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Spring 2020

## Lecture Notes: Multi-Objective Optimization

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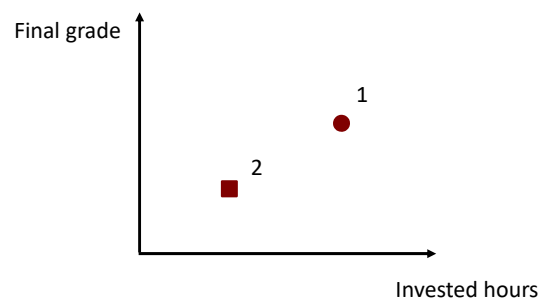


# Multi-objective optimization

Lecture 19  
Tuesday, April 14

CE385D Spring 2020 Lecture 19

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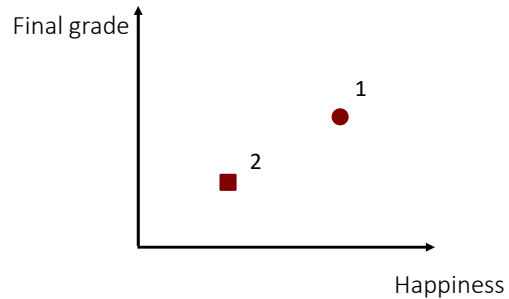


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## Multi-Objective Optimization (MOO)

- Involve **more than one objective** function that are to be minimized or maximized
- Objectives are **competing**
- Multi-objective analysis is used to reveal the **tradeoff** among different objectives
- Find the set of solutions that define the **best tradeoff** between competing objectives



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## Single-Objective Optimization

$$\begin{aligned} &\text{minimize} && f(x) \\ &\text{subject to} && g_i(x) \leq 0 \quad i = 1, \dots, n \\ &&& h_j(x) = 0 \quad j = 1, \dots, k \end{aligned}$$

- $f(x)$  **scalar objective function**
- $x_n$  vector of decision variables
- $g_i(x)$   $i = 1, \dots, k$  inequality constraint functions
- $h_j(x)$   $i = 1, \dots, m$  equality constraint functions

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## Multi-Objective Optimization

$$\begin{aligned}
 &\text{minimize} && f_1(x) \\
 &\text{maximize} && \vdots \\
 &\text{minimize} && f_m(x) \\
 &\text{subject to} && \\
 &&& g_i(x) \leq 0 \\
 &&& h_j(x) = 0
 \end{aligned}$$

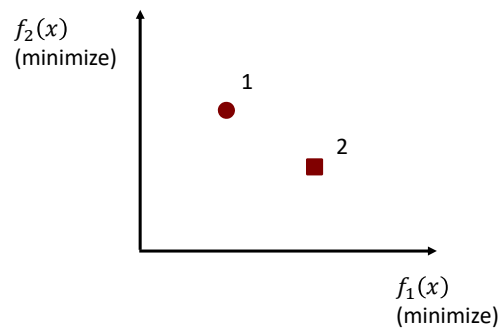
- $f(x)$ :  $M$  scalar objective **functions**
- $x_n$  vector of decision variables
- $g_i(x)$   $i = 1, \dots, k$  inequality constraint functions
- $h_j(x)$   $i = 1, \dots, m$  equality constraint functions

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## Dominance

- In **single-objective** optimization problem, the **superiority** of a solution over other solutions is easily determined by comparing their **objective function** values
- In **multi-objective** optimization, the **goodness** of a solution is determined by the **dominance**



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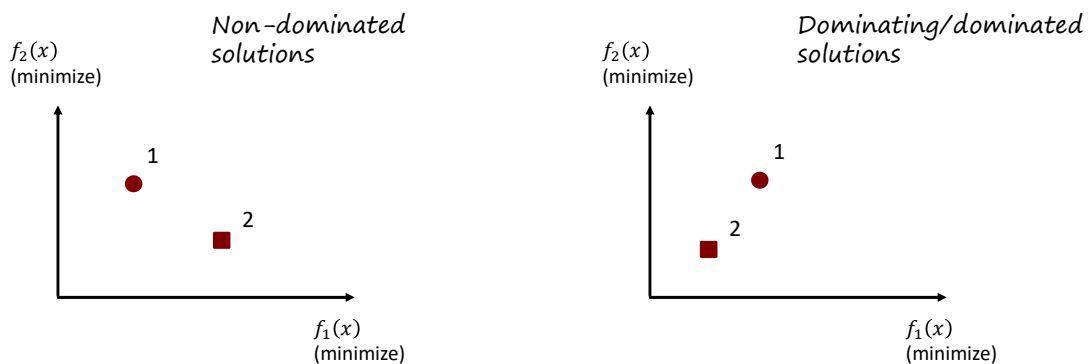
## Dominance test

- $x_1$  dominates  $x_2$ , if:
  - ✓ Solution  $x_1$  is **no worse** than  $x_2$  in **all** objectives
  - ✓ Solution  $x_1$  is **strictly better** than  $x_2$  in **at least one** objective
- If  $x_1$  **dominates**  $x_2$ , then  $x_2$  is **dominated** by  $x_1$
- If  $x_1$  does not dominate  $x_2$  and is not dominated by  $x_2$ , then  $x_1$  and  $x_2$  are **non-dominated** solutions

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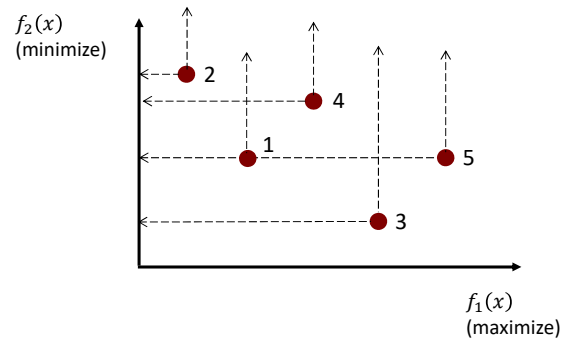
## Dominance test



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## Dominance test



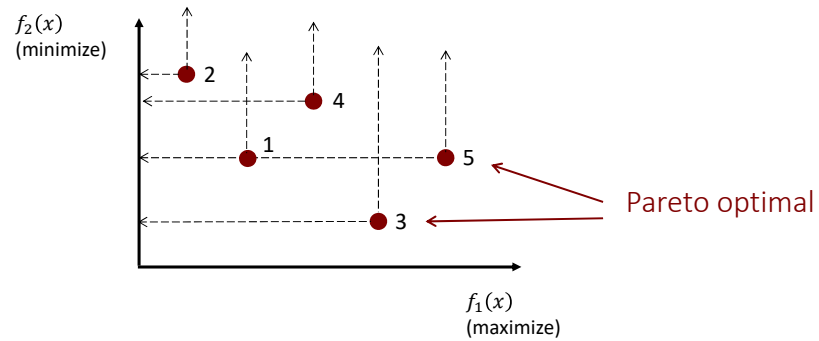
- 1 vs. 2: 1 dominates 2
- 1 vs. 5: 1 is dominated by 5
- 1 vs. 4: 1 & 4 are non-dominated solutions

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## Pareto optimal solutions

- Given a set of solutions, the **non-dominated solution set** is a set of all the solutions that are **not dominated** by any member of the solution set.
- The non-dominated set of the entire feasible decision space is called the **Pareto-optimal set**.
- The boundary defined by the set of all points mapped from the Pareto optimal set is called the **Pareto-optimal front**.

## Dominance test

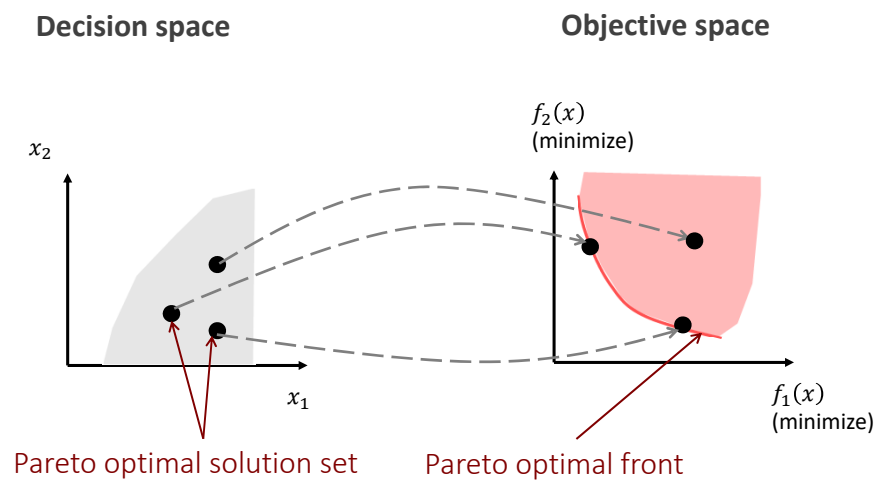


- 1 vs. 2: 1 dominates 2
- 1 vs. 5: 1 is dominated by 5
- 1 vs. 4: 1 & 4 are non-dominated solutions

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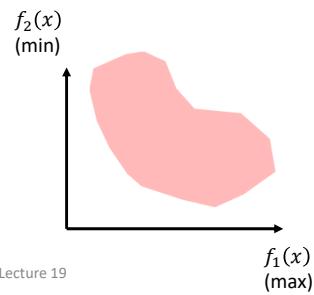
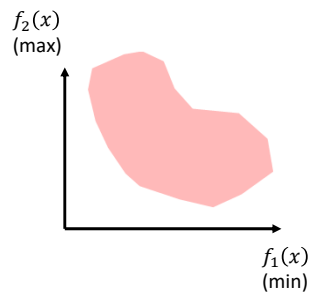
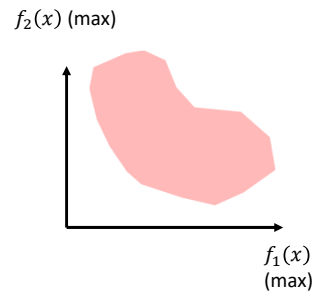
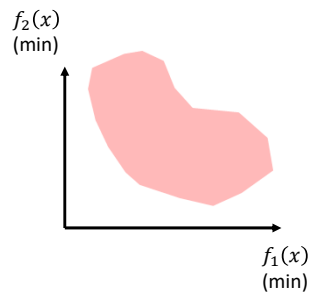
## Pareto optimal



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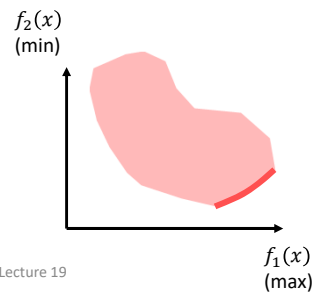
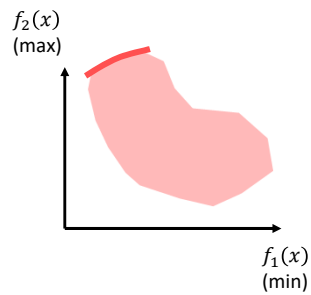
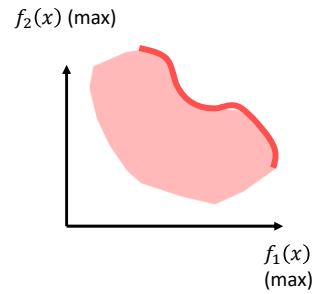
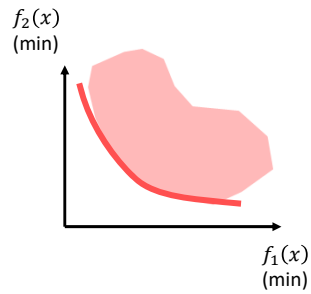
## Examples



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## Examples



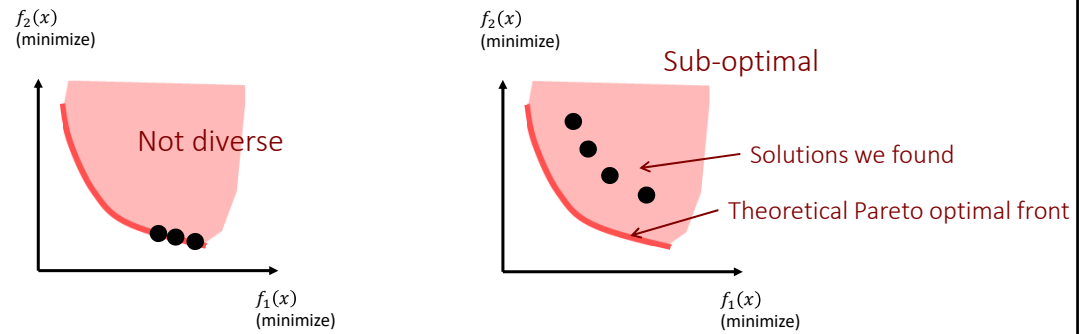
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## Goals of MOO

- Find a set of solutions as **close** as possible to Pareto-optimal front
- Find a set of solutions as **diverse** as possible



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## Classic MOO methods

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## Classic MOO methods

- **Weighted sum method**
- **$\varepsilon$ -Constraint method**
- Weighted metric method
- Rotated weighted metric method
- Dynamically changing the ideal solution
- Benson's method
- Value function method

For further reading:

- *Multi-objective optimization using evolutionary algorithms*. Kalyanmoy Deb.
- *Nonlinear multiobjective optimization*. Kaisa Miettinen.

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## Weighted sum method

- Scalarize a set of objectives into a **single** objective by adding each objective multiplied by a user-specified **weight**
- Weight of an objective is chosen in proportion to the relative importance of the objective

$$\begin{array}{l}
 \text{minimize } f_1(x) \\
 \text{maximize } \vdots \\
 \text{minimize } f_m(x)
 \end{array}
 \left. \vphantom{\begin{array}{l} \text{minimize } f_1(x) \\ \text{maximize } \vdots \\ \text{minimize } f_m(x) \end{array}} \right\}
 \quad \text{minimize } w_1 f_1(x) + w_2 f_2(x) + \dots + w_m f_m(x)$$

## Weighted sum method

$$\begin{aligned} \text{minimize} \quad & F(x) = \sum_{m=1}^M w_m f_m(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \quad i = 1, \dots, n \\ & h_j(x) = 0 \quad j = 1, \dots, k \end{aligned}$$

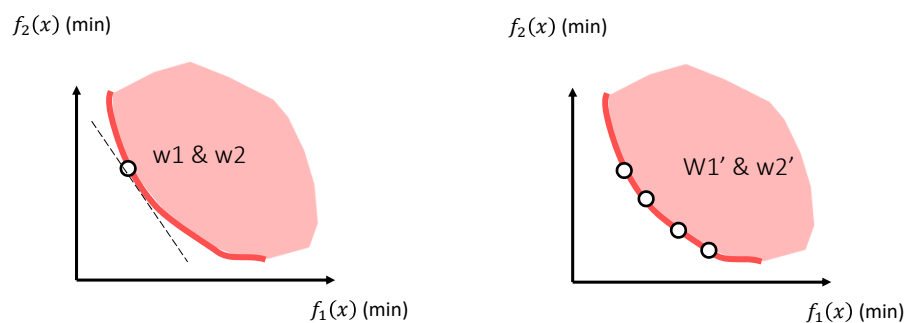
- convert all the objectives into one type
- the objectives are normalized
- $w_m \in [0,1]$  is the weight of the  $m$ -th objective function
- it is usual practice to choose weights such that:  $\sum w_m = 1$
- **Nonlinear** problems are **sensitive** to scale

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## Weighted sum method

$$\begin{aligned} \text{minimize} \quad & w_1 f_1(x) + w_2 f_2(x) \\ \text{subject to} \quad & g_i(x) \leq 0 \\ & h_j(x) = 0 \end{aligned}$$



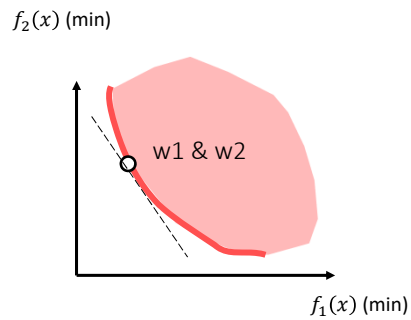
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## Weighted sum method

$$\begin{aligned} &\text{minimize} && w_1 f_1(x) + w_2 f_2(x) \\ &\text{subject to} && g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$

$$\begin{aligned} \text{Example:} & & f_1(x) = 5x_1 + \exp(x_2) & & f_2(x) = -3x_1 \\ & & & & \Downarrow \\ & & F(x) = w_1(5x_1 + \exp(x_2)) + w_2 \cdot -3x_1 \end{aligned}$$



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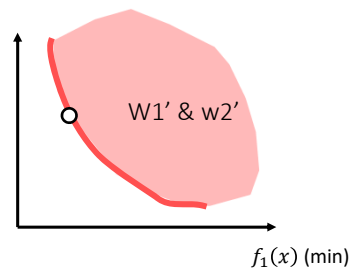
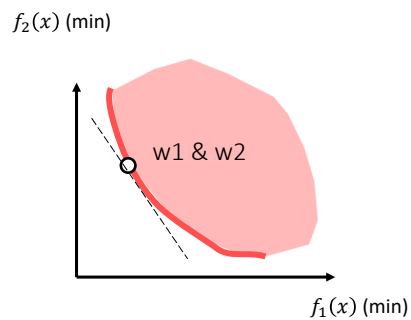
## Weighted sum method

$$\begin{aligned} &\text{minimize} && w_1 f_1(x) + w_2 f_2(x) \\ &\text{subject to} && g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$

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Choose weights:

$$w_1 = 0.3 \quad w_2 = 0.7 \quad F(x) = -0.6x_1 + 0.3\exp(x_2)$$



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## Weighted sum method

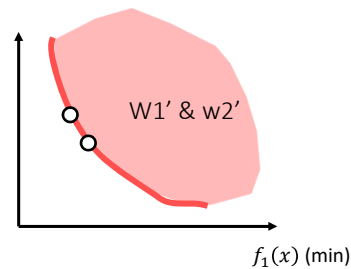
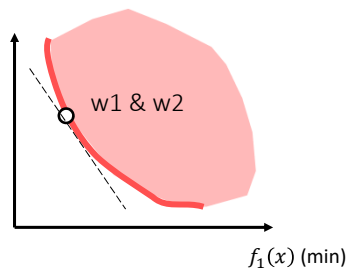
$$\begin{aligned} &\text{minimize} && w_1 f_1(x) + w_2 f_2(x) \\ &\text{subject to} && g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$

$$\begin{aligned} \text{Example:} & \quad f_1(x) = 5x_1 + \exp(x_2) \quad f_2(x) = -3x_1 \\ & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\ & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad F(x) = w_1(5x_1 + \exp(x_2)) + w_2 \cdot -3x_1 \end{aligned}$$

Choose weights:

$$w_1 = 0.7 \quad w_2 = 0.3 \quad F(x) = 2.6x_1 + 0.7 \exp(x_2)$$

$f_2(x)$  (min)



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## Weighted sum method

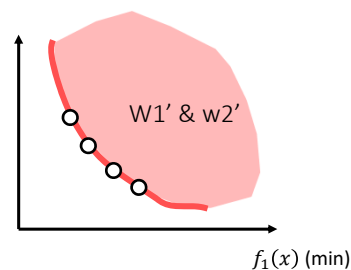
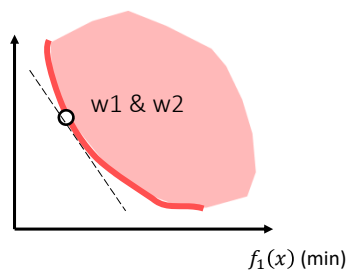
$$\begin{aligned} &\text{minimize} && w_1 f_1(x) + w_2 f_2(x) \\ &\text{subject to} && g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$

$$\begin{aligned} \text{Example:} & \quad f_1(x) = 5x_1 + \exp(x_2) \quad f_2(x) = -3x_1 \\ & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \downarrow \\ & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad F(x) = w_1(5x_1 + \exp(x_2)) + w_2 \cdot -3x_1 \end{aligned}$$

Choose weights:

$$w_1 = \dots \quad w_2 = \dots \quad F(x) = \dots$$

$f_2(x)$  (min)



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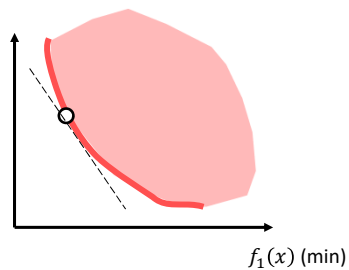
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## Weighted sum method

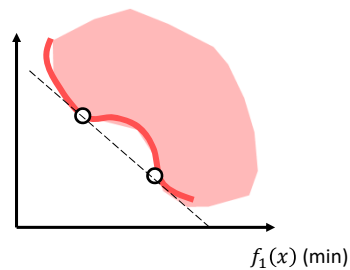
$$\begin{aligned} &\text{minimize} && w_1 f_1(x) + w_2 f_2(x) \\ &\text{subject to} && g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$

Two different set of weights not necessarily lead to two different Pareto-optimal solutions

$f_2(x)$  (min)



$f_2(x)$  (min)



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## Weighted sum method

- **Advantages**
  - ✓ Simple and easy to use
  - ✓ For convex problems it guarantees to find solutions on the entire Pareto-optimal set

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## Weighted sum method

- **Disadvantages**

- ✓ For mixed problems (min-max), we need to convert all the objectives into one type
- ✓ Uniformly distributed set of weights does not guarantee a uniformly distributed set of Pareto-optimal solutions
- ✓ Two different set of weights not necessarily lead to two different Pareto-optimal solutions
- ✓ Cannot find certain Pareto-optimal solutions in the case of a nonconvex objective space

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## $\varepsilon$ -Constraint method

- Keep just **one** of the objectives
- Treat the rest as **constraints** and modify the constraint value.

$$\begin{array}{ll}
 \text{minimize} & f_1(x) \\
 \text{maximize} & \vdots \\
 \text{minimize} & f_m(x) \\
 \text{subject to} & \\
 & g_i(x) \leq 0 \\
 & h_j(x) = 0
 \end{array}
 \quad \longrightarrow \quad
 \begin{array}{ll}
 \text{minimize} & f_l(x) \\
 \text{subject to} & f_m(x) \leq \varepsilon_m \quad m = 1, \dots, M; m \neq l \\
 & g_i(x) \leq 0 \quad i = 1, \dots, n \\
 & h_j(x) = 0 \quad j = 1, \dots, k
 \end{array}$$

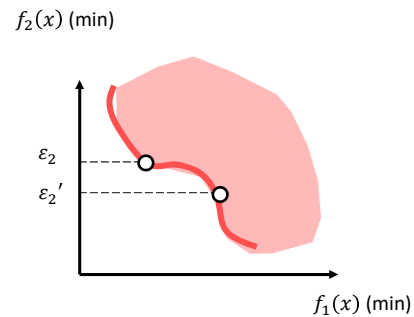
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## $\varepsilon$ -Constraint method

$$\begin{aligned} &\text{minimize} && f_1(x) \\ &\text{minimize} && f_2(x) \\ &\text{subject to} && g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$

$$\begin{aligned} &\text{minimize} && f_1(x) \\ &\text{subject to} && f_2(x) \leq \varepsilon_2 \\ &&& g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$



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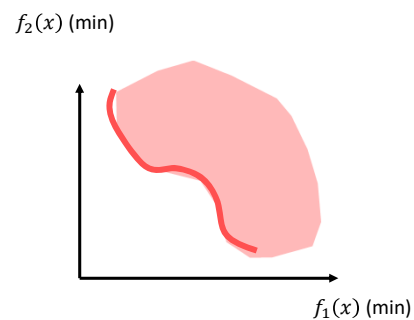
## $\varepsilon$ -Constraint method

$$\begin{aligned} &\text{minimize} && f_1(x) \\ &\text{minimize} && f_2(x) \\ &\text{subject to} && g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$

Example:

$$f_1(x) = 5x_1 + \exp(x_2) \quad f_2(x) = -3x_1$$

$$\begin{aligned} &\text{minimize} && f_1(x) \\ &\text{subject to} && f_2(x) \leq \varepsilon_2 \\ &&& g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$



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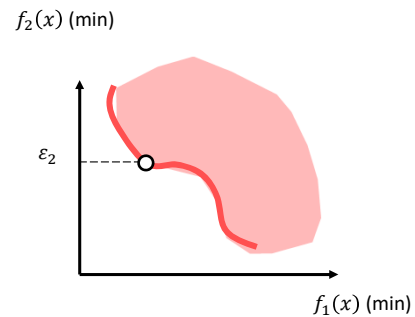
## $\varepsilon$ -Constraint method

$$\begin{aligned} &\text{minimize} && f_1(x) \\ &\text{minimize} && f_2(x) \\ &\text{subject to} && g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$

Example:

$$\begin{aligned} f_1(x) &= 5x_1 + \exp(x_2) & f_2(x) &= -3x_1 \\ \varepsilon_2 &= 10 & -3x_1 &\leq 10 \end{aligned}$$

$$\begin{aligned} &\text{minimize} && f_1(x) \\ &\text{subject to} && f_2(x) \leq \varepsilon_2 \\ &&& g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$



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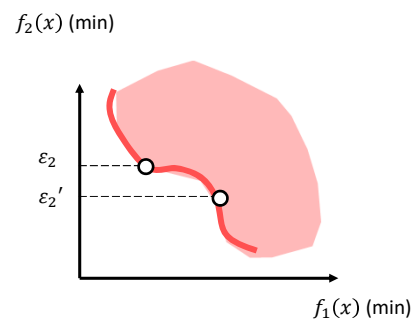
## $\varepsilon$ -Constraint method

$$\begin{aligned} &\text{minimize} && f_1(x) \\ &\text{minimize} && f_2(x) \\ &\text{subject to} && g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$

Example:

$$\begin{aligned} f_1(x) &= 5x_1 + \exp(x_2) & f_2(x) &= -3x_1 \\ \varepsilon'_2 &= 1 & -3x_1 &\leq 1 \end{aligned}$$

$$\begin{aligned} &\text{minimize} && f_1(x) \\ &\text{subject to} && f_2(x) \leq \varepsilon_2 \\ &&& g_i(x) \leq 0 \\ &&& h_j(x) = 0 \end{aligned}$$



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## $\varepsilon$ -Constraint method

- **Advantages**

- ✓ Different Pareto-optimal solutions can be found using different  $\varepsilon$  values
- ✓ Applicable to either convex or non-convex problems

- **Disadvantages**

- ✓ The vector has to be chosen carefully so that it is within the minimum and the maximum values of the individual objective function

Making decisions

## Making decisions

- You are working on a water resources plan for a large river basin that includes the objectives of municipal and industrial (M&I) water supply, irrigation of cotton, and hydropower.
- The planning committee has proposed 13 planning alternatives for the basin.
- You and your technical staff have assessed the material benefits expected in each objective under each plan.
- Here are the results:

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## Making decisions

Alternative	Hydropower	M&I Supply	Agriculture
1	0	500	0
2	250	400	0
3	360	50	0
4	0	400	5000
5	150	375	5000
6	150	300	5000
7	290	50	5000
8	250	300	5000
9	0	200	9000
10	50	175	9000
11	95	135	9000
12	50	135	9000
13	150	50	9000

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## Making decisions

- Are any of these planning alternatives sub-optimal? If so, how would you explain the meaning of sub-optimality to a member of the planning committee?
- You need to determine and explain the trade-off between the alternatives to the planning committee. Suggest different visualization plots you might use to illustrate the trade-off.