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SCALAR TRANSPORT
IN
TURBULENT SHEAR FLOWS

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Lecture 1. General principles of turbulent diffusion in shear flows.

1.1 Basic definitions and notation

Turbulent diffusion can be defined as the study of how a fluid in turbulent motion transports foreign substances that it contains. There are many examples, including smoke, acid rain, and other pollution in the atmosphere, salt in the sea and estuaries, and hot water in factory cooling systems. The foreign substance may have properties (especially its density and overall volume) that affect the motion of the ambient fluid, but it is often the case that the contaminant (as the foreign substance will be called in these lectures) is passive, i.e. the motion of the ambient fluid is the same as it would be in the absence of the contaminant. In practice nearly all cases of atmospheric dispersion fall into this category, as do many industrial applications. On the other hand the presence of dissolved salt has a profound effect on the behaviour of estuaries. These lectures will deal primarily with passive contaminants, although much of what is said applies qualitatively or, sometimes with minor modifications, even quantitatively to non-passive diffusion.

Host fluid flows are, in practice, incompressible, i.e. the density of each fluid element is invariant during the motion (or can be regarded as invariant for all practical purposes). Variations of density from fluid element to fluid element are, of course, quite consistent with incompressible behaviour (e.g. in the atmosphere), but they will not be relevant in these lectures. The ambient fluid will therefore be taken as having constant uniform density p. The velocity field at position x and time t in the ambient fluid will be denoted by T(x,t), where (by incompressibility)

$$\nabla . T = 0 , \qquad (1.1)$$

and T is the appropriate solution of the Navier-Stokes equations. Since the flow is turbulent, T is a random field, i.e. it is unpredictable and has to be described in statistical terms.

Attention will be restricted to contaminants that undergo only two processes, viz. advection by the ambient fluid, and molecular diffusion. In particular chemical change will be assumed not to occur. All contaminants considered will be such that they can be fully described by a single scalar field T(x,t). Usually *T* will be the concentration (in units of kg m^{-3}), but this term is, at first sight, inappropriate when the "contaminant" is heat and T will then be the temperature difference between the dispersing substance and the ambient. (In the latter case, however, it will be noted that changes in Γ due to sources such as viscous heating or radiative heat flux are explicitly excluded.) For clarity Γ will always be referred to *as* the concentration and it is, of course, inherently non-negative. Mass conservation requires Γ to satisfy

$$\frac{\partial\Gamma}{\partial t} + (T.\nabla)\Gamma = \nabla.(k\nabla\Gamma). \qquad (1.2)$$

where k(normally assumed constant) is the molecular diffusivity. Equation (1.2) does not require the contaminant to be passive, but that restriction ensures that T is independent of T and, therefore, that (1.2) is linear in Γ. since T is a random field, so also is Γ.

Initial and boundary conditions must be applied to (1.2) and these will, of course, be determined from the specific problem. However the emphasis in this short course of lectures will be on fundamental principles and it is then most convenient to assume no contaminant is lost at the boundaries where, therefore,

$$n.\nabla\,\Gamma = 0\ , \qquad (1.3a)$$

where n is the normal to the boundary. In directions where the ambient fluid extends to infinity 1.3a) has to be replaced by

$$\Gamma \to 0 \text{ as } |x| \to \infty \qquad (1.3b)$$

In situations in which the total quantity of contaminant is finite, and equal to Q, say, it is convenient (and possible) to choose the time t = 0 so that $\Gamma(x,0)$ is known. Then it follows from (1.1), (1.2) and (1.3) that

$$\frac{d}{dt}\left\{\int \Gamma(x,\, t)d\nabla(x)\right\} = 0 \to \int \Gamma(x,\, t)d\nabla(x) = Q, \qquad (1.4)$$

where the integrals are over the whole space occupied by the ambient fluid. For conciseness, situations in which (1.4) holds will be referred to as the dispersion of a cloud. An appropriate model for many practical problems (and laboratory experiments) is one in which a source of contaminant, e.g. a factory chimney, can be regarded as emitting continuously, i.e. from t = -∞ onwards. The total quantity of contaminant in such a model is not finite, i.e. (1.4) does not hold. In addition to the source geometry it is then necessary to prescribe the emission characteristics, including the emission rate (which could be any function of t but can often be taken to be constant). Such a situation will be referred to as the dispersion of a plune.

1.2 The statistical description of turbulent diffusion

Given that $\Gamma(x,t)$ is a random variable, the concepts and terminology of statistics must be used in any respectable study of turbulent diffusion. There is insufficient time in these lectures to discuss this point in full, but reference can be made to Monin and Yaglom (1971, pp.6-9, 205-256, 579-591; 1975, pp.743-763) for a substantial treatment. Here it is appropriate only to make some fundamental remarks.

A. central concept is that of the ensemble, which is the set of realisations of the dispersion phenomenon to which the statistical description applies. For example, one could consider the ensemble of all possible releases of a cloud of given contaminant in a given laboratory wind tunnel operating under prescribed conditions, with the initial position, size and shape of the cloud fixed. Or one could consider all possible emissions from a given factory chimney. In the latter case one could consider the ensemble in which there is no restriction on atmospheric conditions or on the time (of the day or year), or one could consider the ensemble - a subensemble of the first - in which emission takes place during the day under given atmospheric conditions (e.g. stability class and mean wind direction). It will be clear from these examples that the choice of ensemble requires prior specification of the causes of variability that are of concern in the investigation, and these causes will differ from investigation to investigation. From the point of view of the statistical and mathematical description of turbulent diffusion the choice of ensemble is (very largely) arbitrary; from the practical point of view what is essential is that the ensemble be defined with sufficient precision for it to be clear whether a particular realisation belongs to the ensemble, or not. Further discussion of the ensemble concept is given by Chatwin (1982).

For a given ensemble, consider the values of $\Gamma(x,t)$ at n arbitrary points in space-time. This set of n values has a probability distribution that depends both on the ensemble and on the n points. To determine fully all properties of the ensemble requires full knowledge of the family of such probability distributions for all n and for all possible choices of the n points (Monin and Yaglom 1971, p.213). An equivalent statement is that full knowledge of the characteristic functional is required (Monin and Yaglom 1971, pp.218-222; Monin and Yaglom 1975, pp.743-812).

I think it will be unsurprising that the technical problems associated with obtaining such knowledge are severe, and perhaps insoluble, even for very low values of n. Host research has concentrated on the case with n = 1, and so will these lectures. For a given ensemble, define $P(\theta;x,t)$ by

$$P(\theta;x,t) = \mathrm{prob}\{\Gamma(x,t) \leq \theta\} \ . \tag{1.5}$$

Clearly $P(\theta;x,t) = 0$ for all $\theta < 0$ since Γ is non-negative. The probability density function, to be abbreviated henceforth to p.d.f., $p(\theta;x,t)$ is defined by

$$p(\theta;x,t) = \frac{d}{d\theta} p(\theta;x,t) \tag{1.6}$$

where it may be necessary for the derivative to be defined in a generalized sense. The p.d.f. has the significance that, neglecting powers of $\delta\theta$ greater than 1, $p(\theta;x,t)\delta\theta$ is the probability that $\theta \leq \Gamma(x,t) < \theta + \delta\theta$, hence

$$\int_0^\infty p(\theta;x,t)\,d\theta = 1. \tag{1.7}$$

(In practice there will of course always be a maximum possible concentration $\theta_{\max}$ determined by the ensemble, and $p(\theta;x,t)$ will be identically zero for $\theta > \theta_{\max}$.) Figure 1 shows some experimental estimates of the p.d.f., and illustrates two important points: (i) as indicated by the notation, $p(\theta;x,t)$ does, in general, depend significantly on position x and - though not for the experiments in Figure 1 - on time t; (ii) in general $p(\theta;x,t)$ does not have a "simple" (e.g. as in a Normal or exponential p.d.f.) dependence on θ and may, in particular, have more than one peak (mode).

Let $f[\Gamma(x,t)]$ be any function of $\Gamma(x,t)$. The expectation {or expected value or probability mean or ensemble mean) of $f\,[\Gamma(x,\,t)]$ will be denoted by $E\{f[\Gamma(x,t)]\}$ and is defined by

$$E\{f \mid \Gamma(x,t) \mid\} = \int_0^\infty f(\theta(\theta)p(x,t)d\theta . \tag{1.8}$$

Three special choices of f will be of particular importance in these lectures. The first is $f[\Gamma(x,t)J = \delta[\Gamma(x,t)-\theta)$, where δ denotes the Dirac delta function, and (1.8) then gives the identity

$$P(\theta;x,t)=E\{\delta[\Gamma(x,t)-\theta)\} \tag{1.9}$$

The second choice is $f[\Gamma(x,t)] = \Gamma(x,t)$. It would be standard in statistics to use the notation $\mu(x,t)$ for $E\{\Gamma(x,t)\}$ but, unfortunately, this practice is rare in research papers on turbulence and turbulent diffusion. In these lectures the notation used will be $C(x,t)$. Thus

$$c(x,t) = E\{\Gamma(x,t) = \int_0^\infty \theta p(\theta;x,t)d\theta . \tag{1.10}$$

and $C(x, t)$ will be referred to as the ensemble mean concentration or, more commonly, simply as the mean concentration. Finally there is the case when $f[\Gamma(x,t)] = [\Gamma(x,t) - C(x,t)]^2$, and the expected value is then the variance of $\Gamma(x,t)$, for which the usual statistical notation is $\sigma^2(x,t)$, with σ being the standard deviation. In these lectures I will use this notation, although I recognise that this is a revolutionary step in turbulence circles! Thus

$$\sigma^2(x,t) = E\{\Gamma(x,t)^2\} = \int_0^\infty [\theta - C(x,t)]^2 p(\theta(x,t)d\theta) \tag{1.11}$$

Use of (1.10) shows that $o^2(x,t) = E\{\Gamma^2(x,t)\} - C^2(x,t)$.

From (1.9) it follows that

$$\frac{\partial p}{\partial t} = E\left\{\frac{\partial \Gamma}{\partial t}\delta'|\,\Gamma(x,t) - \theta)\right\} \tag{1.12}$$

where δ' denotes the first derivative of the delta function {to be interpreted in a generalized sense). Use of (1.2) in (1.12) then gives

$$\frac{\partial p}{\partial t} = -E\left\{T_j\frac{\partial \Gamma}{\partial x_j}\delta'[\Gamma(x,t)-\theta]\right\} + E\left\{k\frac{\partial}{\partial x_j}\left[\frac{\partial \Gamma}{\partial x_j}\right]\delta'[\Gamma(x,t)-\theta]\right\} \tag{1.13}$$

where the summation convention has been used and κ has been assumed constant. Thus

$$\frac{\partial p}{\partial t} + E\left\{T_j\frac{\partial \Gamma}{\partial x_j}\delta'[\Gamma(x,t)-\theta]\right\} = k\nabla^2 p - kE\{\delta''[\Gamma(x,t)-\theta](\nabla\Gamma)^2\}$$

$$= k\nabla^2 p - k\frac{\partial^2}{\partial\theta^2}E\{(\nabla\Gamma)^2\delta[\Gamma(x,t)-\theta]\} \tag{1.14}$$

Equation (1.14) is more usually seen in another form. Suppose $U(x,t) = E(T(x,t)\}$ is the mean velocity field (defined by an equation like (1.10) but with the p.d.f. for the velocity rather than the concentration), and write

$$T(x,t) - U(x,t) + u(x,t) \ , \qquad (1.15)$$

so that u(x,t) is the velocity fluctuation (and has zero mean). Substitution of *(1.15) in* (1.14) gives

$$\frac{\partial p}{\partial t} , U.\nabla p.\nabla.E\{u\delta \mid \Gamma(x,t) - \theta]) = k\nabla^2 p - k\frac{\partial^2}{\theta\theta^2} E\{(\nabla\Gamma)^2\delta \mid \Gamma(x,t) - \theta]\} \qquad (1.16)$$

where use has been made of the result v.u = 0 which, together with v.u = 0, is an elementary deduction from (1.1) and the definition of U(x,t). Note that the two expected values in (1.16) are defined by equations like (1.8), but involving the appropriate joint p.d.f.s (of u(x,t) and T(x,t), and of $(\nabla\Gamma)^2$ and Γ(x,t))rather than p(θ;x,t).

Use of (1.10) in (1.16) gives

$$\frac{\partial C}{\partial t} \rightarrow u.\nabla C + \nabla.E(uc) = k\nabla^2 c \qquad (1.17)$$

as the equation governing the mean concentration, where c(x,t) is the concentration fluctuation defined by

$$\Gamma(x,t) = C(x,t) + c(x,t) \qquad (1.18)$$

an equation analogous to (1.15). Similarly, use of (1.11) in (1.16) gives, after some algebra,

$$\frac{\partial\sigma^2}{\partial t} \rightarrow U.\nabla\sigma^2 + \nabla.E\{uc^2\} + 2\nabla C.E\{uc\} = k\nabla^2\sigma^2 - 2kE\{(\nabla c)^2\}. \qquad (1.19)$$

Both (1.17) and (1-19) exhibit the closure problem explicitly, namely they contain extra unknowns. For example, (1.17) contains E{uc} as well as terms in C These terms arise, of course, from the terms in (1.16) which depend on p.d.f.s other than p(θ;x,t). As is well-known (and notorious), attempts to resolve this difficulty by deriving equations for the extra unknowns do not lead to success, at least in a fundamental sense, since new extra unknowns occur, thus leading to an infinite hierarchy of equations. However, since quantities like C and 0^2 are of crucial importance in many practical problems, there have been many attempts to resolve the closure problem empirically by truncating the hierarchy and using closure hypotheses to express as many unknown terms as necessary in terms of the remaining unknowns. For example, the oldest and simplest such hypothesis is that often applied to (1.17). Noting that $-V.E\{uc\}$ represents the contribution to $\partial C/\partial t$ from the turbulent fluctuations in T and Γ, just as - qualitatively - $\kappa\nabla^2 C$ represents that from molecular fluctuations, it is common to postulate that there exists an eddy diffusivity K(x, t) such that

$$E(\mathrm{UC}) - -K(x,t)VC \tag{1.20}$$

and then (1.17) becomes

$$\frac{\partial C}{\partial t} + U.\nabla C = \nabla.\{(K + k)\nabla C\}, \tag{1.21}$$

which is a closed equation for C{x,t) provided K(x,t) is specified. Such specification is an integral part of the closure hypothesis. While (1.21) has been widely applied with some success, and an example will be discussed later, it is not a fundamental equation and is known to be incorrect when the time since release is comparable with (or smaller than) certain time scales characteristic of the velocity field. See Monin and Yaglom (1971, pp.606-614) for further discussion of this point.

I have preferred to derive (1.17) and (1.19) in the way I have since it emphasizes rather better than the usual way (substituting (1.15) and (1.18) into (1.2) directly) how turbulent diffusion can be described using conventional statistical techniques and language, and, in particular, it highlights the role of p.d.f.s like $p(\theta;x,t)$. As will be illustrated later, there is now increasing research emphasis on p.d.f.s in their own right. *X* want also to point out here that the only averages that have so far been used, or defined, are ensemble means defined by equations like (l.8). In general these are not the same as time averages, even when the length of the averaging period tends to infinity. Only when p(9;x,t) and all the other p.d.f.s associated with the ensemble are independent of t, does ergodic theory ensure the equality of these two different types of mean. This never occurs with a cloud because, in a statistical sense, the cloud is continually expanding. It does occur often in the laboratory when contaminant is released at a steady rate into a shear flow, and approximately with some real-life plumes from sewage outfalls and factory chimneys. However, in general, ensemble means can be estimated experimentally only by taking arithmetic means over a sufficiently large number of realisations of the ensemble; it will be recognised that this may be extremely costly, both in terms of time and money. Further discussion of the points in this paragraph is given by Chatwin (1982), Sullivan (1984), Cam and Chatwin (1985), Chatwin and Allen (1985a, 1985b) and Carn, Sherrell and Chatwin (1988).

1.3 The importance of p(θ;x,t) in practical problems

It is obvious that there is insufficient time available in these lectures to give a comprehensive account of scalar transport in turbulent shear flows, and I have therefore chosen to concentrate on certain aspects of the subject that interest me. However, within the constraints, I hope there is some coherence in the set of lectures. This will be achieved by emphasis on p(θ;x,t) and on dependent fields like C(x,t) and $o^2(x,t)$. Historically, research on the mean concentration C(x,t) predated and predominated that on p(θ;x,t) and $o^2(x,t)$. Partly as a result of this undue emphasis, many research papers often give the impression that determining C is the only (or the most important) problem of scalar transport; this impression is reinforced by models such as Gaussian plume models which appear in legislation or quasi-legal guidelines. Such papers, and such models, are unphysical and must, therefore, ultimately prove unsatisfactory from a practical viewpoint. For one thing what is observed is T(x,t), governed by (1.2) and (1.3), and not C(x,t), governed by the different equation (1.17); for another thing exclusive concern with C gives no indication of the variability that, as Figure 1 shows, can be substantial and will cause practical models based on C to be, on occasion, significantly in error.

An important experimental verification of these remarks is provided by Birch, Brown and Dodson (1980). A mixture of air and methane is ignitable at a point x, i.e. can support a flame at that point, if the concentration by volume of methane is between 5% and 15%. Thus the probability of ignitable conditions is p_1{x,t), where

$$P_I(x,t) = \int_{0.05}^{0.15} p(\theta;x,t)\,d\theta \qquad (1.22)$$

{Here, contrary to the standard convention adopted in these lectures where concentration is expressed as a mass per unit volume, p(8;x,t) is the p.d.f. of the concentration of methane expressed as a volume ratio.) Birch, Brown and Dodson (1980) estimated $p_I(x,t)$ at many points in a statistically steady methane jet (so that, in this case, $p_I(x,t) = p_I(x)$) by counting, for each x, the proportion of 400 occasions on which the flame from a spark-generating device was supported. The solid points (and associated error-bars) in Figure 2 are typical of the results obtained, and it will be seen that these points are fitted well by the solid curve obtained by using (1.22) with previous estimates of $p(\theta;x,t)$ due to Birch, Brown, Dodson and Thomas (1978). (Some of these estimates are shown in Figure 1.) The good agreement between the two different methods of finding P_I is convincing support for the practical value of models based on a physically realistic statistical viewpoint. By contrast the dashed curve in Figure 2 is of the mean concentration $C(x)$, and this does not fit $p_I(x)$, even in general shape.

The complicated structure of equation (1.16) for $p(\theta;x,t)$ suggests that there is little hope of real success if it is attacked directly; Pope (1979,1985) summarizes the limited progress that has been made. From the practical point of view there is another approach that is currently popular. This is based on the hypothesis (probably too optimistic) that $p(\theta;x,t)$ can be adequately approximated in many circumstances by "simple" p.d.f.s (e.g. lognormal, truncated normal, beta,...), or by linear combinations of such p.d.f.s. The parameters required to specify such p.d.f.s precisely are, of course, simply related to such properties as $C(x,t)$ and $o^2(x,t)$, and this provides one justification for the direct study of these variables. More mundanely, these are two of the simplest properties associated with the concentration field, and the equations (1.17) and (1.19) governing them are more amenable to analysis than that [(1.16)] governing $p(\theta;x,t)$.

1.4 Some basic dynamics of $C(x,t)$ and $\sigma^2(x,t)$

I want to begin this section by re-emphasizing that C and o^2 are properties of the ensemble, i,e. they are obtained by performing appropriate statistical operations on the results of many experiments and are not direct observables in the normal meaning. Only if $p(\theta;x,t)$ is independent of t (or a coordinate of x) can each be obtained as an average over time (or the coordinate) of the results of a single experiment, and it is in my opinion dangerous to ignore these basic facts. It follows, at least for me, that it is often more appropriate and revealing to apply intuition and simple physical arguments to $\Gamma(x,t)$, the actual observed concentration, and only then to consider the consequences for such ensemble properties as $C(x,t)$ and $o^2(x,t)$.

Superficially, however, there is little to say about $\Gamma(x,t)$. Fluid elements are advected by the velocity field $\Gamma(x,t)$ and would retain the contaminant they began with were it not for molecular diffusion. However molecular diffusion causes contaminant to be transferred from one fluid element to an adjoining one with flux -KvT; the minus sign is necessary, of course, because transfer is from regions of high concentration to regions of low concentration. Since neither advection nor molecular diffusion create or destroy matter, (1.4) holds throughout each realisation of the ensemble (provided this is one in which the same finite quantity Q of contaminant is released in each realisation).

But these simple considerations ignore the fact that the advection is a random process. Moreover, the nature of this random advection is not only to transport fluid elements as a whole in a random manner but also to distort them. The nature of the distortion is shown schematically in Figure 3, which is a famous sketch due to Corrsin (1959); an alternative way of describing the process is that an initially spherical fluid element becomes, as time increases, a more and more increasingly tangled ball of wool (Chatwin and Sullivan 1979c). The details of the tangling are of course random; however the total volume of each fluid element is invariant because of incompressibility. Some attention will be given later in these lectures to the effects of random advection on fluid elements. For the moment it is sufficient to note that it causes the surface area of the fluid element to increase rapidly (on average) and the thickness of strands of the fluid element to decrease rapidly (again on average). Both of these linked effects lead to a rapid increase in the degree of molecular transfer from the element.

The mean concentration C(x,t), obtained by averaging Γ(x,t) over all realisations of the ensemble, depends on the magnitude of the probability that the point x at time t lies in the (highly distorted) wool compared with that of the probability that it lies in the ambient fluid between the strands of wool, and it depends on the average effect of the consequent (enhanced) molecular diffusion. As (1.17) shows, the first of these effects makes a contribution -U.∇C - V.E{uc} = -v.E{TT} to ∂C/∂t. The term -U.∇C represents advection by the mean velocity(which could be interpreted - were C(x,t) a "real" physical quantity – as advection of the fluid element as a whole) and the term -∇.E{uc} = -E[uvc} represents the effect of distortion over and above that contributing to the bulk advection. The term E{uc} is sometimes referred to as the turbulent flux and could be interpreted, with the above proviso, as the contribution from the turbulent velocity fluctuations. It is then natural, despite the fundamental difference in the underlying physical processes, to compare the turbulent flux with the molecular flux $-\kappa VC$. Despite the substantial enhancement of the latter by advection, there Is no doubt that, for the mean concentration C(x,t), the molecular term in (1.17), viz. $K\nabla^2 C$, has negligible magnitude compared with -V.E{uc}. If the latter term can be modelled by an eddy diffusivity K(x,t), as in (1.21), this statement can be written $K >> \kappa$. Arguments supporting this conclusion are given by Monin and Yaglom (1971, pp.591-606), Thus κ can be ignored in calculating C(x,t).

For a cloud the ensemble average of (1.) gives

$$\int C(x,t)dV(x) = Q, \qquad (1.23)$$

and this can also be obtained from (1.17)For future use I define $L_{ij}(t)$ (i.j = 1,2,3) by

$$L_{ij}(t) = \frac{1}{Q}\int x_i x_j c(x,t)dV(x), \qquad (1.24)$$

and I shall refer to $L_{ij}(t)$as the absolute dispersion tensor. Since L_{ij} is symmetric, principal axes can be chosen for each t (but note that in general these will not have constant directions in an arbitrary shear flow) and I shall then write

$$L_{ij} = \begin{bmatrix} L_1^2 & 0 & 0 \\ 0 & L_2^2 & 0 \\ 0 & 0 & L_3^2 \end{bmatrix} \text{and } L^2 = L_{ii} = L_1^2 + L_2^2 + L_3^2. \qquad (1.25)$$

Note that L^2 has a value independent of whether L_{ij} is expressed in principal axis form.

In the equation for $\sigma^2(x,t)$, viz. (1.19), the terms $U.V\sigma^2$, $V.E\{uc^2\}$ and $k\nabla^2\sigma^2$ are analogues of corresponding terms in equation (1.17) for C(x,t) and I need make no further comment about their significance. The term $2\nabla C.E\{uc\}$ is the so-called production term and its interpretation is interesting. If equation (1-17) is multiplied by 2C there results, after some algebra.

$$\frac{\partial C^2}{\partial t} + U.VC^2 + 2V.)CE\{uc)) - 2vC.E\{uc) = k\nabla^2 C^2 - 2k(\nabla C)^2 \qquad (1.26)$$

This equation contains the production term in the equation for $a^2(x,t)$, but with opposite sign. Noting that

$$E(\Gamma^2\} = c^2 + \sigma^2 \qquad (1.27)$$

it follows that the production term has no net effect on $E(\Gamma^2\}$, and therefore represents a transfer of "Γ^2-stuff", without overall loss, from the $C^2(x,t)$ field to the $o^2(x,t)$ field. In general, the term "production", rather than "destruction", is appropriate for the term as it appears in equation (1.19) for $\sigma^2(x,t)$ and two differing arguments will be given. The simplest is that involving an eddy diffusivity K; from (1.20), $2\nabla C.E\{uc\} = -2K(VC)^2$ and is negative assuming that (as befits an eddy diffusivity) K is positive. Thus the production term makes a positive contribution to $\partial a^2/\partial t$. Another argument, better because it makes no appeal to empiricism, follows after adding equations (1.19) and (1.26) to obtain, using (1.1), (1.27) and some algebra.

$$\frac{\partial}{\partial t} E\{\Gamma^2\} + U.\nabla E\{\Gamma^2\} + \nabla.E\{u\Gamma^2\} = K\nabla^2 E\{\Gamma^2) - 2kE\{(\nabla\Gamma)^2\}. \qquad (1.28)$$

Integrating (1.28) over all space, and using (1.1) and (1.3), gives

$$\frac{d}{dt} E\left\{\int \Gamma^2(x,t)\,dV(x)\right\} = -2kE\left\{\int (V\Gamma)^2 dV(x)\right\}. \qquad (1.29)$$

From the point of view of interpreting the production term, it is legitimate for the moment to ignore the term on the right-hand side of 1.29). Then the integral on the left-hand side would be constant, so that

$$\int C^2(x,t)\,dV(x) + \int \sigma^2(x,t)\,dV = \frac{Q^2}{\ell_0^3}, \qquad (1.30)$$

where Q is the total quantity of contaminant, defined in (1.4), and ℓ_0 is a length which must be representative of the source size. For $t > 0$, let $\ell(t)$ be a length characteristic of the extent of the distribution of $C(x,t)$. Then (1.23) requires C to be of order Q/ℓ^3; hence the first integral on the left-hand side of (1.30) is of order Q^2/ℓ^3. Provided, as will almost always be the case, $\ell(t) \to \infty$ as $t \to \infty$, equation (1.30) can be satisfied only by the integral of $o^2(x,t)$ being of order Q^2/ℓ^3_0 Assuming that $i(t)$ is also a measure of the spatial extent of the $o^2(x,t)$ field, this requires σ^2 to be of order $Q^2/\ell^3\ell^3$ (Chatwin and Sullivan 1979a). Thus

$$\frac{\sigma}{c} - \left[\frac{\ell}{\ell_0}\right]^{3/2} >> 1. \qquad (1.31)$$

The argument can be extended to plumes (Chatwin and Sullivan 1979b).

Neglect of the terra on the right-hand side of (1.30) is not justified; hence the estimate in (1.31) will not be accurate, certainly not for large t. Nevertheless the argument does show convincingly that there is a mechanism,, which can only be that described by the production term, for transfer from the C^2 field to the σ^2 field. In simple physical terms, this is due to the increasing randomness as the cloud becomes more tangled during the course of one realisation. Incidentally there has been much controversy (see e.g. Lumley and van Cruyningen 1985; Thomson 1986, 1989) about the behaviour of σ/C as $t \rightarrow \infty$ in real clouds, i.e. with the effects of molecular diffusivity included. I do not want to contribute (further!) to the arguments, but I will note that (1.30) shows that the source size is important, at least potentially, in the determination of the $\sigma^2(x,t)$ field. The last term in equation (1.19), viz. $-2\kappa E\{(\nabla c)^2)$, is inherently negative, as is the corresponding term in each of (1.26) and (1.28). This is the dissipation term and it can be seen, particularly from (1.29), that, in the end, the values of $\Gamma^2\{x,t)$, and therefore of both $C^2(x,t)$ and $\sigma^2(x,t)$, must tend to zero everywhere as the result of molecular diffusion. In the equation (1.19) for $\sigma^2(x,t)$ there is no other term that produces this dramatic effect; accordingly proper modelling of the dissipation term in that equation is essential. (By contrast the other term involving κ, viz. $K\nabla^2\sigma^2$, can be neglected by comparison with $V.E\{uc^2\}$, just as with the corresponding terms in the equation for $C(x,t)$.)

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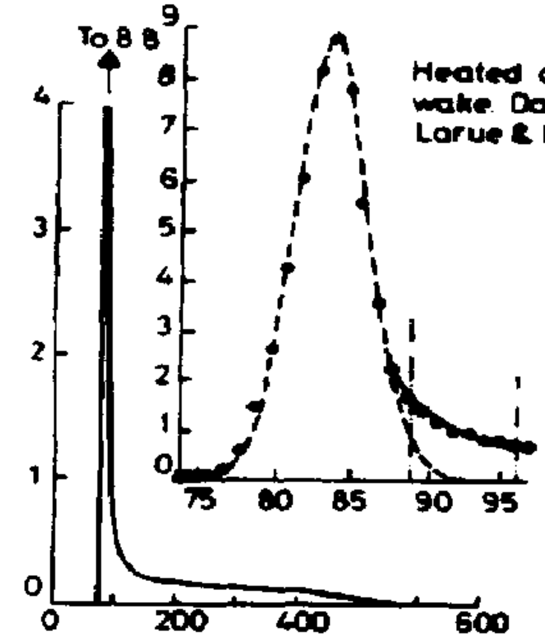

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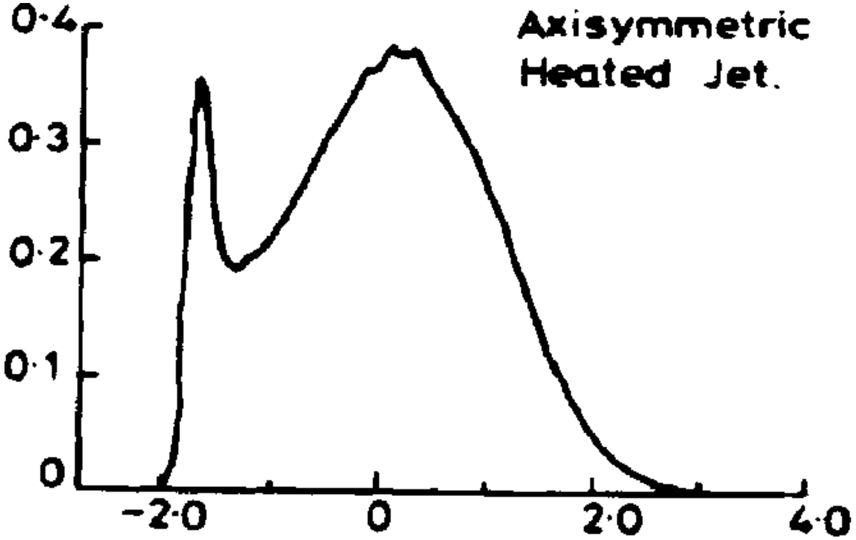
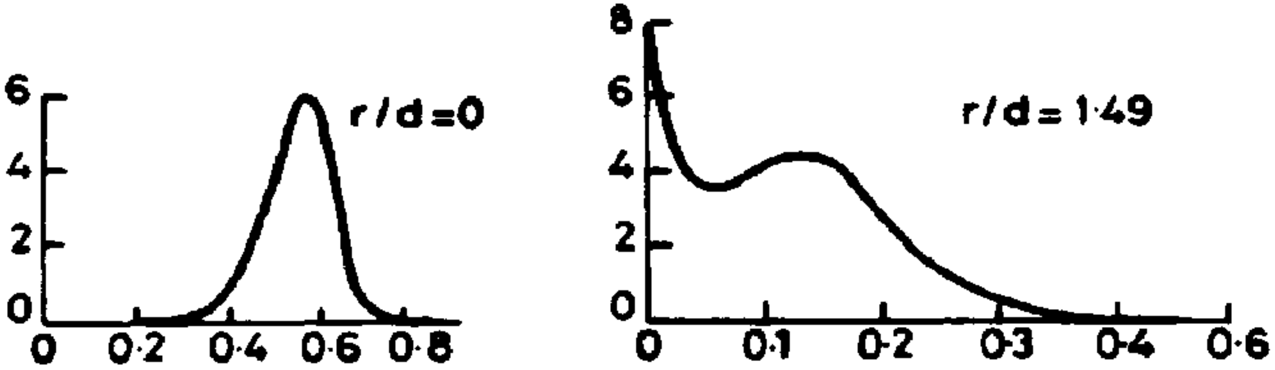


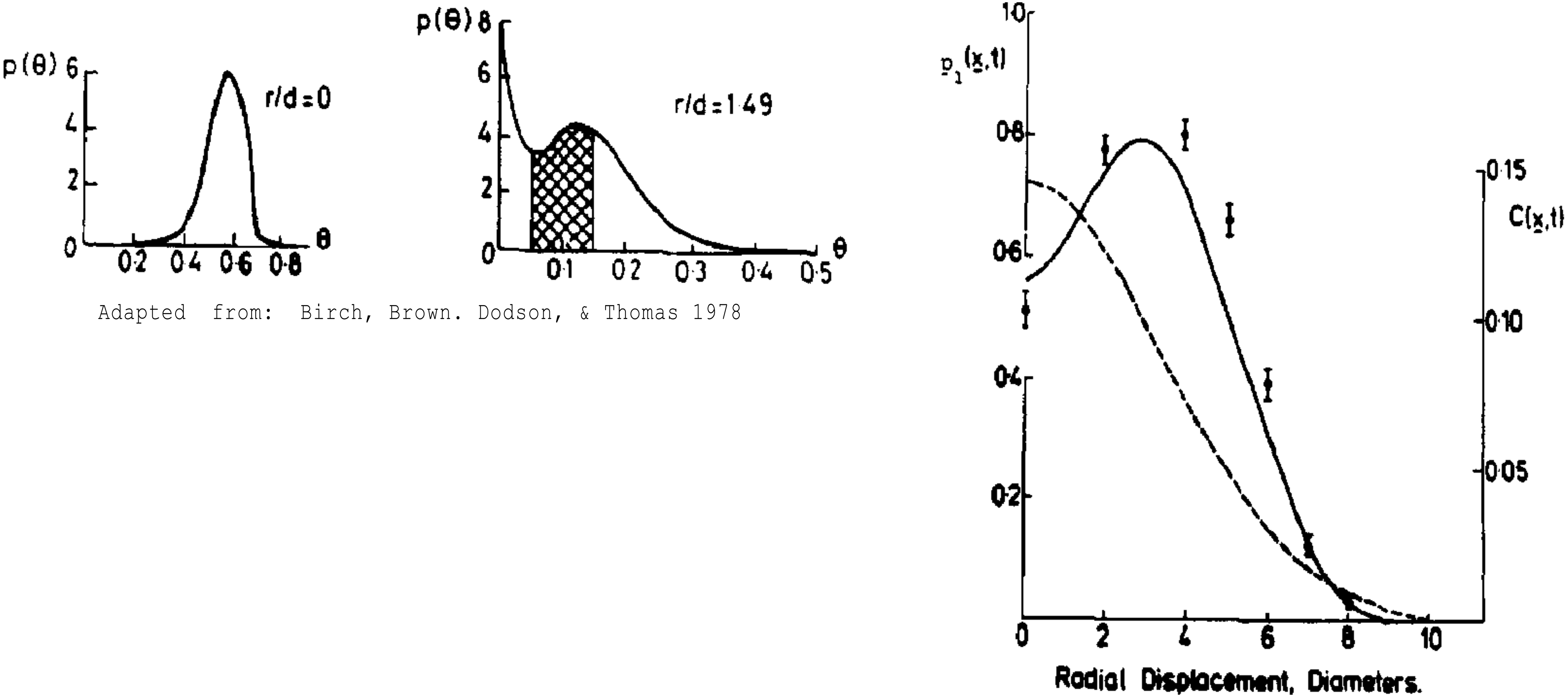
From: Bilger, Antonia & Sreenrvasan 1976

Methane Jet.



Adapted from:Birch,Brown.Dodson & Thomas 1978.

Figure 1, Some experimental determinations of $p(\theta;x,t)$



Adapted from: Birch, Brown. Dodson, & Thomas 1978

Figure 2. The two left-hand diagrams are two of the p.d.f.s shown in Figure 1, and the hatched area on that for r/d = 1.49 is p_I , defined in (1.22). The solid line in the right-hand diagram is a curve - fit to values of p_I determined in this way. The solid points with associated error-bars are direct measurements of p_I (see text) and the dashed curve is the mean concentration.

$$p(\theta;X,t)=P\delta(\theta-QL_0^{-3})+(1-P)\delta(\theta)$$

Adapted from: Corrsin 1959

Figure 3. A schematic sketch of the effect of the advection process on a marked fluid element. (Note that the indicated form of $p(\theta;x,t]$ is (2.3) in a different notation.)

Lecture 2. The turbulent diffusion of fluid elements

2. 1 Introduction

It was shown An the first lecture that molecular diffusion has little importance for certain properties of the concentration field, particularly the mean concentration C(x,t), Conversely, it plays an essential role in other properties, including the variance $\sigma^2(x, t)$. But, if turbulent diffusion is regarded as a branch, of physics, and this will be the primary approach in these lectures, quantification of this role requires prior investigation and understanding of the effects of random advection; this is clear from the discussion following Figure 3. There is therefore a strong case for investigating turbulent diffusion in the hypothetical situation in which $\kappa = 0$, provided the limitations of the results (severe in many circumstances) are recognized. Furthermore, this was, historically, the basis of most early research into the subject.

when $\kappa = 0$, there is no mechanism that can transfer contaminant from one material volume (large or small) to another. Consequently each fluid element of volume δV retains, throughout the dispersion process, the same mass of contaminant that it began with and, therefore (since incompressibility ensures that δV itself is constant), the same concentration Γ. It follows that $f(\Gamma)\delta v$ is also invariant for each fluid element for every function f and, integrating over the whole volume occupied "by fluid, that

$$\int f \mid \Gamma(x,t)]\,dV(x) = \text{constant}. \qquad (2.1)$$

A consequence of (2.1), but a much weaker result, is that

$$E\left\{\int f \mid \Gamma(x,t)]\,dV(x)\right\} = E\{\text{constant}\} = \text{constant}. \qquad (2.2)$$

(Chatwin and Sullivan 1979). The special cases of (2.2) with $f(\Gamma) = \Gamma$ and $f(\Gamma) = \Gamma^2$ are (1.23) and (1-30) respectively, but little other use of 12.1) or (2.2) appears to have been attempted, although that effort could be worthwhile-

As is common in practice consider an ensemble in which the initial concentration within the cloud is the same in each realisation, and uniform, with value θ_1. Subsequently the concentration $\Gamma(x,t)$ has one of only two values, viz, θ_1 if the point x at time t lies in a fluid element emanating from the original cloud, and 0 otherwise. Consequently $p(\theta;x,t)$ has a simple form $P_0(\theta;x,t)$, where

$$p_Q(\theta;x,t) = \pi_Q(x,t)\delta(\theta-\theta_1) + [1-\pi_0(x,t)]\delta(\theta) \ , \qquad (2.3)$$

and

$$\pi_0\ (x,t) = \mathrm{prob}\{\Gamma_0(x,t) > 0\} \ , \qquad (2.4)$$

is the intermittency factor in this hypothetical situation, which will subsequently be denoted by the zero subscript, where appropriate for clarity, as in (2.3) and (2,4). As usual the δs in (2.3) denote Dirac delta functions. Use will be made of these results in a later lecture.

2.2 The Taylor (1921) analysis

So far in these lectures the discussion and, certainly, the equations, have been presented in an Eulerian framework, i.e. the variable x has referred to a point in space and not to a fluid element. Arguably the most famous result in turbulent diffusion, viz. (2.11) below, due to Taylor (1921), was derived from an analysis in a Lagrangian framework, i.e. following fluid particles, this approach has proved very illuminating and I want to discuss some results obtained with it, beginning with Taylor's result.

Let X(a;t), to be shortened to X(t) (or even X) on occasion, be the position vector at time t of the fluid particle that was known to be at a at time t = 0. Thus X(a;0) = a for all a. The velocity V(a;t) of this fluid particle is of course simply $\partial X(a;t)/\partial t$ and, though this will not be relevant, T[X(a;t),t), where T(x,t) is the Eulerian velocity field. The random variables X and V are related by

$$x(t) - a = \int_0^t v(\tau)\,d\tau. \qquad (2.5)$$

The ensemble mean of (2.5) is

$$E\{x(t)\} - a = \int_0^t E\{V(\tau)d\tau\} \qquad (2.6)$$

Taylor considered the situation where V(t) is a stationary random process. This is defined to be a process in which the probability distribution of $\{V(t_1), V(t_2) \ldots . , V(t_n)\}$ for each n depends only on (t_r-t_1) $(r = 2,3,. \ldots,n)$, i.e. it depends only on the time differences and is independent of a translation in time. In particular the p.d.f. of V(t) is independent of t so that (2.6) becomes

$$E\{X(t)\} = a + E\{V\}t \; , \qquad (2.7)$$

and the p.d.f. of $\{V(t_1), V(t_2)>$ depends only on (t_2-t_1). Thus

$$E\{[v_i(t_1)-E(v_i\})[v_j(t_2)-E\{v_j\}]) = R_{ij}\;(t_2-t_1) \qquad (2.8)$$

where $R_{IJ}[T)$ is termed the Lagrangian autocorrelation tensor. Define $\Sigma ij(a;t)$ by

$$\Sigma_{ij}.(a;t) - E\{[X_i-a_j.-E\{V.\}t][X.-a_j.-E\{V_j.\}t]\} \qquad (2.9)$$

Than

$$\frac{d\sum ij}{dt} = E\{[V_i - E\{V_i\}][X_j - a_j - E\{V_j\}t\} + E\{[X_i - a_i - E\{V_i\}t][V_j - E\{V_j\}]\}$$

$$= \int_0^t [R_{ji}(t-\tau) + R_{ij}(t-\tau)d\tau \, , \qquad (2.10)$$

on using (2.5) and (2.8). It follows, since $\Sigma_{ij}((a;0) = 0$, that

$$\sum_{ij}(a;t) = \int_0^t (t-\tau)[R_{ij}(\tau\tau + R_{ji}(\tau\tau)d\tau \qquad (2.11)$$

Taylor gave, in effect, the version of (2.11) with i = j = 1 and noted that, for large t, this becomes

$$\sum_{11} = 2D_{11} t \text{ where } D_{11} = \int_0^{\sim} R_{11}(\tau\tau)d\tau \qquad (2.12)$$

and it is assumed that the infinite integral converges. The derivation above is similar to, but not identical with, that in Monin and Yaglom (1971, pp.540-547).

There are many important applications of Taylor's result (but not - it has to be said - as many as some people appear to think!). One important one is to longitudinal dispersion; this area of research was also originated by Taylor (1953, 1954) but, as he acknowledges in his 1954 paper, he did not realize the connection with his 1921 paper until it was pointed out to him by Batchelor! For simplicity consider a long pipe of constant cross-section, and suppose that a finite cloud of contaminant is released in this pipe at time t = 0. It is intuitively obvious that the contaminant particles will eventually "forget" where they have started from and that, then, the analysis above will apply. In particular, if the 1 axis is taken along the axis of the pipe

$$E\{V_1 = U_0 \ , \ E(V_2) = E\{V_3\} = 0 \ , \tag{2.13}$$

where Uo is the discharge velocity (Batchelor, Binnie and Phillips 1955), and (2.12) holds. Thus (Taylor 1953, 1954) the centroid of the contaminant cloud eventually moves at the discharge velocity and, relative to its centroid, spreads axially. Since $\Sigma_{11}(t)$ is the variance of the displacement of a fluid particle, and since all fluid particles have, eventually, the same variance to highest order in t, the axial extent of the mean cloud grows as $t^{½}$. There is not time here to make more than three further comments, Taylor showed that, in this case, D_{11} in (2.12) could be interpreted as a longitudinal dispersion coefficient in the sense that the average of C(x,t) over the pipe cross-section, say $C_o(x_1,t)$, obeyed {for large t) the diffusion equation

$$\frac{\partial C_0}{\partial t} + u_o \frac{\partial C_o}{\partial x_1} = D_{11} \frac{\partial^2 c_0}{\partial x_1^2}. \tag{2.14}$$

However, a solution of (2.14) is the Gaussian form

$$C_o = \frac{Q}{\sqrt{2 = \Sigma 11}} \exp\left\{ - \frac{(x_1 - u_0 t)^2}{2\Sigma_{11}} \right\}, \tag{2.15}$$

and this can be derived independently of (2.14) by a plausible application of the central limit theorem to (2.5); see Monin and Yaglom (1971, p.541). Thus (2.14) has little significance in itself. This conclusion is reinforced by my second comment on the application of Taylor's analysis to longitudinal dispersion in a long pipe which is that deviations of observed distributions of $C_0(X_1,t)$ from (2.15) are not described by (2.14) (Chatwin 1972). It has been emphasized that (2.12) is valid only for large t, and other deviations from (2.15) occur because of the legacy of the initial conditions (Chatwin 1970). Indeed it now appears that, for times of practical interest, (2.15) is usually inadequate, and much attention has been paid to methods of improving (2.15) in practical situations (Chatwin 1970; Smith 1981a,b; 1987). Finally I note that the analysis leading to (2.11) and (2.14) gives no method for calculating the value of D_{11} Taylor (1954) used an eddy diffusivity argument for a pipe of circular cross-section to obtain the result D_{11} = 10.lau*, where a is the pipe radius and u* is the friction velocity.

There is another point of interest in pipe dispersion that is not connected with longitudinal dispersion. It is obvious that Taylor’s analysis applies in directions perpendicular to the axis as well. Thus, for example, (2.11) gives

$$\Sigma_{22}(t) = 2t \int_0^t R_{22}(\tau)\, d\tau - 2\int_0^t \tau R_{22}(\tau)\, d\tau \qquad (2.16)$$

Since Σ_{22} (unlike Σ_{11}) is bounded for large t because the pipe has finite cross-section, it follows that the integral of $R_{22}(\tau)$ from $\tau = 0$ to ∞ has value zero. Indeed, in this case, it is easy to calculate the finite value of $\Sigma_{22}(\infty)$, given the shape of the cross-section, since the fluid particles must eventually be distributed uniformly in the directions normal to the axis. Thus, for a pipe of circular cross-section, I find

$$\Sigma_{22}(\infty) = \frac{1}{\pi a^2}\iint r^2 \cos^2\theta . r dr d\theta = \frac{1}{4} a^2. \qquad (2.17)$$

2.3 Some other one-particle Lagrangion analyses

It is possible to apply dimensional arguments to the starting point of Taylor's analysis, viz. (2.5), even when V(t) is not a stationary random process. I shall discuss two statistically steady shear flows.

Consider first a round turbulent jet generated by the injection of momentum at rate F into a fluid of density P- For large enough t, i.e. when the details of the source geometry no longer influence the motion of a fluid particle, the statistical properties of X{t) can depend only on F, p and t. Here, for convenience, it is assumed that X(t) is the displacement of the fluid particle relative to the source. The dimensions of (F/*p)* are L^4T^{-2} and, choosing the 1 axis to be in the direction of the injected momentum, it follows by simple dimensional and symmetry arguments that, for large t.

$$E\{X_i(t)\} = \left[\frac{Ft^2}{0}\right]^{\frac{1}{4}} \{\alpha_1, 0, 0\}; \quad E\{X_i(t) x_j(t)\} = \left[\frac{Ft^2}{0}\right]^{\frac{1}{2}} \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{22} \end{bmatrix}, \qquad (2.18)$$

where α_1, α_{11} and α_{22} are universal numerical constants. The dependence on t, for large enough t, of all statistical properties of X can obviously be written down in the same way. In particular, let p(ε;a,t) be the p,d.f. of X{a,t); thus p(ε;a,t)δV(ε) is the probability that X(a,t) is in the small volume element δV{ε) surrounding ε Clearly

$$\int p(\Sigma ; a, t)\, dv(\Sigma) = 1 \qquad (2.19)$$

For all a. For large t, p becomes independent of a and the dimensional arguments above give

$$p(\Sigma ; a, t) = p(\Sigma ; t) = \left[\frac{Ft^2}{p}\right]^{-\frac{1}{4}} \varphi[\Sigma \left[\frac{Ft^2}{p}\right]^{-\frac{1}{4}} \qquad (2.20)$$

where the function φ is the same for all round jets. The arguments above do not lead to values for constants like a_1 or to the precise form of φ but. then, neither do those employed by Taylor for the case when the Lagrangian velocity is a stationary random function.

The second shear flow I want to consider is the neutral planetary boundary layer (Batchelor 1964). When effects due to stratification are negligible, the statistical properties of X(t) can depend only on the friction velocity u*, the roughness height z_0 and t {when, as usual, the effects of the initial position of the particle have been forgotten). However the influence of Z_0 in this long-term situation must be trivial since its only effect on the Eulerian dynamics is to add a constant speed -{u*/k)lnzo to the component of velocity in the direction of the mean wind (Galilean invariance). Here k(= 0.41) is von Karman's constant. choose axes with Ox parallel to the mean wind and Oz vertically upwards; write

$$X(t) = (X(t), Y(t), Z(t)) \ . \qquad (2.21)$$

By symmetry E{Y(t)} is a constant, which can be taken as zero; E{dX/dt) must be the sum of $-(u_*/k)\ln z_o$, and a function of u_* and t, and E{dZ/dt) a function of u_* and t. Dimensional arguments then give

$$E\left\{\frac{dx}{dt}\right\} = \frac{u_*}{k}\, \mathrm{In}\left[\frac{cu,\, t}{z_o}\right] ;\ E\left\{\frac{dz}{dt}\right\} = bu_* \ , \qquad (2.22)$$

where b and c are universal dimension less numbers. On integrating it follows that, to leading order for large t.

$$E\{X(t)\} = \frac{u_* t}{k}\left[\mathrm{In}\left[\frac{cu*t}{z_o}\right] - 1\right] ;\ E\{Z(t)\} = bu_* t \ . \qquad (2.23)$$

The result for E{dZ/dt} deserves comment since, if b ≠ 0, it is non-zero, whereas the vertical component of the Eulerian velocity has zero ensemble mean everywhere. While this result may seem paradoxical to some at first sight, it is easily seen to be reasonable. The result for E{dZ/dt} requires an average, over all realisations, of the vertical velocity of fluid particles released from the source at t = 0 and the fact that this is non-zero, indeed positive, is not then surprising given that fluid particles cannot travel below z = 0 whereas their upward motion is unlimited (at least in the present simplified model). By contrast the mean vertical Eulerian velocity is an average over all fluid particles that are at a given point in space and must be zero if mass conservation is not to be violated.

Given the importance of atmospheric dispersion, there have inevitably been attempts to estimate the constants b and c in (2.22) and (2.23). Chatwin (1968) used (1.21) with the eddy diffusivity K(z) equal to ku*z. This was chosen because of Reynolds analogy, an empirical law which asserts that the eddy diffusivity equals the eddy viscosity (and - by implication - that both exist). Since the mean horizontal Eulerian velocity is $(U_*/k)\ln(z/z_0)$, the latter must be ku_*z. It then follows from (1.21) after algebra that

$$b = k = 0.41\ ,\quad c = 0.23\ . \tag{2.24}$$

Hunt and Weber (1979) obtained almost identical results by a totally different type of argument. Unfortunately it is very difficult to estimate b and c from experiments, essentially because the centroid of the cloud ascends without limit (although, nevertheless, the maximum mean concentration is always at the ground). What observations there are do not contradict (2.24) but nor would they for substantially different estimates. Yaglom (1976) showed that the use of an eddy diffusivity tensor with non-zero off-diagonal terms, i.e. with (1.20) replaced by

$$E\{u_i C\} = -K_{ij}(x,t)\,\frac{\partial c}{\partial x_j} \tag{2.25}$$

with $K_{ij} \neq K\delta ij$, could change (2.24) significantly.

The results (2.22) and (2.23) are due to Batchelor (1964), who discussed other consequences of the methods including the decay of the mean concentration C(x) due to a steady line source with downwind distance x. (See also Monin and Yaglom 1971, pp.559-569, 640-655; these pages give references to work extending the ideas above to stratified atmospheres.)

The methods discussed in this section can be applied to many other shear flows in which simple dimensional arguments are valid (Monin and Yaglom 1971, pp.547-556).

2,4 Material line and surface elements

A study of the effect of random advection on material line elements and material surface elements is illuminating, not least in trying to understand (later!) the role of molecular diffusivity in real situations. The term "material element" denotes a set of fluid particles whose linear dimensions are small. In particular it will be supposed that the elements are small enough for the scales of the turbulent velocity field that dominate their distortion to be statistically isotropic, i.e. without directional preference. That this is a realistic condition was shown by Kolmogorov {1941) in, arguably, the most important paper ever written in turbulence. In any turbulent shear flow at high Reynolds number, the statistical properties of the small scale velocity fluctuations are isotropic and stationary, and are determined by the values of the mean energy dissipation rate ε and the kinematic viscosity v (Monin and Yaglom 1975, pp.337-368). This condition of local isotropy does not, of course, apply to the large scales of the velocity field that are responsible for the transport of the material element as a whole but this transport is not of present concern.

Consider two fluid particles whose initial positions are a and a + ℓ_o. Subsequently the positions of these particles are X(a;t) and X(a + ℓ_0;t), using the notation of section 2.2. As a vector therefore, the material line element ℓ_0 becomes $\ell = \ell(t)$, with $\ell(0) = \ell_0$ and

$$\ell(t) = X(a + \ell_0;t) - X(a;t) \quad . \qquad (2.26)$$

I shall discuss in some detail the case when $|\ell_0|$ is so small that the material line element remains straight for a substantial period of time, so that (2.26) can be replaced by

$$\ell = \ell(t) = \ell_0.\nabla_a x. \text{ i.e. } \ell_i = \frac{\partial x_i}{\partial a_j}\ell_{oj}. \qquad (2.27)$$

Hence

$$|\ \ell\ |^2 = \ell_i \ell_i = A_{jk}\ell_{oj}\ell_{OK} \quad \text{where} \quad A_{jk}(t) = \frac{\partial x_i}{\partial a_j}\frac{\partial x_i}{\partial a_k} . \qquad (2.28)$$

The quantities A_{jk} are the components of symmetric tensor, albeit one which is random and time-dependent. This tensor therefore has real eigenvalues which must be positive since (2.28) shows that the diagonal elements of A_{jk}

are positive. Denote these eigenvalues by A^2_1, A^2_2 and A^2_3.In Lagrangian coordinates the incompressibility condition (1.1) becomes

$$\det \frac{\partial x_i}{\partial a_j} = 1 \qquad (2.29)$$

(Monin and Yaglom 1971, p. 531). Hence

$$\det(A_{jk}) \qquad (2.30)$$

(since $\det(A_{jk}) = \{\det(\partial x_i/\partial a_j)\}^2$), and therefore

$$A_1^2 A_2^2 A_3^2 = 1. \qquad (2.31)$$

Using coordinates with the eigenvectors of A_{jk} as coordinate axes, (2.28) becomes

$$1\ell(t)1^2 = A_1^2 \ell_{01}^2 + A_2^2 \ell_{02}^2 + A_3^2 \ell_{03}^2. \qquad (2.32)$$

The result that I want to establish relates to the ensemble mean of (2.32). By considering first the mean over the sub-ensemble in which A_1^2, A_2^2 and A_2^2 are fixed so that isotropy can be applied, and finally the average over the values of A_1^2, A_2^2 and A_3^2, it follows that

$$E\{1\ell(t)1^2\} = E\{\tfrac{1}{3}(A_1^2 + A_2^2 + A_3^2)\}1\ell_0 1^2. \qquad (2.23)$$

Since the eigenvalues of A_{jk} are positive, the elementary arithmetic mean-geometic mean inequality gives $\frac{1}{3}(A_1^2 + A_2^2 + A_3^2) \geq (A_1 A_2 A_3)^{2/3} = 1$ with equality only if $A_1^2 = A_2^2 = A_3^2 = 1$.This cannot hold for almost all realisaction;hence $E\{\frac{1}{3}(A_1^2 + A_2^2 + A_3^2)\} > 1$ and

$$E(1\ell(t)1^2\} > 1\ell_0 1^2. \qquad (2.34)$$

This proof is a Lagrangian version of one given by Orszag (1970), but is somewhat more straightforward than his. In turn, Orszag"s proof is a simple variant of one by Cocke (1969). Cocke actually proved the stronger result that $E\{\ln(|\ell| / |\ell_0|)\} > 0$, and thereby established coniectures by Batchelor (1952b).

Thus, for so long as the conditions assumed in the proof remain valid, the Length of a material line element increases on the average. I want briefly to consider these conditions, and how progress may be made when they are violated. Let $q\{\Sigma;\ell_o,t)$ be the p.d.f- of $\ell(t)$ so that, as usual, $q\delta V(\xi)$ is the probability that $\ell(t)$ has one end at 0 and the other end in the small volume element $\delta V(\Sigma)$ surrounding Σ, given that $\ell(0) = \ell_0$. Restrict attention throughout to the case when $E\{|\ell(t)|\} \ll L_E$, where L_E is a length-scale characteristic of the energy-containing eddies; naturally a necessary (but not sufficient) condition for this to be valid is that $|\ell_0| \ll L_E$. Then, as noted above, the scales of the velocity field that determine q are isotropic; in particular a characteristic length-scale is n (the Xolmogorov microscale) and a characteristic time-scale is τ, where

$$\eta = \left[\frac{v^3}{\varepsilon}\right]^{\frac{1}{2}} ; \tau = \left[\frac{v}{\varepsilon}\right]^{\frac{1}{2}} . \tag{2.35}$$

It follows from isotropy and dimensional arguments that

$$q(\Sigma;\ell_0,t) = \eta^{-3}Q\left[\frac{\Sigma\cdot\Sigma}{\eta^2},\frac{\ell_0\cdot\Sigma}{\eta^2},\frac{\ell_0\cdot\ell_0}{\eta^2},\frac{t}{\tau}\right]. \tag{2.36}$$

Monin and Yaglom (1975, pp.536-546, 578-584) give this formula and discuss many special cases. See also Batchelor (1952a). Here I can select only one or two special cases. Suppose first that $|\ell_0| \ll n$, when there will be a range of times for which a power series approximation to (2.36) is valid, as in (2.27). Hence (2.36) reduces to

$$\begin{aligned} q(\Sigma;\ell_0,t) = \eta^{-5}\ell_0\cdot\Sigma\, Q_1\left[\frac{\Sigma\cdot\Sigma}{\eta^2},\frac{t}{\tau}\right] &+ \\ \eta^{-7}(\ell_0\cdot\Sigma)^2 Q_2\left[\frac{\Sigma\cdot\Sigma}{\eta^2},\frac{t}{\tau}\right] &+ \eta^{-5}\,|\ell_0|^2\, Q_3\left[\frac{\Sigma\cdot\Sigma}{\eta^2},\frac{t}{\tau}\right]. \end{aligned} \tag{2.37}$$

Hence, for example.

$$E\{|\ell(t)|^2\} = \int \Sigma\cdot\Sigma\, Q dv(\Sigma) = |\ell_0|^2\, g\left[\frac{t}{\tau}\right], \tag{2.38}$$

where, according to (2.34), $g(0) = 1$ and $g(t/\tau) > 1$. (Note that, by symmetry, the term involving Q_1 in (2.37) does not contribute to $E\{|\ell(t)|^2\}$.)

Equations (2.37) and (2.38) cannot, of course, remain valid indefinitely. More generally (2.38) has to be replaced by

$$E\{|\ \ell(t)\ |^2 = |\ |\ \ell_0\ |^2\ h\left[\frac{|\ \ell_0\ |}{\eta}, \frac{t}{\tau}\right]. \qquad (2.39)$$

Eventually, the influence of ℓ_0 will be lost, and (2.39) becomes

$$E\{|\ \ell(t)\ |^2\} = \eta^2 j\left[\frac{t}{\tau}\right]. \qquad 2.40)$$

This is likely to occur (for $|\ell o| \ll n$) when t is somewhat greater than τ. A further, well-known, simplification of (2.40) is possible for very high Reynolds number when, according to a further result of Kolmogorov (1941), there is a range of length scales for which v is irrelevant. This requires $j(t/\tau) = C(t/\tau)^3$, using (2.35), and (2.40) becomes

$$E\{|\ell(t)|^2\} = C\varepsilon t^3\ , \qquad (2.41)$$

a result known to Obukhov and Landau since the early 1940s. Finally, in this brief "snapshot", I note another well-known formula that will be relevant later. The analogue of (2.40) for the rate of change of $\ln E\{|t(t)|\}$ is

$$\frac{d}{dt} \ln E\{|\ \ell(t)\ |\} = \frac{d}{dt} E\{|\ \ell(t)\ |\}/E\{|\ \ell(t)\ |\} = \tau^{-1} k\left[\frac{t}{\tau}\right]. \qquad (2.42)$$

Batchelor (1952b) - see also Monin and Yaglom (1975, pp.581-584) - argued that there is a range of (t/τ) of order 1 for which $k\ (t/\tau)$ in (2.42) is (approximately) constant. Let the value of this constant be a^*_1 , so that (2.42) gives (after integration)

$$E\{|\ \ell(t)\ |\} = |\ \ell_0\ |\ \exp\left[\frac{\alpha^*_1 t}{\tau}\right]. \qquad (2.43)$$

From (2.34), a^*_1 will be positive. (Strictly t in (2.43) ought to be replaced by $(t - t_0)$ where t_0 is a virtual origin representing the legacy of the initial dispersion period.) For $t \gg \tau$, (2.43) is, of course, invalid and has eventually to be replaced by (2.41).

The argument leading to (2.34) for $E\{\ell(t))$ applies, with obvious changes only, to a material surface element $S(t)$, with $S(0) = S_0$ where

$$s_{0i} = \varepsilon_{ijk}\ \ell_{0j}\ m_{0k} \qquad (2.44)$$

and ℓ_0 and m_0 are two material Line elements. It follows (Batchelor 1952b; Cocke 1969; Orszag 1970) that

$$E\{tS(t)l^2\} > |S_Q|^2 . \tag{2.45}$$

Moreover, there are analogues also to all the results for $\ell(t)$ based on (2.36); in particular there is a range of values of t of order τ for which

$$E\{|S(t)|\} = |S_O| \exp\left[\frac{\left(\alpha_1^* + \alpha_2^*\right)t}{\tau}\right], \tag{2.46}$$

where α_1^* is as in (2.43); and α_2^* is a further constant with $\alpha_1^* + \alpha_2^* > 0$. (In principle α_2^* could be negative but is generally believed to be positive.)

Apart from their intrinsic fascination, the results in this section have importance for the later consideration of the effects of molecular diffusion. In particular (2.45) and (2.46) quantify the remarks following Figure 3 in Lecture 1 that the area of a material volume increases rapidly with time on average, for the surface of any material volume is made up of many material surface elements to each one of which (2.45) and (2.46) apply. Moreover consider two adjacent material surface elements on each of which the concentration Γ is constant. Since the magnitudes of these areas increase (on average) in accordance with (2.46) (say) incompressibility requires the distance between the surfaces to decrease {on average) in proportion to $\exp[-(\alpha_1^* + \alpha_2^*)t/\tau]$; hence vrt increases (on average) in proportion to $\exp[\left(\alpha_1^* + \alpha_2^*\right)t/\tau]$. It will be seen in the next Lecture that this result must be modified in the presence of molecular diffusion. (The argument above is a subtle one since it may not be seen immediately why the distance between two material surface elements should not (on average) increase since it is possible to define a material line element $\ell(t)$ joining fluid particles in the adjacent surface elements such that $\ell_0 = \ell(0)$ is normal to the surface elements at t = 0. While (2.34) and (2.43) apply to this line element, the key point is that it does not remain normal to the surface elements. This argument is given mathematical form in Batchelor (1952b).)

2.5 Absolute and relative diffusion

It will have been noted in the previous section that no attention was paid to the translation of the elements as a whole , since what was of exclusive concern was the statistical properties of the relative velocity of two fluid particles. Further developments of these ideas have been examined by many authors (Richardson 1926; Batchelor 1952a,b; Monin and Yaglom 1975, pp,536-567 etc.). However, important though this subject is, it is essential to relate it to practical problems.

Consider one realisation of the dispersion of a cloud. As the result of random advection of the cloud as a whole (meandering), a fixed point in space will sometimes be in the cloud and sometimes not. The same comment obviously applies to a plume. For the ensemble as a whole, there will be a probability distribution associated with the event that at time t the fixed point in space with position vector x lies in the cloud. (To make this precise, a definition of the phrase "in the cloud" is required.) This meandering will make a contribution (usually substantial) to the mean and variance of $T(x,t)$. Gifford (1959) considered a fluctuating plume model in which it was assumed that, at each downwind distance x, the centroid (Y,Z) of the instantaneous concentration distribution in the cross-section was a random variable with p.d.f.

$$p(\eta(\varsigma;x) = \frac{1}{2\pi\Sigma_y\Sigma_z}\exp\left\{-\frac{1}{2}\left[\frac{\eta^2}{\Sigma_y^2}+\frac{\varsigma^2}{\Sigma_z^2}\right]\right\}. \qquad (2.47)$$

i.e- Y and *Z* are independent Normal random variables,each with zero mean and variances $\Sigma_x^2(z)$ and $\Sigma_z^2(x)$ respectively. Gifford assumed further that the concentration distribution $\Gamma(x,y,z)$ was of the form

$$\tau(x,y,z) = \frac{\dot{Q}}{2\pi\cup\Sigma^2}\exp-\left\{\frac{1}{2\Sigma^2}\left[(y-y)^2+(z-z)^2\right]\right\}, \qquad (2.48)$$

where Q is the steady rate of emission of material into the plume, U is the mean wind and $\Sigma(x)$ is a measure of the plume radius. Note that, in the model, Y and Z are the only random variables. I do not pause to evaluate $C(x)$ and $o^2(x)$ for this widely used and influential model, except to note that contributions to both arise from the meandering of the centroid ((2.47)) and from the assumed within - plume distribution [(2.48)]. Further details can be found in the original paper (see also Sykes (1984), Hanna (1984)).

Gifford clearly recognized that (2.48) was not justified for real plumes since the within - plume concentration distribution is itself a random variable. A more general approach follows from the work of Batchelor (1952a), and I shall apply it to clouds (although there is no difficulty in the extension to plumes).. Suppose x denotes position vector relative to an origin fixed in space. In any one realisation of the dispersion, let the centroid of the distribution be Xg(t), so that (by definition),

$$x_g(t) = \frac{1}{Q}\int x\,\Gamma\,(x,t)\,dv(x)\,. \tag{2.49}$$

Here Q is the total quantity of contaminant within the cloud, and is equal to the integral of ɼ (x,t) over all space by (1.4). Now define y by

$$Y = x - x_g\ (t)\ , \tag{2.50}$$

so that y is the position vector relative to a randomly moving origin, and $\Gamma_R(y,t)$ by

$$\Gamma_R(y,t) = r(x,t) = T(Y + x_g(t),t)\ . \tag{2.51}$$

I shall define relative diffusion to be that in which position vectors are measured relative to the randomly moving centroid. By contrast, absolute diffusion occurs when the origin of position vectors is fixed in space. In a relative diffusion ensemble, ensemble averages are over clouds whose centroids coincide at all times; in this way, use of a relative diffusion ensemble eliminates all contributions due to meandering. (As noted in Lecture 1, the choice of ensemble is a free one. However there is still confusion and misunderstanding about the fact that relative diffusion, as here defined, is a perfectly valid ensemble. Remarks in the penultimate sentence of the first paragraph of p.197 of Sykes, Lewellen and Parker (1984)are typical of the errors that abound in the literature.) Define $D_{Rij}(t)$ by

$$D_{Rij}(t) = \frac{1}{Q}\int Y_i T_j \Gamma_R(Y,t)\,dv(y)\,. \tag{2.52}$$

From (2.49), (2.50) and (2.51) it follows that

$$\begin{aligned} D_{Rij}(t) &= \frac{1}{Q}\int (x_i - x_{gi})(x_j - x_{gj})\,\Gamma\,(x,t)\,dv(x) \\ &= \frac{1}{Q}\int x_i x_j \Gamma(x,t)\,dv(x) - x_{gi}x_{gj}. \end{aligned} \tag{2.53}$$

Let the ensemble average of $D_{Rij}(t)$ be the relative dispersion tensor

L_{Rij} (t); the ensemble average of (2.53) gives

$$L_{Rij} = L_{ij} - \overline{x_{gi}x_{gj}}, \text{ i,e. } L_{ij} = L_{Rij} + \overline{x_{gi}x_{gj}}, \tag{2.54}$$

where L_{ij} is the absolute dispersion tensor defined in (1.24). Equation (2.54) is a kind of "parallel-axis" theorem, which shows that the contributions of meandering and relative dispersion to the random spatial spreading in an absolute frame are directly additive. When the extension of (2.54) to plumes is applied to Gifford's model, equations (2.48) and (2.47) give, respectively.

$$L_{Rij} = \begin{bmatrix} \Sigma^2 & 0 \\ 0 & \Sigma^2 \end{bmatrix}; \overline{x_{gi}x_{gj}} = \begin{bmatrix} \Sigma_y^2 & 0 \\ 0 & \Sigma_z^2 \end{bmatrix}, \tag{2.55}$$

where the tensors are (2x2) since only diffusion in the yz plane is relevant, and I have written y_1 for y and Y_2 for z. Thus

$$L_{ij} = \begin{matrix} \Sigma^2 + \Sigma_y^2 & 0 \\ 0 & \Sigma^2 + \Sigma_z^2 \end{matrix} \tag{2.56}$$

Further results on relative diffusion are given in the next Lecture.

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Lecture 3. Recent research on turbulent diffusion in shear flows

3.1 Introduction

In the remaining two Lectures, I shall some recent research, and also indicate some important problems that, in my view, merit attention. Given the time restrictions, severe selection has been necessary and I did not think it was possible, or even desirable, to attempt a comprehensive review. Rather I have tried to produce a quasi-coherent account of some work on my own interests, mainly because that is what I think I understand best !

Since, as noted in Lecture 1, early work in turbulent diffusion was, overwhelmingly, on the properties of the mean concentration C(x,t), I shall (perversely?!) emphasize investigations on $o^2(x,t)$, p(;x,t) and related quantities. Obviously nearly all such investigations are relatively recent; for example, it is remarkable now to read on p.583 of Monin and Yaglom (1971) that the equation [viz. (1.19)] governing $o^2(x,t)$ is "rarely investigated"!

3.2 Some early work on ($o^2(x,t)$

Following the results of Kolmogorov (1941) and their application to the small scales of the velocity field, it woe recognized that the same ideas could be applied to the small scales of a contaminant field (Monin and Yaglom 1975, pp.377-387, 395-448). For example Obukhov (1949) considered the concentration structure function D(x,r,t), defined by

$$D(x,r,t) = E\{[c(x+r,t) - c(x,t)]^2\} , \qquad (3.1)$$

where c(x,t) is the concentration fluctuation defined in (1.18). For sufficiently high Reynolds number there is an inertial subrange governing the velocity differences on length scales r, where

$$n \ll r \ll L ; \qquad (3.2)$$

for this subrange the statistical properties of the velocity differences are homogeneous, stationary and isotropic, and dependent only on r and ε. In (3.2) η, is the Kolmogorov microscale defined in (2.35) and L is a length characteristic of the energy-containing eddies. Using the same ideas there will be, for sufficiently high Péclet numbers (where the Péclet number is defined like the Reynolds number, but with κ replacing v), a Convective subrange governing quantities like D(x,r,t)on length scales r(=|r|), where

$$n_o \ll r \ll L \ , \qquad (3.3)$$

and η_o is to be determined. Within this subrange the statistical properties of concentration differences, i.e. quantities like D(x,r,t), will also be homogeneous, stationary and isotropic, and dependent only on r, ε and ε Γ, where

$$\varepsilon_\Gamma = 2\kappa \, E\{\{Vc)^2\} \qquad (3.4)$$

is the mean dissipation rate of concentration fluctuations discussed at the end of Lecture 1. Furthermore, the linearity of (1.2) for Γ(x,t) shows that D must be proportional to ε_Γ; dimensional arguments then give (Obukhov 1949)

$$D(x,r.t) = A \ \varepsilon_\Gamma \varepsilon^{-1/3} r^{2/3} \ , \qquad (3.5)$$

where A is a universal constant. Corrsin (1951) gave the corresponding result for the spectrum of c(x,t).

A more difficult, but associated, problem is the range of values of r for which (3.5) applies, i.e. the value of the scale η_o in (3.3). Batchelor (1959) argued that the mechanics of the convection process, represented by the term V.E{uc²} in (1.19), is not affected by molecular diffusion, i.e. it can be described by the analysis of material elements discussed in Lecture 2. Clearly, and as Batchelor pointed out, this assumes molecular diffusion becomes important only on scales λ whose order of magnitude is much less than n. In turn this obviously requires $v \gg \kappa$, and implies that, when $v \gg \kappa$, η_0 in (3.3) is of order η. For

$$\lambda \ll r \ll \eta \ , \qquad (3.6)$$

viscosity has an effect on the velocity field and therefore on D; thus (3.5) does not then hold. Batchelor (1959) gave a form for D valid when (3.6) holds but this will not be discussed here. Of relevance, however, is the order of magnitude of λ when $v \gg \kappa$. This can be estimated by requiring the order of magnitude of $V.E\{uc^2\}$, viz. $\eta^{-1}(\eta/\tau)0(o^2)$ under the given conditions, to be the same as that of the molecular diffusion terms in (1.19), viz. $K\lambda^{-2}0(o^2)$. Thus, from (2.35),

$$\lambda^2 = k\tau \rightarrow \lambda = \left[\frac{vk^2}{\varepsilon}\right]^{\frac{1}{4}} \text{ for } v >> k. \qquad (3.8)$$

When $v \sim k$, viscous and molecular diffusion terms become important at the same length scale; also, and consistently, η and λ are then both of order $(K^3/\varepsilon)^{\frac{1}{4}}$. Batchelor, Howells, and Townsend (1959) discuss the less practically important case of $v \ll \kappa$. The scale λ, given by (3.8), will be referred to as the conduction cut-off length; its magnitude in the atmosphere is usually of order 10^{-3}m (Monin and Yaglom 1975, p.494), and in oceans and large lakes is typically one order of magnitude smaller (Chatwin and Sullivan 1979c).

The only other substantial work on $o^2(x,t)$ prior to 1970 was by Csanady (1967a,b). He considered (1.19) as it applied to a steady plume, and looked for self-similar solutions by assuming the fltix terms, viz. $E\{uc^2\} - \kappa\nabla o^2$, could be modelled by an eddy diffusivity equal to that for $E\{uc\}$ in the equation for the mean concentration, and by some other, apparently rather arbitrary, assumptions. Although the resulting profiles had the same shape as some observed profiles, the model did not allow independent determination of the Magnitude of o^2 on the plume axis, and this appears to be a serious shortcoming, given the dependence of the magnitude of o^2 on such parameters as source size; see the discussion following (1.31) above.

3.3 An exact solution for $o^2(x,t)$

This section will deal exclusively with the dispersion of a cloud in a relative diffusion ensemble as defined in section 2.5. More particularly, it will consider only clouds of sufficiently small linear dimensions that the fluid velocity relative to that of the centroid of the instantaneous distribution of contaminant is linear in the position vector x of a point relative to the centroid. In these circumstances the relative velocity field is the sum of a solid-body rotation and a uniform rate-of-strain; the former of these will play no part in the statistical properties of ɼ(x,t) that are considered here and will henceforth be ignored. Then, at any time t, axes can be chosen so that the relative velocity field, to be denoted from now on, without possibility of confusion, by T(x,t), has coordinates such that

$$T = (\alpha_1 x_1 \alpha_2 x_2, \alpha_3 x_3), \qquad (3.9)$$

where x has coordinates (x_1,x_2,x_3). Incompressibility, viz. (1.1), requires

$$\alpha_1+\alpha_2+\alpha_3=0 \quad ; \tag{3.10}$$

without loss of generality, I can insist that

$$\alpha_1 \le \alpha_2 \le \alpha_3 \ , \ \alpha_1 < 0 \ , \ \alpha_3 > 0 \tag{3.11}$$

The aim of the calculations summarized in this section is to illustrate certain simple physical mechanisms and it will not be claimed that the results have exact practical significance. For this reason it will be supposed that the principal rates of strain α_1, α_2, α_3 are constant throughout each realisation, and that they are the same for each realisation of the ensemble. (Relaxation of these conditions would (perhaps greatly) increase algebraic/computational complexity without altering in any important way the conclusions reached,) The single random feature of the motion will then be the distribution in space of the directions of the principal axes. The linearisation of the velocity field in terms of the coordinates of x has justification provided the cloud dimensions are not much greater than the Kolmogorov microscale η; accordingly it is reasonable to suppose that the principal axes are distributed isotropically in space. The random velocity field described above has often been used to illustrate aspects of the small-scale structure of contaminant fields (Townsend 1951; Batchelor 1959; Saffman 1963); the work described below is that of Chat win and Sullivan (1979a) and its extension to plumes is described in Chatwin and Sullivan (1979b).

It is convenient first to consider rather briefly the case when the effects of κ are ignored, and when the initial distribution of contaminant is uniform over a sphere of radius a_o. Thus, in each realisation.

$$\Gamma(x,o) = \frac{3Q}{4\pi\alpha_0^3} \text{ for } |x| \le a_0 \quad \text{and} \quad \Gamma(x,o) = o \text{ for } |x| > a_0 \ . \tag{3.12}$$

According to (1.30),

$$\int \Gamma^2(x,o)\,dv(x) = \frac{Q^2}{\ell_0^3}\,, \tag{3.13}$$

and (3.12) gives

$$\ell_0 = \left[\frac{4\pi}{3}\right]^{1/3} a_0. \tag{3.14}$$

When k = 0, the initial sphere of radius a_o is transformed into an ellipsoid with semi-axes $a_i(t)$ (i = 1,2,3), where

$$a_i = a_0 \exp(\alpha_i t); \qquad (3.15)$$

(cf. the discussion of the stretching of material line elements in section 2.4). Thus, from (3.11), the maximum dimension of the ellipsoid is $2a_3$ and the minimum dimension is $2a_1$. When $\kappa^{-} = 0$ the concentration is uniform (and equal to its initial valueQ/ℓ_0^3) within the ellipsoid. Since the ensemble mean is obtained by considering all possible orientations of the ellipsoid in space to be distributed with equal probability, it follows immediately that the ensemble mean concentration $C(x,t)$ is equal to $T(x,0) = Q/\ell_0^3$) for (xl ≤ a and is zero for $|x| \le a_3$. For $a_1 < |x| < a_3$, the mean concentration changes continuously from Q/ℓ_0^3)to zero. It is also geometrically evident that L(t), defined in (1.25), is of order a_3 (t) since a_3 is exponentially greater than a ; moreover a_1 approaches zero exponentially quickly as $t \to \infty$ and, therefore.

$$C \sim \frac{Q}{\ell^3(t)} \text{ for } |\ x\ | >> a1, \text{ where } l(t) \sim L(t) \sim a3(t) \qquad (3.16)$$

The distribution of a^2 (x,t)is obtained by considering the average of $[\Gamma - C]$ over all possible orientations of the axes in space. It is, for example, obvious that $a^2 \equiv 0$ for $|x| \le a_1$ and for $|x| \ge a_3$.For $a_1 << |x| << a_3$, o^2 is of order $Q^2/\ell^3 \ell^3_0$ as predicted by the analysis that immediately precedes (1-31) in Lecture 1 (see detailed calculations in Chatwin and Sullivan (1979a)).

The preceding results imply, unambiguously, that the spatial structures of C and o^2 have a "core-bulk" structure irrespective of the initial distribution. Thus $C \sim Q/\ell^3_0$ and $0^2 \sim Q^2/\ell^6_0$ for $|x| \le a_1$, and $C \sim Q/\ell^3$ and $o^2 \sim Q^2/\ell^3 \ell^3_0$ for $|x| \gg a_1$ .The "core" radius of order a_1 tends to zero as $t \to \infty$in the hypothetical case when $k \ne 0$ that has just been considered.

Before going forward to consider the key issue,namely the effects of molecular diffusion,I wish to draw attention to the consequences of the above arguments for Richardson's distance-neighbour function. Reference should be made,of course,to Richardson (1926) and Batchelor (1952). The most important experimental paper is by Sullivan (1971),and calculations are given in Chatwin and Sullivan (1979a).

In the real case when $k \ne 0$, the algebra is more complicated than that given above. Although the axes used are not inertial (since the origin is the randomly-moving centroid), this does not change the equation governing T(x,t), viz. *(1,2);* this is simply a statement of mass conservation and does not involve an equation of motion. To avoid unnecessarily complicated algebra adding nothing to physical understanding I shall here consider the special case of the relative velocity field (3.9) with

$$\alpha_2 = \alpha_3 > 0 \ , \quad \alpha_1 = -2\ \alpha_3 < 0 \ , \qquad (3.17)$$

and the special initial distribution of concentration with

$$\Gamma(x, O) = \frac{2\sqrt{2Q}}{\ell_0^3} \exp\left[- \frac{2\pi r^2}{\ell_0^2} \right], r = |x| \qquad (3.18)$$

Note that (3.18) is consistent with (3.13). (Saffman (1963) gives the general solution of (1.2) with an arbitrary initial distribution of concentration when T is given by (3.9).) With (3.9), (3.17) and (3.18), the solution of (1.2) is (Chatwin and Sullivan 1979a)

$$\Gamma(x,\ t) = \frac{2\sqrt{2Q}}{ab^2} \exp\left[- 2\pi \frac{x_1^2}{a^2} + \frac{x_2^2}{a^2} + \frac{x_3^2}{b^2} \right] , \qquad (3.19)$$

where

$$a^2(t) = \left[\ell_0^2 - \frac{2\pi k}{\alpha_3} \right] e^{-4\alpha_3 t} + \frac{2\pi k}{\alpha_3} ; \qquad (3.20a)$$

$$b^2(t) = \left[\ell_0^2 + \frac{4\pi k}{\alpha_3} \right] e2\alpha_3 t - \frac{4\pi k}{\alpha_3} ; \qquad (3.20b)$$

In view of the isotropy, the ensemble mean concentration C(x, t) satisfies

$$C(x,t) = \frac{1}{4\pi r^2} \iint \Gamma(x,t) r^2 \sin\theta in\theta\varphi \ , \qquad (3.21)$$

where (r,θ,φ) are spherical polar coordinates with $X_1 = r\cos\theta$, $X_2 = r\sin\theta\cos\varphi$, $x_3 = r\sin\theta\sin\varphi$. Hence, substituting and integrating.

$$C(x,t) = c(r,t) = \frac{(2\pi)^{\frac{1}{2}} Q}{ab^2} e^{-2\pi r^2/b^2} \frac{\mathrm{erf}(r/d)}{(r/d)} \ , \qquad (3.22)$$

where

$$d(t) = \frac{ab}{\left\{2\pi\left(b^2 - a^2\right)\right\}^{\frac{1}{2}}} \tag{3.23}$$

and

$$\mathrm{erf}\left[\frac{r}{d}\right] = \frac{2}{\sqrt{\pi}} \int_0^{(r/d)} e - x^2_{dx}. \tag{3.24}$$

Also o^2 (x,t) satisfies

$$0^2\{x,t) = \frac{1}{4\pi\ r^2} \iint \{\Gamma(x,t) - c(r,t)\}^2 r^2\ \sin\theta in\theta\varphi \tag{3.25}$$

and, after integration o^2 (x,t) = o^2 (r,t), where

$$o^2(r,\ t) = \frac{Q^2}{a^2 b^4}\ e\ -\ 4\pi\ r^2/b^2 \left\{ 4\pi^{\frac{1}{2}} \frac{\mathrm{erf}\left(2^{\frac{1}{2}} r/d\right)}{\left(2^{\frac{1}{2}} r/d\right)} - 2\pi\left\{\frac{\mathrm{erf}(r/d)}{(r/d\ )}\right\}^2 \right\} \tag{3.26}$$

Finally, before discussion and interpretation of these results, note that the length L(t) defined in (1.25) satisfies

$$L^2(t) = \frac{1}{Q}\int_0^{\infty} 4\pi\ r^4 C(r,t)\,dr = \frac{1}{4\pi}(a^2 + 2b^2). \tag{3.27}$$

When k = 0 and a_3t is of order 1 (or greater), (3.27) and (3.23) give, using the formulae in (3.20):

$$L(t) = \frac{\ell_0}{(2\pi)^{\frac{1}{2}}}\ e^{\alpha_3 t} >> \ell_0;\ d(t) = \frac{\ell_0}{(2\pi)^{\frac{1}{2}}}\ e\ -\ 2\alpha_3 t << \ell_0 \tag{3.28}$$

It follows from (3.22) and (3.26) that $C \sim Q/\ell_0^3$ and $o^2 \sim Q^2/\ell_0^6$ for $r \leq d(t)$. Noting from (3.28) and (3.17) that $d \sim \ell_0 e^{\alpha_1 t}$, these "core" estimates are the same as those derived above in the case of a uniform initial distribution. Similarly, it follows that $C \sim Q/\ell^3$ and $\sigma^2 \sim Q^2/\ell^3\ell_0^3$ for $r \gg d(t)$, where $\ell(t) = L(t) \sim \ell_0 e^{\alpha_3 t}$; again these "bulk" estimates are the same as those for the case of a uniform initial distribution. In either case, L(t) is the order of magnitude of the spatial extent of the mean cloud, and d(t) is the order of magnitude of the thinnest part of the cloud in one realisation. It will be clear also that these results are consistent with the discussion in section 2.4 of Lecture 2 and, moreover, that there is quantitative order of magnitude agreement between the present results and those in (2.46) and the subsequent discussion. (Note that this presumes $\alpha_i^\star/\tau$ in section 2.4 is the present α_1.)

There is a single, but crucial, difference in the real case when $k \neq 0$. This is that d(t), defined in (3.23), now approaches the non-zero value $(k/\alpha_3)^{\frac{1}{3}}$ Since $\alpha_3 \sim \tau^{-1}$ because it is determined by Kolmogorov microscaling, it follows that, for $\kappa \neq 0$,

$$d(t) \sim \lambda = \left[\frac{vk^2}{\varepsilon}\right]^{\frac{1}{4}} \text{ as } t \to \infty \tag{3.29}$$

Thus the thinnest part of the cloud in one realisation approaches the conduction cut-off length, defined in (3.8) and obtained by a different argument, as the length scale at which the statistics of the concentration differences are affected by molecular diffusion. In {3.19), $ab^2 \sim L^2\lambda$ for large a_3t, where the estimate of L(t) is as in (3.28), i.e. it is unaffected by molecular diffusion. Also $a(t) \sim d(t) \sim\lambda$ and $b(t) \sim L(t)$. Thus the distribution of concentration in any one realisation given by (3.19) occupies a flat ellipsoid of revolution (a "discus") with (ultimately) constant thickness of order A but expanding cross-section. The volume of the ellipsoid when $k \neq 0$ is of order $ab^2 \sim L^2\lambda$; consequently the maximum concentration, and the order of magnitude of the overall concentration, is of order $Q/L^2\lambda$ i.e. it decreases with t (unlike the case when $\kappa = 0$). This occurs entirely as the result of the constancy of the minimum dimension of the ellipsoid which, as the mathematics makes clear, is due to the balance between advection and diffusion in this: direction.

It is now geometrically obvious, on considering an isotropic superposition of the ellipsoidal distributions just discussed, that there is still a "core" with, now, constant thickness of order λ, and a "bulk" with the same dimensions, viz. L(t), as when $\kappa = 0$. Thus the order of magnitude of C in the bulk is still Q/L^3,i.e. it is unaffected by molecular diffusion; this confirms the conclusions reached by general considerations in section 1.4 of Lecture 1- By contrast the size of o^2 is affected by molecular diffusion. Its bulk order of magnitude can be estimated from 3.26), or by the following simple, but illuminating, argument based on 1.27), viz.

$$E\{r'^{2}\} = C^2 + a^2 \quad . \tag{3.30}$$

Since r^2 is of order $Q^2/L^4\lambda^2$ over a region of space of volume of order $L^2\lambda$

and since this region is isotropiclly distributed within a volume of space of order L^3,it follows that $E\{\Gamma^2\}\sim\{Q^2/L^4\lambda/L^2).(L^2\lambda/L^3)= Q^2/L^5\lambda$. Since $C^2\sim Q^2/L^6 << Q^2/L^5\lambda$,(3.30) gives the bulk magnitude of 0^2 as of order

$$o^2 \sim \frac{Q^2}{L^5\lambda} \tag{3.31}$$

The work in this section is a surnnary of some of that in Chatwin and Sullivan (1979a); that paper includes some discussion of other cases and discusses whether the "core-bulk" structure that the above results exhibit exists in flows more general than (3.9). This particular aspect will not be considered further here except to note that experimental verification of the separate core structure would be extremely difficult because of the tiny size of the core, illustrated by the typical values of λ given after (3.8). What the above calculations do illustrate clearly include the reasons why:

(a) molecular diffusion does not affect the magnitude of the mean concentration C;

(b) molecular diffusion does affect the magnitude of the concentration variance o^2;

(c) the conduction cut-off length λ is the appropriate measure of the scales of the contaminant field that are affected by molecular diffusion.

The velocity field (3.9) has also been used to construct exact solutions for the p.d.f. $p(\theta;x,t)$ defined in (1.6) (Kowe and Chatwin 1985).

3.4 Random walk models

In simple terms the difficulty with turbulent diffusion is in making quantitative predictions for real flows that have some reliability. In the remaining two sections of this Lecture, I want to comment on two of the currently most popular methods that are being used in an attempt to improve this situation. (Because of the limitations to small scales the work summarized in the previous two sections cannot be extended in any obvious way to real flows; it is the physical insight they provide that is valuable.) I do not have the time to do more than provide a few remarks and references.

The first method can be motivated by considering the statistics of the motion of a contaminant molecule. Let $X(t)$ be the position of the molecule at time t and let $p(x,t,y,s)$ be the p.d.f. of $X(t)$ conditional on $X(s) = y$, where $s \leq t$. Suppose that the concentration field $r(y,s)$ is known, e.g. by release occurring in a specified way at time s. Then it is obvious (Monin and Yaglom 1971, p.584; Thomson 1987) that

$$C(x,t) = E\{\Gamma(x,t) = \int p(x,t \mid y,s)\Gamma(y,s)\,dv(y) \qquad (3.23)$$

(As usual I consider the dispersion of a cloud.) Therefore, if $p(x,t|y,s)$ can be evaluated, or estimated, the mean concentration can be found. One method of doing this is to construct a stochastic model for the evolution of X(t), and random walk models are one example for this. A simple such model is provided by the Langevin equation for $V(t)= \partial x/\partial t$; this can be written

$$dv_i = -\mu V_i dt + \sigma d\Sigma_i \text{ with } dx_i = V_i\,dt\,, \qquad (3.33)$$

where μ and o are constants, and $d^{\Sigma}{}_i$ are the increments of a Wiener process $\Sigma_i(t)$ (Feller 1971, pp.99, 181). The Wiener process has, among its properties, those that $\xi_i(t + dt) - \xi_i(t)$ is a Normally distributed random variable with mean zero and variance dt (Thomson 1987, 1989b); the Langevin equation was proposed (in, 1908) to describe Brownian motion. For many reasons (Thomson 1987,1989b) this model is inadequate in shear flows, and is replaced by the more general model:

$$dv_i = a_i(X,v,t)dt + b_{ij}(x,v,t)d\Sigma_j \;;\; dx_i = v_i dt\;, \qquad (3.34)$$

where a_i and b_{ij} are deterministic, and $d\Sigma_i$ is as in (3.33). Van Dop, Nieuwstadt and Hunt (1985) and Thomson (1987) discuss (3.34) and give examples. In particular Thomson shows how a_i and b_{ij} ought to be chosen to satisfy certain desirable criteria.

The emphasis in this Lecture is on concentration fluctuations. Fortunately it is easy to generalize the results above to two molecules with trajectories $X^{(1)}(t)$, $X^{(2)}(t)$ by writing X(t) for the six-dimensional vector $(x(^1)(t),X^{(2)}(t))$, with identical extensions for x, y and V. The two-molecule extension of <3.32) is then (Thomson 19&9a,b):

$$E\{\Gamma_2(x,t)\} = \int p(x,t,y,s)\Gamma_2(y,s)\;dv(y), \qquad (3.35)$$

where

$$x = (x^{(1)}, x^{(2)});\Gamma_2(x,t) = \Gamma(x^{(1)}),t)\Gamma)\Gamma^{(2)},t). \qquad (3.36)$$

Formally, equation (3.34) applies as a model of the (now) six-dimensional process X(t). Thomson (1989a,b) extends his discussion to cover the choice of a_i and b_{ij} in the present two molecule case, and discusses earlier work by others, including Durbin (1980). Note that $E\{T^2(x,t)\}$ is obtained from (3.35) by putting $x^{(1)} = x^{(2)}$, and that this can be combined with the estimate of $E\{T(x,t)\} = C(x,t)$ obtained from (3.32) to yield an estimate of $o^2(x,t) = E\{T^2(x,t)\} - C^2(x,t)$. In practice, reasonably reliable estimates of p(x,tly,s) for both processes require a large number of numerical simulations of the two random walks. For example Thomson (1989a,b) followed 30000 pairs of molecules; he also used a "particle-splitting" technique for obtaining more reliable estimates in the especially difficult cases when the two molecules are close together.

These techniques have produced good comparisons with some datasets including those of Fackrell and Robins (1982) and Warhaft (1984). It is not yet clear how useful they will be in practice for more complex flows given (a) the need for large computing capacity, and (b) certain rather subtle inconsistencies in the above formulation of the two molecule random walk. These are exposed and discussed by Thomson (1989a,b), but will not be considered further here. Reference should also be made to Pope (1987).

It is however interesting in view of earlier material in these lectures to consider certain differences between the properties of T(x,t) and $r_2(x,t)$ that are relevant to the construction and results of random walk models. If, following the notational device introduced above, the six-dimensional random velocity field $T_2(x,t)$ is defined by

$$T_2(x,t) = [T(x^{(1)},t),T(x^{(2)},t)] \ , \tag{3.37}$$

then $r_2(x,t)$ satisfies

$$\frac{\partial\Gamma_2}{\partial t} + \Gamma_2.V_2\Gamma_2 = k\nabla_2^2\Gamma_2, \tag{3.38}$$

where V_2 is the six-dimensional gradient operator with components $[\partial/\partial x_i^{(1)}$ $[\partial/\partial x_i^{(2)}$Despite the formal identity of (3.38) with (1.2), there are hidden complications due to the structure of $T_2(x,t)$ in (3.37). For example this field is inhomogeneous, even in homogeneous turbulence, since the joint distribution of $T(x^{(1)},t)$ and $T(x^{(2)},t)$ always depends on $x^{(1)}$ - $x^{(2)}$Also, for initially coincident pairs of molecules, the effect of T_2 does not separate them; thus molecular diffusion must be important in determining r_2, and hence o^2. This confirms what is already known from work summarized above.

The comments above have been based on work by Thomson, since this what is most familiar to me. However this type of model is currently undergoing rapid development, particularly for the two particle case and there is some controversy about approach, and formulation. 'Reference should be made, for example, to Kaplan and Dinar (1988).

3.5 Remarks on closure and related hypotheses

In a sense, random walk models constitute a closure hypothesis because they replace the infinite hierarchy of eqations for $c(x,t)$ or $c^{\overline{2}}(x,t)$ (or any other statistical property of the concentration) that was discussed in Lecture 1 by an approximate, but closed, calculation scheme. However the term closure hypothesis is normally restricted to an approximation (or approximations) applied to equations, like (1.17) or (1.19), that are more directly related to (1.2) than the equations used in random walk models. In principle closure hypotheses can be devised for situations of any geometrical and physical complexity; in practice most emphasis has been placed (as with random walk models) on relatively simple flows, and. this is understandable.

Much work has been done in modelling data by Warhaft and Lumley (1978) who measured the decay of o^2 for temperature fluctuations in statistically steady grid turbulence. Newman, Launder and Lumley (1981) modelled the observed decay beginning with the equations:

$$U \frac{do^2}{dz} = -\varepsilon_\Gamma \; ; \tag{3.39a}$$

$$U \frac{d\varepsilon_\Gamma}{dz} = -\, 4kE \left\{ \frac{\partial c}{\partial x_i} \frac{\partial c}{\partial x_j} \frac{\partial u_i}{\partial x_j} \right\} - 4k^2 E \left\{ \left[\frac{\partial^2 c}{\partial x_i \partial x_j} \right]^2 \right\} . \tag{3.39b}$$

In these equations U is the magnitude of the mean velocity in the direction of the z axis, and the other symbols have been defined earlier; in particular ε_Γ is the mean dissipation rate of the fluctuations defined in (3.4). Note that (3.39a) is the exact form of (1.19) under the assumed conditions (homogeneous except for the constant and uniform mean wind), while (3.39b) is the exact form of the equation for ε_Γ under the same conditions. By analogy with the successful procedure used in modelling the decay of the turbulent kinetic energy, the right-hand side of (3.39b) was written in the form - $k_\Gamma\{\varepsilon_\Gamma^2/\sigma^2\}$where k_Γ is a dimensionless decay rate. In turn k_T was expressed in terms of E_T, o and corresponding properties of the turbulent velocity field. This gave four coupled ordinary differential equations forming a closed system, but there remains a choice of certain constants. Appropriate choices of these gave good agreement with the data, and with other comparable datasets. The principles involved in choosing the details of the closure, e.g. the expression for k_Γ in the above example, are interesting and are discussed by e.g. Lumley (1978) and Lumley and van Cruyningen (1985). Nevertheless, to a non-expert in this field (like me), the procedure is not totally satisfactory since it needs so much "tuning", much more for example than in applications of random walk models to the same, or comparable, datasets (Thomson 1989b).

A totally different type of approximation to the one-point closure and second-order modelling technique illustrated above is to apply two-point closures, and this has primarily been applied to the equations in spectral form for situations that are homogeneous and isotropic, or approximately so. (It is only in these circumstances that the use of spectra confers algebraic advantages.) Such methods are discussed by Lareheveqrue, Chollet. Herring, Lesieur, Newman and Schertzer (1980) and by Lesieur, Montmory and Chollet (1987), and, of course, elsewhere. The former of these papers discusses the Test Field Model (TFM) and the eddy-damped. Marrovian procedure (EDM), and models with reasonable agreement the data of Warhaft and Lumley (1978) and others. There is unfortunately insufficient space to discuss any details; however these models also need tuning to give numerical predictions and it is not clear how successful, or how practical, they will be for complex real flows.

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Lecture 4. The interpretation of data and some datasets

4.1 Introduction

This last Lecture attempts to redress a serious imbalance in the course so far, namely an overemphasis on theory at the expense of data. Turbulence and turbulent diffusion, like all branches of fluid mechanics and perhaps more than most, cannot be understood (let alone predicted) without both types of investigation. Perhaps, in an ideal world, each of us would be both experimenter and theoretician!

The first part of the Lecture will consider many datasets from turbulent shear flows, and present and discuss a framework that enables them to be understood rather simply. The results should certainly be useful to modellers. The second (and final) part will discuss some problems that occur in using, and modelling, data; some of these are connected with the instrument smoothing that is normally unavoidable.

4.2 Some data on C *and* o^2

Table 1 gives information about some typical experiments on steady scalar dispersion conducted in a variety of steady self-similar turbulent shear flows. It will be clear that these include several different contaminants, including heat and smoke, and a range of measurement techniques. Despite the differences in the experimental configurations, all the experiments in Table 1 (and nearly all similar ones that have been studied) have two common features.

First, in each case, the profiles of C and o^2 are themselves self-similar (or approaching self-similarity). Let z denote distance downstream from the (effective) source in the direction of mean flow and let $C_M(z)$ denote the maximum value of C{x) at each cross-section. (As noted above, all the statistical properties considered in this section will be independent of time t.) In cases like jets and wakes, $C_M(z)$ is the centreline value and, for turbulent boundary layers with the source at the wall, it is the wall value. The self-similar structure observed in the profiles of C and o^2 shows that there is a transverse length-scale a(z) which, when used to non-dimensionalise the transverse coordinate(s) in plane or cylindrical geometry, produces a dimensionless transverse coordinate η such that, for sufficiently large z.

$$C(x) = C_M(z)F(\eta)\,;\ o^2(x) = C_M^2(z)f(\eta) \qquad (4.1)$$

NO	WORKERS	FLOW	CONTAMINANT	MEASUREMENT TECHNIQUE
(1)	Becker, Hottel and Williams (1967)	Round jet	Oil smoke	Light-scatter
(3)	LaRue and Libby (1974)	Plane wake	Heat	Platinum wire resistance thermometer
(3)	Antonia, Prabhu and Stephenson {1975)	Round jet with coflowing stream	Heat	Platinum/10% rhodium wire
(4)	Sreenivasan, Danh and Antonia (1976)	Smooth-walled boundary-layer	Heat	Wollaston wire
(5)	Shaughnessy and Morton (1977)	Round jet exhausting into a secondary airflow	Smoke particles	Light-scatter
(6)	Birch, Brown, Dodson and Thomas (1978)	Round methane jet	Methane	Raman scattering of laser light
(7)	Gad-el-Hak and Morton (1979)	Grid turbulence	Smoke particles	Laser anemometer
(8)	Fackrell and Robins (1982)	Rough-walled boundary-layer	Propane/helium mixture	Flame ionisation detector system
(9)	Antonia, Browne, Chambers and Rajagopalan (1983)	Plane jet	Heat	Wollaston wire

TABLE I. Experiments in which profiles of C *and* σ^2 *were measured.*

with F(0) = 1 because of the definition of $C_M(z)$. In most flows, including all those in Table 1, F(n) is Gaussian, or approximately so.

The second feature that all the experiments in Table 1 have in common is that f(n) in (4.1) has a maximum at a non-zero value of n of order 1- Figure 4, adapated from Antonia, Prabhu and Stephenson (1975) is entirely typical of all the datasets. Most other measurements have this property but not all; one exception is the work of Nakamura, Sakai and Miyata (1987) on the dispersion of dye in grid-generated water turbulence using a light absorption probe, in which (4.1) held but the maximum of $f(\eta)$ was at n = u.

4.3 A simple framework and its development

Use of a simple framework turns out to explain the robust features noted above; it is particularly noticeable that it does not seem to matter whether the contaminant is passive or not. The framework is once more based on consideration of the hypothetical case when $\kappa = 0$, and aspects of it were discussed in many of the papers listed in Table 1 and elsewhere (Chevray and Tutu 1977; Chatwin and Sullivan 1987a,b). In all the experiments in Table 1, the concentration at the source was uniform; if its value is denoted by θ_1, the p.d.f. $p(\theta;x)$ is given by $p_0(\theta;x)$ in (2.3), viz.

$$P_0(\theta;x) = \pi_0(x)\delta(\theta-\theta_1) + [1-\pi_0(x)]\delta(\theta) , \qquad (4.2)$$

where, using the zero subscript notation introduced after (2.4),

$$\pi_0(x) = \mathrm{prob}\{\Gamma_0(x,t) > 0\} \qquad (4.3)$$

is the intermittency factor in this hypothetical situation. Use of (1.10) and (1.11) gives

$$C(x) = C_Q(x) = \theta_1 \pi_0(x) \; ; \; o^2(x) = O^2{}_0(X) = \theta^2{}_1\pi_0(x)[1-\pi_0(x)] \; . \qquad (4.4)$$

Hence, eliminating π_0 ,

$$o_0^2 = C_0(\theta_1 - C_0) = (\frac{1}{2}\theta_1)^2 - (c_0 - \frac{1}{2}\theta_1)^2 \qquad (4.5)$$

Thus, in this hypothetical situation, σ_0^2 has a maximum value of $\frac{1}{4}\theta_1^2$ which it takes at all points on the surface $c_0(x) = \frac{1}{2}\theta_1$. The schematic Figure 5 shows the transverse and axial variations of σ_0^2/θ_1^2 predicted by (4.5) for a case when

$$C_0(x) = C_M(z)e^{-\frac{1}{2}\eta^2}, \frac{C_M(Z)}{\theta_1} = \frac{17.5d}{(z-5d)}, \qquad (4.6)$$

for $z \gg d$, where d is the source diameter. While (4.6) is typical of observed behaviour in jets (Becker, Hottel and Williams 1967), the main features of Figure 5 are independent of (4.6); indeed (4.5) - and therefore the general properties of the curves in Figure 5 - does not require self-similar or even steady behaviour, or that the scalar be passive.

A brief examination of the datasets listed in Table 1 shows, unsurprisingly, that they are not quantitatively consistent with (4.5). Nevertheless, there are suggestive points of qualitative agreement. It has already been noted that all the observed transverse profiles of o^2 have the off-axis minimum at $n \sim 1$ that is observed at Station A in Figure 5(b) - refer to Figure 4. Also those papers showing the axial variation of σ^2/θ_1^2 (Becker, Hottel and Williams 1967; Birch, Brown, Dodson and Thomas 1978; Fackrell and Robins 1982; Pitts and Kashiwagi 1984) show a maximum as in Figure 5(c). However the measured maximum values of σ^2/θ_1^2 are significantly less than $\underline{1}$, predicted by (4.5). For example, Becker, Hottel and Williams (1967; Figure 7) observe a maximum value of σ^2/θ_1^2 on the centre-line of order 0.018, occurring when $C/\theta_1 = 0.77$ (not $\underline{1}$ as predicted by (4.5)), and $z/d = 13.2$. Molecular diffusion has several interrelated effects that invalidate (4.5) for real flows. Host fundamentally perhaps, the maximum concentration occurring at any point during any realisation is less than the source concentration θ_1 by a factor that increases with z. Also molecular diffusion causes o^2 to be dissipated in a way which depends not only on this reduction of the maximum concentration, but also on the statistical properties of the velocity field inasmuch as these determine the geometrical properties of the scalar - containing volumes emanating from the source. For well understood reasons, the magnitudes of $C(x)$ and $o^2(x)$ in the types of flow being considered do not depend to any measurable degree on the value of the molecular diffusivity κ; it will be noted that there are wide variations in the values of κ in the experiments of Table 1. Given the observed self-similarity of the data, it now seems natural to attempt to account for the effects of k on the maximum concentration by replacing θ_1 in (4.5) by a local concentration scale $\alpha C_M(z)$, where $C_M(z)$ is the maximum mean concentration at downstream distance z defined in (4.1) and a is a constant of order unity. If this hypothesis is correct the value of a can be expected to depend on factors like the type of flow and the source geometry, but not on κ. In the first instance, therefore, the hypothesis is that (4.5) should be replaced by $o^2 = C(\alpha C_M - C)$. (From now on, following work in the earlier Lectures C_0 will be written C, since κ

has no effect on the mean concentration.) But this relationship requires the value of a to account for both effects of *κ* described above, and this seems likely to be unreasonable in general. There is no conceptual difficulty in extending the hypothesis very simply by allowing a constant reduction factor ß in the equation for o^2, and it happens that this is easy to accommodate practically. Therefore the main work of the remainder of this section will be to investigate whether the datasets from Table 1 satisfy

$$\sigma^2 = ßC(\alpha C_M-C)\ , \qquad (4.7)$$

with $C = C(z,\eta)$, $\sigma^2 = \sigma^2(z,n)$, and a and ß constant.

Before giving the results of this investigation, it is helpful to mention two points that will be discussed later. Although (4.5) was derived from the exact form {4.2) for $p0\{\theta;x)$, no hypothesis has yet been made here concerning the p.d.f. in real flows that may lead to {4.7). Secondly, increasing attention is being given to the actual and perceived (i.e. measured) values of $o^{2\prime}$ and other statistical properties of $r'(x,t)$, that may occur due to instrument smoothing (Sullivan 1984; Carn and Chatwin 1985; Chatwin and Sullivan 1987a; Derksen and Sullivan 1987; Hole and Chatwin 1987; Mole 1989), If there are such differences they may well affect the values of α and/or ß, but they do not affect the data analysis and will, for the moment, be ignored.

After proposing (4.7) and drawing several of the graphs in Figure 6 , it was discovered that Becker, Hottel and Williams (1967, Figure 8) had shown that their data followed (4.7) "highly accurately" with a = 1.31 and ß = 0.156. However no physical explanation of its validity was then attempted; thus its status until now was purely empirical.

The procedure adopted to test whether (4.7) described the other datasets in Table 1 was, first, to determine a by the location of the maximum in the self-similar transverse profile of o^2; according to (4.7) this occurs when $C/C_M = \frac{1}{2}\alpha$ Graphs of $C(\alpha(_{M-C})/\sigma^2$ against η were then drawn.

It can be seen from Figure θ that this quantity is a constant, within experimental error, for each dataset for n less than about 1.75. Thus each dataset in Table 1 is consistent with (4.7), and the value of the constant is $ß^{-½}$. Table 2 lists the values obtained in this way.

NO	(1)[1]	(2)	(3)	(4)[2]	(5)	(6)[3]	(7)	(8)[4]	(9)
α	1.31	1.17	1.16	1.09	1.24	1.27	1.52	1.35	1.15
ß	0.16	0.46	0.34	0.46	0.12	0.14	0.37	0.72	0.20

[1] Values from empirical relationship in paper. [2] Data for $z/d = 102$.
[3] Data for $z/d = 40$. [4] Data for $z/H = 5.00, 5.92$.

TABLE 2. Values of a and ß for Table 1 datasets.

The method used in Figure 6 artificially highlights measurement, and graph-reeding, errors in the low values of C and o^2 at higher values of η. (In fact, graph-reading errors account for a substantial proportion of the scatter in the diagrams in Figure 6; these are relatively greatest at the larger values of n.) Despite this, the evidence from Figure 7 is that such errors are likely to be of little practical significance; equation (4.7) provides at least as good a fit (and arguably better) to the data of Fackrell and Robins (1982) as a complicated, and specifically developed, formula due to Wilson, Robins and Fackrell (1982).

It will be observed that all the entries in Table 2 satisfy $1 < \alpha < 2$, and $ß < 1$. From (4.7) it follows that

$$\sigma^2 = \beta\{\frac{1}{2}\alpha^2 C_M^2 - (c - \frac{1}{2}\alpha C_M)^2\} \qquad (4.8)$$

and, since $c \leq C_M$ everywhere at any fixed cross-section, an off-axis (i.e. for $n \neq 0$) maximum in the profile of o^2 occurs only if $\alpha. < 2$. There is no fundamental reason why this shoud be so. It has already been noted that no off-axis maximum occurred for the profiles of o^2 measured by Nakamura, Sakai and Miyats (1987), and it is therefore very interesting that these authors showed (figure 16 of their paper) that their profiles could be well fitted by a curve which is exactly (4.7) with $\alpha = 3$ (and $ß = 1$). Two further points about these experiments are relevant. Nakamura, Sakai and Miyata attributed the differences between their results and those, also from grid turbulence, of Gad-el-Hak and Morton (1979) - dataset No. (7) in Tables 1 and 2, and Figure 6 - to the fact that their fluctuating concentration field was highly intermittent everywhere unlike that recorded by Gad-el-Hak and Morton, In view of later material in this Lecture it would be interesting to know whether this difference in intermittency behaviour was genuine, or due to differences in the degree of instrument smoothing. The second point is that the observed values of o^2 were shown to be strongly dependent on source geometry consistent with the theoretical work of Chatwin and Sullivan (1979); see the discussion in Lecture 1 centred on equations (1.30) and (1.31).

4.4 The development of the framework to higher moments

Given the remarkable agreement between 4.7) and the data, it is natural to extend the hypothesis Leading to (4.7) to further statistical properties such as the higher moments $E\{c^n(x,t))$ and shape parameters like the skewness and flatness factor (or kurtosis). The moment generating function M(s) is defined by

$$M(s) = E\{e^{(\Gamma-C)s}\} = \int_0^\infty e(\theta - c)^s p d\theta = 1 + \sum_{n=2}^\infty E\{c^n\}\frac{s^n}{n!}. \qquad (4.9)$$

Substituting in for p the simple form P_0 in (4.2) (on which (4.7) was based) gives

$$1 + \sum_{n=2}^\infty E\{c^n\}\frac{s^n}{n!} = \pi_0 e^{(\theta_1 - c)s} + (1 - \pi_0)e^{-cs} \qquad (4.10)$$

and then, on expanding the right-hand side and eliminating π_0 using (4.4)

$$E\{C^n\} = \frac{C}{\theta_1}(\theta_1 - C)^n + (-1)^n\left[1 - \frac{C}{\theta 1}\right]C^n \qquad 4.11)$$

It is elementary to check that 4.11) vanishes as $\theta_1 \to 0$ there is then no scalar) and when $C = \theta_1$ (all the fluid is then occupied by scalar of concentration θ_1, i.e. $\pi_0 = 1$ everywhere).

The hypothesis leading to (4.7) can be extended in an obvious way to all n by replacing in (4.11) (a) θ_1 by the local concentration scale aC_M, and (b) equality by proportionality. Thus (4.11) is replaced by

$$E\{C^n\} \propto \frac{C_M^n}{\alpha}\{\varphi(\alpha - \varphi)^n + (-1)^n(\alpha - \varphi)\varphi^n\}; \varphi = \frac{c}{c_M}. \qquad (4.12)$$

In particular

$$E\{C^3\} = \gamma c_M^3\varphi(\alpha - \varphi)(\alpha - 2\varphi); E\{C^4\} = \delta C_M^4\varphi(\alpha - \varphi)(\alpha^2 - 3\alpha\varphi + 3\varphi^2\}, \qquad (4.13)$$

for some constants γ and δ. Furthermore the shape parameters S (skewness) and F (flatness factor) then satisfy

$$S = \frac{E\{c^3\}}{\sigma^3} = \frac{\gamma}{\beta^{3/2}} \frac{(\alpha - 2\varphi)}{\sqrt{\{\varphi(\alpha - \varphi)\}}} ; \qquad (4.14a)$$

$$F = \frac{E\{c^4\}}{\sigma^4} = \frac{\delta}{\beta^2} \frac{(\alpha^2 - 3\alpha\varphi + 3\varphi^2)}{\varphi(\alpha - \varphi)} , \qquad (4.14b)$$

(Chevray and Tutu (1977) give the formulae corresponding to (4.14) that follow from (4.11) rather than (4.12).)

There is unfortunately insufficient space and time to discuss the results of testing datasets against the formulae (4.13) and (4.14), although this has been done by myself and Sullivan. Very briefly, the agreement is reasonable given the difficulty of making measurements of higher moments (which results in large experimental scatter); the principal discrepancy is that the experimental results are flatter in the central region (near n = 0) than (4.13) and (4,14) predict. Table 3 gives values of y and δ that we have determined from three of the datasets in Table 1. Also shown are figures demonstrating that, within experimental error for these datasets.

$$\beta^{1/2} = \gamma^{1/3} = \delta^{1/4} \qquad (4.15)$$

NO	γ	&	$ß^{1/2}$	$\gamma^{1/3}$	$\delta^{1/4}$
(2)	0.3	0.2	0.68	0.67	0.67
(3)	0.15	0.08	0.58	0.53	0.53
(6)	0.034	0.035	0.37	0.32	0.43

TABLE 3. Values of certain constants for three datasets in Table I.

A theoretical justification of (4.15) is based on the reasons for the introduction of the constants a and ß into (4.7). The constant ß allowed for possible overall reduction to measured values of σ^2 (relative to the hypothetical $\kappa = 0$ result) additional to that due to reduction, by molecular diffusion, of the maximum concentration from θ_1 to order aC_M. However conservation of mass requires this reduction to be accompanied by on overall increase in the background concentration (from zero when $k = 0$), and therefore a further reduction in the range of concentration values. Since all moments are estimated from the *same* data record, there is an immediate inference that each of the quantities in (4.15) measures this range reduction at least to an order of magnitude. The evidence presented in Table 3 suggests that the argument has much more precision than this!

A first (and partial) account of the work described in sections 4.2, 4.3 and 4,4 is given by Chatwin and Sullivan (1987b), However the work has been written up in full (Chatwin and Sullivan 1990), and is expected to appear in the March 1990 edition of Journal of. Fluid Mechanics which will consist entirely of invited papers in honour of the 70th birthday of Professor G.K. Batchelor, F.R.S. In the meantime, I should be happy to supply copies of the manuscript of this paper to anyone who writes to me. We believe that the simplicity and apparent generality of the results should make them of great practical value even, perhaps, in more complex flows such as atmospheric dispersion.

4.5 Some remarks on the intermittency factor

The only intermittency factor so far considered in these Lectures has been that defined in (4.3) for the hypothetical case when $k = 0$. Given the physical ideas behind such models as Gifford's fluctuating plume model (see section 2.5 of Lecture 2), it is natural to consider models of $p(\theta;x,t)$ in atmospheric dispersion in which meandering is (at least partly) accounted for by an intermittency factor measuring the probability of being in the plume. Recent work relevant to this is given in e.g. Hanna (1984) and Ride (1984a,b; 1988). It has also been "fashionable" for some years to measure conditional statistics of scalar concentrations in the laboratory; see e.g. LaRue and Libby (1974) and Antonia, Prabhu and Stephenson (1975). These statistics are those determined from measurements of $\Gamma(x,t)$ conditional on it being non-zero.

Obviously, therefore, the measurement of conditional statistics presumes the existence of an intermittency factor $\pi(x,t)$ in real dispersion situations with $\kappa \neq 0$, with $\pi(x,,t)$ defined by

$$\pi(x,t) - \text{prob}\{\Gamma(x,t) > 0\} \qquad (4.16)$$

The distinction between (4.3) and (4.16) is important. The definition of π_0 in (4.3) was applied only to the hypothetical situation in which there is no molecular diffusion, but users of (4.16) intend it to apply in real flows. In view of the importance of the concept of intermittency in both the different situations described - practical predictions of atmospheric dispersion, and conditional statistics - its modelling is an obvious challenge for theoreticians. This section, based on Chatwin and Sullivan (1989), considers this problem.

It is well-known that experimental determination of π is difficult because of its dependence on the fine-scale structure of the velocity and contaminant fields; the achievement of finer spatial and temporal resolution is invariably accompanied by relatively increasing random noise. For example Shaughnesey and Morton (1977) measured π in a smoke-filled turbulent air jet, and Figure 16 of their paper shows a sensitive dependence of the perceived distribution of π on the diameter of the laser beam used in the transducer system. Also Bilger, Antonia and Sreenivasan (1976) illustrated the subjective element involved in measuring π by showing that different workers estimated different values from the same data.

In many experiments such difficulties are explicitly recognised by replacing (4.16) by

$$\pi(x,t) = \mathrm{prob}\{\Gamma\ (x,t) > \theta_T)\ , \qquad (4.17)$$

where $\theta_T(p(>0)$ is a threshold value whose choice inevitably involves arbitrariness. Use of (4.17) gives perceived values of π that are strongly dependent on θ_T (Sreenivasan 1985), and one important practical conclusion is that quantitative} interpretations of published data are impossible unless each individual measurement of π is accompanied by the corresponding value of θ_T.

However from the points of view of both basic understanding and of theoretical modelling, there is an even more serious difficulty with the definitions of π in (4.16) and (4.17) and this is the role of molecular diffusivity. It has been noted in these Lectures on several occasions that the mean concentration is not affected by molecular diffusion but that the variance is. There is no doubt that π, as defined in (4.16), is affected by molecular diffusion. In fact, solutions of the equation governing Γ (x,t), viz.

$$\frac{\partial \Gamma}{\partial t} + (\tau\,.\nabla)\Gamma = k\nabla^2\Gamma, \qquad (4.18)$$

have the well-known property that $\Gamma(x,t) > 0$ everywhere in the flow regime for all $t > t_1$, where t_1 is the time when dispersion began (with a continuous release represented by $t_1 = -\infty$). Use of (4.16) then gives

$$\pi(x,t) = 1 \quad \text{for} \quad t > t_1 \quad \text{and for all } x \, . \tag{4.19}$$

It is common, perhaps, to dismiss this point as one that has no practical importance (presumably because molecular diffusion is a "weak" process). Such a dismissal has no foundation whatsoever (unless its advocates are claiming (without, it must be said, any evidence at all) that (4.18) is not the equation governing the concentration). Molecular diffusion is a real physical process, and its existence makes (4.16) a meaningless definition. (Perhaps "trivial definition" would be more accurate.) In practice, (4.17) is always used instead of (4.16) so that values of Γ less than θ_T are excluded and perceived values of π less than 1 are reported. However it is obvious from the reasoning leading to (4.19) that such values are sensitively dependent on θ_T and this is consistent, as noted above, with experiments. Such sensitive dependence means that theoretical predictions using (4.17) would not have any meaning insofar as the dispersion process itself were concerned.

The conceptual importance of the intermittency of the distribution of a scalar is in no way diminished by the total inadequacy of either (4.16) or (4.17). It therefore seems essential to seek a definition of intermittency factor that is theoretically sensible and can be reliably measured. It is just as important that any new definition of π should have clear physical interpretation; in particular it would be desirable to have a definition that separated the (essentially) rapid and large-scale process of advection from the (relatively) slow and small-scale process of molecular diffusion in an unambiguous way. Only the former process motivates the original concept, but (4.16) involves the latter process also, and (4.17) brings in the irrelevant (from the point of view of understanding the dispersion process) dependence on threshold.

Fortunately there is one, almost obvious, definition. As noted several times, for example in (4.4), the intermittency factor $\pi_o(x,t)$ in the hypothetical dispersion process that is identical in all respects with the real dispersion process except that $\kappa = 0$ satisfies

$$\pi_0(x,t) = \frac{C_0(x,t)}{\theta_1}, \tag{4.20}$$

and the key point is that C, and therefore π_0, depend only on the advection process (in addition to factors like the source geometry) since there is now no molecular diffusion. Since the advection process is, for a passive contaminant, unchanged by the presence of a contaminant undergoing molecular diffusion, $\pi_0(x,t)$ in (4.20) meets the criteria laid down earlier for the new definition of intennittency factor, except that I have not yet considered whether it can be reliably measured. Fortunately this does not pose a problem since, again as noted several times in these Lectures, the mean concentration $C(x,t)$ in real flows is believed to be indistinguishable from $C_0(x,t)$. Thus $\pi_o(x,t)$ can be well estimated by $\pi(x,t)$, where

$$\pi(x,t) = \frac{C(x,t)}{\theta_1} \tag{4.21}$$

It is therefore proposed that (4.20) be adopted as the new definition of intennittency factor, and that it be estimated by (4.21). It is reassuring that (4.21) involves only the source concentration θ_1 and the mean concentration $C(x,t)$, which are the two most easily measured properties of the contaminant field, and the most robust.

Scalars are often used in an attempt to separate the "turbulent regime" from the "non-turbulent (or irrotational) regime" in the velocity field. While my subject is not turbulent flow (and while I certainly do not intend to involve myself in sterile discussions about what turbulence is!), I think a comment or two would be allowed. In fact the presence of viscosity has the same effect on the vorticity field as molecular diffusion has on the concentration field, namely that its diffusive action causes the vorticity to be non-zero everywhere. There is no "irrotational regime" in real flow fields! It would of course be of interest to consider the vorticity analogue of (4.20) with the mean vorticity magnitude replacing $C_0(x.t)$; this may be unnecessary in practice since use of (4.20) as it stands and a passive scalar tracer ensures marking of the original vortical fluid emanating from the source.

4.6 The structure of $p(\theta;x,t)$

The new definition $\pi_0(x,t)$ of intermittency factor in (4.20) has an interesting role in the structure of $p(\theta;x,t)$, the p.d.f. of the concentration in real flows defined in equations (1.5) and (1.6). To see this, it is illuminating to begin with the joint p.d.f. $P_2(\theta,\theta o;x,t)$ of $\Gamma(x,t)$ and $T_0(x,t)$ where, following the notation introduced in section 2.1, $\Gamma_o(x,t)$ satisfies (4.18) with $k = 0$, but all the other conditions satisfied by the real concentration $\Gamma(x,t)$. Thus $p2(\theta,\theta_0;x,t)$ satisfies [cf. (1.5) and (1.6)]

$$P_2(\theta,\theta_0;x,t) = \frac{\partial^2}{\partial\theta\partial\theta_0}[\text{prob}\{\Gamma(x,t) \le \theta \text{ and } \Gamma_0(x,t) \le \theta_0\}] \quad (4.22)$$

In the normal way $P(\theta;x.t)$ can be obtained from $P_2(\theta,\theta o;x,t)$ bY integration:

$$p(\theta,\theta_0;x,t) = \int_0^\infty p_2(\theta,\theta_0;x,t)\,d\theta_0. \quad (4.23)$$

As we have already observed *on* several occasions, Γ_0 takes the value θ_1 with probability π_0 and the value 0 with probability $(1-\pi_0)$. Let $f(\theta;x,t)$ and $g(\theta;x,t)$ be the p.d.f.s of $\Gamma(x,t)$ conditional, respectively, on $\Gamma_0 = \theta_1$ and $\Gamma_0 = 0$. Then, by standard probability laws.

$$p_2(\theta,\theta_0;x,t) = \pi_o(x,t)\delta(\theta_o-\theta_1)f(\theta;x,t) + [1-\pi_0(x,t)]\delta(\theta_0)g(\theta;x,t) \;. \quad (4.24)$$

Use of (4.23) then gives the exact representation (Chatwin and Sullivan 1989)

$$p(\theta;x,t) = \pi_0(x,t)f(\theta;x,t) + [1-\pi_0(x,t)]g(\theta;x,t) \;, \quad (4.25)$$

in which the new definition of intermittency factor appears naturally. The conditional p.d.f.s f and g have clear physical interpretations, and can be modelled. For example $C_f(x,t)$, where

$$C_f(x,t) = \int_0^\infty \theta f(\theta;x,t)\,d\theta) \quad (4.26)$$

is the mean concentration in those fluid particles emanating from the source, and the extent to which it is below θ_1 provides a precise measure of the degree of molecular mixing that has occurred.

Although formulations like (4.25) have appeared in the literature, it is believed that this is the first time they have been given a sound foundation.

4.7 The assessment of instrument smoothing

The discussion of interrnittency has highlighted one of the statistical properties for which proper consideration of molecular diffusion is essential- More generally there are other properties, e.g. the higher moments of c(x,t), for which proper resolution of the small-scale structure of the concentration field is important. In practice it is not normally possible in experiments to resolve accurately both in time and, simultaneously, in all three space dimensions; this appears to be inevitable given the smallness of λ, the conduction cut-off length defined in (3.8). Attention is therefore being paid to the assessment of the degree of instrument smoothing in datasets and; reference shoud. be made to the papers cited following (4.7). Mole (1989) has discussed the principles of deconvolution techniques that may be applied to "raw" time-series of concentration measurements to yield more accurate estimates of parameters like $o^{2\cdot}$ These techniques make no assumptions about the structure of the real concentration field, but do (naturally) require estimates of noise and of the smoothing mechanism. They have been applied to various datasets including recent experiments by Griffiths and Jones in the atmosphere using the ion technique described by Jones (1979, 19&3) and Jones and Griffiths (1984). The results suggested; in fact that instrument smoothing played little role as far as o^2 was concerned; this is undoubtedly due to the very rapid response time of that particular instrumentation system. Unfortunately I have no opportunity here to discuss this important research topic further.

4.8 Concluding remarks

It appears to me that the awesome resources nowadays available for researchers in turbulent dispersion and turbulence (whether they be theoreticians or experimenters) , though of great, potential benefit, do not always result in important papers - unfortunately. Certainly some of the work I read, while full of (doubtlessly worthy) calculations or data, contains few good ideas and adds little to physical understanding. There seems to be an inertia in the research community which somewhat inhibits receptiveness to new ideas or lines of inquiry, and there is (conversely) rather too much emphasis on fashionable research areas which take rather more resources than their results merit (or than they will produce).

I expect it will be clear, therefore, that I support all research based on good (but not "half-baked"!) ideas, particularly that with novel viewpoints. Our subject is still not well enough understood for there to be no place for these. In this last Lecture I have indicated some problems that currently interest me and my colleagues, and which I intend to pursue in the short-term future.

I *am* grateful to Professors Olivari and Benocci for inviting me to give these Lectures. My knowledge of turbulent dispersion has benefitted from contact with many people, but I particularly want to mention Professor Paul J. Sullivan of the University of Western Ontario with whom I have enjoyed a stimulating collaboration over many years. I thank my colleague Dr. Nils Mole for commenting on these Lectures, and my secretary Mrs. Molly Demmar for her efficient conversion of my manuscript into readable form. Finally I want to apologize to my fellow researchers if I have misrepresented, or inappropriately forgotten, their work.

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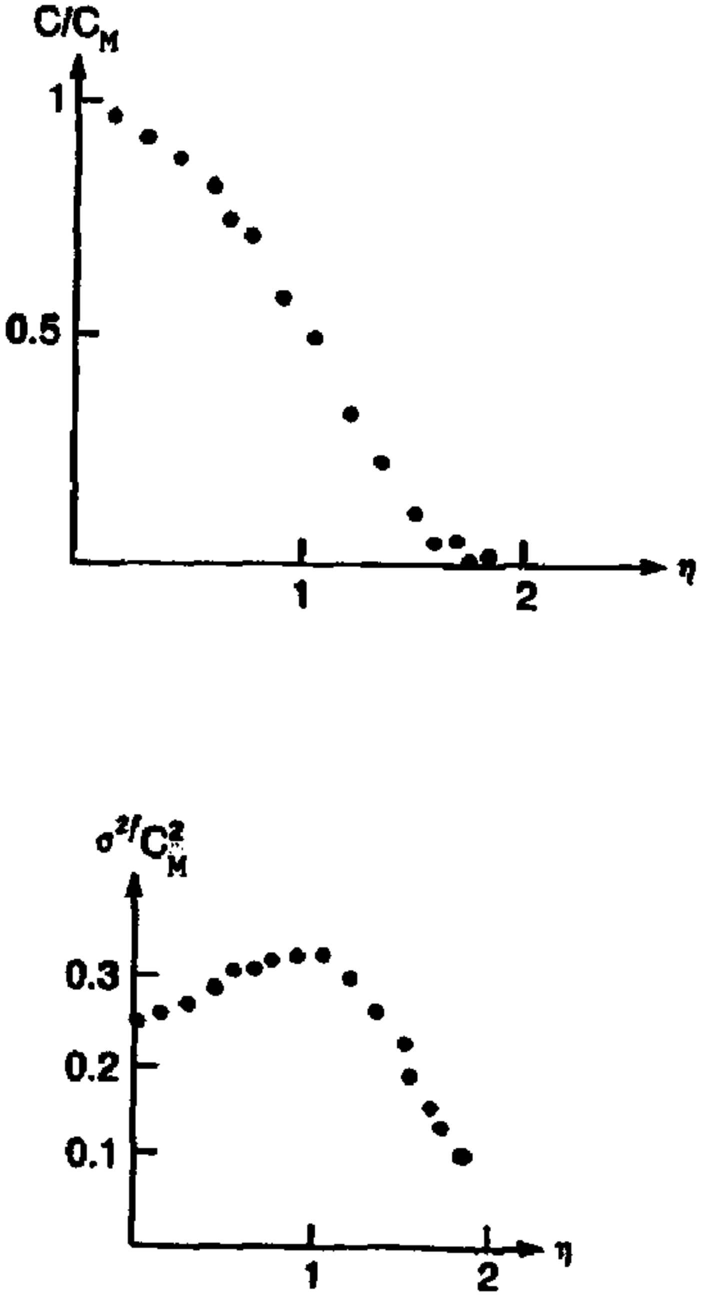

Figure 4. Measurements of C and σ^2 taken in a round jet by Antonio, Prabhu and Stephenson (1975).

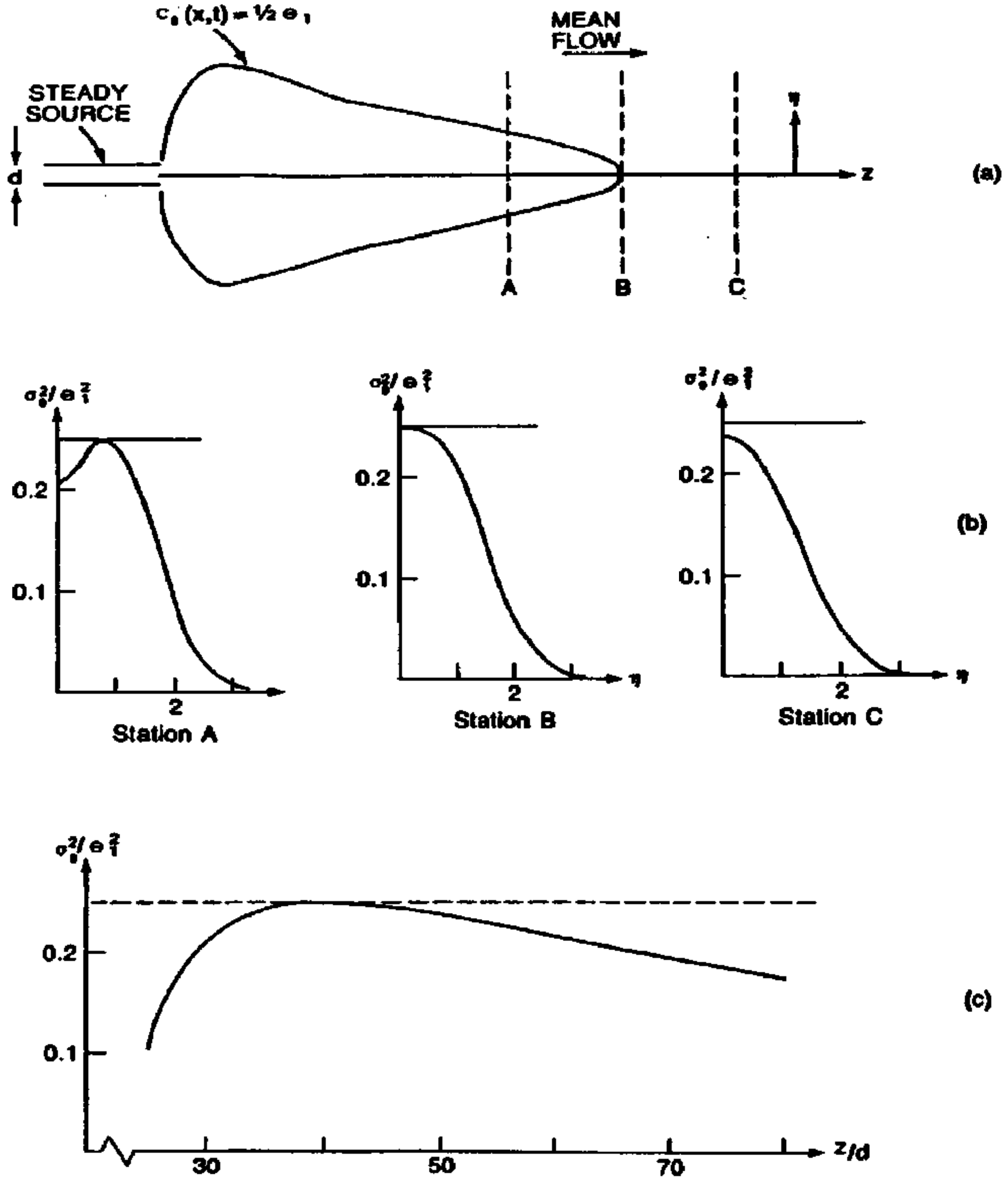

Figure 5. Schematic diagram showing profiles of $\sigma^2_0/\ \theta^{\,2}_1$ given by (4.5) and (4.6). (a) geometry; (b) transverse profiles; (c) axial profile (η 0 0). Note that $0^2_0/\ \theta^{\,2}_1 < \frac{1}{4}$ everywhere except on the surface indicated in (a),and that maxima occur on crossing this surface.

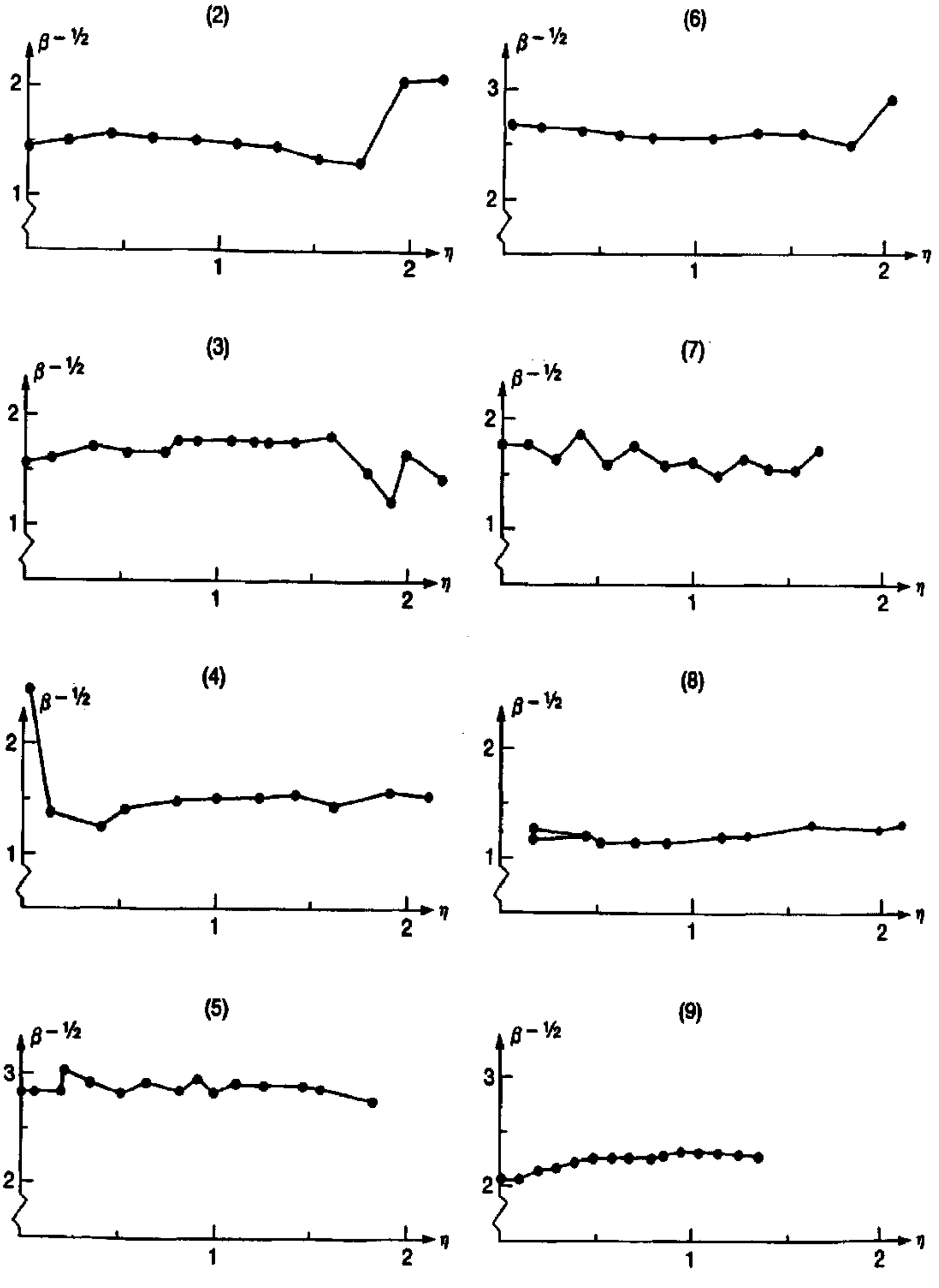

Figure 6. Graphs of $\beta^{-\frac{1}{2}}$ versus n for the datasets in Table 1,where α and ß are defined in (4.7).

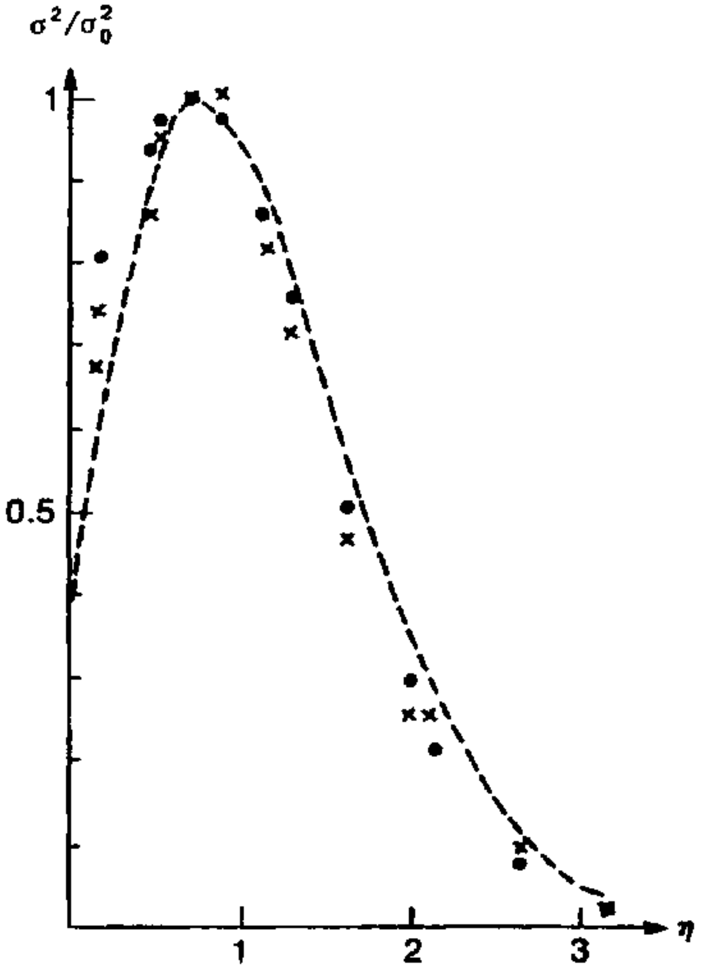

Figure 7. Comparison of data (X) by Fackrell and Robins (1982) with (i) an empirical formula (--) developed for this dataset by Wilson, Robins and Fackrell (1982), and (ii) equation (4.7) (ϕ). (Note that σ^2_0 is the maximum value of σ^2.