ORIGINAL ARTICLE

Bipartite Dot Product Graphs

Sean Bailey[a] and David Brown[b]

[a]College of Business, Engineering, and Technology, Texas A& M University Texarkana, 7101 University Ave, Texarkana, TX, USA; [b]Department of Mathematics and Statistics, Utah State University, Logan, Utah, USA

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ABSTRACT
Given a bipartite graph $G = (X, Y, E)$, the bipartite dot product representation of G is a function $f : X \cup Y \to \mathbb{R}^k$ and a positive threshold t such that for any $x \in X$ and $y \in Y$, $xy \in E$ if and only if $f(x) \cdot f(y) \geq t$. The minimum k such that a bipartite dot product representation exists for G is the bipartite dot product dimension of G, denoted $bdp(G)$. We will show that such representations exist for all bipartite graphs as well as give an upper bound for the bipartite dot product dimension of any graph. We will also characterize the bipartite graphs of bipartite dot product dimension 1 by their forbidden subgraphs.

KEYWORDS
Graph theory; dot product graphs; bipartite graphs; graph representations; subgraph categorization

AMS CLASSIFICATION
05C62, 05C75

1. Introduction

[1]Dot product graphs were independently developed by Reiterman et al [13] and Schienerman et al [6]. A graph $G = (V, E)$ is a *k-dot product graph* if there exists a function $f : V \to \mathbb{R}^k$ with a real number $t > 0$ such that for any $x, y \in V$ $xy \in E$ if and only if $f(x) \cdot f(y) \geq t$. The function f is called a *k dot product representation* of G. The minimum k such that G is a k-dot product graph is called the *dot product dimension* of G, $\rho(G)$.

A *bipartite graph*, $G = (V, E)$, is an undirected graph where the set of vertices V can be partitioned into two sets, X and Y, such that X and Y are disjoint independent sets. The relation of bipartite graphs to hypergraphs and directed graphs, as well as their applications to matching problems, leads us to consider bipartite dot product representations.

Let $G = (V, E)$ be a bipartite graph. As we previously mentioned, we may write $G = (X, Y, E)$ if $V = X \cup Y$ with $X \cap Y = \emptyset$ and for any vertices x, y if $xy \in E(G)$, then $x \in X$ and $y \in Y$. A *bipartite dot product representation* of G is a function

CONTACT Sean Bailey. Email: sbailey@tamut.edu

[1]This paper is derived from the first author's Ph.D. thesis under the supervision of the second author.

$f : \{X, Y\} \to \mathbb{R}^k$ and a threshold $t > 0$ such that for any $x \in X$ and $y \in Y$ $xy \in E(G)$ if and only if $f(x) \cdot f(y) \geq t$. Since $t > 0$, we can use $t = 1$ without loss of generality. The minimum k such that a bipartite dot product representation exists for a bipartite graph G is the *bipartite dot product dimension* of G, denoted $bpd(G)$.

This definition of bipartite dot product graphs is similar to dot product graphs. This new definition allows us to examine classes of graphs where a relaxation on the minimum dimension can be given due to ignoring possible adjacencies. This definition also led to several general results for bipartite dot product graphs.

2. General Results for Bipartite Dot Product Dimension

The relaxation of dot product dimension that motivated our examination of bipartite dot product representations allows us to use the dot product dimension as an upper bound on the bipartite dot product dimension, as proven in Theorem 2.1. This bound also means that for any bipartite graph G there is bipartite dot product representation, since there is a dot product representation by Theorem 2.1.

Theorem 2.1. *Let G be a bipartite graph. Then $bpd(G) \leq \rho(G)$.*

Proof. Let G be a bipartite graph and $\rho(G) = k$. Then there exists $f : V \to \mathbb{R}^k$ and such that for any $x, y \in V$ $xy \in E$ if and only if $f(x) \cdot f(y) \geq 1$. But since $V = X \cup Y$ and $xy \in E$ if and only if $x \in X$ and $y \in Y$, then f is also a bipartite dot product representation of G. Thus $bpd(G) \leq \rho(G)$. □

This upper bound for the bipartite dot product dimension can also be found using graph structures, namely bicliques. A *biclique* of a graph G is a subgraph H of G with $V(H) = X \cup Y$ where X and Y are each independent sets and every vertex $x \in X$ is adjacent to every vertex $y \in Y$. Thus the minimum number of bicliques of G such that each edge of G is contained in at least one biclique of G is the *biclique cover number* of G, denoted $bc(G)$. Biclique cover number has been extensively studied, as seen in [1, 3–5, 11, 12].

Theorem 2.2 proves that the biclique cover number of G bounds the bipartite dot product dimension of G.

Theorem 2.2. *Let G be a bipartite graph. Then $bpd(G) \leq bc(G)$.*

Proof. Suppose $bc(G) = k$. Without loss of generality, we can label the bicliques $B_1, \cdots, B_k$. Define $f : X \cup Y \to \mathbb{R}^k$ such that for any $v \in X \cup Y$ $f(v)_i = 1$ if $v \in B_i$ and 0 otherwise. Then for any $x \in X$ and $y \in Y$, $f(x) \cdot f(y) \geq 1$ if and only if $x, y \in B_i$ for some i. Thus f is a bipartite dot product dimension of G. □

One of the key characteristics of dot product representations is their hereditary property. As shown in Theorem 2.3, bipartite dot product representations also have the hereditary property. This property allows us to characterize graphs with bipartite dot product dimension k via forbidden induced subgraphs.

Theorem 2.3. *Let $G = (X, Y, E)$ be a bipartite graph and G' be an induced subgraph of G. Then $bpd(G') \leq bpd(G)$.*

Proof. Let $bpd(G) = k$. Let $f : X \cup Y \to \mathbb{R}^k$ be a k-dot product representation of G and $G' = (X', Y', E')$ be an induced subgraph of G. Then $f' : X' \cup Y' \to \mathbb{R}^k$ defined by f restricted to $X' \cup Y'$ is still a k-bipartite dot product representation. □

3. Characterization of 1-Bipartite Dot Product Graphs

In addition to the general results, bipartite dot product dimension can be determined for specific graphs. In particular, we will characterize the graphs of bipartite dot product dimension 1 by their forbidden subgraphs. We will show that our list is both necessary and sufficient.

The subsequent subsections will accomplish this characterization. We will first identify the structures that prohibit bipartite dot product dimension of 1. Then we will algorithmically show that, provided that there are no forbidden substructures in G, we can create a 1-bipartite dot product representation of G.

3.1. Forbidden Subgraphs of 1-Bipartite Dot Product Graphs

Our characterization will begin by showing a list of two graphs that have bipartite dot product dimension of 2. These graphs are $3K_2$ and P_5. These forbidden subgraphs can be seen in Figure 1. The proof that their bipartite dot product dimension is not 1 is shown in Lemmas 3.1 and 3.2.

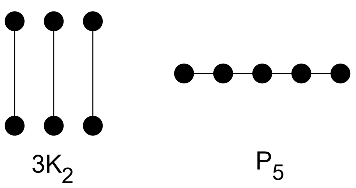

Figure 1. Forbidden Subgraphs of 1-Bipartite Dot Product Graphs.

Lemma 3.1. *Let $G = 3K_2$. Then the $bpd(G) = 2$.*

Proof. Suppose that $bpd(G) = 1$. Then there exists a function $f : X \cup Y \to \mathbb{R}$ such that $f(x) \cdot f(y) \geq 1$ if and only if $xy \in E(G)$. Label the vertices of G as shown in Figure 2.

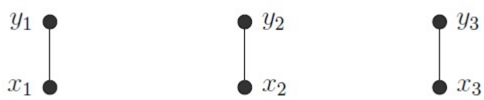

Figure 2. A Labeling of $3K_2$.

By the definition of f, $f(x_i)\cdot f(y_j) \geq 1$ if $i = j$ and $f(x_i)\cdot f(y_j) < 1$ if $i \neq j$. Without loss of generality, we assume that $f(x_1) > 0$ and $f(x_1) \geq f(x_k)$ for $k \in \{2, 3\}$. This also implies that $f(y_1) \geq \frac{1}{f(x_1)} > 0$.

If $f(x_2) > 0$, then $f(y_2) \geq \frac{1}{f(x_2)} > 0$. But $f(x_1) > f(x_2)$ so $f(y_2) \geq \frac{1}{f(x_2)} \geq \frac{1}{f(x_1)}$.

This implies that $f(x_1) \cdot f(y_2) \geq 1$, which is a contradiction since $x_1y_2 \notin E(G)$. This implies that $f(x_2), f(y_2) < 0$. Similarly it can be shown that $f(x_3), f(y_3) < 0$.

Without loss of generality, we now assume that $f(x_2) \geq f(x_3)$. So $f(y_2) \leq \frac{1}{f(x_2)} \leq \frac{1}{f(x_3)}$, which implies that $f(y_2) \cdot f(x_3) \geq 1$. But this is another contradiction of $x_3y_2 \notin E(G)$. Thus our initial assumption that $bpd(G) = 1$ is false, so $bpd(G) \geq 2$.

To prove that $bpd(G) \leq 2$, define $g : X \cup Y \to \mathbb{R}^2$ such that

$$\begin{aligned} x_1, y_1 &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ x_2, y_2 &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ x_3, y_3 &= \begin{bmatrix} -1 \\ 0 \end{bmatrix} \end{aligned}$$

A brief examination shows that this bipartite representation is valid.

Thus $bpd(G) = 2$. □

Lemma 3.2. *Let $G = P_5$. Then the $bpd(G) = 2$.*

Proof. Suppose that $bpd(G) = 1$. Then there exists a function $f : X \cup Y \to \mathbb{R}$ such that $f(x) \cdot f(y) \geq 1$ if and only if $xy \in E(G)$. Label the vertices of G as shown in Figure 3.

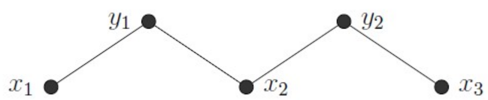

Figure 3. P_5 Labeling

Let $f(y_1) = \alpha$. Then $f(x_1), f(x_2) \geq \frac{1}{\alpha}$ in order for $f(x_1) \cdot f(y_1), f(x_2) \cdot f(y_1) \geq 1$. Similarly since $x_3y_1 \notin E$, then $f(x_3) < \frac{1}{\alpha}$. For $f(y_2) \cdot f(x_2) \geq 1$, it is necessary that $f(y_2) > \alpha$. But then $f(y_2) \cdot f(x_1) \geq 1$, which is a contradiction of $x_1y_2 \notin E$. Thus our initial assumption that $bpd(G) = 1$ is false, so $bpd(G) \geq 2$.

To prove that $bpd(G) \leq 2$, we just need to note that the induced subgraphs on $\{x_1, x_2, y_2\}$ and $\{x_2, x_2, y_2\}$ are both $K_{1,2}$. Therefore the biclique cover number of a P_5 is 2, and thus $bpd(G) \leq 2$ by Theorem 2.2.

Thus $bpd(G) = 2$. □

3.2. Constructive Algorithm for 1-Bipartite Dot Product Graphs

Based on the forbidden subgraphs shown in Lemmas 3.1 and 3.2, we developed Algorithm 1. This algorithm takes a bipartite graph $G = (X, Y, E)$ along with the vertex degrees and returns a 1-bipartite dot product representation of G, namely F. The returned representation will fail to be valid only if $bpd(G) \geq 2$.

Algorithm 1 is built on identifying the vertex of G with maximum degree. This vertex is assigned a value equivalent to its degree. The other vertices in the same partite set as the vertex of maximum degree are assigned values relative to the cardinality of the intersection of the neighborhood of each vertex and the neighborhood of the vertex of maximum degree. The vertices in the other partite set are assigned values based on the values assigned the vertices adjacent to each one.

The validity of Algorithm 1 is shown in Theorem 3.3. An example of how Algorithm 1 works and an example of how Algorithm 1 fails if P_5 is present are given after the proof of Theorem 3.3.

Theorem 3.3. *Let G be a bipartite graph. If G does not contain $3K_2$ or P_5 as induced subgraph, then Algorithm 1 returns a 1-bipartite dot product representation of G.*

Proof. Let $G = (X, Y, E)$ be a bipartite graph with no induced $3K_2$ or P_5.

First we will consider when $\hat{x}\hat{y} \in E(G)$ for some $\hat{x} \in X$ and $\hat{y} \in Y$. There are three cases to consider:

Case 1: Suppose $\hat{y} \in N(x_1)$.

Then $N(x_1) \cap N(\hat{x}) \neq \emptyset$. So $F(\hat{y}) = \frac{1}{\min(F(x_j))}$ such that $\hat{y} \in N(x_1) \cap N(x_j)$. But since $F(\hat{x}) \geq \min(F(x_j))$, $F(\hat{y}) \geq \frac{1}{F(\hat{x})}$. Thus $F(\hat{x}) \cdot F(\hat{y}) \geq 1$.

Case 2: Suppose $\hat{y} \notin N(x_1)$ and $N(x_1) \cap N(\hat{x}) \neq \emptyset$.

Since $N(x_1) \cap N(\hat{x}) \neq \emptyset$, there exists $y_i \in Y$ such that $y_i \in N(x_1) \cap N(\hat{x})$. By definition of x_1, $deg(x_1) \geq deg(\hat{x})$. That implies that $|N(x_1)| \geq |N(x_1) \cap N(\hat{x})| + 1$. So there exists $y_k \in Y$ such that $y_k \in N(x_1)$ and $y_k \notin N(\hat{x})$. Thus $y_k x_1 y_i \hat{x}\hat{y}$ is a P_5. So G has a P_5 as an induced subgraph, which is a contradiction.

Case 3: Suppose that $\hat{y} \notin N(x_1)$ and $N(x_1) \cap N(\hat{x}) = \emptyset$.

If $\hat{y} \notin N(x_1)$, then there exists an x_k explained in Lines 10-12 such that $\hat{y} \in N(x_k)$. So $F(\hat{y}) = \frac{1}{\max(F(x_j))}$ such that $\hat{y} \in N(x_k) \cap N(x_j)$. But since $F(\hat{x}) \geq \max(F(x_j))$, $F(\hat{y}) \leq \frac{1}{F(\hat{x})}$. Thus $F(\hat{x}) \cdot F(\hat{y}) \geq 1$.

Now consider when $\hat{x}\hat{y} \notin E(G)$ for some $\hat{x} \in X$ and $\hat{y} \in Y$.

Case 1: Suppose either $\hat{y}$ or $\hat{x}$ is an isolated vertex.

In either case, $F(\hat{x}) \cdot F(\hat{y}) = 0 < 1$.

Case 2: Suppose $\hat{y} \in N(x_1)$ and $N(x_1) \cap N(\hat{x}) \neq \emptyset$.

First suppose that there exists $x_j \in X$ such that $\hat{y} \in N(x_j)$ and $|N(x_1) \cap N(x_j)| \leq |N(x_1) \cap N(\hat{x})|$. Then there exists $y_k \in N(x_1) \cap N(\hat{x})$ such that $y_k \notin N(x_j)$. Then $x_j \hat{y} x_1 y_k \hat{x}$ is P_5 that is an induced subgraph of G. This is a contradiction.

So for any $x_j \in X$ such that $\hat{y} \in N(x_j)$, $|N(x_1) \cap N(x_j)| > |N(x_1) \cap N(\hat{x})|$.

Then $F(\hat{y}) \leq \frac{1}{|N(x_1) \cap N(x_j)|} < \frac{1}{|N(x_1) \cap N(\hat{x})|} = \frac{1}{F(\hat{x})}$. Thus $F(\hat{x})F(\hat{y}) < 1$.

Case 3: Suppose $\hat{y} \in N(x_1)$ and $N(x_1) \cap N(\hat{x}) = \emptyset$.

This involves the reordering from Lines 10-12 in the algorithm. If that reordering occurred at least twice, there exists non-isolated vertices $x_{k_1}, x_{k_2} \in X$ such that $N(x_1) \cap N(x_{k_1}) = N(x_1) \cap N(x_{k_2}) = N(x_{k_1}) \cap N(x_{k_2}) = \emptyset$. Then $3K_2$ is an induced subgraph of G, which would be a contradiction. Thus the reordering can be done at most once.

In this case, $F(\hat{x}) < 0$ and $F(\hat{y}) > 0$. So $F(\hat{x}) \cdot F(\hat{y}) < 0 < 1$.

Case 4: Suppose $\hat{y} \notin N(x_1)$ and $N(x_1) \cap N(\hat{x}) \neq \emptyset$.

Since $\hat{y}$ is not adjacent to either x_1 or $\hat{x}$ and $\hat{y}$ is not an isolated vertex, there exists $x_i \in X$ such that $\hat{y}x_i \in E$. Similarly there exists $y_i \in N(x_1) \cap N(\hat{x})$ because $N(x_1) \cap N(\hat{x}) \neq \emptyset$.

If $N(x_1) \cap N(x_i) \neq \emptyset$ or $N(x_i) \cap N(\hat{x}) \neq \emptyset$, then P_5 is an induced subgraph of G, which would be a contradiction. Thus $N(x_i) \cap (N(x_1) \cup N(\hat{x})) = \emptyset$. Then we can label x_i as x_k and $F(\hat{y}) < 0$ by the lines 20 and 27 of the algorithm. But $F(\hat{x}) > 0$ since

Table 1. **Algorithm 1:**Returns a 1-bipartite dot product representation of G (F), and fails if $bpd(G) \geq 2$.

INPUT: G=(X,Y,E) and $deg(v)$ for all $v \in X \cup Y$
Vertices begin with no labels.
Define F as a 1-bipartite dot product representation of G.
if A vertex v has $deg(v) = 0$ **then**
  Define $F(v) = 0$.
else
  for An unlabeled vertex of maximum degree **do**
    Label this vertex as x_1 and the independent set X.
    Label the remaining vertices in X such that for $i > j$,
      $|N(x_i) \cap N(x_1)| \geq |N(x_j) \cap N(x_1)|$.
    if There exists k such that for $i \geq k$, $N(x_i) \cap N(x_1) = \emptyset$, and $deg(x_i) > 0$
    then
      Label the vertex of maximum degree of such vertices as x_k
      Label the vertices in X with $N(x_i) \cap N(x_k) \neq \emptyset$ such that for $i > j$,
        $|N(x_i) \cap N(x_k)| \geq |N(x_j) \cap N(x_k)|$.
    end if
    for All vertices in Y **do**
      Label the vertices $y_1, y_2, \cdots, y_n$ where $n = |Y|$.
    end for
  end for
  for Each $x \in X$ **do**
    Define $F(x_1) = deg(x_1)$ and $F(x_i) = |N(x_i) \cap N(x_1)|$ for $i < k$.
    Define $F(x_k) = -deg(x_k)$ and $F(x_j) = -|N(x_k) \cap N(x_j)|$ for $j \geq k$.
  end for
  for $y \in Y$ **do**
    for $y_i \in N(x_1)$ **do**
      Define $F(y_i) = \frac{1}{\min(F(x_j))}$ where $y_i \in N(x_j) \cap N(x_1)$.
    end for
    for $y_m \in N(x_k)$
      Define $F(y_m) = \frac{1}{\max(F(x_j))}$ where $y_m \in N(x_j) \cap N(x_k)$.
    end for
  end for
end if
return $F(v)$ for all $v \in V$

$N(x_1) \cap N(\hat{x}) \neq \emptyset$. Therefore $F(\hat{x}) \cdot F(\hat{y}) < 0 < 1$.

Case 5: Suppose $\hat{y} \notin N(x_1)$ and $N(x_1) \cap N(\hat{x}) = \emptyset$.

Since $\hat{y}$ is not isolated, there exists x_k such that $\hat{y} \in N(x_k)$. Similarly there exists $\bar{y} \in N(\hat{x})$ since $\hat{x}$ is not isolated. It can also be noted that since x_1 is the vertex of maximum degree there exists $y_1 \in N(x_1)$ such that $y_1 \notin N(x_k)$.

If $\bar{y} \notin N(x_k)$, then $3K_2$ is an induced subgraph of G, which would be a contradiction. Thus $\bar{y} \in N(x_k)$. In this case, we need to consider the $deg(x_k)$ and $deg(\hat{x})$. If $deg(\hat{x}) \geq deg(x_k)$, then there exists $y_2 \in N(\hat{x})$ and $y_2 \in N(x_k)$. But this means that P_5 is an induced subgraph of G, which would be a contradiction. Thus $deg(\hat{x}) < deg(x_k)$. We can then assume that x_k is the x_k in line 8 of the algorithm. Thus $F(\hat{x}) = -|N(x_k) \cap N(\hat{x})| > -deg(x_k)$ and $F(\hat{y}) = \frac{1}{F(x_k)} = -\frac{1}{deg(x_k)}$. Therefore $F(\hat{x}) \cdot F(\hat{y}) = -|N(x_k) \cap N(\hat{x})| \cdot -\frac{1}{deg(x_k)} < \frac{deg(x_k)}{deg(x_k)} = 1$. $\square$

3.3. Example and Nonexample of How Algorithm 1 Works

For an example of how Algorithm 1 works, we will let $G = (X, Y, E)$ be the bipartite graph H_1 in Figure 4. In H_1, the grey vertices are in X and the black vertices are in Y.

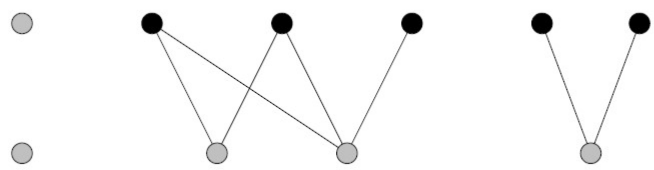

Figure 4. An Example Graph for Algorithm 1. Vertices unlabeled.

First, the isolated vertices can be assigned vectors [0], as designated in Lines 4 and 5. We will also label the vertices, as designated Lines 7-16. Figure 5 shows these assignments and labels. It can be noted that x_3 is x_k.

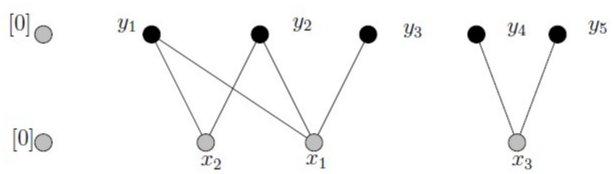

Figure 5. An Example Graph for Algorithm 1. Vertices labeled.

Next we will assign the vertices in $x \in X$ the vectors as explained in Lines 19-20. These assignments are seen in Figure 6.

Next we will assign the vertices in $y \in Y$ the vectors as explained in Lines 22-28. These assignments are seen in Figure 7.

A brief examination of these vectors shows that this 1-dot product representation of H_1 is valid.

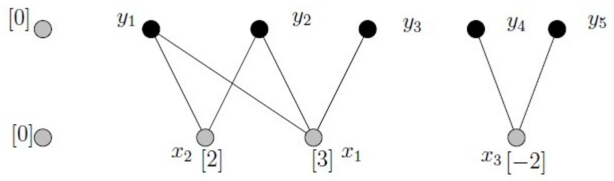

Figure 6. An Example Graph for Algorithm 1. Vectors assigned for X.

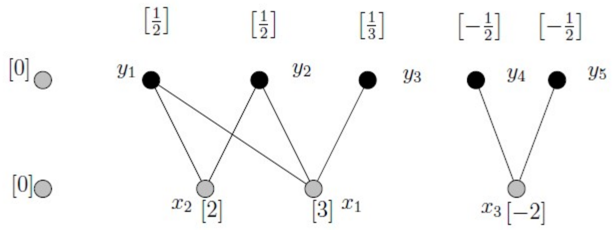

Figure 7. An Example Graph for Algorithm 1. Representation given.

For a nonexample of how Algorithm 1 works, we will let $G = (X, Y, E)$ be the bipartite graph H_2 in Figure 8. In H_1, the grey vertices are in X and the black vertices are in Y. An examination of H_2 shows that $3K_2$ and P_5 are both induced subgraphs.

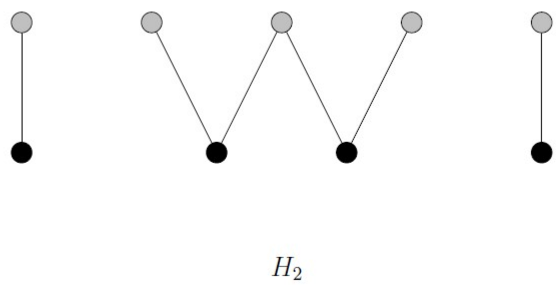

Figure 8. A Nonexample Graph for Algorithm 1. Vertices unlabeled.

There are no isolated vertices in H_2 so we can skip to Line 7. We will label the vertices in X, as designated Lines 7-16. Figure 9 shows these assignments and labels. It can be noted that x_4 is x_k.

Next we will assign the vertices in $x \in X$ the vectors as explained in Lines 19-20. These assignments are seen in Figure 10. For x_5, we can assign $F(x_5) = 0$ because $|N(x_4) \cap N(x_5)|$ and $5 > 4$.

Next we will assign the vertices in $y \in Y$ the vectors as explained in Lines 22-28. These assignments are seen in Figure 11. However, there is no assignment for y_5 since it is not in $N(x_1)$ or $N(x_4)$.

F fails first because y_5 has no assignment. Further, any assignment of y_5 will not result in a dot product with [0] to be greater than or equal to 1. Next it can be noted

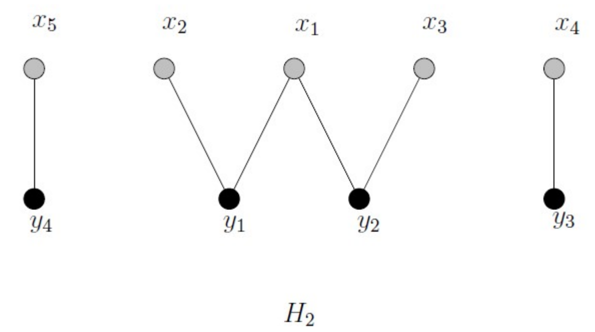

Figure 9. A Nonexample Graph for Algorithm 1. Vertices labeled.

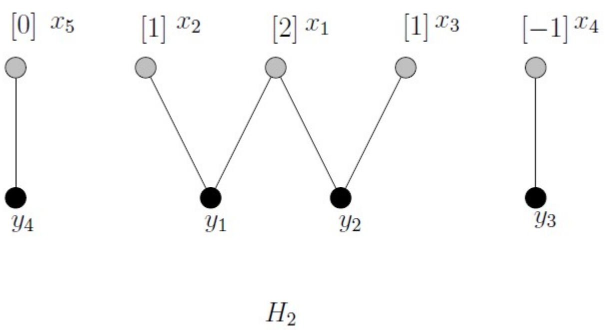

Figure 10. A Nonexample Graph for Algorithm 1. Vectors assigned for X.

that it $F(x_3) \cdot F(y_1) = 1$, which is a contradiction of the nonadjency of those vertices. Thus F is not a valid representation when either a $3K_2$ or P_5 are induced subgraphs of the graph.

3.4. Primary Theorem

Our results combine to given us a forbidden induced subgraph characterization of 1-bipartite dot product graphs, as seen in Theorem 3.4.

Theorem 3.4. *A bipartite graph G is a 1-bipartite dot product graph if and only if G has no induced $3K_2$ or P_5.*

Proof. The necessity of the forbidden subgraphs are given from Lemmas 3.1 and 3.2. The sufficiency of the forbidden subgraphs is given by Theorem 3.3. □

4. Linear Algebra Relation to Bipartite Dot Product Dimension

Any bipartite graph can be thought of as a rectangular $(0, 1)$-matrix. Let $G = (X, Y, E)$ be a bipartite graph with $X = \{x_1, \cdots, x_n\}$ and $Y = \{y_1, \cdots, y_m\}$. Then the *bipartite*

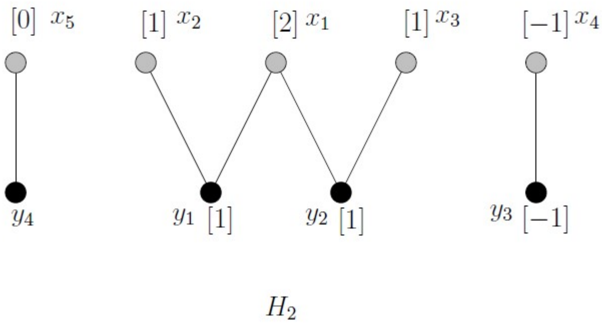

Figure 11. A Nonexample Graph for Algorithm 1. Representation given.

adjacency matrix of G, denoted $B(G)$ or simply B, is an $n \times m$ $(0,1)$-matrix, where b_{ij} is 1 if and only if $x_i y_j \in E(G)$. This perspective allows for combinatorial analysis of linear algebraic or other properties of $(0,1)$-matrices. To understand these matrices, consider the bipartite graph in Figure 12.

The bipartite adjacency matrix of Figure 12 is

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

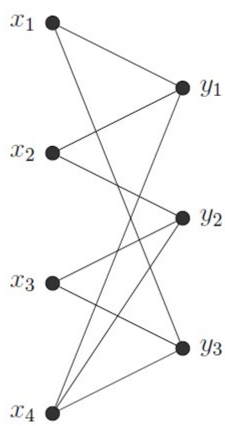

Figure 12. An Example Bipartite Graph.

Because our bipartite graphs can be viewed as matrices, linear algebra and its theory can be used to analyze these graphs. We will use $M_{m,n}(\mathbb{R})$ to denote the set of all $m \times n$ matrices with real entries.

We will utilize the linear algebra parameter real rank to analyze the bipartite adjacency matrix. If $A \in M_{m,n}(\mathbb{R})$, the *real rank* of A, denoted $rank(A)$, is the largest number of columns of A that constitute a linearly independent set [9]. This set of columns is not unique, but the cardinality of this set is unique.

The real rank of a matrix is also equivalent to the factor rank of the matrix. The *factor rank* of $A \in M_{m,n}(\mathbb{R})$ is the minimum integer k such that $A = CF$, where $C \in M_{m,k}(\mathbb{R})$ and $F \in M_{k,n}(\mathbb{R})$. The comparison of factor rank and the bipartite dot product dimension leads to the following theorem.

Theorem 4.1. *Let G be a bipartite graph and B the bipartite adjacency matrix of G. Then $bpd(G) \leq rank(B)$.*

Proof. Suppose that the bipartite sets of G are X and Y, with $|X| = m$ and $|Y| = n$. Suppose that $rank(B) = k$. Then there exists real matrices $R \in M_{m,k}(\mathbb{R})$ and $S \in M_{k,n}(\mathbb{R})$ such that

$$B = RS = \begin{bmatrix} r_{1,1} & r_{1,2} & \cdots & r_{1,k} \\ r_{2,1} & r_{2,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ r_{m,1} & \cdots & \cdots & r_{m,k} \end{bmatrix} \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\ s_{2,1} & s_{2,2} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ s_{k,1} & \cdots & \cdots & s_{k,n} \end{bmatrix}.$$

Since B requires an arbitrary assignment of the vertices such that $X = x_1, \cdots, x_m$ and $Y = y_1, \cdots, y_n$, we can assign the vertices $x_i \in X$ the vector $\vec{x_i} = (r_{i,1}, \cdots, r_{i,k})^T$ and the vertices $y_i \in X$ the vector $\vec{y_i} = (s_{1,i}, \cdots, s_{k,i})^T$. By definition of matrix multiplication, $\vec{x_i} \cdot \vec{y_j} = B_{i,j}$, which is 1 if $x_i y_j \in E$ and 0 otherwise. Thus there exists a k-bipartite dot product representation of G and $bpd(G) \leq k$. $\square$

This bound however is not tight for all graphs. An example of this is the graph $2K_2$. This graph and a 1-bipartite dot product representation of it can be seen in Figure 13.

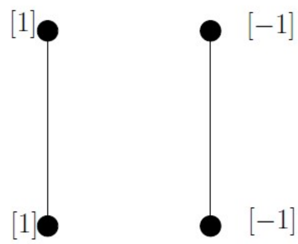

Figure 13. A $2K_2$ and its 1-Bipartite Dot Product Representation.

However the bipartite adjacency matrix of $2K_2$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The real rank of B is 2, which is greater than the bipartite dot product dimension.

5. Further Work

We showed the forbidden subgraph characterization of 1-bipartite dot product graphs in Theorem 3.4. We propose using this characterization to develop a 2-SAT recognition algorithm for 1-bipartite dot product graphs. This algorithm could then be used to determine the complexity of determining if a given graph is a 1-bipartite dot product graph.

We also propose finding a forbidden induced subgraph characterization of 2-bipartite dot product graphs. Since a characterization of 2-dot product graphs has yet to be found, this particular characterization may also be elusive [10]. If such is the case, we propose finding a partial characterization of 2-bipartite dot product graphs similar to the proof that a bipartite claw has dot product dimension 3 [6].

We also propose determining the dot product dimension of other classes of bipartite graphs. such as interval bigraphs [8]. We propose creating a representation similar to the cap-capture graphs used by Scheinerman to show that interval bigraphs have bipartite dot product dimension 2 [6].

Another class of graphs to compare bipartite dot product graphs with is difference graphs. A graph $G = (V, E)$ is a *difference graph* if there exists $f : V \to \mathbb{R}$ with $|f(v)| < T$ for each $v \in V$ such that $uv \in E$ if and only if $|f(u) - f(v)| > T$. It has been shown that difference graphs are bipartite [7]. This representation is related to threshold graphs, which were generalized by dot product graphs [13]. Thus we believe that bipartite dot product graphs are a generalization of difference graphs.

In Theorem 4.1, we showed that the real rank of the bipartite adjacency matrix of a graph G is an upper bound on the bipartite dot product dimension of G. But the multiple ranks of matrices. We propose considering alternate ranks such as non-negative integer rank and boolean rank.

Finally, we propose finding the maximum bipartite dot product dimension of a graph on n vertices, similar to the dot product dimension bound conjecture in [6]. A related question is what graphs on n vertices have attain this maximum bipartite dot product dimension.

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