Maximum principle and numerical method for the multi-term time-space Riesz-Caputo fractional differential equations

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Abstract

The maximum principle for the space and time-space fractional partial differential equations is still an open problem. In this paper, we consider a multi-term time-space Riesz-Caputo fractional differential equations over an open bounded domain. A maximum principle for the equation is proved. The uniqueness and continuous dependence of the solution are derived. Using a fractional predictor-corrector method combining the L1 and L2 discrete schemes, we present a numerical method for the specified equation. Two examples are given to illustrate the obtained results.

Keywords: multi-term time-space fractional differential equation; Riesz-Caputo fractional derivative; maximum principle; predictor-corrector method; L1/L2-approximation method
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1. Introduction

One of the most useful and best known tools employed in the study of ordinary and partial differential equations is the maximum principle. It enables to obtain information about their solutions without explicit knowledge of the solutions themselves [1].

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Recently, with the development of fractional differential equations (FDEs), the maximum principles for FDEs have started to draw attention. Since fractional derivatives are non-local operators, they have been used to model processes with non-local dynamics and display a certain degree of memory. This motivates us to consider the maximum principle and numerical method for the multi-term time-space Riesz-Caputo fractional differential equations. In [3], Nieto presented two new maximum principles for a linear fractional ordinary differential equation with initial or periodic boundary conditions. Luchko [4] proved a maximum principle for the generalized time-fractional diffusion equation over an open bounded domain, based on an extremum principle for the Caputo-Dzherbashyan fractional derivative. The maximum principle was then applied to show some uniqueness and existence results for the initial-boundary-value problem for the generalized time-fractional diffusion equation [5]. He also investigated the initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation [6] and the diffusion equation of distributed order [7], and obtained some existence results for the generalized solutions in [6-7]. For the one dimensional time-fractional diffusion equation, the generalized solution to the initial-boundary-value problem with Dirichlet boundary conditions was shown to be a solution in the classical sense [8].

However, all of the aforementioned maximum principles were restricted to time-fractional equations. In fact, many practical problems involve space-fractional derivatives, in particular, Riesz space fractional derivatives. In Luchko's survey paper [9] and recent research paper [8], he proposed an important and interesting problem: extend the maximum principle to the space and time-space fractional partial differential equations.

There are several different definitions to the space-fractional derivatives, such as space-fractional Laplacian operation [10], Riesz space-fractional derivatives [11], etc. The Riesz fractional derivatives allow the modeling of flow regime impacts from either side of the domain [11]. Benson, Wheatcraft and Meerschaert [12, 13] studied the fractional advection-dispersion equation with the Riesz fractional derivatives for solute transport in a subsurface material. Also, there were some interesting developments concerning the analytical solutions and numerical methods of fractional partial differential equations with the Riesz space fractional derivatives on a finite domain [14-16]. Note that all the Riesz fractional derivatives in [11-16] are in the framework of the Riemann-Liouville fractional derivative.

As Pandey et al. [17] pointed out, Caputo definition can avoid (i) mass

balance error, (ii) hyper-singular improper integral, (iii) non-zero derivative of constant, and (iv) fractional derivative involved in the initial condition which is often ill-posed. In contrast to the Riemann-Liouville fractional derivative, the Caputo fractional derivative was shown to possess a suitable generalization of the extremum principle [9]. Therefore, in this paper, the space-fractional derivative is defined as the Riesz fractional derivative, in the framework of the Caputo sense, also known as Riesz-Caputo fractional derivative [18]. The time-fractional derivative is defined as Caputo fractional derivative including the multi-term fractional derivative.

The main contributions of this paper are summarized as follows. We consider the multi-term time-space Riesz-Caputo fractional differential equations over an open bounded domain. Notice that the maximum principle developed by Luchko [4-9] can only be applied to time-fractional equations, where the derivative with respect to the spatial variable is defined as integer order. First, we extend Luchko's maximum principle for the case of the multi-term time-space Riesz-Caputo fractional differential equations. Then, applying the maximum principle, we derive a priori estimate for the solutions of initial-boundary-value problems for the equation. We prove the initial-boundary-value problem for the equation possessing at most one classical solution and the solution - if exists - continuously depends on the initial and boundary condition. Furthermore, we present a numerical method for the multi-term time-space fractional differential equations. Two examples for fractional differential equations are given and the results are compared with the exact solutions.

The rest of the paper is organized as follows. Section 2 shows the maximum principle. In Section 3, we derive the uniqueness and continuous dependence of the solutions. The numerical method and two examples are given in Section 4 and Section 5, respectively. The conclusions are summarized in Section 6.

2. Maximum principle

In this section, the maximum principle developed by Luchko [4] is extended for the multi-term time-space Riesz-Caputo fractional differential equations (MT-TSRC-FDE) over an open bounded domain $\Omega = (0, L) \times (0, T)$.

Consider the following MT-TSRC-FDE

$$P(D_t)u(x,t) = p(x)R_x^\beta u(x,t) + q(x)R_x^\gamma u(x,t) - h(x)u(x,t) + F(x,t) \quad (1)$$

in an open bounded domain $0 < x < L, 0 < t < T$. Here x and t are the space and time variables, $p, q, h \in C^1[0, L]$, $0 < p(x), 0 \le q(x), 0 \le h(x), x \in [0, L]$, and $1 < \beta \le 2, 0 < \gamma < 1$.

The operator $P(D_t)$ is defined as

$$P(D_t) = {}_0^c D_t^\alpha + \sum_{i=1}^{m} \lambda_i \, {}_0^c D_t^{\alpha_i}, \tag{2}$$

where $0 < \alpha_m < \cdots < \alpha_1 < \alpha \le 1, 0 \le \lambda_i, i = 1, \cdots, m, m \in N, {}_0^c D_t^{\alpha_i}$ is a left-side Caputo fractional derivative of order α_i with respect to t, which is defined as

$$
{}_0^c D_t^{\alpha_i} f(t) = \begin{cases} \frac{1}{\Gamma(1-\alpha_i)} \int_0^t (t - \tau)^{-\alpha_i} f'(\tau) d\tau, & 0 < \alpha_i < 1, \\ f'(t) & , \quad \alpha_i = 1. \end{cases} \tag{3}
$$

The space fractional derivatives R_x^β and R_x^γ are Riesz-Caputo fractional derivatives of order β, γ, respectively. They are defined by Definition 1.

Definition 1. *(See [18]) The Riesz-Caputo fractional operator for $n \in \mathbf{N}$, $n - 1 < \beta \le n$, on a finite interval $0 \le x \le L$ is defined as*

$$R_x^\beta u(x, t) = -c_\beta ({}_0^c D_x^\beta + {}_x^c D_L^\beta) u(x, t), \tag{4}$$

where the coefficient $c_\beta = \frac{1}{2\cos(\frac{\beta\pi}{2})}, \beta \ne 1, , 3, 5, \cdots$ and

$$
{}_0^c D_x^\beta u(x, t) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^x (x - \xi)^{n-\beta-1} \frac{\partial^n}{\partial \xi^n} u(\xi, t) d\xi, & n - 1 < \beta < n, \\ \frac{\partial^n}{\partial x^n} u(x, t) & , \quad \beta = n, \end{cases} \tag{5}
$$

$$
{}_x^c D_L^\beta u(x, t) = \begin{cases} \frac{(-1)^n}{\Gamma(n-\beta)} \int_x^L (\xi - x)^{n-\beta-1} \frac{\partial^n}{\partial \xi^n} u(\xi, t) d\xi, & n - 1 < \beta < n, \\ (-1)^n \frac{\partial^n}{\partial x^n} u(x, t) & , \quad \beta = n, \end{cases} \tag{6}
$$

are the left-side and right-side Caputo fractional derivatives, respectively. In particular, when $\beta = 2, R_x^\beta u(x, t) = \frac{\partial^2}{\partial x^2} u(x, t)$.

In this paper, the initial-boundary-value problem

$$u|_{t=0} = \phi(x), \quad 0 \le x \le L, \tag{7}$$

$$u(0, t) = \mu_1(t), \quad u(L, t) = \mu_2(t), \qquad 0 \le t \le T \tag{8}$$

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for Eq. (1) is considered.

Let $\bar{\Omega} = [0, L] \times [0, T]$ and

$$G(\Omega) = \{u(x,t)|\ \frac{\partial^2 u}{\partial x^2} \in C(\bar{\Omega})\ \ and\ \ \frac{\partial u}{\partial t} \in C(\bar{\Omega})\}.$$

We introduce the notion of the classical solution of the problem (1), (7)-(8).

Definition 2. *A function u is said to be a classical solution of the problem (1), (7), (8) if u is defined in the domain $\bar{\Omega}$ that belongs to the space $G(\Omega)$ and satisfies both Eq.(1) and the initial and boundary conditions (7)-(8).*

If a classical solution to the initial-boundary-value (7), (8) for Eq.(1) exists, then the functions $F, \phi(x)$ and $\mu_i, i = 1, 2$ given in the problem have to belong to the space $C(\Omega), C([0, L]), C([0, T])$, respectively. In the further discussions, we always suppose these inclusions to be valid.

For our maximum principle, we make use of the following extremum principles for the Caputo and Riesz-Caputo fractional derivatives, respectively.

Lemma 1. *(See [4].) Let a function $f \in C^1([0, T])$ attain its maximum over the interval $[0, T]$ at the point $\tau = t_0, t_0 \in (0, T]$. Then the left-side Caputo fractional derivatives of the function f is non-negative at the point t_0 for any $\alpha, 0 < \alpha \le 1$. That is,*

$$0 \le {}^c_0 D^\alpha_{t_0} f(t_0), \quad 0 < \alpha \le 1. \tag{9}$$

Lemma 2. *Let a function $f \in C^1([0, L])$ attain its maximum over the interval $[0, L]$ at the point $x_0, x_0 \in (0, L)$. Then the right-side Caputo fractional derivatives of the function f is non-negative at the point x_0 for any $\gamma, 0 < \gamma \le 1$. That is,*

$$0 \le {}^c_{x_0} D^\gamma_L f(x_0), \quad 0 < \gamma \le 1. \tag{10}$$

Furthermore, the Riesz-Caputo fractional derivatives of the function f is non-positive at the point x_0 for any $\gamma, 0 < \gamma < 1$. That is,

$$R^\gamma f(x_0) \le 0, \quad 0 < \gamma < 1. \tag{11}$$

PROOF. Since

$$R^\gamma f(x_0) = -C_\gamma ({}^c_0 D^\gamma_{x_0} + {}^c_{x_0} D^\gamma_L) f(x_0), \quad 0 < \gamma < 1,$$

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where $C_\gamma = \frac{1}{2cos(\frac{\gamma\pi}{2})} > 0$, it remains only to show the inequality (10). If $\gamma = 1$, it is clear that the inequality (10) is true. Consider the case $0 < \gamma < 1$. Following the idea in [4-5], we introduce an auxiliary function

$$g(x) = f(x_0) - f(x), \quad x \in [0, L],$$

that implies $g(x) \geq 0$, $x \in [0, L]$ and $^c_x D^\gamma_L g(x) = -^c_x D^\gamma_L f(x)$, $x \in [0, L]$. Therefore, inequality (10) will be verified if we show that

$$^c_{x_0} D^\gamma_L g(x_0) \leq 0, \quad 0 < \gamma < 1.$$

Notice that

$$
\begin{aligned}
^c_{x_0} D^\gamma_L g(x_0) &= -\frac{1}{\Gamma(1-\gamma)} \int_{x_0}^{L} (\xi - x_0)^{-\gamma} g'(\xi) d\xi \\
&= -\frac{1}{\Gamma(1-\gamma)} \int_{x_0}^{L-\varepsilon} (\xi - x_0)^{-\gamma} g'(\xi) d\xi - \frac{1}{\Gamma(1-\gamma)} \int_{L-\varepsilon}^{L} (\xi - x_0)^{-\gamma} g'(\xi) d\xi \\
&= I_1 + I_2
\end{aligned}
$$

is valid for any fixed value of $\varepsilon, 0 < \varepsilon < L - x_0$.

Since $f \in C^1([0, L])$, we have $g' \in L([0, L])$. It follows that $\forall \delta > 0$, $\exists \varepsilon > 0$ such that $|I_2| \leq \delta$. Also

$$
\begin{aligned}
I_1 &= -\frac{1}{\Gamma(1-\gamma)} \int_{x_0}^{L-\varepsilon} (\xi - x_0)^{-\gamma} g'(\xi) d\xi \\
&= -\frac{1}{\Gamma(1-\gamma)} (L - \varepsilon - x_0)^{-\gamma} g(L - \varepsilon) - \frac{\gamma}{\Gamma(1-\gamma)} \int_{x_0}^{L-\varepsilon} (\xi - x_0)^{-\gamma-1} g(\xi) d\xi \\
&\leq 0.
\end{aligned}
$$

Therefore, we obtain $^c_{x_0} D^\gamma_L g(x_0) \leq 0$, $0 < \gamma < 1$ and the lemma is proved.

Lemma 3. *(See [19].) Let a function $f \in C^2([0, L])$ attain its maximum over the interval $[0, L]$ at the point $x_0, x_0 \in (0, L)$, and $f'(0) \geq 0$. Then the left-side Caputo fractional derivatives of the function f is non-positive at the point x_0 for any $\beta, 1 < \beta < 2$. That is,*

$$^c_0 D^\beta_{x_0} f(x_0) \leq 0, \quad 1 < \beta < 2. \tag{12}$$

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Lemma 4. *Let a function $f \in C^2([0, L])$ attain its maximum over the interval $[0, L]$ at the point $x_0, x_0 \in (0, L)$. If $f'(0) \geq 0$ and $f'(L) \leq 0$, then the Riesz-Caputo derivative of the function f is non-positive at the point x_0 for any $\beta, 1 < \beta \leq 2$:*

$$R^\beta f(x_0) \leq 0, \quad 1 < \beta \leq 2. \tag{13}$$

PROOF. The case $\beta = 2$ is obvious, so we suppose $1 < \beta < 2$.

By Definition 1,

$$R^\beta f(x) = -c_\beta ({}_0^c D_x^\beta + {}_x^c D_L^\beta) f(x), \tag{14}$$

where the coefficient $c_\beta = \frac{1}{2cos(\frac{\beta\pi}{2})} < 0$. By Lemma 3,

$${}_0^c D_{x_0}^\beta f(x_0) \leq 0, \quad 1 < \beta < 2.$$

By modifying the arguments used in [19], we obtain

$${}_{x_0}^c D_L^\beta f(x_0) \leq 0, \qquad x_0 \in (0, L), \quad 1 < \beta < 2. \tag{15}$$

Therefore the assertion for the lemma is true and the proof is complete.

We are now in a position to prove a maximum principle.

Theorem 1. *Let a function $u \in G(\Omega)$ be a solution of the multi-term time-space Riesz-Caputo fractional differential equations (1) in the domain Ω and $F(x, t) \leq 0, (x, t) \in \Omega$. Suppose $\frac{\partial u}{\partial x}|_{x=0} \geq 0$ and $\frac{\partial u}{\partial x}|_{x=L} \leq 0$. Then either $u(x, t) \leq 0, (x, t) \in \bar{\Omega}$ or the function u attains its positive maximum on the bottom or back-side parts $S = ([0, L] \times \{0\}) \cup (\{0\} \times [0, T]) \cup (\{L\} \times [0, T])$ of the boundary of the domain Ω, i.e.,*

$$u(x, t) \leq \max\{0, \max_{0 \leq x \leq L} \phi(x), \max_{0 \leq t \leq T} \mu_1(t), \max_{0 \leq t \leq T} \mu_2(t)\}, \quad \forall (x, t) \in \bar{\Omega}. \tag{16}$$

PROOF. Suppose the assertion of the theorem is false; that is, there is a point $(x_0, t_0), x_0 \in (0, L), 0 < t_0 \leq T$ such that

$$u(x_0, t_0) > \max\{0, \max_{0 \leq x \leq L} \phi(x), \max_{0 \leq t \leq T} \mu_1(t), \max_{0 \leq t \leq T} \mu_2(t)\} = M \geq 0.$$

Let us define the number $\varepsilon = u(x_0, t_0) - M > 0$ and introduce the auxiliary function

$$w(x, t) = u(x, t) + \frac{\varepsilon}{2} \frac{T - t}{T}, \quad (x, t) \in \bar{\Omega}.$$

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Using the definition of the function w and the conditions of the theorem, we obtain

$$w(x,t) \le u(x,t) + \frac{\varepsilon}{2}, \quad (x,t) \in \bar{\Omega},$$

$$w(x_0,t_0) \ge u(x_0,t_0) = \varepsilon + M \ge \varepsilon + u(x,t) \ge \frac{\varepsilon}{2} + w(x,t), \quad (x,t) \in S.$$

The latter property implies that the function w cannot attain its maximum on S. Let (x_1,t_1) be the maximum point of the function w over the domain $\bar{\Omega}$. Then $x_1 \in (0,L), 0 < t_1 \le T$ and

$$w(x_1,t_1) \ge w(x_0,t_0) \ge \varepsilon + M > \varepsilon.$$

A direct application of Lemma 1, Lemma 2 and Lemma 4 yields

$$\begin{cases} {}^c_0D_t^\alpha w \mid_{(x_1,t_1)} \ge 0 & , \quad 0 < \alpha \le 1, \\ R_x^\gamma w \mid_{(x_1,t_1)} \le 0, & , \quad 0 < \gamma < 1, \\ R_x^\beta w \mid_{(x_1,t_1)} \le 0, & , \quad 1 < \beta \le 2. \end{cases} \tag{17}$$

By the definitions of the function w and properties of the Caputo fractional derivative, we have

$$P(D_t)u = P(D_t)w + \frac{\varepsilon}{2T}\left(\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \sum_{i=1}^m \lambda_i \frac{t^{1-\alpha_i}}{\Gamma(2-\alpha_i)}\right)$$

where $\lambda_i \ge 0, i = 1, \cdots, m$. Thus, it is easily seen that

$$\begin{aligned} &\left(P(D_t)u - p(R_x^\beta u) - q(R_x^\gamma u) + hu - F\right)\mid_{(x_1,t_1)} \\ =\ & P(D_t)w|_{(x_1,t_1)} + \frac{\varepsilon}{2T}\left(\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} + \sum_{i=1}^m \lambda_i \frac{t_1^{1-\alpha_i}}{\Gamma(2-\alpha_i)}\right) - p(x_1)R_x^\beta w|_{(x_1,t_1)} \\ & -q(x_1)R_x^\gamma w|_{(x_1,t_1)} + h(x_1)\left(w(x_1,t_1) - \frac{\varepsilon}{2}\frac{T-t_1}{T}\right) - F(x_1,t_1) \\ \ge\ & \frac{\varepsilon}{2T}\left(\frac{t_1^{1-\alpha}}{\Gamma(2-\alpha)} + \sum_{i=1}^m \lambda_i \frac{t_1^{1-\alpha_i}}{\Gamma(2-\alpha_i)}\right) + h(x_1)\varepsilon\left(1 - \frac{T-t_1}{2T}\right) \\ >\ & 0, \end{aligned}$$

which is a contradiction. The proof of the theorem is therefore complete.

Let us substitute $-u$ instead of u in the above theorem, the minimum principle can be established as follows.

Theorem 2. *Let a function $u \in G(\Omega)$ be a solution of the multi-term time-space Riesz-Caputo fractional differential equations (1) in the domain Ω and $F(x,t) \geq 0, (x,t) \in \Omega$. Suppose $\frac{\partial u}{\partial x}|_{x=0} \leq 0$ and $\frac{\partial u}{\partial x}|_{x=L} \geq 0$. Then either $u(x,t) \geq 0, (x,t) \in \bar{\Omega}$ or the function u attains its negative minimum on the bottom or back-side parts $S = ([0,L] \times \{0\}) \cup (\{0\} \times [0,T]) \cup (\{L\} \times [0,T])$ of the boundary of the domain Ω, i.e.,*

$$u(x,t) \geq \min\{0, \min_{0 \leq x \leq L} \phi(x), \min_{0 \leq t \leq T} \mu_1(t), \min_{0 \leq t \leq T} \mu_2(t)\}, \quad \forall (x,t) \in \bar{\Omega}. \quad (18)$$

3. Uniqueness and continuous dependence of the solution

First, we derive an estimate on how the solution $u(x,t)$ of Eq.(1),(7)-(8) depends on ϕ, μ_1, μ_2 and F by applying the maximum principle and the minimum principle.

Theorem 3. *Let u is a classical solution of the problem (1),(7)-(8) and F belongs to the space $C(\bar{\Omega})$. Suppose $\frac{\partial u}{\partial x}|_{x=0} = 0$ and $\frac{\partial u}{\partial x}|_{x=L} = 0$. Then*

$$\|u\|_{C(\bar{\Omega})} \leq \max\{M_0, M_1, M_2\} + 2\frac{T^\alpha}{\Gamma(1+\alpha)}M, \quad (19)$$

where $M_0 = \|\phi\|_{C(\bar{G})}, \ M_i = \|\mu_i\|_{C([0,T])}, (i=1,2), \ M = \|F\|_{C(\bar{\Omega})}$.

PROOF. Following [4], we introduce an auxiliary function w:

$$w(x,t) = u(x,t) - \frac{M}{\Gamma(1+\alpha)}t^\alpha, \quad (x,t) \in \bar{\Omega}.$$

If u is a solution of the problem (1), (7)-(8), then w is easily shown to be a solution of the same problem with the functions

$$F_1(x,t) = F(x,t) - M - M\sum_{i=1}^{m} \lambda_i \frac{1}{\Gamma(1-\alpha+\alpha_i)}t^{\alpha-\alpha_i} - h(x)\frac{M}{\Gamma(1+\alpha)}t^\alpha,$$

$$v_i(t) = \mu_i(t) - \frac{M}{\Gamma(1+\alpha)}t^\alpha, \quad i = 1,2,$$

instead of F and $\mu_i, i = 1,2$, respectively. Since $F_1(x,t) \leq 0$, the maximum principle to the solution w implies the estimate

$$w(x,t) \leq \max\{M_0, M_1 + \frac{M}{\Gamma(1+\alpha)}T^\alpha, M_2 + \frac{M}{\Gamma(1+\alpha)}T^\alpha\}.$$

Therefore,

$$u(x,t) \leq \max\{M_0, M_1, M_2\} + 2\frac{T^\alpha}{\Gamma(1+\alpha)}M, \quad (x,t) \in \bar{\Omega}.$$

In the same way, defining the auxiliary function as

$$w(x,t) = u(x,t) + \frac{M}{\Gamma(1+\alpha)}t^\alpha, \quad (x,t) \in \bar{\Omega},$$

and applying the minimum principle, we obtain the estimate

$$u(x,t) \geq -\max\{M_0, M_1, M_2\} - 2\frac{T^\alpha}{\Gamma(1+\alpha)}M, \quad (x,t) \in \bar{\Omega}.$$

This proves the estimate (19).

Next, we show the uniqueness and continuous dependence of the solution. Using the above results, we have

Theorem 4. *Suppose $\frac{\partial u}{\partial x}|_{x=0} = 0$ and $\frac{\partial u}{\partial x}|_{x=L} = 0$. Then the problem (1), (7)-(8) possesses at most one classical solution. This solution - if it exists - continuously depends on the data given in the problem in the sense that if*

$$\|F - \tilde{F}\|_{C(\bar{\Omega})} \leq \varepsilon, \quad \|\phi - \tilde{\phi}\|_{C(\bar{G})} \leq \varepsilon_0, \quad \|\mu_i - \tilde{\mu}_i\|_{C([0,T])} \leq \varepsilon_i, i = 1, 2,$$

then the estimate

$$\|u - \tilde{u}\|_{C(\bar{\Omega})} \leq \max\{\varepsilon_0, \varepsilon_1, \varepsilon_2\} + 2\frac{T^\alpha}{\Gamma(1+\alpha)}\varepsilon$$

for the corresponding classical solutions u and $\tilde{u}$ holds true.

4. Numerical method for MT-TSRC-FDE

There have existed many papers on numerical methods for multi-term fractional differential equations. For example, Diethelm and Ford [20] proposed convergent and stable numerical methods for multi-term fractional ordinary differential equations (FODEs). Garrappa [21] investigated the numerical approximation of linear multi-tesrm FODEs. Li et al. [22] studied the equivalent system for a multiple-rational-order fractional differential system. We also notice that an Adams-type predictor-corrector method is an

effective means to give numerical solutions for fractional ordinary differential equations (FODEs) [23-24]. It may be used both for linear and for nonlinear problems, and it may be extended to multi-term FODEs. Recently, Liu et al. [25] proposed a fractional predictor-corrector method for solving the multi-term time-fractional wave-diffusion equations. In this paper we use the method combining the L1 and L2 discrete schemes for the numerical solutions of the multi-term time-space Riesz-Caputo fractional differential equations (1), (7)-(8).

Firstly, we rewrite the given multi-term fractional differential equations (1), (7)-(8) in the form of a system of single-term equations (see [28]). Let $u(x,t) = z_1(x,t)$. Then

$$
\begin{cases}
{}_0^c D_t^{y_1} z_1(x,t) = {}_0^c D_t^{\alpha_m} z_1(x,t) = z_2(x,t), \\
{}_0^c D_t^{y_2} z_2(x,t) = {}_0^c D_t^{\alpha_{m-1}-\alpha_m} z_2(x,t) = z_3(x,t), \\
\vdots \\
{}_0^c D_t^{y_m} z_m(x,t) = {}_0^c D_t^{\alpha_1-\alpha_2} z_m(x,t) = z_{m+1}(x,t), \\
{}_0^c D_t^{y_{m+1}} z_{m+1}(x,t) = {}_0^c D_t^{\alpha-\alpha_1} z_{m+1}(x,t) \\
\qquad = p(x) R_x^\beta z_1(x,t) + q(x) R_x^\gamma z_1(x,t) - h(x) z_1(x,t) + F(x,t) \\
\qquad -(\lambda_1 z_{m+1} + \lambda_2 z_m + \cdots + \lambda_m z_2).
\end{cases}
\tag{20}
$$

These are subject to the initial conditions

$$
z_l(x,0) = \begin{cases} \phi(x) & , \quad if \ \ l = 1, \\ 0 & , \quad else. \end{cases}
\tag{21}
$$

We assume that we are working on a uniform grid $t_k = k\tau, \ k = 0, 1, \cdots, M$; $M\tau = T$; $x_i = ih, i = 0, 1, \cdots, N; Nh = L$. Let $z_l^{i,k+1}$ be the numerical approximation to $z_l(x_i, t_{k+1})$.

By Definition 1, for $1 < \beta < 2$,

$$
R_{x_i}^\beta z_1(x_i, t_j) = -\frac{1}{2\cos\frac{\beta\pi}{2}} ({}_0^c D_{x_i}^\beta + {}_{x_i}^c D_L^\beta) z_1(x_i, t_j).
\tag{22}
$$

Using the L2-algorithm proposed by [26, 27], we can obtain the following numerical discretization scheme for space-fractional derivatives:

$$
{}_0^c D_{x_i}^\beta z_1(x_i, t_j) = \frac{1}{\Gamma(2-\beta)} \int_0^{x_i} (x_i - \xi)^{1-\beta} \frac{\partial^2 z_1}{\partial \xi^2}(\xi, t_j) d\xi
$$

$$
\approx \frac{1}{\Gamma(2-\beta)} \sum_{\mu=0}^{i-1} \frac{z_1((\mu+2)h, t_j) - 2z_1((\mu+1)h, t_j) + z_1(\mu h, t_j)}{h^2}
$$

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$$\cdot \int_{\mu h}^{(\mu+1)h} (ih - \xi)^{1-\beta} d\xi$$

$$= \frac{h^{-\beta}}{\Gamma(3 - \beta)} \sum_{\mu=0}^{i-1} [z_1((i - \mu + 1)h, t_j) - 2z_1((i - \mu)h, t_j) + z_1(i - \mu - 1)h, t_j)]$$

$$\cdot \left[(\mu + 1)^{2-\beta} - \mu^{2-\beta} \right], \tag{23}$$

and

$$_{x_i}^{c} D_L^{\beta} z_1(x_i, t_j) = \frac{1}{\Gamma(2 - \beta)} \int_{x_i}^{L} (\xi - x_i)^{1-\beta} \frac{\partial^2 z_1}{\partial \xi^2}(\xi, t_j) d\xi$$

$$\approx \frac{h^{-\beta}}{\Gamma(3 - \beta)} \sum_{\mu=0}^{N-i-1} [z_1((\mu + i - 1)h, t_j) - 2z_1((\mu + i)h, t_j) + z_1((\mu + i + 1)h, t_j)]$$

$$\cdot \left[(\mu + 1)^{2-\beta} - \mu^{2-\beta} \right]. \tag{24}$$

It follows that

$$R_{x_i}^{\beta} z_1^{i,j} = -\frac{h^{-\beta}}{2 \cos \frac{\beta \pi}{2} \Gamma(3 - \beta)}$$

$$\cdot \{ \sum_{\mu=0}^{i-1} \left[z_1^{i-\mu+1,j} - 2z_1^{i-\mu,j} + z_1^{i-\mu-1,j} \right] \left[(\mu + 1)^{2-\beta} - \mu^{2-\beta} \right]$$

$$+ \sum_{\mu=0}^{N-i-1} \left[z_1^{\mu+i-1,j} - 2z_1^{\mu+i,j} + z_1^{\mu+i+1,j} \right] \left[(\mu + 1)^{2-\beta} - \mu^{2-\beta} \right] \}, \tag{25}$$

where $i = 1, 2, \cdots, N - 1$; $z_1^{0,j} = \mu_1(t_j)$; $z_1^{N,j} = \mu_2(t_j)$.

Note that if $\beta = 2$, then

$$R_{x_i}^{\beta} z_1^{i,j} = \frac{z_1^{i-1,j} - 2z_1^{i,j} + z_1^{i+1,j}}{h^2}. \tag{26}$$

Similarly, for $0 < \gamma < 1$, using the L1 discrete scheme in [26], we have

$$R_{x_i}^{\gamma} z_1^{i,j} = -\frac{h^{-\gamma}}{2 \cos \frac{\gamma \pi}{2} \Gamma(2 - \gamma)} \{ \sum_{\mu=0}^{i-1} \left[z_1^{i-\mu,j} - z_1^{i-\mu-1,j} \right] \left[(\mu + 1)^{1-\gamma} - \mu^{1-\gamma} \right]$$

$$+ \sum_{\mu=0}^{N-i-1} \left[z_1^{\mu+i,j} - z_1^{\mu+i+1,j} \right] \left[(\mu + 1)^{1-\gamma} - \mu^{1-\gamma} \right] \}, \tag{27}$$

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where $i = 1, 2, \cdots, N - 1$; $z_1^{0,j} = \mu_1(t_j)$; $z_1^{N,j} = \mu_2(t_j)$.

Next, we apply the fractional predictor-corrector method (see [25]). The predicted value $z_{l,P}^{i,k+1}$ is determined by the fractional Adams-Bashforth method $(l = 1, \cdots, m)$:

$$z_{l,P}^{i,k+1} = z_l^{i,0} + \frac{1}{\Gamma(y_l)} \sum_{j=0}^{k} b_{j,k+1}^{y_l} z_{l+1}^{i,j}, \tag{28}$$

$$z_{m+1,P}^{i,k+1} = \frac{1}{\Gamma(y_{m+1})} \sum_{j=0}^{k} b_{j,k+1}^{y_{m+1}} \{ p_i R_{x_i}^{\beta} z_1^{i,j} + q_i R_{x_i}^{\gamma} z_1^{i,j} - h_i z_1^{i,j} + F_{i,j}$$
$$- (\lambda_1 z_{m+1}^{i,j} + \lambda_2 z_m^{i,j} + \cdots + \lambda_m z_2^{i,j}) \}, \tag{29}$$

where

$$b_{j,k+1}^{y_l} = \frac{\tau^{y_l}}{y_l} [(k + 1 - j)^{y_l} - (k - j)^{y_l}]. \tag{30}$$

The corrector formula is given by the fractional Adams-Moulton method $(l = 1, \cdots, m)$:

$$z_l^{i,k+1} = z_l^{i,0} + \frac{1}{\Gamma(y_l)} \left(\sum_{j=0}^{k} a_{j,k+1}^{y_l} z_{l+1}^{i,j} + a_{k+1,k+1}^{y_l} z_{l+1,P}^{i,k+1} \right), \tag{31}$$

$$z_{m+1}^{i,k+1} = \frac{1}{\Gamma(y_{m+1})} \{ \sum_{j=0}^{k} a_{j,k+1}^{y_{m+1}} [p_i R_{x_i}^{\beta} z_1^{i,j} + q_i R_{x_i}^{\gamma} z_1^{i,j} - h_i z_1^{i,j} + F_{i,j}$$
$$- (\lambda_1 z_{m+1}^{i,j} + \lambda_2 z_m^{i,j} + \cdots + \lambda_m z_2^{i,j})]$$
$$+ a_{k+1,k+1}^{y_{m+1}} [p_i R_{x_i}^{\beta} z_{1,P}^{i,k+1} + q_i R_{x_i}^{\gamma} z_{1,P}^{i,k+1} - h_i z_{1,P}^{i,k+1} + F_{i,k+1}$$
$$- (\lambda_1 z_{m+1,P}^{i,k+1} + \lambda_2 z_{m,P}^{i,k+1} + \cdots + \lambda_m z_{2,P}^{i,k+1})] \}, \tag{32}$$

where

$$a_{j,k+1}^{y_l} = \frac{\tau^{y_l}}{y_l(y_l + 1)}$$
$$\times \begin{cases} k^{y_l+1} - (k - y_l)(k + 1)^{y_l} & , \quad j = 0, \\ (k - j + 2)^{y_l+1} + (k - j)^{y_l+1} - 2(k - j + 1)^{y_l+1} & , \quad 1 \leq j \leq k, \\ 1 & , \quad j = k + 1. \end{cases} \tag{33}$$

5. Numerical results

In this section, two examples are presented to illustrate the practical application of our numerical method.

Example 1. *Consider the two-term time-space Riesz-Caputo fractional diffusion equation:*

$$
{}_0^c D_t^\alpha u(x,t) + {}_0^c D_t^{\alpha/2} u(x,t) = R_x^\beta u(x,t) + F(x,t), \tag{34}
$$
$$
0 < \alpha \le 1,\ 1 < \beta < 2,\ 0 < t \le T,\ 0 < x < 1,
$$

with the boundary and initial conditions given by

$$
u(0,t) = 1 + t^2, \qquad u(1,t) = 0, \qquad 0 \le t \le T, \tag{35}
$$

$$
u(x,0) = 1 - x^2, \qquad 0 \le x \le 1, \tag{36}
$$

where

$$
\begin{aligned}
F(x,t) &= 2(1 - x^2) \left[\frac{t^{2-\alpha}}{\Gamma(3 - \alpha)} + \frac{t^{2-\alpha/2}}{\Gamma(3 - \alpha/2)} \right] \\
&\quad - \frac{1 + t^2}{\cos\frac{\beta\pi}{2}\Gamma(3 - \beta)} \left[x^{2-\beta} + (1 - x)^{2-\beta} \right].
\end{aligned} \tag{37}
$$

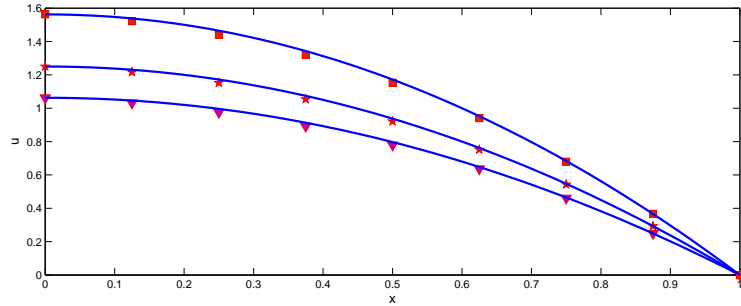*The exact solution of the equations (35)-(37) is $u(x,t) = (1 + t^2)(1 - x^2)$.*

Fig. 1. Exact solutions (lines) and numerical solutions (symbols) at t=0.25 (triangles), t=0.5 (stars) and t=0.75 (squares).

In Fig. 1, we compare the exact solution and the numerical solution obtained by using the numerical method described in Sec. 5 to solve the time-space Riesz-Caputo fractional diffusion equation for $\alpha = 0.95$ and $\beta = 1.25$. It is apparent that the solution $u(x,t)$ attains its positive maximum on the left-side of the boundary of the domain $\Omega = [0,1] \times [0,T]$ and the numerical solution is in good agreement with the exact solution.

Example 2. *Consider the following MT-TSRC-FDE:*

$$\frac{\partial u}{\partial t}(x,t) + {}_0^c D_t^\alpha u(x,t) = R_x^\beta u(x,t) + R_x^\gamma u(x,t) - u(x,t) + F(x,t), \qquad (38)$$

$$0 < x < 1, \qquad 0 < t < T,$$

with initial and boundary conditions

$$u(x,0) = x^2(1-x)^2, \qquad 0 \le x \le 1, \qquad (39)$$

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le T, \qquad (40)$$

where $0 < \alpha < 1, \ \beta = 2, \ 0 < \gamma < 1,$ *and*

$$
\begin{aligned}
F(x,t) \ = \ & -2x^2(1-x)^2 \left[t + \frac{1}{\Gamma(3-\alpha)} t^{2-\alpha} \right] \\
& -2(1-6x+6x^2)(1-t^2) + x^2(1-x)^2(1-t^2) \\
& + \frac{1}{2\cos(\frac{\gamma\pi}{2})}(1-t^2)\Big\{ \frac{2}{\Gamma(3-\gamma)}[x^{2-\gamma} + (1-x)^{2-\gamma}] \\
& - \frac{12}{\Gamma(4-\gamma)}[x^{3-\gamma} + (1-x)^{3-\gamma}] + \frac{24}{\Gamma(5-\gamma)}[x^{4-\gamma} + (1-x)^{4-\gamma}] \Big\}.
\end{aligned}
$$

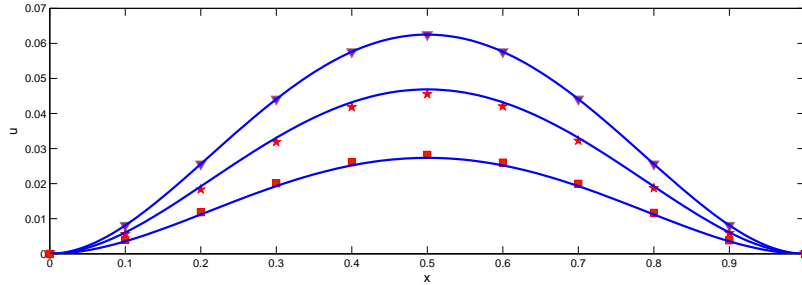*The exact solution of the equations (38)-(40) is*

$$u(x,t) = x^2(1-x)^2(1-t^2).$$

Fig.2. Exact solutions (lines) and numerical solutions (symbols) at t=0 (triangles), t=0.5 (stars) and t=0.75 (squares).

A comparison of the exact solution and the numerical solution for $\alpha = 0.5$, $\gamma = 0.3$ at $t = 0$ (triangles), $t = 0.5$ (stars) and $t = 0.75$ (squares) is shown in Fig. 2, respectively. From Fig. 2, it can be seen that the solution $u(x,t)$ attains its positive maximum on the bottom of the boundary of the domain $\Omega = [0,1] \times [0,T]$ and the numerical solution is in good agreement with the exact solution.

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6. Conclusions

In this paper, we have proved two extremum principles for the right-side Caputo fractional derivatives and Riesz-Caputo fractional derivatives as a generalization of the extremum principle in [4, 19]. Based on these extremum principles, a maximum principle is derived for the multi-term time-space Riesz-Caputo fractional differential equations over an open bounded domain. Also the uniqueness and continuous dependence of the solution are shown. Furthermore, the numerical method for the multi-term time-space fractional differential equations is presented and two examples are given to illustrate the obtained results. The methods discussed in this paper can also be extended for the n-dimensional problems.

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