Seiberg-Witten tau-function on Hurwitz spaces

Meghan White

A Thesis
in
The Department
of
Mathematics and Statistics

Presented in Partial Fulfillment of the Requirements
For the Degree of Master of Science (Mathematics)
Concordia University
Montréal, Québec, Canada

January 2020

Concordia University

School of Graduate Studies

This is to certify that the thesis prepared

By: **Meghan White**

Entitled: **Seiberg-Witten tau-function on Hurwitz spaces**

and submitted in partial fulfillment of the requirements for the degree of

Master of Science (Mathematics)

complies with the regulations of this University and meets the accepted standards with respect to originality and quality.

Signed by the final examining commitee:

Dr. Marco Bertola	Chair
Dr. Alexey Kokotov	Examiner
Dr. Dmitry Korotkin	Examiner
Dr. Marco Bertola	Examiner
Dr. Dmitry Korotkin	Supervisor

Approved Dr. Galia Dafni

Graduate Program Director

Dean of Department

Date

Abstract

Seiberg-Witten tau-function on Hurwitz spaces

Meghan White

We provide a proof of the form taken by the Seiberg-Witten tau-function on the Hurwitz space of N-fold ramified covers of $\mathbb{C}P^1$ by a compact Riemann surface of genus g, a result derived in [10] for a special class of monodromy data. To this end we examine the Riemann-Hilbert problem with $N \times N$ quasi-permutation monodromies, whose corresponding isomonodromic tau-function contains the Seiberg-Witten tau-function as one of three factors. We present the solution of the Riemann-Hilbert problem following [11]. Along the way we give elementary proofs of variational formulas on Hurwitz spaces, including the Rauch formulas.

Acknowledgments

I would like to thank my supervisor Professor Korotkin for his guidance and patience. I would also like to thank Professor Bertola for his illuminating course on theta functions on Riemann surfaces.

Contents

Chapter 1

Introduction

The tau-function is a central object in the theory of integrable systems, which for our purposes appears in the context of a Riemann-Hilbert problem. To describe the Riemann-Hilbert problem we start with a linear system of ordinary differential equations defined on $\mathbb{C}P^1$,

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi(\lambda), \quad \Psi(\lambda_0) = I, \tag{1.0.1}$$

where $A(\lambda)$ is an $N \times N$ matrix-valued function with entries given by meromorphic functions on $\mathbb{C}P^1$, with poles at the set of points $\lambda_1, ..., \lambda_M$; since the system is linear, the singularities of any solution $\Psi(\lambda)$ are confined to this set of points. By Cauchy's theorem we can always find local solutions away from the singularities $\lambda_1, ..., \lambda_M$. Global solutions necessitate analytic continuation, resulting in solutions which are multi-valued under analytic continuation around a singularity [15].

Working in a neighbourhood of a point $\lambda_0 \notin \{\lambda_1, ..., \lambda_M\}$, we find a local basis of N solutions having nonzero Wronskian, and re-define Ψ to be the $N \times N$ matrix whose columns are the N solutions. It is then simple to show using properties of the determinant that outside of the points $\lambda_1, ..., \lambda_M$ the Wronskian is given by

$$W(\lambda; \Psi) = W(\lambda_0; \Psi) \exp\left[\int_{\lambda_0}^{\lambda} \mathrm{tr}(A(\lambda))\right], \tag{1.0.2}$$

and hence $\Psi(\lambda)$ is non-singular (as a matrix) outside of the points $\lambda_1, ..., \lambda_M$. It follows that under analytic continuation $\Psi(\lambda) \to \Psi(\lambda + \ell_m)$ around a contour ℓ_m containing only the singularity λ_m, the columns of $\Psi(\lambda + \ell_m)$ form a second basis of solutions for the system 1.0.1. Hence $\Psi(\lambda + \ell_m)$ and $\Psi(\lambda)$ are related by right-multiplication by some $M_m \in GL_N(\mathbb{C})$. In this way we obtain a monodromy representation of the fundamental group of $\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}$ [15].

The Riemann-Hilbert problem is the problem of finding differential systems of the form 1.0.1 which give rise to a given complex representation of the fundamental group $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$ [10]. We can always find such a system as long as we allow the poles $A(\lambda)$ to have arbitrary order (we must exclude the possibility of essential singularities), however, a system with poles of prescribed orders having the desired monodromy representation may or may not exist [15]. In this work we will examine the Fuchsian case, the case where $A(\lambda)$ has only simple poles.

The Riemann-Hilbert problem is closely related to the theory of isomonodromic deformations of systems of differential equations, that is, deformations which preserve the monodromy representation. Isomonodromic deformation of the system 2.1.4 produces a family of linear systems of partial differential equations

$$\frac{\partial \Psi(\lambda, a)}{\partial \lambda} = A(\lambda, a)\Psi(\lambda, a), \tag{1.0.3}$$

which depend analytically on a deformation parameter a. The dependence of $\Psi(\lambda, a)$ on this deformation parameter is described by a second system of linear PDE's. The compatibility condition for the two systems of PDE's yields a system of nonlinear integrable PDE's for $A(\lambda, a)$, called isomonodromy equations [13].

In the case of a Fuchsian system, the resulting isomonodromy equations are the Schlesinger equations. The isomonodromic tau-function was first introduced by Jimbo and Miwa for the Schlesinger system. For each solution of the Schlesinger system they found a corresponding closed 1-form given by the (exterior) logarithmic derivative of a holomorphic function on the space of deformation parameters: the Jimbo-Miwa tau-function τ_{JM} [9].

More generally the isomonodromy equations form an extensive family of nonlinear integrable systems which is of fundamental importance as a source of transcendents, such as the six Painlevé transcendents. The full family of isomonodromic equations arising from systems with arbitrary pole structure was written down by Jimbo, Miwa and Ueno in [9], originally motivated by the appearance of Painlevé transcendents in the correlation functions of certain quantum field theories. At the same they defined the isomonodromic tau-function associated to a solution of a system of isomonodromic equations.

Tau-functions are themselves important transcendents, appearing for example as partition functions of integrable quantum theories [12]. They arise as part of the natural geometric structure of isomonodromy equations: all isomonodromy equations can be formulated as completely integrable, non-autonomous Hamiltonian systems, with isomonodromic tau-functions acting as generating functions for commuting Hamiltonians. For more details see [16].

More generally a tau-function can be defined in the context of a Riemann-Hilbert problem, where its vanishing indicates that the problem has no solution. In this thesis we compute the Seiberg-Witten tau-function, which appears as a factor in the Jimbo-Miwa tau-function. Working in the context of the Riemann-Hilbert problem for the case of quasi-permutation monodromies, we use the solution of this problem as given in [12] to write down the solutions of the Schlesinger system, and ultimately the Jimbo-Miwa tau-function following the method in [10].

The natural context for the N-dimensional Riemann-Hilbert problem with quasi-permutation monodromies is an N-fold ramified covering of $\mathbb{C}P^1$ by a compact Riemann surface $\mathcal{L}$, with ramification points at the singularities $\lambda_1, ..., \lambda_M$: the solution of such a problem is given by a meromorphic function on $\mathcal{L}$. The cover is described by a pair $(\mathcal{L}, f)$ in the Hurwitz space $H_{g,N}$ consisting of equivalence classes of degree N meromorphic functions f defined on a compact Riemann surface of genus g, with the branch points of the cover given by the critical points of f.

We start in chapter 2 by introducing the Riemann-Hilbert problem with quasi-permutation monodromies. We describe the system of partial differential equations satisfied by a solution Ψ of this problem, which encode how the solution depends on the positions of the singularities $\lambda_1, ..., \lambda_M$ under isomonodromic deformation. The corresponding nonlinear system of isomonodromy equations is the Schlesinger system; the corresponding isomonodromic tau-function, the Jimbo-Miwa tau-function, is defined by the system of equations

$$\frac{\partial \ln \tau_{JM}}{\partial \lambda_m} = \frac{1}{2} \operatorname*{res}_{\lambda=\lambda_m} \frac{\operatorname{tr}(d\Psi\Psi^{-1})^2}{d\lambda}. \tag{1.0.4}$$

We end chapter 2 by describing in detail the correspondence between $N \times N$ quasi-permutation monodromy representations of $\mathbb{C}P^1 \backslash \{\lambda_1, ..., \lambda_M\}$, and compact Riemann surfaces $\mathcal{L}$ given by an N-fold ramified covering of $\mathbb{C}P^1$.

In chapter 3 we review some important background on compact Riemann surfaces, including the Abel map which embeds a compact Riemann surface into its Jacobian variety. We discuss the theta function, a holomorphic function associated to a compact connected Riemann surface and defined on its Jacobian variety. In particular we discuss the divisor of the theta function, and the conditions under which the theta function is not identically zero.

These considerations will lead us to define a spinor, or half-differential, on a compact connected Riemann surface, given by the square root of a holomorphic differential whose only zeros are double zeros at each point in the theta function divisor. Using this spinor we are then able to define the prime form, a bidifferential $E(P,Q)$ holomorphic in both arguments. Together the prime-form and theta function are the objects needed to define the solution of our Riemann-Hilbert problem.

Next we discuss the Bergman kernel, a meromorphic bidifferential $B(P,Q)$ with double poles on the diagonal. We prove that if $\omega_1,...,\omega_g$ are a normalized basis of holomorphic differentials on the covering space $\mathcal{L}$ and $a_1,...,a_g,b_1,...,b_g$ a canonical basis for $H_1(\mathcal{L},\mathbb{Z})$ then the Bergman kernel satisfies

$$\int_{P\in b_\alpha} B(P,Q) = 2\pi i\omega_\alpha(Q). \tag{1.0.5}$$

We also introduce the Szegö kernel $S(P,Q)$, a meromorphic bidifferential with simple poles on the diagonal. The Szegö kernel, or rather a modified version, is the main component of the solution of the Riemann-Hilbert problem.

Using the Bergman kernel we can write down variational formulas, called Rauch formulas, which describe the dependence of the normalized holomorphic differentials $\omega_1,...,\omega_g$ on the positions of the singularities $\lambda_1,...,\lambda_M$. Specifically,

$$\begin{aligned}\partial_{\lambda_m}\omega_\alpha(P) &= \operatorname*{res}_{\lambda=\lambda_m}\frac{1}{d\lambda}\sum_{j=1}^{N}\omega_\alpha(\lambda^{(j)})B(P,\lambda^{(j)}),\\ \partial_{\lambda_m}\mathbf{B}_{\alpha\beta} &= -\operatorname*{res}_{\lambda=\lambda_m}\frac{4\pi i}{d\lambda}\sum_{j<k}\omega_\alpha(\lambda^{(j)})\omega_\beta(\lambda^{(k)}),\quad \mathbf{B}_{\alpha\beta}=\int_{P\in b_\alpha}\omega_\beta(P).\end{aligned} \tag{1.0.6}$$

We provide a proof of these formulas for the case where the singularities $\lambda_1,...,\lambda_M$ have arbitrary order. Finally, we prove a similar variational formula for $B(P,Q)$, namely

$$\partial_{\lambda_m}B(P,Q) = \operatorname*{res}_{\lambda=\lambda_m}\frac{1}{d\lambda}\sum_{j=1}^{N}B(P,\lambda^{(j)})B(\lambda^{(j)},Q). \tag{1.0.7}$$

In chapter 4, we put these formulas to use in finding explicit forms for the differential equations satisfied by τ_{JM} and τ_{SW}. We show that the Jimbo-Miwa tau-function is composed of three factors

$$\tau_{JM} = \tau_{SW}\tau_B^{-1/2}\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega), \tag{1.0.8}$$

where τ_{SW} is the Seiberg-Witten tau-function; $\Theta[^{\mathbf{p}}_{\mathbf{q}}]$ is the theta function with characteristics $\mathbf{p}, \mathbf{q}$ associated to the Riemann surface $\mathcal{L}$, and Ω is a constant which depends on the monodromies; and for $(\mathcal{L}, f) \in H_{g,N}$, $\tau_B(\mathcal{L}, f)$ is the Bergman tau-function satisfying [10]

$$\frac{\partial \ln \tau_B(\mathcal{L}, f)}{\partial \lambda_m} = - \operatorname*{res}_{P=P_m} \frac{B^{df}_{reg}(P)}{df}, \tag{1.0.9}$$

where $B^v_{reg}(P)$ is given by regularizing the Bergman kernel on the diagonal:

$$B^v_{reg}(P) = \left(B(P,Q) - \frac{v(P)v(Q)}{\left(\int_P^Q v\right)^2} \right)_{P=Q}, \tag{1.0.10}$$

for some choice of meromorphic differential v on $\mathcal{L}$.

Finally, we prove that τ_{SW} satisfies the equation

$$\frac{\partial \ln \tau_{SW}}{\partial \lambda_m} = \frac{1}{2} \operatorname*{res}_{\lambda=\lambda_m} \sum_{j=1}^{N} \frac{W^2(\lambda^{(j)})}{d\lambda}, \qquad W(P) = \sum_{m=1}^{M}\sum_{j=1}^{N} r_m^{(j)} d_P \ln E(P, \lambda^{(j)}), \tag{1.0.11}$$

where $r_m^{(j)}$ are constants determined by the monodromies. To this end we prove several variational formulas describing the dependence of the prime form $E(P,Q)$ on the positions of $\lambda_1, ..., \lambda_M$ for arbitrary points P, Q, including branch points.

Ultimately we prove that the Seiberg-Witten tau-function takes the form

$$\tau_{SW}^2 = \prod_{\lambda_m^{(i)} \neq \lambda_n^{(j)}} E(\lambda_m^{(i)}, \lambda_n^{(j)})^{r_m^{(i)} r_n^{(j)}}, \tag{1.0.12}$$

for arbitrary quasi-permutation monodromy data.

Chapter 2

The Riemann-Hilbert problem and tau-function

2.1 The Riemann-Hilbert problem

We start by describing the Riemann-Hilbert problem, laid out in [12], that provides the context for our tau-function. The problem is to find a function $\Psi(\lambda) \in GL(N, \mathbb{C})$ defined on the universal cover of $\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}$, such that

1. $\Psi(\lambda)$ is normalized to satisfy $\Psi(\lambda_0) = I$ at a point λ_0 of the universal cover,
2. $\Psi(\lambda)$ has prescribed right holonomy M_γ along each $\gamma \in \pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$,
3. $\Psi(\lambda)$ has regular singularities at each of the points $\lambda_1, ..., \lambda_M$.

By regular singularities, we mean that $\Psi(\lambda)$ increases no faster than a power of $\lambda - \lambda_m$ for λ in a neighbourhood of λ_m.

Let $\ell_1, ..., \ell_M$ be a basis of generators for $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$ with basepoint λ_*, chosen such that λ_m is the only singular point interior to ℓ_m (with the convention that $\lambda = \infty$ is exterior to any closed contour on $\mathbb{C}P^1$). Let $\mathcal{M}_1, ..., \mathcal{M}_M$ be the corresponding monodromy matrices in the representation of the fundamental group.

We assume that $\ell_M \ell_{M-1} ... \ell_1$ is the identity in $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$, so that

$$\mathcal{M}_M \mathcal{M}_{M-1} ... \mathcal{M}_1 = I. \tag{2.1.1}$$

We also require that the singularities of Ψ are of the form

$$\Psi(\lambda) = \{G_m + O(\lambda - \lambda_m)\}(\lambda - \lambda)^{T_m} C_m, \tag{2.1.2}$$

for $G_m, C_m \in GL(N, \mathbb{C})$ and $T_m = \text{diag}(t_m^{(1)}, ..., t_m^{(N)})$; as a result, the monodromy matrices $\mathcal{M}_m$ are diagonalizable (not necessarily simultaneously), with

$$\mathcal{M}_m = C_m^{-1} e^{2\pi i T_m} C_m. \tag{2.1.3}$$

We call the set $\{\lambda_m, \mathcal{M}_m, T_m\}_{m=1}^M$ the monodromy data for our Riemann-Hilbert problem, as it encodes all the monodromy properties of the solution $\Psi(\lambda)$, namely the monodromies and local expansions of $\Psi(\lambda)$ at each λ_m. As we shall see in chapter 4 the solution $\Psi(\lambda)$ of this Riemann-Hilbert problem satisfies the system of linear differential equations

$$\frac{d\Psi}{d\lambda} = \sum_{m=1}^{M} \frac{A_m}{\lambda - \lambda_m} \Psi(\lambda), \tag{2.1.4}$$

where $A_m = G_m T_m G_m^{-1}$; in other words the system is Fuchsian.

2.2 The Schlesinger system and isomonodromic tau-function

Closely related to our Riemann-Hilbert problem is the question of how the monodromy data $\{\lambda_m, \mathcal{M}_m, T_m\}_{m=1}^M$ depends on the positions of the singularities $\lambda_1, ..., \lambda_M$. We therefore take $a = (\lambda_1, ..., \lambda_M) \in \mathbb{C}^N$ as our deformation parameter; under isomonodromic deformation, the system 2.1.4 produces the family of systems of PDE's

$$\frac{\partial \Psi(\lambda, a)}{\partial \lambda} = \sum_{m=1}^{M} \frac{A_m(a)}{\lambda - \lambda_m} \Psi(\lambda, a) \tag{2.2.1}$$

depending analytically on $(\lambda_1, ..., \lambda_M) \in \mathbb{C}^N$. We assume that $\lambda = \infty$ is not a singular point, so that $\sum_{m=1}^M A_m = 0$.

The requirement that the monodromies $\mathcal{M}_1, ..., \mathcal{M}_M$ be independent of the choice of $(\lambda_1, ..., \lambda_M) \in \mathbb{C}^N$ leads to a second system of partial differential equations, describing the dependence of $\Psi(\lambda, a)$ on each of the singularities λ_m:

$$\frac{\partial \Psi(\lambda, a)}{\partial \lambda_m} = \left(\frac{A_m(a)}{\lambda_0 - \lambda_m} - \frac{A_m(a)}{\lambda - \lambda_m} \right) \Psi(\lambda, a). \tag{2.2.2}$$

Writing down the compatibility condition $\partial_{\lambda_m} \partial_a \Psi = \partial_a \partial_{\lambda_m} \Psi$ for the two systems 2.2.1 and 2.2.2 and taking residues at $\lambda = \lambda_m$ and $\lambda = \lambda_n$, we immediately get a nonlinear system of partial differential equations satisfied by the coefficients $A_m(a)$:

$$\begin{aligned}\frac{\partial A_n(a)}{\partial \lambda_m} &= \frac{[A_n(a), A_m(a)]}{\lambda_n - \lambda_m} - \frac{[A_n(a), A_m(a)]}{\lambda_0 - \lambda_m}, \quad m \neq n, \\ \frac{\partial A_m(a)}{\partial \lambda_m} &= -\sum_{n\neq m}\left(\frac{[A_n(a), A_m(a)]}{\lambda_n - \lambda_m} - \frac{[A_n(a), A_m(a)]}{\lambda_0 - \lambda_m}\right).\end{aligned} \tag{2.2.3}$$

These are the Schlesinger equations. The solution $\Psi(\lambda)$ of the Riemann-Hilbert problem with monodromy data $\{\lambda_m, \mathcal{M}_m, T_m\}_{m=1}^{M}$ satisfies equation 2.2.2 if and only if the corresponding coefficients A_m satisfy the Schlesinger equations. Indeed, we can identify the space of coefficients

$$\big\{A_1, ..., A_M \in \mathrm{End}(\mathbb{C}^N) : A_1 + ... + A_M = 0\big\} \tag{2.2.4}$$

with the space of monodromy representations

$$\{\mathcal{M}_1, ..., \mathcal{M}_M \in GL_N(\mathbb{C}) : \mathcal{M}_1 \cdot ... \cdot \mathcal{M}_M = I\} \tag{2.2.5}$$

of $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$ using the following holomorphic map, which is in fact a locally analytic isomorphism (for more details see [2]). The coefficients $A_1, ..., A_M$ define an Ehresmann connection on the trivial holomorphic vector bundle of rank n on $\mathbb{C}P^1$, given by

$$\nabla : d - \left(\frac{A_1}{\lambda - \lambda_1} + ... + \frac{A_M}{\lambda - \lambda_M}\right) d\lambda. \tag{2.2.6}$$

This connection is holomorphic on $\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}$. Moreover the Schlesinger equations are equivalent to the vanishing of the curvature of the connection, therefore taking its monodromy around each of the singularities $\lambda_1, ..., \lambda_M$ defines a monodromy representation of $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$.

Next, for a given solution of the Schlesinger system we can define a meromorphic one-form, the Jimbo-Miwa-Ueno form:

$$\omega_{JMU} = \frac{1}{2}\sum_{n\neq m} \mathrm{tr}(A_m A_n)\delta \ln(\lambda_m - \lambda_n), \quad \delta = \sum_{m=1}^{M} \partial_{\lambda_m} d\lambda_m, \tag{2.2.7}$$

The Schlesinger equations imply that $d\omega_{JMU} = 0$. Moreover ω_{JMU} has only simple poles- see theorem 2 of [14]. Hence we can find a holomorphic function such that

$$\omega_{JMU} = \delta \ln \tau_{JM}. \tag{2.2.8}$$

This is the original tau-function discovered by Jimbo and Miwa and generalized to non-Fuchsian isomonodromic systems by Jimbo, Miwa and Ueno in [9].

Let $\Psi(\lambda)$ solve the Riemann-Hilbert problem outlined above. Since $\Psi(\lambda)$ satisfies the Fuchsian system 2.1.4 we can express the coefficients A_m as

$$A_m = \operatorname*{res}_{\lambda=\lambda_m} \Psi_\lambda \Psi^{-1} d\lambda. \tag{2.2.9}$$

We can therefore rewrite equation 2.2.7 to obtain the following differential equation for the Jimbo-Miwa tau-function:

$$\frac{\partial}{\partial \lambda_m} \ln \tau_{JM} = \frac{1}{2} \operatorname*{res}_{\lambda=\lambda_m} \operatorname{tr} \frac{(d\Psi\Psi^{-1})^2}{d\lambda}, \qquad \frac{\partial \tau_{JM}}{\partial \bar{\lambda}_m} = 0. \tag{2.2.10}$$

Let (Θ) denote the zero divisor of τ in the universal covering of the space $\{\{\lambda_m\}_{m=1}^M \in \mathbb{C}^M : m \neq n \Rightarrow \lambda_m \neq \lambda_n\}$. Then as we shall see in chapter 4, if $\lambda_m \in (\Theta)$ for any $1 \leq m \leq M$ the solution of the Schlesinger system is singular. In this case the Riemann-Hilbert problem corresponding to the monodromy data $\{\lambda_m, \mathcal{M}_m, T_m\}_{m=1}^M$ does not have a solution.

2.3 Branched coverings and quasi-permutation monodromies

Our setting for all that follows will be a branched covering from a compact connected Riemann surface $\mathcal{L}$ to the Riemann sphere $\mathbb{C}P^1$- see, e.g., chapter 1 of [6]. A branched covering is a continuous surjection $\Pi : \mathcal{L} \to \mathbb{C}P^1$ such that any $\lambda \in \mathbb{C}P^1$ has an open neighbourhood U whose preimage $\Pi^{-1}(U)$ is a disjoint union of N open sets each homeomorphic to U, with the exception of finitely many ramification points $\lambda_1, ..., \lambda_M$.

Any such branched cover corresponds to a meromorphic function of degree N on $\mathcal{L}$, with critical points at the branch points of the covering. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ both have genus g, then two covers $p_1 : \mathcal{L}_1 \to \mathbb{C}P^1$ and $p_2 : \mathcal{L}_2 \to \mathbb{C}P^1$ given by meromorphic functions p_1 and p_2 of degree N are equivalent if there exists a biholomorphic function $f : \mathcal{L}_1 \to \mathcal{L}_2$ such that $p_2 \circ f = p_1$. The space of equivalence classes of degree N meromorphic functions on Riemann surfaces of genus g is the Hurwitz space $H_{g,N}$.

The result of the cover is that $\mathcal{L}$ consists of N copies of $\mathbb{C}P^1$, called sheets, glued together at a finite set of branch points, each of which lies in the preimage of one of the points $\lambda_1, ..., \lambda_M$. Note that $\mathcal{L}$ need not be connected. For any curve γ in $\mathbb{C}P^1$ starting at a point λ and $P \in \mathcal{L}$ such that $\Pi(P) = \lambda$, there is a unique curve Γ- the lift of γ- starting at P and projecting to γ.

The lift of a closed curve $\gamma \in \pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$ need not be a closed curve in $\pi_1(\mathcal{L})$. However the startpoint and endpoint of the lifted curve must project to the same point $\lambda \in \mathbb{C}P^1$, with the endpoint uniquely determined by the homology class of γ. In this way a closed curve γ with basepoint λ acts on the set $\Pi^{-1}(\lambda)$: for each $P \in \Pi^{-1}(\lambda)$ we take the corresponding lifted curve Γ with startpoint P, and we map P to the endpoint of Γ.

We thus obtain a $GL(N, \mathbb{C})$ representation of the group $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$. Specifically, for any point $\lambda \in \mathbb{C}P^1$ let $\lambda^{(j)} \in \Pi^{-1}(\lambda)$ be its preimage lying on the jth sheet. Then for each of the generators ℓ_m of $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$, the lifted curve $\Pi^{-1}(\ell_m)$ in $\mathcal{L}$ is a union of non-intersecting curves $\ell_m^{(1)}, ..., \ell_m^{(N)}$. Each of the components $\ell_m^{(j)}$ has start and endpoints in the preimage of λ_*. We denote by $\ell_m^{(j)}$ the component starting at $\lambda_*^{(j)}$, and denote by $\lambda_*^{(j_m[j])}$ its endpoint. Then $j = j_m[j]$ if and only if $\lambda_m^{(j)}$ is not a branch point, so that the matrix corresponding to ℓ_m is

$$(\mathcal{M}_m)_{jl} = \delta_{j_m[j]l}. \tag{2.3.1}$$

The matrices $\mathcal{M}_1, ..., \mathcal{M}_M$ have only one non-zero entry per row or column, and all nonzero entries equal to 1; such matrices are called $N \times N$ permutation matrices and are in one-to-one correspondence with elements of the symmetric group S_N.

Definition 2.3.1. *We call the representation $\mathcal{M}$ a permutation representation if M_γ is a permutation matrix for all $\gamma \in \pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$.*

In this way we can associate a unique $N \times N$ permutation representation to any N-fold branched covering $\Pi : \mathcal{L} \to \mathbb{C}P^1$. The converse is also true. Given an $N \times N$ permutation representation $\mathcal{M}$ of $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$, we can glue N copies of $\mathbb{C}P^1$ together to obtain a unique compact Riemann surface $\mathcal{L}$ (not necessarily connected) and covering map $\Pi : \mathcal{L} \to \mathbb{C}P^1$ with monodromies corresponding to $\mathcal{M}$. For a description of this construction see page 257 of [7]. In other words,

Lemma 2.3.1. *There is a one-to-one correspondence between $N \times N$ permutation representations of $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$ and N-fold ramified coverings of the Riemann sphere by a compact Riemann surface with branch points projecting to $\lambda_1, ..., \lambda_M$.*

If $\mathcal{L}$ is connected- we can guarantee this by requiring that $\mathcal{M}$ cannot be decomposed into a direct sum of two or more permutation representations with respect to the same basis in $\mathbb{C}^N$- it is easy to determine the branch points of the corresponding cover and genus of $\mathcal{L}$ from the representation $\mathcal{M}$. Each $\mathcal{M}_m$ naturally acts as a permutation of a basis $e_1, ..., e_N$ for $\mathbb{C}^N$. Let $O(e_{i_1}), ..., O(e_{i_{n(m)}})$ be the orbits under the action of $\mathcal{M}_m$ and define $|O(e_{i_j})| = \mathbf{k}_m^{(j)}$ for $j = 1, ..., n(m)$. Then the branched cover must have $n(m)$ distinct points lying over the point λ_m; the $\mathbf{k}_m^{(j)}$ form a partition of N specifying the number of sheets meeting at each of the $n(m)$ points.

For each $\mathbf{k}_m^{(j)} \geq 2$ the corresponding point over λ_m is a branch point of order $\mathbf{k}_m^{(j)}$. The Riemann-Hurwitz formula for the branched cover therefore implies that the compact connected Riemann surface $\mathcal{L}$ must have genus

$$g = \sum_{m=1}^{M} \sum_{j=1}^{n(m)} \frac{k_m^{(j)} - 1}{2} - N + 1. \qquad (2.3.2)$$

Note that we can sum over all points lying over the ramification points λ_m instead of just branch points, since the terms $\mathbf{k}_m^{(j)} = 1$ do not contribute. In the following chapters we will continue to treat ordinary points lying over λ_m like branch points of order $\mathbf{k}_m^{(j)} = 1$. With this convention we can allow the possibility that $\mathcal{M}_m = I$ for some $1 \leq m \leq M$, in which case all points lying above λ_m have order $\mathbf{k}_m^{(j)} = 1$.

Next, we call a matrix a quasi-permutation matrix if it has only one non-zero entry per row or column, which need not equal 1. Such matrices form a subgroup of $GL(N, \mathbb{C})$, therefore it is possible to have a representation $\mathcal{M}$ in which $\mathcal{M}_\gamma$ is a quasi-permutation matrix for any $\gamma \in \pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$. In this case we call the representation $\mathcal{M}$ a quasi-permutation representation.

To any quasi-permutation representation $\mathcal{M}$ we can associate a unique permutation representation $\mathcal{M}^0$, by replacing all non-zero entries of matrices $\mathcal{M}_\gamma$ by 1. We can then construct the branched covering corresponding to the representation $\mathcal{M}^0$. Hence there is a natural correspondence between $N \times N$ quasi-permutation representations of $\pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$, and N-fold branched coverings of $\mathbb{C}P^1$.

Unlike the $N \times N$ permutation representation, we clearly have more than one choice of $N \times N$ quasi-permutation representation associated to the same branched covering. From lemma 2.3.1 we can see that two quasi-permutation representations correspond to the same branched covering iff we obtain the same permutation representation by replacing all nonzero matrix entries with 1.

It is easy to see that under a change of basis $\Psi \to \Psi D$ for some $D \in GL_N(\mathbb{C})$ the monodromy matrices transform as $\mathcal{M}_\gamma \to D\mathcal{M}_\gamma D^{-1}$ for all $\gamma \in [\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$. Under such a transformation the tau-function is unchanged, since the term $d\Psi\Psi^{-1}$ on the right-hand side of equation 2.2.10 is invariant. Since multiplication by a diagonal matrix preserves locations of nonzero entries, we conclude that two quasi-permutation monodromy representations are equivalent if they are related by conjugation by a diagonal matrix $D \in GL_N(\mathbb{C})$.

Moreover two matrices D and D' related by scalar multiplication effect the same transformation since $\det(D)$ cancels out from the determinant of $\mathcal{M}_\gamma$. Therefore two quasi-permutation representations are equivalent if there exists a diagonal matrix D with $\det(D) = 1$ such that

$$\mathcal{M}'_\gamma = D\mathcal{M}_\gamma D^{-1} \quad \forall\ \gamma \in \pi_1[\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]. \tag{2.3.3}$$

We make two further assumptions about the quasi-permutation representation in our Riemann-Hilbert problem.

Assumption 1. *The representation $\mathcal{M}$ cannot be decomposed into a direct sum of two other quasi-permutation representations with respect to the same basis in $\mathbb{C}^N$.*

In this case the permutation representation $\mathcal{M}^0$ also does not have such a decomposition, and consequently the associated branched covering $\mathcal{L}$ is connected. This condition is weaker than the assumption that $\mathcal{M}$ is irreducible, since $\mathcal{M}$ may possess a non-trivial subrepresentation so long as it is not a quasi-permutation representation. Recall the representation is reducible if there is a non-trivial subspace of $\mathbb{C}^N$ invariant under the action of the representation matrices; we then obtain a non-trivial subrepresentation by restricting the action of the representation matrices to this subspace.

Assumption 2. *The monodromy matrices cannot be simultaneously diagonalized.*

If this were not the case, the matrix Riemann-Hilbert problem could be decomposed into a direct sum of N independent scalar Riemann-Hilbert problems.

We are now in a position to determine the dimension of the space of $N \times N$ quasi-permutation matrices corresponding to a given branched covering $\Pi : \mathcal{L} \to \mathbb{C}P^1$ by a compact, connected Riemann surface $\mathcal{L}$.

Definition 2.3.2. *Let $\mathcal{M}(\mathcal{L})$ denote the space of all irreducible quasi-permutation monodromy representations corresponding to the branched cover $\mathcal{L}$, and let $\mathcal{Q}(\mathcal{L})$ denote the space of orbits of $\mathcal{M}(\mathcal{L})$ under the diagonal conjugation action of $GL_N(\mathbb{C})$.*

Lemma 2.3.2. *The orbit space $\mathcal{Q}(\mathcal{L})$ is a manifold of dimension $MN - 2N + 1$.*

Proof. The space $\mathcal{M}(\mathcal{L})$ is $(M-1)N$-dimensional: each $\mathcal{M}_m$ has N nonzero entries, and specifying $M-1$ of the monodromy matrices uniquely determines the Mth according to equation 2.1.1. Meanwhile the space of matrices D is parameterised by $N-1$ diagonal entries of D, the Nth being uniquely determined by the normalization $\det(D) = 1$.

Going back to equation 2.1.4 note that under the transformation $\Psi \to \Psi D$ the matrices $A_1, ..., A_M$ transform as $A_m \to D^{-1} A_m D$. Exploiting the fact that the spaces of monodromy matrices $\mathcal{M}_1, ..., \mathcal{M}_M$ and connection coefficients $A_1, ..., A_M$ are locally isomorphic, we can determine the dimension of the orbits of $\mathcal{M}(\mathcal{L})$ by the dimension of the corresponding orbits of connection coefficients.

The Lie group $GL_N(\mathbb{C})$ acting on its Lie algebra $\mathrm{End}(\mathbb{C}^N)$ gives a coadjoint representation of $GL_N(\mathbb{C})$ (in the case of a matrix Lie group the Lie algebra is equal to its own dual), and hence the orbits of $\mathrm{End}(\mathbb{C}^N)$ under this action are symplectic manifolds. Moreover the tangent space to the orbit $O(A_m)$ at $A_m \in \mathrm{End}(\mathbb{C}^N)$ is isomorphic to $\mathrm{End}(\mathbb{C}^N)/\mathrm{stab}(A_m)$, where $\mathrm{stab}(A_m)$ is the Lie algebra of the subgroup $\mathrm{Stab}(A_m) \leq GL_N(\mathbb{C})$ of matrices commuting with A_m.

Now let the M-tuple $(A_1, ..., A_M)$ correspond to some $(\mathcal{M}_1, ..., \mathcal{M}_M) \in \mathcal{M}(\mathcal{L})$, and let $O(A_1, ..., A_M)$ be the orbit under the action 2.3.3 which contains $(A_1, ..., A_M)$. The tangent space to the orbit at $(A_1, ..., A_M)$ is isomorphic to the Lie algebra $\mathfrak{g}$ of the subgroup $G = \{D \in GL_N(\mathbb{C}) : D \text{ diagonal}, \det(D) = 1\}$, modulo the matrices in $\mathfrak{g}$ which simultaneously commute with each of $A_1, ..., A_M$. Note that such a matrix cannot exist, since otherwise $A_1, ..., A_M$ and consequently $\mathcal{M}_1, ..., \mathcal{M}_M$ would be simultaneously diagonalizable.

Therefore the orbits have dimension $\dim(\mathfrak{g}) = \dim(G) = N-1$, and the dimension of $\mathcal{Q}(\mathcal{L})$ is given by subtracting this dimension from the dimension of $\mathcal{M}(\mathcal{L})$. □

Chapter 3

Variational formulas on compact Riemann surfaces

3.1 Compact Riemann surfaces

We start by reviewing some basic facts of compact Riemann surfaces, following [1]. Let $\mathcal{L}$ be a compact Riemann surface of genus $g \geq 1$. The first homology of $\mathcal{L}$ is $H_1(\mathcal{L}, \mathbb{Z}) = \mathbb{Z}^{2g}$ and we can find a canonical basis of cycles $a_1, ..., a_g, b_1, ..., b_g$ with intersection indices $a_\alpha \# a_\beta = 0, b_\alpha \# b_\beta = 0$ and $a_\alpha \# b_\beta = 1$, whose homology classes generate $H_1(\mathcal{L}, \mathbb{Z}) = \mathbb{Z}^{2g}$. We can cut $\mathcal{L}$ open to obtain a simply connected domain $\hat{\mathcal{L}}$ by picking a point $P_0 \in \mathcal{L}$ and representatives in $\pi_1(\mathcal{L}, P_0)$ of (the homotopy classes of) $a_1, ..., a_g, b_1, ..., b_g$, and removing this set of cycles from $\mathcal{L}$.

Next, the first cohomology has dimension $2g$; we can find a basis of holomorphic differentials $\omega_1, ..., \omega_g$ normalized such that

$$\int_{a_\alpha} \omega_\beta = \delta_{\alpha\beta}, \quad \int_{b_\alpha} \omega_\beta = \mathbf{B}_{\alpha\beta}. \tag{3.1.1}$$

The resulting matrix $\mathbf{B}$, which is symmetric and has positive definite imaginary part, is called the normalized period matrix. Now to any $\gamma \in \pi_1(\mathcal{L})$ we can associate a vector

$$\int_\gamma \omega_1, ..., \int_\gamma \omega_g \in \mathbb{C}^g. \tag{3.1.2}$$

It follows from the Riemann bilinear relations that the set of such vectors form a nondegenerate lattice Λ. Namely, for any $\omega, \eta \in \mathbb{C}(\omega_1, ..., \omega_g)$ the Riemann bilinear relations state [1]

$$\int_{\mathcal{L}} \omega \wedge \eta = \sum_{\alpha=1}^{g} \int_{a_\alpha} \omega \int_{b_\alpha} \eta - \int_{a_\alpha} \eta \int_{b_\alpha} \omega. \tag{3.1.3}$$

We therefore define the quotient space $J(\mathcal{L}) = \mathbb{C}^g/\Lambda$; this space is an abelian variety called the Jacobian variety of $\mathcal{L}$. Next, for some choice of basepoint $P_0 \in \mathcal{L}$ we define the Abel map $U : \mathcal{L} \to J(\mathcal{L})$,

$$U_\alpha(P) = \int_{P_0}^{P} \omega_\alpha \mod \Lambda, \quad 1 \leq \alpha \leq g. \tag{3.1.4}$$

The Abel map is independent of the path of integration, since the integrals over any two choices of path differ by an element of Λ. Moreover the differential $dU = (\omega_1, ..., \omega_g)$ of the Abel map is nowhere vanishing: there cannot exist a point in $\mathcal{L}$ where all holomorphic differentials simultaneously vanish. Otherwise by the Riemann-Roch theorem there would exist a meromorphic function with only a single simple pole at this point, contradicting that $g > 0$ [1].

More generally, for any divisor $D = \sum_i k_i P_i$ where $k_i \in \mathbb{Z}, P_i \in \mathcal{L}$ we define $U(D) = \sum_i k_i U(P_i)$. The Abel theorem states that $U(D) = 0$ if and only if $D = 0$, or equivalently D is a principal divisor (the divisor of a meromorphic function on $\mathcal{L}$) [1]. Consequently if $U(P) = U(Q)$ for points $P \neq Q$ in $\mathcal{L}$, their difference $P - Q$ is a principal divisor; as before, the existence of a meromorphic function with this divisor contradicts $g > 0$.

Hence the Abel map embeds $\mathcal{L}$ into its Jacobian variety $J(\mathcal{L})$. The Jacobian variety will provide the setting to define a theta function associated to $\mathcal{L}$ in the next section, and from there to define the prime-form. We also define the vector of Riemann constants

$$\mathcal{K}_\alpha = \frac{\mathbf{B}_{\alpha\alpha}}{2} - \sum_{\beta=1}^{g} \left[\int_{P=P_0}^{P=P_0+a_\beta} U_\alpha(P)\omega_\beta(P) \right], \quad 1 \leq \alpha \leq g. \tag{3.1.5}$$

The vector $-2\mathcal{K}$ is equal to the Abel map of the canonical divisor class on $\mathcal{L}$: a divisor D_{2g-2} of degree $2g - 2$ is the divisor of a differential iff $U(D_{2g-2}) = -2\mathcal{K}$ [1]. Note that $\mathcal{K}$ depends on both the basepoint P_0, and the choice of basis for $\pi_1(\mathcal{L})$. However differentiating equation 3.1.5 with respect to P_0 gives

$$d_{P_0}\mathcal{K} = (g-1) \sum_{\alpha=1}^{g} \omega_\alpha(P_0). \tag{3.1.6}$$

Using the above equation it is easy to see that the vector $\mathbf{e} = U(D_{g-1}) + \mathcal{K}$ is independent of P_0.

3.2 The theta function and the prime-form

To any compact connected Riemann surface $\mathcal{L}$ we can associate a theta function, a quasi-periodic holomorphic function $\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}] : \mathbb{C}^g \to \mathbb{C}$ with periodicity properties defined by the normalized period matrix $\mathbf{B}_{\alpha\beta}$. The result is that the theta function composed with the Abel map gives a multi-valued function on $\mathcal{L}$, periodic around the a- and b-cycles up to a non-vanishing multiplicative constant.

Given a symmetric matrix $\mathbf{B} \in GL(g, \mathbb{C})$ whose imaginary part is a symmetric, positive definite real matrix, and a choice of vectors called characteristics $\mathbf{p}, \mathbf{q} \in \mathbb{C}^g$, the corresponding theta function is defined by its periodicity relations

$$\begin{aligned}\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z} + \mathbf{e}_\alpha|\mathbf{B}) &= \exp\left(\pi i \mathbf{p}_\alpha\right)\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z}|\mathbf{B}),\\ \Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z} + \mathbf{B}\cdot\mathbf{e}_\alpha|\mathbf{B}) &= \exp\left(-\pi i \mathbf{q}_\alpha - \pi i \mathbf{B}_{\alpha\alpha} - 2\pi i \mathbf{z}_\alpha\right)\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z}|\mathbf{B}),\end{aligned} \tag{3.2.1}$$

for $1 \leq \alpha \leq g$ [1]. The theta function also satisfies the heat equation

$$\begin{aligned}\frac{\partial^2\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z})}{\partial\mathbf{z}_\alpha\partial\mathbf{z}_\beta} &= 2\pi i\frac{\partial\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z})}{\partial\mathbf{B}_{\alpha\beta}}, \alpha \neq \beta,\\ \frac{\partial^2\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z})}{\partial\mathbf{z}_\alpha^2} &= 4\pi i\frac{\partial\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z})}{\partial\mathbf{B}_{\alpha\alpha}}.\end{aligned} \tag{3.2.2}$$

We can define a smooth function $\mathcal{L} \to \mathbb{C}$ by taking $\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](U(P)|\mathbf{B})$; then by the periodicity relations for $\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z}|\mathbf{B})$, under analytic continuation

$$\begin{aligned}\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](U(P + a_\alpha)|\mathbf{B}) &= \exp(\pi i \mathbf{p}_\alpha)\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z}|\mathbf{B}),\\ \Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](U(P + b_\alpha)|\mathbf{B}) &= \exp(-\pi i \mathbf{q}_\alpha - \pi i \mathbf{B}_{\alpha\alpha} - 2\pi i U_\alpha(P))\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z}|\mathbf{B}).\end{aligned} \tag{3.2.3}$$

We can differentiate the function $\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z}|\mathbf{B})$ by noting that $\frac{d}{dP}U_\alpha(P) = \omega_\alpha(P)$, and hence

$$\frac{\partial}{\partial P}\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](U(P)|\mathbf{B}) = \sum_{\alpha=1}^{g}\left[\partial_{\mathbf{z}_\alpha}\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}](\mathbf{z}|\mathbf{B})\right]_{\mathbf{z}=U(P)}\frac{\omega_\alpha(P)}{dx(P)}, \tag{3.2.4}$$

where $x(P)$ is a local coordinate in a neighborhood of P. From now on we suppress the dependence of $\Theta[\begin{smallmatrix}\mathbf{p}\\\mathbf{q}\end{smallmatrix}]$ on $\mathbf{B}$.

Next, we make some remarks about the role of the characteristics $\mathbf{p}, \mathbf{q}$. To each choice of characteristics we associate a point in $J(\mathcal{L})$, given by

$$\mathbf{e} := \frac{1}{2}\mathbf{p} + \frac{1}{2}\mathbf{B} \cdot \mathbf{q}. \tag{3.2.5}$$

We can now state the following version of the Riemann vanishing theorem, which explains the conditions for which the theta function is non-singular. For more details, see chapter 5 of [1].

Theorem 3.2.1. *We have $\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](U(P)) \equiv 0$ on $\mathcal{L}$ iff $\mathbf{e} = U(D_{g-1}) + \mathcal{K}$ where D_{g-1} is a positive divisor of degree $g-1$ such that $i(P_0 + D_{g-1}) > 0$, for P_0 the basepoint of the Abel map.*

Otherwise, if $\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](U(P)) \not\equiv 0$, then $\mathbf{e} = U(D_{g-1}) + \mathcal{K}$ for some divisor $D_{g-1} = P_1 + \ldots + P_{g-1}$ such that $i(D_{g-1}) = 1$. In this case, the zero divisor of $\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](U(P) - U(Q))$ is equal to $D_{g-1} + Q$.

From now on we assume that $\mathbf{p}, \mathbf{q}$ are chosen such that $\Theta[{}^{\mathbf{p}}_{\mathbf{q}}] \not\equiv 0$. For $\mathbf{p}, \mathbf{q} \in \mathbb{Z}^g$, we have the additional relation [1]

$$\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\text{-}\mathbf{z}) = \exp(\pi i \mathbf{p}^t \cdot \mathbf{q}) \Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z}). \tag{3.2.6}$$

Hence when $\mathbf{p}^t \cdot \mathbf{q}$ is odd (respectively even), the function $\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](z)$ is odd (even) as a function of $\mathbf{z}$ and we call the characteristics $\mathbf{p}, \mathbf{q}$ odd (even) half-integer characteristics.

When $\mathbf{p}^*, \mathbf{q}^*$ are odd half-integer characteristics we denote $\Delta = \frac{1}{2}\mathbf{p}^* + \frac{1}{2}\mathbf{B} \cdot \mathbf{q}^*$. The condition that $(\mathbf{p}^*)^t \cdot \mathbf{q}^*$ is odd guarantees that $\Delta \not\equiv 0$ as a point in the Jacobian, while clearly $2\Delta \equiv 0$. Hence

$$U(2D^{\Delta}_{g-1}) = 2\Delta - 2\mathcal{K} = -2\mathcal{K}, \tag{3.2.7}$$

so that $2D^{\Delta}_{g-1}$ is the divisor of a holomorphic differential [1]. We will show that this differential is

$$\omega_{\Delta}(P) = \Big[d_P \Theta[{}^{\mathbf{p}^*}_{\mathbf{q}^*}](U(P) - U(Q))\Big]_{P=Q} = \sum_{\alpha=1}^{g} \partial_{\mathbf{z}_\alpha} \Theta[{}^{\mathbf{p}^*}_{\mathbf{q}^*}](\mathbf{z})\Big|_{\mathbf{z}=0} \omega_\alpha(P). \tag{3.2.8}$$

Clearly $(\omega_\Delta) \geq D^{\Delta}_{g-1}$, and by Theorem 3.2.1 $i(D^{\Delta}_{g-1}) = 1$. Hence ω_Δ is the unique differential satisfying $(\omega_\Delta) \geq D^{\Delta}_{g-1}$, and the divisor D satisfying $U(D^{\Delta}_{g-1} + D) = -2\mathcal{K}$ is also unique [1]. Since $D = D^{\Delta}_{g-1}$ satisfies the relation, $(\omega_\Delta) = 2D^{\Delta}_{g-1}$.

We therefore define a half-differential whose divisor is D^{Δ}_{g-1} by taking

$$h(P) = \sqrt{\sum_{\alpha=1}^{g} \partial_{\mathbf{z}_\alpha} \Theta[\begin{smallmatrix}\mathbf{p}^*\\ \mathbf{q}^*\end{smallmatrix}](\mathbf{z})\Big|_{\mathbf{z}=0} \omega_\alpha(P)}. \tag{3.2.9}$$

Formally, a half-differential or spinor $f(x)\sqrt{dx}$ is a system of locally holomorphic functions $f_i(x_i)$ on each coordinate neighborhood $x_i : U_i \to \mathbb{C}P^1$ such that $f_i(x)\sqrt{dx_i} = f_j(x)\sqrt{dx_j}$ on all intersections $U_i \cap U_j$, and such that

$$\sqrt{\frac{dx_i}{dx_j}}\sqrt{\frac{dx_j}{dx_k}}\sqrt{\frac{dx_k}{dx_i}} = 1 \tag{3.2.10}$$

on all triple intersections $U_i \cap U_j \cap U_k$ [1]. With this definition we have the natural property that whenever h is a spinor, h^2 is a holomorphic differential with only double zeros, and conversely whenever a holomorphic differential ω has only double zeros, $\sqrt{\omega}$ is a spinor. There are 4^g ways to choose the square roots, corresponding to two possible assignments of half-periods for each of the a- and b-cycles [1].

Lemma 3.2.2. *The spinor $h(P)$ satisfies $h(P+a_\alpha) = \exp(\pi i \boldsymbol{p}^*_\alpha)h(P)$ and $h(P+b_\alpha) = \exp(-\pi i \boldsymbol{q}^*_\alpha)h(P)$.*

Proof. Using equation 3.2.3,

$$\omega_\Delta(P + a_\alpha) = \exp\left(\pi i \mathbf{p}^*_\alpha\right) \left[d_P \Theta[\begin{smallmatrix}\mathbf{p}^*\\ \mathbf{q}^*\end{smallmatrix}](U(P) - U(Q))\right]_{P-a_\alpha=Q} = \exp\left(2\pi i \mathbf{p}^*_\alpha\right) \omega_\Delta(P), \tag{3.2.11}$$

while

$$\begin{aligned}
&\omega_\Delta(P + b_\alpha)\\
&= \left[d_P \Theta[\begin{smallmatrix}\mathbf{p}^*\\ \mathbf{q}^*\end{smallmatrix}](U(P+b_\alpha) - U(Q))\right]_{P-b_\alpha=Q},\\
&= \left[\exp\left(-\pi i \mathbf{q}^*_\alpha - \pi i \mathbf{B}_{\alpha\alpha} - 2\pi i [U_\alpha(P) - U_\alpha(Q)]\right) d_P \Theta[\begin{smallmatrix}\mathbf{p}^*\\ \mathbf{q}^*\end{smallmatrix}](U(P) - U(Q))\right]_{P-b_\alpha=Q},
\end{aligned} \tag{3.2.12}$$

since $[\Theta[{}^{\mathbf{p}^*}_{\mathbf{q}^*}](U(P)-U(Q))]_{P+b_\alpha=Q}=0$. Finally,

$$\begin{aligned}
&\omega_\Delta(P+b_\alpha)\\
&=\left[\exp\left(-\pi i\mathbf{q}^*_\alpha-\pi i\mathbf{B}_{\alpha\alpha}-2\pi i[U_\alpha(P)-U_\alpha(Q)]\right)d_P\Theta[{}^{\mathbf{p}^*}_{\mathbf{q}^*}](U(P)-U(Q+b_\alpha))\right]_{P=Q},\\
&=\exp\left(-2\pi i\mathbf{q}^*_\alpha-2\pi i\mathbf{B}_{\alpha\alpha}\right)\\
&\quad\times\left[\exp\left(-4\pi i[U_\alpha(P)-U_\alpha(Q)]+2\pi i\mathbf{B}_{\alpha\alpha}\right)d_P\Theta[{}^{\mathbf{p}^*}_{\mathbf{q}^*}](U(P)-U(Q)\right]_{P=Q},\\
&=\exp\left(-2\pi i\mathbf{q}^*_\alpha\right)\left[d_P\Theta[{}^{\mathbf{p}^*}_{\mathbf{q}^*}](U(P)-U(Q)\right]_{P=Q}.
\end{aligned}\tag{3.2.13}$$

□

We can now define the prime-form, the $(-1/2,0)\times(-1/2,0)$-form

$$E(P,Q)=\frac{\Theta[{}^{\mathbf{p}^*}_{\mathbf{q}^*}](U(P)-U(Q))}{h(P)h(Q)}.\tag{3.2.14}$$

By oddness of $\Theta[{}^{\mathbf{p}}_{\mathbf{q}}]$ we have $E(P,Q)=-E(Q,P)$. With respect to either P or Q, the numerator has divisor $D^\Delta_{g-1}+Q$ by Theorem 3.2.1 and the denominator has divisor $(h)=D^\Delta_{g-1}$. Therefore the prime-form is holomorphic, with a simple zero at each $P=Q$ and no other zeros. Using 3.2.4 and 3.2.9 we have

$$\begin{aligned}
&\partial_P\left(E(P,Q)\right)\\
&=\frac{\sum_{\alpha=1}^g\partial_{\mathbf{z}_\alpha}\Theta[{}^{\mathbf{p}^*}_{\mathbf{q}^*}](\mathbf{z})\Big|_{\mathbf{z}=U(P)-U(Q)}\omega_\alpha(P)}{h(P)h(Q)dx(P)}-\frac{\Theta[{}^{\mathbf{p}^*}_{\mathbf{q}^*}](U(P)-U(Q))}{h^2(P)h(Q)}\frac{dh}{dP},
\end{aligned}\tag{3.2.15}$$

and therefore $\left[\partial_PE(P,Q)\sqrt{dx(P)}\sqrt{dx(Q)}\right]_{P=Q}=1$. Hence near the diagonal $P=Q$, the prime-form satisfies

$$E(P,Q)=\frac{x(P)-x(Q)}{\sqrt{dx(P)}\sqrt{dx(Q)}}+O\left((x(P)-x(Q))^2\right),\tag{3.2.16}$$

where x is a local coordinate on a neighbourhood containing the points P,Q. Like the theta function, $E(P,Q)$ is not single-valued; using equation 3.2.3 and Lemma 3.2.2,

$$\begin{aligned}
E(P+a_\alpha,Q)&=E(P,Q)=E(P,Q+a_\alpha),\\
E(P+b_\alpha,Q)&=\exp[-\pi i\mathbf{B}_{\alpha\alpha}-2\pi i(U(P)-U(Q))]E(P,Q),\\
E(P,Q+b_\alpha)&=\exp[-\pi i\mathbf{B}_{\alpha\alpha}-2\pi i(U(Q)-U(P))]E(P,Q).
\end{aligned}\tag{3.2.17}$$

Together equation 3.2.17 and Lemma 3.2.16 imply $E(P,Q)$ is independent of the choice of odd characteristics $\mathbf{p}^*, \mathbf{q}^*$: since the a- and b-periods of $E(P,Q)$ do not depend on $\mathbf{p}^*, \mathbf{q}^*$, $E(P,Q)$ must be independent up to multiplication by a holomorphic, hence constant function, and Lemma 3.2.16 implies that the constant must be 1.

3.3 The Bergman and Szegö kernels

We can now introduce the final two ingredients needed for the solution of the Riemann-Hilbert problem. The Bergman kernel is a symmetric meromorphic $(1,0)\times(1,0)$-form, having only a double pole at $P=Q$ with local behaviour

$$B(P,Q) = \left[\frac{1}{(x(P)-x(Q))^2} + H(x(P),x(Q)) + O\left((x(P)-x(Q))\right)\right] dx(P)dx(Q), \tag{3.3.1}$$

where the non-singular part $H(x(P),x(Q))$ depends on the coordinate chart [1].

Lemma 3.3.1. *The normalization condition* $\int_{P\in a_\alpha} B(P,Q)=0$ *implies that*

$$\int_{P\in b_\alpha} B(P,Q) = 2\pi i \omega_\alpha(Q). \tag{3.3.2}$$

Proof. The proof follows from the bilinear relation for a meromorphic differential ω and a differential of the second kind η (a meromorphic differential having zero residue at all its poles). The relation is [1]:

$$\sum_{\beta=1}^{g}\left[\int_{a_\beta}\eta\int_{b_\beta}\omega - \int_{a_\beta}\omega\int_{b_\beta}\eta\right] = 2\pi i \sum_{R\in(\omega\eta)_\infty} \operatorname*{res}_{P=R}\left[\int_{P_0}^{P}\eta(P')\right]\omega(P) \tag{3.3.3}$$

where $(\omega\eta)_\infty$ is the pole divisor of the product $\omega(P)\eta(P)$. Inserting $B(P,Q)$ and $\omega_\alpha(P)$, all but one term on the left hand side vanishes by normalization of B and ω_α, leaving

$$-\int_{P\in b_\alpha} B(P,Q) = 2\pi i \operatorname*{res}_{P=Q}\left[\int_{P'=P_0}^{P'=P} B(P',Q)\right]\omega_\alpha(P). \tag{3.3.4}$$

Using the local expansion for $B(P,Q)$ we have

$$\operatorname*{res}_{P=Q}\left[\int_{P_0}^{P} B(P',Q)\right]\omega_\alpha(P) = \operatorname*{res}_{P=Q}\left[\int_{P'=P_0}^{P'=P}\frac{dx(P')dx(Q)}{(x(P')-x(Q))^2}\right]\omega_\alpha(P) = -\omega_\alpha(Q), \tag{3.3.5}$$

and the result follows. □

The Bergman kernel is related to $E(P,Q)$ by [1]

$$B(P,Q) = d_P d_Q \left(\ln(E(P,Q)\sqrt{dx(P)}\sqrt{dx(Q)}\right). \tag{3.3.6}$$

In chapter 4 we will encounter the Bergman tau-function as one of the factors, alongside the Seiberg-Witten tau-function, making up the Jimbo-Miwa tau-function. The Bergman tau-function on the Hurwitz space $H_{g,N}$ is defined for a meromorphic function $f \in H_{g,N}$. It is defined using the regularization of the Bergman kernel on the diagonal: for an arbitrary Abelian differential v on $\mathcal{L}$, this is the differential

$$B^v_{reg}(P) = \left(B(P,Q) - \frac{v(P)v(Q)}{\left(\int_P^Q v\right)^2} \right)_{P=Q}. \tag{3.3.7}$$

The Bergman tau-function for $f \in H_{g,N}$ is given by the differential equation [10]

$$\frac{\partial \ln \tau_B}{\partial \lambda_m} = - \operatorname*{res}_{\lambda=\lambda_m} \frac{B^{df}_{reg}(\lambda^{(j)})}{d\lambda} = 2 \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{j<k} B(\lambda^{(j)}, \lambda^{(k)}). \tag{3.3.8}$$

Finally let $\mathbf{p}, \mathbf{q}$ be half-integer characteristics such that $\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](0) \neq 0$ (in contrast to the odd characteristics $\mathbf{p}^*, \mathbf{q}^*$ used for the prime-form). We define the Szegö kernel, the $(1/2,0) \times (1/2,0)$-form

$$S(P,Q) = \frac{1}{\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](0)} \frac{\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](U(P) - U(Q))}{E(P,Q)}. \tag{3.3.9}$$

Using equations 3.2.17 and 3.2.3, under analytic continuation the Szegö kernel satisfies

$$\begin{aligned} S(P + a_\alpha, Q) &= \exp(\pi i \mathbf{p}_\alpha) S(P,Q), \\ S(P + b_\alpha, Q) &= \exp(-\pi i \mathbf{q}_\alpha) S(P,Q), \end{aligned} \tag{3.3.10}$$

with the inverse factors for analytic continuation with respect to Q.

Lemma 3.3.2. *Near the diagonal the Szegö kernel satisfies*

$$S(P,Q) = \left[\frac{1}{x(P) - x(Q)} + a_0(P) + O\left(x(P) - x(Q)\right) \right] \sqrt{dx(P)}\sqrt{dx(Q)}. \tag{3.3.11}$$

where

$$a_0(P) = \sum_{\alpha=1}^{g} \partial_{\boldsymbol{z}_\alpha} \Theta[{}^{\boldsymbol{p}}_{\boldsymbol{q}}](\boldsymbol{z}) \Big|_{\boldsymbol{z}=0} \frac{\omega_\alpha(P)}{dx(P)}. \tag{3.3.12}$$

Proof. We have

$$\lim_{P\to Q}\frac{(x(P)-x(Q))\,S(P,Q)}{\sqrt{dx(P}\sqrt{dx(Q)}}=\frac{x(P)-x(Q)}{E(P,Q)\sqrt{dx(P}\sqrt{dx(Q)}}=1, \tag{3.3.13}$$

while

$$\begin{aligned}&\partial_P\left[(x(P)-x(Q))\,S(P,Q)\right]\\ &=S(P,Q)-\frac{\Theta[^{\mathbf{p}}_{\mathbf{q}}](U(P)-U(Q))\partial_P E(P,Q)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](0)E^2(P,Q)}\,(x(P)-x(Q))\\ &\quad+\frac{\sum_{\alpha=1}^{g}\partial_{\mathbf{z}_\alpha}\Theta[^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\Big|_{\mathbf{z}=U(P)-U(Q)}\omega_\alpha(P)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](0)E(P,Q)dx(P)}\,(x(P)-x(Q))\,.\end{aligned} \tag{3.3.14}$$

In the limit $P\to Q$ the first two terms cancel, leaving

$$\begin{aligned}&\lim_{P\to Q}\partial_P\left[\frac{(x(P)-x(Q))\,S(P,Q)}{\sqrt{dx(P)}\sqrt{dx(Q)}}\right]\\ &=\lim_{P\to Q}\left[\frac{S(P,Q)-E^{-1}(P,Q)}{\sqrt{dx(P)}\sqrt{dx(Q)}}\right]+\frac{\sum_{\alpha=1}^{g}\partial_{\mathbf{z}_\alpha}\ln\Theta[^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\Big|_{\mathbf{z}=0}\omega_\alpha(P)}{\sqrt{dx(P)}\sqrt{dx(Q)}}=a_0(P).\end{aligned} \tag{3.3.15}$$

□

We can relate the Szegö and Bergman kernels using the following lemma from Fay- see lemma 2.12 in [3].

Lemma 3.3.3. *For any choice of* $\boldsymbol{e}\in\mathbb{C}^g$*, we have*

$$\begin{aligned}&\frac{\Theta[^{\boldsymbol{p}}_{\boldsymbol{q}}](U(P)-U(Q)+\boldsymbol{e})\Theta[^{\boldsymbol{p}}_{\boldsymbol{q}}](U(P)-U(Q)-\boldsymbol{e})}{\Theta[^{\boldsymbol{p}}_{\boldsymbol{q}}]^2(\boldsymbol{e})E^2(P,Q)}\\ &\qquad=B(P,Q)+\sum_{\alpha=1}^{g}\sum_{\beta=1}^{g}\partial_{\boldsymbol{z}_\alpha}\partial_{\boldsymbol{z}_\beta}\ln\Theta[^{\boldsymbol{p}}_{\boldsymbol{q}}](\boldsymbol{z})\big|_{\boldsymbol{z}=\boldsymbol{e}}\omega_\alpha(P)\omega_\beta(Q).\end{aligned} \tag{3.3.16}$$

Now since $\mathbf{p},\mathbf{q}$ are even characteristics, using equation 4.1.8 we have

$$S(P,Q)S(Q,P)=-B(P,Q)-\sum_{\alpha=1}^{g}\sum_{\beta=1}^{g}\partial_{\mathbf{z}_\alpha}\partial_{\mathbf{z}_\beta}\ln\Theta[^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=0}\omega_\alpha(P)\omega_\beta(Q). \tag{3.3.17}$$

Finally, in the next chapter we will need the following Fay identity, which can be found as corollary 2.19 in [3]. For any two sets of points $P_1,...,P_N$ and $Q_1,...,Q_N$ we have, writing P instead of $U(P)$ inside the argument of the theta function to make the equation more compact,

$$\det\left[\frac{\Theta[^{\mathbf{p}}_{\mathbf{q}}](P_j - Q_k + \Omega)}{E(P_j, Q_k)}\right] = \Theta[^{\mathbf{p}}_{\mathbf{q}}]\left(\sum_{j=1}^{N}(P_j - Q_j) + \Omega\right)\left(\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)\right)^{N-1} \times \frac{\prod_{j<k} E(P_j, P_k)E(Q_k, Q_j)}{\prod_{j,k} E(P_j, Q_k)}. \tag{3.3.18}$$

3.4 Rauch Variational Formulas

The Rauch variational formulas allow us to compute the first variation of the holomorphic one-forms and period matrix $\mathbf{B}_{\alpha\beta}$ with respect to the singular points $\lambda_1, ..., \lambda_M$. We will need these formulas in order to write down the explicit forms of the differential equations satisfied by the Jimbo-Miwa and Seiberg-Witten tau-functions in the next chapter.

Theorem 3.4.1. *The dependence of the holomorphic differentials* $\{\omega_\alpha\}_{\alpha=1}^{g}$ *and the period matrix* $\boldsymbol{B}_{\alpha\beta}$ *on the position of the branch point* λ_m *is given by the Rauch formulas*

$$\begin{aligned}\partial_{\lambda_m}\omega_\alpha(P) &= \underset{\lambda=\lambda_m}{res}\,\frac{1}{d\lambda}\sum_{j=1}^{N}\omega_\alpha(\lambda^{(j)})B(P,\lambda^{(j)}),\\ \partial_{\lambda_m}\boldsymbol{B}_{\alpha\beta} &= -\underset{\lambda=\lambda_m}{res}\,\frac{4\pi i}{d\lambda}\sum_{j<k}\omega_\alpha(\lambda^{(j)})\omega_\beta(\lambda^{(k)}).\end{aligned} \tag{3.4.1}$$

where $\lambda^{(j)}$ *is the point projecting to* λ *on the* j*th sheet.*

To prove this theorem we make use of the fact that if two meromorphic differentials on a compact Riemann surface have matching pole divisors and principal parts at each pole, as well as matching a- and b- periods, then they must be equal. Indeed, in this case taking their difference defines a holomorphic differential with vanishing a- and b-periods, which must therefore be identically zero.

We also make use of the following lemmas.

Lemma 3.4.2. *Let* $\eta(P)$ *be a meromorphic differential on* $\mathcal{L}$*. Then* $\sum_{j=1}^{N}\eta(\lambda^{(j)})$ *is a meromorphic differential on* $\mathbb{C}P^1$ *with residue at* $\lambda = \lambda_m$ *given by*

$$\underset{\lambda=\lambda_m}{res}\sum_{j=1}^{N}\eta(\lambda^{(j)}) = \sum_{Q\in\Pi^{-1}(\lambda_m)}\underset{P=Q}{res}\,\eta(P). \tag{3.4.2}$$

Proof. It is easy to see this from the corresponding contour integrals. Let $\ell_m^{(j)}$ be the lift of ℓ_m to the jth sheet of the cover, where ℓ_m is the closed curve containing λ_m and no other branch points. Then

$$\begin{aligned} \operatorname*{res}_{\lambda=\lambda_m} \sum_{j=1}^{N} \eta(\lambda^{(j)}) &= \frac{1}{2\pi i} \int_{\lambda\in\ell_m} \sum_{j=1}^{N} \eta(\lambda^{(j)}), \\ &= \frac{1}{2\pi i} \sum_{j=1}^{N} \int_{P\in\ell_m^{(j)}} \eta(P). \end{aligned} \tag{3.4.3}$$

Let $\lambda_m^{(1)}, ..., \lambda_m^{n(m)}$ be the set of distinct points projecting to λ_m, and $\mathbf{k}_m^{(1)}, ..., \mathbf{k}_m^{n(m)}$ the number of sheets meeting at each of these $n(m)$ points. If $\lambda_m^{(i)}$ is a ramification point, then taking the union of the curves $\ell_m^{(j)}$ on each of the $\mathbf{k}_m^{(i)}$ sheets meeting at $\lambda_m^{(i)}$ we get a closed curve $L_i := \bigcup_{j=1}^{\mathbf{k}_m^{(i)}} \ell_m^{(j)}$ containing $\lambda_m^{(i)}$ and no other ramification points. Otherwise if $\lambda_m^{(i)}$ is not a ramification point, $\ell_m^{(i)}$ is a closed curve on its own.

Consequently

$$\begin{aligned} \operatorname*{res}_{\lambda=\lambda_m} \sum_{j=1}^{N} \eta(\lambda^{(j)}) &= \frac{1}{2\pi i} \sum_{i=1}^{n(m)} \int_{P\in L_i} \eta(P), \\ &= \sum_{i=1}^{n(m)} \operatorname*{res}_{P=\lambda_m^{(i)}} \eta(P), \end{aligned} \tag{3.4.4}$$

as required. $\square$

Lemma 3.4.3. *Whenever $\omega(P)$ is a holomorphic differential on $\mathcal{L}$, $\sum_{j=1}^{N} \omega(\lambda^{(j)})$ is a holomorphic differential on $\mathbb{C}P^1$, and therefore*

$$\sum_{j=1}^{N} \omega(\lambda^{(j)}) = 0. \tag{3.4.5}$$

Proof of lemma. The only points λ at which $\sum_{j=1}^{N} \omega(\lambda^{(j)})$ might fail to be holomorphic are the branch points. Let $\mathbf{k}_m$ be the number of sheets meeting at P_m, and let $\lambda_m^{(j)}$ be the point projecting to λ_m on the jth sheet, so that P_m appears $\mathbf{k}_m$ times in the sequence of points $\lambda_m^{(1)}, ..., \lambda_m^{(N)}$.

Taking $\lambda = \Pi(P)$, in a neighborhood of $P = P_m$ we define the local parameter $x_m = (\lambda - \lambda_m)^{1/\mathbf{k}_m}$. The holomorphic differential has expansion

$$\omega(P) = \sum_{k=0}^{\infty} A_k \sum_{j=1}^{N} \left[x_m^k \frac{dx_m}{d\lambda} \right]_{P=\lambda^{(j)}} d\lambda, \quad A_k \in \mathbb{C}, \tag{3.4.6}$$

where $dx_m/d\lambda = \frac{1}{\mathbf{k}_m} x_m^{1-\mathbf{k}_m}$. Note that under analytic continuation around $\lambda \to \lambda + \ell_m$ we have $x_m \to e^{2\pi i/\mathbf{k}_m} x_m$. Therefore defining $\gamma = e^{2\pi i/\mathbf{k}_m}$,

$$\omega = \sum_{k=0}^{\infty} A_k \left[\sum_{j=0}^{\mathbf{k}_m - 1} \gamma^{j(n+1)} \right] x_m^k \frac{dx_m}{d\lambda} d\lambda. \tag{3.4.7}$$

The sum over j is zero for each $0 \le k \le \mathbf{k}_m - 2$, and hence ω is $O(1)$. □

Proof of theorem. We begin by proving the first equation in the case that P_m is the sole branch point projecting to λ_m. As in the proof of lemma 3.4.3, let $\mathbf{k}_m$ be the order of the branch point P_m and define the local parameter $x_m = (\Pi(P) - \lambda_m)^{1/\mathbf{k}_m}$ in a neighbourhood of P_m. For P near P_m the holomorphic differential $\omega_\alpha(P)$ has expansion

$$\omega_\alpha(P) = \sum_{k=0}^{\infty} C_k x_m^k(P) dx_m = \sum_{k=0}^{\infty} \frac{C_k}{\mathbf{k}_m} (\Pi(P) - \lambda_m)^{(k+1)/\mathbf{k}_m - 1} d\lambda, \quad C_k \in \mathbb{C}. \tag{3.4.8}$$

Differentiating the above equation with respect to λ_m and re-writing it in terms of x_m,

$$\partial_{\lambda_m} \omega_\alpha(P) = \sum_{k=0}^{\infty} C_k \left(1 - \frac{k+1}{\mathbf{k}_m} \right) x_m^{k - \mathbf{k}_m}(P) dx_m. \tag{3.4.9}$$

The term of degree $k - \mathbf{k}_m = -1$ vanishes, leaving

$$\partial_{\lambda_m} \omega_\alpha(P) \\ = \left[C_0 \left(1 - \frac{1}{\mathbf{k}_m} \right) x_m^{-\mathbf{k}_m} + C_1 \left(1 - \frac{2}{\mathbf{k}_m} \right) x_m^{1-\mathbf{k}_m} + \dots + \frac{C_{\mathbf{k}_m - 2}}{\mathbf{k}_m} x_m^{-2} + O(x_m) \right] dx_m. \tag{3.4.10}$$

Otherwise for P near another branch point P_n we have $\partial_{\lambda_m} \omega_\alpha(P) = O(1) dx_n(P)$. In this case the right-hand side of equation 3.4.1 is clearly also $O(1) dx_n(P)$. Thus $\partial_{\lambda_m} \omega_\alpha(P)$ is a differential of the second kind, with its only pole at $P = P_m$.

Moreover the a-periods of $\partial_{\lambda_m} \omega_\alpha$ are zero as a result of the normalization of ω_α (this fact is consistent with equation 3.4.1 by normalization of $B(P, Q)$). We can therefore compute the b-periods of $\partial_{\lambda_m}\omega_\alpha$ using the same method as Lemma 3.3.1:

$$\begin{aligned} \int_{b_\beta} \partial_{\lambda_m}\omega_\alpha &= \sum_{\gamma=1}^{g}\left[\int_{a_\gamma}\omega_\beta \int_{b_\gamma}\partial_{\lambda_m}\omega_\alpha - \int_{a_\gamma}\partial_{\lambda_m}\omega_\alpha \int_{b_\gamma}\omega_\beta\right] \\ &= 2\pi i \operatorname*{res}_{P=P_m}\left[\int_{P_0}^{P}\frac{C_{\mathbf{k}_m-2}}{\mathbf{k}_m}x_m^{-2}(P')dx_m(P')\right]\omega_\beta(P), \qquad (3.4.11) \\ \therefore \int_{b_\beta}\partial_{\lambda_m}\omega_\alpha &= 2\pi i\frac{C_{\mathbf{k}_m-2}}{\mathbf{k}_m}\frac{\omega_\beta(P_m)}{dx_m}. \end{aligned}$$

Now using the expansion 3.4.8 and lemma 3.4.2 we can write

$$\operatorname*{res}_{\lambda=\lambda_m}\sum_{j=1}^{N}\frac{dx_m}{d\lambda}\omega_\alpha(\lambda^{(j)}) = \operatorname*{res}_{P=P_m}\frac{dx_m}{d\lambda}\omega_\alpha(P) = \frac{C_{\mathbf{k}_m-2}}{\mathbf{k}_m}. \qquad (3.4.12)$$

Hence using lemma 3.3.1 we can re-write equation 3.4.11 as

$$\int_{b_\beta}\partial_{\lambda_m}\omega_\alpha = \int_{P\in b_\beta}\operatorname*{res}_{\lambda=\lambda_m}\frac{1}{d\lambda}\sum_{j=1}^{N}\omega_\alpha(\lambda^{(j)})B(P,\lambda^{(j)}). \qquad (3.4.13)$$

To show that the integrands are equal it therefore suffices to show that they have matching principal parts at $P = P_m$. Note that only the singular term of $B(P,\lambda^{(j)})$ will contribute to the principal part. Hence for P and $\lambda^{(j)}$ sufficiently close to P_m we have

$$\begin{aligned} &\operatorname*{res}_{\lambda=\lambda_m}\frac{1}{d\lambda}\sum_{j=1}^{N}\omega_\alpha(\lambda^{(j)})B(P,\lambda^{(j)}) \\ &= \operatorname*{res}_{\lambda=\lambda_m}\frac{1}{\mathbf{k}_m}\sum_{j=1}^{N}\omega_\alpha(\lambda^{(j)})\left[\frac{dx_m(P)dx_m(\lambda^{(j)})}{\left(x_m(P)-x_m(\lambda^{(j)})\right)^2}+O(1)\right], \\ &= \operatorname*{res}_{\lambda=\lambda_m}\frac{1}{\mathbf{k}_m}\sum_{j=1}^{N}\left[C_0x_m^{1-\mathbf{k}_m}(\lambda^{(j)})+\dots+C_{\mathbf{k}_m-2}x_m^{-1}(\lambda^{(j)})+O(1)\right] \qquad (3.4.14) \\ &\qquad\times\left[\sum_{i=0}^{\infty}(i+1)x_m^{i}(\lambda^{(j)})x_m^{-2-i}(P)+O(1)\right]dx_m(\lambda^{(j)})dx_m(P). \end{aligned}$$

Applying lemma 3.4.2 and discarding any terms that are not $O(x_m^{-1})$,

$$\begin{aligned} &\operatorname*{res}_{\lambda=\lambda_m}\frac{1}{d\lambda}\sum_{j=1}^{N}\omega(\lambda^{(j)})B(P,\lambda^{(j)}) \\ &= \left[\operatorname*{res}_{\lambda^{(j)}=P_m}\frac{dx_m}{x_m}\right]\frac{1}{\mathbf{k}_m}\left[C_0(\mathbf{k}_m-1)x_m^{-\mathbf{k}_m}(P)+\dots+C_{\mathbf{k}_m-2}x_m^{-2}(P)+O(1)\right]dx_m(P), \end{aligned}$$

(3.4.15)

and we recover the principal part of $\partial_{\lambda_m}\omega_\alpha(P)$ in equation 3.4.10.

For the general case let $P_1, ..., P_{n(m)}$ be the set of branch points projecting to λ_m. In this case let $\lambda_m^{(j)}$ be the point projecting to λ_m on the jth sheet, and let $\mathbf{k}_m^{(j)}$ be the number of sheets meeting at $\lambda_m^{(j)}$ so that $P_i = \lambda_m^{(j)}$ appears $\mathbf{k}_m^{(j)}$ times in the sequence of points $\lambda_m^{(1)}, ..., \lambda_m^{(N)}$.

For P in a neighborhood of $\lambda_m^{(j)}$ we take $\lambda = \Pi(P)$ and define the local parameter $x_m^{(j)} = (\lambda - \lambda_m)^{1/\mathbf{k}_m^{(j)}}$. Then $dx_m^{(j)} = (\mathbf{k}_m^{(j)})^{-1}(x_m^{(j)})^{1-\mathbf{k}_m^{(j)}}d\lambda$ so that if $\lambda_m^{(j)}$ is an ordinary point $dx_m^{(j)} = d\lambda$. We have expansions of the form 3.4.8 and 3.4.10 at each of the points $\lambda_m^{(1)}, ..., \lambda_m^{(N)}$.

However this time when we compute the b-period of $\partial_{\lambda_m}\omega_\alpha$ using equation 3.3.3, we must sum over all the points $P_1, ..., P_{n(m)}$ in the pole divisor of $\partial_{\lambda_m}\omega_\alpha$. The result is that

$$\int_{b_\alpha} \partial_{\lambda_m}\omega_\alpha = 2\pi i \sum_{i=1}^{n(m)} \frac{C_{\mathbf{k}_m^{(j)}-2}}{\mathbf{k}_m^{(j)}} \frac{\omega_\alpha(P_i)}{dx_m^{(j)}}, \qquad \lambda_m^{(j)} = P_i, \tag{3.4.16}$$

and lemma 3.4.2 once again gives

$$\sum_{i=1}^{n(m)} \operatorname*{res}_{P=P_i} \frac{dx_m^{(j)}}{d\lambda}\omega_\alpha(P) = \operatorname*{res}_{\lambda=\lambda_m} \sum_{j=1}^{N} \frac{dx_m^{(j)}}{d\lambda}\omega_\alpha(\lambda^{(j)}) = \sum_{i=1}^{n(m)} \frac{C_{\mathbf{k}_m^{(j)}-2}}{\mathbf{k}_m^{(j)}}. \tag{3.4.17}$$

We therefore once again have

$$\partial_{\lambda_m}\omega_\alpha(P) = \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{j=1}^{N} \omega(\lambda^{(j)})B(P, \lambda^{(j)}). \tag{3.4.18}$$

Now computing the b-periods of $\partial_{\lambda_m}\omega_\alpha(P)$ and using Lemma 3.3.1 gives us the variation of the matrix component $\mathbf{B}_{\alpha\beta}$:

$$\partial_{\lambda_m}\mathbf{B}_{\alpha\beta}(P) = 2\pi i \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{j=1}^{N} \omega_\alpha(\lambda^{(j)})\omega_\beta(\lambda^{(j)}). \tag{3.4.19}$$

Finally, using lemma 3.4.3 we have

$$\partial_{\lambda_m}\mathbf{B}_{\alpha\beta}(P) = -4\pi i \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{j<k} \omega_\alpha(\lambda^{(j)})\omega_\beta(\lambda^{(k)}). \tag{3.4.20}$$

□

We end by proving a Rauch formula for the Bergman kernel.

Theorem 3.4.4.

$$\partial_{\lambda_m} B(P,Q) = \underset{\lambda=\lambda_m}{res} \frac{1}{d\lambda} \sum_{j=1}^{N} B(P,\lambda^{(j)})B(\lambda^{(j)},Q). \tag{3.4.21}$$

Proof. Similar to before we show that the differentials on either side of equation 3.4.21 have matching periods, poles and principal parts with respect to P; by symmetry of $B(P,Q)$ the same equivalence will hold with respect to Q. Therefore fix Q.

Note that $\partial_{\lambda_m} B(P,Q)$ has vanishing a-periods by normalization of $B(P,Q)$, while lemma 3.3.1 and theorem 3.4.1 give

$$\int_{P\in b_\alpha} \partial_{\lambda_m} B(P,Q) = \partial_{\lambda_m} \int_{P\in b_\alpha} B(P,Q) = 2\pi i \underset{\lambda=\lambda_m}{\mathrm{res}} \frac{1}{d\lambda} \sum_{j=1}^{N} \omega_\alpha(\lambda^{(j)})B(\lambda^{(j)},Q), \tag{3.4.22}$$

which is clearly consistent with equation 3.4.21.

Next, assume temporarily that our fixed Q satisfies $\Pi(Q) \neq \lambda_m$. Then since $B(P,Q)$ is holomorphic at all points $P \neq Q$, an identical argument to the proof of theorem 3.4.1 shows that $\partial_{\lambda_m} B(P,Q)$ has a pole at each ramification point projecting to λ_m with principal part consistent with 3.4.21. At $P = Q$ we simply have $\partial_{\lambda_m} B(P,Q) = O(1)dx(P)$.

Otherwise if $Q = P_m$ for some ramification point P_m projecting to λ_m, $B(P,P_m)$ is still holomorphic outside of $P = P_m$; the same argument shows that $\partial_{\lambda_m} B(P,P_m)$ has poles at each ramification point $P_n \neq P_m$ projecting to λ_m, and the principal parts are found in an identical manner.

However we must check that the right-hand side of equation 3.4.21 has the correct principal part at $P = P_m$. Expanding $B(P,P_m)$ at $P = P_m$ we have

$$B(P,P_m) = \left[\frac{1}{x_m^2(P)} + \sum_{i=0}^{\infty} D_i x_m^i(P)\right] dx_m(P)dx_m(P_m), \quad D_i \in \mathbb{C}, \tag{3.4.23}$$

and differentiating gives

$$\partial_{\lambda_m} \frac{B(P,P_m)}{dx_m(P_m)} = \left[\left(1+\frac{1}{\mathbf{k}_m}\right) x_m^{-2-\mathbf{k}_m} + D_0\left(1-\frac{1}{\mathbf{k}_m}\right) x_m^{-\mathbf{k}_m} + ... + \frac{D_{\mathbf{k}_m-2}}{\mathbf{k}_m} x_m^{-2} + O(x_m)\right] dx_m(P). \tag{3.4.24}$$

Finally, we set $Q = P_m$ in the right-hand side of equation 3.4.21 and show that the expansion at $P = P_m$ has the same principal part as 3.4.24. For P and $\lambda^{(j)}$ sufficiently close to P_m we have

$$\begin{aligned}
&\operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{j=1}^{N} B(P,\lambda^{(j)})B(\lambda^{(j)},P_m) \\
&= \operatorname*{res}_{\lambda=\lambda_m} \sum_{j=1}^{N} \frac{1}{\mathbf{k}_m} \left[\sum_{i=0}^{\infty} (i+1) x_m^i(\lambda^{(j)}) x_m^{-2-i}(P) + O(1) \right] \\
&\qquad \times x_m^{1-\mathbf{k}_m}(\lambda^{(j)}) \left[x_m^{-2}(\lambda^{(j)}) + \sum_{i=0}^{\infty} D_i x_m^i(\lambda^{(j)}) \right] dx_m(P) dx_m(\lambda^{(j)}) dx_m(P_m),
\end{aligned} \tag{3.4.25}$$

The residue will simplify in the same manner as the residue in equation 3.4.14, with the addition of a term resulting from the singular term in $B(\lambda^{(j)}, P_m)$:

$$\begin{aligned}
&\operatorname*{res}_{\lambda=\lambda_m} \sum_{j=1}^{N} \frac{1}{\mathbf{k}_m} \left[\sum_{i=0}^{\infty} (i+1) x_m^i(\lambda^{(j)}) x_m^{-2-i}(P) + O(1) \right] x_m^{-1-\mathbf{k}_m}(\lambda^{(j)}) \\
&= \frac{\mathbf{k}_m + 1}{\mathbf{k}_m} x_m^{-2-\mathbf{k}_m}(P),
\end{aligned} \tag{3.4.26}$$

and we recover the expansion 3.4.24. □

Chapter 4

The Seiberg-Witten tau-function

4.1 Solution of the Riemann-Hilbert problem

The solution Ψ of the Riemann-Hilbert problem as stated in chapter 2 will depend on several parameters. Firstly, the solution has the Szegö kernel as a main ingredient and therefore depends on the choice of characteristics $\mathbf{p}, \mathbf{q} \in \mathbb{C}^g$.

Secondly, we introduce N parameters $r_m^{(j)} \in \mathbb{C}$ corresponding to the points $\lambda_m^{(j)}$ (with multiplicity) lying over a ramification point λ_m. We require that $r_m^{(j)} = r_m^{(k)}$ if $\lambda_m^{(j)} = \lambda_m^{(k)}$ is a branch point. We further require that

$$\sum_{m=1}^{M} \sum_{j=1}^{N} r_m^{(j)} = 0. \tag{4.1.1}$$

This fact will be justified later on, when we see that the residue of the solution $A(\lambda)$ of the Schlesinger system at $\lambda = \lambda_m$ is given by $\sum_{j=1}^{N} r_m^{(j)}$. As a result the number of independent parameters among the $r_m^{(j)}$ is

$$MN - 1 - \sum_{m=1}^{M} \sum_{j=1}^{N} \mathbf{k}_m^{(j)} - 1 = MN - 2g - 2N + 1, \tag{4.1.2}$$

where we used the Riemann-Hurwitz formula from equation 2.3.2, and as usual $\mathbf{k}_m^{(j)}$ is the number of sheets meeting at the point $\lambda_m^{(j)}$. Together with $\mathbf{p}, \mathbf{q}$ we thus have a total of $MN - 2N + 1$ independent parameters, which according to lemma 2.3.1 is enough to uniquely specify the solution $\Psi(\lambda)$ corresponding to the monodromies $\mathcal{M}_1, ..., \mathcal{M}_m$.

Next, we discuss the contours on $\mathcal{L}$ that will be used to tabulate the monodromies of Ψ around each $\gamma \in \pi_1 [\mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}]$. Let S be a contour on $\hat{\mathcal{L}}$ connecting the basepoint of the Abel map P_0 with each of the points $\lambda_m^{(j)}$ for $1 \leq m \leq M$ and $1 \leq j \leq N$, including both branch points and ordinary points. We require that the basepoint λ_* of the contours $\ell_1, ..., \ell_M$ does not coincide with the projection of S or any of the set of cycles $\{a_\alpha, b_\alpha\}_{\alpha=1}^g$ onto $\mathbb{C}P^1$. Letting $\ell_m^{(j)}$ be the contour projecting to ℓ_m contained in the jth sheet, we define the intersection indices

$$I_{m\alpha}^{(j)} = \ell_m^{(j)} \circ a_\alpha, \quad J_{m\alpha}^{(j)} = \ell_m^{(j)} \circ b_\alpha, \quad K_m^{(j)} = \ell_m^{(j)} \circ S, \tag{4.1.3}$$

for $1 \leq m \leq M, 1 \leq \alpha \leq g$ and $1 \leq j \leq N$.

Finally, our solution for Ψ will involve the prime-form E_0 on $\mathbb{C}P^1$ lifted to $\mathcal{L}$:

$$E_0(\lambda, \mu) = \frac{\lambda - \mu}{\sqrt{d\lambda d\mu}}. \tag{4.1.4}$$

Since $d\lambda$ has a double pole at $\lambda = \infty$, the lifted differential $d\lambda(P)$ will have double poles at each of the N points projecting to ∞. Meanwhile using the local parameter $x_m = (\lambda - \lambda_m)^{1/\mathbf{k}_m}$ clearly $d\lambda(P)$ has zeros of order $\mathbf{k}_m - 1$ at each branch point P_m. Therefore $\sqrt{d\lambda(P)}$ is not a holomorphic section of a spinor bundle on $\mathcal{L}$.

However, it is possible to define the lift such that the function $h(P)/\sqrt{d\lambda}$ has trivial automorphy factors along the basic cycles. This function, which has poles of order $1/2$ at each P_m, will have automorphy factor -1 along the cycles ℓ_m [11].

Next, consider the ratio

$$f(P, Q) = \frac{E_0(\lambda, \mu)}{E(P, Q)}, \quad \lambda = \Pi(P), \mu = \Pi(Q). \tag{4.1.5}$$

As a result of our definition of the lift of $\sqrt{d\lambda}$, the function $f(P, Q)$ will have holonomies $\exp(\pi i \mathbf{p}_\alpha^*)$ around a_α, $\exp(-\pi i \mathbf{q}_\alpha^* - 2\pi i[U_\alpha(P) - U_\alpha(Q)])$ around b_α, and $\exp(2\pi i(\mathbf{k}_m - 1))$ around ℓ_m.

Now we can finally describe how the solution $\Psi(\lambda_0, \lambda)$ given in [12] is constructed. Taking λ sufficiently close to λ_0, we define the germ of the component functions $\Psi_{jk}(\lambda_0, \lambda) : \hat{\mathcal{L}} \to \mathbb{C}P^1$ in terms of a bi-meromorphic function $\psi : \hat{\mathcal{L}} \times \hat{\mathcal{L}} \to \mathbb{C}P^1$:

$$\Psi_{kj}(\lambda_0, \lambda) = \psi(\lambda^{(j)}, \lambda_0^{(k)}), \qquad 1 \leq k, j, \leq N, \tag{4.1.6}$$

where as usual $\lambda^{(k)}$ is the point on the kth sheet projecting to λ. The function $\psi(P, Q)$ is given by

$$\psi(P, Q) = \hat{S}(P, Q) E_0(\lambda, \mu), \qquad \lambda = \Pi(P), \;\; \mu = \Pi(Q), \tag{4.1.7}$$

where $\hat{S}(P,Q)$ is the modified Szegö kernel

$$\hat{S}(P,Q) = \frac{\Theta[^{\mathbf{p}}_{\mathbf{q}}](U(P)-U(Q)+\Omega)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)E(P,Q)} \prod_{m=1}^{M}\prod_{\ell=1}^{N}\left[\frac{E(P,\lambda_m^{(\ell)})}{E(Q,\lambda_m^{(\ell)})}\right]^{r_m^{(\ell)}}, \tag{4.1.8}$$

and $\Omega \in \mathbb{C}^g$ is the vector $\Omega_\alpha = \sum_{m=1}^{M}\sum_{j=1}^{N} r_m^{(j)} U_\alpha(\lambda_m^{(j)})$. Similar to the original Szegö kernel, $\mathbf{p},\mathbf{q}$ are even half-integer characteristics chosen to satisfy $\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega) \neq 0$.

Under a shift of basepoint from P_0 to P_0',

$$\Omega_\alpha = \sum_{m=1}^{M}\sum_{j=1}^{N} r_m^{(j)} \int_{P_0'}^{P_0} \omega_\alpha + \sum_{m=1}^{M}\sum_{j=1}^{N} r_m^{(j)} \int_{P_0}^{\lambda_m^{(j)}} \omega_\alpha. \tag{4.1.9}$$

The first term vanishes since the integral factors out of the sum, giving a constant multiplied by $\sum_{m=1}^{M}\sum_{j=1}^{N} r_m^{(j)} = 0$. Therefore Ω is independent of the choice of basepoint P_0.

Taking $\mathbf{e} = \Omega$ in Lemma 3.3.17 gives the following relation between the modified Szegö kernel and Bergman kernel:

$$\hat{S}(P,Q)\hat{S}(Q,P) = -B(P,Q) - \sum_{\alpha=1}^{g}\sum_{\beta=1}^{g} \partial_{\mathbf{z}_\alpha}\partial_{\mathbf{z}_\beta} \ln \Theta[^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=\Omega}\omega_\alpha(P)\omega_\beta(Q). \tag{4.1.10}$$

The Szegö kernel has the same automorphy factors as $S(P,Q)$ under analytic continuation around a- and b-cycles. These relations follow from the holonomy properties of the theta function and prime-form in equations 3.2.3 and 3.2.17, and from the fact that $\sum_{m=1}^{M}\sum_{\ell=1}^{N} r_m^{(\ell)} = 0$.

We are finally ready to state the solution of the Riemann-Hilbert problem that we defined in chapter 2.

Theorem 4.1.1. *Let $\Psi(\lambda_0,\lambda)$ be the function given by analytic continuation of the germ defined by equation 4.1.6 from a neighbourhood of λ_0 to the universal cover $\hat{T}$ of $\mathbb{C}P^1 \setminus \{\lambda_1,...,\lambda_M\}$. Then Ψ is non-singular and non-degenerate on $\hat{T}$ with regular singularities at each λ_m, and satisfies the normalization $\Psi(\lambda_0,\lambda_0) = I$. It solves the Riemann-Hilbert problem outlined in section 1 with monodromy around λ_n given by the quasi-permutation matrix*

$$(\mathcal{M}_n)_{jk} = \delta_{j_n(j)k} \times \exp\left[2\pi i\left[\left[\boldsymbol{k}_n^{(j)}\left[r_n^{(j)}+\frac{1}{2}\right]-\frac{1}{2}\right]K_n^{(j)} + \sum_{\alpha=1}^{g}\left[J_{n\alpha}^{(j)}(\boldsymbol{p}_\alpha+\boldsymbol{p}_\alpha^*) - I_{n\alpha}^{(j)}(\boldsymbol{q}_\alpha+\boldsymbol{q}_\alpha^*)\right]\right]\right], \tag{4.1.11}$$

where $j_n(j)$ is the sheet on which the contour ℓ_n ends.

Proof of theorem. First, we check the normalization condition $\Psi_{jk}(\lambda_0, \lambda_0) = \delta_{jk}$. In the limit $\lambda \to \lambda_0$, equations 4.1.7 and 4.1.8 give

$$\lim_{\lambda\to\lambda_0} \psi(\lambda^{(j)}, \lambda_0^{(k)}) = \lim_{\lambda\to\lambda_0} \frac{E_0(\lambda, \lambda_0)}{E(\lambda^{(j)}, \lambda_0^{(k)})} \prod_{m=1}^{M} \prod_{\ell=1}^{N} \left[\frac{E(\lambda^{(j)}, \lambda_m^{(\ell)})}{E(\lambda_0^{(k)}, \lambda_m^{(\ell)})} \right]^{r_m^{(\ell)}} . \tag{4.1.12}$$

We require that λ_0 be an ordinary point; therefore we have the local parameter $x(\lambda^{(j)}) = \lambda - \lambda_0$ in a small neighbourhood of $\lambda^{(j)} = \lambda_0^{(j)}$, and the local expansion 3.2.16 implies that

$$\lim_{\lambda\to\lambda_0} \frac{E_0(\lambda, \lambda_0)}{E(\lambda^{(j)}, \lambda_0^{(k)})} = \delta_{jk}. \tag{4.1.13}$$

None of the factors in the product over m, ℓ are zero for λ close to λ_0, and for $j = k$ the product equals 1. Therefore $\psi(\lambda_0^{(j)}, \lambda_0^{(k)}) = \delta_{jk}$.

Next, setting setting $P_j = \lambda^{(j)}$ and $Q_k = \lambda_0^{(k)}$ in Fay's identity 3.3.18 and dividing both sides by $\Theta[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}](\Omega)^N$ we have

$$\begin{aligned} &\det \left[\frac{\Theta[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}](\lambda^{(j)} - \lambda_0^{(k)} + \Omega)}{\Theta[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}](\Omega) E(\lambda^{(j)}, \lambda_0^{(k)})} \right] \\ &= \frac{\Theta[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}] \left(\sum_{j=1}^{N} \left(\lambda^{(j)} - \lambda_0^{(j)} \right) + \Omega \right)}{\Theta[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}](\Omega)} \frac{\prod_{j<k} E(\lambda^{(j)}, \lambda^{(k)}) E(\lambda_0^{(k)}, \lambda_0^{(j)})}{\prod_{j,k} E(\lambda^{(j)}, \lambda_0^{(k)})}. \end{aligned} \tag{4.1.14}$$

Therefore using equations 4.1.7 and 4.1.8 we have

$$\begin{aligned} &\det \left[\Psi_{kj}(\lambda_0, \lambda) \right] \\ &= \frac{\Theta[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}] \left(\sum_{j=1}^{N} \left(\lambda^{(j)} - \lambda_0^{(j)} \right) + \Omega \right)}{\Theta[\begin{smallmatrix}\mathbf{p}\\ \mathbf{q}\end{smallmatrix}](\Omega)} \frac{\prod_{j<k} E(\lambda^{(j)}, \lambda^{(k)}) E(\lambda_0^{(k)}, \lambda_0^{(j)})}{\prod_{j,k} E(\lambda^{(j)}, \lambda_0^{(k)})} \\ &\qquad \times \prod_{m=1}^{M} \prod_{\ell=1}^{N} \left[\det \left[\frac{E(\lambda^{(j)}, \lambda_m^{(\ell)})}{E(\lambda_0^{(k)}, \lambda_m^{(\ell)})} \right] \right]^{r_m^{(\ell)}} \left[E_0(\lambda, \lambda_0) \right]^N. \end{aligned} \tag{4.1.15}$$

However note that for any $\lambda_0^{(j)}, \lambda^{(j)} \in \mathcal{L}$,

$$\sum_{j=1}^{N} \left[U_\alpha(\lambda^{(j)}) - U_\alpha(\lambda_0^{(j)}) \right] = \int_{\lambda_0}^{\lambda} \sum_{j=1}^{N} \omega_\alpha(\lambda^{(j)}) = 0, \tag{4.1.16}$$

by holomorphicity of ω_α (lemma 3.4.3), and therefore the theta function terms will simplify to 1.

As a function of λ, equation 4.1.15 is holomorphic and nonzero outside the points λ_m since in particular we have already shown $\det(\Psi_{jk}(\lambda_0)) = 1$. Moreover this function has zeros at each λ_m, with each distinct branch point P_n lying over λ_m contributing a factor $x_m^{\mathbf{k}_n^{(j)} r_n^{(j)}}$. To see this, consider the grouping of factors

$$\frac{\prod_{j<k} E(\lambda^{(j)}, \lambda^{(k)}) E(\lambda_0^{(k)}, \lambda_0^{(j)})}{\prod_{j,k} E(\lambda^{(j)}, \lambda_0^{(k)})} [E_0(\lambda, \lambda_0)]^N \tag{4.1.17}$$

for λ in a neighbourhood of λ_m. This expression is nonzero since each branch point P_n lying over λ_m contributes a factor of $x_m^{N(1-\mathbf{k}_n)/2}$ resulting from the half-differential $\sqrt{d\lambda}$, and a factor $x_m^{N(\mathbf{k}_n-1)/2}$ resulting from the terms such that $\lambda_m^{(k)} = \lambda_m^{(j)}$ and $j < k$. Therefore the only zeros result from the factor

$$\prod_{m=1}^{M} \prod_{\ell=1}^{N} \left[\det \left[\frac{E(\lambda^{(j)}, \lambda_m^{(\ell)})}{E(\lambda_0^{(k)}, \lambda_m^{(\ell)})} \right] \right]^{r_m^{(\ell)}}, \tag{4.1.18}$$

which has the required zeros.

We will show that

$$\det\left[\Psi_{jk}(\lambda_0, \lambda)\right] = \prod_{m=1}^{M} \prod_{j,k=1}^{N} \left[\frac{E(\lambda^{(j)}, \lambda_m^{(k)})}{E(\lambda_0^{(j)}, \lambda_m^{(k)})} \right]^{r_m^{(k)}}. \tag{4.1.19}$$

The right-hand side of equation 4.1.19 is clearly holomorphic and has zeros of the required orders at each λ_m. Therefore we need only show that it has the same holonomies as the right-hand side of equation 4.1.15 to show that the two expressions must be equivalent up to multiplication by a holomorphic, hence constant function. Then since 4.1.19 agrees with $\det[\Psi_{jk}(\lambda_0, \lambda_0)] = 1$ we will have shown the two expressions are equivalent.

Using equation 3.2.17, the right-hand side of 4.1.19 is invariant under analytic continuation around a_α; around b_α, 4.1.19 has automorphy factor

$$\exp\left[-\pi i \sum_{m=1}^{M} \sum_{\ell=1}^{N} r_m^{(\ell)} \left[\mathbf{B}_{\alpha\alpha} + 2\left[U_\alpha(\lambda^{(j)}) - U_\alpha(\lambda_m^{(\ell)}) \right] \right] \right] = \exp\left(2\pi i \Omega\right). \tag{4.1.20}$$

Using equation 4.1.15 we compute the holonomies of $\det(\Psi_{jk})$ and show that they match. Under analytic continuation $\lambda_\alpha^{(i)} \to \lambda_\alpha^{(i)} + a_\alpha$ the theta function term contributes a factor $\exp(\pi i \mathbf{p}_\alpha^*)$, while for analytic continuation around b_α we get a factor

$$\exp\left[-\pi i \mathbf{q}_\alpha^* - \pi i \mathbf{B}_{\alpha\alpha} - 2\pi i \sum_{j=1}^{N} \left[U_\alpha(\lambda^{(j)}) - U_\alpha(\lambda_0^{(j)})\right]\right] = \exp\left[-\pi i \mathbf{q}_\alpha^* - \pi i \mathbf{B}_{\alpha\alpha}\right]. \tag{4.1.21}$$

Moreover the automorphy factors of the term $1/\sqrt{d\lambda}$ in the lift of $E_0(\lambda, \lambda_0)$ will cancel the factor of $\exp(\pi i \mathbf{p}_\alpha^*)$ under analytic continuation around a_α and the factor $\exp(-\pi i \mathbf{q}_\alpha^*)$ under analytic continuation around b_α. Consequently equation 4.1.15 is invariant under analytic continuation around a_α, since the prime-forms on $\mathcal{L}$ are invariant.

Next, consider the determinant in the right-hand side of 4.1.15. Under analytic continuation $\lambda^{(i)} \to \lambda^{(i)} + b_\alpha$ the row corresponding to $j = i$ picks up an overall factor which we can therefore factor out; this factor is

$$\exp\left[-\pi i \sum_{m=1}^{M} \sum_{\ell=1}^{N} r_m^{(\ell)} \left[\mathbf{B}_{\alpha\alpha} + 2[U_\alpha(\lambda^{(i)}) - U_\alpha(\lambda_m^{(\ell)})]\right]\right] = \exp(2\pi i \Omega). \tag{4.1.22}$$

Finally, the remaining prime-form terms have holonomy around b_α given by

$$\begin{aligned}&\exp\left[-\pi i \sum_{i\neq j} \left[\mathbf{B}_{\alpha\alpha} + 2[U_\alpha(\lambda^{(i)}) - U_\alpha(\lambda^{(j)})]\right] + \pi i \sum_{j=1}^{N} \left[\mathbf{B}_{\alpha\alpha} + 2[U_\alpha(\lambda^{(i)}) - U_\alpha(\lambda_0^{(j)})]\right]\right] \\ &= \exp\left[\pi i \mathbf{B}_{\alpha\alpha} + 2\pi i \sum_{j=1}^{N} [U_\alpha(\lambda^{(j)}) - U_\alpha(\lambda_0^{(j)})]\right] = \exp(\pi i \mathbf{B}_{\alpha\alpha}).\end{aligned} \tag{4.1.23}$$

This factor cancels the remaining factor from the theta function that was not canceled by $E_0(\lambda, \lambda_0)$. Hence $\det(\Psi_{jk}(\lambda_0, \lambda))$ has the correct automorphy factors of 1 around a_α and $\exp(2\pi i \Omega)$ around b_α, proving equation 4.1.19.

Next, any singularities of $\Psi(\lambda_0, \lambda)$ result from the factors $[E(\lambda^{(j)}, \lambda_m^{(\ell)})]^{r_m^{(\ell)}}$; and from the lifted spinor $1/\sqrt{d\lambda}$, since $d\lambda$ has zeros at all branch points. These occur only at the points $\lambda = \lambda_m$ and are clearly regular. Therefore Ψ is holomorphic and non-degenerate on $\hat{T}$.

Finally equation 4.1.11 for the monodromy around λ_n follows from the definitions of the intersection indices, properties of the theta function and prime-form, and the properties of the lifted spinor $\sqrt{d\lambda}$ discussed at the beginning of the section. □

We end this section by showing that the differential system satisfied by $\Psi(\lambda)$ is indeed Fuchsian. To this end we need the following lemma, which follows from the Fay identity 3.3.18 [12].

Lemma 4.1.2. *For any $\lambda, \lambda_0 \in \mathbb{C}P^1$ the function $\Psi_{jk}(\lambda_0, \lambda)$ satisfies*

$$\Psi_{jk}(\lambda_0, \lambda) = \Psi_{jk}^{-1}(\lambda, \lambda_0), \tag{4.1.24}$$

Theorem 4.1.3. *The function $\Psi(\lambda)$ defined by equations 4.1.7 and 4.1.8 satisfies a Fuchsian system, that is,*

$$\frac{d\Psi}{d\lambda} = A(\lambda)\Psi(\lambda). \tag{4.1.25}$$

where $A(\lambda)$ has only simple poles.

Proof. Using Lemma 4.1.2, the meromorphic function $\Psi_\lambda \Psi^{-1}$ is given by

$$\left(\Psi_\lambda \Psi^{-1}\right)_{ij} = \sum_{k=1}^{N} \psi_\lambda(\lambda^{(k)}, \lambda_0^{(i)}) \psi(\lambda_0^{(j)}, \lambda^{(k)}). \tag{4.1.26}$$

Clearly $\Psi_\lambda \Psi^{-1}$ is non-singular outside of the points λ_n. We will show that this function has simple poles at each λ_n, with residues $\sum_{j=1}^{N} r_n^{(j)}$ corresponding to each branch point $\lambda_n^{(j)}$ lying over λ_n. For any $P \in \hat{\mathcal{L}}$ equations 4.1.7 and 4.1.8 give

$$\begin{aligned}\psi_\lambda(\lambda^{(k)}, \lambda_0^{(i)}) = &\left[\sum_{\alpha=1}^{g} \partial_{\mathbf{z}_\alpha} \ln \Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\Big|_{\mathbf{z}=U(\lambda^{(k)})-U(\lambda_0^{(i)})+\Omega} \frac{\omega_\alpha(\lambda^{(k)})}{d\lambda} - \partial_\lambda \ln E(\lambda^{(k)}, \lambda_0^{(i)})\right.\\ &\left. + \sum_{m,\ell} r_m^{(\ell)} \partial_\lambda \ln E(\lambda^{(k)}, \lambda_m^{(\ell)}) + \partial_\lambda \ln E_0(\lambda - \lambda_0)\right] \psi(\lambda^{(k)}, \lambda_0^{(i)}).\end{aligned} \tag{4.1.27}$$

Let P_n be a branch point of order $\mathbf{k}_n^{(k)}$ lying over λ_n, with corresponding residue $r_n^{(k)}$. For $\lambda^{(k)}$ in a neighbourhood of P_n the first term and final terms in the brackets are holomorphic, as are the terms in the sum over m, ℓ corresponding to $\lambda_m^{(\ell)} \neq P_n$. Next, since $E(\lambda^{(k)}, \lambda_0^{(i)})$ is holomorphic and has its only zero at $\lambda_0^{(k)}$,

$$E(\lambda^{(k)}, \lambda_0^{(i)}) dx_n(\lambda^{(k)}) dx(\lambda_0^{(i)}) = a_0(\lambda_0^{(i)}) + a_1(\lambda_0^{(i)}) x_n(\lambda^{(k)}) + a_2(\lambda_0^{(i)}) x_n^2(\lambda^{(k)}) + \ldots \tag{4.1.28}$$

for some complex coefficients such that $a_0(\lambda_0^{(i)}) \neq 0$, and hence

$$\partial_\lambda \ln E(\lambda^{(k)}, \lambda_0^{(i)}) = \frac{\partial_{x_n} \left[a_0 + a_1 x_n + a_2 x_n^2 + ...\right]}{a_1 + a_2 x_n + ...} \frac{dx_n}{d\lambda}, \\ = x_n^{1-\mathbf{k}_n^{(k)}} \left[b_0(\lambda_0^{(i)}) + b_1(\lambda_0^{(i)}) x_n + ...\right], \tag{4.1.29}$$

for some complex coefficients such that $b_0(\lambda_0^{(i)}) \neq 0$. For the remaining term, expansion 3.2.16 with local parameter $x_n(\lambda^{(k)}) = (\lambda - \lambda_n)^{1/\mathbf{k}_n^{(k)}}$ gives

$$\partial_\lambda \ln E(\lambda^{(k)}, P_n) = \left[\frac{\partial_P E(P, P_n)}{E(P, P_n)}\right]_{P=\lambda^{(k)}} \frac{dx_n}{d\lambda} = \frac{1}{\mathbf{k}_n^{(k)}} \frac{1}{\lambda - \lambda_n}. \tag{4.1.30}$$

In the sum over m, ℓ in 4.1.27 there will be $\mathbf{k}_n^{(k)}$ terms $\lambda_m^{(\ell)} = P_n$, cancelling the factor of $(\mathbf{k}_n^{(k)})^{-1}$. Hence for $\lambda^{(k)}$ sufficiently close to P_n equation 4.1.27 takes the form

$$\psi_\lambda(\lambda^{(k)}, \lambda_0^{(i)}) = \left[\frac{r_n^{(k)}}{\lambda - \lambda_n} - \sum_{s=0}^{\infty} b_s(\lambda_0^{(i)}) x_n^{s+1-\mathbf{k}_n^{(k)}}(\lambda^{(k)}) + O(1)\right] \psi(\lambda^{(k)}, \lambda_0^{(i)}). \tag{4.1.31}$$

Now choose a subsequence $P_1 = \lambda_n^{(k_1)}, ..., P_L = \lambda_n^{(k_L)}$ of distinct points lying above λ_n of orders $\mathbf{k}_n^{(k_1)}, ..., \mathbf{k}_n^{(k_L)}$. Then for λ sufficiently close to λ_n equation 4.1.26 becomes

$$\left(\Psi_\lambda \Psi^{-1}\right)_{ij} \\ = \sum_{l=1}^{L} \left[\frac{\mathbf{k}_n^{(k_l)} r_n^{(k_l)}}{\lambda - \lambda_n} - \sum_{s=0}^{\infty} \left[\sum_{s=0}^{\mathbf{k}_n^{(k_l)}-1} \gamma^{k(s+1)}\right] b_s(\lambda_0^{(i)}) x_n^{s+1-\mathbf{k}_n^{(k)}}(\lambda^{(k)}) + O(1)\right] \tag{4.1.32}$$

where $\gamma = e^{2\pi i/\mathbf{k}_n^{(k_l)}}$. The sum over powers of γ is zero for $s = 0, ..., \mathbf{k}_n^{(k_l)} - 2$, so that

$$\left(\Psi_\lambda \Psi^{-1}\right)_{ij} = \sum_{l=1}^{L} \frac{\mathbf{k}_n^{(k_l)} r_n^{(k_l)}}{\lambda - \lambda_n} + O(1) = \sum_{j=1}^{N} \frac{r_n^{(j)}}{\lambda - \lambda_n} + O(1). \tag{4.1.33}$$

□

4.2 Solution of the Schlesinger system

With our solution of the Riemann-Hilbert problem in hand we turn to the Schlesinger system. Assuming that the monodromies $\mathcal{M}_1, ..., \mathcal{M}_M$ do not depend on $\lambda_1, ..., \lambda_M$, $\Psi(\lambda)$ will satisfy equation 2.2.2. Equivalently $A(\lambda)$ will satisfy the Schlesinger equations 2.2.3.

Theorem 4.2.1. *Let $\boldsymbol{p}, \boldsymbol{q}$ and $r_m^{(k)}$ be independent of $\{\lambda_m\}_{m=1}^M$, and let $\Psi(\lambda)$ be defined by 4.1.7. Then the coefficients*

$$A_n\left(\{\lambda_m\}_{m=1}^M\right) = \underset{\lambda=\lambda_n}{res}\{\Psi_\lambda \Psi^{-1}\} \tag{4.2.1}$$

satisfy the Schlesinger system 2.2.3 outside of the hyperplanes $\lambda_m = \lambda_n$ and the zero-divisor of the tau-function. The latter is a submanifold of codimension 1 in $\{\lambda_m\}$-space defined by the condition

$$\boldsymbol{Bp}+\boldsymbol{q}+\Omega \in (\Theta) \tag{4.2.2}$$

where (Θ) is the divisor of the theta function with characteristics $\boldsymbol{p}, \boldsymbol{q}$.

Proof. Let $\{\lambda_m, \mathcal{M}_m\}_{m=1}^M \notin (\Theta)$, and let $\Psi(\lambda)$ correspond to the particular choice of $\{\lambda_m\}_{m=1}^M$. Set $\lambda_0 = \infty$. Then $\Psi(\lambda)$ is invariant under an infinitesimal translation $\lambda \to \lambda + \epsilon$ of all $\lambda \in \mathbb{C}P^1$, since such a translation fixes $\lambda_0 = \infty$. Hence

$$0 = \partial_\epsilon \Psi(\lambda) = \partial_\lambda \Psi(\lambda) + \sum_{m=1}^M \partial_{\lambda_m} \Psi(\lambda) \tag{4.2.3}$$

Setting $\epsilon = 0$ gives $\Psi_\lambda + \Psi_{\lambda_1} + ... + \Psi_{\lambda_M} = 0$ and hence multiplying both sides of this equation by $\Psi^{-1}(\lambda)$ gives

$$A(\lambda) = \Psi_\lambda \Psi^{-1}(\lambda) = -\sum_{m=1}^M \Psi_{\lambda_m} \Psi^{-1}. \tag{4.2.4}$$

In other words the coefficients A_n in 4.2.1 satisfy

$$\frac{\partial \Psi}{\partial \lambda_n} = -\frac{A_n}{\lambda - \lambda_n}, \tag{4.2.5}$$

which is equation 2.2.1 for $\lambda_0 = \infty$. The result now extends to arbitrary choice of normalization point λ_0 since we are free to move λ_0 by a gauge-transformation of Ψ. Equivalently, the coefficients A_n satisfy the Schlesinger system 2.2.2.

Finally, the Schlesinger equations 2.2.3 are clearly non-singular so long as $\lambda_m \neq \lambda_n$ for all $m \neq n$, and so long as the factor $\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)$ in the denominator of 4.1.8 is non-zero. □

4.3 Tau-function for the Schlesinger System

We can now use our solution for $\Psi_{kj}(\lambda_0, \lambda)$ to explicitly write out the differential equation 2.2.10 satisfied by the tau-function. Since $\Psi(\lambda_0, \lambda)$ is independent of the normalization point, for fixed $\lambda \in \mathbb{C}P^1 \setminus \{\lambda_1, ..., \lambda_M\}$ we can take $\lambda_0 = \lambda$; then $\psi(\lambda^{(j)}, \lambda_0^{(k)}) = \delta_{jk}$ and

$$\left(\Psi_\lambda \Psi^{-1}\right)_{ij} = \lim_{\lambda_0 \to \lambda} \psi_\lambda(\lambda^{(j)}, \lambda_0^{(i)}). \tag{4.3.1}$$

To this end we prove the following lemma.

Lemma 4.3.1. *Fix $\lambda \in \mathbb{C}P^1$, and for each $\lambda^{(j)}$ projecting to λ let $\boldsymbol{k}^{(j)}$ be the number of sheets to which $\lambda^{(j)}$ belongs. Then for λ_0 sufficiently close to λ, the matrix elements $\Psi_{kj}(\lambda, \lambda_0)$ satisfy*

$$\begin{aligned} \Psi_{kj}(\lambda, \lambda_0) &= \frac{\lambda_0 - \lambda}{d\lambda} \hat{S}(\lambda^{(j)}, \lambda^{(k)}), \quad \lambda^{(j)} \neq \lambda^{(k)}, \\ \Psi_{kj}(\lambda, \lambda_0) &= \frac{1}{\boldsymbol{k}^{(j)}} + \frac{1}{\boldsymbol{k}^{(j)}} \frac{\lambda_0 - \lambda}{d\lambda} \left[W_1(\lambda^{(j)}) + W_2(\lambda^{(j)})\right], \quad \lambda^{(j)} = \lambda^{(k)}, \end{aligned} \tag{4.3.2}$$

where $W_1(P)$ is the holomorphic 1-form

$$W_1(P) = \frac{1}{\Theta[^{\boldsymbol{p}}_{\boldsymbol{q}}](\Omega)} \sum_{\alpha=1}^{g} \partial_{z_\alpha} \Theta[^{\boldsymbol{p}}_{\boldsymbol{q}}](\boldsymbol{z}) \Big|_{\boldsymbol{z}=\Omega} \omega_\alpha(P), \tag{4.3.3}$$

and $W_2(P)$ is the meromorphic 1-form

$$W_2(P) = \sum_{m=1}^{M} \sum_{j=1}^{N} r_m^{(j)} d_P \ln E(P, \lambda_m^{(j)}). \tag{4.3.4}$$

Proof. When $\lambda^{(j)} \neq \lambda^{(k)}$, $\hat{S}(\lambda^{(j)}, \lambda^{(k)})$ is non-singular so the first equation is an immediate consequence of the definitions. Next, let $\lambda^{(j)} = \lambda^{(k)}$ and define the local parameter $x(\lambda_0^{(j)}) = (\lambda_0 - \lambda)^{1/\mathbf{k}^{(j)}}$ near $\lambda_0^{(j)} = \lambda^{(j)}$. Equation 3.3.11 gives us

$$\begin{aligned} \lim_{\lambda_0 \to \lambda} \Psi_{kj}(\lambda, \lambda_0) &= \lim_{\lambda_0 \to \lambda} \left[\frac{1}{(\lambda_0 - \lambda)^{1/\mathbf{k}^{(j)}}} - a_0(\lambda^{(j)}) + O(x(\lambda_0^{(j)}))\right] \\ &\qquad \times \sqrt{dx(\lambda_0^{(j)}) dx(\lambda^{(j)})} \frac{\lambda_0 - \lambda}{\sqrt{d\lambda_0 d\lambda}}, \\ &= \lim_{\lambda_0 \to \lambda} \left[\frac{1}{(\lambda_0 - \lambda)^{1/\mathbf{k}^{(j)}}} - a_0(\lambda^{(j)}) + O(x(\lambda_0^{(j)}))\right] \frac{1}{\mathbf{k}^{(j)}} (\lambda_0 - \lambda)^{1/\mathbf{k}^{(j)}}, \\ \therefore \lim_{\lambda_0 \to \lambda} \Psi_{kj}(\lambda, \lambda_0) &= \frac{1}{\mathbf{k}^{(j)}}. \end{aligned} \tag{4.3.5}$$

Next, we have

$$\lim_{\lambda_0\to\lambda}\Psi_{kj}(\lambda,\lambda_0)=\lim_{\lambda_0\to\lambda}\left[\frac{dx(\lambda_0^{(j)})}{d\lambda_0}\frac{\partial}{\partial\lambda_0^{(j)}}\hat{S}(\lambda_0^{(j)},\lambda^{(j)})E_0(\lambda_0,\lambda)+\frac{\hat{S}(\lambda_0^{(j)},\lambda^{(j)})}{\sqrt{d\lambda_0 d\lambda}}\right]. \quad (4.3.6)$$

If Q is not a branch point, then

$$\begin{aligned}\partial_P\hat{S}(P,Q)=&\frac{\sum_{\alpha=1}^{g}\partial_{\mathbf{z}}\Theta[^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})|_{U(P)-U(Q)+\Omega}\omega_\alpha(P)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)E(P,Q)dx(P)}\prod_{\ell=1}^{M}\prod_{k=1}^{N}\left[\frac{E(P,\lambda_\ell^{(k)})}{E(Q,\lambda_\ell^{(k)})}\right]^{r_\ell^{(k)}},\\ &-\frac{\Theta[^{\mathbf{p}}_{\mathbf{q}}](U(P)-U(Q)+\Omega)\partial_P\ln E(P,Q)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)E(P,Q)}\prod_{m=1}^{M}\prod_{j=1}^{N}\left[\frac{E(P,\lambda_\ell^{(k)})}{E(Q,\lambda_\ell^{(k)})}\right]^{r_\ell^{(k)}},\\ &+\frac{\Theta[^{\mathbf{p}}_{\mathbf{q}}](U(P)-U(Q)+\Omega)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)E(P,Q)dx(P)}\sum_{m=1}^{M}\sum_{j=1}^{N}r_m^{(j)}d_P\ln E(P,\lambda_m^{(j)})\prod_{\ell=1}^{M}\prod_{k=1}^{N}\left[\frac{E(P,\lambda_\ell^{(k)})}{E(Q,\lambda_\ell^{(k)})}\right]^{r_\ell^{(k)}},\end{aligned} \quad (4.3.7)$$

and hence

$$\lim_{P\to Q}\partial_P\hat{S}(P,Q)=\lim_{P\to Q}\frac{1}{E(P,Q)dx(P)}\left[W_1(P)+W_2(P)-\frac{1}{E(P,Q)}\right]. \quad (4.3.8)$$

Otherwise, if $Q=\lambda_m^{(j)}$ is a branch point of ramification index $\mathbf{k}^{(j)}$,

$$\begin{aligned}\partial_P\hat{S}(P,Q)=&\frac{\sum_{\alpha=1}^{g}\partial_{\mathbf{z}}\Theta[^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})|_{U(P)-U(Q)+\Omega}\omega_\alpha(P)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)E(P,Q)dx(P)}\prod_{\ell=1}^{M}\prod_{k=1}^{N}\left[\frac{E(P,\lambda_\ell^{(k)})}{E(Q,\lambda_\ell^{(k)})}\right]^{r_\ell^{(k)}}\\ &+\frac{\Theta[^{\mathbf{p}}_{\mathbf{q}}](U(P)-U(Q)+\Omega)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)E(P,Q)dx(P)}\sum_{\lambda_n^{(i)}\neq\lambda_m^{(j)}}r_n^{(i)}d_P\ln E(P,\lambda_n^{(i)})\prod_{\ell=1}^{M}\prod_{k=1}^{N}\left[\frac{E(P,\lambda_\ell^{(k)})}{E(Q,\lambda_\ell^{(k)})}\right]^{r_\ell^{(k)}}\\ &+\frac{\Theta[^{\mathbf{p}}_{\mathbf{q}}](U(P)-U(Q)+\Omega)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)E(P,Q)dx(P)}\sum_{\lambda_n^{(i)}=\lambda_m^{(j)}}(r_m^{(j)}-1)d_P\ln E(P,\lambda_m^{(j)})\prod_{\ell=1}^{M}\prod_{k=1}^{N}\left[\frac{E(P,\lambda_\ell^{(k)})}{E(Q,\lambda_\ell^{(k)})}\right]^{r_\ell^{(k)}}.\end{aligned} \quad (4.3.9)$$

We can re-write the last two terms as

$$\begin{aligned}&\frac{\Theta[^{\mathbf{p}}_{\mathbf{q}}](U(P)-U(Q)+\Omega)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)E(P,Q)dx(P)}\sum_{n=1}^{M}\sum_{i=1}^{N}r_n^{(i)}d_P\ln E(P,\lambda_n^{(i)})\prod_{\ell=1}^{M}\prod_{k=1}^{N}\left[\frac{E(P,\lambda_\ell^{(k)})}{E(Q,\lambda_\ell^{(k)})}\right]^{r_\ell^{(k)}}\\ &\qquad-\frac{\Theta[^{\mathbf{p}}_{\mathbf{q}}](U(P)-U(Q)+\Omega)}{\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)E(P,Q)dx(P)}\mathbf{k}^{(j)}d_P\ln E(P,\lambda_m^{(j)})\prod_{\ell=1}^{M}\prod_{k=1}^{N}\left[\frac{E(P,\lambda_\ell^{(k)})}{E(Q,\lambda_\ell^{(k)})}\right]^{r_\ell^{(k)}}.\end{aligned} \quad (4.3.10)$$

Therefore

$$\lim_{P\to Q} \partial_P \hat{S}(P,Q) = \lim_{P\to Q} \frac{1}{E(P,Q)dx(P)} \left[W_1(P) + W_2(P) - \frac{\mathbf{k}^{(j)}}{E(P,Q)} \right], \tag{4.3.11}$$

which agrees with our equation for the case where Q has ramification index $\mathbf{k}^{(j)} = 1$. Returning to equation 4.3.5 we have

$$\begin{aligned} &\lim_{\lambda_0\to\lambda} \Psi_{kj}(\lambda,\lambda_0) \\ &= \lim_{\lambda_0\to\lambda} \frac{E_0(\lambda_0,\lambda)}{E(\lambda_0^{(j)},\lambda^{(k)})} \frac{1}{d\lambda_0} \left[W_1(\lambda_0^{(j)}) + W_2(\lambda_0^{(j)}) - \frac{\mathbf{k}^{(j)}}{E(\lambda_0^{(j)},\lambda^{(k)})} \right] + \lim_{\lambda_0\to\lambda} \frac{\hat{S}(\lambda_0^{(j)},\lambda^{(k)})}{\sqrt{d\lambda_0 d\lambda}}. \end{aligned} \tag{4.3.12}$$

Note that $\lim_{P\to Q} \hat{S}(P,Q) = \lim_{P\to Q} 1/E(P,Q)$ since all terms which differ go to 1 in the limit. Using equation 4.3.5 we therefore have

$$\lim_{\lambda_0\to\lambda} \frac{E_0(\lambda_0,\lambda)}{E(\lambda_0^{(j)},\lambda^{(k)})} = \lim_{\lambda_0\to\lambda} \Psi_{kj}(\lambda,\lambda_0) = \frac{1}{\mathbf{k}^{(j)}}. \tag{4.3.13}$$

Thus, finally,

$$\begin{aligned} \lim_{\lambda_0\to\lambda} \Psi_{kj}(\lambda,\lambda_0) = \lim_{\lambda_0\to\lambda} \frac{1}{\mathbf{k}^{(j)} d\lambda_0} \left[W_1(\lambda_0^{(j)}) + W_2(\lambda_0^{(j)}) - \mathbf{k}^{(j)} \hat{S}(\lambda_0^{(j)},\lambda^{(k)}) \right] \\ + \lim_{\lambda_0\to\lambda} \frac{\hat{S}(\lambda_0^{(j)},\lambda^{(k)})}{\sqrt{d\lambda_0 d\lambda}}. \end{aligned} \tag{4.3.14}$$

The terms involving $\hat{S}$ cancel, leaving

$$\lim_{\lambda_0\to\lambda} \Psi_{kj}(\lambda,\lambda_0) = \frac{1}{\mathbf{k}^{(j)} d\lambda} \left[W_1(\lambda^{(j)}) + W_2(\lambda^{(j)}) \right], \tag{4.3.15}$$

and the result follows. □

Using this lemma, we can now compute the right-hand side of 2.2.10. Recalling our requirement that λ_0 be an ordinary point, for λ_0 sufficiently close to an ordinary point λ we have

$$\begin{aligned} \partial_\lambda \Psi_{kj}(\lambda,\lambda_0) &= -\frac{1}{d\lambda} \hat{S}(\lambda^{(j)},\lambda^{(k)}), \quad j \neq k, \\ \partial_\lambda \Psi_{jj}(\lambda,\lambda_0) &= -\frac{1}{d\lambda} \left[W_1(\lambda^{(j)}) + W_2(\lambda^{(j)}) \right], \end{aligned} \tag{4.3.16}$$

hence equation 4.3.1 gives

$$\operatorname{tr}\left(\Psi_\lambda \Psi^{-1}\right)^2 (d\lambda)^2 = 2\sum_{j<k} \hat{S}(\lambda^{(j)}, \lambda^{(k)})\hat{S}(\lambda^{(k)}, \lambda^{(j)}) + \sum_{j=1}^{N}[W_1(\lambda^{(j)}) + W_2(\lambda^{(j)})]^2. \tag{4.3.17}$$

Since $W_1(P)$ is holomorphic we have $\sum_j W_1(\lambda^{(j)}) = 0$ and therefore

$$\begin{aligned}\sum_{j=1}^{N} W_1^2(\lambda^{(j)}) &= -2\sum_{j<k} W_1(\lambda^{(j)})W_1(\lambda^{(k)}),\\ &= -2\sum_{j<k}\sum_{\alpha,\beta=1}^{g} \partial_{\mathbf{z}_\alpha}\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=\Omega}\partial_{\mathbf{z}_\beta}\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=\Omega}\omega_\alpha(\lambda^{(j)})\omega_\beta(\lambda^{(k)}).\end{aligned} \tag{4.3.18}$$

Next, using Lemma 3.3.17,

$$\hat{S}(\lambda^{(j)}, \lambda^{(k)})\hat{S}(\lambda^{(k)}, \lambda^{(j)}) = -B(\lambda^{(j)}, \lambda^{(k)}) - \sum_{\alpha=1}^{g}\sum_{\beta=1}^{g} \partial_{\mathbf{z}_\alpha}\partial_{\mathbf{z}_\beta}\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=\Omega}\omega_\alpha(\lambda^{(j)})\omega_\beta(\lambda^{(k)}). \tag{4.3.19}$$

Therefore we can re-write equation 4.3.17 as

$$\begin{aligned}\frac{1}{2}\operatorname{tr}\left(\Psi_\lambda \Psi^{-1}\right)^2 (d\lambda)^2 &= -\sum_{j<k} B(\lambda^{(j)}, \lambda^{(k)})\\ &- \sum_{j<k}\sum_{\alpha=1}^{g}\sum_{\beta=1}^{g} \partial_{\mathbf{z}_\alpha}\partial_{\mathbf{z}_\beta}\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=\Omega}\omega_\alpha(\lambda^{(j)})\omega_\beta(\lambda^{(k)})\\ &- \sum_{j<k}\sum_{\alpha,\beta=1}^{g} \partial_{\mathbf{z}_\alpha}\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=\Omega}\partial_{\mathbf{z}_\beta}\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=\Omega}\omega_\alpha(\lambda^{(j)})\omega_\beta(\lambda^{(k)}) + \tfrac{1}{2}\textstyle\sum_{j=1}^{N} W_2^2(\lambda^{(j)})\\ &+ \sum_{j=1}^{N}\sum_{\alpha=1}^{g}\sum_{m=1}^{M}\sum_{\ell=1}^{N} \partial_{\mathbf{z}_\alpha}\ln\Theta[{}^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=\Omega}\omega_\alpha(\lambda^{(j)})r_m^{(\ell)} d_{x_m}\ln E(\lambda^{(j)}, \lambda_m^{(\ell)}).\end{aligned} \tag{4.3.20}$$

Now, using the Rauch formula for ω_α from equation 3.4.1, we have

$$\begin{aligned}\partial_{\lambda_m}\Omega &= \sum_{m=1}^{M}\sum_{\ell=1}^{N}\sum_{\alpha=1}^{g} r_m^{(\ell)}\left[\omega_\alpha(\lambda_m^{(\ell)}) + \operatorname*{res}_{\lambda=\lambda_m}\sum_{j=1}^{N}\int_{P_0}^{\lambda_m^{(\ell)}} \frac{1}{d\lambda}\omega_\alpha(\lambda^{(j)})d_{\lambda^{(j)}}d_{\lambda_m^{(\ell)}}E(\lambda^{(j)}, \lambda_m^{(\ell)})\right],\\ &= \sum_{m=1}^{M}\sum_{\ell=1}^{N}\sum_{\alpha=1}^{g}\sum_{j=1}^{N} r_m^{(\ell)}\operatorname*{res}_{\lambda=\lambda_m}\omega_\alpha(\lambda^{(j)})d_{x_m}\ln E(\lambda^{(j)}, \lambda_m^{(\ell)}),\end{aligned} \tag{4.3.21}$$

since $\sum_{\ell=1}^N \omega(\lambda_m^\ell) = 0$. Hence using the Rauch formula for $\mathbf{B}_{\alpha\alpha}$ from equation 3.4.1 and the heat equation 3.2.2, we have

$$\frac{\partial}{\partial \lambda_m}\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega)\big|\mathbf{B}) = -\operatorname{res}_{\lambda=\lambda_m} \tfrac{1}{d\lambda}\textstyle\sum_{\alpha=1}^g \sum_{\beta=1}^g \sum_{j<k} \partial_{\mathbf{z}_\alpha}\partial_{\mathbf{z}_\beta}\Theta[^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=\Omega}\omega_\alpha(\lambda^{(j)})\omega_\beta(\lambda^{(k)})$$
$$+ \operatorname*{res}_{\lambda=\lambda_m} \sum_{j=1}^N\sum_{\alpha=1}^g\sum_{m=1}^M\sum_{\ell=1}^N \partial_{\mathbf{z}_\alpha} \ln \Theta[^{\mathbf{p}}_{\mathbf{q}}](\mathbf{z})\big|_{\mathbf{z}=\Omega}\omega_\alpha(\lambda^{(j)})r_m^{(\ell)} d_{x_m} \ln E(\lambda^{(j)}, \lambda_m^{(\ell)}). \tag{4.3.22}$$

Therefore upon taking residues, we have

$$\begin{aligned}&\frac{1}{2}\operatorname*{res}_{\lambda=\lambda_m} \operatorname{tr}\left(\Psi_\lambda\Psi^{-1}\right)^2(d\lambda)^2\\ &= -\operatorname*{res}_{\lambda=\lambda_m}\sum_{j<k} B(\lambda^{(j)},\lambda^{(k)}) + \frac{1}{2}\operatorname*{res}_{\lambda=\lambda_m}\sum_{j=1}^N W_2^2(\lambda^{(j)}) + \partial_{\lambda_m}\Theta[^{\mathbf{p}}_{\mathbf{q}}](\Omega).\end{aligned} \tag{4.3.23}$$

We have thus shown the following.

Theorem 4.3.2. *The solution of the equation 4.3.23 is*

$$\tau_{JM} = \tau_{SW}\tau_B^{-1/2}\Theta[^{\boldsymbol{p}}_{\boldsymbol{q}}](\Omega), \tag{4.3.24}$$

where τ_B is the Bergman tau-function from equation 3.3.7, and τ_{SW} is the Seiberg-Witten tau-function.

4.4 Computation of Seiberg-Witten tau-function

We are finally ready to show that the Seiberg-Witten tau-function is given by the expression

$$\tau_{SW}^2 = \prod_{\lambda_m^{(i)} \neq \lambda_n^{(j)}} E(\lambda_m^{(i)}, \lambda_n^{(j)})^{r_m^{(i)} r_n^{(j)}}. \tag{4.4.1}$$

To this end we need to prove variational formulas for $E(P, Q)$, corresponding to the cases where neither, one or both of the points P and Q coincide with a branch point.

Lemma 4.4.1. *For fixed $\Pi(P), \Pi(Q) \in \mathbb{C}P^1$, we have*

$$\frac{\partial}{\partial\lambda_m}\ln(E(P,Q)\sqrt{dx(P)}\sqrt{dx(Q)}) = -\frac{1}{2}\sum_{j=1}^N \operatorname*{\mathit{res}}_{\lambda=\lambda_m}\frac{1}{d\lambda}\left[d_{x_m}\ln\frac{E(P,\lambda^{(j)})}{E(Q,\lambda^{(j)})}\right]^2. \tag{4.4.2}$$

Proof. We begin by taking $d_P d_Q$ of each side of the equation, leaving only those terms which depend on both P and Q, and showing equivalence. Using equation 3.3.6 for the Bergman kernel, the left-hand side becomes

$$\frac{\partial}{\partial\lambda_m} d_P d_Q \ln(E(P,Q)\sqrt{dx(P)}\sqrt{dx(Q)}) = \frac{\partial}{\partial\lambda_m} B(P,Q). \tag{4.4.3}$$

The right-hand side is

$$-\frac{1}{2}\sum_{j=1}^{N} \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} d_P d_Q \left[d_{x_m} \ln E(P,\lambda^{(j)}) - d_{x_m} \ln E(Q,\lambda^{(j)})\right]^2. \tag{4.4.4}$$

Only the cross-terms of the quadratic will survive differentiation with respect to both P and Q, therefore the right-hand side becomes

$$\sum_{j=1}^{N} \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} d_P d_{x_m} \ln E(P,\lambda^{(j)}) d_Q d_{x_m} \ln E(Q,\lambda^{(j)}) = \sum_{j=1}^{N} \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} B(P,\lambda^{(j)}) B(Q,\lambda^{(j)}). \tag{4.4.5}$$

The required equivalence now follows from equation 3.4.4. Therefore equation 4.4.2 is valid up to addition of a function of the form $f(P)+g(Q)$, holomorphic in P and Q. Note that for $P=Q$, both sides of equation 4.4.2 vanish, and so $f(P) = -g(P)$ for any fixed $P \in \mathcal{L}$.

Next, we compare the a- and b-periods of both sides of equation 4.4.2. The a-periods of both sides are zero since $E(P,Q)$ is single-valued under analytic continuation around a_α. Under analytic continuation around b_α, $E(P,Q)$ gains a factor $\exp(-\pi i \mathbf{B}_{\alpha\alpha} - 2\pi i(U(P) - U(Q)))$. Therefore the left-hand side has b_α period equal to $-\partial_{\lambda_m}(\pi i \mathbf{B}_{\alpha\alpha} + 2\pi i(U(P) - U(Q)))$.

We will show that the right-hand side of equation 4.4.2 has matching b_α periods. Under analytic continuation around b_α,

$$\begin{aligned} &\frac{1}{d\lambda}\left[d_{x_m} \ln \frac{E(P+b_\alpha,\lambda^{(j)})}{E(Q,\lambda^{(j)})}\right]^2 \\ &= \frac{1}{d\lambda}\left[-d_{x_m}\left(\pi i \mathbf{B}_{\alpha\alpha} + 2\pi i \int_{\lambda^{(j)}}^{P} \omega_\alpha\right) + d_{x_m} \ln \frac{E(P,\lambda^{(j)})}{E(Q,\lambda^{(j)})}\right]^2, \\ &= \frac{1}{d\lambda}\left[\pi i \frac{d\lambda_m}{dx_m} \partial_{\lambda_m} \mathbf{B}_{\alpha\alpha} dx_m + 2\pi i \omega_\alpha(\lambda^{(j)}) + \int_{P_0}^{P} B(P',\lambda^{(j)}) - \int_{Q_0}^{Q} B(Q',\lambda^{(j)})\right]^2, \end{aligned} \tag{4.4.6}$$

where we have used equation 3.3.6 for the Bergman kernel. Consider the first term in the brackets; using the Rauch formula for $\mathbf{B}_{\alpha\alpha}$,

$$\frac{d\lambda_m}{dx_m}\partial_{\lambda_m}\mathbf{B}_{\alpha\alpha} = \mathbf{k_m} x_m^{\mathbf{k}_m-1} \operatorname*{res}_{\lambda=\lambda_m} \frac{4\pi i}{d\lambda}\sum_{j<k}^{N}\omega_\alpha(\lambda^{(j)})\omega_\beta(\lambda^{(k)}). \tag{4.4.7}$$

This function is $O(1)$ at $\lambda^{(j)} = \lambda_m^{(j)}$, and so will not contribute any nonzero terms when we take the residue at $\lambda = \lambda_m$ of 4.4.6. Therefore

$$\begin{aligned}&\frac{1}{d\lambda}\left[d_{x_m}\ln\frac{E(P+b_\alpha,\lambda^{(j)})}{E(Q,\lambda^{(j)})}\right]^2 - \frac{1}{d\lambda}\left[d_{x_m}\ln\frac{E(P,\lambda^{(j)})}{E(Q,\lambda^{(j)})}\right]^2 \\ &= 2\pi i\frac{1}{d\lambda}\left[2\pi i\omega_\alpha^2(\lambda^{(j)}) + 2\int_{P_0}^{P}\omega_\alpha(\lambda^{(j)})B(P',\lambda^{(j)}) - 2\int_{Q_0}^{Q}\omega_\alpha(\lambda^{(j)})B(Q',\lambda^{(j)})\right].\end{aligned} \tag{4.4.8}$$

Multiplying by $-1/2$, summing over $j = 1, ..., N$, and taking residues at $\lambda = \lambda_m$, we arrive at the b_α-period of the right-hand side of equation 4.4.2:

$$\begin{aligned}&-\operatorname*{res}_{\lambda=\lambda_m}\frac{2\pi i}{d\lambda}\sum_{j=1}^{N}\left[\pi i\omega_\alpha^2(\lambda^{(j)}) + \int_{P_0}^{P}\omega_\alpha(\lambda^{(j)})B(P',\lambda^{(j)}) - \int_{Q_0}^{Q}\omega_\alpha(\lambda^{(j)})B(Q',\lambda^{(j)})\right], \\ &= -\pi i\partial_{\lambda_m}\mathbf{B}_{\alpha\alpha} - 2\pi i\left[\int_{P_0}^{P}\partial_{\lambda_m}\omega_\alpha(\lambda^{(j)}) - \int_{Q_0}^{Q}\partial_{\lambda_m}\omega_\alpha(\lambda^{(j)})\right], \\ &= -\partial_{\lambda_m}\left(\pi i\mathbf{B}_{\alpha\alpha} + 2\pi i(U_\alpha(P) - U_\alpha(Q))\right),\end{aligned} \tag{4.4.9}$$

where we have used the Rauch formula 3.4.1 for $\partial_{\lambda_m}\boldsymbol{B}_{\alpha\alpha}$. Therefore $f(P)$ must be single-valued as well as holomorphic, which is only possible if $f(P) \equiv$ constant $\Rightarrow$ $f(P) - f(Q) \equiv 0$. □

Lemma 4.4.2. *For fixed $\Pi(P) \in \mathbb{C}P^1$ and P_m a point projecting to λ_m,*

$$\frac{\partial}{\partial\lambda_m}\ln\left(E(P,P_m)\sqrt{dx(P)}\sqrt{dx_m(P_m)}\right) = -\frac{1}{2}\operatorname*{\mathit{res}}_{\lambda=\lambda_m}\frac{1}{d\lambda}\sum_{j=1}^{N}\left[d_{x_m}\ln\frac{E(P,\lambda^{(j)})}{E(P_m,\lambda^{(j)})}\right]^2. \tag{4.4.10}$$

Proof. We start by fixing Q in a neighborhood of P_m, before eventually setting $Q = P_m$. Define the local coordinates $x(P) = \Pi(P)$ and $x(Q) = \Pi(Q)$, and the local parameter $x_m(Q) = (\lambda - \lambda_m)^{1/\mathbf{k}_m}$ where $\lambda = \Pi(Q)$. Then $dx_m(Q) = \mathbf{k}_m^{-1}x_m^{1-\mathbf{k}_m}dx(Q)$

and $dx(Q) = d\lambda$. Note that P_m may or may not be a branch point; if P_m is an ordinary point, $\mathbf{k}_m = 1$ and $dx_m(Q) = dx(Q)$. Now

$$\begin{aligned}
&\ln\left(E(P,Q)\sqrt{dx(P)}\sqrt{dx_m(Q)}\right) \\
&= \ln\left(E(P,Q)\sqrt{dx(Q)}\sqrt{dx(P)}\right) + \frac{1}{2}\ln\left(\frac{x_m^{1-\mathbf{k}_m}(Q)}{\mathbf{k}_m}\right).
\end{aligned} \tag{4.4.11}$$

Using equation 4.4.2 we have

$$\begin{aligned}
&\partial_{\lambda_m}\ln\left(E(P,Q)\sqrt{dx(P)}\sqrt{dx_m(Q)}\right) \\
&= -\frac{1}{2}\operatorname*{res}_{\lambda=\lambda_m}\frac{1}{d\lambda}\sum_{j=1}^{N}\left[d_{x_m}\ln\frac{E(P,\lambda^{(j)})}{E(Q,\lambda^{(j)})}\right]^2 + \frac{1-\mathbf{k}_m}{2}\partial_{\lambda_m}\ln\left(x_m(Q)\right), \\
&= -\frac{1}{4\pi i}\int_{\gamma}\frac{1}{d\lambda}\left[d_{x_m}\ln\frac{E(P,\lambda^{(j)})}{E(Q,\lambda^{(j)})}\right]^2 + \frac{1}{2}\frac{\mathbf{k}_m-1}{\mathbf{k}_m}x_m^{-\mathbf{k}_m}(Q).
\end{aligned} \tag{4.4.12}$$

where γ is a closed loop containing P_m as an interior point, but not Q or P. If we set $Q = P_m$ the second term will either be zero or infinite. However, as we will see, the integral gains an additional term that exactly cancels it.

In order to take the limit $Q \to P_m$, we must integrate around a loop $\tilde{\gamma}$ which contains both γ and Q. By Cauchy's theorem,

$$\begin{aligned}
&-\frac{1}{4\pi i}\int_{\tilde{\gamma}}\frac{1}{d\lambda}\left[d_{x_m}\ln\frac{E(P,\lambda^{(j)})}{E(Q,\lambda^{(j)})}\right]^2 \\
&= -\frac{1}{2}\operatorname*{res}_{\lambda=\lambda_m}\frac{1}{d\lambda}\sum_{j=1}^{N}\left[d_{x_m}\ln\frac{E(P,\lambda^{(j)})}{E(Q,\lambda^{(j)})}\right]^2 - \frac{1}{2}\operatorname*{res}_{R=Q}\frac{1}{d\lambda}\left[d_R\ln\frac{E(P,R)}{E(Q,R)}\right]^2.
\end{aligned} \tag{4.4.13}$$

In the second term only the cross-terms of the quadratic will have non-zero residues, since near $R = Q$ we have

$$\partial_R\ln\left(E(Q,R)\sqrt{dx(Q)}\sqrt{dx(R)}\right) = \frac{1}{x(Q)-x(R)} + O\left((x(Q)-x(R))^2\right). \tag{4.4.14}$$

Hence

$$
\begin{aligned}
&-\frac{1}{2}\operatorname*{res}_{R=Q}\frac{1}{d\lambda}\left[d_R\ln\frac{E(P,R)}{E(Q,R)}\right]^2, \\
&=\operatorname*{res}_{R=Q}\frac{1}{d\lambda}d_R\ln\left(E(P,R)\sqrt{dx(P)}\sqrt{dx(R)}\right)d_R\ln\left(E(Q,R)\sqrt{dx(Q)}\sqrt{dx(R)}\right), \\
&=-\frac{1}{d\lambda}d_Q\ln\left(E(P,Q)\sqrt{dx(P)}\sqrt{dx(Q)}\right), \\
&=-\frac{dx_m}{d\lambda}\partial_Q\ln\left(E(P,Q)\sqrt{dx(P)}\sqrt{dx_m(Q)}\right)-\frac{\mathbf{k}_m-1}{2}\frac{dx_m}{d\lambda}\frac{d}{dx_m}\ln(x_m(Q)), \\
&=-\frac{dx_m}{d\lambda}\partial_{x_m}\ln\left(E(P,Q)\sqrt{dx(P)}\sqrt{dx_m(Q)}\right)-\frac{1}{2}\frac{\mathbf{k}_m-1}{\mathbf{k}_m}x_m^{-\mathbf{k}_m}(Q).
\end{aligned}
\tag{4.4.15}
$$

Next, we set $Q=P_m$, noting that

$$
\begin{aligned}
\partial_{\lambda_m}\ln\left(E(P,P_m)\sqrt{dx(P)}\sqrt{dx_m(P_m)}\right)=&\left[\partial_{\lambda_m}\ln\left(E(P,Q)\sqrt{dx(P)}\sqrt{dx_m(Q)}\right)\right]_{Q=P_m} \\
&+\frac{dx_m}{d\lambda_m}\partial_{x_m}\ln\left(E(P,P_m)\sqrt{dx(P)}\sqrt{dx_m(P_m)}\right).
\end{aligned}
\tag{4.4.16}
$$

Now taking $Q=P_m$ in equation 4.4.12, the integrals over γ and $\tilde{\gamma}$ are equal and we have

$$
\begin{aligned}
\left[\partial_{\lambda_m}\ln\left(E(P,Q)\sqrt{dx(P)}\sqrt{dx_m(Q)}\right)\right]_{Q=P_m}=&-\frac{1}{2}\operatorname*{res}_{\lambda=\lambda_m}\frac{1}{d\lambda}\sum_{j=1}^{N}\left[d_{x_m}\ln\frac{E(P,\lambda^{(j)})}{E(P_m,\lambda^{(j)})}\right]^2 \\
&-\frac{dx_m}{d\lambda_m}\partial_{x_m}\ln\left(E(P,P_m)\sqrt{dx(P)}\sqrt{dx_m(P_m)}\right).
\end{aligned}
\tag{4.4.17}
$$

Substituting this expression into equation 4.4.16 gives the required equality. □

For the case where $P=P_n$ and $Q=P_m$ for distinct branch points P_n,P_m projecting to λ_m, the proof is analogous. In place of the singular term in equation 4.4.12 we obtain two singular terms, while in equation 4.4.13 the contour integral will yield residues at both $R=P$ and $R=Q$ which ultimately cancel the two singular terms.

Suppose $P=P_\ell$ and $Q=P_n$ are distinct branch points not projecting to λ_m. Then as long as $\Pi(P_\ell),\Pi(P_n)$ are held fixed,

$$
\partial_{\lambda_m}\ln\left(E(P,Q)\sqrt{dx_\ell(P)}\sqrt{dx_n(Q)}\right)=\partial_{\lambda_m}\ln\left(E(P,Q)\sqrt{dx(P)}\sqrt{dx(Q)}\right), \tag{4.4.18}
$$

since the additional terms will disappear when we differentiate with respect to λ_m. Therefore we can set $P = P_\ell$, $Q = P_n$ in Lemma 4.4.1, and it does not matter if we use the local parameters x_n, x_ℓ or the local coordinates given by $x(P) = \Pi(P)$.

We finally come to the main result of this thesis.

Theorem 4.4.3. *The solution of the system of equations*

$$\frac{\partial \ln \tau_{SW}}{\partial \lambda_m} = \frac{1}{2} \underset{\lambda=\lambda_m}{res} \sum_{j=1}^{N} \frac{W_2^2(\lambda^{(j)})}{d\lambda}, \tag{4.4.19}$$

where

$$W_2(P) = \sum_{m=1}^{M} \sum_{j=1}^{N} r_m^{(j)} d_P \ln E(P, \lambda_m^{(j)}), \tag{4.4.20}$$

is given by

$$\tau_{SW}^2 = \prod_{\lambda_m^{(i)} \neq \lambda_n^{(j)}} E(\lambda_m^{(i)}, \lambda_n^{(j)})^{r_m^{(i)} r_n^{(j)}}. \tag{4.4.21}$$

Proof. Using equation 4.4.21,

$$\frac{\partial \ln \tau_{SW}}{\partial \lambda_m} = \frac{1}{2} \sum_{\lambda_\ell^{(i)} \neq \lambda_n^{(j)}} r_\ell^{(i)} r_n^{(j)} \partial_{\lambda_m} \ln E(\lambda_\ell^{(i)}, \lambda_n^{(j)}). \tag{4.4.22}$$

Therefore substituting the variational formula from lemmas 4.4.1 and 4.4.2 and expanding the quadratic we have

$$\begin{aligned} \frac{\partial \ln \tau_{SW}}{\partial \lambda_m} &= \frac{1}{2} \sum_{\lambda_\ell^{(i)} \neq \lambda_n^{(j)}} r_\ell^{(i)} r_n^{(j)} \underset{\lambda=\lambda_m}{\text{res}} \frac{1}{d\lambda} \sum_{k=1}^{N} d_{x_\ell} \ln E(\lambda_\ell^{(i)}, \lambda^{(k)}) d_{x_n} \ln E(\lambda_n^{(j)}, \lambda^{(k)}) \\ &- \frac{1}{4} \sum_{\lambda_\ell^{(i)} \neq \lambda_n^{(j)}} r_\ell^{(i)} r_n^{(j)} \underset{\lambda=\lambda_m}{\text{res}} \frac{1}{d\lambda} \sum_{k=1}^{N} \left[\left(d_{x_\ell} \ln E(\lambda_\ell^{(i)}, \lambda^{(k)}) \right)^2 + \left(d_{x_n} \ln E(\lambda_n^{(j)}, \lambda^{(k)}) \right)^2 \right]. \end{aligned} \tag{4.4.23}$$

Note that since $\sum_n \sum_j r_n^{(j)} = 0$, we have

$$\sum_{\lambda_\ell^{(i)} \neq \lambda_n^{(j)}} r_n^{(j)} = - \sum_{\lambda_\ell^{(i)} = \lambda_n^{(j)}} r_n^{(j)}, \tag{4.4.24}$$

and hence we can re-write the second set of terms:

$$\begin{aligned}
&\sum_{\lambda_\ell^{(i)}\neq\lambda_n^{(j)}} r_\ell^{(i)} r_n^{(j)} \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{k=1}^{N} \left(d_{x_n} \ln E(\lambda_n^{(j)}, \lambda^{(k)})\right)^2 \\
&= \sum_{n,j} \left[\sum_{\substack{i,\ell \text{ s.t.} \\ \lambda_\ell^{(i)}\neq\lambda_n^{(j)}}} r_\ell^{(i)} \right] r_n^{(j)} \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{k=1}^{N} \left(d_{x_n} \ln E(\lambda_n^{(j)}, \lambda^{(k)})\right)^2, \\
&= -\sum_{n,j} \left[\sum_{\substack{i,\ell \text{ s.t.} \\ \lambda_\ell^{(i)}=\lambda_n^{(j)}}} r_\ell^{(i)} \right] r_n^{(j)} \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{k=1}^{N} \left(d_{x_n} \ln E(\lambda_n^{(j)}, \lambda^{(k)})\right)^2, \\
&= -\sum_{\lambda_\ell^{(i)}=\lambda_n^{(j)}} r_\ell^{(i)} r_n^{(j)} \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{k=1}^{N} \left(d_{x_n} \ln E(\lambda_n^{(j)}, \lambda^{(k)})\right)^2.
\end{aligned} \tag{4.4.25}$$

Thus, finally,

$$\begin{aligned}
\frac{\partial \ln \tau_{SW}}{\partial \lambda_m} &= \frac{1}{2} \sum_{\lambda_\ell^{(i)}\neq\lambda_n^{(j)}} r_\ell^{(i)} r_n^{(j)} \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{k=1}^{N} d_{x_\ell} \ln E(\lambda_\ell^{(i)}, \lambda^{(k)}) d_{x_n} \ln E(\lambda_n^{(j)}, \lambda^{(k)}) \\
&\quad + \frac{1}{2} \sum_{\lambda_\ell^{(i)}=\lambda_n^{(j)}} (r_n^{(j)})^2 \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{k=1}^{N} \left(d_{x_n} \ln E(\lambda_n^{(j)}, \lambda^{(k)})\right)^2, \\
&= \frac{1}{2} \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{k=1}^{N} \sum_{m,n=1}^{M} \sum_{i,j=1}^{N} r_\ell^{(i)} r_n^{(j)} d_{x_\ell} \ln E(\lambda_\ell^{(i)}, \lambda^{(k)}) d_{x_n} \ln E(\lambda_n^{(j)}, \lambda^{(k)}), \\
\therefore \frac{\partial \ln \tau_{SW}}{\partial \lambda_m} &= \frac{1}{2} \operatorname*{res}_{\lambda=\lambda_m} \frac{1}{d\lambda} \sum_{k=1}^{N} \left[\sum_{n=1}^{M} \sum_{j=1}^{N} r_n^{(j)} d_{x_n} \ln E(\lambda_n^{(j)}, \lambda^{(k)}) \right]^2.
\end{aligned} \tag{4.4.26}$$

□

Chapter 5

Conclusion

The main result of this thesis was the proof that the Seiberg-Witten tau-function associated to the $N \times N$ Riemann-Hilbert problem with arbitrary quasi-permutation monodromy data takes the form

$$\tau_{SW}^2 = \prod_{\lambda_m^{(i)} \neq \lambda_n^{(j)}} E(\lambda_m^{(i)}, \lambda_n^{(j)})^{r_m^{(i)} r_n^{(j)}}. \tag{5.0.1}$$

To this end we provided elementary proofs of the Rauch formulas for the normalized holomorphic differentials and period matrix, as well as variational formulas for the Bergman kernel and prime-form.

The Seiberg-Witten tau-function can be interpreted as a conformal block of a conformal field theory with W_N symmetry. Equation 5.0.1 was first derived in this context in [5]. This link is only one example of a correspondence between isomonodromic tau functions and conformal blocks of CFTs currently under investigation.

Other examples include the Jimbo-Miwa tau-function; the $N \times N$ Riemann-Hilbert problem with quasi-permutation monodromies is expressible in terms of correlation functions of a conformal field theory describing N free massless chiral fermions, leading to a similar representation for the Jimbo-Miwa tau-function [4].

To give another example, more recently the solution of the Riemann-Hilbert problem for $SL(2, \mathbb{C})$ monodromies was found to be given by linear combinations of conformal blocks of Liouville theory for $c = 1$. As a result the corresponding isomonodromic tau-function was written as a linear combination of conformal blocks, and the Painleve VI tau-function was shown to act as generating function for $c = 1$ conformal blocks of Liouville theory [8][4].

Bibliography

[1] M. Bertola. *Riemann Surfaces and Theta Functions.* URL: `http://www.mathstat.concordia.ca/faculty/bertola/ThetaCourse/ThetaCourse.pdf`.

[2] P. Boalch. "Symplectic manifolds and isomonodromic deformations". In: *Advances in Mathematics* 163.2 (2001), pp. 137–205.

[3] J.D. Fay. *Theta Functions on Riemann Surfaces.* Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2006. ISBN: 9783540378150.

[4] O. Gamayun, N. Iorgov, and O. Lisovyy. "Conformal field theory of Painlevé VI". In: *Journal of High Energy Physics* 2012.10 (2012), p. 38.

[5] P. Gavrylenko and A. Marshakov. "Exact conformal blocks for the W-algebras, twist fields and isomonodromic deformations". In: *Journal of High Energy Physics* 2016.2 (2016), p. 181.

[6] B. Gilligan and O. Forster. *Lectures on Riemann Surfaces.* Graduate Texts in Mathematics. Springer New York, 2012. ISBN: 9781461259619.

[7] P. Griffiths and J. Harris. "Riemann Surfaces and Algebraic Curves". In: *Principles of Algebraic Geometry.* John Wiley & Sons, Ltd, 2011. Chap. 2, pp. 212–363. ISBN: 9781118032527.

[8] N. Iorgov, O. Lisovyy, and J. Teschner. "Isomonodromic tau-functions from Liouville conformal blocks". In: *Communications in Mathematical Physics* 336.2 (2015), pp. 671–694.

[9] M. Jimbo, T. Miwa, and K. Ueno. "Monodromy preserving deformation of linear ordinary differential equations with rational coefficients: I. General theory and τ-function". In: *Physica D: Nonlinear Phenomena* 2.2 (1981), pp. 306–352.

[10] D. Korotkin. "Bergman tau function: from Einstein equations and Dubrovin-Frobenius manifolds to geometry of moduli spaces". In: *arXiv preprint arXiv:1812.03514* (2018).

[11] D. Korotkin. "Matrix Riemann-Hilbert problems related to branched coverings of $\mathbb{C}P^1$". In: *arXiv preprint math-ph/0106009* (2001).

[12] D. Korotkin. "Solution of matrix Riemann-Hilbert problems with quasi-permutation monodromy matrices". In: *arXiv e-prints*, math-ph/0306061 (June 2003), math–ph/0306061. arXiv: `math-ph/0306061 [math-ph]`.

[13] G. Mahoux. "Introduction to the theory of isomonodromic deformations of linear ordinary differential equations with rational coefficients". In: *The Painlevé Property*. Springer, 1999, pp. 35–76.

[14] T. Miwa. "Painlevé property of monodromy preserving deformation equations and the analyticity of τ functions". In: *Publications of the Research Institute for Mathematical Sciences* 17.2 (1981), pp. 703–721.

[15] J. Reignier. "Singularities of Ordinary Linear Differential Equations and Integrability". In: *The Painlevé Property*. Springer, 1999, pp. 1–33.

[16] D. Yamakawa. "Tau functions and Hamiltonians of isomonodromic deformations". In: *Josai Mathematical Monographs* 10 (2017). ISSN: 13447777.