
s_1	s_2	s_3	s_4	s_5
i_2	i_3	i_1	i_4	.
i_1	i_6	i_6	i_3	.
i_5	i_4	i_2	i_6	.
i_6	i_1	i_3	.	
i_4	.	.	.	.
i_3	.	.	.	.

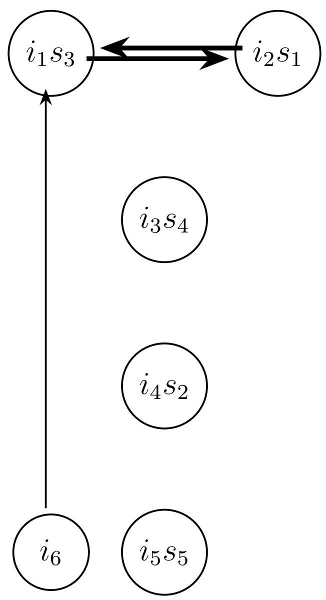
Step 1					
Round	s_1	s_2	s_3	s_4	s_5
1	$i_4, \underline{i_5}$	$\underline{i_1}$	$\underline{i_2}, i_3$	$\underline{i_6}$	$\emptyset$
2	$\underline{i_5}$	$i_1, \underline{i_4}$	$\underline{i_2}$	$\underline{i_3}, i_6$	$\emptyset$
3	$\boxed{i_5}, \underline{i_1}, i_6$	$\underline{i_4}$	$\underline{i_2}$	$\underline{i_3}$	$\emptyset$
4	$\underline{i_1}$	$\underline{i_4}$	$\boxed{i_2}, \underline{i_6}$	$\underline{i_3}$	$\underline{i_5}$
5	$\underline{i_2}, \boxed{i_1}$	$\underline{i_4},$	$\underline{i_6}$	$\underline{i_3}$	$\underline{i_5}$
6	$\underline{i_2}$	$\underline{i_4},$	$\underline{i_1}, \boxed{i_6}$	$\underline{i_3}$	$\underline{i_5}$
7	$\underline{i_2}$	$\underline{i_6}, \boxed{i_4}$	$\underline{i_1}$	$\underline{i_3}$	$\underline{i_5}$
8	$\underline{i_2}$	$\underline{i_6}$	$\underline{i_1}$	$\underline{i_4}, \boxed{i_3}$	$\underline{i_5}$
9	$\underline{i_2}$	$\underline{i_3}, \boxed{i_6}$	$\underline{i_1}$	$\underline{i_4}$	$\underline{i_5}$
10	$\underline{i_2}$	$\underline{i_3}$	$\underline{i_1}$	$\underline{i_4}$	$\underline{i_5}$

Round 1



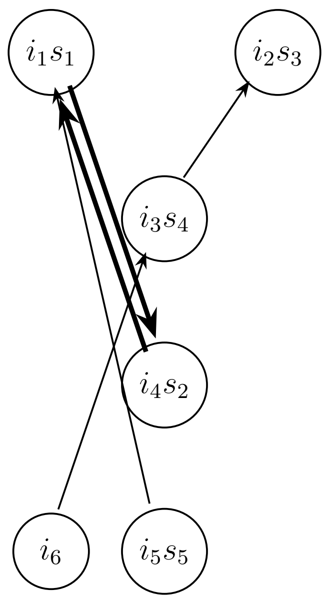
students i_3 and i_4 trade. The interrupting pair is (i_6, s_2).

Round 2



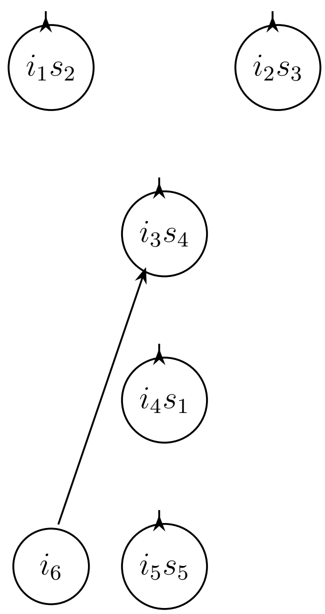
students i_1 and i_2 trade. The interrupting pair is (i_6, s_3).

Round 3



students i_1 and i_4 trade. The interrupting pair is (i_5, s_1).

Round 4



There are no more trading cycles. Thus, the DA-BCR rule leads to the same matching as EADAM. In this example it also has the same rounds and the same interrupting pairs in each round as the EADAM, but this need not be the case in general.

Next, we introduce another trading cycle rule which starts with the DA endowments, the DA with Bottom Trading Cycles (DA-BTC) rule. This rule may appear similar to the DA-BCR rule, but we will show via a counterexample below that this is not the case.

DA with Bottom Trading Cycles (DA-BTC)

Step 1 - DA step: Run the DA algorithm.

Step 2 - Efficiency adjustment step: Bottom Trading Cycles (BTC)

> If the outcome in Step 1 is Pareto-efficient, it becomes the final outcome. If the outcome in Step 1 is not Pareto-efficient, let this be the endowments of the students, and let each student point to the student who is endowed with their next-preferred school compared to their current endowment. That is, instead of pointing for their top-ranked school, as in the DA-TTC, students point to the next best school that is ranked above their current endowment, if any. Identify the "bottom trading cycles" this way and let students trade their schools accordingly.
>
> Since trading once does not result in a final assignment (unlike in the TTC), keep both the students and schools in the market after trading, but update the endowments of the trading students to reflect the trade that has just been carried out. Continue similarly with the next highest-ranked school for each student, until no more bottom trading cycles can be found.

Note that the main difference between the DA-BCR and DA-BTC algorithms is that the DA-BCR follows the order of rejections in the DA-table starting from the last round of the DA and working backwards, while the DA-BTC does not rely on the DA algorithm (apart from the initial endowments), and can be carried out directly based on the preference profile of the students, which is similar for the DA-TTC.

The following example shows that DA-BTC is not equivalent to the DA-BCR rule, and thus to EADAM.

Example 5. Consider the following matching problem with five students and five schools. The preferences and priorities are as follows:

Student Preferences				
1	2	3	4	5
b	c	b	a	d
a	a	c	c	e
	b		d	

School Priorities				
a	b	c	d	e
1	2	3	4	5
2	3	4	5	
4	1	2	3	

The DA-BTC results in the following matching:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ \text{b} & \text{a} & \text{c} & \text{d} & \text{e} \end{bmatrix}$$

However, the DA-BCR (and EADAM) matching is the following:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ \text{a} & \text{c} & \text{b} & \text{d} & \text{e} \end{bmatrix}$$

4.4 Deferred Acceptance with Improvement Trading Cycles

We now introduce a new set of rules, called **Deferred Acceptance with Improvement Trading Cycles**, or **DA-ITC** rules, as we will refer to them. This set of rules is parameterized by $k \in \{0, 1, \ldots,$ where k is the number of the rounds in the initial DA algorithm from which cycles are recovered in the Backward Cycle Restoration step (Step 2) in the DA-BTC procedure, which is shown to be equivalent to EADAM by Theorem 1.

k-DA with Improvement Trading Cycles (k-DA-ITC)

Step 1 - DA step: Run the DA algorithm.

Step 2 - Efficiency adjustment step: Improvement Trading Cycles (ITC)

If the DA outcome in Step 1 is Pareto-efficient, it becomes the final outcome. If the DA outcome in Step 1 is not Pareto-efficient, let the DA assignments be the endowments of the students, and find the graph of trading by moving backward in the DA-table (i.e., from a table which indicates the rounds of the initial DA algorithm) in the order of rejections as they occurred in Step 1 in the Deferred Acceptance procedure, where each rejected student points to the school that rejected the student in that round. Move back up to at most k rounds in the DA algorithm, but do not let any student point to an school that rejected it in a round that is more than k rounds counting from the end of the algorithm. Once a cycle is found, carry out the corresponding trade. If multiple cycles are completed in the same round of the DA procedure while moving backwards through the DA rounds, then carry out all such cycles simultaneously.

Since trading once does not result in a final assignment (unlike in the TTC), keep both the students and schools in the market after trading but update the endowments of the trading students to reflect the trade that has just been carried out. Continue similarly backwards in the DA-table by having students point to the school that rejected them in the previous round, and keep finding pointing students until a cycle is completed, but making sure that no more than k rounds in the DA algorithm in Step 1 from the end of the algorithm are taken into account for pointing students. Carry out the resulting trading cycles, and keep repeating until k rounds are reached.

Once k rounds counting from the end of the algorithm are reached, stop the BCR procedure. Given the current endowments of the students in this round, apply the TTC algorithm to this market to complete the procedure.

One extreme member of the class of DA-ITC rules is EADAM, when k is the maximum number of rounds that is possible for the DA for any particular model with N

and M, that is, when all rounds of the DA are taken into account for Bottom Cycle Restoration, in which case the DA-ITC rule is identical to the DA-BCR rule and is therefore equivalent to EADAM by Theorem 1. Another extreme member of the class of DA-ITC rules is DA-TTC, when $k = 0$, since in this case Step 2 of the procedure is simply the TTC, which is applied to the DA endowments, given that no trading cycles are restored from the DA algorithm when $k = 0$. Other members of the DA-ITC class are "hybrid" members that have some features in common with both EADAM and DA-TTC.

4.5 Properties of DA-ITC Rules

The following proposition immediately follows from the construction of the DA-ITC rules.

Proposition 3. *For all $k \geq 0$, the k-DA-ITC rule is Pareto-efficient and weakly Pareto-dominates the DA rule.*

Based on already existing results on the strategyproofness of rules which Pareto-dominate the DA, we can state that none of the DA-ITC rules are strategyproof. This follows from Abdulkadiroğlu et al. (2009, [1]) directly, and from a more general result by Alva and Manjunath (2019, [8]) as well. Kesten (2010, [37]) shows that that there is no Pareto-efficient matching rule which selects the fair and Pareto-efficient matching at every profile where it exists, which also implies the negative result below, given that all k-DA-ITC rules select the DA outcome at each profile where it is Pareto-efficient.

Proposition 4. *For all $k \geq 0$, the k-DA-ITC rule is not strategyproof.*

Although DA-ITC rules are manipulable, a weaker incentive property introduced by Troyan and Morrill (2020, [53]), called obvious manipulability, does not hold for DA-ITC rules.

Proposition 5. *For all $k \geq 0$, the k-DA-ITC rule is not obviously manipulable.*

Turning to the fairness properties of DA-ITC rules, the following conjecture implies that the closer the rule to EADAM, the fewer justified envy instances will there be for those students who might be Pareto-improvable.

The **envy graph** of a matching μ at preference profile $R \in \mathcal{R}$ is a directed graph in which the students are the nodes and there is a directed edge from student i to student j if and only if $\mu_j P_i \mu_i$. We will call a student **Pareto-unimprovable** at a matching μ and preference profile $R \in \mathcal{R}$ if the student is not in a cycle in the envy graph of μ at R.

Conjecture 1. *For each preference profile $R \in \mathcal{R}$, disregard all justified envy instances (i.e., blocking pairs of students and schools) that involve a Pareto-unimprovable student at the DA outcome at profile R. Then for all $k \geq 0$, the k-DA-ITC rule has at least as many justified envy instances remaining as does EADAM at each preference profile R.*

Given a fixed priority profile $\succ$, a matching rule f **has less justified envy** than matching rule g at preference profile $R \in \mathcal{R}$ if for any justified envy instance (i, x) at $f(R)$, (i, x) is also a justified envy instance at $g(R)$, and f **has strictly less justified envy** than g at preference profile R if in addition there exists (j, y) that is a justified envy instance at $g(R)$ but not at $f(R)$. Given a class of rules, a matching rule f is **justified-envy minimal** in this class if there is no other rule in the class that has strictly less justified envy than f at some profile R' and less justified envy at all preference profiles $R \in \mathcal{R}$. Note that this comparison of justified envy instances is based on an inclusion relation between two sets of justified envy instances, instead of counting them. This notion was introduced by Abdulkadiroğlu et al. (2017, [3]).

Our last conjecture concerns the justified-envy minimality of DA-ITC rules.

Conjecture 2. *For all $\bar{k} \geq 0$, the $\bar{k}$-DA-ITC rule is justified-envy minimal in the class of k-DA-ITC rules with $0 \leq k \leq \bar{k}$.*

4.6 Conclusion

In this paper we first revisit EADAM and show that a simple algorithm which re-traces cycles in the DA procedure in a backward order of the rejections, following the DA rounds starting from the last round, and restores these cycles in this backward order, is equivalent to the EADAM rule. We also introduce a new class of matching rules, called Deferred Acceptance with Improvement Trading Cycles (DA-ITC), which start with the DA, and if the DA outcome is not Pareto-efficient then there is an iterated improvement trading cycle phase which allows for Pareto-improvements until a Pareto-efficient outcome is reached. This family of matching rules includes the EADAM and DA-TTC rules as the two extreme members of this class, and this makes it a novel set of rules, since EADAM and DA-TTC are the only two rules that have been identified in the literature so far among rules that are Pareto-improvements over the DA. This class of rules, the DA-ITC rules, are Pareto-efficient and non-obviously manipulable, and have interesting fairness criteria according to our conjectures. DA-ITC rules allow the market designer to improve the efficiency when the DA is not Pareto-efficient by choosing from a menu of different options that span the well-studied EADAM rule, and the not so well-studied but conceptually simple DA-TTC rule.

Chapter 5

Degrees of Stability in Matching

5.1 Introduction

The standard stability notion in matching requires that if agent i prefers an object to the object she is assigned at the given matching then agent j who is assigned the preferred object has a higher priority for this object than i does, when priorities are strict. This stability notion also corresponds to fairness in the model where on one side of the market there are active strategic students with preferences over objects, and on the other side there are objects to be consumed, which have mandated priorities (as opposed to preferences) over students. In this context fairness means the lack of justified envy for any agent at any preference profile, where an agent i has justified envy if i prefers an object to the object she is assigned at the given matching and any agent j who is assigned the preferred object has a lower priority for this object than i does. Given this equivalence of the notions of stability and fairness in the two contexts (which often includes, additionally, the standard axioms of individual rationality and non-wastefulness), we will use the terminology of stability in this paper.

It is well-known that stability and Pareto-efficiency cannot be reconciled in this setting, and there are various strands of the literature which study the inevitable trade-offs. One particular approach is to relax stability and thereby increase efficiency. The standard stability concept can be relaxed in many ways. One interesting and natural way to relax it is to simply consider "small" priority violations acceptable. For example, the priorities might be based on some criterion that is subject to random variation or inaccuracy of measurement, such as a test score. Or consider the idea of consent introduced by Kesten (2010, [37]). Instead of unconditional consent (which may be ideal to increase efficiency), one can easily imagine that a potentially consenting party may agree to consent only if the priority violation is deemed small. This type of weakening of the standard stability notion is the focus of this paper. This more relaxed stability depends on the size of the priority violation measured by the distance in the priority ranking, assuming strict priorities to begin with. For example, test scores of 88 and 89 out of 100 may be considered close enough to deem the difference insignificant and thus allow for a priority violation in the matching, by letting the applicant with 88 points be matched to a university over the applicant with 89 points, while the latter applicant prefers this university to the university he is matched to, but if this applicant had, say, 92 points then a similar violation would not be allowed, since a 4-point difference is no longer considered small. We refer to this as the degree of stability and define the concept of k-stability which specifies the allowed size of the priority violation. In the above example, 4-stability would be violated if an applicant with 92 points had justified envy in the described situation, but 4-stability would be satisfied if this applicant's test score was 91. In other words, k-stability means that a gap of $k-1$ is allowed for priority violation. Thus, 1-stability coincides with standard stability, while 2-stability requires a more relaxed degree of stability that allows for a small priority violation, namely, when two students' priority rankings are adjacent for the object in question. Clearly, a larger degree of stability is associated with a smaller k-stability (minimally 1), and a smaller degree of stability requires a larger k-stability. At one extreme we have standard stability when $k = 1$ and at the other extreme $k = n$, where n is the number of students, and this means that there is no stability implication since any priority violation is acceptable.

Our paper consists of studying two different classes of matching rules satisfying k-stability. In the first part of our paper, we assume that each object has one copy only and we introduce the k-Consent rules, which are generalizations of Kesten's (2010, [37]) EADAM rule using different degrees of stability. The papers that are closest to the first part of our paper are Kesten (2010, [37]), Tang and Yu (2014,[51], and Dur et al.(2019, [21]). Kesten (2010, [37]) proposes the idea of consent in school choice, in which each agent is asked whether she consents to give up her priority at a desired object, or equivalently whether she allows other students to violate her priorities if doing so does not hurt her own assignment but potentially improves the assignment of other students. Our work differs in that we consider the case where an agent does not necessarily give up her entire priority even if she consents but, rather, she consents to have her priority violated up to a $k-1$ priority gap at each object. Kesten (2010, [37]) proposes the EADAM rule which is efficient when all students consent, and constrained efficient otherwise. Tang and Yu (2014, [51]) introduces the notion of underdemanded objects and defines the Simplified EADAM rule, which is shown to be outcome equivalent to Kesten's EADAM rule. It is simpler and more intuitive than Kesten's original definition. Dur et al.(2019, [21]) seeks to improve agent welfare by allowing certain priority violations, relaxing the standard notion of stability (or fairness) and study partial stability. Partial stability generalizes the idea of consent in Kesten (2010, [37]) by allowing each agent to consent to a violation of her priorities at select objects, instead of consenting at all objects. They define the Top Priority rule and prove that it is also outcome equivalent to EADAM. We define equivalent k-Consent rules corresponding to all three of the above definitions of EADAM.

In the matching model with multiple copies, that is, in the school choice model of Abdulkadiroğlu and Sönmez (2003, [2]), we define the k-DA rules, which generalize the celebrated Deferred Acceptance (DA) rule of Gale and Shapley (1962, [30]). At one extreme, when $k=1$, the k-DA rule coincides with the DA (and so does the k-Consent rule), and at the other extreme, when $k = n$, it becomes the Boston rule (Abdulkadiroğlu and Sönmez, 2003, [2]). While k-Consent rules are constrained efficient, the k-DA rules are not, which is due to the fact that a k-DA rule may satisfy a lower-degree (i.e., higher) stability than k. However, these are intuitive rules which satisfy k-stability and provide a new class of rules which bridge the DA and the Boston rules. Known classes of matching rules that do the same are the Application-Rejection rules of Chen and Kesten (2017, [16]) and the First Preference First rules of Pathak and Sönmez (2013, [46]).

There are a number of papers on weakening stability in some manner, such as Klijn and Massó (2003, [38]), Kesten (2004, [36]), Alcalde and Romero-Medina (2015, [7]), Cantala and Papai (2014, [14]), Papai (2015, [43]), Morrill (2014, [41], 2016, [42]), without intending to provide an exhaustive list. The only one that uses a concept closely related to k-stability is Huang et al. (2017, [33]). They study school choice problems where students have ordinal preferences over schools, but their priorities are cardinal, such as scores, and this way the intensity of priority violations is quantified. Thus, their epsilon-stability concept is inspired by the same intuition as k-stability, the only difference being that they measure the stability relaxation in terms of cardinal intensity, while we measure it in terms of the distance in ordinal rankings. Theirs is in the same spirit as our concept, since Huang et al. (2017, [33]) also consider the case of conditional consent where a student who becomes Pareto-unimprovable instead of consenting to give up her priorities at schools that she desires to all students with lower scores than hers, she consents only to priority violations by students whose scores are lower but are within ϵ difference. Huang et al. (2017, [33]) calls this concept ϵ-stability, which requires that when student j violates student $i's$ priority at a school, consent is only given if the score of j at this school is no less than $i's$ minus ϵ. They propose the ϵ-EADAM which is a variation of the EADAM rule, and Huang et al. (2017, [33]) prove that ϵ-EADAM is constrained efficient within the set of ϵ-stable matching rules. Our results in the first part of the paper on the k-Consent rules partially correspond to theirs in the respective settings, but we also look at two additional representations of k-Consent rules, based on the Simplified EADAM rule and the Top Priority rule.

5.2 Model

Let S denote the set of n students and let C be the set of m schools. Each student $i \in S$ has a strict preference relation P_i over $C \cup \{0\}$, where 0 represents being unassigned for both students and schools. Relation R_i denotes the weak counterpart of P_i and, since the preferences are strict, cR_ic' implies either cP_ic' or $c = c'$. Let $\mathcal{R}$ denote the set of all preference relations and let $\mathcal{R}^n = \mathcal{R} \times .... \times \mathcal{R}$ (n-fold Cartesian product of $\mathcal{R}$). A preference profile of the students is denoted by $R = (R_i)_{i \in S}$, where $R \in \mathcal{R}^n$. Each school has a strict priority ordering $\succ_c$ of all the students in S, and the priorities for all schools $(\succ_c)_{c \in C}$ are denoted by $\succ$. The priority table $\succ$ is assumed to be fixed.

The outcome of a matching problem is an assignment of students to schools: a **matching** is $\mu : S \to C$ where $\mu(i)$ indicates the assignment of student i. For ease of notation we let μ_i denote $\mu(i)$ and μ_c denote $\mu^{-1}(c)$. Let the set of matchings be denoted by M.

A matching μ is **blocked by student** $s \in S$ if s prefers being single to being matched to μ_s, that is, $sP_s\mu_s$. A matching is **individually rational** if it is not blocked by any student. A matching μ is **non-wasteful** if no student prefers an unassigned school to his assignment at μ, that is, for all $s \in S$ and $c \in C$, if $cP_s\mu_s$ then $\mu_c \neq 0$.

Student i has **justified envy** at μ, given R, if there exist school $c \in C$ and student $j \in S$ such that:

- $cP_i\mu_i$
- $i \succ_c j$
- $\mu_j = c$

We will say that student i has justified envy for c, given that j is matched to c.

A matching μ is **stable** at μ, given R, if it is individually rational, non-wasteful and there is no student $i \in S$ who has justified envy at μ, given R.

Let $i \succ_c^k j$ denote that i is at least k places above j in the priority ranking $\succ_c$ of school $c \in C$, where $k \in \{1, \dots n\}$. Thus, $i \succ_c^1 j$ is identical to $i \succ_c j$.

Student i has **k-justified envy** at μ, given R, if there exist school $c \in C$ and student $j \in S$ such that:

- $cP_i\mu_i$
- $i \succ_c^k j$
- $\mu_j = c$

A matching μ is ***k*-stable** at R if it is individually rational, non-wasteful and there is no student $i \in S$ who has k-justified envy at μ, given R. Note that if a matching is 1-stable it means that it is stable (i.e., 1-stability is simply standard stability), while the requirement of n-stability has no implication. As k increases, k-stability becomes less and less demanding.

Remark 1. By the definition of k-stability, for all $k \in \{1, \dots, n-1\}$, if a matching is k-stable at R then it is also k'-stable at R for all $k' > k$.

Given the fixed priority table $\succ$, a **matching rule** assigns a matching to each preference profile $f : R \to M$. A matching rule is **stable** if it assigns a stable matching to each preference profile. A matching rule is ***k*-stable** if it assigns a k-stable matching to each preference profile.

A matching μ is **Pareto-efficient** if it is not Pareto-dominated by another matching. Matching $\mu \in M$ is Pareto-dominated by another matching $\eta \in M$ if for all $i \in S$, $\eta_i R_i \mu_i$ and for some $j \in S$, $\eta_j P_j \mu_j$. A matching is ***k*-constrained efficient** if it is not Pareto-dominated by another k-stable matching. A matching rule is **Pareto-efficient** if it assigns a Pareto-efficient matching to each preference profile. A matching rule is ***k*-constrained efficient** if it assigns a k-constrained-efficient matching to each preference profile.

5.3 The k-Consent Rules

Following the definition of Kesten (2010, [37]), if student s is tentatively accepted by school c in some round t in the DA procedure, and if s is later rejected by school c in some round $t' > t$, and at least one other student is rejected by school c in some round l such that $t' > l \geq t$, then student s is an **interrupter** for school c in round t', and the pair (s, c) is an interrupting pair in round t'.

k-Consent Rules

Fix a preference profile R, and recall that the fixed priority table is $\succ$.

Let $\succ^1 := \succ$

Let $t := 1$ and do step t.

Step t:

Run the DA using $\succ^t$.

- If the DA matching is Pareto-efficient, stop, and let this matching be the final outcome.
- Otherwise, check for interrupters and find all of them in the last step of the DA in which there are any interrupters.
- Apply k-consent to each interrupting pair (s, c) in this step: move interrupter s down in $\succ_c$ by $k-1$ positions, (or less, if moving s down by $k-1$ positions is not possible). Skip any interrupter who is an interrupter for the second time for the same school. Call this updated priority table $\succ^{t+1}$.
- If $\succ^{t+1} = \succ^t$, stop, and let this DA matching be the final outcome. Otherwise let $t := t+1$ and repeat step t.

Extreme cases of the k-Consent rule:

$k = 1$ implies that the k-Consent is the DA. Since 1-consent means no consent: run the DA once, stop.

$k = n$ implies that the k-Consent is the EADAM. Each student fully consents.

The following example illustrates how the k-Consent rule works.

Example 6. Consider a matching problem with five students and five schools. The preferences and priorities are as follows:

Student Preferences

R_1	R_2	R_3	R_4	R_5
b	c	a	a	c
c	a	b	c	d
a	b	c	d	e
d				

School Priorities

a	b	c	d	e
1	2	3	1	1
4	3	4	4	2
3	1	1	5	5
2	5	5	3	4
5	4	2	2	3

Assume $k = 2$.

***k*-Consent procedure:**

Step 1

Round	a	b	c	d	e
1	$3, \underline{4}$	$\underline{1}$	$2, \underline{5}$	0	0
2	$\underline{4}, 2$	$1, \underline{3}$	$\underline{5}$	0	0
3	$\underline{4}$	$\underline{2}, \boxed{3}$	$\underline{1}, \boxed{5}$	0	0
4	$\underline{4}$	$\underline{2}$	$\boxed{1}, \underline{3}$	$\underline{5}$	0
5	$\underline{1}, \boxed{4}$	$\underline{2},$	$\underline{3}$	$\underline{5}$	0
6	$\underline{1}$	$\underline{2},$	$\underline{3}, 4$	$\underline{5}$	0
7	$\underline{1}$	$\underline{2},$	$\underline{3}$	$\underline{4}, 5$	0
8	$\underline{1}$	$\underline{2},$	$\underline{3}$	$\underline{4}$	$\underline{5}$

Note that the interrupters are indicated by the squares. In Round 5 of the DA algorithm student 4 was an interrupter for school a, so we move student 4 down in the priorities for school a by 1 position, given $k = 2$. The school priority table becomes the following:

School Priorities

a	b	c	d	e
1	2	3	1	1
3	3	4	4	2
$\boxed{4}$	1	1	5	5
2	5	5	3	4
5	4	2	2	3

Step 2

Round	a	b	c	d	e
1	$\underline{3}, 4$	$\underline{1}$	$2, \underline{5}$	0	0
2	$\underline{3}, 2$	$\underline{1}$	$\underline{4}, 5$	0	0
3	$\underline{3}$	$1, \underline{2}$	$\underline{4}$	$\underline{5}$	0
4	$\underline{1}, \boxed{3}$	$\underline{2}$	$\underline{4}$	$\underline{5}$	0
5	$\underline{1}$	$\underline{2}, 3$	$\underline{4}$	$\underline{5}$	0
6	$\underline{1}$	$\underline{2}$	$\underline{3}, \boxed{4}$	$\underline{5}$	0
7	$\underline{1}$	$\underline{2},$	$\underline{3}$	$\underline{4}, 5$	0
8	$\underline{1}$	$\underline{2},$	$\underline{3}$	$\underline{4}$	$\underline{5}$

In round 6 of the DA algorithm student 4 was an interrupter for school c, so move student 4 down in the priorities for school c by 1 position. The school priority table now becomes the following:

School Priorities

a	*b*	*c*	*d*	*e*
1	2	3	1	1
3	3	1	4	2
2	1	$\boxed{4}$	5	5
4	5	5	3	4
5	4	2	2	3

Step 3

Round	*a*	*b*	*c*	*d*	*e*
1	$\underline{3}, 4$	$\underline{1}$	$2, \underline{5}$	0	0
2	$\underline{3}, 2$	$\underline{1}$	$\underline{4}, \boxed{5}$	0	0
3	$\underline{3}$	$1, \underline{2}$	$\underline{4}$	$\underline{5}$	0
4	$\underline{3}$	$\underline{2}$	$\underline{1}, \boxed{4}$	$\underline{5}$	0
5	$\underline{3}$	$\underline{2}$	$\underline{1}$	$\underline{4}, 5$	0
7	$\underline{3}$	$\underline{2}$	$\underline{1}$	$\underline{4}$	$\underline{5}$

In round 4, student 4 is an interrupter for school c again. Thus, the final matching is the one found in Step 3. Note that applying k-consent in this case would violate k-stability. Accordingly, the k-Consent rule gives the following matching:

$$\left\{ \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \\ \text{c} & \text{b} & \text{a} & \text{d} & \text{e} \end{array} \right\}$$

When the same student is an interrupter for the second time for the same school, we skip the student (student 4 in the example) and move to the next interrupter, student 5, we find the following. Student 5 is an interrupter for school c, so moving 5 down by 1 position the priority table becomes the following:

School Priorities

a	b	c	d	e
1	2	3	1	1
3	3	1	4	2
2	1	$\boxed{4}$	5	5
4	5	2	3	4
5	4	$\boxed{5}$	2	3

Step 3

Round	a	b	c	d	e
1	$\underline{3}, 4$	$\underline{1}$	$\underline{2}, 5$	0	0
2	$\underline{3}$	$\underline{1}$	$\boxed{2}, \underline{4},$	$\underline{5}$	0
3	$\underline{3}, 2$	$\underline{1}$	$\underline{4}$	$\underline{5}$	0
4	$\underline{3}$	$\underline{2}, 1$	$\underline{4}$	$\underline{5}$	0
5	$\underline{3}$	$\underline{2}$	$\underline{1}, \boxed{4}$	$\underline{5}$	0
6	$\underline{3}$	$\underline{2}$	$\underline{1}$	$\underline{4}, 5$	$\underline{5}$
7	$\underline{3}$	$\underline{2}$	$\underline{1}$	$\underline{4}$	$\underline{5}$

In this example we can see that stopping and skipping gives the same outcome.

We will say that $\tilde{\succ}_c$ **satisfies k-priority** with respect to $\succ_c$ if for all $i, j \in N$, $i \succ_c^k j$ implies $i \tilde{\succ}_c j$. Let $\tilde{\succ} = (\tilde{\succ}_c)_{c \in C}$ and $\succ = (\succ_c)_{c \in C}$. If $\tilde{\succ}_c$ satisfies k-priority with respect to $\succ_c$ for all schools $c \in C$ then we will say that $\tilde{\succ}$ satisfies k-priority respect to $\succ$.

Proposition 6. *For all $k \in \{1, \ldots, n\}$, the k-Consent rule is k-stable.*

Proof. By the construction of k-Consent rules, if the same student is an interrupter again for a school that she was an interrupter for in a previous round in the last step of the DA in which there are any interrupters, then modifying the priorities for this student would violate k-priority with respect to $\succ_c$ since if $k \geq 2$ then $2(k-1) \geq k$. Thus, by stopping the procedure in such a case the priority table $\succ^T$ used in the last round T satisfies k-priority with respect to $\succ$. Thus the DA applied to $\succ^T$ will satisfy k-stability with respect to $\succ$. They are also individually rational since there is no student such that $sP_s\mu_s$. Finally, k-Consent rules are non-wasteful. This follows from the standard DA being non-wasteful. Hence, k-Consent rules satisfy k-stability. $\square$

Matching $\eta \in M$ **weakly Pareto-dominates** another matching $\mu \in M$ if for all $i \in S$, $\eta_i R_i \mu_i$. We will say that a matching rule weakly Pareto-dominates another matching rule if the matching assigned by one weakly Pareto-dominates the matching assigned by the other one at every preference profile.

A student is in a Pareto-improving cycle with respect to a particular assignment μ if there is a way to exchange assignments among students such that the resulting assignment Pareto-dominates the original assignment. A student is *inactive* in a particular round of the DA procedure if the student is not in any Pareto-improving cycle, considering the temporary assignments of all students in that round of the procedure.

Proposition 7. *For all $k \in \{1, \ldots, n\}$, the k-Consent rule weakly Pareto-dominates the DA.*

Proof. Suppose, by contradiction, that there is a preference profile R, a step $t \geq 1$ of the k-Consent rule, and a student $i \in S$ such that the school student i receives in step t is worse for him than the school he receives in step $t-1$. We consider two cases:

1. *Student i is not a last interrupter in step $t-1$.*

   Assume without loss of generality that student i is the first student to apply to a school which is worse for him than the school he receives in step $t-1$. Let b be i's school in step $t-1$. Since student i is not a last interrupter in step $t-1$, his priorities are the same in step t and step $t-1$. Thus, since the DA is non-wasteful, there exists a student j who is placed at b in step t. However, since there is no justified envy in the DA in Step t, this implies that $j \bar{\succ}_b i$, where $\bar{\succ}$ is the priority table in step t. Let c be the school that j had in step $t-1$.

   We consider the following cases:

(a) bP_jc: Then $j\tilde{\succ}_b i$, where $\tilde{\succ}_b$ is the priority table in step $t-1$, given that i is not a last interrupter in Step $t-1$. Thus, student j has justified envy for b, since j is assigned b in step t. This is a contradiction, since the DA has no justified envy.

(b) cP_jb: This contradicts the assumption that student i is the first student to apply to a school which is worse for him than the school he receives in step $t-1$.

2. *Student i is a last interrupter in step $t-1$.*

We show in Proposition 6 that a last interrupter is an inactive student in that step. This implies that i receives the same school in steps $t-1$ and t, a contradiction.

Thus, we have a contradiction in both cases. □

Next, we will show that the k-Consent rules is not strategyproof for $k \geq 2$. This is not too surprising, since it is well known that EADAM is not strategyproof Kesten (2010, [37]). Note that $k = 1$ is not included in the statement, since the DA is strategyproof.

It is well-known that there is no strategyproof rule which Pareto-dominates the DA (Abdulkadiroğlu et al., 2009, [1]). Thus, the previous Proposition implies that the k-Consent rule is not strategyproof for $k \geq 2$. We will demonstrate this in the following example.

Example 7. Consider a matching problem with three students and three schools. The preferences and priorities are as follows:

Student Preferences			School Priorities		
R_1	R_2	R_3	c_1	c_2	c_3
c_1	c_1	c_3	3	2	2
c_2	c_2	c_1	1	3	1
c_3	c_3	c_2	2	1	3

Let $k \geq 2$.

Since, there are no interrupters, the k-Consent rule produces the following matching:

$$\left\{\begin{array}{ccc} 1 & 2 & 3 \\ c_1 & c_2 & c_3 \end{array}\right\}$$

Now consider the following preference profile:

Student Preferences

R_1	R'_2	R_3
c_1	c_1	c_3
c_2	c_3	c_1
c_3	c_2	c_2

Step 1

Round	c_1	c_2	c_3
1	$\underline{1}, 2$	0	$\underline{3}$
2	$\underline{1}$	0	$\underline{2}, 3$
3	$\underline{3}, \boxed{1}$	0	$\underline{2}$
4	$\underline{3}$	1	$\underline{2}$

Student 1 is the only interrupter in round 3, and since he is an interrupter for school c_1, we move student 1 down in the priorities for school c_1 by $k-1$ positions which means moving 1 to the last position in $\succ_c$ if $k \geq 2$. The school priority table becomes the following:

School Priorities

c_1	c_2	c_3
3	2	2
2	3	1
1	1	3

Applying the DA to this priority table, the k-Consent rule produces the following matching:

Step 2

Round	c_1	c_2	c_3
1	$1, \underline{2}$	0	$\underline{3}$
2	$\underline{2}$	$\underline{1}$	$\underline{3}$

Thus, student 2 can manipulate. The intuition is that student 2 can render student 1 an interrupter for c, and then student 2 gets higher priority than 1 for c_1. □

Note that we could easily extend this example to include any number of schools by making additional schools ranked below the current schools, and by adding students who have lower priority for each of the schools than the current students. This would constitute an independent proof of the manipulability of k-Consent rules for $k \geq 2$.

5.4 The k-Top Priority (k-TP) Rules

Dur et al. (2019b, [21]) give an alternative definition of EADAM, called the Top Priority (TP) rule, which selects trading cycles to be carried out when the endowments are the DA assignments. We give a similar definition of k-TP rules, which extends the TP rule by allowing consent to be partial, that is, k-consent.

Let $G = (V, E)$ be a directed graph where the set of vertices is the set of students ($V = N$), and the set of directed edges E consists of a set of ordered pairs of V. For s_i, $s_j \in N$, let $s_i s_j \in E$ denote that there is a directed edge in G from s_i to s_j. A **trail** is an ordered set of distinct directed edges $(s_1 s_2, s_2 s_3, ...s_{v-1} s_v)$. We will call a trail a **path** if the vertices $s_1, s_2, ..., s_v$ are distinct. A path is called a **cycle** if all vertices are distinct except that $s_1 = s_v$.

We will show how the k-TP rule selects the matching at preference profile R for a fixed k. The k-TP rule determines the final matching at R as a result of the following iterative procedure. At step 0 the initial matching μ^0 is given by the DA outcome at R. At each step $t \geq 1$, there exists a matching μ^{t-1} which is determined in the previous step $t-1$.

For all $t = 1....$ we define the **Envy Graph** $G_t = (N, E_t)$ based on μ^{t-1} as follows: for all $i, j \in N$, $i \neq j$, let $ij \in E_t$ if $\mu_j^{t-1} P_i \mu_i^{t-1}$.

A directed edge $ij \in E_t$ is **active** in G_t if ij is contained in a cycle in E_t. Otherwise ij is **inactive** in G_t.

A vertex $j \in N$ is **active** in G_t if there exists $i \in N$ such that $ij \in E_t$ is active in G_t. Otherwise, the vertex (student) is inactive in G_t.

We will define the **Trade Graph** $\Gamma_t = (\bar{V}_t, \bar{E}_t)$ for all $t = 1....$ based on the Envy Graph G_t. Γ_t is a subgraph of G_t in the sense that its vertex set is $\bar{V}_t \subseteq N$ and its edge set is $\bar{E}_t \subseteq E_t$. The vertex set $\bar{V}_t$ of Γ_t is the set of active vertices (students) in G_t. We will identify the set of directed edges $\bar{E}_t$ in Trade Graph Γ_t as follows. Let $ij \in E_t$ be a directed edge in G_t. Then $ij \in \bar{E}_t$ if and only if for all $l \in N$ such that $lj \in E_t$ and l is inactive in G_t, we do not have $l \succ^k_{\mu_j^t} i$. Thus, if there exists $l \in V_t$ such that $lj \in E_t$ and l is inactive in G_t, and $l \succ^k_{\mu_j^t} i$, then $ij \notin \bar{E}_t$, and otherwise $ij \in \bar{E}_t$. Note that this determines which directed edges point to each active vertex $j \in \bar{V}_t$, if any, in Γ_T and thus it defines the set of directed edges $\bar{E}_t$, where $\Gamma_t = (\bar{V}_t, \bar{E}_t)$.

Now that we have defined for all t both the Envy Graph $G_t = (N, E_t)$ and the Trade Graph $\Gamma_t = (\bar{V}_t, \bar{E}_t)$, we are ready to define the iterative algorithm for the k-TP rule.

k-TP Rules

Fix a preference profile R, and recall that the fixed priority table is $\succ$.
Let μ^0 be the DA outcome. For arbitrary $t \geq 0$ we will specify how to find μ^{t+1} at step t.

Step t:

- Given the matching μ^t, let each student $i \in N$ be endowed with μ_i^t.

- If there is no cycle in the envy graph G_t based on μ^t then let $\mu^{t+1} = \mu^t$.

- Otherwise, if there is at least one cycle in G_t, then consider the Trade Graph Γ_t based on G_t and do a Top Priority cycle selection in Γ_t as follows.

- Construct $\Gamma_t^c = (\bar{V}_t, \bar{E}_t^c)$, which is a subgraph of $\Gamma_t = (\bar{V}_t, \bar{E}_t)$ in the sense that $\bar{E}_t^c \subseteq \bar{E}_t$. For each $j \in \bar{V}_t$, let $ij \in \bar{E}_t^c$ if and only if $ij \in \bar{E}_t$ and for all $l \in \bar{V}_t$ with $lj \in \bar{E}_t$, where $l \neq i$, $i \succ_{\mu_j^{t-1}} l$. Note that this implies that each vertex in $\bar{V}_t$ has an in-degree (the number of incoming edges) of at most one in Γ_t^c. Thus, cycles do not intersect in Γ_t^c. Select all the cycles in Γ_t^c, if there are any, and carry out all the corresponding trades of endowments, given μ^{t-1}. Update μ^{t-1} accordingly, and call the resulting matching μ^t.

- If there is no cycle in Γ_t^c, then $\mu^t = \mu^{t-1}$.

- Repeat until there is a $T > 0$ such that $\mu^{T-1} = \mu^T$. The final matching is μ^T.

The following example illustrates how the k-TP rule works.

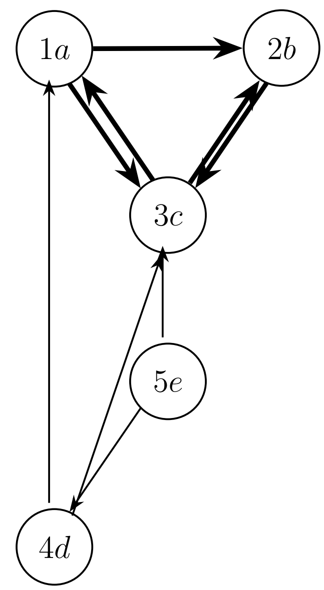
Example 8. Consider the matching problem with five students and five schools from Example 6.

The DA rule produces the following matching:

$$\begin{Bmatrix} 1 & 2 & 3 & 4 & 5 \\ a & b & c & d & e \end{Bmatrix}$$

Assume $k = 2$.

Envy Graph



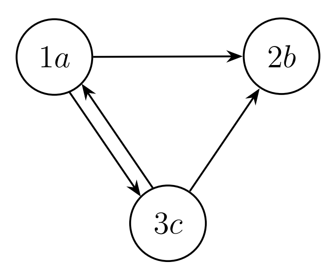
Active students:

1-a:

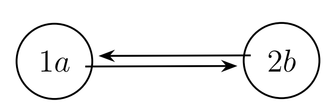
2-b:

3-c: $2 \to 3$ (remove), since this violates the k-priority of student 4 for school c.

Trade Graph



Envy Graph



Active students:

1-c:

2-b: $2 \rightarrow 1$ (remove), since this violates the k-priority of student 4 for school c. There is no trade and the procedure stops.

The k-TP rule produces the following matching:

$$\begin{Bmatrix} 1 & 2 & 3 & 4 & 5 \\ \text{c} & \text{b} & \text{a} & \text{d} & \text{e} \end{Bmatrix}$$

This is the same as for the k-Consent rule, as seen in Example 6.

Note that EADAM yields the following matching:

$$\begin{Bmatrix} 1 & 2 & 3 & 4 & 5 \\ \text{b} & \text{c} & \text{a} & \text{d} & \text{e} \end{Bmatrix}$$

Proposition 8. *For all $k \in 1, ..., n$, the k-TP rule is k-stable.*

Proof. Suppose not. Then there exists a preference profile R such that the k-TP rule results in μ at R, where μ is not k-stable, given a fixed $k \in (1, \ldots, n-1)$. Thus, there exists $i, j \in N$ such that $\mu_j P_i \mu_i$ and $i \succ^k_{\mu_j} j$. This means that $i \succ_{\mu_j} j$.

Let $t \in (0, 1, ...)$ be the first step in the k-TP algorithm at which $\mu^t_j = \mu_j$. Since $\mu_i R_i \mu^t_i$, we have $\mu^t_j P_i \mu^t_i$. Then $i \succ_{\mu_j} j$ implies that $t \neq 0$, since μ^0 is the DA outcome which is stable. Thus, $t > 0$ and $\mu^{t-1}_j \neq \mu^t_j = \mu_j$

Let $s_l \in N$ such that $\mu^{t-1}_l = \mu^t_j$. Then $l \neq j$ and since $\mu^t_j P_i \mu^t_i R_i \mu^{t-1}_i$, $l \neq i$. Moreover since $\mu^{t-1}_l P_i \mu^{t-1}_i$, $il \in E_{t-1}$

If i is inactive in G_{t-1} then $i \succ^k_{\mu^{t-1}_l} j$ implies that $jl \notin \bar{E}_{t-1}$ and thus, given $\bar{E}^c_{t-1} \subseteq \bar{E}_{t-1}$, $jl \notin \bar{E}^c_{t-1}$. If i is active in G_{t-1} then $i \succ_{\mu^{t-1}_l} j$ implies that $jl \notin \bar{E}^c_{t-1}$. Therefore, in both cases we have a contradiction to $\mu^{t-1}_l = \mu^t_j$. This proves the proposition. □

Remark 2. By the construction of the k-TP rule, for $k \geq 2, \ldots, n$ this rule weakly Pareto-dominates the DA.

Extreme cases of the $\boldsymbol{k}$-TP rule:

$k = 1$ implies that the k-TP is the DA, that is, the 1-TP rule is the DA rule.

Suppose $k = 1$ and $\mu^T \neq \mu^0$. Then, given the definition of the k-TP rule, μ^T Pareto dominates μ^0. Since $k = 1$, μ^T is 1-stable, that is, μ^T is stable. By Gale and Shapley (1962), μ^0 is the student-optimal stable matching and cannot be Pareto-dominated by another stable matching. Hence, we have a contradiction and $\mu^T = \mu^0$. This means that 1-TP is the DA, as desired.

$k = n$ implies that the k-TP is EADAM, that is, the n-TP rule is the EADAM rule.

If $k = n$, the inactive students in G_t will not play a role for any $t \geq 0$ and thus $E_t = \bar{E}_t$. Then the k-TP algorithm reduces to the TP algorithm of Dur et al (2018), and they show that the TP rule is equivalent to EADAM.

Theorem 6. *: The k-TP rule is constrained efficient within the set of k-stable matching rules.*

Proof. Let μ be the matching obtained by the k-TP algorithm. We will show that there does not exist a k-stable matching which Pareto-dominates μ. Suppose, by contradiction, that there exists a k-stable matching η which Pareto dominates μ. That is, $\eta_i R_i \mu_i$ for all $i \in S$ and $\eta_j P_j \mu_j$ for some $j \in S$. Then in the last step of the k-TP there is at least one cycle which transforms μ to η. However, none of the cycles will be carried out in the k-TP procedure. This implies that for each cycle there exist students $i, j \in S$ such that $\mu_i \neq \eta_i$ and student j is inactive, while $j \succ_{\eta_i}^k i$. Thus, since j is an inactive student, $\mu_j = \eta_j$ and η is not k-stable, which is a contradiction. □

Student i gives k**-consent** in the k-TP procedure if he allows l-justified envy with $l < k$ but not l'-justified envy with $l' \geq k$ when consent is called for in the procedure. We will show next that the k-TP rule is k-consent-proof in the following sense.

Proposition 9. *Given any preference profile R, for each $i \in S$ and $c \in C$, if student i gives k-consent for c then the student does not get a better assignment by not giving k-consent for c in the k-TP rule.*

Proof. By the definition of the k-TP rule, at each step t k-consent is only relevant for inactive students at μ_{t-1}. Let student i be an inactive student in step $t-1$ and be assigned to c. Since student i is inactive at step $t-1$ of the k-TP procedure, at each step $t \geq t-1$ she is assigned to c, as she is not a part of any cycle. Hence, each inactive student is assigned the same school under the k-TP rule in each step after the student becomes inactive. Thus, the decision about consent does not affect the student's assignment in the k-TP rule. □

5.5 The Simplified k-Consent Rules

Following Tang and Yu (2014, [51]), we also define a set of rules based on k-consent that simplifies the original EADAM procedure.

Simplified k-Consent Rules

Fix a preference profile R, and recall that the fixed priority table is $\succ$.

Round 0: Run the DA and let the resulting matching be denoted by μ^0. Let $P^0 := P$.

For $k \geq 1$, given μ^{k-1} and P^{k-1}, determine μ^k and P^k as follows.

Round k: ($k \geq 1$)

- Identify all the inactive students at round $k-1$ based on μ^{k-1} and R^{k-1}, and let μ_i^{k-1} be ranked first by all inactive students i.

- Call the schools of the active students active. For each inactive student i, active school c, and active student j such that $i \succ_c^k j$ and i prefers c to his assigned school μ^{k-1} at the end of round $k-1$, let $\mu_j^{k-1} R_j^t c$ such that c is ranked directly below μ_j^{k-1} and all other orderings for j remain the same (i.e., let R_j^k be the same otherwise as R_j^{k-1}). For all remaining students $h \in N$, let $R_h^k = R_h^{k-1}$.

- Finally, run the DA for preference profile R^k and let the resulting assignment be μ^k.

The procedure ends when there are no more active students, that is, the trade graph has no cycles.

Remark 3. In the Simplified k-Consent rule, in the envy graph there will be at most the same or fewer incoming edges to each student at step t compared to step $t-1$, since active students can only get the same or higher-ranked schools at step t compared to their matchings at step $t-1$. Since none of the inactive students at step $t-1$ were in a cycle at step $t-1$, this implies that they cannot be in a cycle at step t. Thus, inactive students remain inactive from step $t-1$ to step t. Also, active students may become inactive from step $t-1$ to step t if they were in a cycle at step $t-1$ but not at step t. Therefore, there exists a finite $T \geq 1$ such that none of the students are in a cycle in the envy graph at the end of step T, and hence the simplified k-Consent rule terminates in a finite number of steps.

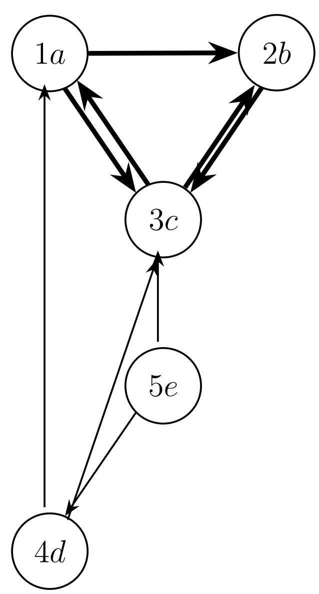
The following example illustrates how a Simplified k-Consent rule works.

Example 9. Consider the matching problem in Example 1.

Step 1

Round	a	b	c	d	e
1	$3, \underline{4}$	$\underline{1}$	$\underline{2}$	$\underline{5}$	0
2	$\underline{4}$	$1, \underline{3}$	$\underline{2}$	$\underline{5}$	0
3	$\underline{4}$	$\underline{3}$	$\underline{1}, 2$	$\underline{5}$	0
4	$\underline{4}$	$\underline{2}, \boxed{3}$	$\underline{1}$	$\underline{5}$	0
5	$\underline{4}$	$\underline{2},$	$\boxed{1}, \underline{3}$	$\underline{5}$	0
6	$\underline{1}, \boxed{4}$	$\underline{2},$	$\underline{3}$	$\underline{5}$	0
7	$\underline{1}$	$\underline{2},$	$\underline{3}, 4$	$\underline{5}$	0
8	$\underline{1}$	$\underline{2},$	$\underline{3}$	$\underline{4}, 5$	0
9	$\underline{1}$	$\underline{2},$	$\underline{3}, 5$	$\underline{4}$	0
10	$\underline{1}$	$\underline{2},$	$\underline{3}$	$\underline{4}$	$\underline{5}$

Envy Graph



Active students: $1, 2, 3$

Inactive students: $4, 5$

Next we change the preferences for inactive students 4 and 5, as well as for student 2 (since student 4 prefers c to d and consents for 2-priority only). The student preferences become the following:

Student Preferences

R_1	R'_2	R_3	R'_4	R'_5
b	b	a	d	e
c	c	b	a	d
a		c	c	c
d				

Step 2

Round	a	b	c	d	e
1	$\underline{3}$	$\underline{2}, 1$	0	$\underline{4}$	$\underline{5}$
2	$\underline{3}$	$\underline{2}$	$\underline{1}$	$\underline{4}$	$\underline{5}$

Note that this yields the same matching as the k-Consent and the k-TP rules, as seen in Examples 1 and 3.

5.6 Equivalence Results

Before we state the equivalence of the different matching rules, we will prove a key result that relates interrupters to inactive students, which will be used in the equivalence proofs.

Proposition 10. *Given preference profile R, the lastly rejected interrupters in the DA are inactive students at* (R, μ^{DA}).

Proof. Let round t be the last round in the DA procedure at which interrupters are rejected. Assume that student i is an interrupter for school a and the pair (i, a) is an interrupting pair in round t. Suppose i is active at μ^{DA}.

Let β be the matching where the trading cycle for S is carried out where $S = (i_1, \ldots, i_{\bar{v}})$. Let$i_1 : i$ and $\forall v \in 1, \ldots, \bar{v}$, $\beta_{i_v} = \mu^{DA}_{i_{v+1}}$, mod $\bar{v}$, $\mu^{DA}_{i_{v+1}} P_{i_t} \mu^{DA}_{i_v}$. Thus, $\forall j \in N \setminus S, \beta_j = \mu^{DA}_j$.

Specifically, $\mu^{DA}_{i_1} P_{i_{\bar{v}}}$. $\mu^{DA}_{i_{\bar{v}}}$. Thus, $i_{\bar{v}}$ is rejected by μ^{DA}_i. As shown, $i_{\bar{v}}$ is rejected by μ^{DA}_i in round $t_{\bar{v}} \geq t'$. We have $\mu^{DA}_{i_2} P_{i_1} \mu^{DA}_{i_1}$. Since i_2 is matched to $\mu^{DA}_{i_2}$, student i_1 was rejected by $\mu^{DA}_{i_2}$ before round t' in favor of some other student l, say in round t_1, so $t_1 < t'$. We have $t_{\bar{v}} \geq t' > t_1$ and thus $t_{\bar{v}} > t_1$.

For all $v \in (1, \ldots, \bar{v})$, student i_v is rejected by $\mu^{DA}_{i_{v+1}}$ in step t_v. Let $\mu^{DA}_{\bar{v}} P_{\bar{v}-1} \mu^{DA}_{\bar{v}-1}$ and assume that $t_{\bar{v}} > t_{\bar{v}-1}$. This implies that there exist some other student l such that $l \succ_{\mu^{DA}_{\bar{v}}} \bar{v} - 1$. Hence the acceptance of $\bar{v}$ at $\mu^{DA}_{\bar{v}}$ implies that student l is an interrupter at round $t' > t$. A contradiction that round t is the last round at which interrupter is rejected.Thus $t_{\bar{v}-1} > t_{\bar{v}}$. Applying the same argument iteratively to each pair in the cycle as to $t_{\bar{v}}$ and $t_{\bar{v}-1}$. We have $t_1 > t_2 > t_3 > \ldots > t_{\bar{v}} > t_1$. A contradiction. □

Theorem 7. *For all $k \in \{1, \ldots, n\}$, the k-Consent rule is outcome equivalent to the simplified k-Consent rule.*

Proof. We define the **Modified k-Consent rule**, which can be shown to be equivalent to both the k-Consent rule and the Simplified k-Consent rule.

Modified k-Consent rule

Fix a preference profile R. Let $t := 1$ and do step t.

Step t:

- Run the DA using R^t: the DA matching is Pareto-efficient, stop, and let this matching be the final outcome. Specifically, if there are no more interrupters, stop. Similarly, if $P^t = P^{t+1}$ for $t \geq 2$ then let this matching be the final outcome.

- Otherwise check for interrupters and find all of them in the last round of the DA in which there is any. If (i, c) is an interrupting pair, let μ^{t-1}_i be ranked first by i. For all other students j such that $i \succ^k_a j$ for all a such that $a P^t \mu^{t-1}_i$ remove a from j's preferences, that is let $\mu_j^{t-1} P_j^t a$, move a down to below μ_j^{t-1} and let P_j^t be the same otherwise as P_j^{t-1}. Call the updated preference profile R^{t+1}.

Let $t := t + 1$ and repeat step t.

Note that by the construction of the modified k-Consent rule, the preference profile R^T used in the last step T satisfies k-priority with respect to R. Thus, the DA applied to R^T will satisfy k-stability with respect to R. Hence, this rules satisfies k-stability.

It is straightforward to verify that the Modified k-Consent rule is equivalent to the k-Consent rule. In order to see that the Modified k-Consent rule is also equivalent to the Simplified k-Consent rule, we need to apply Proposition 10 to each step of the Modified k-Consent rule. □

Theorem 8. *For all $k \in \{1, \ldots, n\}$, the Simplified k-Consent rule is outcome equivalent to the k-TP rule.*

Proof. Fix a preference profile P and let μ and η, respectively denote the matching obtained by the simplified k-Consent rule and the k-TP rules respectively. We need to show that $\mu(R) = \eta(R)$. Let $IA_t \subseteq S$ denote the set of inactive students in round t.

For $t = 0$, since μ and η Pareto-dominate the DA, $\mu_j(R) = \eta_j(R) = DA_j(R)$ for all $j \in IA_0$. We will now show that for each $t \geq 1$ if, for all $j \in IA_0 \cup IA_1 \cup \ldots IA_{t-1}$, $\mu_j(R) = \eta_j(R)$, then for all $j' \in IA_t$, $\mu_{j'}(R) = \eta_{j'}(R)$, for all $t \geq 0$.

Let $t = 1$. Since $j \in IA_0$, j cannot be in a cycle at step 1 and hence $j \in IA_1$. In the Simplified k-Consent rule for each inactive student j, active student i such that $j \succ_c^k i$ for some and j prefers c to his assigned school, we remove c from i's preferences, that is, we let $\mu_i^0 P_i^1 c$. In the k-TP algorithm for each inactive student j we remove the active edges for which $j \succ_c^k i$. Therefore, the set of inactive students at the end of step 1 is the same for the Simplified k-Consent and the k-TP rules. The same argument can be applied iteratively to each step.

Note that in the last step t of the k-Consent rule (as well as the k-TP) all students are inactive. Therefore, none of them are in a cycle in the Envy Graph at the end of step t and hence the trade graph is the empty graph. We can thus conclude that $\mu_{j'}(R) = \eta_{j'}(R)$. Therefore, $\mu(R) = \eta(R)$. □

As a corollary to previous results, we will summarize our main findings in the theorem below.

Theorem 9. *For all $k \in \{1, \ldots, n\}$, the k-Consent, the Simplified k-Consent, and the k-TP rules are equivalent. Moreover, all three matching rules are k-stable and constrained efficient within the set of k-stable matching rules.*

5.7 The k-Deferred Acceptance Rules

In this section we introduce the k-DA rules in a model where schools are allowed to have multiple seats. Accordingly, we let $q_c \geq 1$ be the capacity of school $c \in C$.

Let $G = ((S_p, S_t)E)$ be a bipartite graph with vertex set $S_p \cup S_t$ and edge set E. S_p represents the newly proposing students, S_t represents the tentatively matched students, and E is the set of edges which are given by the pairs where a tentatively matched student will be rejected in favor of a newly proposing student, given the k-constraint, that is, a student i can only be rejected in favor of another student j by school c if $j \succ_c^k i$.

We can find the maximum cardinality matching in G using the **max (min)** method which is defined as follows. Start with the highest (lowest) priority newly proposing student and hypothetically reject the highest (lowest) priority tentatively assigned student that satisfies the k-constraints. Do this repeatedly in order to determine how many students can be rejected at most. Let that number be $m_{max}(m_{min})$.

Lemma 1 clarifies how to find the maximum number of matchings in G.

Lemma 1. *The maximum cardinality of matching in the bipartite graph $G = (S_p, S_t, E)$ has cardinality $m_{max} = m_{min} = m$.*

Proof. Let $E_{max} \subseteq E$ such that E_{max} is the set of edges found by the max method in G. Then m_{max} matches in E_{max} will be incident to the m_{max} highest priority newly proposing students in S_p. Let's suppose that m_{max} is less than the maximum number of matchings in G. This means that there is at least one more edge that we can find which is independent of the edges in E_{max}. It must be the case that the $m_{max} + 1^{st}$ highest priority student in S_p would have an edge to a tentatively matched student in S_t satisfying the k-constraints. Let this student in S_t be the highest priority such that in E_{max} there is no edge that is incident to this student. But then this edge would have been found by the max method and would be in E_{max}. Hence, we have a contradiction. Similarly, let $E_{min} \subseteq E$ such that E_{min} is the set of edges found by the min method in G. Then m_{min} matches in E_{min} will be incident to the m_{min} lowest tentatively matched students in S_t. Let's suppose that m_{min} is less than the maximum number of matchings in G. This means that there is at least one more edge that we can find which is independent of the edges in E_{min}. It must be the case that $m_{min} + 1^{st}$ lowest priority student in S_t would have an edge to a newly proposing student S_p satisfying the k-constraints. Let this student in S_p be the the lowest priority such that in E_{min} there is no edge that is incident to this student. But then this edge would have been found by the min method and would be in E_{min}. Hence, we have a contradiction. □

k-DA Rules

Fix a preference profile R, and recall that the fixed priority table is $\succ$.

Step 1: Each student applies to his first-ranked school. Each school tentatively accepts as many students as possible, up to its capacity, among the current applicants, following its priority order, and rejects the rest.

Step t: Each rejected student in the previous step applies to his highest-ranked school that has not rejected him yet. For each school $c \in C$:

- Determine m_c, the maximum cardinality matching in G.
- Given $m_c \leq |S_p|$, let $S'_p = \{s \in S_p : \ s$ is among the m highest priority students in $S_p\}$. Note that $|S'_p| = m_c$.

- Let $\bar{S} = S'_p \cup S_t$ and choose the q_c highest priority students in $\bar{S}$ to be tentatively matched at c. Note, that this is the same as rejecting the m_c lowest priority students.

The algorithm terminates when every student is tentatively assigned to some school or has been rejected from each of his acceptable schools.

Using S'_p can be interpreted as "pre-screening". This pre-screening gets rid of the lower-ranked newly proposing students who will not be accepted.

Remark 4. For $k = 1$ this procedure yields the same as the standard DA. The reason is that the pre-screening does not affect the outcome when $k = 1$, that is, the top q_c students are the same in $S_p \cup S_t$ and $S'_p \cup S_t$. Note, however, that $S_p \neq S'_p$ in general.

The following example illustrates how the k-DA rule works.

Example 10. Consider a school choice problem with eleven students and two schools. Let $q_1 = 3$ and $q_2 = 1$. Assume $k = 5$. The preferences and priorities are as follows:

Student Preferences

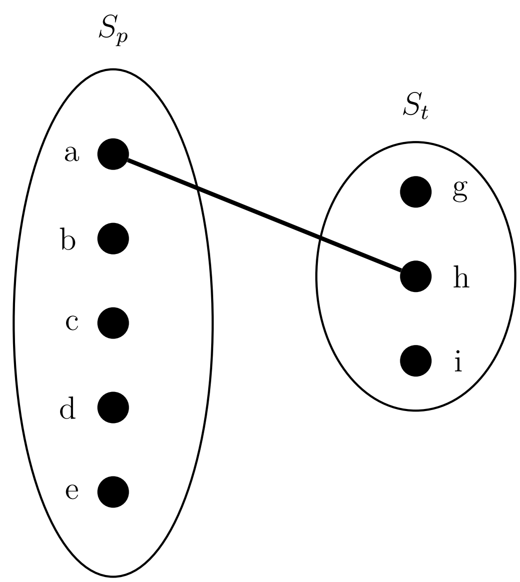
a	b	c	d	e	f	g	h	i	j	k
c_2	c_2	c_2	c_2	c_2	c_2	c_1	c_1	c_1	c_1	c_1
c_1	c_1	c_1	c_1	c_1	c_1	c_2	c_2	c_2	c_2	c_2

School Priorities

$q_1 = 3$	$q_2 = 1$
c_1	c_2
a	k
f	f
g	g
b	a
h	b
c	c
i	i
d	j
e	d
j	e
k	h

Step 1 Students g, h, i, j, k apply to school c_1 and as a result the lowest-priority students j and k are rejected . Similarly, students a, b, c, d, e, f apply to school c_2 and as a result the lowest-priority students a, b, c, d, e are rejected.

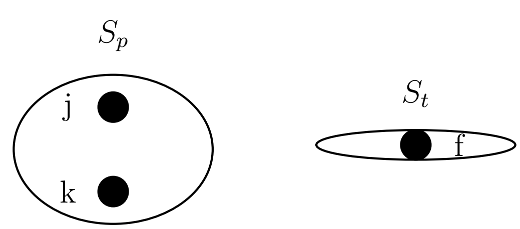
Step 2 Students a, b, c, d and e apply to school c_1. Determine m_1 for school c_1:



$m_1 = 1$

Thus, for school c_1, students b, c, d, e are rejected. Since $\bar{S} = \{a, g, h, i\}$, student i is also rejected.

Students j and k apply to school c_2. Determine m_2 for school c_2:



Since $m_2 = 0$ for school c_2, students j and k are rejected.

Thus, the k-DA rule yields the following assignment:

$$\left\{ \begin{array}{cc} c_1 & c_2 \\ \text{a, g, h} & \text{f} \end{array} \right\}$$

The following can be verified directly from the construction of the k-DA rules.

Theorem 10. *For all $k \in \{1, \dots, n\}$, the k-DA rule is k-stable.*

Unfortunately the k-DA rule is not constrained efficient, that is, there may be a preference profile at which the k-DA matching is Pareto-dominated by a k-stable matching.

Example 11. Consider a school choice problem with five students and five schools, each with one seat. The preferences and priorities are as follows.

Student Preferences

1	2	3	4	5
b	a	a	a	e
a	c	c	d	
	b	b		
		d		

School Priorities

a	b	c	d	e
1	2	3	4	5
5	3	2		
4	1	5		
3				
2				

Assume $k = 2$. The k-DA rule yields the following assignment:

$$\left\{\begin{array}{ccccc} a & b & c & d & e \\ 1 & 2 & 3 & 4 & 5 \end{array}\right\}$$

However, this is Pareto-dominated by the following k-stable matching.

$$\left\{\begin{array}{ccccc} a & b & c & d & e \\ 3 & 1 & 2 & 4 & 5 \end{array}\right\}$$

□

However, we can show the following.

Theorem 11. *The k-DA rule is not Pareto dominated by the DA.*

Proof. Fix a preference profile R and let μ be the matching obtained by the k-DA algorithm at this profile. Let γ be the matching obtained by the DA algorithm at R. We will show that γ does not Pareto-dominate μ. Suppose by contradiction that γ Pareto-dominates μ. That is, $\gamma_j R_j \mu_j$ for all $j \in S$ and $\gamma_i P_i \mu_i$ for some $i \in S$.
Let $\gamma_i = c$. Then $\mu_i \neq c$ and N$cP_i\mu_i$. this means that i must have applied to school c before he applied to school μ_i. Since $\mu_i \neq c$, i must have been rejected in step t of the k-DA procedure in favor of some other student j such that $\mu_j = c$ and $j \succ_c^1 i$. Assume without loss of generality that t is the first step in the k-DA algorithm such that a student i gets rejected from γ_i. We consider the following cases:

Case 1: $\gamma_j P_j c$. We have a contradiction since this means that j was rejected in the k-DA algorithm before step t.

Case 2: $cP_j\gamma_j$. Since γ is the matching obtained by the DA algorithm and $\gamma_i = c$ we have $i \succ_c^1 j$. This, however, contradicts $j \succ_c^1 i$.

Thus, we have a contradiction in each case. □

Since it is possible that the outcome of the k-DA rule satisfies k'-stability at some profile where $k' < k$, we conjecture that the k-DA rule is constrained efficient at each preference profile R with respect to the lowest k' for which it is k'-stable.

Example 12. The k-DA rules are not strategy-proof (for $k \geq 2$).

Consider a school choice problem with five students and five schools, each with one seat. The preferences and priorities are as follows:

Student Preferences

R_1	R_2	R_3	R_4	R_5
c_1	c_2	c_1	c_4	c_5
c_2	c_1	c_4	c_3	c_4
c_3	c_5	c_5	c_2	c_1
c_4	c_3	c_2	c_1	c_2
c_5	c_4	c_3	c_5	c_3

School Priorities

c_1	c_2	c_3	c_4	c_5
2	1	4	5	3
3	2	5	4	5
1	3	1	3	2
4	5	2	1	4
5	4	3	2	1

Assume $k = 2$. The k-DA mechanism yields the following assignment:

$$\left\{ \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \\ 3 & 2 & 1 & 4 & 5 \end{array} \right\}$$

Now consider the following preference profile:

Student Preferences

R'_1	R_2	R_3	R_4	R_5
c_2	c_2	c_1	c_4	c_5
c_3	c_1	c_4	c_3	c_4
c_1	c_5	c_5	c_2	c_1
c_4	c_3	c_2	c_1	c_2
c_5	c_4	c_3	c_5	c_3

The k-DA yields the following assignment:

$$\left\{ \begin{array}{ccccc} c_1 & c_2 & c_3 & c_4 & c_5 \\ 3 & 1 & 2 & 4 & 5 \end{array} \right\}$$

Thus, student 1 can manipulate.

We know that the DA rule is k-stable for all $k \in \{i, \ldots, n\}$ and it is strategyproof. Interestingly, when $k = n - 1$, the TTC rule is also k-stable and strategyproof.

Next, we demonstrate that there is no unique optimal k-stable matching.

Example 13. Consider a school choice problem with five students and five schools, each with one seat. The preferences and priorities are as follows:

Students Preferences

1	2	3	4	5
b	a	a	a	e
a	c	c	d	
	b	b		
		d		

Schools Priorities

a	b	c	d	e
1	2	3	4	5
5	3	2		
4	1	5		
3				
2				

Assume $k = 3$. The 2-stable k-DA matching gives:

$$\left\{ \begin{array}{ccccc} a & b & c & d & e \\ 3 & 1 & 2 & 4 & 5 \end{array} \right\}$$

However, this is not Pareto dominated by the following 3-stable matching, and in fact neither matching Pareto-dominates the other.

$$\begin{Bmatrix} a & b & c & d & e \\ 2 & 1 & 3 & 4 & 5 \end{Bmatrix}$$

Since a 2-stable matching is also 3-stable, this demonstrates that there is no unique optimal 3-stable matching.

5.8 Conclusion

We analyzed a set of matching rules between the DA and Kesten's EADAM rule, which exhibit gradually lower degrees of stability but become generally more efficient. The trade-offs are thus clear: at one extreme we can choose the DA which is fully stable and least efficient, and at the other extreme we can choose EADAM which has the lowest degree of stability but is fully efficient. The set of k-Consent rules allows the market designer to choose among a whole range of matching rules between these two extreme rules where the trade-off is between stability and efficiency.

We have provided different definitions of the k-Consent rules that allow for small priority violations, and have shown that these different definitions are equivalent. These equivalences clarify the rules well, and provide us with a rare flexibility when presenting and working with this set of rules.

Furthermore, we have shown that each k-Consent (or Simplified k-Consent or k-TP) rule satisfies k-stability, the corresponding notion of relaxed stability in terms of the maximum size of the acceptable priority violation, where k determines the degree of stability. Moreover, we have also proved that each k-Consent rule is constrained efficient within the set of k-stable matching rules. These result are well-known for the two extreme members: the Deferred Acceptance rule is constrained efficient within the set of stable rules, and EADAM is fully efficient. Our results show that the matching rules between these two rules, which represent gradually lower degrees of stability as they become gradually more efficient, have similar features.

We have also studied k-stability in the school choice model and introduced the class of k-DA rules, which has the DA and Boston rules are the two extreme members. However, unlike our k-Consent rules, the k-DA rules are not constrained efficient within the set of k-stable rules, albeit, not Pareto dominated by the DA. The trade-offs are thus clear again: at one extreme we can choose the DA which is fully stable and least efficient, and at the other extreme we can choose the Boston rule which has the lowest degree of stability but is fully efficient.

We assume that the allowed size of the priority violations is uniform across the schools, which is an assumption that can be relaxed quite easily, we believe, and our current results would generalize.

In future work we are planning to analyze the incentive properties of k-Consent rules. As shown here, these rules are not strategyproof, with the notable exception of the DA rule. But the EADAM rule has some interesting incentive properties which may be carried over to other k-Consent rules as well. We cannot say the same about k-DA rules, since the Boston rule is highly manipulable. However, these rules have a clearer structure and thus could be more popular in applications. Note that to date the EADAM rule is yet to be used, whereas the Boston rule is used widely, even though several negative aspects of the Boston rule have been discussed in the literature. Finally, we would like to provide characterizations of all of these rules in terms of their normative and incentive properties.

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