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Uniform Exponential Stability Result for the Rigid-Body Attitude Tracking Control Problem

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Uniform Exponential Stability Result for the Rigid-Body Attitude Tracking Control Problem

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Uniform Exponential Stability Result for the Rigid-Body Attitude Tracking Control Problem

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This work proves that uniform exponential stability is achieved for the attitude control problem by adopting a PD+ control law that retains the classical proportional-derivative (PD) structure plus feedforward terms associated with tracking the desired attitude state. Previously, this controller was only known to offer the weaker result of uniform asymptotic stability. This thesis parameterizes the kinematics through the three-dimensional Modified Rodrigues Parameter (MRP), assumes perfect measurement of the full-state (i.e., both orientation and angular rate signals) and guarantees a stronger uniform exponential stability (UES) result. It should be emphasized that no additional restrictions on the reference trajectory or high-gain feedback assumptions are placed in achieving this new exponential stability result for the closed loop system. The design of a new Lyapunov function permits this stronger UES result which further allows facilitating robustness analysis in the possible presence of bounded unknown external disturbance torques. Saliently, this new Lyapunov

function naturally extends to the classical Gibbs-Rodrigues parameterization of the attitude kinematics.

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Chapter 1

Introduction

1.1 Literature Review

Attitude tracking controllers implemented through proportional derivative feedback components along with feedforward terms (the so called "PD+" structure) have been extensively studied over the past several decades [1] and find wide applicability for the attitude control of spaceflight vehicles, aircraft and rigid robots. The simple structure of this controller makes it easy to compute the control input at any point of time. Moreover, this PD+ control law does not require knowledge of any system specific inertia parameters for the special case of set-point attitude stabilization (regulation) once the feedback gains are selected by the user (the self-reduction property [1]). Notwithstanding the fact that the structural simplicity of the PD+ controller contributed to its wider adoption for large measure, it has only been shown to deliver uniform asymptotically stability for the closed-loop dynamics. This work preserves the classical PD+ controller structure but upgrades the result to uniform exponential stability (UES) without any additional restrictions on the controller parameters, thereby providing a stronger theoretical foundations for the design and implementation of the PD+ controller for rigid-body attitude dynamics.

The dynamics of the attitude tracking control problem for many of the aforementioned aero-mechanical systems tend to be nonlinear, making the controller design a challenging task. Precise characterization of the stability and robustness characteristics of the controller significantly enhances the opportunities of its successful adoption for any particular application. This work specifically considers the orientation control of a fully actuated rigid body in 3-dimensional space with full state feedback on the angular velocity and attitude components. The dynamics of the rotational velocity are given by Euler's equation, and for this thesis the attitude is chosen to be parameterized using the three-dimensional Modified Rodrigues Parameter (MRP) vector [2][3]. It is well known that MRPs present a non-redundant representation of attitude with a kinematic singularity that occurs when the body experiences a full 360-degree rotation [4]. However, the MRPs are not unique and there exists a shadow set which can be used to avoid the singularity [5]. Other choices for attitude parameterization include Euler angles, quaternions (Euler Parameters) [6], Gibbs-Rodrigues parameter [7], and rotation matrices [8]. Full-state feedback is assumed for this work along with the availability of perfect measurements (no noise corruption). The attitude tracking problem can be shown to be exponentially stable by feedback linearization [9], but it requires the control input to cancel the nonlinear terms in the dynamics which is not required by the simple PD+ controller.

Note that PD+ controllers have been shown to provide uniform asymptotic stability using both MRP [10] and Gibbs-Rodigues parameters [3]. Wen et al. [6] use the same PD+ controller with quaternions and demonstrate uniform exponential stability but lay lower bounds on the feedback gains to ensure the non-linear terms within the closed-loop dynamics are adequately dominated. This high feedback gain condition, however, can potentially induce large control inputs leading to actuator saturation besides potential hazards due to excitation of unmodeled dynamics. Tsiotras [11] presents attitude stabilization control results without angular velocity feedback which Akella [12] extended to trajectory tracking control using the MRP kinematics. Junkins et al. [13] [9] consider problems in which the system parameters such as inertia are not perfectly determined but can be estimated online through an adaptive control law for maneuvering spacecraft. A sliding mode based controller for systems with disturbances and uncertainties is presented in Ref. [14] and a back-stepping based control law is described by Krogstad et al.[15] using both quaternions and MRPs. Attitude control of distributed systems is discussed in Ref. [10]. Adaptive tracking control of the attitude motion of spacecraft with uncertain inertia matrices is handled using a dynamic compensator by Bernstein et al. [16].

This thesis retains the classical PD+ structure, but achieves exponential convergence of the attitude states to their specified reference values. Moreover, the controller can be designed without any prior knowledge of the bounds on the reference trajectory or body inertia, since no additional conditions are imposed upon the control gains. This stronger stability property of the closed loop system is a previously unknown result and therefore presents itself as the major contribution of this thesis[1]. Our work here is aided by the choice of MRPs for kinematics representation along with the judicious design of a novel Lyapunov function for closed-loop stability analysis. This chosen Lyapunov function builds upon the logarithmic term from Ref. [3] alongside the rotational kinetic energy terms, all of which are blended into an exponential function. Using Lyapunov direct method, the closed-loop system is found to exhibit uniform exponential stability, which interestingly also extends quite readily to the Gibbs-Rodrigues attitude representation. Further the effect of bounded disturbance torques on the controller performance is analyzed and a rigorous characterization of input-to-state (ISS) stability characteristics in the presence of bounded disturbances is provided. While the exponential stability property of the origin can be proven by linearizing the closed-loop system, this is only local stability whereas the result presented in this thesis holds in the large (almost global), without having to restrict the initial conditions over some compact set. The importance of this result is further amplified for the general case of attitude tracking wherein the linearization approach cannot be readily involved to claim local UES. This is because the closed-loop dynamics due to the application of the PD+ control law followed by linearization yields a linear time-varying (LTV) system. It is rather well known that LTV systems cannot be guaranteed to be stable even if their pointwise-in-time eigenvalues remain restricted to the open left-half of the complex plane.

[1]A preliminary version of this result was presented as a conference paper in Ref. [17] and the main result itself appeared as a journal paper in Ref. [18].

The remainder of the thesis is organized as follows: the following section reviews certain mathematical properties and stability definitions that will be useful for the remainder of the thesis. The governing equations are stated and the tracking error dynamics are derived in 2.1. This is followed by a statement of the control law and a quick review of the classical results pertaining to asymptotic stability analysis in 2.2. Chapter 3 then provides the proof for uniform exponential stability before further analyzing other attitude representations and disturbance rejection in Chapter 4. Bold face variables are used to represent vector and matrix quantities.

1.2 Preliminary Definitions

The following classical definitions for signal properties and stability results are reviewed here [19, section 4]:

1. Class $\mathcal{K}$ function: A continuous function $\phi\colon [0,\infty) \to \Re^+$ with $\phi(0) = 0$ and strictly increasing on $[0,\infty)$.

2. Class $\mathcal{KR}$ function: Any function $\phi \in \mathcal{K}$ with $\lim\limits_{r\to\infty} \phi(r) = \infty$.

3. Same order of magnitude: Two functions $\phi_1, \phi_2 \in \mathcal{K}$ on $[0,\infty)$ are said to be of same order of magnitude if there exist positive constants k_1 and k_2 such that $k_1\phi_1(r_1) \leq \phi_2(r_2) \leq k_2\phi_1(r_1)\ \forall r_1 \in [0,\infty)$.

4. Positive definite function: A function $V(t,\boldsymbol{x})\colon \Re^+ \times B_r \to \Re$ with $B_r \doteq \{\boldsymbol{x} \in \Re^n \ni \|\boldsymbol{x}\| < r\}$ for some $r > 0$ and $V(t,0) = 0\ \forall t \in \Re^+$ is said to be positive definite if there exists a function $\phi \in \mathcal{K}$ such that $V(t,\boldsymbol{x}) \geq \phi(\|\boldsymbol{x}\|)\ \forall t \in \Re^+$, for all $\boldsymbol{x} \in B_R$.

5. Decrescent function: A function $V(t,\boldsymbol{x})\colon \Re^+ \times B_r \to \Re$ with $B_r \doteq \{\boldsymbol{x} \in \Re^n \ni \|\boldsymbol{x}\| < r\}$ for some $r > 0$ and $V(t,0) = 0\ \forall t \in \Re^+$ is said to be decrescent if there exists a function $\phi \in \mathcal{K}$ such that $|V(t,\boldsymbol{x})| \leq \phi(\|\boldsymbol{x}\|)\ \forall t \in \Re^+$ and $\forall \boldsymbol{x} \in B_r$.

6. Radially unbounded function: A function $V(t,\boldsymbol{x})\colon \Re^+ \times \Re^n \to \Re$ with $V(t,0) = 0\ \forall t \in \Re^+$ is said to be radially unbounded if there exists a function $\phi \in \mathcal{KR}$ such that $|V(t,\boldsymbol{x})| \geq \phi(\|\boldsymbol{x}\|)\ \forall t \in \Re^+$ and $\forall \boldsymbol{x} \in \Re^n$.

7. Lyapunov stability theorem [19, pp. 154]: Suppose there exists a decrescent and radially unbounded function $V(t, \boldsymbol{x}) : \Re^+ \times \Re^n \rightarrow \Re^+$ with continuous first order partial derivatives with respect to t and $\boldsymbol{x}$ and $V(t, 0) = 0 \ \forall t \in \Re^+$. If there exist $\phi_1, \phi_2, \phi_3 \in \mathcal{KR}$ of the same order of magnitude such that $\phi_1(\|\boldsymbol{x}\|) \leq V(t, \boldsymbol{x}) \leq \phi_2(\|\boldsymbol{x}\|)$ and $\dot{V}(t, \boldsymbol{x}) \leq -\phi_3(\|\boldsymbol{x}\|)$ then the equilibrium point $\boldsymbol{x} = 0$ of $\dot{\boldsymbol{x}} = f(t, \boldsymbol{x})$ is global UES.

8. Class $\mathcal{KL}$ function: A continuous function $\beta\colon [0, r) \times [0, \infty] \rightarrow [0, \infty)$ is said to be class $\mathcal{KL}$ if $\beta(\boldsymbol{x}, \boldsymbol{y})$ satisfies:

   - For each fixed $\boldsymbol{y}$, $\beta(\boldsymbol{x}, \boldsymbol{y})$ is class $\mathcal{K}$ with respect to $\boldsymbol{x}$.
   - $\beta(\boldsymbol{x}, \boldsymbol{y})$ is decreasing with $\boldsymbol{y}$
   - $\lim\limits_{y \rightarrow \infty} \beta(\boldsymbol{x}, \boldsymbol{y}) = 0$

9. Input-to-state stability (ISS): The system $\dot{\boldsymbol{x}} = f(t, \boldsymbol{x}, \boldsymbol{u})$ is said to be input-to-state stable if $\exists \beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for any initial state $\boldsymbol{x}(t_0) \in \mathbb{R}^n$ and any bounded input $\boldsymbol{u}(t) \in \mathbb{R}^m$, the solution $\boldsymbol{x}(t)$ exists for all $t \geq t_0$ and satisfies

$$\|\boldsymbol{x}(t)\| \leq \beta\left(\|\boldsymbol{x}(t_0)\|, t - t_0\right) + \gamma\left(\sup_{t_0 \leq \tau \leq t} \|\boldsymbol{u}(\tau)\|\right)$$

10. ISS Theorem: Let $V : [0, \infty) \times \Re^n \rightarrow \Re$ be a continuously differentiable function such that

$$\alpha_1\left(\|\boldsymbol{x}\|\right) \leq V(t, \boldsymbol{x}) \leq \alpha_2(\|\boldsymbol{x}\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial \boldsymbol{x}} f(t, \boldsymbol{x}, \boldsymbol{u}) \leq -W_3(\boldsymbol{x}), \ \forall \|\boldsymbol{x}\| \geq \rho(\|\boldsymbol{u}\|) > 0$$

$\forall (t, \boldsymbol{x}, \boldsymbol{u}) \in [0, \infty) \times \Re^n \times \Re^m$, where α_1, α_2 are class $\mathcal{KR}$ functions, ρ is a class $\mathcal{K}$ function, and $W_3(\boldsymbol{x})$ is a continuous positive definite function on $\Re^n$. Then, the system $\dot{\boldsymbol{x}} = f(t, \boldsymbol{x}, \boldsymbol{u})$ is input-to-state stable (ISS).

Chapter 2

System Dynamics and Controller Design

2.1 Rotational Kinematics and Dynamics Equations

Our system is a rigid body rotating in three-dimensional space with the Euler principal axis (unit-vector) $\hat{\boldsymbol{e}}$ and the principal rotation angle of the attitude ϕ. To kinematically describe the motion of this system, Modified Rodrigues Parameters are used, given by

$$\boldsymbol{\sigma} = \hat{\boldsymbol{e}} \tan \frac{\phi}{4} \tag{2.1}$$

The kinematics and dynamics are given by:

$$\dot{\boldsymbol{\sigma}} = G(\boldsymbol{\sigma})\boldsymbol{\omega} \tag{2.2}$$

$$J\dot{\boldsymbol{\omega}} = -\boldsymbol{\omega} \times J\boldsymbol{\omega} + \boldsymbol{\tau} + \boldsymbol{d} \tag{2.3}$$

Note that $\boldsymbol{\omega}$ is the angular velocity of the rigid-body with respect to the inertial frame expressed in the body frame,

$$G(\boldsymbol{\sigma}) = \frac{1}{2}\left(S(\boldsymbol{\sigma}) + \boldsymbol{\sigma}\boldsymbol{\sigma}^{\boldsymbol{T}} + \frac{(1 - ||\boldsymbol{\sigma}\|^2)}{2} I_{3\times 3}\right) \tag{2.4}$$

$S(.)$ is the 3×3 skew-symmetric matrix representing the vector cross product, $\|\boldsymbol{\sigma}\| = \sqrt{\boldsymbol{\sigma}^{\boldsymbol{T}}\boldsymbol{\sigma}}$, $\boldsymbol{J} \in R^{3\times 3}$ is the inertia tensor (symmetric positive-definite matrix), $\boldsymbol{\tau} \in \mathbb{R}^3$ is the control torque and $\boldsymbol{d}$ is the unknown external disturbance in the torque vector assumed to be bounded with a maximum norm value of $d_{\max}$, i.e., $\sup_{t\geq 0} \|\boldsymbol{d}(t)\| \leq d_{\max}$.

Given any MRP vector $\boldsymbol{\sigma}$ and desired attitude in terms of MRP $\boldsymbol{\sigma_d}$, the relative attitude can be parameterized through the error MRP vector (Eq. 3.153 in [4])

$$\boldsymbol{\sigma_e} = \frac{(1 - \|\boldsymbol{\sigma_d}\|^2)\boldsymbol{\sigma} - (1 - \|\boldsymbol{\sigma}\|^2)\boldsymbol{\sigma_d} + 2\boldsymbol{\sigma} \times \boldsymbol{\sigma_d}}{1 + \|\boldsymbol{\sigma}\|^2\|\boldsymbol{\sigma_d}\|^2 + 2\boldsymbol{\sigma_d^T}\boldsymbol{\sigma}} \tag{2.5}$$

Let $\boldsymbol{A(\sigma)}$ denote the direction cosine matrix associated with the body frame, which can be stated in terms of the MRP vector σ as

$$\boldsymbol{A(\sigma)} = \boldsymbol{I_{3\times 3}} + \frac{8S(\boldsymbol{\sigma})^2 - 4(1 - \|\boldsymbol{\sigma}\|^2)S(\boldsymbol{\sigma})}{(1 + \|\boldsymbol{\sigma}\|^2)^2} \tag{2.6}$$

Note that $A\left(\boldsymbol{\sigma_e}\right) = \boldsymbol{A(\sigma)A^T(\sigma_d)}$

Similarly, relative angular velocity in body frame $\boldsymbol{\omega_e} = \boldsymbol{\omega} - \boldsymbol{\omega_d^b}$, wherein $\boldsymbol{\omega_d}$ is the desired angular velocity in the desired frame and $\boldsymbol{\omega_d^b} = A(\boldsymbol{\sigma_e})\boldsymbol{\omega_d}$ is the angular velocity of the desired frame expressed in the body frame. Let the desired reference for the rotational velocity $\boldsymbol{\omega_d^b}$ have a maximum value of $\boldsymbol{\delta} = \sup_{t\geq 0}(\|\boldsymbol{\omega_d^b}\|)$, the inertia matrix J has largest eigenvalue of J_{M} i.e., $\|J\| = J_{\mathrm{M}}$ and smallest eigenvalue J_{m}. The error dynamics are given by

$$\dot{\boldsymbol{\sigma}}_{\boldsymbol{e}} = G(\boldsymbol{\sigma_e})\boldsymbol{\omega_e} \tag{2.7}$$

$$\begin{aligned}
\boldsymbol{J}\dot{\boldsymbol{\omega}}_{\boldsymbol{e}} &= J\dot{\boldsymbol{\omega}} - J\dot{\boldsymbol{\omega}}_{\boldsymbol{d}}^{\boldsymbol{b}} \\
&= -\boldsymbol{\omega} \times J\boldsymbol{\omega} + \boldsymbol{\tau} + \boldsymbol{d} - J\dot{\boldsymbol{\omega}}_{\boldsymbol{d}}^{\boldsymbol{b}} \\
&= -\boldsymbol{\omega} \times J\boldsymbol{\omega} + \boldsymbol{\tau} + \boldsymbol{d} - JA(\boldsymbol{\sigma_e})\dot{\boldsymbol{\omega}}_{\boldsymbol{d}} + \boldsymbol{\omega_e} \times \boldsymbol{\omega_d^b}
\end{aligned} \tag{2.8}$$

Given the foregoing tracking error dynamics, the control design would provide commanded torques to be generated by the actuators in order for the attitude and angular velocity states to track the desired trajectory. For the proof of exponential convergence, the perfect case of zero external disturbances is considered, i.e., $\boldsymbol{d} = 0$ in Eq. 2.3 and Eq. 2.8.

2.2 Controller Design

Let us consider the candidate Lyapunov function

$$\begin{aligned}
V_0 &= \frac{1}{2}\boldsymbol{\omega_e}^T J\boldsymbol{\omega_e} + 2k_p \ln(1+\boldsymbol{\sigma_e}^T\boldsymbol{\sigma_e}), \qquad \text{any } k_p > 0,\\
\frac{V_0}{2k_p} &= \ln\exp\left(\frac{1}{4k_p}\boldsymbol{\omega_e}^T J\boldsymbol{\omega_e}\right) + \ln(1+\boldsymbol{\sigma_e}^T\boldsymbol{\sigma_e})\\
&= \ln\left[(1+\boldsymbol{\sigma_e}^T\boldsymbol{\sigma_e})\exp\left(\frac{1}{4k_p}\boldsymbol{\omega_e}^T J\boldsymbol{\omega_e}\right)\right]
\end{aligned} \tag{2.9}$$

which is a class $\mathcal{KR}$ function, decrescent and radially unbounded.

For the PD+ control torque given by

$$\boldsymbol{\tau} = -k_p\boldsymbol{\sigma_e} - k_v\boldsymbol{\omega_e} + JA(\boldsymbol{\sigma_e})\dot{\boldsymbol{\omega}}_{\boldsymbol{d}} + \boldsymbol{\omega_d^b} \times (J\boldsymbol{\omega_d^b}), \qquad \text{any } k_v > 0 \tag{2.10}$$

the time-derivative of V_0 in Eq. 2.9, along the trajectories of the closed-loop system becomes

$$\begin{aligned}
\dot{V}_0 &= \boldsymbol{\omega_e}^T\left(\boldsymbol{J}\dot{\boldsymbol{\omega}} - \boldsymbol{J}\dot{\boldsymbol{\omega}}_{\boldsymbol{d}}^{\boldsymbol{b}}\right) + \frac{4k_p\boldsymbol{\sigma_e}^T(\dot{\boldsymbol{\sigma}}_{\boldsymbol{e}})}{1+\boldsymbol{\sigma_e}^T\boldsymbol{\sigma_e}}\\
&= \boldsymbol{\omega_e}^T\left[-\boldsymbol{\omega}\times(\boldsymbol{J\omega}) + \boldsymbol{\tau} - \boldsymbol{J}\dot{\boldsymbol{\omega}}_{\boldsymbol{d}}^{\boldsymbol{b}}\right]\\
&\quad + \frac{2k_p\boldsymbol{\sigma_e}^T}{1+\boldsymbol{\sigma_e}^T\boldsymbol{\sigma_e}}\left[S(\boldsymbol{\sigma_e}) + \boldsymbol{\sigma}_e\boldsymbol{\sigma}_e^T + \frac{1-\|\boldsymbol{\sigma}_e\|^2}{2}\right]\boldsymbol{\omega_e}\\
&= \boldsymbol{\omega_e}^T\left[-\boldsymbol{\omega}\times(\boldsymbol{J\omega}) + \boldsymbol{\tau} - \boldsymbol{J}\dot{\boldsymbol{\omega}}_{\boldsymbol{d}}^{\boldsymbol{b}}\right] + \frac{2k_p}{1+\boldsymbol{\sigma_e}^T\boldsymbol{\sigma_e}}\left[\boldsymbol{\sigma}_e^T\boldsymbol{\omega}_e\frac{1+\|\boldsymbol{\sigma}_e\|^2}{2}\right]\\
&= \boldsymbol{\omega_e}^T\left[-\boldsymbol{\omega}\times(\boldsymbol{J\omega}) - k_p\boldsymbol{\sigma_e} - k_v\boldsymbol{\omega_e} + \boldsymbol{J}A(\boldsymbol{\sigma_e})\dot{\boldsymbol{\omega}}_{\boldsymbol{d}} + \boldsymbol{\omega_d^b}\times(\boldsymbol{J}\boldsymbol{\omega_d^b}) - \boldsymbol{J}\dot{\boldsymbol{\omega}}_{\boldsymbol{d}}^{\boldsymbol{b}}\right]\\
&\quad + k_p\boldsymbol{\sigma_e}^T\boldsymbol{\omega_e}\\
&= \boldsymbol{\omega_e}^T\left[-\boldsymbol{\omega}\times(\boldsymbol{J\omega}) + \boldsymbol{J}(\boldsymbol{\omega_e}\times\boldsymbol{\omega_d^b}) + \boldsymbol{\omega_d^b}\times(J\boldsymbol{\omega_d^b})\right] - k_v\|\boldsymbol{\omega_e}\|^2\\
&= \boldsymbol{J\omega}^T(-\boldsymbol{\omega_e}\times\boldsymbol{\omega_e}) - \boldsymbol{\omega_e}^T\underbrace{\left[S(\boldsymbol{\omega_d^b})J + J(S(\boldsymbol{\omega_d^b})\right]}_{Q}\boldsymbol{\omega_e} - k_v\|\boldsymbol{\omega_e}\|^2
\end{aligned}$$

Since $Q^T = -Q$, we immediately have

$$\dot{V}_0 = -k_v \|\boldsymbol{\omega_e}\|^2 \tag{2.11}$$

This choice of the Lyapunov function in Eq. 2.9 is classical and it is sufficient to prove uniform asymptotic stability for the closed-loop system (as shown in 3). In the following chapter, a new stability proof is constructed to demonstrate that the same control law from Eq. 2.10 actually ensures UES for the closed-loop system.

Chapter 3

Uniform Exponential Stability Result

Having proven asymptotic stability of the system, we not proceed to the novel uniform exponential stability proof. In this chapter, we will define a new Lyapunov function whose derivative will then be shown to satisfy the UES conditions (7). Using V_0 in Eq. 2.9 as a building block, and the "cross-term" $N \doteq \boldsymbol{\sigma_e^T J \omega_e}$, another Lyapunov-like candidate function is proposed

$$\begin{aligned} V &= c[\exp\left(\frac{V_0}{2k_p}\right) - 1] + N \\ &= c\left[\left(1 + \boldsymbol{\sigma_e^T \sigma_e}\right)\exp\left(\frac{1}{4k_p}\boldsymbol{\omega_e^T J \omega_e}\right) - 1\right] + \boldsymbol{\sigma_e^T J \omega_e} \end{aligned} \tag{3.1}$$

wherein $c > 0$ is a sufficiently large finite constant. The precise conditions for selection of c would be specified in the sequel. For V to be a valid Lyapunov-like function, it needs to be positive definite. Toward that goal, note that

$$\begin{aligned} V &\geq c\left[\left(1 + \|\boldsymbol{\sigma_e}\|^2\right)\left(1 + \frac{\boldsymbol{\omega_e^T J \omega_e}}{4k_p}\right) - 1\right] + \boldsymbol{\sigma_e^T J \omega_e} \\ &\geq c\left[\|\boldsymbol{\sigma_e}\|^2 + \frac{\boldsymbol{\omega_e^T J \omega_e}}{4k_p}\right] - \frac{J_{\mathrm{M}}}{2}\left[\|\boldsymbol{\sigma_e}\|^2 + \|\boldsymbol{\omega_e}\|^2\right] \\ &\geq \left(c - \frac{J_{\mathrm{M}}}{2}\right)\|\boldsymbol{\sigma_e}\|^2 + \left(\frac{cJ_{\mathrm{m}}}{4k_p} - \frac{J_{\mathrm{M}}}{2}\right)\|\boldsymbol{\omega_e}\|^2 \end{aligned} \tag{3.2}$$

Thus if

$$c > \frac{J_{\mathrm{M}}}{2}\ \max\left\{1, \frac{4k_{\mathrm{p}}}{J_{\mathrm{m}}}\right\} \tag{3.3}$$

there exists a finite positive constant μ such that

$$V \geq \mu \left(\|\boldsymbol{\sigma_e}\|^2 + \|\boldsymbol{\omega_e}\|^2 \right) \tag{3.4}$$

ensuring V is radially unbounded. Since $V \in \mathcal{KR}$, it can also be proven to be decrescent by showing $|V| \leq kV$ for any $k > 1$.

The time-derivative of V in Eq. 3.1 is

$$\begin{aligned}
\dot{V} &= c \left[\frac{1}{k_p} \exp\left(\frac{V_0}{2k_p} \right) \dot{V}_0 \right] + \dot{N} \\
&= \frac{-ck_v \|\boldsymbol{\omega_e}\|^2}{2k_p} \exp\left(\frac{V_0}{2k_p} \right) + \dot{N} \\
&= \frac{-ck_v \|\boldsymbol{\omega_e}\|^2 \left(1 + \|\boldsymbol{\sigma_e}\|^2\right)}{4k_p} \exp\left(\frac{\boldsymbol{\omega_e^T J \omega_e}}{4k_p} \right) - \frac{ck_v \|\boldsymbol{\omega_e}\|^2}{4k_p} \exp\left(\frac{V_0}{2k_p} \right) + \dot{N} \\
&= S - \frac{ck_v \|\boldsymbol{\omega_e}\|^2}{4k_p} \exp\left(\frac{V_0}{2k_p} \right) + \dot{N}
\end{aligned} \tag{3.5}$$

wherein the quantity S defined by

$$S = \frac{-ck_v \|\boldsymbol{\omega_e}\|^2 \left(1 + \|\boldsymbol{\sigma_e}\|^2\right)}{4k_p} \exp\left(\frac{\boldsymbol{\omega_e^T J \omega_e}}{4k_p} \right) \tag{3.6}$$

is introduced for convenience of algebra and notation.

Next, differentiate the cross term $N = \boldsymbol{\sigma_e^T J \omega_e}$ with respect to time to get,

$$
\begin{aligned}
\dot{N} &= (\boldsymbol{J\omega_e})^T \dot{\boldsymbol{\sigma}}_{\boldsymbol{e}} + \boldsymbol{\sigma_e^T J} \dot{\boldsymbol{\omega}}_{\boldsymbol{e}} \\
&= \boldsymbol{\omega_e^T J} G(\boldsymbol{\sigma}_e) \boldsymbol{\omega_e} + \boldsymbol{\sigma}_{\boldsymbol{e}}^T \left[-\boldsymbol{\omega} \times (\boldsymbol{J\omega}) + \boldsymbol{\tau} - \boldsymbol{J} \dot{\boldsymbol{\omega}}_{\boldsymbol{d}}^{\boldsymbol{b}} \right] \\
&= \boldsymbol{\omega_e^T J} G(\boldsymbol{\sigma}_e) \boldsymbol{\omega_e} \\
&\quad + \boldsymbol{\sigma_e^T} \left[-\boldsymbol{\omega} \times (\boldsymbol{J\omega}) - \boldsymbol{J} A(\boldsymbol{\sigma}_e) \dot{\boldsymbol{\omega}}_{\boldsymbol{d}} + \boldsymbol{J}(\boldsymbol{\omega_e} \times \boldsymbol{\omega_d^b}) + -k_p \boldsymbol{\sigma_e} - k_v \boldsymbol{\omega_e} \right] \\
&\quad + \boldsymbol{\sigma_e^T} \left[\boldsymbol{J A(\sigma_e)} \dot{\boldsymbol{\omega}}_{\boldsymbol{d}} + \boldsymbol{\omega_d^b} \times (\boldsymbol{J\omega_d^b}) \right] \\
&= \boldsymbol{\omega_e^T J} G(\boldsymbol{\sigma}_e) \boldsymbol{\omega_e} + \boldsymbol{\sigma_e^T} \left[-k_p \boldsymbol{\sigma_e} - k_v \boldsymbol{\omega_e} - \boldsymbol{\omega_e} \times (J \boldsymbol{\omega_e}) \right] \\
&\quad - \boldsymbol{\sigma_e^T} \left[\boldsymbol{\omega_e} \times (J \boldsymbol{\omega_d^b}) + \boldsymbol{\omega_d^b} \times (J \boldsymbol{\omega_e}) + J(\boldsymbol{\omega_e} \times \boldsymbol{\omega_d^b}) \right]
\end{aligned}
$$

Taking the two-norm for terms on the right-hand side, and using $x \leq (1+x^2)/2$,

$$
\begin{aligned}
\dot{N} \leq\ & \boldsymbol{\omega_e^T J} G(\boldsymbol{\sigma_e}) \boldsymbol{\omega_e} - k_p \|\boldsymbol{\sigma_e}\|^2 - k_v \boldsymbol{\sigma_e^T \omega_e} + \frac{J_{\mathrm{M}}}{2}(1 + \|\boldsymbol{\sigma_e}\|^2) \|\boldsymbol{\omega_e}\|^2 \\
& + 3 J_{\mathrm{M}} \boldsymbol{\delta} \|\boldsymbol{\sigma_e}\| \|\boldsymbol{\omega_e}\|
\end{aligned}
$$

Using the fact that $\|G(\boldsymbol{\sigma_e})\| = (1 + \boldsymbol{\sigma_e^T \sigma_e})/4$ as shown in [9] provides

$$
\begin{aligned}
\dot{N} &\leq -k_p \|\boldsymbol{\sigma_e}\|^2 + \frac{J_{\mathrm{M}}}{4}(1 + \|\boldsymbol{\sigma_e}\|^2) \|\boldsymbol{\omega_e}\|^2 - k_v \boldsymbol{\sigma_e^T \omega_e} + \frac{J_{\mathrm{M}}}{2}(1 + \|\boldsymbol{\sigma_e}\|^2) \|\boldsymbol{\omega_e}\|^2 \\
&\quad + 3 J_{\mathrm{M}} \boldsymbol{\delta} \|\boldsymbol{\sigma_e}\| \|\boldsymbol{\omega_e}\| \\
&\leq -k_p \|\boldsymbol{\sigma_e}\|^2 + \frac{3 J_{\mathrm{M}}}{4}(1 + \|\boldsymbol{\sigma_e}\|^2) \|\boldsymbol{\omega_e}\|^2 + (k_v + 3 J \boldsymbol{\delta}) \|\boldsymbol{\sigma_e}\| \|\boldsymbol{\omega_e}\| \\
&\leq -\frac{k_p}{2} \|\boldsymbol{\sigma_e}\|^2 + \left[\frac{3 J_{\mathrm{M}}}{4} + \frac{(k_v + 3 J_{\mathrm{M}} \boldsymbol{\delta})^2}{2 k_p} \right] (1 + \|\boldsymbol{\sigma_e}\|^2) \|\boldsymbol{\omega_e}\|^2 \\
\dot{N} &\leq -\frac{k_p}{2} \|\boldsymbol{\sigma_e}\|^2 + k_0 (1 + \|\boldsymbol{\sigma_e}\|^2) \|\boldsymbol{\omega_e}\|^2 \qquad (3.7)
\end{aligned}
$$

wherein

$$
k_0 \doteq [3 J_{\mathrm{M}}/4 + (k_v + 3 J_{\mathrm{M}} \boldsymbol{\delta})^2 / (2 k_p)] \qquad (3.8)
$$

is introduced for notational convenience. For the stabilization special case wherein $\boldsymbol{\omega}_d(t) \equiv 0$ for all $t \geq 0$, the expression is simpler $k_0 = [3J_{\mathrm{M}}/4 + k_v^2/(2k_p)]$, since $\boldsymbol{\delta} = 0$ for the stabilization special case.

Further analysis for the term S defined in Eq. 3.6 provides the following

$$\begin{aligned} S &= \frac{-ck_v(1+\|\boldsymbol{\sigma_e}\|^2)}{k_p} \exp\left(\frac{\boldsymbol{\omega}_{\boldsymbol{e}}^T J \boldsymbol{\omega_e}}{4k_p}\right)\left(\frac{\|\boldsymbol{\omega_e}\|^2 J_{\mathrm{M}}}{4J_{\mathrm{M}}}\right) \\ &\leq -\frac{ck_v}{J_{\mathrm{M}}}(1+\|\boldsymbol{\sigma_e}\|^2) \exp\left(\frac{\boldsymbol{\omega_e^T} J \boldsymbol{\omega_e}}{4k_p}\right)\left(\frac{\boldsymbol{\omega_e^T} J \boldsymbol{\omega_e}}{4k_p}\right) \end{aligned}$$

Using $-x\exp(x) < -(\exp(x) - 1)$, for any real-valued scalar variable x, it follows that

$$\begin{aligned} S &\leq -\frac{ck_v}{J_{\mathrm{M}}}\left[(1+\|\boldsymbol{\sigma_e}\|^2)\exp(\frac{\boldsymbol{\omega_e^T} J \boldsymbol{\omega_e}}{4k_p}) - 1\right] + \frac{ck_v}{J_{\mathrm{M}}}\|\boldsymbol{\sigma_e}\|^2 \\ &\leq -\frac{k_v}{J_{\mathrm{M}}}\left[c\left[(1+\|\boldsymbol{\sigma_e}\|^2)\exp(\frac{\boldsymbol{\omega_e^T} J \boldsymbol{\omega_e}}{4k_p}) - 1\right] + \boldsymbol{\sigma_e^T} J \boldsymbol{\omega_e}\right] + \frac{ck_v}{J_{\mathrm{M}}}\|\boldsymbol{\sigma_e}\|^2 \\ &\quad + \frac{k_v}{J_{\mathrm{M}}}\sigma_e^T J \omega_e \\ &\leq -\frac{k_v}{J_{\mathrm{M}}}V + \frac{ck_v}{J_{\mathrm{M}}}\|\boldsymbol{\sigma_e}\|^2 + \frac{k_v}{J_{\mathrm{M}}}\boldsymbol{\sigma_e^T} J \boldsymbol{\omega_e} \end{aligned} \tag{3.9}$$

Returning to $\dot{V}$ in Eq. 3.5,

$$\dot{V} \leq S - \frac{ck_v}{4k_p}(1+\|\boldsymbol{\sigma_e}\|^2)\exp\left(\frac{\boldsymbol{\omega_e} J \boldsymbol{\omega_e}}{4k_p}\right)\|\boldsymbol{\omega_e}\|^2 + \dot{N}$$

Since S is non-positive by its definition, $\alpha \in (0,1)$ can be chosen such that

$$\begin{aligned}
\dot{V} &\leq \alpha S - \frac{ck_v}{4k_p}(1+\|\boldsymbol{\sigma_e}\|^2)\|\boldsymbol{\omega_e}\|^2 - \frac{k_p}{2}\|\boldsymbol{\sigma_e}\|^2 + k_0(1+\|\boldsymbol{\sigma_e}\|^2)\|\boldsymbol{\omega_e}\|^2 \\
&\leq \alpha S - \frac{ck_v}{8k_p}\|\boldsymbol{\omega_e}\|^2 - \frac{k_p}{2}\|\boldsymbol{\sigma_e}\|^2 - \left(\frac{ck_v}{8k_p} - k_0\right)\left(1+\|\boldsymbol{\sigma_e}\|^2\right)\|\boldsymbol{\omega_e}\|^2 \\
&\leq \alpha S - \frac{ck_v}{8k_p}\|\boldsymbol{\omega_e}\|^2 - \frac{k_p}{2}\|\boldsymbol{\sigma_e}\|^2
\end{aligned}$$

provided c, combined with Eq.(3.3) is chosen such that

$$c \geq \max\left\{\frac{8k_pk_0}{k_v}, \frac{J_{\mathrm{M}}}{2}, \frac{2J_{\mathrm{M}}k_p}{J_{\mathrm{m}}}\right\} \tag{3.10}$$

Using the upper bound upon S established in Eq. 3.9 gives

$$\begin{aligned}
\dot{V} &\leq \alpha\left(-\frac{k_v}{J_{\mathrm{M}}}V + \frac{ck_v}{J_{\mathrm{M}}}\|\boldsymbol{\sigma_e}\|^2 + \frac{k_v}{J_{\mathrm{M}}}\sigma_e^T J\omega_e\right) - \frac{ck_v}{8k_p}\|\boldsymbol{\omega_e}\|^2 - \frac{k_p}{2}\|\boldsymbol{\sigma_e}\|^2 \\
&\leq -\frac{\alpha k_v}{J_{\mathrm{M}}}V - \left(\frac{k_p}{2} - \frac{\alpha ck_v}{J_{\mathrm{M}}}\right)\|\boldsymbol{\sigma_e}\|^2 - \frac{ck_v}{8k_p}\|\boldsymbol{\omega_e}\|^2 + \frac{\alpha k_v}{J_{\mathrm{M}}}\sigma_e^T J\omega_e \\
&\leq -\frac{\alpha k_v}{J_{\mathrm{M}}}V - \left(\frac{k_p}{2} - \frac{\alpha ck_v}{J_{\mathrm{M}}}\right)\|\boldsymbol{\sigma_e}\|^2 - \frac{ck_v}{8k_p}\|\boldsymbol{\omega_e}\|^2 + \frac{\alpha k_v}{2}(\|\boldsymbol{\sigma_e}\|^2 + \|\boldsymbol{\omega_e}\|^2) \\
&\leq -\frac{\alpha k_v}{J_{\mathrm{M}}}V + \left(\frac{\alpha k_v}{2}(1+\frac{2c}{J_{\mathrm{M}}}) - \frac{k_p}{2}\right)\|\boldsymbol{\sigma_e}\|^2 + \frac{k_v}{2}\left(\alpha - \frac{c}{4k_p}\right)\|\boldsymbol{\omega_e}\|^2 \\
&\leq \frac{-\alpha k_v}{J_{\mathrm{M}}}V
\end{aligned} \tag{3.11}$$

if α is chosen such that

$$0 < \alpha < \min\left[\frac{c}{4k_p}, \frac{k_p}{k_v\left(1+\frac{2c}{J_{\mathrm{M}}}\right)}, 1\right] \tag{3.12}$$

Eq. 3.12 and Eq. 3.10 represent the conditions on α and c for Eq.3.11 to be true.

Thus it is ensured that $\dot{V} \leq \beta V$ for some constant $\beta \doteq (\alpha k_v / J_{\mathrm{M}})$, and therefore, from comparison lemma [19],

$$V(t) \leq V(0) \exp(-\alpha k_v t / J_{\mathrm{M}}), \quad \text{for all } t \geq 0 \tag{3.13}$$

Combining this with the fact that $V \in \mathcal{KR}$, is decrescent and radially unbounded (Eq. 3.4), the Lyapunov theorem (7) proves UES for the origin of the closed-loop system with the PD+ controller in Eq. 2.10 for tracking any specified reference trajectory. It can also be seen from just Eq. (3.4) that since V is an upper bound on the norm of the error in the states, the errors too converge exponentially to zero.

It is important to emphasize that parameters c and α are used only for analysis and they do not affect the controller design itself. More specifically, the control law in Eq. (2.10) provides UES for the closed-loop system so long as k_p and k_v are chosen to be positive, irrespective of the body inertia and reference trajectory characteristics. The stability condition shown here can be considered to be an *almost global* result, since the controller itself provides exponential stability for all initial conditions but the presence of a singularity at $\phi = \pm 2\pi$ in the MRP attitude representation (Eq. 2.1) hinders the claim of global uniform exponential stability. More precisely, the new result derived here in this work shows global UES in the $(\boldsymbol{\sigma}_e, \boldsymbol{\omega}_e)$ space using the PD+ control law in Eq. 2.10. Essentially, this implies uniform exponential stability over an open and dense set in the configuration space of the attitude motion SO(3), which is not a contractible space. In the literature, this type of stability result is often endowed with the qualifier *almost global* [20]. In this context, it is pertinent to recall that the topological structure of SO(3) does not allow for globally continuously stabilizing control laws and accordingly, no claims of global UES are made as a result of application of the PD+ controller in Eq. 2.10. To be more specific, the results of this thesis allow us to claim almost global UES for the closed loop system due to the control law in Eq. 2.10.

Chapter 4

Implications for Gibbs-Rodrigues Parameterization and Robustness Analysis

In this chapter we discuss further the implications of the new UES result to quaternions and Gibbs-Rodrigues based attitude parameterizations, and the case when we have bounded non-zero disturbances in the torque input. The following remarks are now in order.

(a) From the fact that $\lim_{t\to\infty} V(t) = 0$ exponentially fast(from Eq. 3.13), taken together with Eq. (3.4), provides the MRP error state

$$\boldsymbol{\sigma}_e = \hat{\boldsymbol{e}} \tan(\phi_e/4)$$

where ϕ_e is the error in the principal rotation angle of the attitude, also decaying to zero exponentially fast. This result readily implies exponential convergence for the vector part of the quaternion error $\hat{\boldsymbol{e}} \sin(\phi_e/2)$ as can be seen from the trigonometric identity:

$$|\sin(\phi_e/2)| = \frac{2|\tan(\phi_e/4)|}{1 + \tan^2(\phi_e/4)} \leq 2|\tan(\phi_e/4)| \tag{4.1}$$

In other words, the vector part of the quaternion error is upper bounded by twice the magnitude of the MRP error vector and thus exponential convergence also holds for the vector part of the quaternion as a result of using the PD+ controller from Eq. (2.10) in terms of the MRP vector for attitude kinematics parameterization.

(b) The UES result for the PD+ controller in terms of the MRP attitude kinematics parameterization extends *mutatis mutandis* when using the Classical/Gibbs Rodrigues vector for attitude parameterization. Specifically, the Gibbs Rodrigues parameter is given by

$$\boldsymbol{q} = \hat{\boldsymbol{e}} \tan \frac{\phi}{2} \tag{4.2}$$

whereas the kinematic differential equation for this attitude representation is given by [4, pp. 115]

$$\dot{\boldsymbol{q}} = B(\boldsymbol{q})\boldsymbol{\omega} \tag{4.3}$$

with

$$B(\boldsymbol{q}) = \frac{1}{2}[S(\boldsymbol{q}) + \boldsymbol{q}\boldsymbol{q}^{\boldsymbol{T}} + I_{3x3}] \tag{4.4}$$

The parameterization for the direction cosine matrix in terms of the Gibbs vector is

$$\bar{A}(\boldsymbol{q}) = \frac{(1 - \boldsymbol{q}^{\boldsymbol{T}}\boldsymbol{q})I_{3x3} + 2\boldsymbol{q}\boldsymbol{q}^{\boldsymbol{T}} - 2S(\boldsymbol{q})}{1 + \boldsymbol{q}^{\boldsymbol{T}}\boldsymbol{q}} \tag{4.5}$$

Similar to (2.10), the control torque can be chosen to have the PD+ structure given by

$$\boldsymbol{\tau} = -k_p\boldsymbol{q_e} - k_v\boldsymbol{\omega_e} + J\bar{A}(\boldsymbol{q_e})\boldsymbol{\omega_d} + \boldsymbol{\omega_d^b} \times (J\boldsymbol{\omega_d^b}) \tag{4.6}$$

wherein $\boldsymbol{q}_e$ represents the attitude error in terms of the Gibbs Rodrigues vector and the feedback gains k_p and k_v being positive constants. Simply replacing V_0 with

$$\bar{V}_0 = \frac{1}{2}\boldsymbol{\omega}_{\boldsymbol{e}}^T J\boldsymbol{\omega}_{\boldsymbol{e}} + k_p \ln(1 + \boldsymbol{q}_{\boldsymbol{e}}^T \boldsymbol{q}_{\boldsymbol{e}}), \qquad \text{any } k_p > 0 \tag{4.7}$$

the parameter k_0 in Eq. (3.8) with

$$\bar{k}_0 = [J_{\mathrm{M}} + (k_v + 3J_{\mathrm{M}}\boldsymbol{\delta})^2/(2k_p)] \tag{4.8}$$

and $\boldsymbol{\sigma}_e$ within V with $\boldsymbol{q}_e$, allows the stability analysis to proceed along identical lines, ultimately resulting in UES at the origin for the closed-loop system with the adoption of the control torque given in Eq. (4.6).

(c) The disturbance in torque input term $\boldsymbol{d}$ was neglected in the preceding analysis (Eq. 2.3 and Eq. 2.8). However, if an upper bound on the disturbances $\boldsymbol{d}_{\max}$ is taken to exist, these disturbances can be compensated for, using some of the non-positive terms that appear in the foregoing Lyapunov analysis. Returning to Eq. 3.11 , this results in slight modifications in the choice of α and analysis of $\dot{V}$:

$$\begin{aligned}\dot{V} \leq \frac{-\alpha k_v}{\boldsymbol{J}_M} V + \underbrace{\|\boldsymbol{\sigma}_e\|\boldsymbol{d}_{\max} - \frac{k_p}{4}\|\boldsymbol{\sigma}_e\|^2}_{\text{Term 1}} \\ + \underbrace{(1-\alpha)P + \frac{c}{k_p}\exp\left(\frac{V_0}{2k_p}\right)\|\boldsymbol{\omega}_e\|\boldsymbol{d}_{\max}}_{\text{Term 2}}\end{aligned} \tag{4.9}$$

which will be of the form $\dot{V} \leq -kV$ for some $k > 0$, provided Terms 1 and 2 are non-positive and α is chosen such that

$$0 < \alpha < \min\left[\frac{c}{4k_p}, \frac{k_p}{2k_v\left(1 + \frac{2c}{\boldsymbol{J}_M}\right)}, 1\right] \tag{4.10}$$

We will now consider the conditions for Terms 1 and 2 to be non-positive.

Term 1: $\|\boldsymbol{\sigma}_e\| \geq (4\boldsymbol{d}_{\max})/k_p$ would make $\|\boldsymbol{\sigma}_e\|\boldsymbol{d}_{\max} - (k_p/4)\|\boldsymbol{\sigma}_e\|^2 \leq 0$

Term 2: $\|\boldsymbol{\omega}_e\| \geq (4\boldsymbol{d}_{\max})/(k_v(1-\alpha))$ would make

$(1-\alpha)P + (c/k_p)\exp\left(V_0/2k_p\right)\|\boldsymbol{\omega}_e\|\boldsymbol{d}_{\max} \leq 0$

Next, we express the state to be $x = \left[\sigma_e^T, \omega_e^T\right]^T$ and the disturbance torque $\boldsymbol{d}$ to be the input to the closed loop system with the control law in Eq. (2.10). Since it is known that $V \in \mathcal{KR}$ and $\dot{V} \leq -kV$ for $k > 0$, using the ISS theorem (10) with $W_3(x) = kV$, the closed loop system with disturbances is known to be input-to-state stable, indicating that the system states will remain bounded for bounded disturbances.

Thus if

$$\mu = \sqrt{\left(\frac{4}{k_p}\right)^2 + \left(\frac{4}{k_v(1-\alpha)}\right)^2} \tag{4.11}$$

then,

$$\|x\| > \mu\boldsymbol{d}_{\max} \tag{4.12}$$

ensures satisfaction of the $\|x\| \geq \rho(\|u\|)$ condition in (10). This represents a hyper-sphere centered about the origin in 6-dimensional space and outside the boundary of this region, the closed loop system with bounded disturbances is input-state stable.

Chapter 5

Conclusions

The proportional derivative control structure augmented with feed-forward terms (PD+) has received considerable attention in the attitude control literature albeit the fact that the resulting closed-loop system is thus far only shown to be uniform asymptotically stable. Aided by the construction of a new Lyapunov function, this thesis significantly strengthens the closed-loop stability conditions to establish uniform exponential stability for the PD+ controller placing no additional restrictions on the feedback gains. This stronger result reaffirms the effectiveness of the PD+ controller and provides a compelling motivation for further investigation of the robustness properties for the closed-loop system in the presence of measurement errors and uncertain inertia parameters.

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