Numerical treatment of a two-dimensional variable-order fractional nonlinear reaction-diffusion model

F. Liu, P. Zhuang, I. Turner, V. Anh and K. Burrage

***Abstract*—A two-dimensional variable-order fractional nonlinear reaction-diffusion model is considered. A second-order spatial accurate semi-implicit alternating direction method for a two-dimensional variable-order fractional nonlinear reaction-diffusion model is proposed. Stability and convergence of the semi-implicit alternating direct method are established. Finally, some numerical examples are given to support our theoretical analysis. These numerical techniques can be used to simulate a two-dimensional variable order fractional FitzHugh-Nagumo model in a rectangular domain. This type of model can be used to describe how electrical currents flow through the heart, controlling its contractions, and are used to ascertain the effects of certain drugs designed to treat arrhythmia.**

***Index Terms*—variable-order operator, alternating direction method, second-order spacial accuracy, fractional nonlinear reaction-diffusion model, stability and convergence.**

I. Introduction

Reaction-diffusion models have found numerous applications in patten formation in biology, chemistry, physics and engineering (see [1]). These applications show that diffusion can produce the spontaneous formation of spatial-temporal patterns. The simplest reaction-diffusion model can be described by

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} + f(u,t), \tag{1}$$

where K is a diffusion constant and $f(u,t)$ is a nonlinear function representing the reaction kinetics. It is interesting to observe that for $f(u) = u(1-u)$, Eq. (1) reduces to the Fisher-Kolmogorov equation. If we set $f(u) = u(1-u^2)$, it reduces to the real Ginsburg-Landau equation and if we set $f(u) = u(1-u)(u-a)$, it reduces to the FitzHugh-Nagumo model. Here a is a model parameter.

Considerable interest in fractional differential equations has arisen over the last decade due to their ability to model anomalous transport phenomena (see [12], [13], [14]). Fractional-order models provide an excellent instrument for describing long memory and non-Gaussian behaviours of various processes (see [2], [3], [4], [12]).

F. Liu: School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Qld. 4001, Australia. email: f.liu@qut.edu.au

P. Zhuang: School of Mathematical Sciences, Xiamen University, Xiamen 361005, China.

I. Turner and V. Anh: School of Mathematical Sciences, Queensland University of Technology, GPO Box 2434, Brisbane, Qld. 4001, Australia.

K. Burrage: Department of Computer Science and OCISB, University of Oxford, OXI 3QD, UK/School of Mathematical Sciences, Queensland University of Technology, Brisbane, Qld. 4001, Australia

A random walk with a Lévy distribution of jump lengths is described on large scales by a diffusion equation with a Riesz fractional derivative. Some space Riesz fractional diffusion equations have been proposed and used in natural systems including heterogeneous soils, aquifers, and rivers, where the underlying transport phenomena are typically observed to be non-Fickian (see [15]).

A two-dimensional space Riesz fractional diffusion equation with a nonlinear reaction term (2D-RSFDE-NRT) has been considered in [17], [18], [19]:

$$\frac{\partial u}{\partial t} = K_x\frac{\partial^\alpha u}{\partial |x|^\alpha} + K_y\frac{\partial^\alpha u}{\partial |y|^\alpha} + f(u,x,y,t), \tag{2}$$

where $\frac{\partial^\alpha u}{\partial |x|^\alpha}$ is a space Riesz fractional operator (see [16]), is defined for a finite domain [a,b] as

$$\frac{\partial^\alpha u}{\partial |x|^\alpha} = c_\alpha\left(\frac{\partial^\alpha u}{\partial x^\alpha} + \frac{\partial^\alpha u}{\partial (-x)^\alpha}\right),$$

$$\frac{\partial^\alpha u(x,y,t)}{\partial x^\alpha} = \frac{1}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_a^x \frac{u(\xi,y,t)d_\xi}{(x-\xi)^{\alpha-1}},$$

$$\frac{\partial^\alpha u(x,y,t)}{\partial (-x)^\alpha} = \frac{(-1)^2}{\Gamma(2-\alpha)}\frac{\partial^2}{\partial x^2}\int_x^b \frac{u(\xi,y,t)d_\xi}{(\xi-x)^{\alpha-1}}$$

$1 < \alpha \le 2$, K_x, K_y are diffusion coefficients, $c_\alpha = \frac{-1}{2\cos(\frac{\pi\alpha}{2})}$. Underlying this description is the assumption of zero Dirichlet boundary conditions.

Similarly, the space Riesz fractional derivative $\frac{\partial^\alpha u}{\partial |y|^\alpha}$ of order α is defined with respect to y for a finite domain [c,d]. The fractional reaction diffusion equation based on the fractional Laplacian has also been considered in [29].

In various recent applications in science and engineering, variable-order fractional partial differential equations have been studied [5], [20], [21]. In order to motivate the use of variable fractional order operators, a little more we fours on several examples. Glockle and Nonnenmacher [6] studied the relaxation processes and reaction kinetics of proteins that are described by fractional differential equations of order α. The order α was found to have a temperature dependence. Electroviscous or electrorheo-logical fluids [7] and polymer gels [8] are known to change their properties in response to changes in imposed electric field strength. The properties of magnetorheological elastomers respond to magnetic field strength [9]. From the field of damage modelling, it is noted that as the damage accumulates (with time) in a structure, the nonlinear stress/strain behavior changes. This behavior may be better described with variable-order calculus.

One of the most fundamental problems in cardiac science is understanding the electrophysiological activity of the heart, and, in particular understanding the mechanisms that can induce arrhythmias with system biology approaches [10] increasingly used. It is known that the myocardium has a laminar fibrous structure, with cardiac cell being arranged into fibres, which are then arranged into branching networks of sheets, separated by collagen and extracellular pores [11]. While the bidomain models can capture the anisotropic structure of cardiac microstructure through the conductivity tensors, it cannot capture the heterogeneous nature of the extracellular region, nor deal with the fact that ventricular tissue structure can be very different from one region to another. In addition, the nature of these heterogeneities seem to change significantly due to the ageing of the heart. Bueno-Orovio et al. [30] have developed a new approach to the bidomain model in which the Laplacian operator is replaced by the fractional Laplacian and show that this approach better explains many experimental and observational electrophysiological real data. Thus we propose a fundamental rethink of the variable-order fractional model in which we capture the spatia heterogeneities in the extrarellular domain through the use of fractional derivatives. Finally, we note that Cusimano et al. [17] have developed variable order fractional operator to describe cell movement in the development of the neural crest.

Numerical methods for the variable-order model are still incipient. Lin et al. established an equality between the variable-order Riemann–Liouville fractional derivative and its Grünwald–Letnikov expansion (see [22]. Using this relationship, they proposed an explicit finite difference approximation scheme for a one-dimensional variable-order fractional nonlinear reaction-diffusion equation. The convergence and stability of this approximation are proved. Zhuang et al. [23] presented explicit and implicit Euler approximations for one-dimensional variable-order fractional advection-diffusion equation with a nonlinear source term on a finite domain. Stability and convergence of the methods are discussed. They also investigated a fractional method of lines, a matrix transfer technique, and an extrapolation method for the equation.

In this paper a new two-dimensional variable-order fractional nonlinear reaction-diffusion model (2D-VOFNRDM) is considered:

$$\frac{\partial u}{\partial t} = K_x \frac{\partial^{\alpha(x,y)} u}{\partial |x|^{\alpha(x,y)}} + K_y \frac{\partial^{\alpha(x,y)} u}{\partial |y|^{\alpha(x,y)}} + f(u,x,y,t), \quad (3)$$

with initial condition:

$$u(x,y,0) = \psi(x,y), \quad a \le x \le b, \quad c \le y \le d, \quad (4)$$

and zero Dirichlet boundary conditions:

$$u(a,y,t) = 0; \quad u(b,y,t) = 0, \quad (5)$$
$$u(x,c,t) = 0; \quad u(x,d,t) = 0, \quad (6)$$

where $1 < \alpha(x,y) \le 2$, K_x, K_y are diffusion coefficients. The nonlinear source term $f(u,x,y,t)$ is assumed to be locally Lipschitz continuous. The space Riesz fractional operator $\frac{\partial^{\alpha(x,y)} u}{\partial |x|^{\alpha(x,y)}}$ on a rectangular domain $[a,b] \times [c,d]$ is defined as (see [23]):

$$\frac{\partial^{\alpha(x,y)} u}{\partial |x|^{\alpha(x,y)}} = c_{\alpha(x,y)} \left(\frac{\partial^{\alpha(x,y)} u}{\partial x^{\alpha(x,y)}} + \frac{\partial^{\alpha(x,y)} u}{\partial (-x)^{\alpha(x,y)}} \right),$$

where $c_{\alpha(x,y)} = \frac{-1}{2\cos(\frac{\pi \alpha(x,y)}{2})}$, $1 < \alpha(x,y) \le 2$, and

$$\frac{\partial^{\alpha(x,y)} u(x,y,t)}{\partial x^{\alpha(x,y)}} = \left[\frac{1}{\Gamma(2-\alpha(x,y))} \frac{\partial^2}{\partial \xi^2} \int_a^{\xi} \frac{u(\eta,y,t) d\eta}{(\xi-\eta)^{\alpha(x,y)-1}} \right]_{\xi=x}, \quad (7)$$

$$\frac{\partial^{\alpha(x,y)} u(x,y,t)}{\partial (-x)^{\alpha(x,y)}} = \left[\frac{(-1)^2}{\Gamma(2-\alpha(x,y))} \frac{\partial^2}{\partial \xi^2} \int_{\xi}^{b} \frac{u(\eta,y,t) d\eta}{(\eta-\xi)^{\alpha(x,y)-1}} \right]_{\xi=x}. \quad (8)$$

Similarly, we can define the space Riesz fractional derivative $\frac{\partial^{\alpha(x,y)} u}{\partial |y|^{\alpha(x,y)}}$ of order $\alpha(x,y)$ with respect to y.

Remark: We say that $f : X \to X$ is globally Lipschitz continuous if for some $L > 0$, we have $\|f(u) - f(v)\| \le L \|u - v\|$ for all $u, v \in X$, and is locally Lipschitz continuous, if the latter holds for $\|u\|, \|v\| \le M$ with $L = L(M)$ for any $M > 0$ (see [24], [25]).

The remainder of this article is organized as follows. In Section 2, a spatially second-order accurate semi-implicit alternating direction method (SIADE) for the (2D-VOFNRDM) in a square domain is proposed. The stability and convergence of the SIADE in a square domain are discussed in Section 3. Finally, some numerical results are presented and these techniques are also used to simulate a two-dimensional variable order fractional FitzHugh-Nagumo model in a rectangular domain.

II. Semi-implicit alternating direction method

We consider the numerical approximation of Eq. (3) in a rectangular domain. Let $h_x = (b-a)/m_1$ and $h_y = (d-c)/m_2$ be the spatial grid size in the x-direction and in the y-direction, respectively; $\tau = T/n$ be the time step; $x_i = a + i h_x, i = 0, 1, \cdots, m_1$; $y_j = c + j h_y, j = 0, 1, \cdots, m_2$; $t_k = k\tau, k = 0, 1, \cdots, n$. Define $u_{i,j}^k$ as the numerical approximation to $u(x_i, y_j, t_k)$. The initial conditions are set by $u_{i,j}^0 = \psi(x_i, y_j)$.

Firstly, we use the backward Euler difference scheme for the first order time derivative

$$\frac{\partial u}{\partial t}\Big|_{(x_i,y_j,t_k)} \simeq \frac{u_{i,j}^k - u_{i,j}^{k-1}}{\tau} + O(\tau). \quad (9)$$

Secondly, adopting the fractional centered difference scheme (see [26], [28]), we can discretize the Riesz fractional derivative as

$$\frac{\partial^{\alpha(x_i,y_j)}}{\partial |x|^{\alpha(x_i,y_j)}} u(x_i,y_j,t_k) \simeq -\frac{1}{h_x^{\alpha(x_i,y_j)}} \sum_{p=-m_1+i}^{i} g_{\alpha(x_i,y_j)}^{p} u(x_{i-p}, y_j, t_k) + O(h_x^2),$$

where the coefficients $g^p_{\alpha(x_i,y_j)}$ are defined by

$$g^p_{\alpha(x_i,y_j)} = g^p_{\alpha_{i,j}}$$
$$= \frac{(-1)^p\Gamma(\alpha_{i,j}+1)}{\Gamma(\frac{\alpha_{i,j}}{2}-p+1)\Gamma(\frac{\alpha_{i,j}}{2}+p+1)}, \quad p=0,\mp 1,\mp 2,\cdots.$$

Similarly,

$$\frac{\partial^{\alpha(x_i,y_j)}}{\partial |y|^{\alpha(x_i,y_j)}}u(x_i,y_j,t_k)$$
$$= -\frac{1}{h_y^{\alpha(x_i,y_j)}}\sum_{q=-m_2+j}^{j} g^q_{\alpha(x_i,y_j)}u(x_i,y_{j-q},t_k)+O(h_y^2).$$

Lemma 1. (see [26], [28]) For all non-negative i,j, the coefficients $g^p_{\alpha_{i,j}}$ $p=0,\mp 1,\mp 2,\cdots$ satisfy:

(1) $g^0_{\alpha_{i,j}}\geq 0$, $g^k_{\alpha_{i,j}}=g^{-k}_{\alpha_{i,j}}\leq 0 \ for \ all \ |k|\geq 1$;

(2) $\sum\limits_{p=-\infty}^{\infty} g^p_{\alpha_{i,j}}=0$;

(3) For any positive integers n,m with $n<m$, we have $\sum\limits_{p=-m+n}^{n} g^p_{\alpha_{i,j}}>0$.

The nonlinear source term can be treated either explicitly or implicitly. In this paper, we use an explicit method and evaluate the nonlinear source term at the previous time step:

$$\begin{aligned} &f(u(x_i,y_j,t_k),x_i,y_j,t_k)\\ &=f(u(x_i,y_j,t_{k-1}),x_i,y_j,t_{k-1})+O(\tau). \end{aligned} \tag{10}$$

Therefore the semi-implicit numerical method for the 2D-VOFNRDM is determined by the following finite difference equation:

$$\frac{u^k_{i,j}-u^{k-1}_{i,j}}{\tau} = -\frac{K_x}{h_x^{\alpha_{i,j}}}\sum_{p=-m_1+i}^{i} g^p_{\alpha_{i,j}}u^k_{i-p,j}$$
$$- \frac{K_y}{h_y^{\alpha_{i,j}}}\sum_{q=-m_2+j}^{j} g^q_{\alpha_{i,j}}u^k_{i,j-q}+f(u^{k-1}_{i,j},x_i,y_j,t_{k-1}). \tag{11}$$

The semi-implicit numerical method defined by (11) is consistent with order $O(\tau+h_x^2+h_y^2)$.

Define the following fractional partial difference operators:

$$\delta_x^{\alpha_{i,j}}u^k_{i,j} = -\frac{K_x}{h_x^{\alpha_{i,j}}}\sum_{p=-m_1+i}^{i} g^p_{\alpha_{i,j}}u^k_{i-p,j},$$

$$\delta_y^{\alpha_{i,j}}u^k_{i,j} = -\frac{K_y}{h_y^{\alpha_{i,j}}}\sum_{q=-m_2+j}^{j} g^q_{\alpha_{i,j}}u^k_{i,j-q}.$$

With these operator definitions, the semi-implicit Euler method for the 2D-VOFNRDM with homogeneous Dirichlet boundary conditions may be written in the following operator form:

$$\begin{aligned} (1-\tau\delta_x^{\alpha_{i,j}}-\tau\delta_y^{\alpha_{i,j}})u^k_{i,j} = u^{k-1}_{i,j}+\tau f^{k-1}_{i,j}(u), \\ 1\leq i\leq m_1-1,\ 1\leq j\leq m_2-1, \end{aligned} \tag{12}$$

where $f^{k-1}_{i,j}(u)=f(u^{k-1}_{i,j},x_i,y_j,t_{k-1})$.

We introduce an additional perturbation error equal to $(\tau)^2\left(\delta_x^{\alpha_{i,j}}\delta_y^{\alpha_{i,j}}\right)u^k_{i,j}$. Eq. (12) is written in the following directional separation product form:

$$\left(1-\tau\delta_x^{\alpha_{i,j}}\right)\left(1-\tau\delta_y^{\alpha_{i,j}}\right)u^k_{i,j} = u^{k-1}_{i,j}+\tau f^{k-1}_{i,j}(u). \tag{13}$$

Here we have dropped the τ^2 term on the right in Eq. (13). The additional perturbation error is not large compared to the approximation errors for the other terms in (12), and hence (13), which is called a semi-implicit alternating direction method (SIADM), is consistent with order $O(\tau+h_x^2+h_y^2)$.

Computationally, the SIADM defined by (13) can now be solved by the following iterative scheme. At time t_k:

Step 1: Solve the problem in the x-direction (for each fixed y_j) to obtain an intermediate solution $u^*_{i,j}$ in the form

$$(1-\tau\delta_x^{\alpha_{i,j}}))u^*_{i,j} = u^{k-1}_{i,j}+\tau f^{k-1}_{i,j}(u). \tag{14}$$

Step 2: Then solve in the y-direction (for each fixed x_i)

$$\big(1-\tau\delta_y^{\alpha_{i,j}}\big))u^k_{i,j} = u^*_{i,j}. \tag{15}$$

The initial and boundary conditions for the numerical solution $u^k_{i,j}$ and $u^{k-1}_{i,j}$ are defined from the given initial and boundary conditions. Prior to carrying out step one of solving (14), the boundary conditions for the intermediate solution $u^*_{i,j}$ should be set from Eq. (15) (which incorporates the values of $u^k_{i,j}$ at the boundary), otherwise the order of convergence will be adversely affected. Specifically, for homogeneous Dirichlet boundary conditions (5) and (6), we have

$$u^k_{0,j}=u(a,y_j,t_k)=0;\quad u^k_{m_1,j}=u(b,y_j,t_k)=0;$$
$$u^k_{i,0}=u(x_i,c,t_k)=0;\quad u^k_{i,m_2}=u(x_i,d,t_k)=0.$$

Thus, we compute the boundary values for u^* from

$$u^*_{0,j}=\left(1-\tau\delta_y^{\alpha_{i,j}}\right)u^k_{0,j},\qquad u^*_{m_1,j}=\left(1-\tau\delta_y^{\alpha_{i,j}}\right)u^k_{m_1,j}.$$

III. Stability and convergence

In this section, we discuss the stability and convergence of the SIADM (13). We first need to rewrite (14), (15) and (13) in matrix form.

Let

$$r^x_{i,j}=\frac{\tau K_x}{(h_x)^{\alpha_{i,j}}},\qquad r^y_{i,j}=\frac{\tau K_y}{(h_y)^{\alpha_{i,j}}}.$$

Then Eq. (14) may be written as

$$A^{(v)}V^*_v = V^{k-1}_v+\tau F^{k-1}_v,\qquad v=1,2,\cdots,m_2-1, \tag{16}$$

where $V^{k-1}_v = (u^{k-1}_{1,v},u^{k-1}_{2,v},\cdots,u^{k-1}_{m_1-1,v})^T$, $V^*_v = (u^*_{1,v},u^*_{2,v},\cdots,u^*_{m_1-1,v})^T$,

$$F^{(k-1)}_v = (f^{k-1}_{1,v}(u),f^{k-1}_{2,v}(u),\cdots,f^{k-1}_{m_1-1,v}(u))^T,$$

and $A^{(v)}=(a^{(v)}_{i,j})_{(m_1-1)\times(m_1-1)}$,

$$a^{(v)}_{i,j} = \begin{cases} r^x_{i,v}g^{i-j}_{\alpha_{i,v}}, & \text{for } j<i;\\ 1+r^x_{i,v}g^0_{\alpha_{i,v}}, & \text{for } j=i;\\ r^x_{i,v}g^{-j+i}_{\alpha_{i,v}}, & \text{for } j>i. \end{cases} \tag{17}$$

Similarly, Eq. (15) may be written as

$$B^{(w)}\bar{W}^k_w = \bar{W}^*_w,\qquad w=1,2,\cdots,m_1-1, \tag{18}$$

where $\bar{W}^k_w = (u^k_{w,1},u^k_{w,2},\cdots,u^k_{w,m_2-1})^T$, $\bar{W}^*_w = (u^*_{w,1},u^*_{w,2},\cdots,u^*_{w,m_2-1})^T$, $B^{(w)}=(b^{(w)}_{i,j})_{(M_2-1)\times(m_2-1)}$,

$$b^{(w)}_{i,j} = \begin{cases} r^y_{w,j}g^{i-j}_{\alpha_{w,j}}, & \text{for } j<i;\\ 1+r^y_{w,j}g^0_{\alpha_{w,j}}, & \text{for } j=i;\\ r^y_{w,j}g^{-j+i}_{\alpha_{w,j}}, & \text{for } j>i. \end{cases} \tag{19}$$

Hence, Eq. (13) may be written in the matrix form

$$SHU^k = U^{k-1} + F^{k-1}, \tag{20}$$

where the matrices S and H represent the operators $\left(1-\tau\delta_x^{\alpha_{i,j}}\right)$ and $\left(1-\tau\delta_y^{\alpha_{i,j}}\right)$, respectively,

$$\begin{aligned} U^k &= [u_{1,1}^k, \cdots, u_{m_1-1,1}^k, u_{1,2}^k, \cdots, u_{m_1-1,2}^k, \\ &\quad \cdots, u_{1,m_2-1}^k, \cdots, u_{m_1-1,m_2-1}^k]^T \end{aligned}$$

and the vector F^{k-1} is the source term.

The matrix S is a block diagonal matrix of the form $S = diag(A^{(1)}, A^{(2)}, \cdots, A^{(m_2-1)})$, where each block is of size m_1-1.. Similarly, the matrix H is a block matrix of $(m_2-1)\times(m_2-1)$ blocks are such that $H_{i,j}^{(w)}$ is an $(m_1-1)\times(m_1-1)$ matrix, where $H_{i,j}^{(w)}$ is a diagonal matrix $H_{i,j}^{(w)} = diag(b_{i,j}^{(w)}, b_{i,j}^{(w)}, \cdots, b_{i,j}^{(w)})$, and where the notation $b_{i,j}^{(w)}$ refers to the (i,j)th entry of the matrix $B^{(w)}$ defined previously.

To prove the stability and convergence of the numerical method, we need the following lemma from [27].

Lemma 2. Let $X = [x_1, x_2, \cdots, x_m]^T$ and $\|X\|_\infty = \max_{1\le i\le m} |x_i|$. If the matrix $D = (d_{i,j})_{m\times m}$ satisfies the conditions

$$\sum_{j=1,j\neq i}^{m} |d_{i,j}| \le |d_{i,i}| - 1, \quad (i=1,2,\cdots,m), \tag{21}$$

then

$$\|X\|_\infty \le \|DX\|_\infty. \tag{22}$$

Let $u_{i,j}^k$ and $\widetilde{u}_{i,j}^k$ be the numerical and approximate solutions of the SIADM (13), respectively, and set $\mathbf{E}^n = [\varepsilon_{1,1}^k, \varepsilon_{2,1}^k, \cdots, \varepsilon_{m_1-1,m_2-1}^k]^T$, where $\varepsilon_{i,j}^k = u_{i,j}^k - \widetilde{u}_{i,j}^k$, and let $\widetilde{f}_{i,j}^k(u)$ be the approximation of $f_{i,j}^k(u)$.

Theorem 1. The SIADM defined by (13) is unconditionally stable, and there is a positive constant C_1^* such that

$$\|\mathbf{E}^k\|_\infty \le C_1^* \|\mathbf{E}^0\|_\infty, \quad k = 0,1,2,\cdots.$$

Proof. Let $\bar{F}_{i,j}^{k-1} = f_{i,j}^{k-1}(u) - \widetilde{f}_{i,j}^{k-1}(u)$, then the error $\mathbf{E}^k$ satisfies the following equation:

$$SH\mathbf{E}^k = \mathbf{E}^{k-1} + \bar{F}^{k-1}, \tag{23}$$

where $\bar{F}^k = \left[\bar{F}_{1,1}^k(u), \cdots, \bar{F}_{M_1-1,M_2-1}^k(u)\right]^T$.

From Lemma 1, we have $a_{i,i}^{(v)} > 0, \quad a_{i,j}^{(v)} < 0 \quad (i \neq j)$. Therefore

$$\begin{aligned} \sum_{j=1,j\neq i}^{M_1-1} |a_{i,j}^{(v)}| &= \sum_{p=-m_1+i+1}^{i-1} |r_{i,v}^x g_{\alpha_{i,j}}^p| \\ = \quad &- \sum_{p=-m_1+i+1}^{i-1} r_{i,v}^x g_{\alpha_{i,j}}^p < r_{i,v}^x g_{\alpha_{i,j}}^0 = |a_{i,i}^{(v)}| - 1. \end{aligned}$$

Similarly, we have

$$\sum_{j=1,j\neq i}^{m_2-1} |b_{i,j}^{(w)}| < |b_{i,i}^{(w)}| - 1.$$

Therefore $A^{(v)}$ and $B^{(w)}$ satisfy the conditions of Lemma 2.

Since $f(u,x,y,t)$ is locally Lipschitz continuous, then $\|f_{i,j}^{k-1}(u) - \widetilde{f}_{i,j}^{k-1}(u)\|_\infty \le L_k \|\varepsilon_{i,j}^{k-1}\|_\infty$, i.e., $\|\tilde{F}^{k-1}\|_\infty \le L_k \|\mathbf{E}^{k-1}\|_\infty$.

As $A^{(v)}$ and $B^{(w)}$ satisfy the conditions of Lemma 2, then according to the relationship between the matrices S and $A^{(v)}$ and the relationship between the matrices H and $B^{(v)}$, we see that S and H also satisfy the conditions of Lemma 2. Therefore

$$\begin{aligned} \|\mathbf{E}^k\|_\infty &\le \|H\mathbf{E}^k\|_\infty \le \|SH\mathbf{E}^k\|_\infty \\ &\le \|\mathbf{E}^{k-1}\|_\infty + \tau L_k \|\mathbf{E}^{k-1}\|_\infty \\ &\le (1+\tau L)^k \|\mathbf{E}^0\|_\infty \\ &\le C_1^* \|\mathbf{E}^0\|_\infty, \end{aligned}$$

where $L = \max_{1\le l\le k} L_l$ and $C_1^* = e^{LT}$.

Now let us consider the convergence of the SIADM defined by (15). Let $u(x_i, y_j, t_k)$ be the exact solution of the 2D-VOFNRDM (3)-(5), $u_{i,j}^k$ be the numerical solution of SIADM defined by (13). Let $\eta_{i,j}^k = u(x_i,y_j,t_k) - u_{i,j}^k$ and $\mathbf{Y}^n = [\eta_{1,1}^k, \eta_{2,1}^k, \cdots, \eta_{m_1-1,m_2-1}^k]^T$.

Theorem 2. The SIADM as defined by (13) is convergent and there is a positive constant C_2^* such that

$$\|\mathbf{Y}^k\|_\infty \le C_2^*(\tau + h_x^2 + h_y^2), \quad k = 0,1,2,\cdots.$$

Proof.

The error $\mathbf{Y}^k$ satisfies the following equation:

$$SH\mathbf{Y}^k = \mathbf{Y}^{k-1} + \tau\bar{F}^{k-1} + \tau\bar{R}^k, \tag{24}$$

where $\|\bar{F}^{k-1}\|_\infty \le L_k \mathbf{Y}^{k-1}$ and $\|\bar{R}^k\|_\infty \le C_2(\tau + h_x^2 + h_y^2)$.

Similar to the proof of Theorem 1, we have

$$\begin{aligned} \|\mathbf{Y}^k\|_\infty &\le \|H\mathbf{Y}^k\|_\infty \le \|SH\mathbf{Y}^k\|_\infty \\ &\le \|\mathbf{Y}^{k-1}\|_\infty + \tau\|\bar{R}^k\|_\infty + \tau L_k \|\mathbf{Y}^{k-1}\|_\infty \\ &\le (1+\tau L)\|\mathbf{Y}^{k-1}\|_\infty + \tau C_2(\tau + h_x^2 + h_y^2). \end{aligned}$$

Using the discrete Gronwall inequality, we obtain

$$\begin{aligned} \|\mathbf{Y}^k\|_\infty &\le C_2 k\tau e^{Lk\tau}(\tau + h_x^2 + h_y^2) \\ &\le C_2 T e^{LT}(\tau + h_x^2 + h_y^2) \\ &= C_2^*(\tau + h_x^2 + h_y^2). \end{aligned}$$

Therefore the SIADM defined by (13) is convergent.

IV. Numerical Results

In this section, two numerical examples are presented to evaluate our theoretical analysis from Section III.

Example 1. Consider the following variable-order fractional nonlinear reaction-diffusion

$$\begin{aligned} &\frac{\partial u}{\partial t} = 1.5\frac{\partial^{\alpha(x,y)}u}{\partial|x|^{\alpha(x,y)}} + 1.5\frac{\partial^{\alpha(x,y)}u}{\partial|y|^{\alpha(x,y)}} + u - u^3 + f(x,y,t), \\ &\qquad\qquad t>0, \quad 0<x,y<1 \\ &u(0,y,t) = u(1,y,t) = 0, \quad 0\le y\le 1, \quad t\ge 0, \\ &u(x,0,t) = u(x,1,t) = 0, \quad 0\le x\le 1, \quad t\ge 0, \\ &u(x,y,0) = x^2(1-x)^2y^2(1-y)^2, \quad 0\le x,y\le 1 \end{aligned} \tag{25}$$

where

$$f(x,y,t) = e^{3t}x^6(1-x)^6y^6(1-y)^6$$
$$+g_1(x,y)\left[x^{2-\alpha(x,y)}f_1(x,y)+(1-x)^{\alpha(x,y)}f_2(x,y)\right]$$
$$+g_2(x,y)\left[y^{2-\alpha(x,y)}f_3(x,y)+(1-y)^{\alpha(x,y)}f_4(x,y)\right],$$

$$\begin{aligned}
g_1(x,y) &= \frac{e^t y^2(1-y)^2}{\Gamma(3-\alpha(x,y))\cos\frac{\pi\alpha(x,y)}{2}},\\
g_2(x,y) &= \frac{e^t x^2(1-x)^2}{\Gamma(3-\alpha(x,y))\cos\frac{\pi\alpha(x,y)}{2}},\\
f_1(x,y) &= 1-\frac{6x}{3-\alpha(x,y)}-\frac{12x^2}{(3-\alpha(x,y))(4-\alpha(x,y))},\\
f_2(x,y) &= 1-\frac{6(1-x)}{3-\alpha(x,y)}-\frac{12(1-x)^2}{(3-\alpha(x,y))(4-\alpha(x,y))},\\
f_3(x,y) &= 1-\frac{6y}{3-\alpha(x,y)}-\frac{12y^2}{(3-\alpha(x,y))(4-\alpha(x,y))},\\
f_4(x,y) &= 1-\frac{6(1-y)}{3-\alpha(x,y)}-\frac{12(1-y)^2}{(3-\alpha(x,y))(4-\alpha(x,y))},\\
\alpha(x,y) &= 1+\sin^2\pi xy.
\end{aligned}$$

The exact solution of (25) is $u(x,y,t)=e^t(x-x^2)^2(y-y^2)^2$.

TABLE I
For $\tau = h^2$, the error between the exact solution and the numerical solution at $t = 1.0$

h	$\max_{1\le i,j\le m-1}\lvert u_{i,j}^n - u(x_i,y_j,t_n)\rvert$	Rate of Convergence
$1/5$	7.711E-004	
$1/10$	2.542E-004	1.601 ≈ 10/5
$1/20$	7.366E-005	1.787 ≈ 20/10
$1/40$	1.934E-005	1.929 ≈ 40/20

Table 1 shows the maximum error between the exact solution and the numerical solution obtained by the SIADM in Example 1 at time $t = 1$. From the above results, it can be seen that the numerical results are in good agreement with the theoretical results.

These numerical techniques are now employed to simulate the two-dimensional variable order fractional FitzHugh-Nagumo model in a square domain.

Example 2 Consider the following two-dimensional variable order fractional FitzHugh-Nagumo model in a square domain $\Omega_S : 0 \le x, y \le 2.5$:

$$\begin{aligned}
\frac{\partial u}{\partial t} &= K_x\frac{\partial^{\alpha(x,y)}u}{\partial|x|^{\alpha(x,y)}}+K_y\frac{\partial^{\alpha(x,y)}u}{\partial|y|^{\alpha(x,y)}}\\
&\quad +u(1-u)(u-\sigma)-v,\\
\frac{\partial v}{\partial t} &= \epsilon(\beta u-\gamma v-\delta),
\end{aligned}$$

where u is a normalized transmembrane potential and v is a dimensionless time-dependent recovery variable; K_x and K_y are the diffusion coefficients. We consider the following choice of model parameters, $\sigma = 0.1,\quad \epsilon = 0.01,\quad \beta = 0.5,\quad \gamma = 1,\quad \delta = 0$, which is known to generate stable patterns in the system in the form of reentrant spiral waves. The model parameters have been taken from [29].

In this simulation, we consider the initial conditions as

$$u(x,y,0)=\begin{cases}1.0, & 0<x\le 1.25,\quad 0<y<1.25,\\ 0.0, & 1.25\le x<2.5,\quad 0<y<1.25,\\ 0.0, & 0<x<2.5,\quad 1.25\le y<2.5\end{cases}$$

$$v(x,y,0)=\begin{cases}0.0, & 0<x\le 2.5,\quad 0<y<1.25,\\ 0.1, & 1.25\le x<2.5,\quad 0<y<2.5\end{cases}$$

with homogeneous Dirichlet boundary conditions

$$u(0,y,t)=u(2.5,y,t)=0,\quad 0\le y\le 2.5,\quad t\ge 0,$$
$$u(x,0,t)=u(x,2.5,t)=0,\quad 0\le x\le 2.5,\quad t\ge 0.$$

In this simulation, we take $h_x = h_y = 2.5/256$ and $\tau = 0.1$. The contour of the stable rotating solution in the FitzHugh-Nagumo model (i.e., $\alpha = 2$) is shown in Figure 1(a), and the contours of the stable rotating solution in the fractional FitzHugh-Nagumo model ($\alpha = 1.8$ and $\alpha = 1.6$) are shown in Figure 1(b) and 1(c), respectively. From this simulation, it can be seen that spiral waves in the space Riesz fractional FitzHugh-Nagumo model generate a curve and rotate clockwise as observed in [29]. From Figures 1(a)-1(c), we find that as expected, the wave travels more slowly as fractional order α decreases. The behaviour of the solution is particularly interesting for the case $\alpha(x,y) = 1.5 + 0.5\sin\frac{2\pi x}{5}\sin\frac{2\pi y}{5}$. The contour of the stable rotating solution is shown in Figure 1(d), where we observe that the model describes a smooth and continuous transition in the considered domain.

V. Conclusions

In this paper, a novel semi-implicit alternating direction method is proposed for approximating a new two-dimensional variable-order fractional nonlinear reaction-diffusion model subject to homogeneous Dirichlet boundary conditions in a rectangular domain. Stability and convergence of the method have been established. These techniques are also used for simulating a two-dimensional variable-order fractional FitzHugh-Nagumo model in a rectangular domain. The numerical results demonstrate the effectiveness of these techniques. The variable-order models can be used to capture complex spatial dynamics in heterogeneous media. We plan to apply this new approach to cardiac electrophysiology in any irregular domain in our future work.

References

[1] L. M. Pismen, Patterns and Interfaces in Dissipative Systems, *Springer Verlag, Berlin*, (2006), 163-169.

[2] F. Liu, V. Anh and I. Turner, Numerical solution of the space fractional Fokker-Planck Equation, *J. Comp. Appl. Math.*, 166(2004), 209-219.

[3] F. Liu, P. Zhuang, V. Anh, I. Turner and K. Burrage, Stability and Convergence of the difference Methods for the space-time fractional advection-diffusion equation, *Appl. Math. Comp.*, 191, (2007), 12-20.

[4] R. Metzler and J. Klafter, The random walk's guide to anomalous diffusion: a fractional dynamics approach, *Phys. Rep.* 339, (2000), 1-77.

[5] C.F. Lorenzo and T.T. Hartley, Initialization, conceptualization, and application in the generlized fractional calculus, *NASA/TP-1998-208415*, 1998.

[6] W.G. Glockle and T.F. Nonnenmacher, A fractional calculus approach to self-similar protein dynamics, *Biophysical J.*, 68, (1995), 46-53.

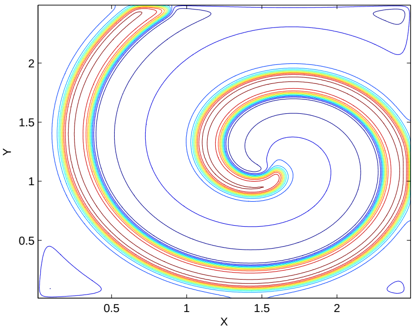
[7] D.L. Klass, and T.W. Martinek, Electroviscous fluids. I. Rheological properties, J. Appl. Phys. 38(1), (1967), 67-74.

[8] T. Shiga, Deformation and viscoelastic behavior of polymer gel in electric fields, *Proceedings of the Japanese Academy, Series B, Physical and Biological Sciences*, 74, (1998), 6-11.

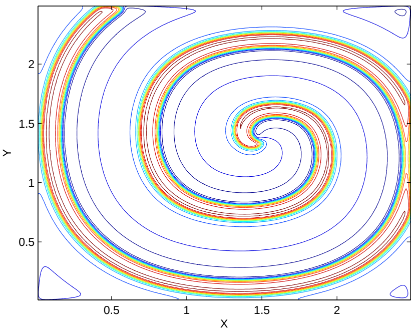
[9] L.C. Davis, Model of magnetorheological elastomers, *J. Appl. Phys.*, 85(6),(1999), 3342-3351.

[10] B. Rodriguez, K.Burrage, D. Gavaghan, V. Grau P. Kohl, and D.Noble, (2010): Cardiac Applications of the Systems Biology Approach to Drug Development, *Clinical Pharmacology & Therapeutics*, (2010) 88 1, 130134. doi:10.1038/clpt.2010.95.

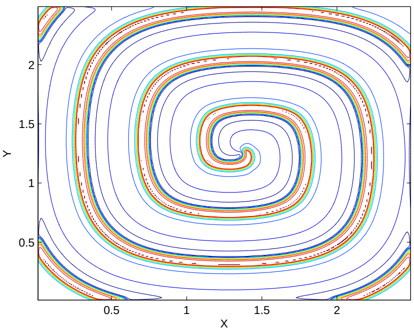
[11] C.H. Luo and Y. Rudy, A model of the ventricular cardiac action potential. Depolarization, repolarization, and their interaction, *Circ. Res.*, 68, 1991, 1501-1526.



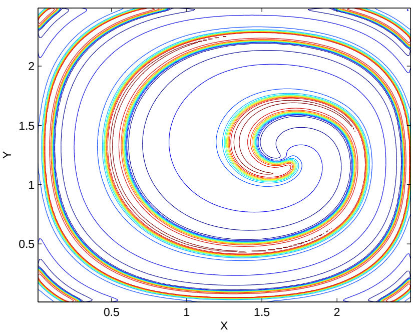
$(a)\ \alpha = 2.0$



$(b)\ \alpha = 1.8$



$(c)\ \alpha = 1.6$



$(d)\ \alpha(x,y) = 1.5 + 0.5 \sin \frac{2\pi x}{5} \sin \frac{2\pi y}{5}$

Fig. 1. Spiral waves in the space Riesz fractional Fitzhugh-Nagumo model with zero Dirichlet boundary conditions at $t = 1000$, where $K_x = K_y = 10^{-4}$ and with different α .

[12] I. Podlubny, Fractional Differential Equations. *Academic Press*, 1999.

[13] J. Ochoa-Tapia, F. Valdes-Parada and J. Alvarez-Ramirez, A fractional-order Darcys law, *Physica A*, 374, (2007), 1-14.

[14] F.J. Valdes-Parada, J.A. Ochoa-Tapia and J. Alvarez-Ramirez, Effective medium equations for fractional Fick's law in porous media, *Physica A*, 373 (2007) 339-353.

[15] Y. Zhang, D. Benson and D. Reeves, Time and space nonlocalities underlying fractional-derivative models: Distinction and literature review of field applications, *Adv. Water Resour*, 32, (2009), 561-581.

[16] Q. Yang, F. Liu and I. Turner. Numerical methods for fractional partial differential equations with Riesz space fractional derivatives. Appl. Math. Modelling, 34(1), (2010), 200-218.

[17] N. Cusimano, K. Burrage and P. Burrage, Fractional models for the migration of biological cells in complex spatial domains, *ANZIAM*, 54, (2013), C250-C270.

[18] F. Liu, S. Chen, I. Turner, K. Burrage and V. Anh, Numerical simulation for two-dimensional Riesz space fractional diffusion equations with a nonlinear reaction term, *Cent. Eur. J. Phys.*, 54, (2013), 1-12.

[19] F. Liu, I. Turner, V. Anh, Q. Yang and K. Burrage, A numerical method for the fractional Fitzhugh-Nagumo monodomain model, *ANZIAM J.*, 154, (2013), C608-C629.

[20] H.G. Leopold, Embedding of function spaces of variable order of differentiation, *Czechoslovak Math. J.*, 49 (1999), 633-644.

[21] M.D. Ruiz-Medina, V.V. Anh, and J.M. Angulo, Fractional generalized random fields of variable order, *Stochastic Analysis and Applications*, 22(2) (2004), pp. 775-799.

[22] R. Lin, F. Liu, V. Anh and I. Turner, Stability and convergence of a new explicit finite-difference approximation for the variable-order nonlinear fractional diffusion equation, *Appl. Math. Comput.*, 212, (2009), 435-445.

[23] P. Zhuang, F. Liu, V. Anh and I. Turner, Numerical methods for the variable-order fractional advection-diffusion with a nonlinear source term, *SIAM J. Numer. Anal.*, 47(3), (2009), 1760-1781.

[24] B. Baeumer, M. Kovaly and M. Meerschaert, Fractional reproductiondispersal equations, *Bulletin of Mathematical Biology*, 69, (2007), 2281-2297.

[25] B. Baeumer, M. Kovaly and M. Meerschaert, Numerical solutions for fractional reaction-diffusion equations, *Comput. and Math. with Appl.*, 55, (2008), 2212-2226.

[26] C. Celik and M. Duman, Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative, *J. Comp. Phys.*, 231, (2012), 1743-1750.

[27] S. Chen, F. Liu, I. Turner and V. Anh, An implicit numerical method for the two-dimensional fractional percolation equation, *Appl. Math. Comp.*, 219, (2013), 4322-4331.

[28] F. Liu, V. Anh, I. Turner, S. Shen and J. Chen, A novel numerical approximation for the Riesz space fractional advection-dispersion equation, *IMA J. Appl. Math.*, (2012), 1-14.

[29] A. Bueno-Orovio, D. Kay and K. Burrage, Fourier spectral methods for fractional-in-space reaction-diffusion equations, BIT, April 2014, DOI 10.1007/s10543-014-0484-2.

[30] A. Bueno-Orovio, D. Kay, V. Grau, B. Rodriguez and K. Burrage, Fractional diffusion models of cardiac electrical propagation reveal structural heterogeneity effects on dispersion of repolarization, submitted, (2014).