8-6-2020

Unifying AoI Minimization and Remote Estimation — Optimal Sensor/Controller Coordination with Random Two-way Delay

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***Abstract*—The ubiquitous usage of communication networks in modern sensing and control applications has kindled new interests on the timing coordination between sensors and controllers, i.e., how to use the "waiting time" judicially to improve the system performance. Contrary to the common belief that a zero-wait policy is optimal, Sun *et al.* showed that a controller can strictly improve the data freshness, the so-called Age-of-Information (AoI), by postponing transmission in order to lengthen the duration of staying in a good state. The optimal waiting policy for the *sensor* side was later characterized in the context of remote estimation. Instead of focusing on the sensor and controller sides separately, this work develops the jointly optimal sensor/controller waiting policy in a Wiener-process system. This work generalizes the above two important results in the sense that not only do we consider joint sensor/controller designs (as opposed to sensor-only or controller-only schemes), but we also assume random delay in both the forward and feedback directions (as opposed to random delay in only one direction). In addition to provable optimality, extensive simulation is used to verify the performance of the proposed scheme.**

***Index Terms*—Age-of-information, remote estimation, optimal sampling, stochastic control, data freshness, information update system, infinite-horizon Markov decision process.**

I. Introduction

The omnipresence of portable devices has led to increasing focus on systems with multiple sensors and controllers interconnected by wireless communication networks. Many new research directions have been initiated, including healthcare, energy management systems, cloud data infrastructure (see [1]–[3]). In this work, we study the question: How to optimally coordinate the sensor and the controller when there is random delay in both the forward and backward directions? We begin the analysis by observing there are two distinct ways of timing-based system optimization: *data-freshness control* and *state-based sampling*.

Data-freshness control: In this approach, the controller is the one who actively maintains the data-freshness of the system. For example, say the goal is to lower the risk of heart attacks of the patients. One way is for the hospital (controller) to make sure that the blood pressure (BP) or the heart rate (HR) records of the patients are as fresh as possible. To this end, the hospital should intermittently request the patients (sensors) to measure their latest BP or HR and send in the reports. In practice, any sensor-to-controller measurement packet inevitably experiences some delay and is thus always "stale" to some degree. The controller (hospital) must decide how to optimize its request schedule in order to optimize the data-freshness of its records.

One breakthrough of the data-freshness control is the introduction of a new metric, Age-of-Information (AoI) [4], the corresponding minimization algorithms [5], and its numerous follow-up results [6]–[8]. For instance, a "generate-at-will" model was studied in [9], [10], which has the potential of considerable energy savings.

In general, AoI minimization behaves differently from throughput maximization. For example, the zero-wait policy [11] was provably throughput optimal but can be strictly suboptimal in terms of the average AoI [12]. In [12] Sun *et al.* also characterized the optimal "waiting time" policy at the controller side that can provably minimize the average AoI, i.e., the optimal policy when a hospital (controller) should request its patient (sensor) to submit his/her BP/HR report.

State-based sampling: Unlike the data-freshness control, in this line of research, it is the sensor that actively optimizes the overall system.[1] Continue from the aforementioned hospital-patient example. The state-based sampling approach is for the patient (sensor) to measure his/her own BP or HR continuously and report it when and only when the BP/HR shows elevated risk. Once the hospital (controller) receives the report, some treatment (action) is prescribed to bring the BP/HR back to normal. The patient will stay inactive afterwards and only send in new reports if his/her BP or HR starts to exhibit new concerns.

This work was supported in parts by NSF under Grant CCF-1422997, Grant CCF-1618475, and Grant CCF-1816013.

Part of this work was presented in 2020 IEEE International Conference on Computer Communications (INFOCOM).

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[1]The best way to determine whether a scheme is *controller-based* or *sensor-based* is to examine in which physical location the decision is made, since their distinct locations naturally lead to *asymmetric access to the underlying random states and timing information*. Also see our discussion in Sec. II. However, such a definition does not apply to many existing results. The reason is that with the assumption of instantaneous ACK feedback, one node has complete and instantaneous access to the information available at the other node, which breaks the information asymmetry and thus blends the roles of sensors and controllers. The second way of classification is thus to see whether the algorithm has instantaneous access to the (random) value of the measurement and whether it explicitly uses the measurement to decide when to transmit. If so, it is a senor-side algorithm, e.g., the remote estimation scheme in [13]. Otherwise, it is a controller-side algorithm. Under this methodology, the AoI minimization scheme in [12] is classified as controller-based even though it is actually executed by the sensor. That is, one can envision "the controller" being a separate computer program within the physical sensor that tells the sensor when to transmit *without using the actual measurement data*.

The focus of this direction is thus to design schemes that detect the changes in signal/measurement values and opportunistically send the updates when the need arises. This direction is often termed the (state-based) sampling schemes for remote estimation. An early work [14] showed that a *threshold policy* can lower the estimation error. Later it was shown that the *threshold policy* is optimal for a variety of settings, including cellular networks [15], noisy channels [16] and multi-dimensional Wiener processes in [17]. In [13], Sun *et al.* generalized the setting of [17] by adding a queue with random service time between the sampler and estimator, and showed that the optimal waiting time at the sensor side again takes the form of a threshold policy. Further discussion of the threshold policy will be provided in Sec. IV.

The main motivation of this work is two-fold. Firstly, as shown in two important studies [12], [13], since either controller (hospital) or sensor (patient) alone can significantly improve the system performance, one cannot help but wonder how much improvement one may experience with a globally jointly optimal sensor/controller policy. Secondly, since we are interested in remote systems with non-collocated sensors and controllers, there is likely to be random delay for both the sensor-to-controller and the controller-to-sensor directions [18]. Nonetheless, existing results [12] and [13] and all the aforementioned works assume random delay in one direction, plus idealized zero-delay acknowledgement (ACK) for the other direction. It is thus of paramount interest to study new optimal schemes under a more realistic 2-way delay model. Our key contributions are summarized as follows.

(i) We propose a new framework that unifies the controller-side AoI minimization problem [12] and the sensor-side remote estimation problem [13].

(ii) Our framework allows for arbitrary random 2-way delay distributions, does not rely on idealized instantaneous ACK, and thus would be more suitable for practical applications where random delay is present in both directions.

(iii) We derive the jointly optimal sensor/controller policy under the proposed new setting. For comparison, existing works focus either on the sensor [13] or on the controller [12] and take into account random delay in only one direction. The double relaxation from a single-node policy to a joint policy and from 1-way delay to 2-way delay represents a significant advancement over the state of the art.

(iv) When evaluated numerically, our findings show that blindly applying the existing instant-ACK schemes [12], [13] to practical systems with random 2-way delay could lead to significant performance loss and the results can sometimes be worse than a naive zero-wait policy.

(v) The new unified framework includes many existing results as special cases, and we have used it to derive a new, optimal remote estimation scheme with 2-way delay, a strict generalization of [13]. (The optimal 2-way-delay AoI minimization results can be found in [19].)

The remainder of the paper is organized as follows. In Sec. II, our detailed system model and problem formulation are presented. Our main results are outlined in Secs. III and IV. Sec. V uses the proposed framework to solve the remote estimation problem with random two-way delay. Numerical results are reported in Sec. VI. We conclude our work in Sec. VII. Most of the proofs will be provided in the appendices.

II. Model and Formulation

A. System Model

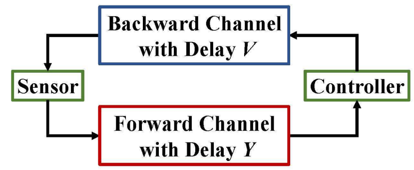


Fig. 1: A sensor/controller system with 2-way delay.

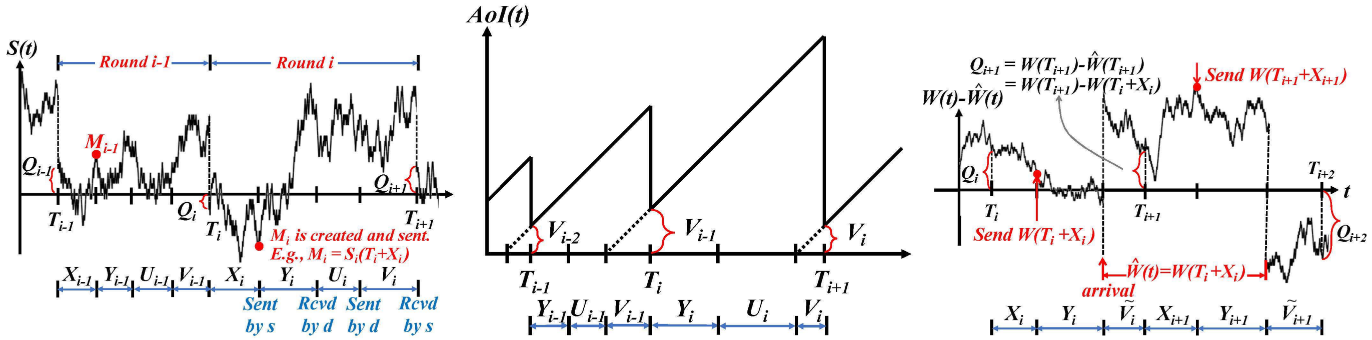
Our system model is best depicted in Fig. 1, which consists of a sensor, a controller, a forward sensor-to-controller channel and a backward controller-to-sensor channel. It is worth noting that we use the terms of sensor and controller in their broadest sense. The sensor node is not limited to a physical sensor that measures the location/temperature of the environment. Instead, it can be any data-generating node, e.g., a database server, a video-streaming source, etc. Also, the controller is not restricted to a node directly commanding an actuator. Instead, it can be any decision making component, e.g., computation of the inferred status of the remote database, or the video processing applications that render the actual video.

Each of the two channels incurs random transmission delay. With two-way delay in the communication loop, the timing information at the sensor and the controller is inherently unsynchronized. Specifically, the waiting policy of the sensor (resp. controller) does not have instantaneous access to the status of the underlying network and has to wait for the delayed response from the controller (resp. sensor). *This two-way delay model and the resulting double time asynchrony where neither the sensor nor the controller has the perfect global timing information is the most distinguishing feature of this work.* For comparison, most existing works [6]–[8], [12], [13], [20] assume one node has perfect network-wide timing information, which may not hold in practice where random delay is universally present.

We now explain our system model. We denote the system state as $S(t)$, for which we shift/relabel the values so that the origin $S(t) = 0$ is the most desired system state. The value of $S(t)$ may drift away from zero as time proceeds. We assume the evolution of $S(t)$ is related to a Wiener process $W(t)$ [21], a widely used (though idealized) model of the system state.[2] The detailed system evolution is defined as follows, and the corresponding illustration is provided in Fig. 2a.

Time sequences: The system consists of four discrete-time real-valued random processes X_i, Y_i, U_i, and V_i for all i. X_i is the i-th waiting time at the sensor; Y_i is the random delay for the i-th use of the sensor-to-controller channel; U_i is the i-th waiting time at the controller; V_i is the random delay for the i-th use of the controller-to-sensor channel.

[2]Some applications of the Wiener process model include unmanned aerial vehicles (UAVs) [22], biosensing schemes [23] and mobile networks [24].



(a) Our setting.

(b) Evolution of AoI.

(c) Evolution of $W(t) - \hat{W}(t)$ for remote estimation.

Fig. 2: Illustration of system evolution for different problem formulations.

The values of $\{X_i\}$ to $\{V_i\}$ can be used to derive another time sequence $\{T_i\}$ as follows: $T_1 \triangleq 0$ and $T_{i+1} \triangleq T_i + X_i + Y_i + U_i + V_i$ for all i. We call the interval $[T_i, T_{i+1})$ as the *i-th round,* which consists of the i-th waiting time of the sensor, the delay of the i-th use of the forward channel, the i-th waiting time of the controller, and the delay of the i-th use of the backward channel. Clearly, the value T_i is the beginning of the i-th round.

We now describe the system behavior in the i-th round.

Reset-to-Q_i at the sensor: At time T_i, the sensor has received the message from the controller in the previous $(i-1)$-th round. It is very convenient to view the message as a *reset command.* We assume that upon receiving the reset command, the system state at time T_i will be reset to a random value Q_i, which is the (random) *initial value* of the i-th round. For example, with thermal noise it may be impossible to set the state value to be exactly 0 with infinite precision. The value Q_i thus models the residual randomness after reset, if there is any. We assume $\{Q_i\}$ is i.i.d. with $\mathbb{E}\{Q_i\} = 0$.

Remark 1: Again the term *reset* is used in the broadest sense. For example, in terms of data freshness control, *reset* could simply mean that the system state is changed from "stale" back to "fresh", not necessarily referring to a physical reset operation.

After reset-to-Q_i, the system state will evolve according to a Wiener process $W(t)$, until it is once again reset to Q_{i+1} at time T_{i+1}. The state value in the i-th round, denoted by $S_i(t)$, is thus described by

$$S_i(t) = W(t) - W(T_i) + Q_i, \text{ for } t \in [T_i, T_{i+1}). \quad (1)$$

We sometimes drop the subscript i and simply use $S(t)$.

Waiting time at the sensor: The sensor has the ability of waiting for an arbitrary amount of time $X_i \geq 0$, also see [9], [10], [12]. The random variable X_i is a *stopping time* with respect to the filtration generated by $\{S(\tau) : \tau \leq t\}$ and the past acknowledgement packets. That is, the sensor observes the evolution of the system state and causally decides when to stop waiting and start transmission.

Upon transmission, the sensor sends (T_i, X_i, M_i) to the controller, where (T_i, X_i), defined in the previous paragraphs, serves as the time stamp(s) while M_i is the additional message/payload generated based on the past system states.

Random delay in the forward direction: The tuple (T_i, X_i, M_i) sent by the sensor at time $T_i + X_i$ will arrive at the controller at time $T_i + X_i + Y_i$. The transmission delay Y_i is i.i.d. and is independent from the rest of the system.

Waiting time at the controller: Since the message is time-stamped (containing (T_i, X_i)), the controller can infer the value of the forward transmission delay Y_i by subtracting $T_i + X_i$ from the actual arrival time $T_i + X_i + Y_i$. The waiting time $U_i \geq 0$ at the controller is then a function of all the previous messages and timing information $\{(T_j, X_j, Y_j, M_j) : j \leq i\}$.

Random delay in the backward direction: At time $T_i + X_i + Y_i + U_i$, the controller sends a reset signal, which will reach the sensor at time $T_{i+1} \triangleq T_i + X_i + Y_i + U_i + V_i$. The $(i+1)$-th round then begins, and we go back to reset-to-Q_{i+1} at the sensor. Again, we assume the backward delay V_i is i.i.d. and is independent from the rest of the system.

Technical assumptions: Similar to [12], [13], we assume the statistics of $\{Q_i\}$, $\{Y_i\}$, and $\{V_i\}$ are known to both the sensor and the controller and $0 < \mathbb{E}\{Y_i\} + \mathbb{E}\{V_i\} < \infty$, and $\mathrm{Var}\{Q_i\} + \mathrm{Var}\{Y_i\} + \mathrm{Var}\{V_i\} < \infty$.

Remark 2: The non-negativity $X_i \geq 0$ (resp. $U_i \geq 0$) prohibits the sensor (resp. controller) to transmit before receiving the reset command (resp. message packet) from the controller (resp. sensor). This complies with the spirits of most TCP-based control protocols [25] where the transmitter sends a new packet *after* receiving the ACK. It is possible to design an even better scheme that transmits anticipatively before receiving any ACK, which, however, is beyond the scope of this work.

B. The Objective

For any given scheme $\{X_i\}$ and $\{U_i\}$, we define the *cost-aware L2 norm* (CAL2N) in the i-th round as

$$\mathbb{E}\left\{ \int_{T_i}^{T_{i+1}} |S_i(t)|^2 \, dt \right\} + c_0 \quad (2)$$

where $S_i(t)$ is defined in (1) and we use its L2 norm to characterize how far it has drifted away from 0. The constant $c_0 \geq 0$ is the *cost of reset* in the end of the round. The value of c_0 is chosen by the system designer and can be set to $c_0 = 0$ if desired.

Our goal is to minimize the *long-term average* CAL2N defined as follows.

$$\beta^*_{\text{CAL2N}} \triangleq \min_{\{X_i,M_i,U_i\}} \limsup_{n\to\infty} \frac{\sum_{i=1}^{n}\left(\mathbb{E}\left\{\int_{T_i}^{T_{i+1}} |S_i(t)|^2\, dt\right\} + c_0\right)}{\sum_{i=1}^{n}\left(\mathbb{E}\left\{T_{i+1}-T_i\right\}+c_1\right)} \quad (3)$$

where the non-negative constant $c_1 \geq 0$ serves as a knob that determines whether we are biased towards a *duration-based* or *round-based* averaging. When $c_1 = 0$, the denominator is the total duration and (3) becomes the time-averaged CAL2N. For sufficiently large c_1, the denominator is approximately $c_1 n$ and (3) is proportional to the CAL2N averaged over n rounds, with each round having equal weight regardless how long/short it is. The value of c_1 is chosen by the system designer and can be set to 0 if desired.

To simplify (3), we notice that the optimization problem is a Markov decision problem with i.i.d.[3] Q_i, Y_i and V_i. As a result, it is sufficient to first find the optimal policy for the *single-round* optimization problem, assuming both the sensor and controller have access to some common randomness.[4] We can then apply the optimal single-round solution to every round. Following this reasoning, the equivalent single-round optimization problem becomes

$$\beta^*_{\text{CAL2N}} = \min_{(X,M,U)} \frac{\mathbb{E}\left\{\int_0^{X+Y+U+V} |S(t)|^2\, dt\right\} + c_0}{\mathbb{E}\left\{X+Y+U+V\right\}+c_1} \quad (4)$$

where we drop the subscript i for notational simplicity.

C. AoI Minimization Setting with Random Two-way Delay

Our setting can be viewed as a strict generalization of the AoI minimization problem with random two-way delay described as follows. (Also see [12] for more details and for illustration.) The source sends packets to the destination through a queue that is not collocated with the source. We use T_i to denote the time instant at which the queue becomes empty for the i-th time. At time T_i, a notification packet will be sent from the queue (or equivalently from the destination) back to the source, which takes Y_i time to arrive. After receiving the notification, the source imposes a waiting time $U_i \geq 0$ and after that injects a new packet to the queue, which takes V_i time to be serviced. Once it is serviced, the queue becomes empty again (the $(i+1)$-th time). We thus have $T_{i+1} = T_i + Y_i + U_i + V_i$ and the process starts over. If we assume instantaneous feedback ($Y_i = 0$ with probability 1), the above problem formulation is identical[5] to that of [12].

Suppose each packet is time-stamped and the AoI is defined as the current time minus the time stamp of the latest received packet [4]. Then the AoI grows linearly over time and is intermittently reset to V_i at time $T_{i+1} = T_i + Y_i + U_i + V_i$ since the time stamp of the latest arrival packet is $T_i+Y_i+U_i$. See [12] for more details and see Fig. 2b for illustration. The goal is to minimize the long-term average AoI:

$$\beta^*_{\text{AoI}} \triangleq \min_{\{U_i\}} \limsup_{n\to\infty} \frac{\sum_{i=1}^{n}\mathbb{E}\left\{\int_{V_{i-1}}^{V_{i-1}+Y_i+U_i+V_i} t dt\right\}}{\sum_{i=1}^{n}\mathbb{E}\left\{Y_i+U_i+V_i\right\}} \quad (5)$$

$$= \min_{U_i} \frac{\mathbb{E}\left\{\int_{V_{i-1}}^{V_{i-1}+Y_i+U_i+V_i} t dt\right\}}{\mathbb{E}\left\{Y_i+U_i+V_i\right\}} \quad (6)$$

where (6) is based on the equivalent single-round optimization.

We now show that (6) is a special case of our setting defined in Secs. II-A and II-B by (i) assuming $c_0 = c_1 = 0$, (ii) hardwiring $X_i = M_i = 0$, i.e., forgoing the possibility of designing better X_i and M_i, and (iii) choosing $Q_i = W(T_i) - W(T_{i-1}+Y_{i-1}+U_{i-1})$. As will be seen later, it is as if we let the controller (resp. the sensor) assume the role of the source (resp. the destination) and the L2 norm of the system state $\mathbb{E}\{S(t)^2\}$ captures the linearly growing AoI metric.

Define the filtration until time $T_i + Y_i$ as $\mathcal{F}_i \triangleq \{(T_j, Y_j), : j \leq i\}$. We then have for any $t \in [T_i, T_{i+1})$,

$$\begin{aligned}
&\mathbb{E}\left\{|S_i(t)|^2 \middle| \mathcal{F}_i\right\} \\
&= \mathbb{E}\left\{|W(t) - W(T_{i-1}+Y_{i-1}+U_{i-1})|^2 \middle| \mathcal{F}_i\right\} && (7)\\
&= t - (T_{i-1}+Y_{i-1}+U_{i-1}) = V_{i-1} + (t - T_i) && (8)
\end{aligned}$$

where (7) follows from (1) and our choice of Q_i in (iii); and (8) follows from the strong Markov property of the Wiener process. We can then rewrite the numerator of (4) as

$$\begin{aligned}
&\mathbb{E}\left\{\int_{T_i}^{T_i+Y_i+U_i+V_i} \mathbb{E}\left\{|S_i(t)|^2 \middle| \mathcal{F}_i\right\} dt\right\} && (9)\\
&= \mathbb{E}\left\{\int_{T_i}^{T_i+Y_i+U_i+V_i} (V_{i-1}+t-T_i) dt\right\} && (10)\\
&= \mathbb{E}\left\{\int_{V_{i-1}}^{V_{i-1}+Y_i+U_i+V_i} t dt\right\} && (11)
\end{aligned}$$

where (9) follows from Wald's lemma [26] and the facts (i) Y_i is deterministic once conditioning on $\mathcal{F}_i$; (ii) In Sec. II-A, the waiting time U_i at the controller is defined as a function of $\{(T_j, X_j, Y_j, M_j) : j \leq i\}$. Since we set $X_i = M_i = 0$, it is clear that U_i is also deterministic once conditioning on $\mathcal{F}_i$; and (iii) V_i is independent of $\mathcal{F}_i$ and $|S_i(t)|^2$. Eq. (10) follows from

[3] More precise requirements are: (i) $\{Y_i\}$ and $\{V_i\}$ are i.i.d. and independent from the rest of the system; (ii) For any i, Q_i is independent of the waiting times $\{X_j, U_j : j < i\}$ in the previous rounds.

[4] The common randomness enables us to convert the temporal average over many rounds to the probabilistic average over a single round. In this work we implicitly assume the availability of common randomness when discussing any single-round optimization problem.

[5] [12] contained the results of more advanced settings, e.g., arbitrary AoI penalty functions, non-i.i.d. noises, etc. In this work, we focus on their simpler i.i.d. setting with linear AoI penalty function and without the *maximum update frequency constraint*, which captures the simplest and most fundamental findings of the results in [12].

(8); and (11) follows from the change of variables. This proves that our objective function (4) collapses to the one for AoI minimization with random two-way delay (6) once we set c_0, c_1, Y_i, X_i, M_i, and Q_i properly. The high-level intuition is that the controller cannot directly observe the system state and has to make its decisions based on the *expected* cost $\mathbb{E}\{|S(t)|^2\}$, which has a linear growth rate with respect to the elapsed time (AoI) after the last reset. The CAL2N minimization problem thus includes the AoI minimization [12] as a special case.

D. Remote Estimation Setting with Random Two-way Delay

Next, we show that our setting is also a strict generalization of the remote estimation problem with random two-way delay. Also see [13] for details and for illustration. Consider a system in which a sampler sends packets to an estimator through a queue that is not collocated with the sampler. Whenever the queue becomes empty, the queue sends a notification packet back to the sampler. We use T_i to denote the time instant at which the i-th notification packet has arrived at the sampler. After time T_i, the sampler continuously monitors an external random process $W(T_i + t)$, which is assumed to be a Wiener process. After some waiting time $X_i \geq 0$, which is a stopping time of $W(T_i + t)$, the sampler injects the latest observed value $W(T_i + X_i)$ to the queue, which takes Y_i time to be serviced. Once it is serviced, the queue becomes empty and the estimator has received the latest observation. Then a new notification packet is sent back to the sampler, which experiences some random delay $\tilde{V}_i$. Once the sampler receives the new $(i+1)$-th notification after delay $\tilde{V}_i$, the process starts over. It is clear that we have $T_{i+1} = T_i + X_i + Y_i + \tilde{V}_i$ in this system model.

We now describe the *estimation error* of this remote estimation system. Specifically, at time $t = T_i+X_i+Y_i$ the estimator receives the latest observed value $W(T_i+X_i)$ and uses it as an estimate of the external process $\hat{W}(t) = W(T_i+X_i)$ until the arrival of the next update packet at time $T_{i+1} + X_{i+1} + Y_{i+1}$. As a result, the estimation error $W(t) - \hat{W}(t)$ jumps to a new (smaller) initial value $W(T_i + X_i + Y_i) - W(T_i + X_i)$ at time $t = T_i + X_i + Y_i$. See Fig. 2c for illustration. Otherwise it evolves as a Wiener process until the arrival of the next observation $W(T_{i+1} + X_{i+1})$ at time $t = T_{i+1} + X_{i+1} + Y_{i+1}$.

If we again use the single-round problem formulation, the optimization problem becomes:

$$\beta^*_{\text{MMSE}} \triangleq \min_{X_i} \frac{\mathbb{E}\left\{\int_{T_i}^{T_i+X_i+Y_i+\tilde{V}_i} \left(W(t) - \hat{W}(t)\right)^2 dt\right\}}{\mathbb{E}\left\{X_i + Y_i + \tilde{V}_i\right\}}. \quad (12)$$

The numerator of (12) can be rewritten as

$$\begin{aligned}
&\mathbb{E}\left\{\int_{T_i}^{T_i+X_i+Y_i} \left(W(t) - \hat{W}(t)\right)^2 dt\right\} \\
&+ \mathbb{E}\left\{\int_{T_i+X_i+Y_i}^{T_i+X_i+Y_i+\tilde{V}_i} \left(W(t) - W(T_i + X_i)\right)^2 dt\right\} \\
&= \mathbb{E}\left\{\int_{T_i}^{T_i+X_i+Y_i} \left(W(t) - \hat{W}(t)\right)^2 dt\right\} \\
&\quad + \mathbb{E}\left\{\int_0^{\tilde{V}_i} (W(Y_i + t) - W(0))^2 dt\right\} \qquad (13)
\end{aligned}$$

where (13) uses the strong Markov property of the Wiener process and the assumption that $(Y_i, \tilde{V}_i)$ is independent from the Wiener process.

We notice that the latter half of (13) can be further simplified as follows. Given $\tilde{V}_i = v$ and $Y_i = y$, we have

$$\begin{aligned}
&\mathbb{E}\left\{\int_0^{\tilde{V}_i} (W(Y_i + t) - W(0))^2 dt \middle| \tilde{V}_i = v, Y_i = y\right\} \\
&= \int_0^v \mathbb{E}\left\{(W(y + t) - W(0))^2\right\} dt \qquad (14) \\
&= \int_0^v (y + t) dt = yv + \frac{v^2}{2} \qquad (15)
\end{aligned}$$

where (14) follows from the fact that $\tilde{V}_i$ and Y_i are independent from the rest of the system; and (15) follows from the strong Markov property of the Wiener process.

By further taking the expectation of (15) over the i.i.d. $\tilde{V}_i$ and Y_i, the numerator of (12) can be rewritten as

$$\begin{aligned}
&\mathbb{E}\left\{\int_{T_i}^{T_i+X_i+Y_i} \left(W(t) - \hat{W}(t)\right)^2 dt\right\} \\
&+ \mathbb{E}\{Y_i\}\mathbb{E}\{\tilde{V}_i\} + \frac{\mathbb{E}\{(\tilde{V}_i)^2\}}{2} \qquad (16)
\end{aligned}$$

We now show that the above remote estimation problem is a special case of our setting in Secs. II-A and II-B by (i) hardwiring $M_i = U_i = V_i = 0$, i.e., short-circuiting the controller and the backward delay and using a dummy message $M_i = 0$, and (ii) choosing $c_0 = \mathbb{E}\{Y_i\}\mathbb{E}\{\tilde{V}_i\} + \frac{\mathbb{E}\{(\tilde{V}_i)^2\}}{2}$, $c_1 = \mathbb{E}\{\tilde{V}_i\}$, $Q_i = W(T_i) - W(T_{i-1} + X_{i-1})$. As will be seen later, it is as if we let the sensor (resp. the controller) assume the role of the sampler (resp. the estimator) and the L2 norm of the system state $\mathbb{E}\{S(t)^2\}$ captures the estimation error of the remote estimation system.

Note that by (1) and the choices of $U_i = V_i = 0$, we can rewrite the numerator of (4) as

$$\mathbb{E}\left\{\int_{T_i}^{T_i+X_i+Y_i} (W(t) - W(T_i) + Q_i)^2 dt\right\} + c_0 \qquad (17)$$

Then by the special choices of c_0 and Q_i in (ii) and because $\hat{W}(t) = W(T_{i-1} + X_{i-1})$ during the time interval $[T_i, T_i + X_i+Y_i]$, it is straightforward to verify that the numerator of (4) (i.e., (17)) is identical to the numerator of (12) (i.e., (13)). It is also straightforward to verify that the denominators of (12) and (4) are identical. Our objective function (4) thus collapses to the one for remote estimation with two-way delay (12) once we set c_0, c_1, Y_i, V_i, U_i, Q_i, and M_i properly. The high-level intuition is that having direct observation of the system state $S(t)$, the sensor naturally has the same role as the *sampler* in the context of remote estimation of a Wiener process [13].

Remark 3: [13] considered remote estimation with *one-way* delay, which is a special case of the one in this subsection by further setting controller-to-sensor delay $\tilde{V}_i = 0$ (and hence $c_0 = c_1 = 0$).

Remark 4: In some sense, the two important papers [12], [13] form a perfect pair, where the former AoI minimization work focuses on the controller action (assuming $X_i = Y_i = 0$) without using the state information while the latter remote-estimation work studies the sensor action (assuming $U_i = V_i = 0$) that directly observes the state $S(t)$. A main contribution of this work is to unify these two results and study the optimal sensor/controller scheme that jointly optimizes X_i and U_i.

III. Main Results — The Optimal Policies

In this section, we will present three policies: (i) the jointly optimal policy (X^*, M^*, U^*), (ii) the optimal No-Wait-At-Sensor (NWAS) policy which imposes $X_i = 0$ and optimizes the rest of the system, and (iii) the optimal No-Wait-At-Controller (NWAC) policy which imposes $U_i = 0$ and optimizes the rest. Policies (ii) and (iii) are meant for scenarios in which either the sensor or the controller is forced to adopt a suboptimal zero-wait policy due to other system-level considerations.

A. An Auxiliary Minimization Problem

Given the distributions of the i.i.d. $\{Y_i\}$, $\{V_i\}$, $\{Q_i\}$ and any constant values $c_0, c_1 \geq 0$, for any $\beta \in (-\infty, \infty)$ we define $p(\beta)$ as the optimal value of the following minimization problem:

$$p(\beta) \triangleq \inf_{(X,M,U)} \mathbb{E}\left\{\int_0^{X+Y+U+V} |S(t)|^2 \, dt\right\} + c_0 - \beta\left(\mathbb{E}\{X+Y+U+V\} + c_1\right) \quad (18)$$

where we drop the subscript i for simplicity. We then have

Proposition 1: (i) The function $p(\beta)$ is concave, continuous, and strictly decreasing, (ii) there exists a unique $\beta^* \in [0, \beta_{\text{max}}]$ such that $p(\beta^*) = 0$, where

$$\beta_{\text{max}} \triangleq \frac{\mathbb{E}\{Y\}\mathbb{E}\{V\} + \frac{1}{2}\mathbb{E}\{Y^2 + V^2\} + c_0}{\mathbb{E}\{Y+V\} + c_1} + \frac{\mathbb{E}\{Q^2\} \cdot \mathbb{E}\{Y+V\}}{\mathbb{E}\{Y+V\} + c_1} \quad (19)$$

(iii) the unique solution β^* is identical to the β^*_{CAL2N} defined in (4), and (iv) The (X, M, U) scheme that attains $p(\beta^*)$ also achieves the β^*_{CAL2N} in (4).

Proof: See Appendix A. ■

By Proposition 1, the minimization problem (4) can be solved in the following steps: For any given β, we first find the optimal (X, M, U) that minimizes (18) and the corresponding $p(\beta)$ value. We then find the optimal $\beta^* = \beta^*_{\text{CAL2N}}$ by a bisection search over $[0, \beta_{\text{max}}]$. In the sequel, we discuss how to find the optimal (X, M, U) solution of $p(\beta)$ in (18) for any given β.

B. Optimal Waiting Time at the Controller

Define $\overline{M}_i = (T_i, X_i, M_i)$. The following proposition holds for any arbitrary message scheme $\{M_i\}$.

Proposition 2: Given any arbitrary payload $\{M_i\}$ and any $\beta > 0$, the optimal waiting time $U^*_{i|M}$ at the controller that minimizes (18) is as follows.

$$U^*_{i|M} = \max\left(\beta - \left(Y_i + \mathbb{E}\left\{(S_i(T_i + X_i))^2 \,\middle|\, \overline{M}^{(i)}\right\} + \mathbb{E}\{V_i\}\right), 0\right) \quad (20)$$

where $\overline{M}^{(i)} \triangleq \{\overline{M}_j : j \leq i\}$.

Proof: See Appendix B. ■

That is, the optimal controller is a *water-filling policy* that calculates the difference between β and $\left(Y_i + \mathbb{E}\{(S_i(T_i + X_i))^2 \,|\, \overline{M}^{(i)}\} + \mathbb{E}\{V_i\}\right)$.

C. Optimal Message Sent by the Sensor

Proposition 3: The optimal message that minimizes (18) is $M^*_i = S_i(T_i + X_i)$, the latest state value at the time of transmission $T_i + X_i$.

This result follows directly from the fact that the system state is a strong Markov process and thus the latest system state consists of all the information the controller can possibly need. We omit the proof due to the space limit. Combining Propositions 2 and 3, we immediately have

Corollary 1: With the optimal message M^*_i in Proposition 3, the optimal waiting time at the controller becomes

$$U^*_i \triangleq U^*_{i|M^*} = \max\left(\beta - \left(Y_i + (S_i(T_i + X_i))^2 + \mathbb{E}\{V_i\}\right), 0\right). \quad (21)$$

D. Optimal Waiting Time at the Sensor

The design of the sensor waiting time X has to take into account the sent message M and the controller waiting time U. In the sequel we exclusively assume M^*_i is used. Two different controller schemes $U = U^*_i$ and $U = 0$ are considered. When U^*_i is used, we denote the corresponding optimal sensor scheme by X^*_i, i.e., the tuple (X^*_i, M^*_i, U^*_i) represents the jointly optimal sensor/controller policy. When $U = 0$ is used, we denote the corresponding optimal sensor scheme by $X^*_{i|\text{ZW}}$, which represents the best-possible X if the controller employs a zero-wait (ZW) policy. The tuple $(X^*_{i|\text{ZW}}, M^*_i, U_i = 0)$ is thus what we previously referred to as the optimal No-Wait-At-Controller (NWAC) policy. Note that the optimal No-Wait-At-Sensor (NWAS) policy is the combination of $(X_i = 0, M^*_i, U^*_i)$, which was previously described in Proposition 3 and Corollary 1 and is thus not the focus of this subsection.

We first describe how to find jointly optimal X_i^*, and later present how the procedure of finding X_i^* can be modified to find $X_{i|\mathsf{ZW}}^*$.

For any $s \in (-\infty, \infty)$, define two functions $g_\beta(s)$ and $h_\beta(s)$ by

$$g_\beta(s) \triangleq a_{s,4} \cdot s^4 + a_{s,2} \cdot s^2 + a_{s,0} + a_0 \quad (22)$$

$$h_\beta(s) \triangleq g_\beta(s) - (\beta s^2 - \frac{1}{6}s^4) \quad (23)$$

where

$$a_{s,4} \triangleq -\frac{\mathbb{P}\left(s^2 + Y \leq \beta - \mathbb{E}\{V\}\right)}{2} \quad (24)$$

$$a_{s,2} \triangleq \mathbb{E}\{Y+V\} + \mathbb{E}\left\{\mathbb{1}_{\{s^2+Y\leq\beta-\mathbb{E}\{V\}\}} \cdot (\beta - \mathbb{E}\{V\} - Y)\right\} \quad (25)$$

$$a_{s,0} \triangleq -\frac{\mathbb{E}\left\{\mathbb{1}_{\{s^2+Y\leq\beta-\mathbb{E}\{V\}\}} \cdot (\beta - \mathbb{E}\{V\} - Y)^2\right\}}{2} \quad (26)$$

$$a_0 \triangleq -\beta\left(\mathbb{E}\{Y+V\} + c_1\right) + \mathbb{E}\{Y\}\mathbb{E}\{V\} + \frac{1}{2}\mathbb{E}\left\{Y^2+V^2\right\} + c_0 \quad (27)$$

and $\mathbb{1}_{\{\cdot\}}$ is the indicator function. Note that $g_\beta(s)$ and $h_\beta(s)$ are not fourth-order polynomials since the coefficients $a_{s,4}$, $a_{s,2}$, and $a_{s,0}$ also depend on s.

Lemma 1: For any $\beta \geq 0$ and any distributions of Y and V, which can be discrete, continuous, or hybrid, both functions $g_\beta(s)$ and $h_\beta(s)$ are even and continuous. Furthermore, $h_\beta(s)$ is lower bounded by a shifted 4-th order polynomial of s, i.e., $h_\beta(s) \geq \frac{1}{12}s^4 + (a_0 - 3\beta^2)$ for all $s \in (-\infty, \infty)$.

For any two functions f_1 and f_2, we say $f_1 \prec f_2$ if $f_1(s) \leq f_2(s), \forall s \in (-\infty, \infty)$. The *convex hull* (also called *the lower convex envelope*) of the function $h_\beta(s)$ is defined as

$$\mathrm{Cnvx}\big(h_\beta(s)\big) \triangleq \sup\{f(s) : f \text{ is convex}, f \prec h_\beta\}. \quad (28)$$

Corollary 2: For any $\beta \geq 0$, the lower convex envelope $\mathrm{Cnvx}\big(h_\beta(s)\big)$ is finite for all $s \in (-\infty, \infty)$.

Corollary 2 follows directly from Lemma 1. The corresponding proofs are relegated to Appendix D.

We now describe the optimal sensor waiting time X_i^*.

Proposition 4: For any given β, the optimal X_i^* that minimizes (18) is the hitting time:

$$X_i^* = \inf\{t \geq 0 : S_i(T_i + t) \in \mathcal{S}_{\mathsf{tx},\beta}\} \quad (29)$$

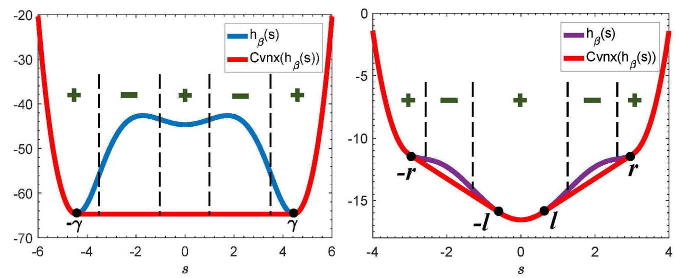
where the set $\mathcal{S}_{\mathsf{tx},\beta}$, called *the transmission set*, is the collection of all state values s satisfying

$$\mathrm{Cnvx}\left(h_\beta(s)\right) = h_\beta(s) \quad (30)$$

i.e., the set of s whose corresponding values of the convex hull function are equal to those of the original function $h_\beta(s)$.

Proof: See Appendix E. ■

Fig. 3a illustrates a (piecewise) even function $h_\beta(s)$, which contains 5 pieces with the corresponding second-order derivatives being $+-+-+$ if we scan the s values from $-\infty$ to ∞. Fig. 3a also plots the convex hull function $\mathrm{Cnvx}\left(h_\beta(s)\right)$. One can see that in this example, $\mathrm{Cnvx}\left(h_\beta(s)\right) = h_\beta(s)$ iff $|s| \geq \gamma$ for some threshold γ. As a result, $\mathcal{S}_{\mathsf{tx},\beta} = \{s : |s| \geq \gamma\}$. The optimal X_i^* is thus the first time when $|S_i(T_i + t)|$ hits γ.



(a) (b)

Fig. 3: Examples of $h_\beta(s)$ and $\mathrm{Cnvx}(h_\beta(s))$.

Another example of $h_\beta(s)$ is plotted in Fig. 3b, for which $\mathrm{Cnvx}\left(h_\beta(s)\right) = h_\beta(s)$ iff s belongs to neither of the two (symmetric) intervals $(-r, -l)$ and (l, r). In this example, the sensor transmits if either $|S(T_i + t)| \geq r$ or $|S(T_i + t)| \leq l$.

Proposition 4 describes the X_i^* in the jointly optimal sensor/controller scheme. In the following we elaborate how we derive $X_{i|\mathsf{ZW}}^*$ for the optimal NWAC policy.

Define a new $g_{\mathsf{NWAC},\beta}(s)$ by

$$g_{\mathsf{NWAC},\beta}(s) \triangleq \mathbb{E}\{Y+V\}s^2 + \mathbb{E}\{Y\}\mathbb{E}\{V\} + \frac{1}{2}\mathbb{E}\left\{Y^2+V^2\right\} - \beta\left(\mathbb{E}\{Y+V\} + c_1\right) + c_0. \quad (31)$$

Note that the $g_{\mathsf{NWAC},\beta}(s)$ is a second-order polynomial of s since its coefficients do not depend on s.

By substituting $g_\beta(s) = g_{\mathsf{NWAC},\beta}(s)$ in (22) and repeating the steps listed (23), (28), and Proposition 4, we can find the optimal waiting time $X_{i|\mathsf{ZW}}^*$ of the best NWAC policy. Specifically, for any given $\beta \in (-\infty, \infty)$, define

$$h_{\mathsf{NWAC},\beta}(s) \triangleq g_{\mathsf{NWAC},\beta}(s) - (\beta s^2 - \frac{1}{6}s^4). \quad (32)$$

Since $g_{\mathsf{NWAC},\beta}(s)$ has a nice form of being a second-order polynomial, by simple calculus one can verify that

$$\mathrm{Cnvx}(h_{\mathsf{NWAC},\beta}(s)) = \begin{cases} h_{\mathsf{NWAC},\beta}(s) & \text{if } s^2 \geq \gamma_{\mathsf{NWAC}} \\ h_{\mathsf{NWAC},\beta}(\sqrt{\gamma_{\mathsf{NWAC}}}) & \text{if } s^2 < \gamma_{\mathsf{NWAC}} \end{cases} \quad (33)$$

where

$$\gamma_{\mathsf{NWAC}} \triangleq \max(3 \cdot (\beta - \mathbb{E}\{Y+V\}), 0) \quad (34)$$

is a constant threshold.

Proposition 5: Using the definition of γ_{NWAC} in (34), the optimal waiting time $X_{i|\mathsf{ZW}}^*$ is the hitting time:

$$X_{i|\mathsf{ZW}}^* = \inf\{t \geq 0 : |S_i(T_i + t)|^2 \geq \gamma_{\mathsf{NWAC}}\}. \quad (35)$$

Proof: See Appendix H. ■

Intuitively, the difference between X_i^* and $X_{i|\mathsf{ZW}}^*$ is due to different schemes used at the controller, and the sensor thus has to react differently. Propositions 4 and 5 prove that the effects of different controller schemes can be summarized either as the function $g_\beta(s)$ in (22) or $g_{\mathsf{NWAC},\beta}(s)$ in (31). The actual optimization mechanisms at the sensor remain the same and are described by the steps of finding $h_\beta(s)$ and the convex hull $\mathrm{Cnvx}\big(h_\beta(s)\big)$, and comparing $h_\beta(s)$ and $\mathrm{Cnvx}\big(h_\beta(s)\big)$ to decide the corresponding transmission set $\mathcal{S}_{\mathsf{tx},\beta}$.

E. Finding the Optimal β^*

All the previous discussions assume an arbitrarily given β. We now describe how to find β^*. We first discuss the case for the jointly optimal policy, and then describe the cases for the optimal NWAC and NWAS policies.

Proposition 1 shows that the optimal β^* for the jointly optimal scheme (X_i^*, M_i^*, U_i^*) is the unique root of $p(\beta) = 0$ defined in (18). We now describe how to find $p(\beta)$. Recall that Q_i describes the random state value in the beginning of the i-th round (right after reset). We then have

Proposition 6: For any $\beta \geq 0$, define $\phi(\beta, s)$ as

$$\phi(\beta, s) \triangleq \text{Cnvx}\left(h_\beta(s)\right) + (\beta s^2 - \frac{1}{6}s^4) \tag{36}$$

where $h_\beta(s)$ was first defined in (22) to (27). The optimal $p(\beta)$ in (18) can be computed by

$$p(\beta) = \mathbb{E}_Q\{\phi(\beta, Q)\} \tag{37}$$

i.e., we first assume the s value in $\phi(\beta, s)$ is randomly distributed with distribution Q and then evaluate $p(\beta)$ by finding the expectation in (37). See Appendix E for the proof.

Proposition 7: For any $\beta \geq 0$, define $\phi_{\text{NWAC}}(\beta, s)$ and $p_{\text{NWAC}}(\beta)$ as

$$\phi_{\text{NWAC}}(\beta, s) \triangleq \text{Cnvx}(h_{\text{NWAC},\beta}(s)) + (\beta s^2 - \frac{1}{6}s^4) \tag{38}$$

$$p_{\text{NWAC}}(\beta) \triangleq \mathbb{E}_Q\{\phi_{\text{NWAC}}(\beta, Q)\} \tag{39}$$

where $h_{\text{NWAC},\beta}(s)$ was first defined in (31) and (32). The optimal β^* for the optimal NWAC scheme (i.e., $(X_{i|\text{ZW}}^*, M_i^*, U_i = 0)$) is the unique root of $p_{\text{NWAC}}(\beta) = 0$. See Appendix H for the proof.

Proposition 8: For any $\beta \geq 0$, define $\phi_{\text{NWAS}}(\beta, s)$ and $p_{\text{NWAS}}(\beta)$ as

$$\phi_{\text{NWAS}}(\beta, s) \triangleq g_\beta(s) \tag{40}$$

$$p_{\text{NWAS}}(\beta) \triangleq \mathbb{E}_Q\{\phi_{\text{NWAS}}(\beta, Q)\} \tag{41}$$

where g_β was first defined in (22). The optimal β^* for the optimal NWAS scheme (i.e., $(X_i = 0, M_i^*, U_i^*)$) is the unique root of $p_{\text{NWAS}}(\beta) = 0$. See Appendix I for the proof.

Comparing Propositions 6 and 8, we notice that both propositions are very similar in the sense that they first find a function $\phi(\beta, s)$ and then evaluate the corresponding $p(\beta)$ by taking the expectation over Q. The differences are that the $\phi(\beta, s)$ in Proposition 6 is obtained by applying a sequence of convex-hull-based operations to $g_\beta(s)$ in (22), (23), (28), and (36), whereas Proposition 8 directly sets $\phi_{\text{NWAS}}(\beta, s) = g_\beta(s)$. The intuition is that since in the NWAS policy, the sensor always chooses $X_i = 0$ without any optimization/minimization. Therefore the initial function $g_\beta(s)$, which is the objective function based on the optimal controller waiting time U_i^*, will be directly used as the $\phi_{\text{NWAS}}(\beta, s)$. This essentially skips the intermediate optimization/minimization steps in (22), (23), (28), and (36) that compute $\phi(\beta, s)$ in (36) from $g_\beta(s)$, which captures the effects of using optimal X_i^*. Once we swap (36) with (40), the steps of (37) and (41) are identical.

We also compare Propositions 6 and 7. Since $g_\beta(s)$ represents the effect of the optimal controller waiting time U_i^*, when shifting from optimal U_i^* to zero $U_i = 0$ in NWAC, the only change is to replace $g_\beta(s)$ in (22) by the $g_{\text{NWAC},\beta}(s)$ in (31). The remaining steps, i.e., {(23), (36), (37)} versus {(32), (38), (39)}, are identical.

F. Complexity of Finding the Jointly Optimal Scheme

We first summarize the detailed steps of finding the jointly optimal sensor/controller policy and the corresponding β^*.

Step 1: For any β, compute the functions $g_\beta(s)$, $h_\beta(s)$, and $\phi(\beta, s)$ by (22), (23), and (36), and then compute the value of $p(\beta)$ by (37).

Step 2: Repeatedly use Step 1 and the bisection search over $\beta \in [0, \beta_{\max}]$ to find the unique β^* satisfying $p(\beta^*) = 0$.

Step 3: Substitute $\beta = \beta^*$ in Secs. III-B and III-D to derive the respective optimal policies for the controller and the sensor.

We note that the bisection steps, i.e., Steps 2 and 3, also appear in [12], [13] and thus do not incur additional complexity. For some special delay distributions Y_i and V_i and the reset distribution Q_i, say, exponential, it is possible to derive closed-form expressions of $g_\beta(\cdot)$, $h_\beta(\cdot)$, $\phi(\beta, \cdot)$, and $p(\beta)$ by calculus. For arbitrary Y_i, V_i, and Q_i distributions, we can compute $g_\beta(\cdot)$ and $h_\beta(\cdot)$ by quantizing the continuous s values into discrete points. Then we can use existing *linear-time* algorithms, e.g., [27], [28], to compute the convex hull $\text{Cnvx}(h_\beta(s))$. The expectation step in (37) can subsequently be computed in linear time as well. Overall, the complexity of our algorithm is identical to [12], [13]. That is, all being linear-time in terms of the number of quantization points.

IV. Further Examination of The Optimal Policy

In this section, we prove some properties of the jointly optimal sensor/controller scheme (X_i^*, M_i^*, U_i^*).

Lemma 2: $\mathcal{S}_{\text{tx},\beta^*}$ is symmetric over $s = 0$, i.e., for any $s \in (-\infty, \infty)$, $s \in \mathcal{S}_{\text{tx},\beta^*}$ if and only if $(-s) \in \mathcal{S}_{\text{tx},\beta^*}$.

Proof: This lemma follows directly from Lemma 1 and the definition of $\mathcal{S}_{\text{tx},\beta}$ in Proposition 4. ■

Define $\mathcal{S}_{\text{tx},\beta^*}^c \triangleq (-\infty, \infty)\backslash\mathcal{S}_{\text{tx},\beta^*}$ as the complement of $\mathcal{S}_{\text{tx},\beta^*}$. We then have the following self-explanatory lemma.

Lemma 3: $\mathcal{S}_{\text{tx},\beta^*}^c$ must be a collection of disjoint open intervals (l_i, r_i), namely,

$$\mathcal{S}_{\text{tx},\beta^*}^c = \bigcup_{i=1}^{\mu}(l_i, r_i) \tag{42}$$

where μ is the total number of disjoint open intervals, and $\{(l_i, r_i) : i\}$ satisfies $-\infty < l_i < r_i < \infty$ for all $i \in [1, \mu]$.

Lemmas 2 and 3 imply that if $\mu = 1$, then $\mathcal{S}_{\text{tx},\beta^*} = \{s : |s| \geq \gamma\}$ for some $\gamma > 0$, which is termed *the threshold policy* in [13]. Similarly, if $\mu = 0$, then $\mathcal{S}_{\text{tx},\beta^*} = (-\infty, \infty)$ and the optimal policy is a zero-wait policy. In the sequel, we examine the value of μ, calculated by (42), for various scenarios.

A. Deterministic Forward Transmission Delay $Y_i = y_0$

Proposition 9: If there exists a constant y_0 such that $\mathbb{P}(Y_i = y_0) = 1$, then we always have $\mu \leq 1$ and $\mathbb{P}(U_i^* = 0) = 1$.

Proof: See Appendix J. ■

In other words, with deterministic forward transmission delay Y_i, the optimal sensor policy is either a zero-wait policy ($\mu = 0$) or a threshold policy ($\mu = 1$), and the optimal controller strategy is always a zero-wait policy regardless of the distribution of backward delay V_i.

B. Exponential Forward Transmission Delay Y_i

Proposition 10: If $Y_i \sim \text{Exp}(\lambda_Y)$ is exponentially distributed with service rate $\lambda_Y > 0$, then we always have $\mu \leq 2$.

Proof: See Appendix K. ■

If we choose $Y \sim \text{Exp}(\lambda_Y)$ with $\lambda_Y = 0.2$, $V \sim \text{Exp}(\lambda_V)$ with $\lambda_V = 6$, $c_0 = 20$, $c_1 = 0$, and $Q \sim \mathcal{N}(0, \sigma^2)$ with $\sigma = 0.125$, then we can numerically compute $\beta^* = 7.236$ using the 3 steps in Sec. III-E. The resulting $\mathcal{S}_{\text{tx},\beta^*} = \{s : |s| \leq 0.012 \text{ or } 0.082 \leq |s|\}$ indeed has $\mu = 2$. The upper bound $\mu \leq 2$ in Proposition 10 is thus tight.

We call the $\mu = 2$ policy an *interval policy*. The reason is that with $\mu = 2$ the transmission set $\mathcal{S}_{\text{tx},\beta^*}$ is of the form

$$\mathcal{S}_{\text{tx},\beta^*} = \{s : |s| \leq l \text{ or } r \leq |s|\} \tag{43}$$

for a pair of $0 < l < r < \infty$. That is, the optimal sensor scheme should transmit when the system state is either too large $|s| \geq r$ or too small $|s| \leq l$. At the first glance, this strategy seems counterintuitive due to the following reason: Our goal is to minimize the average value of $|S_i(t)|^2$. Therefore, large $|s|$ is considered to be "bad" and small $|s|$ is considered to be "good". An intuitive strategy inspired by [12] and [13] is to hold off transmission (i.e., to wait) when the state is good (when $|s|$ is small) in order to prolong the duration of staying in a good state. Our results show that under the setting of joint sensor/controller coordination, the sensor sometimes should transmit when the state becomes *too good* (when $|s| \leq l$).

One explanation of this surprising phenomenon is as follows. The goal of CAL2N minimization in (4) is for the sensor and the controller to jointly design their strategies and *one thus has to decide how to split the waiting time between the sensor and the controller.* A deeper look shows that each of them has its unique advantages and disadvantages. In particular, the sensor is able to observe the full system state $S_i(t)$ continuously and use it to make its decision X_i. The controller cannot observe $S_i(t)$ directly, but instead can directly observe the realization of the random sensor-to-controller delay Y_i, a valuable piece of information known exclusively to the controller. Therefore, when the system state is very good, $|s|$ being small, there is a bigger chance that the controller will see a good expected system state[6] $Y_i + s^2 + \mathbb{E}\{V_i\}$ in (20). As a result, the sensor should transmit so that the controller, which has the additional observation of Y_i, can make a better informed decision U_i^* to further extend the duration of staying in a good system state. *One of the main contributions of this work is to uncover this unexpected sensor/controller coordination that is critical to achieving the optimal performance.*

[6]Since the optimal $M_i^* = S_i(T_i + X_i)$ is used, the term $\mathbb{E}\{(S_i(T_i + X_i))^2 | \overline{M}^{(i)}\} = (S_i(T_i + X_i))^2 = s^2$ is directly related to the value of $|s|$.

The above discussion also explains the intuition of Proposition 9. With deterministic Y_i, the controller has a strictly inferior set of information since the observed Y_i is a constant. Hence, all the waiting time should be allocated to the sensor, i.e., $\mathbb{P}(U_i^* = 0) = 1$, and the sensor transmits if and only if the system state is bad (either a zero-wait or a threshold policy $|s| \geq \gamma$), which corresponds to $\mu \leq 1$.

C. A Special Case of $\mu = 6$

The coordination between the sensor and the controller can sometimes be very subtle and beyond the high-level intuition discussed previously. Consider the following example.

Example 1: Consider the distribution of Y being

$$\begin{cases} \mathbb{P}(Y = 6) = 0.35, & \mathbb{P}(Y = 45) = 0.06, \\ \mathbb{P}(Y = 51) = 0.08, & \mathbb{P}(Y = 53) = 0.08, \\ \mathbb{P}(Y = 54) = 0.23, & \mathbb{P}(Y = 90) = 0.2, \end{cases} \tag{44}$$

$\mathbb{P}(V = 20) = 1$, $c_0 = 45$, $c_1 = 0$, and the initial random variable $Q \sim \mathcal{N}(0, \sigma^2)$ with $\sigma = 6.6$. We can numerically compute $\beta^* = 80.049$ using the 3 steps in Sec. III-E, and the resulting $\mathcal{S}_{\text{tx},\beta^*}$ is

$$\begin{aligned} \mathcal{S}_{\text{tx},\beta^*} = \{s : |s| \leq 1.803 \text{ or } 3.100 \leq |s| \leq 3.795 \\ \text{or } 3.858 \leq |s| \leq 6.767 \text{ or } 7.305 \leq |s|\} \end{aligned} \tag{45}$$

which has $\mu = 6$.

The reason of having a highly fractured transmission set $\mathcal{S}_{\text{tx},\beta^*}$ is due to the delicate probabilistic balance between the benefits of observing $S_i(t)$ at the sensor and observing Y_i at the controller.

V. Solving Two-way Remote Estimation Problem

In this section, we derive[7] the optimal Wiener-process remote estimation scheme with random delay Y_i and $\tilde{V}_i$ in forward and feedback directions, respectively, a generalization of the results in [13].

In Sec. II-D, we have shown that the remote estimation problem with random two-way delay is a special case of our setting with $U_i = V_i = 0$, $M_i = S_i(T_i + X_i)$, and non-negative $c_0 = \mathbb{E}\{Y_i\}\mathbb{E}\{\tilde{V}_i\} + \frac{\mathbb{E}\{(\tilde{V}_i)^2\}}{2}$ and $c_1 = \mathbb{E}\{\tilde{V}_i\}$. We now apply the best NWAC scheme ($U_i = 0$) in Propositions 5 and 7 to this particular set of parameter values. Specifically, by (31), we have

$$\begin{aligned} g_{\text{NWAC},\beta}(s) \triangleq & \mathbb{E}\{Y\} s^2 + \frac{1}{2}\mathbb{E}\{Y^2\} - \beta\left(\mathbb{E}\{Y\} + \mathbb{E}\{\tilde{V}\}\right) \\ & + \mathbb{E}\{Y\}\mathbb{E}\{\tilde{V}\} + \frac{\mathbb{E}\{(\tilde{V})^2\}}{2}. \end{aligned} \tag{46}$$

By Proposition 5, the optimal transmission policy $X_{i|\text{ZW}}^*$ for the sensor is to transmit whenever $s^2 \geq \gamma_{\text{NWAC}}^*$, where

$$\gamma_{\text{NWAC}}^* = \max(3\,(\beta_{\text{NWAC}}^* - \mathbb{E}\{Y\}), 0) \tag{47}$$

and the β_{NWAC}^* used in (47) can be computed by finding the root of the $p_{\text{NWAC}}(\beta)$ defined in Proposition 7.

[7]One can use similar techniques to solve AoI minimization problem with random two-way delay. Due to space limits, we refer the readers to [19] for the final results.

In the following, we show the above description of how to compute the optimal threshold γ^*_{NWAC} can be further simplified to the following equivalent form.

Lemma 4: The γ^*_{NWAC} defined in (47) and Proposition 7 can also be computed by finding the root of

$$\frac{\mathbb{E}\left\{\max(\gamma^2,(W(Y))^4)\right\}+3\mathbb{E}\{(\tilde{V})^2\}}{2\gamma} = \mathbb{E}\left\{\max\left(\gamma,(W(Y))^2\right)\right\}+\mathbb{E}\{\tilde{V}\}. \quad (48)$$

Proof: See Appendix L. ∎

Corollary 3: If we further limit ourselves to the 1-way delay setting, i.e., $\tilde{V}_i = 0$, then the optimal X_i policy described in (47) and Lemma 4 reproduces the optimal 1-way-delay remote-estimation scheme in [13, Theorem 1].

Proof: By specializing (48) in Lemma 4 with the one-way delay setting $\tilde{V}_i = 0$, the optimal remote estimation scheme in (47) has the corresponding γ^*_{NWAC} value being a root of

$$\frac{\mathbb{E}\left\{\max(\gamma^2,(W(Y))^4)\right\}}{2\gamma} = \mathbb{E}\left\{\max(\gamma,(W(Y))^2)\right\} \quad (49)$$

which is identical to the Eq. (15) in [13]. Hence, the optimal scheme in (47) and Lemma 4 reproduces [13, Theorem 1]. ∎

VI. Simulation Results

We compare the performance of our jointly optimal sensor/controller policy and five other important alternatives.

(i) Zero-wait (ZW) $(X_i = 0, U_i = 0)$ [11]: The zero-wait policy is commonly known as the work-conserving policy in queueing theory [11].

(ii) Optimal No-Wait-At-Sensor (NWAS) policy $(X_i = 0, M^*_i, U^*_i)$, see the discussion in Secs. III-B and III-C.

(iii) Optimal No-Wait-At-Controller (NWAC) policy $(X^*_{i|\text{ZW}}, M^*_i, U_i = 0)$, see Sec. III-D.

(iv) AoI-minimization scheme (AoI-min) [12]: It is related to the NWAS $(X_i = 0)$ scheme. The differences are (i) It does not take into account the reset cost c_0 and the per-round cost c_1, see (3); (ii) It falsely assumes the forward delay $Y_i = 0$ even though the actual Y_i could be non-zero; (ii) It employs the suboptimal message $M_i = 0$ instead of the optimal $M^*_i = S_i(T_i+X_i)$, and (iv) It hardwires $X_i = 0$ and optimizes the U_i under the suboptimal assumptions (i)–(iii). We are interested in measuring the performance loss (compared to the optimal NWAS scheme) due to these suboptimal decisions.

(v) Remote-estimation scheme (RE) [13]: As discussed in Sec. II-D, it is an instance of NWAC schemes. The differences between the RE and the optimal NWAC schemes are (i) RE does not take into account the reset cost c_0 and the per-round cost c_1 in (3) and (ii) RE falsely assumes the backward delay $V_i = 0$ even though the actual V_i could be non-zero.

We report the results for exponential forward and backward delays, while similar behaviors can be observed for log-normal delays as well. The initial value Q is assumed to be Gaussian with zero mean and variance σ^2. The results are presented in Fig. 4.

In Fig. 4a we notice that the larger the σ value, the wider the range of the initial value Q, which models the case of less accurate reset/control. Hence, the CAL2N of all 5 schemes increases as σ goes up.

A more interesting comparison is to calculate the ratio of the CAL2N of any scheme over that of our scheme, i.e., the normalized CAL2N plotted in Fig. 4b. Indeed, the normalized CAL2N of any scheme is always $\geq 100\%$ since our scheme is provably optimal.

In Fig. 4b we also observe that when the reset is accurate (small σ), the performance of the optimal NWAS is identical to that given by the optimal solution, which implies the jointly optimal scheme will allocate all its waiting time to the controller and perform zero-wait at the sensor. On the other hand, when the reset is loose (large σ), the jointly optimal scheme will allocate all its waiting time to the sensor and perform zero-wait at the controller, i.e., the optimal NWAC becomes globally optimal for large σ. In either case, our algorithm optimally splits the waiting time between the sensor and the controller and always attains the best performance.

As shown in Figs. 4c and 4e, we fix $\sigma = 4$ and vary the delay distribution parameters λ_Y and λ_V, respectively. In both figures, similar trends can be observed: When either λ_Y or λ_V increases (namely, when the expected delay is shorter), the CAL2N of any scheme goes down. Interestingly, the performance of the optimal NWAC is as good as the jointly optimal solution in both cases. It appears that in these scenarios, the reset quality σ value, see Figs. 4a and 4b, has stronger impact on whether NWAC is jointly optimal or not than the delay distributions of Y and V.

In Figs. 4g and 4i, we consider different values of c_0 and c_1. As can be seen, the optimal split of the waiting time between the sensor and the controller is heavily dependent on the value of reset cost c_0, see Figs. 4g and 4h, but much less on the per-round cost c_1, see Figs. 4i and 4j. Overall, from Figs. 4a to 4j one can see that each of the five alternatives excels in some scenarios but performs poorly in the others, while our scheme always achieves the optimal performance.

We are particularly interested in the relative performance of the existing 1-way-delay-based AoI-min and RE schemes when there is 2-way delay in the system. Because existing results do not take into account 2-way delay, as expected, the AoI-min scheme (resp. RE scheme) is always worse than the optimal NWAS scheme (resp. NWAC scheme) and is much worse than the jointly optimal solution. Furthermore, considering only 1-way delay (i.e., the AoI-min and RE schemes) and ignoring the delay in the other direction can be quite detrimental. In many cases they perform worse than the naive zero-wait solution. See RE vs. ZW in Fig. 4b and AoI-min vs. ZW in Fig. 4f. In particular, RE is significantly worse the ZW in Fig. 4b, while the difference between AoI-min and ZW in Fig. 4f is much smaller.

VII. Conclusion

We have proposed a new Wiener-process-based framework and characterized the corresponding jointly optimal sensor/controller policy, which unifies AoI minimization and remote estimation, two recent important results that have spawned substantial interests in the literature. The consideration of the two-way delay model and joint sensor-&-controller

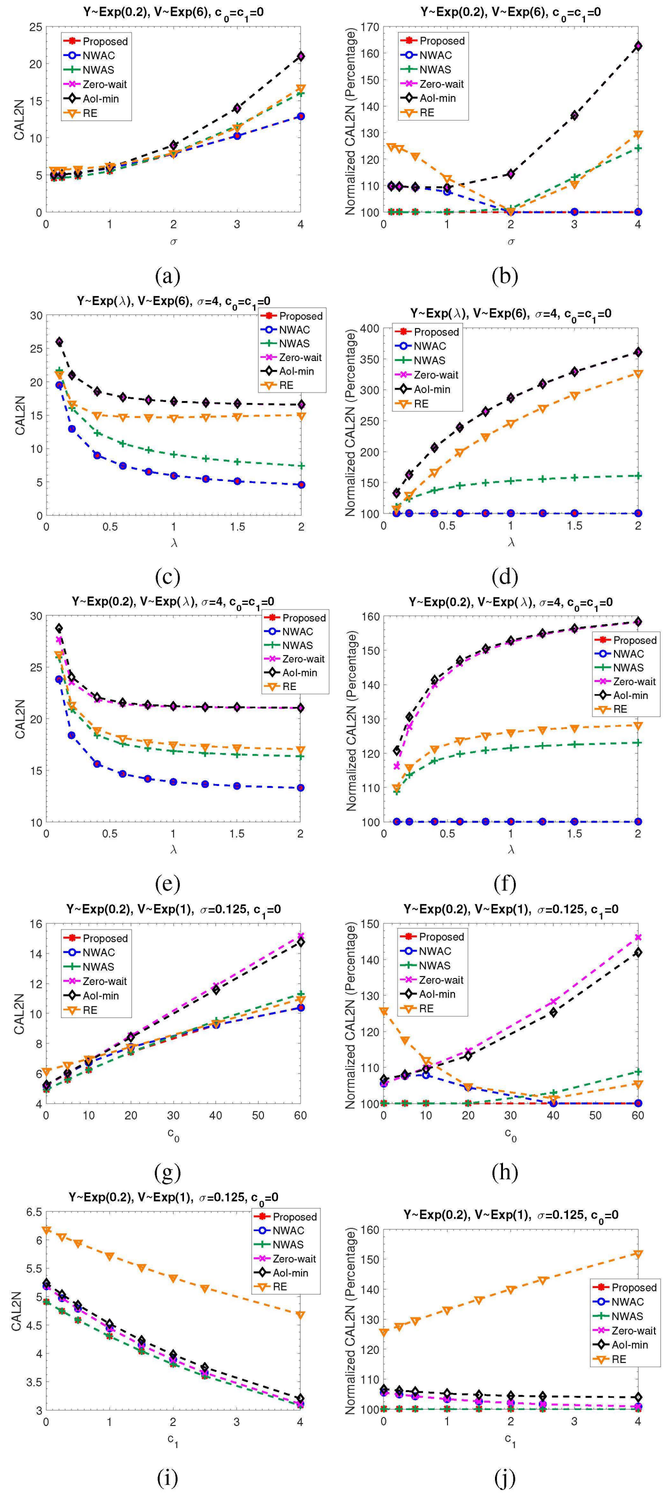


Fig. 4: Long-term average CAL2N for various settings. Those on the left are on absolute scale and those on the right are normalized with respect to the CAL2N of the optimal policy.

design have successfully addressed the time asynchrony of the practical systems and represents a significant improvement over the existing results based on idealized zero-delay acknowledgement feedback.

Appendix A
Proof of Proposition 1

Since $\mathbb{E}\{Y+V\}>0$ and $\mathbb{E}\{Q^2+Y^2+V^2\}<\infty$, $p(\beta)$ *in* (18) *is the infimum of a set of strictly decreasing affine functions of* β. As a result, $p(\beta)$ is concave, strictly decreasing, and continuous over $(-\infty,\infty)$. The proof of Statement (i) is complete.

By the non-negativity of $|S(t)|^2$ in (18), we have $p(0)\geq 0$. We now evaluate $p(\beta_{\max})$. By considering the choice of $X=U=0$ in (18), we have $p(\beta_{\max})\leq 0$. As a result, there exists a unique root $\beta^*\in[0,\beta_{\max}]$ satisfying $p(\beta^*)=0$. Statement (ii) is proven.

By plugging the scheme that attains β_{CAL2N} of (4) into (18), we immediately have $p(\beta_{\text{CAL2N}})\leq 0$, which in turn implies $\beta^*\leq\beta_{\text{CAL2N}}$. On the other hand, plugging the scheme that attains β^* of (18) into (4) will lead to $\beta^*\geq\beta_{\text{CAL2N}}$. Jointly, we thus have $\beta^*=\beta_{\text{CAL2N}}$ and the achievable schemes of β^* and β_{CAL2N} are identical. Statements (iii) and (iv) are proven.

Appendix B
Proof of Proposition 2

To study the optimal waiting time $U^*_{i|M}$ at the controller, we first place a *finite-horizon* constraint such that the random variable $U^*_{i|M}$ must satisfy $U^*_{i|M}\in[0,J]$ for some finite J. Later, we will let J go to infinity to derive the optimal waiting time $U^*_{i|M}$. Since all packets are time stamped, at the time instant at which the controller makes the decision $U^*_{i|M}$, the controller has complete causal observation of $\overline{M}^{(i)}=\{T_j,X_j,M_j:j\leq i\}$, which can be used to derive $\{Y_j:j\leq i\}$ when comparing to its local clock. As a result, the value of U in (18) is a function of $\overline{M}^{(i)}$ and Y_i. Further, the minimization problem (18) can be rewritten in an equivalent form once we remove the terms that do not depend on the choice of U (e.g., c_0, c_1, and $\mathbb{E}\{V|\overline{M}^{(i)},Y_i\}$ are all constants) and write the deterministic values in lower case ($X\to x$) and ($Y\to y$), since the optimization problem is now conditioning on $\overline{M}^{(i)}$ and Y_i.

As a result, we can write the equivalent optimization problem as follows

$$\begin{aligned}&f^{[1]}(\overline{m},y,J)\\&=\min_{U(\overline{m},y)\leq J}\mathbb{E}\left\{\int_{x+y}^{x+y+U+V}|S(t)|^2\,dt\middle|\overline{M}=\overline{m},Y=y\right\}\\&\quad-\beta\cdot\mathbb{E}\{U\}.\end{aligned}\tag{50}$$

We solve the above minimization problem by the following three major steps: joint time-and-space quantization, analytically solving the resulting quantized dynamic programming problem, and converting it back to the original continuous version.

A. Joint time and space quantization

Before proceeding, we first simplify (50) as follows. We notice that

$$\mathbb{E}\{|S(x+y)|^2|\overline{M}=\overline{m}, Y=y\} = \mathbb{E}\{|S(x)|^2|\overline{M}=\overline{m}\}+y \tag{51}$$

since $S(t)$ evolves according to the Wiener process and by the strong Markov property $S(x+y)-S(x)$ has zero-mean and variance y. The first term of (50) can then be further simplified as

$$\mathbb{E}\Bigg\{ (U+V)\cdot\left(\mathbb{E}\left\{|S(x)|^2\,\middle|\overline{M}=\overline{m}\right\}+y\right)\Bigg\} + \mathbb{E}\left\{\int_{t=0}^{U+V}\mathbb{E}\left\{|S(x+y+t)-S(x+y)|^2\right\}dt\right\} \tag{52}$$

which again uses the strong Markov property and $S(x+y+t)-S(x+y)$ having zero mean.

Our time and space quantization is motivated by the following classical result. Let $\mathcal{B}(i)$ be a symmetric binary random walk such that the initial point $\mathcal{B}(0)\in\mathbb{Z}$ can be arbitrarily chosen and $\mathcal{B}(i+1)=\mathcal{B}(i)+(1-2b(i))$, $i\geq 0$, where $\{b(i)\in\{0,1\}:i\}$ are i.i.d. Bernoulli random variables with $p=0.5$.[8] We can use the discrete-time $\mathcal{B}(i)$ to construct a continuous-time process $W^\delta(t)$:

$$W^\delta(t)=\sqrt{\delta}\cdot\mathcal{B}\left(\left\lfloor\frac{t}{\delta}\right\rfloor\right) \tag{53}$$

where $\lfloor\cdot\rfloor$ is the floor function. It is well known that $W^\delta(t)$ converges to a Wiener process[8] $W(t)$ in distribution when $\delta\to 0$ [29]. In the quantized setting, we thus assume the system state $S(t)$ evolves according to $W^\delta(t)$.

For the time domain quantization, we quantize the time-domain variables (e.g., x, y, U, V, β, and J) with quantization step δ that matches the time-domain quantization in (53). Specifically, define $\overline{x}=\lfloor\frac{x}{\delta}\rfloor$, $\overline{y}=\lfloor\frac{y}{\delta}\rfloor$, $\overline{U}=\lfloor\frac{U}{\delta}\rfloor$, $\overline{V}=\lfloor\frac{V}{\delta}\rfloor$, $\overline{\beta}=\left\lfloor\frac{\beta}{\delta}\right\rfloor$, and $\overline{J}=\lfloor\frac{J}{\delta}\rfloor$ as the integer approximations of x, y, U, V, β, and J.

For the state-space quantization, we quantize the state-space values with quantization step size $\sqrt{\delta}$ that matches the space-domain quantization in (53). Specifically, in the original problem, the state, faced by the controller when deciding U_i, is $\mathbb{E}\{S(x)^2|\overline{M}\}$ in (52). As a result, we define the quantized state value i_1 by

$$i_1=\left\lfloor\frac{\sqrt{\mathbb{E}\{S(x)^2|\overline{M}=\overline{m}\}}}{\sqrt{\delta}}\right\rfloor. \tag{54}$$

If we change the Wiener process model $W(t)$ to the binary random walk $W^\delta(t)$ and use the above spatial and temporal quantization, we can write down the following integer-based optimization problem.

[8]In the literature [29], a random walk and a Wiener process typically starts with $\mathcal{B}(0)=0$ and $W(0)=0$. Here we relax this constraint to allow $\mathcal{B}(0)$ and $W(0)$ being of any value, which is notationally convenient since we allow for the Wiener process that is periodically "reset" to Q_i.

$$F^{[1]}(i_1,\overline{y},\overline{J})=\min_{\overline{U}(i_1,\overline{y})\leq\overline{J}} \mathbb{E}\left\{\sum_{k=1}^{\overline{U}+\overline{V}}\left((i_1)^2+\overline{y}+(\mathcal{B}(\overline{x}+\overline{y}+k)-\mathcal{B}(\overline{x}+\overline{y}))^2\right)\right\} - \overline{\beta}\cdot\mathbb{E}\left\{\overline{U}\right\}. \tag{55}$$

which is the integer counterpart of the original problem in (50) and (52). By (50) to (55) we immediately have

$$f^{[1]}(\overline{m},y,J)=\lim_{\delta\to 0}\delta^2F^{[1]}\left(i_1,\overline{y},\overline{J}\right). \tag{56}$$

The roadmap of our subsequent derivation is as follows. We will first solve the quantized problem $F^{[1]}(i_1,\overline{y},\overline{J})$. Since $W^\delta(t)$ is based on the discrete-time $\mathcal{B}(i)$, the minimization problem in (55) becomes a discrete-time finite-horizon Markov decision process (MDP) with an undiscounted objective function, which can be readily solved by dynamic programming (DP). After taking the limit $\overline{J}\to\infty$, the optimizer in (55) will become the optimal integer-based waiting time $\overline{U}^*$. Finally, we take the limit $\delta\to 0$ to convert the discrete $\overline{U}^*$ back to the continuous version $U^*=\lim_{\delta\to 0}\delta\overline{U}^*$ to obtain the optimal waiting time at the controller in Proposition 2, and derive the closed-form expression of $f^{[1]}(\overline{m},y,J)$ using (56).

B. Analytically solving quantized problem

In a dynamic programming solver, the computation is performed backward, from (integer) time index $\overline{J}$ back to time index 0. To simplify the notation, we slightly abuse the notation and use $F^{[1]}(i_1,\overline{y},\overline{J})$ (resp. $F^{[1]}(i_1,\overline{y},0)$) to represent the objective function at time index 0 (resp. time index $\overline{J}$). Namely, we adopt reverse time indexing in a way that we can iteratively compute $F^{[1]}(i_1,\overline{y},j+1)$ from $F^{[1]}(i_1,\overline{y},j)$.

At the beginning of each time slot with reverse time index j, if the controller has not sent the reset packet, it can choose to "Send" or to "Wait". If the controller chooses to wait, then an immediate cost is incurred and we shall proceed to the reverse time index $j-1$. If the controller opts to send, it receives some immediate cost and goes to the terminal state. We now describe the corresponding backward induction.

Define $F^{[1]}(i_1,\overline{y},j)$ as

$$F^{[1]}(i_1,\overline{y},j)\triangleq\min_{\overline{U}(i_1,\overline{y})}\mathbb{E}\Bigg\{\sum_{k=\overline{J}-j+1}^{\overline{U}+\overline{V}}\Big((i_1)^2+\overline{y}+(\overline{J}-j) + \left(\mathcal{B}(\overline{x}+\overline{y}+k)-\mathcal{B}\left(\overline{x}+\overline{y}+(\overline{J}-j)\right)\right)^2\Big)\Bigg|\overline{U}\geq(\overline{J}-j)\Bigg\} - \overline{\beta}\cdot\mathbb{E}\left\{\overline{U}-(\overline{J}-j)\middle|\overline{U}\geq(\overline{J}-j)\right\} \tag{57}$$

$$=\begin{cases}F^{[1]}_{\mathsf{Send}}(i_1,\overline{y},j) & \text{if } j=0\\ \min\left(F^{[1]}_{\mathsf{Send}}(i_1,\overline{y},j),F^{[1]}_{\mathsf{Wait}}(i_1,\overline{y},j)\right) & \text{if } 1\leq j\leq\overline{J}\end{cases} \tag{58}$$

where the definition in (57) implies that when setting $j=\overline{J}$, the definition is consistent with (55). The case for $1\leq j\leq\overline{J}$

in (58) is the standard Bellman equation of dynamic programming, whereas the case for $j = 0$ in (58) is the boundary condition since we set the horizon to be $\overline{U} \leq \overline{J}$. That is, when the reverse index $j = 0$, one must always send. The two functions $F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j)$ an $F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j)$ are defined as below.

$$F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j) = F^{[1]}(i_1, \overline{y}, j)\Big|_{\overline{U}=\overline{J}-j}$$
$$= \mathbb{E}\Bigg\{ \sum_{k=\overline{J}-j+1}^{\overline{J}-j+\overline{V}} \Big((i_1)^2 + \overline{y} + (\overline{J}-j)$$
$$+ \left(\mathcal{B}(\overline{x}+\overline{y}+k) - \mathcal{B}\left(\overline{x}+\overline{y}+(\overline{J}-j)\right)\right)^2 \Big) \Bigg\}$$
$$= \mathbb{E}\left\{\overline{V}\right\} \cdot \left((i_1)^2 + \overline{y} + \overline{J} - j\right) + \frac{1}{2}\mathbb{E}\left\{(\overline{V})^2 + \overline{V}\right\}. \quad (59)$$

where (59) is computed by Wald's lemma [26] and conditioning on choosing to send at the reverse time index j ($\overline{U} = \overline{J} - j$), and

$$F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j) = (i_1)^2 + \overline{y} + \overline{J} - j - \overline{\beta} + F^{[1]}(i_1, \overline{y}, j-1) \quad (60)$$

where (60) is based on the original definition in (57) but replacing all the conditions from $\overline{U} \geq (\overline{J}-j)$ to $\overline{U} > (\overline{J}-j)$, i.e., we condition on choosing to wait at the reverse time index j. Eq. (60) then relates $F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j)$ to the $F^{[1]}(i_1, \overline{y}, j-1)$ value of the previous $j-1$.

Eqs. (57)–(60) are the complete description of the DP problem. We now analytically characterize some properties of this particular DP problem.

Lemma 5: Given i_1 and $\overline{y}$, for any j satisfying $1 \leq j \leq \overline{J}$, if $F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j) \leq F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j)$, i.e., Wait is better for reverse time index j, then $F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, t) \leq F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, t)$, $j \leq t \leq \overline{J}$, i.e., Wait will always be better for subsequent iterations $t \geq j$.

Proof: Suppose $j = j_0$ is the smallest j that satisfies $F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j) \leq F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j)$. We will first prove that for all $j \geq j_0$,

$$F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j) - F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j+1) \geq \mathbb{E}\{V\}$$
$$= F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j) - F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j+1). \quad (61)$$

The equality of (61) can be easily derived by substituting the definition of $F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j)$ in (59) into (61). Note that since we focus on $j \geq j_0$, if (61) holds, we immediately have

$$F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j+1) - F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j+1)$$
$$\geq F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j) - F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j)$$
$$\geq \cdots \geq F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j_0) - F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j_0) \geq 0 \quad (62)$$

which completes the proof of Lemma 5.

We now prove the inequality of (61) by induction. For $j = j_0$, we have

$$F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j_0) - F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j_0+1)$$
$$= 1 + F^{[1]}(i_1, \overline{y}, j_0-1) - F^{[1]}(i_1, \overline{y}, j_0) \quad (63)$$
$$= 1 + F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j_0-1) - F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j_0) \quad (64)$$
$$\geq F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j_0-1) - F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j_0) \quad (65)$$
$$= \mathbb{E}\{V\} \quad (66)$$

where (63) follows from (60); (64) follows from (58) and j_0 being the smallest j satisfying $F^{[1]}(i_1, \overline{y}, j) = F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j)$; (65) follows from $F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j_0) \geq F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j_0)$ and removing the positive constant 1; and (66) follows from (59).

Suppose the inequality of (61) holds for all $j \in [j_0, j_1]$. Then for $j = j_1 + 1$, we have

$$F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j_1+1) - F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j_1+2)$$
$$= 1 + F^{[1]}(i_1, \overline{y}, j_1) - F^{[1]}(i_1, \overline{y}, j_1+1) \quad (67)$$
$$= 1 + F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j_1) - F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j_1+1) \quad (68)$$
$$\geq 1 + \mathbb{E}\left\{\overline{V}\right\} \geq \mathbb{E}\{\overline{V}\} \quad (69)$$

where (67) follows from (60); (68) holds since (62) holds for $j \in [j_0, j_1]$; and (69) holds since the inequality of (61) holds for $j \in [j_0, j_1]$. By induction, the inequality of (61) holds for all $j \geq j_0$. The proof is complete. ■

Using Lemma 5, the decision of the optimal (integer-valued) waiting time $\overline{U}$ can be found by finding the smallest $j \in [1, \overline{J}]$ satisfying the following inequality (i.e., Wait is better):

$$F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j)$$
$$= (i_1)^2 + \overline{y} + \overline{J} - j - \overline{\beta} + F^{[1]}(i_1, \overline{y}, j-1) \quad (70)$$
$$= (i_1)^2 + \overline{y} + \overline{J} - j - \overline{\beta} + F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j-1) \quad (71)$$
$$= (i_1)^2 + \overline{y} + \overline{J} - j - \overline{\beta}$$
$$+ \mathbb{E}\left\{\overline{V}\right\} \cdot \left((i_1)^2 + \overline{y} + \overline{J} - j + 1\right) + \frac{1}{2}\mathbb{E}\left\{(\overline{V})^2 + \overline{V}\right\} \quad (72)$$
$$\leq F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j)$$
$$= \mathbb{E}\left\{\overline{V}\right\} \cdot \left((i_1)^2 + \overline{y} + \overline{J} - j\right) + \frac{1}{2}\mathbb{E}\left\{(\overline{V})^2 + \overline{V}\right\} \quad (73)$$

where (70) follows from (60); (71) follows from that we are searching for the first time $F^{[1]}(i_1, \overline{y}, j) = F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j)$; (72) and (73) follow from (59).

We denote the smallest such $j \in [1, \overline{J}]$ by j_0. If no such j exists, it means that we should always Send and we can thus set the value of $j_0 = \overline{J} + 1$ for notational simplicity. Solving (70)–(73), we have

$$j_0 = \min((i_1)^2 + \overline{y} + \overline{J} - \overline{\beta} + \mathbb{E}\{\overline{V}\}, \overline{J} + 1). \quad (74)$$

Here we assume the finite horizon $\overline{J}$ is sufficiently large such that the index j_0 computed in (74) is no less than 1, i.e., $j_0 \geq 1$. We can then compute the optimal waiting time $\overline{U}^*$ by

$$\overline{U}^* = \overline{J} + 1 - j_0 \quad (75)$$
$$= \max\left(1 + \overline{\beta} - \left((i_1)^2 + \overline{y} + \mathbb{E}\{\overline{V}\}\right), 0\right) \quad (76)$$

where (75) changes the reverse time index j_0 back to the normal time index $\overline{U}^*$ and (76) follows from (74).

C. Converting back to the continuous time/space problem

Since $U^* = \lim_{\delta\to 0} \delta \times \overline{U}^*$, from (76) and (54) we conclude that

$$U^* = \max\left(\beta - \left(y + \mathbb{E}\{S(x)^2 | \overline{M} = \overline{m}\} + \mathbb{E}\{V\}\right), 0\right) \tag{77}$$

for which the constant 1 in (76) vanishes after we multiply δ and $\overline{U}^*$ and then let $\delta \to 0$. The proof of Proposition 2 is thus complete.

We close this section by also evaluating $f^{[1]}(\overline{m}, y, \infty) \triangleq \lim_{J\to\infty} f^{[1]}(\overline{m}, y, J)$ through (56). Firstly we have the following lemma:

Lemma 6: For sufficiently large $\overline{J}$, we have

$$F^{[1]}(i_1, \overline{y}, \overline{J}) = -\frac{(\overline{U}^*)^2}{2} + \frac{\overline{U}^*}{2} + \mathbb{E}\{\overline{V}\} \cdot \left((i_1)^2 + \overline{y}\right) + \frac{1}{2}\mathbb{E}\left\{(\overline{V})^2 + \overline{V}\right\}. \tag{78}$$

Proof: See Appendix C. ■

We thus have

$$\begin{aligned} f^{[1]}(\overline{m}, y, \infty) &= \lim_{J\to\infty} f^{[1]}(\overline{m}, y, J) \\ &= \lim_{\delta\to 0} \lim_{\overline{J}\to\infty} \delta^2 F^{[1]}\left(i_1, \overline{y}, \overline{J}\right) \qquad (79) \\ &= -\frac{(U^*)^2}{2} + \mathbb{E}\{V\} \cdot \left(\mathbb{E}\{S(x)^2 | \overline{M} = \overline{m}\} + y\right) + \frac{1}{2}\mathbb{E}\{V^2\} \qquad (80) \end{aligned}$$

where (79) follows from (56), and (80) follows from (78) for which the terms $\frac{\overline{U}^*}{2}$, 1, and $0.5\mathbb{E}\{\overline{V}\}$ grow at the rate of $O(\frac{1}{\delta})$ and thus diminish to zero once we multiply δ^2 and let $\delta \to 0$.

Appendix C
Proof of Lemma 6

Assume sufficiently large $\overline{J}$ in the discussion. Define

$$u \triangleq 1 + \overline{\beta} - \left((i_1)^2 + \overline{y} + \mathbb{E}\{\overline{V}\}\right). \tag{81}$$

Consider two cases:

Case 1: $u \le 0$. In this case, we have $j_0 = \overline{J}+1$ and $\overline{U}^* = 0$ from (74) and (76). Hence,

$$\begin{aligned} F^{[1]}(i_1, \overline{y}, \overline{J}) &= F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, \overline{J}) \qquad (82) \\ &= \mathbb{E}\{\overline{V}\} \cdot \left((i_1)^2 + \overline{y}\right) + \frac{1}{2}\mathbb{E}\left\{(\overline{V})^2 + \overline{V}\right\} \qquad (83) \end{aligned}$$

where (82) holds since $\overline{U}^* = 0$ (i.e., choosing Send at the reverse time index $\overline{J}$); and (83) follows from (59). Eq. (78) holds in this case.

Case 2: $u > 0$. In this case, we have $j_0 = \overline{J} + 1 - u$ and $\overline{U}^* = u$ from (74) and (76), respectively. Hence,

$$\begin{aligned} F^{[1]}(i_1, \overline{y}, j_0 - 1) &= F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j_0 - 1) \qquad (84) \\ &= \mathbb{E}\{\overline{V}\} \cdot \left((i_1)^2 + \overline{y} + \overline{J} - (j_0 - 1)\right) + \frac{1}{2}\mathbb{E}\left\{(\overline{V})^2 + \overline{V}\right\} \qquad (85) \end{aligned}$$

where (84) holds since j_0 is the smallest j satisfying $F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, j) \le F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j)$; and (85) follows from (59).

In the sequel, we describe how to use the expression of $F^{[1]}(i_1, \overline{y}, j_0 - 1)$ in (85) to derive the end result $F^{[1]}(i_1, \overline{y}, \overline{J})$ in (78). From Lemma 5, for $j_0 \le t \le \overline{J}$, we have $F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, t) \le F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, t)$, and thus

$$\begin{aligned} F^{[1]}(i_1, \overline{y}, t) &= F^{[1]}_{\mathsf{Wait}}(i_1, \overline{y}, t) \\ &= (i_1)^2 + \overline{y} + \overline{J} - t - \overline{\beta} + F^{[1]}(i_1, \overline{y}, t - 1) \qquad (86) \end{aligned}$$

where (86) follows from (60).

By iteratively using (86) for $j_0 \le t \le \overline{J}$, we have

$$\begin{aligned} F^{[1]}(i_1, \overline{y}, \overline{J}) &= \sum_{k=j_0}^{\overline{J}} \left((i_1)^2 + \overline{y} + \overline{J} - k - \beta\right) \\ &\quad + F^{[1]}_{\mathsf{Send}}(i_1, \overline{y}, j_0 - 1) \qquad (87) \\ &= \sum_{l=0}^{u-1} \left((i_1)^2 + \overline{y} + l - \overline{\beta}\right) + \mathbb{E}\{\overline{V}\} \cdot \left((i_1)^2 + \overline{y} + u\right) \\ &\quad + \frac{1}{2}\mathbb{E}\{(\overline{V})^2 + \overline{V}\} \qquad (88) \\ &= -\frac{u^2}{2} + \frac{u}{2} + \mathbb{E}\{\overline{V}\} \cdot ((i_1)^2 + \overline{y}) + \frac{1}{2}\mathbb{E}\{(\overline{V})^2 + \overline{V}\} \qquad (89) \end{aligned}$$

where (87) follows from (85) and (86); and (88) follows from (59), the definition of u in (81), and a change of variable; and (89) follows from the arithmetic sum formula and the definition of u in (81).

Lemma 6 follows from (83) and (89).

Appendix D
Proofs of Lemma 1 and Corollary 2

To prove that $g_\beta(s)$ is continuous, we first prove a similar statement and later show that this statement is equivalent to $g_\beta(s)$ being continuous.

Lemma 7: For any non-negative random variable Y, which may be continuous, discrete, or hybrid, the function $f(x) = \mathbb{E}\{\mathbb{1}_{\{Y \le x\}} \cdot (Y - x)^2\}$ is continuous with respect to x.

Proof: Consider any two values $-\infty < x_l < x_r < \infty$. We have

$$\begin{aligned} &|f(x_r) - f(x_l)| \\ &= \Big| \mathbb{E}\left\{\mathbb{1}_{\{Y \le x_l\}} \cdot \left(2(x_r - x_l)(x_l - Y) + (x_r - x_l)^2\right)\right\} \\ &\quad + \mathbb{E}\left\{\mathbb{1}_{\{Y \in (x_l, x_r]\}} \cdot (Y - x_r)^2\right\} \Big| \qquad (90) \\ &\le 2 \cdot (x_r - x_l) \cdot |x_l| + (x_r - x_l)^2 \qquad (91) \\ &\quad + (x_r - x_l)^2 \qquad (92) \end{aligned}$$

where (90) follows from $\mathbb{1}_{\{Y \le x_r\}} = \mathbb{1}_{\{Y \le x_l\}} + \mathbb{1}_{\{Y \in (x_l, x_r])\}}$; and (91) follows from the triangle inequality and Y being non-negative; and (92) follows from $Y \in (x_l, x_r]$ implying $(Y - x_r)^2 \le (x_r - x_l)^2$. Overall (91) and (92) imply that $f(x)$ is continuous regardless of the distribution of Y. ■

We now prove $g_\beta(s)$ is continuous. To that end, by the definition in (22), it suffices to prove

$$\begin{aligned} \overline{f}(s) &= -0.5\mathbb{E}\left\{\mathbb{1}_{\{Y \le \beta - \mathbb{E}\{V\} - s^2\}}\right\} s^4 \\ &\quad + \mathbb{E}\left\{\mathbb{1}_{\{Y \le \beta - \mathbb{E}\{V\} - s^2\}} \cdot (\beta - \mathbb{E}\{V\} - Y)\right\} s^2 \\ &\quad - 0.5\mathbb{E}\{\mathbb{1}_{\{Y \le \beta - \mathbb{E}\{V\} - s^2\}} \cdot (\beta - \mathbb{E}\{V\} - Y)^2\} \qquad (93) \\ &= -0.5\mathbb{E}\left\{\mathbb{1}_{\{Y \le \beta - \mathbb{E}\{V\} - s^2\}} \cdot \left(Y - (\beta - \mathbb{E}\{V\} - s^2)\right)^2\right\} \qquad (94) \end{aligned}$$

is continuous with respect to s, where (93) follows from removing the coefficients in (22) that do not depend on s, which is equivalent to removing the second order polynomial $\mathbb{E}\{Y+V\}s^2+a_0$ from our consideration and (94) is by simple arithmetic rearrangement.

By applying Lemma 7 to (94), $\overline{f}(s)$ is continuous with respect to s^2 and hence with respect to s. As a result, $g_\beta(s)$ is continuous with respect to s.

We now prove $h_\beta(s) \geq \frac{1}{12}s^4 + (a_0 - 3\beta^2)$ for all $s \in (-\infty, \infty)$. We consider the following two cases.

Case 1: $s^2 \geq \beta - \mathbb{E}\{V\}$. In this case, $a_{s,4} = 0$, $a_{s,2} = \mathbb{E}\{Y+V\}$, and $a_{s,0} = 0$ by the definitions in (24)–(27). We then have

$$h_\beta(s) = \frac{1}{6}s^4 + (\mathbb{E}\{Y+V\} - \beta)s^2 + a_0 \tag{95}$$

$$\geq \frac{1}{12}s^4 + \frac{1}{12}s^4 - \beta s^2 + a_0 \tag{96}$$

$$\geq \frac{1}{12}s^4 - 3\beta^2 + a_0 \tag{97}$$

where (95) follows from the definition of $h_\beta(s)$ in (23); (96) follows from $\mathbb{E}\{Y+V\} > 0$; (97) follows from the inequality $\frac{1}{12}s^4 - \beta s^2 + 3\beta^2 = \frac{1}{12}(s^2 - 6\beta)^2 \geq 0$.

Case 2: $s^2 < \beta - \mathbb{E}\{V\}$. We first notice that from the definitions in (24), (25) and (26), we have

$$|a_{s,4}s^4| \leq |s^4| \leq \beta^2 \tag{98}$$

$$a_{s,2} \geq 0 \tag{99}$$

$$|a_{s,0}| \leq \frac{\beta^2}{2}. \tag{100}$$

where (98) follows from from $|a_{s,4}| \leq 1$ and $0 \leq s^2 < \beta - \mathbb{E}\{V\} \leq \beta$. ; (99) follows from the non-negativity of Y and V, and $s^2 + Y \leq \beta - \mathbb{E}\{V\}$ implying $0 \leq \beta - \mathbb{E}\{V\} - Y$; and (100) follows from $s^2 + Y \leq \beta - \mathbb{E}\{V\}$ implying $0 \leq (\beta - \mathbb{E}\{V\} - Y) \leq \beta$. Then,

$$h_\beta(s) = (\frac{1}{6} + a_{s,4})s^4 + (a_{s,2} - \beta)s^2 + a_{s,0} + a_0 \tag{101}$$

$$\geq \frac{1}{12}s^4 + a_{s,4}s^4 + (a_{s,2} - \beta)s^2 + a_{s,0} + a_0 \tag{102}$$

$$\geq \frac{1}{12}s^4 - \beta^2 - \beta(\beta - \mathbb{E}\{V\}) - \frac{\beta^2}{2} + a_0 \tag{103}$$

$$\geq \frac{1}{12}s^4 - 3\beta^2 + a_0 \tag{104}$$

where (101) follows from the definition of $h_\beta(s)$ in (23); (102) holds since $\frac{1}{6}s^4 \geq \frac{1}{12}s^4$; (103) follows from (98), (99) and (100); (104) follows from $\mathbb{E}\{V\} \geq 0$ and $\beta^2 \geq 0$. Lemma 1 is proved.

Corollary 2 directly follows from Lemma 1 since the supremum is taken over a non-empty set containing at least one convex function $\frac{1}{12}s^4 + (a_0 - 3\beta^2)$.

Appendix E
Proofs of Propositions 4 and 6

The proofs of Propositions 4 and 6 are the most involved and consist of the following components: **Component 1:** The joint time and space quantization and the corresponding dynamic programming (DP) problem after quantization, also see similar discussion in Appendix B; If we call the new DP problem as *quantized-DP*, **Component 2** describes how the controller-side policy affects the initialization of the sensor-side quantized-DP solver; Using components 1 and 2, one can numerically solve the quantized-DP problem assuming a finite-horizon setting. However, the brute-force DP iterations often obscure the physical meaning/interpretation of the optimal decision rules and do not provide any closed-form solution. To uncover further results, **Component 3** focuses on solving quantized-DP analytically through careful convergence analysis when $\overline{J} \to \infty$. Finally, **Component 4** discusses how (i) the initialization, (ii) the closed-form optimal decision rule, and (iii) the closed-form optimal objective values can be seamlessly converted back to the corresponding parts in the original continuous time/space problem and thus completes the proofs of both Propositions 4 and 6.

A. Component 1: Joint time and space quantization

Recall that Proposition 4 assumes the optimal message M^* and waiting time U^* in Proposition 3 and Corollary 1, respectively. Given $S(0) = s$ and a large (but finite) horizon J, we define the optimal *continuous-time cost* for the sensor as

$$\begin{aligned} &f^{[2]}(s, J) \\ &= \min_{X \leq J} \mathbb{E}\left\{\int_0^{X+Y+U^*+V} |S(t)|^2\, dt \middle| S(0) = s\right\} + c_0 \\ &- \beta\left(\mathbb{E}\left\{X + Y + U^* + V \middle| S(0) = s\right\} + c_1\right). \end{aligned} \tag{105}$$

We use the same quantization method as in Appendix B, for which the space and time quantization levels are set to $\sqrt{\delta}$ and δ, respectively. Specifically we set $i = \left\lfloor \frac{S(0)}{\sqrt{\delta}} \right\rfloor$, $\overline{X} = \lfloor \frac{X}{\delta} \rfloor$, $\overline{Y} = \lfloor \frac{Y}{\delta} \rfloor$, $\overline{U} = \lfloor \frac{U}{\delta} \rfloor$, $\overline{V} = \lfloor \frac{V}{\delta} \rfloor$, $\overline{J} = \lfloor \frac{J}{\delta} \rfloor$, $\overline{c}_0 = \lfloor \frac{c_0}{\delta^2} \rfloor$, and $\overline{c}_1 = \lfloor \frac{c_1}{\delta} \rfloor$. Using integer approximations, we then have the following sensor-side optimization problem

$$\begin{aligned} &F^{[2]}(i, \overline{J}) \\ &= \min_{\overline{X} \leq \overline{J}} \mathbb{E}\left\{\sum_{k=1}^{\overline{X}+\overline{Y}+\overline{U}^*+\overline{V}} \left(i^2 + (\mathcal{B}(k) - \mathcal{B}(0))^2\right) \middle| \mathcal{B}(0) = i\right\} \\ &- \overline{\beta}\left(\mathbb{E}\left\{\overline{X} + \overline{Y} + \overline{U}^* + \overline{V} \middle| \mathcal{B}(0) = i\right\} + \overline{c}_1\right) + \overline{c}_0 \end{aligned} \tag{106}$$

which satisfies

$$f^{[2]}(s, J) = \lim_{\delta \to 0} \delta^2 F^{[2]}\left(i, \overline{J}\right). \tag{107}$$

Similar to Appendix B, we solve $F^{[2]}(i, \overline{J})$ by dynamic programming over finite horizon $[0, \overline{J}]$. For convenience, we again use the reverse time index j during the iterative Bellman equation, where $j = 0$ (resp. $j = \overline{J}$) represents the last slot (resp. the first slot). Specifically, define

$$\begin{aligned} F^{[2]}(i,j) &= \min_{\overline{X}} \\ &\mathbb{E}\Bigg\{ \sum_{k=\overline{J}-j+1}^{\overline{X}+\overline{Y}+\overline{U}^*+\overline{V}} \left(i^2 + (\mathcal{B}(k) - \mathcal{B}(\overline{J}-j))^2 \right) \Bigg| \mathcal{B}(\overline{J}-j) = i, \\ &\qquad \overline{X} \geq (\overline{J}-j) \Bigg\} \\ &- \overline{\beta} \left(\mathbb{E}\left\{ \overline{X} - (\overline{J}-j) + \overline{U}^* \Big| \mathcal{B}(\overline{J}-j) = i, \overline{X} \geq (\overline{J}-j) \right\} \right) \\ &- \overline{\beta}\left(\mathbb{E}\left\{ \overline{Y} + \overline{V} \right\} + \overline{c}_1 \right) + \overline{c}_0 \end{aligned} \tag{108}$$

$$= \begin{cases} F^{[2]}_{\mathsf{Send}}(i,j) & \text{if } j = 0 \\ \min\left(F^{[2]}_{\mathsf{Send}}(i,j), F^{[2]}_{\mathsf{Wait}}(i,j) \right) & \text{if } 1 \leq j \leq \overline{J}. \end{cases} \tag{109}$$

It is self-explanatory to verify that the above definition of (108) is consistent with (106) once we set $j = \overline{J}$. Eq. (109) is the boundary condition of the finite-horizon dynamic programming.

B. Component 2: Initialization of the sensor-side quantized-DP solver

The objective function of $F^{[2]}_{\mathsf{Send}}$ in (109) is

$$\begin{aligned} &F^{[2]}_{\mathsf{Send}}(i,j) \\ &= \mathbb{E}\Bigg\{ \sum_{k=\overline{J}-j+1}^{(\overline{J}-j)+\overline{Y}+\overline{U}^*+\overline{V}} \left(i^2 + (\mathcal{B}(k) - \mathcal{B}(\overline{J}-j))^2 \right) \\ &\qquad \Bigg| \mathcal{B}(\overline{J}-j) = i, \overline{X} = \overline{J}-j \Bigg\} \\ &\quad - \overline{\beta}\left(\mathbb{E}\left\{ \overline{Y} + \overline{U}^* + \overline{V} \Big| \mathcal{B}(\overline{J}-j) = i, \overline{X} = \overline{J}-j \right\} + \overline{c}_1 \right) \\ &\quad + \overline{c}_0 \end{aligned} \tag{110}$$

which is obtained from (108) by hardwiring $\overline{X} = \overline{J} - j$. Breaking the summation in (110) into pieces and using the notation $\mathbb{E}_{(i,j)}$ as shorthand for the conditional expectation, we have

$$\begin{aligned} &\mathbb{E}_{(i,j)}\left\{ \sum_{k=\overline{J}-j+1}^{(\overline{J}-j)+\overline{Y}} \left(i^2 + \left(\mathcal{B}(k) - \mathcal{B}(\overline{J}-j) \right)^2 \right) \right\} \\ &+ \mathbb{E}_{(i,j)}\left\{ \sum_{k=\overline{J}-j+\overline{Y}+1}^{(\overline{J}-j)+\overline{Y}+\overline{U}^*+\overline{V}} \left(i^2 + \left(\mathcal{B}(k) - \mathcal{B}(\overline{J}-j) \right)^2 \right) \right\}. \end{aligned} \tag{111}$$

Define

$$\mathcal{K}(j) = \{ k : \overline{J} - j + \overline{Y} + 1 \leq k \leq (\overline{J}-j) + \overline{Y} + \overline{U}^* + \overline{V} \}. \tag{112}$$

Then, the second summation in (111) can be computed as

$$\mathbb{E}\left\{ \mathbb{E}_{(i,j)}\left\{ \sum_{k \in \mathcal{K}(j)} \left(i^2 + \left(\mathcal{B}(k) - \mathcal{B}(\overline{J}-j) \right)^2 \right) \Bigg| \overline{Y} \right\} \right\} \tag{113}$$

using the law of total expectation. We proceed to simplify the inner conditional expectation in (113).

$$\mathbb{E}_{(i,j,\overline{y})}\left\{ \sum_{k=\overline{J}-j+\overline{y}+1}^{(\overline{J}-j)+\overline{y}+\overline{U}^*+\overline{V}} \left(i^2 + \left(\mathcal{B}(k) - \mathcal{B}(\overline{J}-j) \right)^2 \right) \right\}$$

$$= \mathbb{E}_{(i,j,\overline{y})}\left\{ \sum_{t=1}^{\overline{U}^*+\overline{V}} \left(i^2 + \left(\mathcal{B}(\overline{J}-j+\overline{y}+t) - \mathcal{B}(\overline{J}-j) \right)^2 \right) \right\} \tag{114}$$

$$= \mathbb{E}_{(i,j,\overline{y})}\left\{ \sum_{t=1}^{\overline{U}^*+\overline{V}} \left(i^2 + \overline{y} + \left(\mathcal{B}(\overline{J}-j+\overline{y}+t) - \mathcal{B}(\overline{J}-j+\overline{y}) \right)^2 \right) \right\} \tag{115}$$

$$= F^{[1]}(i, \overline{y}, \overline{J}) + \overline{\beta} \cdot \mathbb{E}_{(i,j,\overline{y})}\left\{ \overline{U}^*(i, \overline{y}) \right\} \tag{116}$$

where $\mathbb{E}_{(i,j,\overline{y})}$ is the shorthand for the conditional expectation under the event $\left\{ \mathcal{B}(\overline{J}-j) = i, \overline{X} = \overline{J}-j, \overline{Y} = \overline{y} \right\}$; (114) follows from setting $t = k - (\overline{J} - j + \overline{y})$; (115) follows from the property of the binary symmetric random walk that $\mathcal{B}(t_1 + t) - \mathcal{B}(t_1)$ has zero mean and variance t. Since the optimal message is used, the sensor transmits the latest system state $\mathcal{B}(\overline{J}-j) = i$ to the controller. Hence, i_1 defined in (54) is equal to $\mathcal{B}(\overline{J}-j) = i$. Eq. (116) then follows from the definition of $F^{[1]}(i_1, \overline{y}, \overline{J})$ in (55).

The objective function of $F^{[2]}_{\mathsf{Send}}$ in (110) can then be rewritten as

$$\begin{aligned} &F^{[2]}_{\mathsf{Send}}(i,j) \\ &= \mathbb{E}\left\{ \sum_{k=\overline{J}-j+1}^{(\overline{J}-j)+\overline{Y}} \left(i^2 + (\mathcal{B}(k) - \mathcal{B}(\overline{J}-j))^2 \right) \right\} \\ &\quad + \mathbb{E}\Bigg\{ \sum_{k=\overline{J}-j+\overline{Y}1}^{(\overline{J}-j)+\overline{Y}+\overline{U}^*+\overline{V}} \left(i^2 + (\mathcal{B}(k) - \mathcal{B}(\overline{J}-j))^2 \right) \\ &\qquad \Bigg| \mathcal{B}(\overline{J}-j) = i, \overline{X} = \overline{J}-j \Bigg\} \\ &\quad - \overline{\beta}\left(\mathbb{E}\left\{ \overline{Y} + \overline{U}^* + \overline{V} \Big| \mathcal{B}(\overline{J}-j) = i, \overline{X} = \overline{J}-j \right\} + \overline{c}_1 \right) \\ &\quad + \overline{c}_0 \end{aligned} \tag{117}$$

$$\begin{aligned} &= \mathbb{E}\left\{ \sum_{k=\overline{J}-j+1}^{(\overline{J}-j)+\overline{Y}} \left(i^2 + (\mathcal{B}(k) - \mathcal{B}(\overline{J}-j))^2 \right) \right\} \\ &\quad + \mathbb{E}_{\overline{Y}}\left\{ F^{[1]}(i, \overline{Y}, \overline{J}) + \overline{\beta} \cdot \overline{U}^*(i, \overline{Y}) \right\} \\ &\quad - \overline{\beta}\left(\mathbb{E}\{\overline{Y}\} + \mathbb{E}_{\overline{Y}}\left\{ \overline{U}^*(i, \overline{Y}) \right\} + \mathbb{E}\{\overline{V}\} + \overline{c}_1 \right) + \overline{c}_0 \end{aligned} \tag{118}$$

$$\begin{aligned} &= \mathbb{E}\left\{ \overline{Y} \right\} (i)^2 + \frac{1}{2}\mathbb{E}\left\{ (\overline{Y})^2 \right\} + \frac{1}{2}\mathbb{E}\{\overline{Y}\} \\ &\quad + \mathbb{E}_{\overline{Y}}\left\{ F^{[1]}(i, \overline{Y}, \overline{J}) \right\} - \overline{\beta}\left(\mathbb{E}\{\overline{Y} + \overline{V}\} + \overline{c}_1 \right) + \overline{c}_0 \end{aligned} \tag{119}$$

where (117) follows from (110), (111) and that the conditional expectation of the first term of (111) is identical to the regular (unconditional) expectation due to the strong Markov property of $\mathcal{B}(k)$; (118) follows from (116), (117) and that the delays $\overline{Y}$ and $\overline{V}$ are independent of the conditioning event $\{\mathcal{B}(\overline{J}-j)=i, \overline{X}=\overline{J}-j\}$ and only the controller action $\overline{U}^*(i,\overline{Y})$ depends on the conditional event; (119) uses the strong Markov property of the binary random walk. Note that the final expression of $F^{[2]}_{\mathsf{Send}}(i,j)$ does not depend on j since its definition in (110) only counts the "incremental/future" utility after the Send decision. Therefore, the expression in (119) is both the objective value of the Send action and the objective value of the boundary condition in (109).

$F^{[2]}_{\mathsf{Wait}}(i,j)$ can be defined backwardly based on the next time slot (i.e., reverse time index $j-1$)

$$F^{[2]}_{\mathsf{Wait}}(i,j) = (i)^2 - \overline{\beta} + 0.5\left(F^{[2]}(i+1,j-1) + F^{[2]}(i-1,j-1)\right) \tag{120}$$

where $(i)^2 - \overline{\beta}$ is the immediate cost and the rest is the expected future cost calculated from the previous iteration. Since we consider a symmetric binary random walk, the probability that the discrete state i in next time slot increases (or decreases) by 1 is 0.5.

Eqs. (109), (119), and (120) jointly describe the complete backward iteration of the dynamic programming solver. In the sequel we analyze the properties of the optimal solution.

C. Component 3: Analytically solving the quantized-DP

By the boundary condition in (109), we have

$$F^{[2]}_{\mathsf{Send}}(i,j) = F^{[2]}(i,0). \tag{121}$$

By (120) and (109), we thus have the following more compact form of iteration:

$$F^{[2]}(i,j) = \min\big(F^{[2]}(i,0), i^2 - \overline{\beta} + 0.5(F^{[2]}(i+1,j-1) + F^{[2]}(i-1,j-1))\big), \forall j \geq 1 \tag{122}$$

To analyze (122), we notice that the polynomial

$$A(i) \triangleq -\frac{i^4}{6} + (\overline{\beta} + \frac{1}{6})i^2 \tag{123}$$

satisfies the difference equation

$$A(i) = i^2 - \overline{\beta} + 0.5(A(i+1) + A(i-1)). \tag{124}$$

If we define another function

$$H^{[2]}(i,j) \triangleq F^{[2]}(i,j) - A(i) \tag{125}$$

then $H^{[2]}(i,j)$ must satisfy the following *homogeneous*[9] iterative equation

$$H^{[2]}(i,j) = \min\Big(H^{[2]}(i,0), 0.5\left(H^{[2]}(i+1,j-1) + H^{[2]}(i-1,j-1)\right)\Big). \tag{126}$$

[9]That is, an iterative equation that does not involve any external constants.

due to (122) and (124). Namely, the new function $H^{[2]}(i,j)$ absorbs the $i^2 - \overline{\beta}$ term in (122) and thus follows a homogeneous iterative computation in (126). Our approach is to first compute/evaluate $H^{[2]}(i,j)$ directly using the cleaner iteration (126) and then compute retrospectively $F^{[2]}(i,j) = H^{[2]}(i,j) + A(i)$.

In the following, we analyze $\lim_{\overline{J}\to\infty} H^{[2]}(i,\overline{J})$, i.e., the limiting results of the iterative computation in (126). We first notice that

Lemma 8:

$$H^{[2]}(i,j+1) \leq H^{[2]}(i,j) \tag{127}$$

for any i and $j \geq 0$.

Proof: This is a straightforward result following the homogeneous iterative equation in (126). See Appendix F for details. ■

Before proceeding, we define

Definition 1: A discrete function $f(i) : \mathbb{Z} \mapsto \mathbb{R}$ is said to be *d.convex* if

$$f(i) \leq 0.5\left(f(i+1) + f(i-1)\right), \forall i \in \mathbb{Z}. \tag{128}$$

Lemma 9: There exists a d.convex function $H_{\mathrm{LB}}(i)$ such that

$$-\infty < H_{\mathrm{LB}}(i) \leq H^{[2]}(i,j) \text{ for all } i \text{ and } j. \tag{129}$$

Proof: See Appendix F. ■

By Lemmas 8 and 9, and the Monotone Convergence Theorem [30], the limit $\lim_{\overline{J}\to\infty} H^{[2]}(i,\overline{J})$ exists.

We now derive $\lim_{\overline{J}\to\infty} H^{[2]}(i,\overline{J})$.

We define the *discrete convex hull* (function) of the function $H^{[2]}(i,0)$ as

$$\mathrm{D.Cnvx}(H^{[2]}(i,0)) \triangleq \sup\{f(i) : f \prec H^{[2]}(i,0), f \text{ is d.convex}\}. \tag{130}$$

By Lemma 9, $\mathrm{D.Cnvx}(H^{[2]}(i,0))$, which is a function of i, always exists since the supremum is taken over a non-empty set. Furthermore, $\mathrm{D.Cnvx}(H^{[2]}(i,0))$ is also d.convex due to similar reasons that the (continuous) convex hull is itself convex [31].

We now have the following lemmas:

Lemma 10: For any arbitrarily given integer i, we have

$$\lim_{\overline{J}\to\infty} H^{[2]}(i,\overline{J}) \leq \mathrm{D.Cnvx}(H^{[2]}(i,0)). \tag{131}$$

Proof: By Lemmas 8 and 9, and the Monotone Convergence Theorem [30], we know $\lim_{\overline{J}\to\infty} H^{[2]}(i,\overline{J})$ exists and satisfies $\lim_{\overline{J}\to\infty} H^{[2]}(i,\overline{J}) \leq H^{[2]}(i,0)$ for all i. Furthermore, by (126), the limit must be d.convex. Since $\mathrm{D.Cnvx}(H^{[2]}(i,0))$ in (130) takes the supremum of all such d.convex functions, we must have (131). ■

Lemma 11: $\mathrm{D.Cnvx}(H^{[2]}(i,0)) \leq \lim_{\overline{J}\to\infty} H^{[2]}(i,\overline{J})$.

Proof: By the definition (130), $\mathrm{D.Cnvx}(H^{[2]}(i,0))$ is d.convex and satisfies $\mathrm{D.Cnvx}(H^{[2]}(i,0)) \leq H^{[2]}(i,0)$. In the proof of Lemma 9, we showed that any d.convex function that lower bounds the initial value $H^{[2]}(i,0)$ is also a lower bound of $H^{[2]}(i,j)$ for all j. As a result, we have

D.Cnvx($H^{[2]}(i,0)$) $\leq H^{[2]}(i,j)$ for all j. The proof is thus complete. ■

Lemmas 10 and 11 jointly imply

$$\lim_{\overline{J}\to\infty} H^{[2]}(i,\overline{J}) = \text{D.Cnvx}(H^{[2]}(i,0)). \tag{132}$$

Thus far, we have proven that the dynamic programming iterations in (109), (119), and (120) can be solved by (i) computing $H^{[2]}(i,0) \triangleq F^{[2]}(i,0) - A(i)$ in (125), (ii) finding the D.Cnvx($H^{[2]}(i,0)$), and (iii) finding $\lim_{\overline{J}\to\infty} F^{[2]}(i,\overline{J}) =$ D.Cnvx($H^{[2]}(i,0)$) $+ A(i)$. The final step of the proof is to convert the operations/steps of discrete-time, discrete-space solutions back to the original continuous-time, continuous-space solution.

D. Component 4: Converting back to the continuous time/space problem

Lemma 12: Denote the continuous-time continuous-space state value as $S(t) = s$. We have

$$\lim_{\delta\to 0}\lim_{\overline{J}\to\infty} \delta^2 F^{[2]}_{\mathsf{Send}}\left(\left\lfloor \frac{s}{\sqrt{\delta}}\right\rfloor, \overline{J}\right) = g_\beta(s) \tag{133}$$

which is defined in (22).

Proof: See Appendix G ■

Lemma 12 shows that $g_\beta(s)$ corresponds to the starting point (the last slot) of the dynamic programming solver. In the following we will show that in the continuous-time continuous-space domain, the iterative computation in (109), (119), and (120) corresponds to the computation of $h_\beta(s)$ and the convex hull operations defined in Proposition 4.

By (123), one can easily see that $\lim_{\delta\to 0}\delta^2 A\left(\left\lfloor \frac{s}{\sqrt{\delta}}\right\rfloor\right) = -\frac{s^4}{6} + \beta s^2$. Combining this observation with (133) and (125), we have

$$\lim_{\delta\to} \delta^2 H^{[2]}\left(\left\lfloor \frac{s}{\sqrt{\delta}}\right\rfloor, 0\right) = h_\beta(s) = g_\beta(s) - (\beta s^2 - \frac{1}{6}s^4). \tag{134}$$

Finally by translating the discrete convex hull relationship in (132) to its continuous convex hull counterpart, we have

Lemma 13:

$$\lim_{\delta\to 0}\lim_{\overline{J}\to\infty} \delta^2 H^{[2]}(i,\overline{J}) = \text{Cnvx}\left(g_\beta(s) - \left(\beta s^2 - \frac{1}{6}s^4\right)\right) = \text{Cnvx}(h_\beta(s)). \tag{135}$$

We now describe the optimal sensor policy. Define $H^{[2]}(i,\infty) \triangleq \lim_{\overline{J}\to\infty} H^{[2]}(i,\overline{J})$. Note that by (126) and comparing it to the original versions (109) and (122), the sensor should send if $H^{[2]}(i,\infty) = H^{[2]}(i,0)$ and should wait if

$$H^{[2]}(i,\infty) = 0.5\left(H^{[2]}(i+1,\infty) + H^{[2]}(i-1,\infty)\right) \tag{136}$$

By translating the above discrete-time decision back to its continuous time domain, the optimal sensor waiting policy is to transmit if and only if $S(t) = s$ satisfying $\text{Cnvx}(h_\beta(s)) = h_\beta(s)$.

We close this section by establishing the connection between objective functions of the continuous and discrete domains.

Lemma 14:

$$\lim_{J\to\infty} f^{[2]}(s,J) = \lim_{\delta\to 0}\lim_{\overline{J}\to\infty} \delta^2 F^{[2]}(i,\overline{J})$$
$$= \text{Cnvx}\left(h_\beta(s)\right) + \left(\beta s^2 - \frac{1}{6}s^4\right) \tag{137}$$
$$= \phi(\beta, s) \tag{138}$$

Proof: The first equality is the standard conversion between the continuous domain and the quantized domain. The second equality is a straightforward result of (125) and Lemma 13. The third equality is from the definition of $\phi(\beta, s)$ in (36). ■

Recall that $f^{[2]}(s,J)$, defined in (105), is conditioned on $S(0)$. As a result, $\phi(\beta,s) = \lim_{J\to\infty} f^{[2]}(s,J)$ in Lemma 14 is the continuous infinite-horizon objective function of any given state $S(0) = s$. Since the distribution of $S(0) \sim Q$, by the definition of $p(\beta)$ in (18), the value of $p(\beta)$ can be found by taking the expectation of s over distribution Q, i.e., (37). We have thus proven Proposition 6.

APPENDIX F
PROOFS OF LEMMAS 8 AND 9

A. Proof of Lemma 8

When $j = 0$, (127) holds trivially because of the first half of the minimum operation in (126). Suppose (127) holds all $j \leq j_0$. By plugging in $j = j_0 + 1$ and $j = j_0$ into (126), we have

$$H^{[2]}(i, j_0+1) = \min\Big(H^{[2]}(i,0),$$
$$0.5\left(H^{[2]}(i+1,j_0) + H^{[2]}(i-1,j_0)\right)\Big) \tag{139}$$

and

$$H^{[2]}(i, j_0) = \min\Big(H^{[2]}(i,0),$$
$$0.5\left(H^{[2]}(i+1,j_0-1) + H^{[2]}(i-1,j_0-1)\right)\Big). \tag{140}$$

By the induction hypothesis, each of the three terms in the right-hand side of (139) is no larger than the corresponding term in the right-hand side of (140). We thus have (127) hold for $j = j_0 + 1$ as well. The proof is complete.

B. Proof of Lemma 9

First, from (76) and (78), we have

$$F^{[1]}(i_1, \overline{y}, \overline{J}) \geq -(\overline{\beta}+1)^2. \tag{141}$$

From (141),

$$\mathbb{E}_{\overline{Y}}\left\{F^{[1]}(i,\overline{Y},\overline{J})\right\} \geq -(\overline{\beta}+1)^2. \tag{142}$$

From (109), (119) and (142),

$$F^{[2]}(i,0) \geq L_0 \tag{143}$$

where

$$L_0 \triangleq -(\overline{\beta}+1)^2 - \overline{\beta}(\mathbb{E}\{\overline{Y}+\overline{Y}\}+\overline{c}_1) \tag{144}$$

is a constant that does not depend on i.

Recall that $H^{[2]}(i,j) \triangleq F^{[2]}(i,j) - A(i)$. Hence,

$$\begin{aligned} H^{[2]}(i,0) &= F^{[2]}(i,0) - A(i) \\ &\geq \frac{i^4}{6} - (\overline{\beta}+\frac{1}{6})i^2 + L_0 \end{aligned} \tag{145}$$

where (145) follows from (143). Since (145) is a 4-th order polynomial with the leading coefficient of the i^4 term being strictly positive, we can always lower bound (145) by $\frac{i^4}{12} + L_1$ for a sufficiently small (negative) but finite constant L_1 constant. Since $\frac{i^4}{12} + L_1$ is d.convex, the proof of Lemma 9 is complete.

Appendix G
Proof of Lemma 12

From (119), we obtain

$$\begin{aligned} &\lim_{\delta\to 0}\lim_{\overline{J}\to\infty} \delta^2 F^{[2]}_{\mathsf{Send}}\left(\left\lfloor \frac{s}{\sqrt{\delta}}\right\rfloor, \overline{J}\right) \\ &= \mathbb{E}\{Y\}s^2 + \frac{1}{2}\mathbb{E}\{Y^2\} - \beta\left(\mathbb{E}\{Y+V\}+c_1\right) + c_0 \\ &\quad + \lim_{\delta\to 0}\lim_{\overline{J}\to\infty} \delta^2 \mathbb{E}_{\overline{Y}}\left\{F^{[1]}(i,\overline{Y},\overline{J})\right\}. \end{aligned} \tag{146}$$

By (56), (80) and recall that the optimal message is used in (119), we have

$$\lim_{\delta\to 0}\lim_{\overline{J}\to\infty} \delta^2 \mathbb{E}_{\overline{Y}}\left\{F^{[1]}(i,\overline{Y},\overline{J})\right\} = \mathbb{E}_Y\left\{f^{[1]}(\overline{m}^*,Y,\infty)\right\} \tag{147}$$

where the term $\overline{m}^*$ represents the optimal message is used. The closed-form expression of $\mathbb{E}_Y\{f^{[1]}(\overline{m}^*,Y,\infty)\}$ can be derived from (80) as follows. Since we use the optimal message $M_i^* = S_i(T_i+X_i)$ in (119), the $\mathbb{E}\{S(x)^2|\overline{M}=\overline{m}^*\}$ term in (80) is equal to $S(x)^2$ and we thus have

$$f^{[1]}(\overline{m}^*,y,\infty) = -\frac{(U^*)^2}{2} + \mathbb{E}\{V\}S(x)^2 + \frac{1}{2}\mathbb{E}\{V^2\}. \tag{148}$$

Note that there is still the U^* inside (148). We now plug in the closed-form expression of U^* in (77) to further simplify (148). Let $s = S(x)$. Given $Y = y$, we consider two cases to evaluate (147).

Case 1: $\beta - \left(y + s^2 + \mathbb{E}\{V\}\right) \leq 0$. From (77), in this case we have $U^* = 0$. From (148), we have

$$f^{[1]}(\overline{m}^*,y,\infty) = \mathbb{E}\{V\}(s^2+y) + \frac{1}{2}\mathbb{E}\{V^2\}. \tag{149}$$

Case 2: $\beta - \left(y + s^2 + \mathbb{E}\{V\}\right) > 0$. From (77), in this case we have $U^* > 0$. From (148), we have

$$\begin{aligned} f^{[1]}(\overline{m}^*,y,\infty) &= -\frac{(U^*)^2}{2} + \mathbb{E}\{V\}(s^2+y) + \frac{1}{2}\mathbb{E}\{V^2\} \\ &= -\frac{1}{2}s^4 + \left(\beta - \mathbb{E}\{V\} - y\right)s^2 - \frac{1}{2}\left(\beta - \mathbb{E}\{V\} - y\right)^2 \\ &\quad + \mathbb{E}\{V\}(s^2+y) + \frac{1}{2}\mathbb{E}\{V^2\} \end{aligned} \tag{150}$$

where (150) follows from $U^* = \beta - \left(y + s^2 + \mathbb{E}\{V\}\right)$.

Finally, assembling Cases 1 and 2, we can derive the right-hand side of (147), using (147), (149) and (150). We thus have

$$\begin{aligned} &\lim_{\delta\to 0}\lim_{\overline{J}\to\infty} \delta^2 \mathbb{E}_{\overline{Y}}\left\{F^{[1]}(i,\overline{Y},\overline{J})\right\} = \mathbb{E}_Y\left\{f^{[1]}(\overline{m}^*,Y,\infty)\right\} \\ &= -\frac{\mathbb{P}(s^2+Y\leq\beta-\mathbb{E}\{V\})}{2}s^4 \\ &+ \Big(\mathbb{E}\{V\} + \mathbb{E}\left\{\mathbb{1}_{\{s^2+Y\leq\beta-\mathbb{E}\{V\}\}}\cdot(\beta-\mathbb{E}\{V\}-Y)\right\}\Big)s^2 \\ &- \frac{\mathbb{E}\left\{\mathbb{1}_{\{s^2+Y\leq\beta-\mathbb{E}\{V\}\}}\cdot(\beta-\mathbb{E}\{V\}-Y)^2\right\}}{2} \\ &+ \mathbb{E}\{V\}\mathbb{E}\{Y\} + \frac{1}{2}\mathbb{E}\{V^2\}. \end{aligned} \tag{151}$$

Eq. (133) then follows from (146) and (151) immediately.

Appendix H
Proofs of Propositions 5 and 7

Recall that the optimal NWAC scheme uses $(X^*_{i|\mathsf{ZW}}, M_i^*, U_i = 0)$. Hence, the proofs of Propositions 5 and 7 follow the same manner as the four components described at the beginning of Appendix E, except $U = 0$ instead of $U = U^*$.

In Appendix E, we have proved that the optimal sensor waiting policy is to transmit if and only if $S(t) = s$ satisfying $\mathrm{Cnvx}(h_\beta(s)) = h_\beta(s)$. The same reasoning applies to the optimal NWAC scheme with the only change being that the sensor-side dynamic programming problem has a different initial/boundary point $F^{[2]}_{\mathsf{Send}}(i,j)$ in (109) that needs to take into account a different controller policy $U = 0$. The new initial/boundary point can be easily found by combining (146), (147), and (148), except that we modify (148) by using $U = 0$ instead of the optimal $U = U^*$. In the end, the new initialization point of the NWAC policy becomes

$$\begin{aligned} &\lim_{\delta\to 0}\lim_{\overline{J}\to\infty} \delta^2 F^{[2]}_{\mathsf{Send}}\left(\left\lfloor \frac{s}{\sqrt{\delta}}\right\rfloor, \overline{J}\right) \\ &= \mathbb{E}\left\{Y+V\right\}s^2 + \mathbb{E}\{Y\}\mathbb{E}\{V\} + \frac{1}{2}\mathbb{E}\left\{Y^2+V^2\right\} \\ &\quad - \beta\left(\mathbb{E}\{Y+V\}+c_1\right) + c_0 \qquad (152) \\ &= g_{\mathrm{NWAC},\beta}(s). \qquad (153) \end{aligned}$$

Using this new $g_{\mathrm{NWAC},\beta}(s)$, we can derive the optimal NWAC policy in the way that is outlined in (31) to (35).

We conclude this section by proving the closed-form expression of $\mathrm{Cnvx}(h_{\mathrm{NWAC},\beta}(s))$ that was first provided in (33) and (34). To that end, we first note that $h_{\mathrm{NWAC},\beta}(s)$ is an even function and we thus only need to study the case of $s \geq 0$.

We first write down the closed-form expression of the first and second derivatives of $h_{\mathrm{NWAC},\beta}(s)$.

$$\frac{d}{ds}h_{\mathrm{NWAC},\beta}(s) = \frac{2}{3}s^3 + 2(\mathbb{E}\{Y+V\}-\beta)s \tag{154}$$

$$\frac{d^2}{ds^2}h_{\mathrm{NWAC},\beta}(s) = 2s^2 + 2\left(\mathbb{E}\{Y+V\}-\beta\right) \tag{155}$$

We now consider two cases depending on the value of β in the below.

Case 1: If $\mathbb{E}\{Y+V\}-\beta \geq 0$, then the second order derivative $\frac{d^2}{ds^2}h_{\text{NWAC},\beta}(s) \geq 0$ for all s. That is, the $h_{\text{NWAC},\beta}(s)$ function itself is convex. Therefore $\text{Cnvx}(h_{\text{NWAC},\beta}(s)) = h_{\text{NWAC},\beta}(s)$, $\forall s$ and the sensor should always transmit.

Case 2: If $\mathbb{E}\{Y+V\}-\beta < 0$, we observe that in the range of $s \geq 0$, there are 2 zeros $s=0$ and $s=\sqrt{3(\beta-\mathbb{E}\{Y+V\})}$ in the first order derivative $\frac{d}{ds}h_{\text{NWAC},\beta}(s)$. Furthermore, $\frac{d}{ds}h_{\text{NWAC},\beta}(s) > 0$ if $s > \sqrt{3(\beta-\mathbb{E}\{Y+V\})}$ and $\frac{d}{ds}h_{\text{NWAC},\beta}(s) < 0$ if $0 < s < \sqrt{3(\beta-\mathbb{E}\{Y+V\})}$. As a result, $s=\sqrt{3(\beta-\mathbb{E}\{Y+V\})}$ and its mirror point $s=-\sqrt{3(\beta-\mathbb{E}\{Y+V\})}$ are the *global* minimum of $h_{\text{NWAC},\beta}(s)$. This implies that we can lower bound $h_{\text{NWAC},\beta}(s)$ by $h_{\text{target}}(s)$, namely,

$$h_{\text{NWAC},\beta}(s) \geq h_{\text{target}}(s) \triangleq \begin{cases} h_{\text{NWAC},\beta}(s) & \text{if } s^2 \geq 3(\beta-\mathbb{E}\{Y+V\}) \\ h_{\text{NWAC},\beta}(\sqrt{\gamma_{\text{NWAC}}}) & \text{if } s^2 < 3(\beta-\mathbb{E}\{Y+V\}) \end{cases} \quad (156)$$

where $\gamma_{\text{NWAC}} \triangleq \max(3\,(\beta-\mathbb{E}\{Y+V\})\,,0)$ was first defined in (34).

Also note that in the range of $s \geq \sqrt{3(\beta-\mathbb{E}\{Y+V\})}$ (also when $s \leq -\sqrt{3(\beta-\mathbb{E}\{Y+V\})}$), we have $\frac{d^2}{d^2s}h_{\text{NWAC},\beta} \geq 0$. This implies that the above lower bound $h_{\text{target}}(s)$ is convex. It is easy to see that for any other function $\tilde{h}(s)$ such that there exists s_0 satisfying $\tilde{h}(s_0) > h_{\text{target}}(s_0)$, we either have $\tilde{h}(s_1) > h_{\text{NWAC},\beta}(s_1)$ for some s_1 or $\tilde{h}(\cdot)$ is not convex. As a result, $h_{\text{target}}(s)$ is the tightest convex lower bound of $h_{\text{NWAC},\beta}$. The expression of $\text{Cnvx}\left(h_{\text{NWAC},\beta}(s)\right)$ in (34) is thus proven.

Appendix I
Proof of Proposition 8

To prove Proposition 8, we recall that the optimal NWAS scheme uses $X_i=0$. From the definition of $F^{[2]}_{\mathsf{Send}}(i,j)$ in (110), the continuous objective function for the NWAS scheme for any given state $S(0)=s$ is $\lim_{\delta\to 0}\lim_{\overline{J}\to\infty}F^{[2]}_{\mathsf{Send}}(i,\overline{J})$, which is equal to $g_\beta(s)$ from Lemma 12. Since the distribution of $S(0)\sim Q$ and since $p_{\text{NWAS}}(\beta)$ is defined in (41), by evaluating the expectation in (41) we have proven Proposition 8.

Appendix J
Proof of Proposition 9

For ease of notation, define a constant $y_1 \triangleq \mathbb{E}\{V\}+y_0$. Using the fact that Y_i is deterministic and plugging it into (24) to (27) and thus (22), we have

$$h_\beta(s) = \begin{cases} \frac{1}{6}s^4-(\beta-y_1)s^2+a_0 & \text{if } s^2 > \beta-y_1 \\ -\frac{1}{3}s^4+d_1 & \text{otherwise} \end{cases} \quad (157)$$

where a_0 (defined in (27)) and $d_1 \triangleq a_0-0.5(\beta-y_1)^2$ are constants that do not depend on s.

Recall that we find β^* by solving $p(\beta)=0$. We now consider two cases depending on the value of β^*.

Case 1: If $\beta^*-y_1 \leq 0$, then any s value always falls into the first expression of (157) and one can easiliy verify that $h_{\beta^*}(s)$ is convex for $s\in(-\infty,\infty)$. By Proposition 4, $\mathcal{S}_{\text{tx},\beta^*}=(-\infty,\infty)$ and we thus have $\mu=0$ in this case. Additionally, since $\beta^*-y_1\leq 0$, the first half of (20) is non-positive with probability 1, and we obtain $\mathbb{P}(U_i^*=0)=1$.

Case 2: $\beta^*-y_1 > 0$. In this case, the curve $h^*_\beta(s)$ is still continuous but contains 3 pieces for the three intervals $(-\infty,-\sqrt{\beta^*-y_1})$, $[-\sqrt{\beta^*-y_1},\sqrt{\beta^*-y_1}]$, and $(\sqrt{\beta^*-y_1},\infty)$, respectively. Since the leftmost and rightmost pieces are convex and the center piece is concave, by the statement in Proposition 4 we must have $\mathcal{S}_{\text{tx},\beta^*}=\{s:|s|\geq\gamma\}$ for some $\gamma>0$ satisfying $\gamma^2>\beta^*-y_1$. As a result, we have $\mu=1$ in this case. Moreover, since the Wiener process is continuous and the transmission time X_i being the hitting time (29), when the sender transmits the packet at time T_i+X_i, we either have $(S(T_i+X_i))^2=\gamma^2$ if $X_i>0$ or $(S(T_i+X_i))^2\geq\gamma^2$ if $X_i=0$. As a result,

$$\mathbb{E}\left\{(S_i(T_i+X_i))^2\,\middle|\,\overline{M}^{(i)}\right\}\geq\gamma^2 \quad (158)$$

with probability 1. Therefore, the first half of (20) is non-positive with probability 1 and $\mathbb{P}(U_i^*=0)=1$ in Case 2 as well.

Appendix K
Proof of Proposition 10

When evaluating (22) to (27) using the fact that $Y\sim\text{Exp}(\lambda_Y)$, we have

$$h_\beta(s) = \begin{cases} \frac{1}{6}s^4-(\beta-\mathbb{E}\{Y+V\})\,s^2+c_3 & \text{if } s^2\geq\beta-\mathbb{E}\{V\} \\ -\frac{1}{3}s^4+\frac{1}{\lambda_Y^2}e^{-\lambda_Y(\beta-\mathbb{E}\{Y\}-s^2)}+c_4 & \text{otherwise} \end{cases} \quad (159)$$

where c_3 and c_4 are constants that can be evaluated in (22) to (27). Since the actual values of c_3 and c_4 have little impact on the proof, we do not expand their expressions.

Recall that we find β^* by solving $p(\beta)=0$. We now consider two cases depending on the value of β^*.

Case 1: $\beta^*-\mathbb{E}\{V\}\leq 0$. In this case, only the first expression of (159) is active. By simple calculus, $h_{\beta^*}(s)$ is convex, and by Proposition 4, $\mathcal{S}_{\text{tx},\beta^*}=(-\infty,\infty)$. In sum, in this case the optimal sensor policy is to send immediately ($\mu=0$).

Case 2: $\beta^*-\mathbb{E}\{V\}>0$. In this case, the second expression of (159) is active when $|s|$ is sufficiently small. Note that $h_{\beta^*}(s)$ is always continuous due to Lemma 1. Since $h_{\beta^*}(s)$ is an even function, we focus our discussion only on the range of $s\geq 0$. More specifically, in the range $s\in[0,\infty)$, $h_{\beta^*}(s)$ consists of 2 pieces with the turning point being $s_0 \triangleq \sqrt{\beta^*-\mathbb{E}\{V\}}$. We then compute the second-order derivative of $h_{\beta^*}(s)$ for the two pieces $[0,s_0)$ and (s_0,∞), respectively, and we have

$$\frac{d^2}{ds^2}h_{\beta^*}(s) = \begin{cases} 2s^2-2\left(\beta^*-\mathbb{E}\{Y+V\}\right) & \text{if } s\in(s_0,\infty) \\ -4s^2+\left(\frac{2}{\lambda_Y}+4s^2\right)e^{-\lambda_Y\left(\beta^*-\mathbb{E}\{V\}-s^2\right)} & \text{if } s\in[0,s_0). \end{cases} \quad (160)$$

Note that even though Lemma 1 only guarantees the continuity of $h_{\beta^*}(s)$, when the forward delay distribution Y_i is exponential, from the fact that the two expressions in (160) have the same value when setting $s = s_0 = \sqrt{\beta^* - \mathbb{E}\{V\}}$, we know $h_{\beta^*}(s)$ is *doubly continuously differentiable for all* $s \in (-\infty, \infty)$.

We now consider the number of positive roots of $\frac{d^2}{ds^2}h_{\beta^*}(s)$. By (160), we observe that even though $\frac{d^2}{ds^2}h_{\beta^*}(s)$ contains two pieces, it is strictly positive whenever $s \geq s_0$. Therefore the number of positive roots of $\frac{d^2}{ds^2}h_{\beta^*}(s)$ is the same as the number of roots in the range of $[0, s_0)$. In the sequel, we will prove: **Claim 1:** $\frac{d^2}{ds^2}h_{\beta^*}(s)$ in the range of $[0, s_0)$ (the second expression in (160)) has *at most* two strictly positive roots.

Using Claim 1, $h_{\beta^*}(s)$ being even, and the fact that when evaluating at $s = 0$, $\frac{d^2}{ds^2}h_{\beta^*}(0) = \frac{2}{\lambda_Y} \cdot e^{-\lambda_Y(\beta^* - \mathbb{E}\{V\})} > 0$ is strictly positive, if we scan the s value from $-\infty$ to ∞, then the sign of the second order derivative $\frac{d^2}{ds^2}h_{\beta^*}(s)$ must be either $+ - + - +$ or being always $+$. The first case corresponds to $\frac{d^2}{ds^2}h_{\beta^*}(s)$ having 2 strictly positive roots; the second case corresponds to $\frac{d^2}{ds^2}h_{\beta^*}(s)$ having one double root that is strictly positive or zero root. (As shown earlier, if there is any positive root, it is always between $[0, s_0)$).

Recall the definition of μ in Lemma 3. The above observation of the signs of the second-order derivative implies $\mu \leq 2$ since the value of μ, being the number of intervals for which $\mathrm{Cnvx}(h_{\beta^*}(s)) \neq h_{\beta^*}(s)$, is upper bounded by the number of disjoint intervals of having negative $\frac{d^2}{ds^2}h_{\beta^*}(s)$. For illustration, Fig. 3a has 2 intervals of negative second order derivative and the corresponding $\mu = 1$.

The rest of the proof is to prove Claim 1. Suppose that $\frac{d^2}{ds^2}h_{\beta^*}(s)$ has ≥ 3 positive roots in the range of $[0, s_0)$. By Rolle's theorem [30], this implies $\frac{d^3}{ds^3}h_{\beta^*}(s)$ have at least two roots in $(0, s_0)$. Some simple calculation shows that

$$\frac{d^3}{ds^3}h_{\beta^*}(s) = -8s + (12s + 8\lambda_Y s^3)e^{-\lambda_Y(\beta - \mathbb{E}\{V\} - s^2)}. \tag{161}$$

Therefore, any strictly positive root of $\frac{d^3}{ds^3}h_{\beta^*}(s)$ must satisfy

$$2 = e^{-\lambda_Y(\beta - \mathbb{E}\{V\} - s^2)}(3 + 2\lambda_Y s^2). \tag{162}$$

Since the right-hand side of (162) is strictly increasing with respect to s, $\frac{d^3}{ds^3}h_{\beta^*}(s)$ has at most one positive root. The proof is thus complete by contradiction.

Appendix L
Proof of Lemma 4

We first prove a lemma.

Lemma 15: Let β^*_{NWAC} used in (47) be the root of the $p_{\text{NWAC}}(\beta)$ defined in Proposition 7. We must have $\beta^*_{\text{NWAC}} > \mathbb{E}\{Y\}$.

Proof: Let $c \triangleq \mathbb{E}\{Y\}\mathbb{E}\{\tilde{V}\} + \frac{\mathbb{E}\{(\tilde{V})^2\}}{2}$. Recall that in Sec. II-D, we have shown that by setting $Q_i = W(T_i) - W(T_{i-1} + X_{i-1})$ and $c_0 = c$ defined above, our setting collapses to the one for remote estimation. By the strong Markov property of the Wiener process, the reset random variable $Q_i = W(T_i) - W(T_{i-1} + X_{i-1})$ has the same distribution as $Q_i \sim (W(Y) - W(0)) = W(Y)$ if we use the traditional definition of $W(0) = 0$.

We prove Lemma 15 by contradiction. Suppose that $\beta^*_{\text{NWAC}} \leq \mathbb{E}\{Y\}$. Since $\beta^*_{\text{NWAC}} \leq \mathbb{E}\{Y\}$ the convex hull of $h_\beta(s)$ is $h_\beta(s)$ itself, see (33) and (34). Therefore, we have

$$p_{\text{NWAC}}(\beta^*_{\text{NWAC}}) = \mathbb{E}_Q\{\phi_{\text{NWAC}}(\beta^*_{\text{NWAC}}, Q)\} \tag{163}$$

$$\mathbb{E}_Q\left\{\mathbb{E}\{Y\}Q^2 + \frac{1}{2}\mathbb{E}\{Y^2\} - \beta^*_{\text{NWAC}}(\mathbb{E}\{Y\} + \mathbb{E}\{\tilde{V}\}) + c\right\} \tag{164}$$

$$= \mathbb{E}\{Y\}\mathbb{E}\{Q^2\} + \frac{1}{2}\mathbb{E}\{Y^2\} - \beta^*_{\text{NWAC}}(\mathbb{E}\{Y\} + \mathbb{E}\{\tilde{V}\}) + c \tag{165}$$

$$= \mathbb{E}\{Y\}\mathbb{E}\{Y\} + \frac{1}{2}\mathbb{E}\{Y^2\} - \beta^*_{\text{NWAC}}(\mathbb{E}\{Y\} + \mathbb{E}\{\tilde{V}\}) + c \tag{166}$$

where (163) follows from (39); (164) follows from (46), (32), and (38); (165) simplifies (164); (166) follows from Wald's lemma $\mathbb{E}\{(W(Y))^2\} = \mathbb{E}\{Y\}$ [26].

Since β^*_{NWAC} is a root of $p_{\text{NWAC}}(\beta)$, from (166) the β^*_{NWAC} must also satisfy

$$\beta^*_{\text{NWAC}} = \frac{(\mathbb{E}\{Y\})^2 + \frac{1}{2}\mathbb{E}\{Y^2\} + c}{\mathbb{E}\{Y\} + \mathbb{E}\{\tilde{V}\}}. \tag{167}$$

Recall that $c \triangleq \mathbb{E}\{Y\}\mathbb{E}\{\tilde{V}\} + \frac{\mathbb{E}\{(\tilde{V})^2\}}{2} \geq \mathbb{E}\{Y\}\mathbb{E}\{\tilde{V}\}$, then from (167) we immediately have $\beta^*_{\text{NWAC}} > \mathbb{E}\{Y\}$ (since $\mathbb{E}\{Y^2\} > 0$) and hence we arrive at a contradiction. The proof is complete. ■

With Lemma 15 and (47), we have

$$\gamma^*_{\text{NWAC}} = 3(\beta^*_{\text{NWAC}} - \mathbb{E}\{Y\}) > 0. \tag{168}$$

Recall that β^*_{NWAC} is a root of $\mathbb{E}_Q\{\phi_{\text{NWAC}}(\beta, Q)\}$ and $Q \sim$

$W(Y)$. With (168), γ^*_{NWAC} is then a root of

$$\begin{aligned}
&\mathbb{E}_Q\left\{\phi_{\text{NWAC}}(\frac{\gamma}{3}+\mathbb{E}\{Y\},Q)\right\}\\
&=\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\left(-\frac{1}{6}Q^4+\left(\frac{\gamma}{3}+\mathbb{E}\{Y\}\right)Q^2-\frac{1}{6}\gamma^2\right)\right\}\\
&\quad+\mathbb{E}\left\{\mathbb{1}_{\{Q^2\geq\gamma\}}\cdot\left(\mathbb{E}\{Y\}Q^2\right)\right\}\\
&\quad+\frac{1}{2}\mathbb{E}\{Y^2\}-(\frac{\gamma}{3}+\mathbb{E}\{Y\})(\mathbb{E}\{Y\}+\mathbb{E}\{\tilde{V}\})+c \qquad (169)\\
&=\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\left((\frac{\gamma}{3})Q^2-\frac{1}{6}\gamma^2\right)\right\}\\
&\quad+\mathbb{E}\left\{\mathbb{1}_{\{Q^2\geq\gamma\}}\cdot\left(\frac{1}{6}Q^4\right)\right\}\\
&\quad+\mathbb{E}\{Y\}\mathbb{E}\{Q^2\}-(\frac{\gamma}{3}+\mathbb{E}\{Y\})(\mathbb{E}\{Y\}+\mathbb{E}\{\tilde{V}\})+c \qquad (170)\\
&=\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\left((\frac{\gamma}{3})Q^2-\frac{1}{6}\gamma^2\right)\right\}\\
&\quad+\mathbb{E}\left\{\mathbb{1}_{\{Q^2\geq\gamma\}}\cdot\left(\frac{1}{6}Q^4\right)\right\}\\
&\quad-\frac{\gamma}{3}\mathbb{E}\{Y\}-(\frac{\gamma}{3}+\mathbb{E}\{Y\})(\mathbb{E}\{\tilde{V}\})+c \qquad (171)\\
&=\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\left(-\frac{1}{6}\gamma^2\right)\right\}\\
&\quad+\mathbb{E}\left\{\mathbb{1}_{\{Q^2\geq\gamma\}}\cdot\left(\frac{1}{6}Q^4-\frac{\gamma}{3}Q^2\right)\right\}\\
&\quad-\frac{\gamma}{3}\mathbb{E}\{\tilde{V}\}+\frac{\mathbb{E}\{(\tilde{V})^2\}}{2}=0 \qquad (172)
\end{aligned}$$

where (169) follows from (46), (32), (38) and (168) since the convex hull of $h_{\beta^*_{\text{NWAC}}}(s)$ contains two separate pieces in (32); (170) follows from Wald's lemma $\mathbb{E}\{(W(Y))^4\}=3\mathbb{E}\{Y^2\}$ (and thus $\frac{1}{6}\mathbb{E}\{Q^4\}=\frac{1}{2}\mathbb{E}\{Y^2\}$) [26] and the fact that the first two terms in (169) both have $\mathbb{E}\{Y\}Q^2$; (171) follows from Wald's lemma $\mathbb{E}\{Q^2\}=\mathbb{E}\{(W(Y))^2\}=\mathbb{E}\{Y\}$ [26]; (172) further simplifies (171) by regrouping different terms led by the indicator functions and using $c\triangleq\mathbb{E}\{Y\}\mathbb{E}\{\tilde{V}\}+\frac{\mathbb{E}\{(\tilde{V})^2\}}{2}$ and $\mathbb{E}\{Q^2\}=\mathbb{E}\{Y\}$.

Proceeding from (172), we obtain

$$\begin{aligned}
&\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\gamma^2\right\}\\
&=\mathbb{E}\left\{\mathbb{1}_{\{Q^2\geq\gamma\}}\cdot Q^4\right\}-\mathbb{E}\left\{\mathbb{1}_{\{Q^2\geq\gamma\}}\cdot 2\gamma Q^2\right\}\\
&\quad-2\gamma\mathbb{E}\{\tilde{V}\}+3\mathbb{E}\{(\tilde{V})^2\} \qquad (173)
\end{aligned}$$

which is a direct result of (172). Note that we also have

$$\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\gamma^2\right\}=2\gamma\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\gamma\right\}-\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\gamma^2\right\}. \qquad (174)$$

Eqs. (173) and (174) jointly imply

$$\begin{aligned}
&\frac{\mathbb{E}\left\{\mathbb{1}_{\{Q^2\geq\gamma\}}\cdot Q^4\right\}+\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\gamma^2\right\}}{2\gamma}\\
&=\mathbb{E}\left\{\mathbb{1}_{\{Q^2\geq\gamma\}}\cdot Q^2\right\}+\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\gamma\right\}\\
&\quad+\mathbb{E}\{\tilde{V}\}-\frac{3\mathbb{E}\{(\tilde{V})^2\}}{2\gamma}. \qquad (175)
\end{aligned}$$

Finally we note that since $Q=W(Y)$, we have

$$\begin{aligned}
&\mathbb{E}\left\{\mathbb{1}_{\{Q^2\geq\gamma\}}\cdot Q^4\right\}+\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\gamma^2\right\}\\
&=\mathbb{E}\left\{\max(\gamma^2,(W(Y))^4)\right\} \qquad (176)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left\{\mathbb{1}_{\{Q^2\geq\gamma\}}\cdot Q^2\right\}+\mathbb{E}\left\{\mathbb{1}_{\{Q^2<\gamma\}}\cdot\gamma\right\}\\
&=\mathbb{E}\left\{\max(\gamma,(W(Y))^2)\right\}. \qquad (177)
\end{aligned}$$

Eq. (175) is thus equivalent to (48) in Lemma 4. The proof is complete.

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