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Examining the Specialized Math Content Knowledge of Elementary Teachers in the Age of the Common Core

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A Dissertation Presented

by

STEPHANIE B. PURINGTON

Submitted to the Graduate School of the
University of Massachusetts Amherst in partial fulfillment
of the requirements for the degree of

DOCTOR OF PHILOSOPHY

May 2020

College of Education
Teacher Education and Curriculum Studies

Examining the Specialized Math Content Knowledge of Elementary Teachers
in the Age of the Common Core

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ACKNOWLEDGMENTS

I am very grateful to those who helped me along this journey. Dr. Darrell Earnest guided me through the dissertation process with kindness, while always nudging me to better paths. He also invited me to a graduate student support group that was a lifeline during my years at UMass Amherst. My husband coined the term “group hug” for those regular meetings, as I always came back refreshed and feeling better about life. Dr. Betsy McEneaney was my touchstone through my whole PhD experience, guiding me in my academics, comps research, and allowing me to work on her research. Mike Hayes, whose amazing teaching improves the math identity of so many UMass Amherst students, helped me gain access to a research site and agreed to act as the outside member of my committee. Dr. Martina Nieswandt adopted me when my advisors went on sabbaticals, coached me through research and writing, and introduced me to the iCons program.

My fellow students provided immeasurable support, and without them this journey would have been much lonelier. Alicia Gonzalez has been my rock throughout these five years, always willing to listen and to keep me accountable. Chris McGrail and Ally Hunter have rounded out my amazing “squad,” and reminded me to always keep moving “more up, more forward.” Bona Kang had been a great writing buddy and has offered comfort and perspective. Special thanks to my cohort of one, Ryan Ambuter, who arranged writing groups and encouraged a brisk pace.

Kristin Tyler and Mike Hanna deserve extra thanks for the many questions they answered, forms they processed, rooms they booked, and freak-outs they calmed.

This dissertation would have been impossible to complete without the wonderful teachers who agreed to be participants. That they were willing to give of their time so close to the end of the school year was a true gift.

Last, but certainly not least, I want to thank my family, Dave, Katy, and Elizabeth, for supporting me and believing in me throughout this process.

This work was supported in part by a Graduate School Dissertation Research Grant from the University of Massachusetts Amherst.

ABSTRACT

EXAMINING THE SPECIALIZED MATH CONTENT KNOWLEDGE OF ELEMENTARY TEACHERS IN THE AGE OF THE COMMON CORE

MAY 2020

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Mathematical standards for students have increased with the development of the Common Core State Standards for Mathematics and its accompanying high stakes testing. Teachers need strong conceptual knowledge of the mathematics they teach in order to give students the opportunity to learn that math deeply. An earlier study (Ma, 1999) found that US elementary teachers lack the deep knowledge to teach math conceptually. Given the mathematics standards movements of the last two decades, it is plausible that the knowledge base of teachers has changed. Using the framework of Specialized Content Knowledge (SCK), which is the knowledge required to teach math that extends beyond the knowledge to do math, this study examines the current level of SCK held by practicing elementary teachers. It also examines themes in the explanations they give for the four topics: subtraction with regrouping; multi-digit multiplication; division with fractions; and area, perimeter, and proof.

This study used a multiple-case study design and an interview protocol with current elementary teachers (*N*=18). Analysis of teacher interviews indicates that elementary teacher SCK can vary with the topic being addressed, with all but two of the

participants falling into different SCK levels across the mathematical content areas. This points to the need for assessments that offer topic-level data so we can determine the support individual teachers need. Most of the current teachers studied have strong Specialized Content Knowledge in areas of whole number calculation, such as subtraction with regrouping and multi-digit multiplication. In those topics they are able to create representations and justify the standard algorithms. In the areas of division with fractions and area, perimeter, and proof, however, Specialized Content Knowledge was frequently much lower, and many of the teachers struggled to create representations or explain the mathematics contained in the algorithms. This indicates a need for teacher education and professional development that extends beyond whole number operations and focuses on conceptual understanding of these challenging topics.

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CHAPTER 1

INTRODUCTION

The National Governors Association and the Council Chief State School Officers, through their Common Core State Standards for Mathematics (CCSSM), have instituted rigorous standards for the mathematics students are expected to learn, raising the expectations for teachers to be able to teach to those standards (Selling, Garcia, & Ball, 2016). Because the tests that students take to measure progress in mathematics have repercussions for both the students and their schools, there is pressure on districts and teacher preparation programs to ensure that teachers have strong content knowledge (Zimpher & Jones, 2010; Cochran-Smith et al., 2016). It is logical to claim that teachers cannot be expected to teach well any material that they do not know and understand well. As Shirvani (2015) found, there is an indication of a direct relationship between teachers' content knowledge and the mathematics that students learn.

Consider, for example, how one might solve the following problem: $1\frac{3}{4} \div \frac{1}{2}$. One common, procedural way to solve the problem is to use the reciprocal of ½, 2, to change the problem to multiplication: $1\frac{3}{4} \times 2$. Yet consider how you might teach 10-year-old children to understand – not just compute – the same problem. When teachers do not understand the concepts underlying division with fractions, they do not teach the topic with deep conceptual understanding, and their students will not master division with fraction concepts. It is crucial, therefore, that we find ways to determine what our elementary teachers know deeply and ways to help them deepen knowledge when needed.

For teachers of upper level math, some states have imposed requirements of degrees in mathematics and math-specific licensure tests to address this pressure (Gitomer, 2007). Setting requirements for elementary teachers, who are expected, in the US, to teach all of the subjects, has been more challenging. While teacher preparation programs have generally increased the number of math courses and methods courses required, there is no standard for determining if that material has been deeply understood (Cochran-Smith, 2005; Cochran-Smith et al., 2016). Different states have differing licensure requirements, and in many states, candidates can be certified even if they fail the math portion of the licensure test (Cochran-Smith, 2005; Cochran-Smith et al., 2016; Evertson, Hawley, & Zlotnik, 1985; Epstein & Miller, 2011). Although there is not consensus about how much mathematics is necessary for elementary teachers to know, scholars and policy makers agree that elementary teachers must have knowledge of the mathematics content they teach – yet the field currently lacks in-depth research on whether and how teachers understand fundamental mathematical ideas to teach it.

Changes in math curriculum and standards over the past three decades have been instituted with the goal of deepening conceptual knowledge for students and teachers. As the mathematical and educational communities have worked to change the way that mathematics is taught, and that classroom mathematics is conceptualized, it has led to an evolution in school mathematics standards. (NCTM, 1980; NCTM, 1989; NCTM, 2000; CCSSI, 2010). It is unclear, however if these changes to standards and practices have led to increasing teacher knowledge and capacity to understand and teach to the new mathematical standards. This teacher knowledge, also known as Specialized Content

Knowledge (SCK), has been defined as the mathematics one must know in order to teach math that is beyond the knowledge needed to do math (Ball, Thames, & Phelps, 2008).

Previous studies have indicated that elementary teachers have weak conceptual knowledge of fundamental math concepts (Ma, 1999; Ball, 1990), and these results have driven some of the calls for reforms in teacher education and licensure (Greenberg & Walsh, 2008). These early studies included interview components that gave the researchers insight into the thinking and reasoning of teachers that is impossible to garner with strictly multiple-choice and content tests, as valuable as such tests might be for certain purposes (Hill, Ball, & Schilling, 2008; Hill, Umland, Litke, & Kapitula, 2012). The tests that grew out of those studies, however, have measured teacher knowledge almost exclusively with multiple-choice tests, and those results have been used to make claims about the status of teacher understanding (e.g., Hill, Rowan, & Ball, 2005; Ball, Hill, & Bass, 2005; Qian & Youngs, 2016).

Given that some of the strongest evidence for teachers' math content knowledge resulted from interview studies conducted more than three decades ago, and that such studies were conducted prior to the introduction of significant policy documents addressing the teaching of mathematics (CCSSI, 2010; NCTM, 2000, 2006), there is a need for research that uses interview techniques to study the current status of elementary teacher content knowledge of mathematics. Without knowing what our teachers do and do not know, we cannot design the most effective and targeted professional development nor plan appropriate teacher education courses.

1.1 Rationale for the Study

As a long-time math teacher, I first read Liping Ma's book "Knowing and Teaching Elementary Mathematics" (1999) while enrolled in a 'train the trainer' program for the curriculum "Developing Mathematical Ideas" (DMI) in 2001. The data she presented about the state of US elementary teacher mathematics knowledge was alarming, as she concluded that the teachers had little to no conceptual understanding of the mathematics they taught, and were confined to presenting procedural explanations. For example, Ma found that some of the US teachers described the subtraction algorithm, which involves a set of base ten exchanges, through unrelated analogies such as borrowing a cup of sugar from a neighbor. She also found that some US teachers believed that symbols other than a zero were appropriate as placeholders in a multi-digit multiplication problem (Ma, 1999).

Knowing that I wanted to improve the math experiences of elementary students by helping to develop even more knowledgeable math teachers, I decided to pursue my PhD in Teacher Education, focusing on elementary mathematics. I had the opportunity to sit in on a Master's level math methods course with two different cohorts in 2016 and 2017. Ma's book (1999) was required reading, which is not unusual for this type of course. The students were disheartened to read Ma's findings, and they focused on how much better the Chinese teachers in the study seemed to understand the mathematical concepts. I realized that I had read the book fifteen years earlier, and discovered that the interviews Ma based her work on were done in the late 1980s. That meant this research was almost thirty years old and still being seen by some as the current state of teacher knowledge.

Given the mathematical standards reform movements that have taken place over the past thirty years and the changes in textbooks and teacher education programs those movements had triggered, I grew to believe that teacher knowledge, especially conceptual understanding of these elementary school math topics, must have improved. Were teachers really still explaining subtraction with regrouping as borrowing a cup of sugar from a neighbor? Or suggesting that an asterisk could be used as a placeholder in multi-digit multiplication? I could find no studies that seemed to replicate or challenge Ma's work, so decided that I would undertake that for my dissertation research. My goals were to see what Specialized Content Knowledge teachers demonstrated in their explanations of mathematical concepts and to identify important themes in those explanations.

I conducted this study, using teacher interviews, to investigate current elementary teachers' knowledge of mathematics content through the following research questions:

1. How do teachers' explanations of mathematics content demonstrate Specialized Content Knowledge (SCK) for the following topics?
   a. Subtraction with regrouping?
   b. Multiplying multi-digit numbers?
   c. Division with fractions?
   d. The relationship between area and perimeter?
2. What themes are found in teachers' explanations for the following topics?
   a. Subtraction with regrouping?
   b. Multiplying multi-digit numbers?
   c. Division with fractions?

d. The relationship between area and perimeter?

1.2 Theoretical Framework

This study uses the concept of Specialized Content Knowledge (SCK), a subset of Mathematical Knowledge for Teaching (MKT) as a theoretical frame. In this section, I will define the concepts, describe the development of the Mathematical Knowledge for Teaching model, and discuss the role of Specialized Content Knowledge within that model.

1.2.1 Pedagogical Content Knowledge

In the 1970's and 1980's there was de-emphasis on content knowledge and a push toward effective teaching strategies as the most important facet of teacher knowledge (see, for example, Sparks (1983) and Kindsvatter, Ishler, & Wilen (1988)). If a teacher could implement the strategies for planning, delivery of lessons, classroom management, and classroom climate in the prescribed way, she would be an effective teacher and the students would learn, regardless of the content area. This focus on strategies led to the creation of checklists of competencies that administrators could use when observing a teacher, with the understanding that a teacher with more checkmarks was a more competent and effective teacher than one who had fewer (Shulman, 1986; Darling-Hammond, 2016).

Feeling that the pendulum had swung too far in the direction of favoring pedagogical knowledge over content knowledge for teachers, Shulman (1986) posited that good teaching was more than a set of skills to be mastered and more than a body of content to know. Good teaching required knowledge of the content, why it was important, how the concepts fit together, and how to best engage students in learning the content.

Bringing together a focus on teacher knowledge of content with the understanding that teachers must also know their students and teaching methods well, Shulman believed that strong teacher knowledge was a hybrid that he termed Pedagogical Content Knowledge (PCK).

PCK allowed a teacher to transform their knowledge of content into representations, illustrations, and examples for students and understand the conceptions and misconceptions that their students would likely hold and develop about the topic of study. Shulman also stated that a teacher with strong PCK should understand how the material they taught was situated within the other courses their students were taking and should understand the content of their subject before and beyond the course they were teaching. Shulman's PCK (1986) was general in nature and did not address any specific subject area.

1.2.2 Mathematical Knowledge for Teaching

With the concept of PCK in play, scholars worked to define how it looked in different academic subject areas. Leading the efforts in mathematics, Deborah L. Ball (1988) argued that school children who were learning mathematics at that time were unlikely to gain the skills and knowledge to deeply understand the concepts, largely because of the way they were being taught. Math was generally presented as a set of procedures one could follow to get to a right answer, and there was little effort made to explore or understand the concepts that were the foundation of those procedures. Ball also posited that knowing mathematics for oneself was arguably different that knowing it in a way that one could teach it. And, building on Shulman, she theorized that subject matter knowledge and teaching knowledge were intertwined in a way that had not been studied. She defined Mathematical Knowledge for Teaching (MKT) as the set of knowledge necessary for a teacher to possess in order to have true competence for math instruction (Ball, 1988). Ball then worked to further define MKT and its components while developing measures for MKT.

1.2.2.1 Developing and Measuring MKT

Early work on defining and measuring MKT was conducted through the University of Michigan's National Center for Research on Teacher Education (NCRTE), which was formed in 1985 and renamed the National Center for Research on Teacher Learning (NCRTL) in 1991. Between 1986 and 1990, this center undertook a large, multi-site, longitudinal study, called the Teacher Education and Learning to Teach Study (TELT), to examine how teacher knowledge for teaching elementary mathematics and writing changed as participants transitioned from teacher education programs to classroom placements (McDiarmid & Ball,1989; NCRTE, 1991). The mathematics portion of this study included a questionnaire, interviews of focal participants, and classroom observations of selected participants to see how various approaches to teacher education related to mathematics influenced teacher understandings and beliefs, while also trying to parse out the impact of elementary preservice teacher preparation programs compared with other influences. As part of her doctoral program, Ma worked with Ball's team to transcribe and analyze the teacher interviews, and her dissertation, which became the book "Knowing and Teaching Elementary Mathematics" (1999), grew out of that research.

As the researchers attempted to design measures that would test not only content, but the pedagogical knowledge that they conceptualized as how teachers saw the relationship between student and concept, they found that their model could be further refined, as shown in Figure 1.1 (Hill, Schilling, & Ball, 2004; Hill et al., 2008; Ball et al., 2008). In this conceptualization, mathematical content knowledge is divided further into 6 types of knowledge, three of which fall under subject matter knowledge: common content knowledge (CCK), which is knowledge that anyone with a solid math background should hold; horizon content knowledge (HCK), which is knowledge about how the mathematics one is teaching is situated within the larger field and within the scope and sequence of school mathematics (Jakobsen, Thames, Ribeiro, & Delaney, 2012); and Specialized Content Knowledge (SCK), which is the knowledge of mathematics that is specific to being able to teach it. SCK houses the knowledge to answer the questions students have about how and why the mathematics functions as it does.

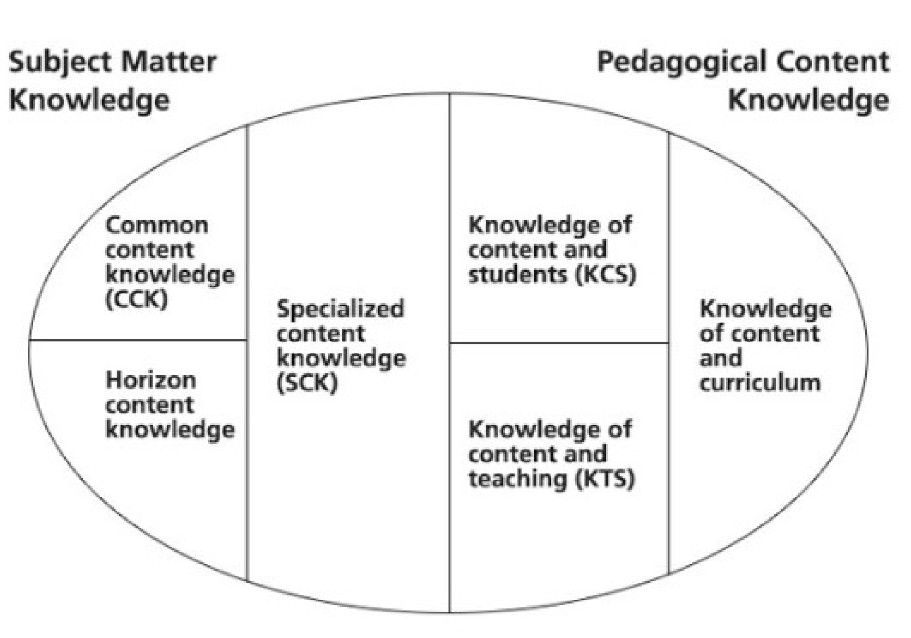

Figure 1.1. Domains of Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008).

The other half of the model contains the elements of pedagogical content knowledge, which acknowledge the relationships among knowledge of content, students, curriculum, and teaching. As stated earlier, what good is it if one understands how to calculate $1 (3)/4 \div 1/2$ if one does not understand how students learn such a concept, what tools are available to help teach the concept, or when it is appropriate to teach the concept? The elements of PCK in this MKT model address those types of knowledge and decisions.

This dissertation focuses on the Specialized Content Knowledge domain of MKT, which I describe in greater detail below.

1.2.3 Specialized Content Knowledge

As noted above, Specialized Content Knowledge (SCK) is defined by Ball and her colleagues as the knowledge of mathematics that is specific to being able to teach it (Ball et al., 2008). Teachers with strong SCK can design, justify, and evaluate mathematical explanations and conjectures; be able to explain how and why algorithms work; generate relevant contexts, examples, or counter examples to illuminate math concepts; evaluate student errors and invented strategies; and know and connect multiple representations for mathematical concepts (Hill et al., 2004; Carreño, Ribeiro, & Climent, 2013; Zembat, 2013; Bair & Rich, 2011; Kazemi, Lesseig, Mumme, Caroll, & Kelley-Petersen, 2009; Selling et al., 2016). They can not only identify an incorrect answer, but can suggest the logical method or faulty reasoning needed to produce it (Markworth, Goodwin, & Glisson., 2009). Lin, Chin, & Chiu (2011) argue that these facets of SCK can be categorized into three areas: “explanation – how to provide mathematical explanations for common rules and procedures; representation – how to choose, make, and use mathematical representations effectively and accurately; and justification – how to explain and justify one's mathematical ideas" (p.1)

These components of SCK are represented in both the NCTM's Principles and Standards for School Mathematics (2000) and in the Common Core State Standards for Mathematics (2010) expectations for students. If we are expecting students to develop the ability to explain, represent, and justify their mathematical ideas and processes, we must make certain that their teachers have the knowledge to teach in ways that develop those skills.

1.2.4 Assessing MKT and SCK

Throughout the development of the MKT framework, researchers worked to develop measures that could test for aspects of MKT in teachers. Ball and her associates, as part of the Learning Mathematics for Teaching (LMT) project, developed a catalog of over 1000 multiple-choice test items intended to measure aspects of MKT in elementary teachers (Hill et al., 2004; Hill et al., 2008). These multiple-choice tests present classroom-based scenarios that involve a mathematical decision a teacher must make or an explanation that must be given, thereby attempting to provide contexts in which responses reflect the interplay of content knowledge with other forms of knowledge needed to teach content to students. Researchers found, however that some teachers could employ test-taking strategies in order to pare down possible responses in ways that led to higher scores, and interviews showed that some teachers who scored well did not necessarily have a strong foundation in the topic being assessed (Hill et al., 2008).

SCK has generally been assessed using items from the LMT (see Strawhecker, 2005; Welder & Simonsen, 2011; McCoy, 2011; Swars, Hart, Smith, Smith, & Tolar, 2007). That test, however, is not intended to be an assessment of individual teachers, but was designed to get an overall sense of teacher knowledge from a group of participants, and can be used to determine if courses or professional development efforts have improved SCK for a group (Selling et al., 2016).

While other researchers have developed their own questionnaires (Zembat, 2013), qualitative tools (Bair & Rich, 2011; Leavy & Hourigan, 2018), and scoring rubrics (Ho & Lai, 2012) to try to assess SCK in teachers of K-12 mathematics, these have generally been applied to single mathematical topics. While it is important to understand the facets of SCK for different mathematical concepts, these single-topic topic studies give us a very limited view of teacher knowledge. Current measures of MKT and SCK, therefore, rely heavily on multiple choice and paper-and-pencil tests. As research has shown that those tests are not sufficient for determining the mathematical knowledge of individual teachers, this study seeks to fill that gap through the use of teacher interviews intended to capture the ways that current teachers think about and explain several mathematical concepts (Hill et al., 2008; Hill et al., 2012).

1.3 Overview of Chapters

Through a review of the literature in Chapter 2, I examine current research related to MKT and SCK, noting what those measures highlight and ignore. I then look at research related to examining research related to the Common Core State Standards for Mathematics, looking specifically at how teacher knowledge of the content standards has been studied. The end of the review in Chapter 2 focuses on the four content areas addressed in this study: subtraction with regrouping; multi-digit multiplication; division with fractions; and area, perimeter, and proof. I examine the methods that have been used to study teacher knowledge and student learning on these topics, as well as what research says about the challenges in teaching and learning them.

In Chapter 3, I present my methodology for the study, including the design, description of the participants, methods for data collection, and techniques for data analysis. Chapters 4 and 5 contain the findings for the two research questions, with Chapter 4 focusing on the Specialized Content Knowledge demonstrated by teacher explanations and Chapter 5 examining the themes found in those explanations. Both chapters contain excerpts from the interviews that illustrate the findings. Chapter 6 contains a summary of major findings and a discussion of those findings, which lead into implications and recommendations for teacher education.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

The purpose of this study was to examine the Specialized Content Knowledge (SCK) demonstrated by current elementary teachers as they explained four different math topics and then to describe the themes found in those explanations. The focus on multiple content areas allows me to look for variation in SCK across individual participants and across different topics.

This chapter focuses first on studies related to defining and measuring Mathematical Knowledge for Teaching and Specialized Content Knowledge, noting that the most common measure is a multiple-choice test that does not necessarily predict teaching quality (Hill et al., 2008). Since the push for defining and measuring MKT and SCK is to allow for stronger teaching, the review then looks at how MKT impacts student learning, noting that there are correlations between MKT level and student achievement (Hill & Lubienski, 2007), and that our least-resourced students tend to have teachers with lower MKT (Ball et al., 2005; Hill et al., 2005; Tatto et al., 2008). I then address the ways in which professional development efforts and teacher education have tried to improve MKT and SCK in teachers. Moving next to how MKT has made its way into educational policy discussions, I note that policy influencers have cited data on teacher knowledge that was decades old (Greenberg & Walsh, 2008).

In order to understand the Specialized Content Knowledge for teaching, we need to know what the field currently holds to be best practices, noting that these may have changed since Ma's study was published in 1999. To that end, I present content background, best practices, and studies related to the four topic areas addressed in the study: subtraction with regrouping; multi-digit multiplication; division with fractions; and area, perimeter, and proof. While these studies do not focus on SCK directly, they do look at the concepts and understandings that would be measured when determining SCK. In these sections, I note that, except for division with fractions, there is little research on how teachers and PSTs understand and engage with these topics. Available research does show a focus on algorithms and a lack of conceptual understanding from both in-service and preservice teachers.

Through these studies, it can be seen that we need more current information about the state of SCK in teachers around these four topics, as that information can help us to determine if our teacher education programs and in-service professional development efforts are sufficiently preparing teachers to instruct students in the ways expected by today's high-stakes standards.

2.2 Mathematical Knowledge for Teaching and Specialized Content Knowledge

As we consider the current state of elementary teacher Mathematical Knowledge for Teaching in light of changing standards, it is important to see what relevant research has found with respect to defining and measuring MKT and SCK. It is also important to understand how MKT has been shown to impact student learning and achievement. And since studies on MKT have sometimes been cited to influence the requirements for teachers and set standards for teacher preparation programs (TPPs), we must examine research on the educational policies that been informed by those studies. As I detail below, these studies show that MKT, as strong as the underlying theory might be, is challenging to measure and many proxies are used for it; it does have an impact on student learning, though the relationship is challenging to generalize; and policy makers often use proxies for MKT that have no research basis in determining the qualification of new teachers.

2.2.1 Defining and Measuring MKT and SCK

Since its conception, researchers have looked for efficient ways to measure Mathematical Knowledge for Teaching (MKT). Recall from Chapter 1 that Ball (1988) developed the concept of Mathematical Knowledge for Teaching from the foundation of Shulman's (1986) Pedagogical Content Knowledge (PCK). It was obvious to them that teachers needed to know more and different mathematics to teach math conceptually than one would need to know to do math. She and her colleagues developed a multiple-choice test, known as the LMT, as part of the Learning Mathematics for Teaching Project (McDiarmid & Ball,1989; NCRTE, 1991). This test was intended to measure MKT, and was also used to determine if there were multiple components to MKT.

The multiple-choice items of the LMT were intended to test for both content and pedagogical knowledge of elementary teachers, and researchers also developed interview questions that could assess how deeply participants understood mathematical content and the strategies for teaching that content (Ball, 1990). Using the multiple-choice instrument, as part of the Teacher Education and Learning to Teach Study (TELT) mentioned previously, Ball (1990) looked at the prior mathematical knowledge of 252 preservice elementary teachers in five different teacher preparation programs and also tracked the change in that knowledge throughout the program. She found that the initial knowledge the participants brought from their high school and college courses was "rulebound and thin." In other words, perhaps adequate for non-teachers, but insufficient for teaching. Through analyzing both the results of the longitudinally-administered questionnaires and the smaller subset of interviews, the researcher found that what is learned in K-12 mathematics is generally not sufficient for teaching that mathematical content, and that majoring in mathematics in college did not adequately prepare participants to teach the content well. She concluded that elementary school mathematics is challenging to teach and to learn, and that we should look beyond content in deciding who is ready to work in a classroom. This argues for assessment of novice teachers beyond content-based licensure tests.

Using results from an even larger, 1500-participant study associated with the teacher questionnaire, researchers attempted to determine if one can measure MKT as a singular construct, or if it is further composed of measurable sub-components that emerge from the analysis (Hill et al., 2004). They argued that the finding of more than one dimension, in this case common content knowledge and Specialized Content Knowledge, supports the development of a teacher preparation curriculum that goes into more depth than content courses and focuses on the specific work teachers must do to develop strong conceptual understanding in their students. While non-teachers can hold common content knowledge (CCK) like finding a decimal that is halfway between 1.1 and 1.11, teachers need more Specialized Content Knowledge (SCK) in order to examine and appreciate multiple representations of a topic, provide clear and correct explanations to students, understand and correct student errors, and evaluate unexpected or uncommon methods that students may use. While the researchers could identify two components to MKT (CCK and SCK), they acknowledged that much more work needed to be done to further identify the elements of the Specialized Content Knowledge that teachers need to best help students learn mathematics.

2.2.1.1 SCK Critiques and Boundaries

Some scholars have questioned the definition of SCK as the knowledge needed to teach math as opposed to the knowledge needed to do math. They ask where the knowledge of mathematicians fits into this, and whether there must be some intention for teaching for the knowledge to qualify as SCK. They also wonder what the benefits are to categorizing math knowledge in these different ways (Flores, Escudero, & Carrillo Yáñez, 2013).

Other scholars have wondered if, by being taught in a conceptual way, K-12 students can develop SCK as well as CCK, and whether the dividing line between those two is blurred or moved (Browning et al., 2013; Selling et al. 2016; Leavy & Hourigan, 2018). They also have questioned some of the borders between SCK and Knowledge of Content and Teaching, and SCK and Knowledge of Students and Teaching, which are two of the pedagogical aspects of MKT (Browning et al., 2013; Markworth et al., 2009; Carreño et al., 2013).

Browning and her colleagues (2013) have worked to better define the boundaries between SCK, CCK, and the pedagogical knowledges, and have developed some examples to show the differences between the categories. For example, when working with fractions, they argue that the CCK would involve understanding and solving problems with fractions, the SCK would involve understanding multiple representations of those problems, and the pedagogical knowledge would include understanding children's struggles and misconceptions around the idea of the unit. Through this type of delineation, it is easy to see that a teacher could have different levels of CCK, SCK, and pedagogical knowledge based on the topic.

SCK appears to be distinctive from CCK and pedagogical content knowledge in terms of what it enables a teacher to do, and how it enables them to teach (Leavy & Hourigan 2018, Carreño et al., 2013). SCK is "knowledge required by the teacher who genuinely wishes their students to understand what they do, and not merely mechanically run through [mathematical] procedures" (Carreño et al., 2013). There is clearly knowledge that is useful to teachers, but not to others. Jakobsen et al. (2013) illustrate this through the topic of factoring trinomials. While we might expect algebra students to be able to factor, knowing that a pair of related trinomials are factorable and how to generate more factorable pairs would only be expected of teachers. And, while most adults remember learning the algorithm for division with fractions, often through a mnemonic with little to no mathematical meaning, such as 'yours is not to question why, just invert and multiply', or 'keep-change-flip,' few can explain why the algorithm works or understand how it relates to the meaning of division and part-whole relationships (Bair & Rich, 2011).

2.2.2 Mathematical Knowledge for Teaching and Teaching Quality

Hill et al. (2008) noted the importance of interviews in learning how teachers make use of their Mathematical Knowledge for Teaching in the classroom. Expanding their work further to determine if higher levels of MKT actually lead to better teaching, researchers again used the questionnaire developed for the TELT study and an instrument to measure Mathematics Quality of Instruction (MQI), looking for a relationship between MKT and MQI (Hill et al., 2008). MQI for the ten participants was measured through a rubric that tried to capture the level of rigor and depth of mathematics in each lesson by looking at the representations a teacher used, the explanations and justifications they offered, and any errors in the teaching. In other words, how well they demonstrated the major facets of Specialized Content Knowledge. Those same participants were also given the LMT test to determine their MKT level. The researchers found a strong, positive relationship between MKT and MQI in this small sample, though there were two teachers who had strong MKT and low MQI. The researchers needed to conduct interviews and observations in order to gauge more fully the knowledge that the teachers held and how that related to their classroom practices. It was not noted if the classroom observations were directly related to the test questions given to determine MKT.

Based on the above findings, the researchers put forth a recommendation that the US consider using teachers with high MKT as math specialists to teach all mathematics in elementary schools. This, they argue, would give more students access to teachers with the level of MKT that is associated with higher-quality instruction. This specialization is practiced successfully in other countries and could be a way to improve the conceptual math knowledge of the next generation of students, possibly leading to improved MKT levels in future preservice teachers. Given that this specialist structure is not currently common in the US, we must work within teacher education to help candidates improve their MKT. Improving baseline MKT is especially crucial because research has shown that schools with students of low socioeconomic status (SES) are more likely to have teachers with low MKT scores (more than one standard deviation below the mean), which are linked with lower student achievement (Hill & Lubienski, 2007). When our poorest students are more likely to be taught by our least able teachers, their opportunities to learn math deeply and conceptually are reduced, leading to fewer opportunities and options.

These studies show that MKT and its components can be measured, but to gain better insight into teacher knowledge and instruction, qualitative interviews and observations need to be included. It is also important to note that none of these studies considered mathematical standards, either state or NCTM, when generating content questions or analyzing findings. Since our students, and our teachers, will be judged on how well they perform on standards-related questions, that aspect seems important.

2.2.3 MKT and Student Learning

The discussion about Mathematical Knowledge for Teaching is only relevant if there is a relationship between MKT and student achievement. Many studies have examined which teacher, school, community, classroom, and student factors affect student achievement, and a 2002 prospectus study found that between three and thirteen percent of variance in student performance is linked to individual classroom teachers (Rowan, Correnti, & Miller, 2002); in other words, is linked to individuals' Mathematical Knowledge for Teaching. Their review of research identified that the largest effects on achievement correlated with the experience level of the teacher, the use of whole-class instruction, and exposure to a rigorous mathematics program. For a rigorous mathematics program to be effective, a teacher must know the material well enough to present it accurately to students, indicating a measure of MKT.

Making the case that teachers cannot teach well what they do not know themselves, Shirvani (2015) tested 87 preservice teachers in Texas using the mathematics portion of the sixth-grade Texas Assessment of Knowledge and Skills (TAKS). He then compared their results to those of the sixth graders who took the same test. More than one-third of the preservice teachers failed the measurement portion of the exam, as did more than half of sixth graders. While these PSTs were not classroom teachers at the time, they were likely representative of those teachers who had entered the field from the same teaching preparation programs. Shirvani makes the case that lack of content knowledge in teachers is likely affecting student achievement in the specific weak areas. This finding is supported by another study, which focused not on teacher knowledge, but on the teacher behaviors that predict student achievement (Blazar, 2015). Studying 111 fourth- and fifth-grade teachers in two school districts, the author found that teachers' unwitting mathematical errors, which are likely strongly linked to teacher knowledge, had a significant and negative effect on student achievement.

Several studies have found that students taught by teachers who scored higher on tests measuring Common Content Knowledge (CCK) and Specialized Content Knowledge (SCK) performed better on assessments than did those whose teachers had lower levels of those aspects of MKT (Ball et al., 2005; Hill et al., 2005). In other words, high levels of MKT let to greater student performance. Questions from the TELT study, related to those used for my dissertation study, were used in two of the related studies, assessing both CCK and SCK. Importantly, researchers found that teacher knowledge as measured by these items was a stronger teacher-level predictor of student gains than either the average time spend each day on math instruction or qualities of teacher background (Hill et al., 2005). In other words, more instruction from teachers with poor SCK does not support children's learning of mathematics. Learning gains for students were on the order of two to four weeks more than those whose teachers were at the median of teacher knowledge scores, and the effect size was as large as that as the effect of socioeconomic status on student gains. Low SES students tend to show a significant achievement gap when they enter school, and that gap increases with each year of schooling (Ball et al., 2005).These studies of about 700 first-and third-grade teachers and 3000 of their students found that higher-knowledge teachers tended to work with non-minority students, leaving minority students with lower gains that would add up over time (Ball et al., 2005).

Echoing these results on an international stage, a cross-national study that used data from the Trends in International Mathematics and Science Study (TIMMS) focused on how teachers around the world are prepared to teach elementary and middle-grades mathematics (Tatto et al., 2008). Using teacher instruments intended to measure what they called mathematical pedagogical content knowledge (MPCK), which was conceptualized very similarly to Mathematical Knowledge for Teaching, the researchers found that students taught by teachers with higher MPCK scores performed significantly better in tenth-grade math than did their peers, when controlling for ninth-grade math achievement.

These studies indicate that we learn much more detail from interviews than from a written assessment alone, indicating the need for studies that include more qualitative components. They also indicate that teacher MKT is a predictor of student achievement in mathematics, which points to the need to make sure all students have access to a teacher with strong content and pedagogical knowledge. To address this need, teacher preparation programs and in-service professional development programs have designed interventions intended to improve MKT or aspects of MKT. Also indicated is that we learn much more Studies of those programs will be discussed in the next section.

2.2.4 Efforts to Improve MKT and SCK

The components of MKT, both Specialized Content Knowledge and the pedagogical practices, take time to develop. Efforts over the past fifteen years have attempted to support and improve in-service and preservice teachers' MKT. As I was examining studies for this literature review, I found much more research focused on preservice teachers (PSTs) than on in-service teachers, who were the participants in my study. I include those PST-based studies as they give us the starting points for teacher SCK and information about the courses, content, and activities that seem to impact SCK development.

Preservice teachers need to be exposed to more facets of actual teaching during their teacher preparation coursework and practica to improve their Specialized Content Knowledge, according to Morris, Hiebart, & Spitzer (2009). In their study, 30 K-8 PSTs who were presented with learning goals for a unit had difficulty identifying subskills or sub-concepts, ideas that go beyond common content knowledge, that would be needed to meet those goals, and they also struggled to plan or implement lessons appropriate to address or assess the goals. The Specialized Content Knowledge required to be successful in those tasks requires knowledge of how to determine what students already know, what activities and lessons can move them from that point to the new concepts, which teaching strategies are most effective, what misconceptions are likely to occur and how to address them, and which strategies students are likely to try. This combined knowledge of content, students, and teaching, the pillars of MKT, takes time to develop that we normally don't have time to give it. This leads to teachers entering the field without firm skills in place.

To address the need to improve SCK in preservice teachers, teacher preparation programs have designed and studied redesigned or new courses. For example, one course for elementary PSTs in a teacher preparation program had participants learning math content and also teaching that content in a math enrichment program for seventh and eighth graders (Jonker, 2008). This allowed the participants to work on content and pedagogy at the same time, seeing how to apply what they were learning and giving it greater context. While participants in the program were eloquent in expressing how well it worked for them, the author did not give the number of participants or provide formal findings. More research into this type of course would be needed to say that it is a model to be emulated.

Preparing and revising videocasts, which are podcasts with a video component, was shown to improve MKT by giving teachers the opportunity to think deeply about both the content of the lesson and the methods of presenting that content. A rigorously studied intervention had both in-service and preservice teachers creating explanatory math videocasts, which were intended to both measure and improve MKT over the semester (Galligan, Hobohm, & Peake, 2017). Studying four cohorts of 40-50 students, the researchers developed surveys and rubrics for assessing both the mathematical content and the pedagogical approaches of the participants. The videocasts were to be presented as if the participants were teaching the material to middle-school students, though the actual grades taught by in-service teachers was not given. Initial results showed that the participants were weak in both content and teaching strategies, but with peer and faculty feedback, the participants showed good growth in their MKT through improved explanations. Missing from this was any interaction with students, but this type of activity could help with the incremental work of building Specialized Content Knowledge and practicing without risk while getting ample feedback from others.

Several studies have explored how to improve SCK in PSTs, especially through specially designed math content and methods courses. Looking at 69 preservice elementary teachers in the first semester of a year-long sequence of elementary-specific math content courses, Welder and Simonsen (2011) found that participant SCK improved more than .6 standard deviations, as measured by the multiple-choice test developed by Ball and her colleagues, the LMT. These gains were significant, as were the students' gains in CCK. The researchers noted that the courses blended content and pedagogy and focused on hands-on activities, manipulatives, and exploring instructional strategies through examining student work and errors.

In a study of similar courses, results indicated that PSTs with higher levels of SCK, as measured with the LMT, were more likely to believe that children can construct their own knowledge, and that mathematical procedures and processes should be taught with conceptual understanding. In this study, researchers looked at teacher beliefs about self-efficacy and effectiveness and how those related to SCK in PSTs. The courses were designed to promote conceptual understanding of mathematical content, to focus on problem solving and representations, and to encourage communication, connections, and proof. The researchers did not indicate what, if any, change in Specialized Content Knowledge occurred over the course of the study (Swars et al., 2007).

Leavy and Hourigan (2018) also found that a focus on a combination of content and pedagogy supported SCK improvement. Using lesson study, a professional development activity that has small groups of teachers collaboratively develop, discuss, and reflect on a lesson, PSTs developed understanding of the complex relationships between early number concepts that lead children to strong number sense, and developed the knowledge necessary for identifying the sources of children's errors. The authors noted that, in Ireland, teacher education programs are very competitive, with admitted students ranking in the top 15% of high school graduates. During their elementary teacher education program, elementary PSTs engage in 100 hours of mathematics education courses which focus extensively on mathematical pedagogy along with content. This study added lesson study to the curriculum for a group of 25 primary-level PSTs. Using qualitative thematic analysis similar to the analysis shared in Chapter 5, they analyzed the Specialized Content Knowledge of the participants over the semester and found that the lesson study enhanced the understanding the preservice teachers had about number concept development.

Also using a qualitative approach to measure SCK, Bair and Rich (2011) found that the ability of preservice teachers to pose problems, a teaching skill that involves both content and pedagogy, was linked to development of Specialized Content Knowledge. The researchers explored the development of SCK in algebraic reasoning and number sense for PSTs in a middle school math specialist program. The content courses focused on having students unpack their mathematical ideas so that they could better understand the conceptual underpinnings of procedures, and on having the PSTs explain and justify their mathematical reasoning. Researchers found that many students could explain what they did, but not why they did it. The level to which the preservice teachers could justify their answers indicated the depth of their understanding. Problem posing, which they defined as being able to formulate new questions relative to a given problem or being able to restate a problem while solving it, was found to be tightly tied to SCK development. Those PSTs who lacked problem-posing skills tended to stagnate in their SCK development, but focused attention on those skills tended to restart the growth in SCK.

Looking at the effects of a mathematics methods course on MKT, Kajander (2010) examined the procedural and conceptual knowledge of preservice teachers at the start and the end of a one-year teacher preparation program (TPP) over the course of three cohorts. Kajander considers procedural knowledge to be a sequence of steps or set method one follows to get to an answer, and conceptual knowledge requires an understanding of the underlying structure and relationships of the mathematical ideas. She found that all of the more than 300 participants were initially weak in both content and pedagogy, indicating that their pre-course mathematics was not sufficient to prepare them to teach. The mathematics course she was teaching and assessing had a strong conceptual focus, and students improved their understandings, but only to what she considered a minimally acceptable level for teachers who would need to teach the concepts to children in the near future. These results led to the creation of an extra (but optional) mathematics course and the institution of a mandatory high-stakes examination for those in the TPP, which seemed to be showing promise in increasing Mathematical Knowledge for Teaching.

These studies indicate that Specialized Content Knowledge can be improved through courses that are specifically designed to promote conceptual understanding of mathematical ideas and development of pedagogical skills. They also demonstrate that researchers have found many ways to measure MKT and its component part SCK, with some using the multiple-choice test, the LMT, developed by Ball and her colleagues and others using qualitative or quantitative methods they have constructed for that purpose.

2.2.5 MKT in Policy Discussions

How we measure MKT and SCK and which proxies we allow to represent them are important because policy makers institute requirements for schools and teachers based on such measures, or their assumptions about such measures. Some who try to influence teacher policy have used Ma's findings of low Specialized Content Knowledge, which she called Profound Understanding of Fundamental Mathematics, of elementary teachers as strong evidence that teacher preparation for elementary teachers in the US needs to change (Greenberg & Walsh, 2008). The National Council on Teacher Quality (NCTQ), a partisan and political research group, has called for higher standards in coursework and GPAs and administration of standardized assessment tests before students are admitted to teacher preparation programs (TPPs) and calls for changes to coursework during those programs. Fuller (2014) looked into the NCTQ review of teacher preparation programs and found many questionable practices. The NCTQ has established a common set of standards for all subject areas and grade levels without sufficient evidence to support those standards, and it did not correlate its quality ratings of programs with available data such as licensure test pass rates or states' value-added measures. Sleeter (2014) expanded the critique to note that the NCTQ based their review of TPPs solely on syllabi and student teaching handbooks, never visiting programs or examining how graduates actually fare in the classroom.

Those who have examined the data on teacher preparation and student achievement find that there is little evidence to support reforms like higher GPAs, nor is there clear research on how much or which math is necessary for those teaching in elementary schools (Evertson, Hawley, & Zlotnik, 1985; Cochran-Smith, 2005; Cochran-Smith et al.,2016). Screening tests, which are largely multiple-choice content tests, have been found to disproportionately exclude minority candidates, calling into question the generalizability of the test results (Evertson, Hawley, & Zlotnik, 1985). Qian and Young (2016) also found little correlation between more college-level mathematics courses for teachers and greater math content knowledge or math pedagogical content knowledge. Using multiple-country data, they did find a statistically significant association between prior math achievement and teacher math content and pedagogical content knowledge across the participating countries. This finding indicates that school mathematics is important in preparing future teachers, and that we need to make sure prospective teachers have access to rich and rigorous mathematics before they attend colleges and universities.

Types of teacher certification (traditional, alternate route, teaching fellow, Teach for America, etc.), have been seen as proxies for teacher knowledge and quality, but researchers have not found that types of certifications or advanced degrees are well-correlated with student achievement (Buddin & Zamarro, 2009; Kane, Rockoff, & Staiger, 2008). A study in a large New York school district found that student achievement did increase with teacher experience, but the link was weak and largely tied to poor outcomes associated with first- and second-year teachers (Buddin & Zamarro, 2009). Kane et al. (2008) found that classroom effectiveness in the first two years of teaching was a more reliable indicator of future effectiveness than was initial certification status. They calculated that raising initial effectiveness of New York teachers by one standard deviation would have the same impact as the improvement associated with eight years of teaching experience. Both studies indicate that we need to find ways to make novice teachers more prepared to teach, so that they are safe to practice with students. Kane et al. (2008) argue for the development of tools to better evaluate teacher performance.

Darling-Hammond (2010, 2016) agrees that we need better tools to assess teaching practices for preservice and novice teachers. Most current licensure tests are multiple-choice content tests that cannot assess teaching skill. Several states have developed portfolio processes that are meant to evaluate novice teachers in their first two years of teaching. These portfolios ask teachers to document their lesson plans, videotape and critique their teaching, and collect and evaluate evidence of student learning. Darling-Hammond argues that these are much more effective in assessing teaching practice than are the checklists of teaching behaviors most schools use. If these types of portfolios could be instituted in TPPS, we could better engage our candidates in the daily practices of teaching and potentially increase their pedagogical skill.

It is challenging to translate research into programs that impact teaching practice. Sleeter (2014) found that only 6% of 196 articles published in 2012 in four leading teacher education journals examined the impact of teacher education on teaching practice or student learning. None of those articles addressed the impact of teacher education on mathematics. We need current data on elementary teacher Specialized Content Knowledge to inform both teacher preparation programs and policy. This data needs to be anchored in more than multiple-choice tests and needs to be rooted in current content standards.

Looking at these studies, it is evident that measuring MKT with multiple choice tests alone does not give us a complete picture of the knowledge that a teacher holds, nor how a teacher intends to put that knowledge into practice in a classroom. Many proxies have been used for MKT, including number of math courses taken and college major, but those proxies have not been shown to align with student achievement in mathematics. There is a need for a more qualitative study of MKT and, more specifically, SCK, in current elementary teachers that replicates the earlier work of Liping Ma. While her work has been cited by policy advocates (Greenberg & Walsh, 2008), her research was conducted prior to the mathematics standards movements of the past three decades. The TELT questions that she used, however, are still well-aligned with current topics and expectations in the elementary classroom, and therefore appropriate for use in this study. This work can help as we set priorities for teacher education programs and in-service professional development.

The next sections of this literature review will shift to research that addresses each of the four topic areas present in this study: subtraction with regrouping, multi-digit multiplication, division with fractions, and area, perimeter, and proof. I will look at how those topics have been studied, what has been uncovered about teaching and learning those topics, and current best practices around those topics. Teachers' instructional practices are nested within a broader educational context that certainly includes state and national policy decisions that districts and schools are required to implement. Research related to MKT, often from the early studies of Ma (1999) and Ball (1990), has played a consequential role in such policy decisions. As I consider and critique here, at times those policy decisions have not appropriately or adequately reflected more recent research on the effects of that policy, nor has there been an update to Ma's work.

How we measure MKT and SCK and which proxies we allow to represent them are important because policy makers institute requirements for schools and teachers based on such measures, or their assumptions about such measures. Some who try to influence teacher policy have used Ma's findings of low Specialized Content Knowledge, which she called Profound Understanding of Fundamental Mathematics, of elementary teachers as strong evidence that teacher preparation for elementary teachers in the US needs to change (Greenberg & Walsh, 2008). The National Council on Teacher Quality (NCTQ), a partisan and political research group that has cited Ma in their proposals, has called for higher standards in coursework and GPAs and administration of standardized assessment tests before students are admitted to teacher preparation programs (TPPs) and calls for changes to coursework during those programs. Fuller (2014) looked into the NCTQ review of teacher preparation programs and found many questionable practices. The NCTQ has established a common set of standards for all subject areas and grade levels without sufficient evidence to support those standards, and it did not correlate its quality ratings of programs with available data such as licensure test pass rates or states' value-added measures. Sleeter (2014) expanded the critique to note that the NCTQ based their review of teacher preparation programs solely on syllabi and student teaching handbooks, never visiting programs or examining how graduates actually fare in the classroom.

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2.3 Specialized Content Knowledge in the Four Mathematical Topic Areas

The next sections of this literature review will shift to research that addresses the different Specialized Content Knowledge embedded in each of the four topic areas present in this study: subtraction with regrouping, multi-digit multiplication, division with fractions, and area, perimeter, and proof. I will look at how those topics have been studied, what has been uncovered about teaching and learning those topics, and current best practices around those topics. It will be seen that each of these topics is complex and has its own unique set of underlying mathematical ideas. As with the prior section, most of the research has been conducted with preservice teachers, giving us a lens into the starting point for SCK for the various topics.

2.3.1 Subtraction with regrouping

Subtraction is when we take one quantity away from another or find the difference between two quantities. Regrouping comes into play when we use the traditional US algorithm to solve problems like 52-25, where there are more ones in the ones place of the subtrahend (the number that is *being subtracted*) than in the ones place of the minuend (the number we are subtracting *from*). To use the standard US algorithm for subtraction with regrouping, we generally rewrite the problem vertically, and exchange one of the tens in the minuend for ten ones. We then subtract the ones and the tens, arriving at the difference (see Figure 2.1).

$$\begin{array}{rrr} & 5 & 2 \\ - & 2 & 5 \\ \hline \end{array} \qquad \begin{array}{rrr} & 4 & \\ & \cancel{5} & 12 \\ - & 2 & 5 \\ \hline \end{array} \qquad \begin{array}{rrr} & 4 & \\ & \cancel{5} & 12 \\ - & 2 & 5 \\ \hline & 2 & 7 \end{array}$$

Figure 2.1. Subtraction with regrouping

There are several places students can make errors in this process that generally underscore misunderstandings. The first is subtracting up in the ones place, so getting an answer of 33 for the given problem. The second is adding one to the ones place in the minuend and not ten, so making 43-25. A third is forgetting to change the “5” in the minuend into a “4,” so getting an answer of 37. Part of the Specialized Content Knowledge for this topic is understanding how each error indicates how a child thought about the underlying mathematics. The first error indicates that the student is not thinking about each number as a whole, but as unrelated digits. The second error indicates that the student thinks of the “borrowed” amount as a one instead of a ten, so is forgetting the values associated with each place in the number. The third error can indicate that a student is not understanding that the ten is being converted into ten ones and now no longer exists. Combatting these misunderstandings requires teachers and students to focus on the relationship of the tens to the ones and to the composition of each number.

Subtraction, with or without regrouping, can also be challenging because it has so many more contexts and representations than addition (Van de Walle, Karp, & Bay-Williams, 2013). While subtraction names a missing part, it can be seen as a “join” problem with either the change or the start unknown; a “separate” problem with the result, change, or start unknown; a “part/part/whole” problem with a part unknown; or a “compare” problem with either the difference, smaller number, or larger number unknown. Providing context and modeling helps students to determine which operation is involved and how to set up the problem. Subtraction can also be thought of as a “think-addition” problem, such as “what would I need to add to 8 to get 14.”

Teachers are encouraged to start with student-invented strategies, as students tend to make fewer errors and develop number sense in creating them. They also promote mental math and can sometimes be faster than the standard algorithm. Van de Walle et al. (2013), who speak to best practices supported by research, suggest using models, including number lines to teach subtraction. They also suggest a focus on splitting numbers apart flexibly, turning 8+5 into 8+2+3, for instance, which can help students develop their mental math skills. Bridging the tens and hundreds is a topic they note can be particularly challenging for children, and they also suggest that teachers anticipate difficulties with zeros, especially when they are in the minuend of the problem, both aspects of the MKT teachers need to hold. Manipulatives, especially base 10 blocks, model in concrete ways the mathematics embodied in the algorithm, and their use is encouraged as teachers introduce the use of the standard algorithm. Using base 10 blocks, we could model the number 52 as shown in Figure 2.2. Each small square represents a one, and is a cube in a physical set of blocks. The long rectangles are exactly the length of ten unit cubes set edge to edge and represent ten. One can exchange one of the tens for ten unit cubes and represent the 52 as four tens and 12 ones (see Figure 2.3). Then it is much easier to act out the subtraction of 25 by removing two of the tens and 5 of the ones, leaving two tens and seven ones behind, and giving a physical representation to the steps of the algorithm.

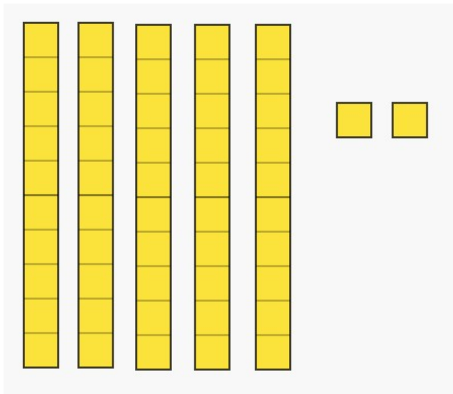


Figure 2.2. 52 modeled in base 10 blocks.

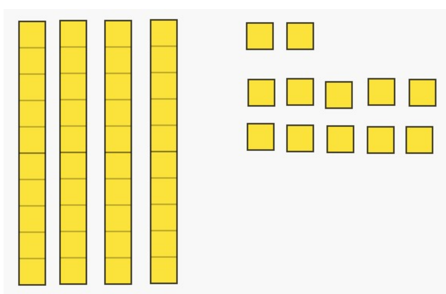


Figure 2.3. Exchanging a ten for ten ones.

As indicated above, subtraction with regrouping involves a coordination of multiple mathematical ideas. To capture the varied ideas needed for this mathematical topic, Ma (1999) identified what she called the knowledge package for subtraction with regrouping and created a web of ideas that build on one another to develop the full concept of subtraction with regrouping. She started with addition and subtraction within 10, then within 20, moved to understanding the decomposition of tens and that our number system is based on groups of tens, and connected all of that to the idea that addition and subtraction are inverse operations. These elements align well with standards from the Common Core State Standards for Math (CCSSI, 2010), which they predate by more than a decade, as shown in Table 2.1.

Table 2.1 Common Core State Standards for Mathematics as they address the aspects of PUFM

Aspect of PUFM – Subtraction with regrouping (Ma, 1999, p. 19)	CCSSM Standard (CCSSI, 2010)
Addition and subtraction within 10 Addition and subtraction within 20	1.OA.1: Use addition and subtraction within 20 to solve word problems involving situations of adding to, taking from, putting together, taking apart, and comparing, with unknowns in all positions, e.g., by using objects, drawings, and equations with a symbol for the unknown number to represent the problem. (p. 15)
Composing and decomposing a higher value unit The rate of composing a higher value unit Composition within 100	K.NBT.1: Compose and decompose numbers from 11 to 19 into ten ones and some further ones, e.g., by using objects or drawings, and record each composition or decomposition by a drawing or equation (e.g., 18 = 10 +8); understand that these numbers are composed of ten ones and one, two, three, four, five, six, seven, eight, or nine ones. (p. 12) 1.NBT.4: Understand that in adding two-digit numbers, one adds tens and tens, ones and ones; and sometimes it is necessary to compose a ten. (p. 16) 2.NBT.1: Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones; e.g., 706 equals 7 hundreds, 0 tens, and 6 ones. Understand the following as special cases: a. 100 can be thought of as a bundle of ten tens — called a "hundred." (p. 19)
Addition and subtraction as inverse operations	1.OA.4: Understand subtraction as an unknown-addend problem. *For example, subtract 10 – 8 by finding the number that makes 10 when added to 8.* (p. 15) 2.NBT.5: Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. (p. 19)

Composition within 10	K.OA.4: For any number from 1 to 9, find the number that makes 10 when added to the given number, e.g., by using objects or drawings, and record the answer with a drawing or equation. (p. 11)
Addition without carrying Subtraction without regrouping	1.OA.6: Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. Use strategies such as counting on; making ten(e.g., 8 + 6 = 8 + 2 + 4 = 10 + 4 = 14); decomposing a number leading to a ten (e.g., 13 – 4 = 13 – 3 – 1 = 10 – 1 = 9); using the relationship between addition and subtraction (e.g., knowing that 8 + 4 = 12, one knows 12 – 8= 4); and creating equivalent but easier or known sums (e.g., adding 6 + 7 by creating the known equivalent 6 + 6 + 1 = 12 + 1 = 13). (p.15)
Subtraction with regrouping between 20 and 100	2.NBT.5: Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction. (p. 19)
Subtraction with regrouping of large numbers	2.NBT.7: Add and subtract within 1000, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method. Understand that in adding or subtracting three-digit numbers, one adds or subtracts hundreds and hundreds, tens and tens, ones and ones; and sometimes it is necessary to compose or decompose tens or hundreds. (p. 19)

Very few studies were found that focused on the topic of subtraction with regrouping. Ma (1999), whose interviews with US teachers were conducted a decade before her book was published, found that 77% of the US teachers in her sample limited their explanations of how to teach subtraction with regrouping to the procedural steps of the algorithm, while Chinese teachers in her sample focused on the concept of decomposing tens into ones. One-third of the Chinese teachers showed non-standard methods of regrouping, that is, breaking apart numbers other than a ten, and they also focused on the relationships between the non-standard methods and the standard algorithm. Where US teachers mentioned manipulatives as a tool they would use to show students why and how the algorithm works, the Chinese teachers would have students do exploratory work with the manipulatives and then have students discuss what they had found and noticed in order to draw out the connections. Ma also found that US teacher explanations of the regrouping process included thinking of it as borrowing a cup of sugar from a neighbor, which removes the idea of place value and the role of ten entirely.

Perhaps linked to Ma's findings that US teachers were lacking conceptual understanding of subtraction, one study conducted around the same time as Ma's work (Van Houten, 1993) looked at whether students learned subtraction facts better by rote or by using rules. The rules were essentially magic tricks with no conceptual underpinnings provided. For example, to subtract nine from a teen number, add one to the number above the nine. So, 15-9=6 because 5+1 = 6. The rules method was found to be more efficient and accurate than learning the facts by rote, but conceptual understanding was not addressed.

One study conducted not long after Ma and Ball collected their data shows a shift from their findings. The study asked preservice (PSTs) and in-service teachers how they would respond to students who forgot to regroup while subtracting, incorrectly calculating 60-28=48 (Fuller, 1996). Fifteen of the 26 PSTs (58%) and 4 of 28 experienced teachers (14%) focused exclusively on the procedure, but the others focused on a more conceptual response, stating that they would have the students use manipulatives to create a representation of the problem. This representation would allow students to understand the regrouping necessary to solve the problem. This is in contrast to Ma’s (1999) study which had 83% of the combined group of PSTs experienced teachers focusing on the procedure.

Thanheiser (2009) found that preservice teachers sometimes struggled to conceptualize the various ways to represent multidigit whole numbers. Two-thirds saw the digits in a number incorrectly in terms of ones at least some of the time. That is, they saw a number such as 253 as having only three ones, rather than having 253 ones, or some other combination of hundreds, tens, and ones. This could lead to challenges in work with regrouping.

I was unable to find current studies focusing on how teachers think about and explain the topic of subtraction with regrouping. None of the cited studies examined using number lines or flexible regrouping, which are among the current best practices for teaching subtraction (Van de Walle et al., 2013). My study should help to fill the gap on how current elementary teachers are engaging their students in subtraction with regrouping and what they see as the important elements to its teaching and learning.

2.3.2 Multi-digit Multiplication

Multiplication can represent repeated addition, a total number if we have X groups with Y elements in each group, and an area of a rectangular figure that is X long by Y wide. Van de Walle, et al. (2013) suggest that multiplication can be modeled as an array or area model, as equal sets, and as repeated jumps on a number line. They encourage flexible methods for computation, with strategies varying with the numbers involved and the contexts presented. In approaching multi-digit multiplication, the authors suggest using partitioning strategies, cluster problems, area models, and partial products, which can then be linked to the standard algorithm. These strategies are all part of the SCK for the topic. They acknowledge that the algorithm is often more efficient in calculating products of multi-digit numbers than area models or partial products, but note that the algorithm should be taught with reference to the other models in order give meaning to the procedure. Efficiency is, of course, relative to the individual. What is efficient for an expert is not necessarily efficient or useful for a learner. The Common Core State Standards for Mathematics expect that, by the end of the fifth grade, students will be able to “Fluently multiply multi-digit whole numbers using the standard algorithm” (CCSSI, 2010, p 35)

An area model is a rectangular diagram in which the side lengths of the rectangle represent the numbers being multiplied. If the factors are multi-digit, they are often broken down by place value, as shown in Figure 2.4, which represents multiplying 15 by 17. When students first start using this model, they generally work with graph paper to create accurate representations of the factors and the related areas. As the numbers get larger, creating scale representations gets more challenging, so the model is often simplified to be a set of boxes that are not in proportion to the factors they represent, as shown in Figure 2.5. This model is sometimes referred to as a box model, but often still called an area model.

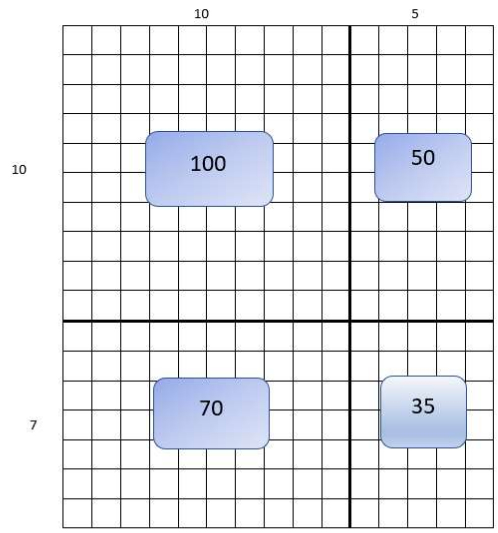


Figure 2.4. An area model representation of 15 x 17.

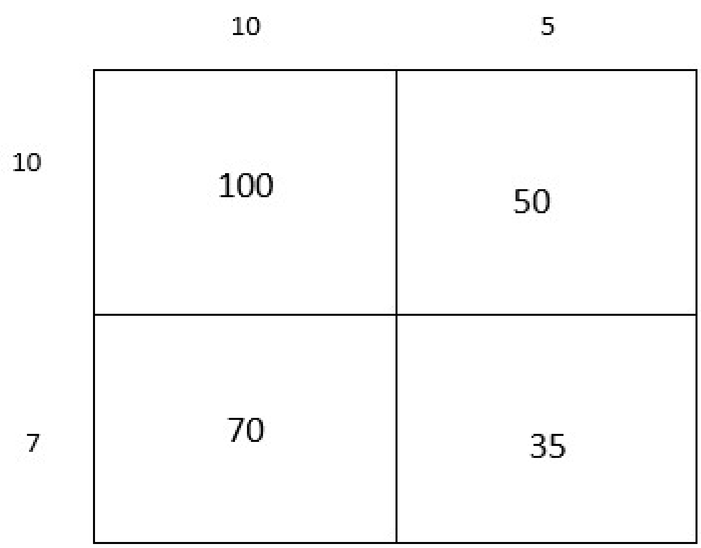


Figure 2.5. An area model that is not to scale.

These models are useful for representing the partial products embodied in the standard algorithm for multi-digit multiplication. In the standard algorithm, the multiplicands, also known as factors or numbers being multiplied, are aligned one on top of the other, with the ones digits aligned. The number with the fewest digits is generally placed on the bottom. The ones digit of the bottom factor is then multiplied by the upper factor, and then the tens digit of the bottom factor is multiplied by the upper factor, as shown in Figure 2.6. One can also multiply each of the component parts of the bottom number by each of the component parts of the upper number to create four products, known as partial products, that align with the products in the separate boxes of the array method.

	1	7
x	1	5
	8	5
1	7	0
2	5	5

	1	7
x	1	5
	3	5
	5	0
	7	0
1	0	0
2	5	5

Figure 2.6. The standard algorithm for multiplying 17 x 15 and the corresponding partial products version of the algorithm.

Lampert, in 1986, argued that partial products should be accepted as a standard algorithm. While the US standard algorithm might be slightly faster, she found that participants working with partial products tended to be more accurate. She argued that, if speed is the objective, have students use a calculator, but if the goal is to understand the process and what it means, a model using partial products seems superior.

Key elements of multi-digit multiplication that were identified by Ma (1999) in her knowledge package included a strong understanding of multiplication by 10 and powers of 10, an understanding of the meaning of multiplication, and a working knowledge of the distributive property. The Chinese teachers also noted that they would focus on 2-digit by 2-digit problems using partial products with their students to help them understand the procedure and then work with larger problems.

Studies on teacher understanding of multi-digit multiplication have focused generally on the standard US algorithm. Do preservice and in-service teachers understand why the algorithm works? Can they create a model to justify the algorithm? Can their models explain the pattern of zeros in the lines of the solution?

Ma, who asked both PSTs and in-service teachers how they would respond to students who right justified all of their partial products (refer to the multi-digit multiplication problem in Chapter 1), found that 77% of US participants said that the problem the students had was with the lining-up procedure, and 30% said the students didn't understand the rationale of the algorithm. Most identified zeros as placeholders, as opposed to indicators of values, and two participants said that you could use an asterisk or another symbol to hold the place, instead of using a zero or leaving a blank. Only 39% of US teachers were able to provide a valid conceptual explanation for the procedure, as compared to 92% of the Chinese teachers. This difference in knowledge between the populations indicate both that Specialized Content Knowledge for this topic is possible, and that US teachers may have inadequate SCK for multi-digit multiplication.

Several studies looked at whether PSTs could relate other strategies to the standard algorithm. When Southwell and Penglase (2005) asked 78 elementary PSTs to calculate 47x25 two different ways, the participants struggled to use the distributive property and partial products or to look for related numbers (47x100/4, for instance). PSTs showed an inflexible idea of multiplication, seeing it only as the standard algorithm. Since modern curricula recommend that teachers encourage natural or invented methods, it is concerning that PSTs aren't showing the flexibility needed to accept or understand student-created methods.

Similarly, a group of PSTs were asked to develop and justify a way to multiply 22x37 using a first step of either 22x10, 37x10, or 20x30 (Lo, Grant, & Flowers, 2008). Many struggled to justify their reasoning based on a flexible understanding of multiplication. They especially struggled to coordinate features (words, context, pictures) in their responses. Most participants focused on an equal groups model of multiplication, and not an area/array model. The researchers found it challenging to tell if the PSTs had an insufficient knowledge of multiplication or if they didn't understand what it means to justify. The authors suggested those were linked and that "inability to explain is frequently tied to incomplete understanding." This lack of ability to justify is indicative of lower SCK.

Also showing issues with justification, another group of PSTs was asked to describe why an invented algorithm worked, and most of their explanations were procedural and focused on memorized definitions. In the solution they were shown, the student had multiplied 25x34 as 5x34 then 20x34 instead of 4x25 and 30x25 (Harkness and Thomas, 2008). They were told that the student's teacher had responded in a belittling way. Less than 1/3 of the participants demonstrated some level of conceptual understanding of the mathematical properties inherent in the solution. As noted above, it is important for teachers to have enough SCK about fundamental topics to understand why a student's methods lead to a correct (or incorrect) answer. If teachers do not, they must withhold judgement on a student's process until they can explore the methods and think about the validity, based on mathematical properties.

Whitacre and Nickerson (2016) found success in working with PSTs to develop a stronger understanding of multidigit multiplication, In a required content course, they led students in work on partial product in one by two digit multiplication, connecting partial products to the area model, understanding partial products in two by two, and then justifying the standard algorithm. They felt that this sequence of explorations helped the PSTs to create the necessary connections between partial products, area models, and the standard algorithm.

These studies of preservice teachers show that focused attention on the conceptual work of multi-digit multiplication can lead to stronger understanding, but most PSTs seem to come to their programs without that conceptual understanding. This study can provide information about current teachers' knowledge and understanding of multi-digit multiplication, allowing us to see if teacher preparation programs and professional development efforts have been implemented and effective for this topic.

2.3.3 Division by a Fraction

Division can be seen as finding the number of groups of size X in a total (quotitive or measurement division), the number in each of Y groups in a total (partitive division). When dividing by fractions, we must consider even more components. Van de Walle et al. (2013) state that there are several ways of thinking about fractions (as part of a whole, ratios, and division), and several models for representing them (area, length, and set models). When looking at division with fractions, they note that people find it challenging to think of fraction contexts, but it is helpful to have a context in order to model a problem, estimate, and solve. They also focus on creating visual representations, such as number lines, sets of counters, and area models, for these contexts that can help students understand the division problems. In noting what makes division with fractions especially challenging, they mention that, in whole number division, the answer to a division problem is generally smaller than the dividend, but in fraction division the answer is generally larger than the dividend. This can be confusing to those who can't contextualize the problem situation.

While we tend to think first of the invert-and multiply algorithm for dividing fractions, Van de Walle et al. (2013) note that there is also a common denominator algorithm that can be developed. Once the divisor and dividend have been rewritten with the same denominator, the denominator is superfluous, and the problem simplifies to dividing the two whole-number numerators. For example, for the problem $1\frac{3}{4} \div \frac{1}{2}$, we could rewrite the dividend as 7/4 and the divisor as 2/4, transforming the problem into $\frac{7}{4} \div \frac{2}{4}$. This is equivalent to $7 \div 2$. Having a common denominator also allows for easier modeling of the division as repeated subtraction.

Ma (1999) found that, given a division with fractions problem to solve, only 39% of US teachers and 72% of Chinese teachers calculated a correct answer. In this part of the question, they were asked only to calculate the answer, not explain it conceptually, which would have been even more challenging. Even more alarming was that only one of the US teachers was able to present a conceptually correct context, and even that had a complication with the units used. Most of the US teachers who attempted a representation confused division by ½ with either division by 2 or multiplication by ½, which are mathematically equivalent to each other. Ma felt that an "inadequate understanding of procedure impedes creating a representation" (p. 69), a sentiment echoed in the CCSSM (2010). The Chinese teachers, on the other hand, most of whom could successfully compute the correct answer, also provided conceptual rationales. They frequently mentioned the definition of division in justifying the algorithmic step of multiplying by the reciprocal of the divisor, and they were able to create many representations and models, both measurement (quotitive) and partitive. Ma suggested that an understanding of division by fractions was built on a foundation of knowing the meaning of multiplication and division with whole numbers, the meaning of multiplication with fractions, and the relationship of multiplication and division as inverse operations. Understanding that $1\frac{3}{4} \div \frac{1}{2}$, can be thought of as the inverse of $\frac{1}{2}$ of what (or $\frac{1}{2}$ times what) is equal to $1\frac{3}{4}$, can lead to the representation shown in Figure 2.7. This representation can make it much clearer why we would multiply $1\frac{3}{4}$ by 2 to solve the problem.

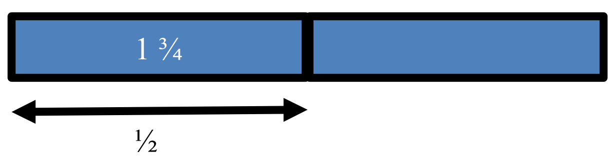


Figure 2.7. Representing the division problem as its multiplicative correlate.

There is a large body of research on student, preservice teacher, and in-service teacher understanding of fractions and division by a fraction, including a 1924 study that indicates US students' struggles with fractions are not a recent phenomenon. In that early study (Morton, 1924), 8th grade students were tested on all four operations (addition, subtraction, multiplication, and division) with common fractions. No context was given for the problems; students were simply asked to perform the calculations. Morton found that many of the students had "inadequate conception of the processes involved." They sometimes performed the wrong operations and they struggled to know when one needed (or didn't need) a common denominator.

Those challenges with understanding how and when to apply the various fraction calculation rules have persisted. Preservice teachers were tested on all four operations prior to a math content course (Jones Newton, 2008). The questions were not only tests of computational skill, but also on basic conceptual understanding, word problems, flexibility, and transfer of skills and knowledge. The PSTs had difficulty adding and subtracting fractions, as many were unsure what to do with the denominators, and some simply added or subtracted the numerators and denominators. Uncertainty was highest around division, then subtraction, multiplication, and addition. The researcher found that errors were mostly related to misconceptions about fractions and how operations with them differ from whole number operations. PSTs seemed to remember procedures, but they used them inappropriately, and they did not have the conceptual knowledge to reason through the procedures. This low-level of conceptual knowledge around fractions would likely lead to teaching that was procedural and focused on the rules without understanding. It would also mean that PSTs would lack awareness of the misconceptions held by students, as they would be shared misconceptions. The author instituted a curriculum intended to boost knowledge of fractions and the four operations. Post-course evaluations showed an increase in conceptual understanding, and an improvement in choosing the correct operation for contextual problems, though calculation errors persisted.

A Belgian study (Van Steenbrugge, Lesage, Valcke, & Desoete, 2014) found that third-year PSTs made a lot of errors on a test of fraction knowledge, and their procedural knowledge scores were significantly higher than their conceptual scores. More than one-third of the PSTs did not indicate an understanding that a fraction represents a single number. When they were asked to explain the rationale for a procedure, the average score was below 25% on the task. This lack of conceptual knowledge would limit the ability of these future teachers to teach well for student understanding.

Experienced teachers have also shown a weakness in the area of fractions. When asked to respond to a student misconception about adding fractions, about half of both PSTs and in-service teachers in a study (Fuller, 1996) gave strictly procedural responses. The experienced teachers showed a much greater conceptual understanding of whole-number operations than they did of fractions.

Fractions are considered challenging for a variety of reasons. As Lortie-Forgues, Tian, and Siegler noted (2015), the conceptual basis of fraction operations and procedures is not always obvious. We need common denominators for addition and subtraction or fractions, but not for multiplication and division of fractions. When multiplying fractions, one can multiply the numerators together and the denominators together, but one cannot just add or subtract the numerators and denominators for those operations. Why can we invert and multiply when solving division with fractions problems? These larger ideas are often linked to algebraic concepts and proofs, which are generally taught after fractions. There are a large number of procedures involved in fraction operations which require prerequisite skills, as Ma (1999) indicated in the knowledge package for division with fractions. Students who lack a conceptual understanding of the procedures cannot then reconstruct one if they have forgotten it. Reasons that teachers continue to focus on procedure and memorization instead of a conceptual understanding include that they learned it that way, they have no incentive to change, they struggle to understand the material themselves and teach it in a conceptual way, and they are avoiding the embarrassment of student questions they can't answer.

In a review of studies on PSTs and fractions (Thanheiser, Browning, Edson, Kastberg, & Lo, 2013), the authors concluded that teachers can perform fraction operation algorithms, but lack conceptual understanding to explain them. They also tend to apply whole number understandings to fraction problems. Use of representations of student thinking can help surface misconceptions and help PSTs to develop stronger math knowledge for teaching (MKT). The reviewers also found that PSTs struggle to model operations with representations, interpret student-generated algorithms, and identify sources of student errors.

Division with fractions seems to be a particularly challenging topic, when compared with the other three operations. PSTs were studied, looking at the connectedness of their procedural and conceptual knowledge for the operation, and their knowledge of unit relationships in division problems (Simon, 2006). Conceptual knowledge was found to be weak in understanding algorithms, the relationship between partitive (fair shares) and quotitive (subtraction or measurement) division, in linking the symbolic representation to the real-world problems, and in identifying units.

In a study looking at why Chinese students might fare better with the topic of division with fractions, as Ma (1999) determined, it was found that Chinese students are expected to learn more than the rote procedure of division by fractions (Li, 2008). Chinese textbooks recommend extensive lessons on the topic, and those lessons include the meaning of and computational rule for division of fractions, application problems, and the interpretation of ratio. Rather than showing the algorithm, the textbook gets to the algorithm by solving a problem that leads to that outcome. Understanding of the meaning of division by fractions as the inverse of multiplying by fractions is a key and primary goal. A comparative study (Son & Senk, 2010) found that US textbooks focused almost exclusively on the algorithm.

That US curriculum does not have the same focus was evident in a study on PST understanding of fraction operations (Li & Kulm, 2008). While 90% of PSTs could identify the incorrectness of fraction addition and subtraction problems, only 2 of 46 (4%) identified that the solution to a fraction division problem was computed correctly. Because it wasn't solved in the "standard" keep-change-flip manner, they did not recognize the solution was valid, which indicates that the PSTs had very low SCK for fraction division. The researchers note that, on a survey used to measure the PSTs' perceptions of their knowledge for teaching, which asked how ready the felt to teach "Representing and explaining computations with fractions using words, numbers, or models?'', 60% of the participants expressed high confidence in their mathematical knowledge. Conversely, in a similar survey, Chinese and Korean teachers PSTs expressed less confidence in their knowledge, but more than 95% answered correctly problems like ''Tell whether 9/11 ÷ 2/3 is greater than or less than 9/11 ÷ 3/4 without evaluating. Explain your reasoning.'' It is clear that people can learn fraction division in a conceptual way, but the authors suggest that we need to look at what approaches US schools and teacher educators need to incorporate to reach that conceptual understanding.

Along with struggling with the operation of division with fractions, PSTs struggle to create contexts for the operation (Unlu & Ertekin, 2012; Alenazi, 2016; Nillas, 2003;

Jansen & Hohensee, 2016; Işik & Kar, 2012; Ma, 1999; Lo & Luo, 2012). Common errors and misconceptions included confusing dividing by ½ and dividing by 2, or dividing by ½ and multiplying by 1/2 (Nillas, 2003; Işik & Kar, 2012; Ma, 1999), inability to accept that the contextual problem involved fractional division (Alenazi, 2016; Jansen & Hohensee, 2016), confusion with units (Alenazi, 2016; Işik & Kar, 2012; Jansen & Hohensee, 2016), assigning natural number meaning to fractions (Işik & Kar, 2012), and the inability to establish part-whole relationships (Işik & Kar, 2012; Jansen & Hohensee, 2016; Alenazi, 2016). Alenazi (2016) noted the vicious cycle that occurs when PSTs enter college with insufficient understanding of fraction division, receive little college instruction on this topic, which they then are unprepared to teach when they enter the classroom, leading to their students having insufficient understanding of fraction division.

In an attempt to improve PST understanding of multiplication and division with fractions, researchers developed a course that focused specifically on those topics (Whitehead & Walkowiak, 2017). Pre- and post-course assessments asked the PSTs to identify errors in student work on fraction problems and provide rationales for why the strategies were faulty. While 98% of the PSTs had taken a Calculus for elementary teachers course, they struggled to explain the common fraction algorithms. Post-course, students could explore problems such as ½ of ¾ is 3/8, but they did not tie their explorations back to the algorithm of multiplying numerator times numerator and denominator times denominator. The professors realized that they had not explicitly identified or named those connections during the course and noted that the explication is likely necessary for tying the procedural to the conceptual.

It is clear that fractions are challenging for students, PSTs, and in-service teachers, and that fraction division is extremely challenging. These studies indicate that conceptual understanding of fraction division is weak for the US population, that it is a struggle for US PSTs and in-service teachers to develop contextual approaches to the topic, and that some other countries seem to have developed more successful curricula to address fraction division. This study will examine if there have been improvements in understanding this topic in current elementary teachers and will seek to identify areas of strength and areas of concern.

2.3.4 Area, Perimeter, and Proof

Area and perimeter are qualities of closed, plane figures called polygons. The perimeter is the length of the outer edge of the figure, and the area is the amount of surface inside the perimeter (see Figure 2.8). Perimeter is a linear, one-dimensional measure and carries a label such as feet or meters. Area is a two-dimensional measure and carries a label such as square feet or square meters. Van de Walle et al. (2013) note that area, perimeter, and volume are related , but not linearly, which is a part of SCK for this topic. They also say that students often confuse area and perimeter concepts, and those topics are generally taught with an overemphasis on formulas and very little conceptual background.

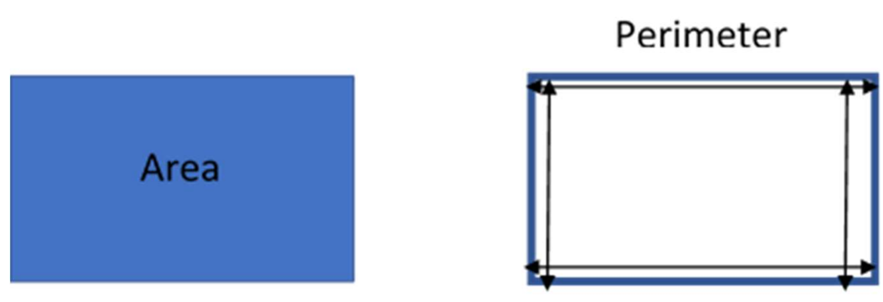


Figure 2.8. Area and perimeter of a rectangle.

Research on teacher knowledge of area and perimeter indicate that both preservice and in-service teachers tend to have only a procedural understanding of both concepts (Menon, 1998; Kellogg, 2010; Reinke, 1997; Livy, Muir, & Maher, 2012; Fuller, 1996; Ma, 1999). In Ma's study (1999) comparing US and Chinese teachers, she asked teachers to respond to an incorrect claim a student made about the relationship between perimeter and area, that as perimeter increases, area also increases. Only 13% of the US teachers investigated the claim mathematically, as compared to 92% of Chinese teachers. The Chinese teachers who explored the claim were much more likely than the US teachers to clarify and explain conditions under which the claim could be true.

Many preservice teachers in a more recent study (Livy et al., 2012) said that the student's claim from the Ma (1999) example was correct, indicating that Ma's decades-old findings related to area and perimeter may have persisted despite changing standards. Those PSTs (72% of 222 participants) had a procedural understanding of area and perimeter and showed similar misconceptions to students, who often think that there is a constant relationship between area and perimeter. This misconception persisted for the PSTs despite a similar problem in tutorials, on a practice exam question, use of an interactive website, and the fact that the test was open note and open book test.

One of the third-grade standards in the Common Core State Standards for Mathematics (CCSSI, 2010) states that students should be able to "[exhibit] rectangles with the same perimeter and different areas or with the same area and different perimeters" (pg. 25.) Being able to show changing perimeter without increasing area should allow students, and teachers, to recognize that the claim from Ma's (1999) example is incorrect.

Preservice teachers in another study were unable to determine that there was sufficient information to solve problems involving area and perimeter, even when they didn't have to actually calculate the answer (Menon, 1998). A study (Fuller, 1996) of both in-service and preservice teachers found that, while all participants attempted to give conceptual answers to a question about the relationship between area and perimeter, the responses focused on showing the students an example instead of exploring the relationship.

Teachers with misconceptions about the relationship between area and perimeter are unable to help their students dispel the same misconceptions (Kow & Yeo, 2008). An intervention designed to correct PST misconceptions about area and perimeter (Kellogg, 2010) was successful in improving their ability to anticipate student ideas and misconceptions, but the PSTs did not see that the intervention strategy and presentation could also be useful in preventing or addressing the same misconceptions with students.

These studies indicate that preservice teachers are entering their programs with shallow understanding of the relationship of area and perimeter, but studies of in-service teachers are few and decades old. This study will update our knowledge about in-service teacher understanding of area and perimeter, and the conditions necessary for proving the relationship between them.

2.4 Conclusion

In the review above, I considered research related to Mathematical Knowledge for Teaching, Specialized Content Knowledge, and the SCK important to the four mathematical topic areas addressed in my study. The studies and reports presented in this literature review show the evolution of thinking about Mathematical Knowledge for

Teaching and Specialized Content Knowledge and how to measure them. MKT is indeed linked to student learning, suggesting that improving MKT and SCK is crucial to giving our students a chance to learn math deeply and conceptually. Yet *how* we determine SCK, as in the actual empirical techniques we use, is often problematic. The multiple-choice instruments used to measure Mathematical Knowledge for Teaching is not intended to be used to evaluate individual teachers, nor does it have the fine grain to investigate individual topics. It is therefore time to revisit SCK in teachers using Ma's (1999) questions from the Teacher Education and Learning to Teach Study (TELT) and conducting interviews to collect data.

Other than division with fractions, there are few studies on the four mathematical topics addressed in Ma's (1999) work, and most of the prior studies in all four topic areas have focused on preservice teachers as opposed to in-service teachers. None of the prior studies found addressed the expectations of the current educational standards for students. By exploring the four topics found in Ma's (1999) work, we will be able to look for evidence that teachers are showing greater understanding of fundamental mathematics and if they seem to have the SCK to teach conceptually in the ways that current standards are expecting if students are to gain mastery.

In Chapter 3, I provide the methodology for this study. I describe the research method, the sample, data collection methods, and data analysis techniques.

CHAPTER 3

METHODOLOGY

In this section I describe the research design, sample, data collection methods, and data analysis methods for the study.

3.1 Research design

This study employed a qualitative, multiple-case study design (Yin, 2017) to answer the following Research Questions:

1. How do teachers' explanations of mathematics content demonstrate Specialized Content Knowledge (SCK) for the following topics? (RQ1)
   a. Subtraction with regrouping?
   b. Multiplying multi-digit numbers?
   c. Division with fractions?
   d. The relationship between area and perimeter?
2. What themes are found in teachers' explanations for the following topics? (RQ2)
   a. Subtraction with regrouping?
   b. Multiplying multi-digit numbers?
   c. Division with fractions?
   d. The relationship between area and perimeter?

A case study is defined as "an empirical method that investigates a contemporary phenomenon (the 'case') in depth and within its real-world context, especially when the boundaries between phenomenon and context may not be clearly evident" (Yin, 2017, p. 13). Case study design was appropriate for this study because I was examining the phenomenon of Specialized Content Knowledge (SCK), which Ma called Profound

Understanding of Fundamental Mathematics (PUFM), in individual participants, or cases. I used cross-case analysis (Yin, 1981), to both look for commonalities across the cases and to compare SCK and themes for my participants with Ma's findings to see if SCK either has improved in consideration of the progression of mathematics standards movements. As shown in the literature review, it is unclear if changing standards have had an impact on teacher SCK. The research questions for this study require an in-depth look at teacher thinking that cannot be captured by the multiple choice, standardized assessments that I highlighted in the literature review. The research questions also require more than one case, as I am interested in the thinking of more than one teacher. Therefore, a multiple-case study design allowed me to see if teacher SCK is keeping pace with the new standards. In order to explore the phenomena of SCK, I conducted interviews that approximated the real-world context of teaching with questions about topics teachers encounter in the classroom. At the same time, this strategy reduced some of the complexity that would have been introduced in the actual classroom, allowing me to focus specifically on Specialized Content Knowledge.

In my analysis, I used the case-study approach of pattern matching, defined by Yin (2017) as "comparing an empirically based pattern…with a predicted one." In this dissertation, I compare the patterns I find in my teachers' responses with those I expect to see based on current mathematical standards and practices to answer both research questions. To address RQ2, I also employ cross-case comparison (Yin, 1981), looking for themes across the participants' responses. This cross-case comparison is particularly appropriate because of the consistency of the contexts and protocols of the interviews I conducted. I compare the themes that I identify to Ma's findings, in order to evaluate if and how response patterns have changed.

I used one main source for my evidence: the video and written work captured during structured interviews that asked teachers to respond to four teaching scenarios by both doing and explaining the mathematics involved (see Appendix A for the interview questions). I augmented this with information I collected from the teachers using a questionnaire (see Appendix B) about their teaching experience and professional development in mathematics.

To answer RQ1, I first developed a coding scheme for analyzing participant responses. I used language and content from the "knowledge packages" conceptualized by Ma (1999), from the Common Core State Standards for Mathematics (CCSSI, 2010), and from the literature on best practices to create criteria for evaluating the SCK teachers demonstrated in their explanations. In the analysis for RQ2, which involved themes from the responses that illuminated what was important and relevant to the participants for each topic, I used thematic coding (Braun & Clarke, 2006), a technique to look for themes in the features of the explanations relevant to Ma's findings (1999), content or practice standards (CCSSI, 2010), and other themes that were not pre-defined.

I elaborate on my sample, data collection, and data analysis in the following sections.

3.2 Sample

The population for this study is current elementary school teachers in grades 1-6. For my sample, I recruited current Massachusetts elementary school teachers who taught math to students in grades 1-6 as part of their daily work, and who held or met the requirements for a Massachusetts teaching license, having passed the requisite state licensing exam. By staying in one state, I was able to study teachers who all met the same minimum standards for certification. The teachers came from a variety of teacher preparation programs and educational backgrounds, and have had different teaching experiences.

I chose to recruit teachers from a group of elementary schools that were part of a regional cooperative, sharing a superintendent, a common curriculum, and common professional development opportunities. In order to gain permission from the office of the Superintendent to conduct the research, I first had to obtain written consent from each of the principals. Three principals agreed to allow me access to the teachers in their school. Two of the elementary schools qualify for Title 1 funds, indicating that they serve a large percentage of low-income students. One of the elementary schools did not qualify for this funding. Demographics for the three schools are shown below (Table 3.1).

Table 3.1. Demographics for the three schools.

		School 1	School 2	School 3
Race	White	77%	43.9%	48.6%
	Hispanic	11%	24.9%	21.6%
	Black	4.4%	9.4%	7.3%
	Asian	2.2%	12.9%	15.2%
	Multi-race	5.2%	8.4%	7.3%
Academic profile	ESL	3%	30.5%	26.7%
	ELL	.7%	18.8%	13.7%
	Disability	22.2%	21.3%	25.1%
	High needs	36.3%	48.2%	51.1 %
	Economically disadvantaged	19.3%	29.4%	37.1%
State rating		Meeting targets	Focused/targeted support	Partially meeting targets
Average math score	(500 state average)	510	492	499

The principals each sent an email on my behalf to their faculties, seeking participants. The email noted that I was looking for teachers currently certified in and teaching math in grades 1-6. A small incentive in the form of a gift card was offered for participation. I also asked colleagues for suggestions of teachers they knew from the district who might be willing to participate and sent the email directly to those teachers. I also used purposive snowball sampling to garner more participants (Devlin, 2018). At the end of each interview, I asked the participants if they could suggest other teachers, and sent emails directly to those who were suggested. I also directly emailed teachers whose email addresses were available on the school websites. In all, 18 teachers agreed to be interviewed. The participants are described below in Table 3.2. Across participants, they ranged in years of experience from 0 to 34, were mostly female, and taught a range of grades. Most had some familiarity with the Common Core State Standards for Mathematics (CCSSI, 2010), but few had received professional development pertaining to the standards.

Table 3.2. Description of Participants

Participant	Years of Experience	Gender	Current Grade level	Familiarity with CCSSM?	PD on CCSSM
1	34	F	3	Pretty	No
2	14	M	4	NA	NA
3	9	F	5	Generally	Standards-based grading
4	2	F	4	Mostly	Graduate school
5	19	M	4	Not very	NA
6	22	F	5	Familiar	No
7	12	F	1	NA	NA
8	30	F	4	Somewhat	No
9	23	F	Math specialist	Very	No
10	28	F	Math specialist	Very	No
11	10	M	4	Mostly	Yes
12	1	F	6	Fairly	No
13	0	F	4	Yes	No
14	7	M	6	NA	NA
15	2	F	3	NA	NA
16	13	F	6	Very	No
17	13	F	2	Somewhat	No
18	10	F	2	Familiar	No

The two math specialists served multiple grade levels and worked with students in classrooms and sometimes one-on-one. The participant with zero years of experience had worked in one of the schools as a paraprofessional for two years at several grades levels, had just completed a year of student teaching at the school, and had met all of the requirements for her elementary certification, which was pending. All participants except Participant 12 held an advanced degree.

The chosen district curriculum was Everyday Mathematics (The University of Chicago Schools Mathematics Project, McGraw-Hill Education) for grades 1-5, and Big Ideas (Big Ideas Learning, LLC) for grade 6. Of the teachers in grades 1-5, ten expressed dissatisfaction with the chosen curriculum. Two teachers had obtained permission to teach a different curriculum, and seven others reported that they supplemented heavily from other resources.

3.3 Data Collection

There were two components of my data collection. First, I asked participants to fill an online Qualtrics questionnaire asking them about their teacher education histories, their teaching experiences, and their participation in math-related professional development. The questions are shown in Figure 3.1 (see also Appendix B).

1. What teaching licenses do you hold?
2. How long have you been teaching?
3. What grade level(s) do you teach/have you taught? How long at each?
4. Where and when did you complete your undergraduate education?
5. What was your major?
6. If you have an advanced degree, where and when did you complete that?
7. What professional development have you participated in for math?
8. Were these school-based, district-based, or from another organization, such as NCTM?
9. Are you a member of NCTM or similar math-education organization?
10. How familiar are you with the current standards for math teaching and learning?
11. Have you gotten specific PD or training on these standards?

Figure 3.1. Items from the Qualtrics questionnaire.

I asked that each participant complete this survey before our in-person interview, so I could clarify if there were any questions about the survey responses. In many cases, the participants had not completed the survey ahead of the scheduled interview, so I reminded them at the interview and then sent follow up emails after the interview. In all, I received completed surveys from 14 of the 18 participants.

To collect the rest of the data, I conducted in-person interviews that asked the TELT-developed questions used by Liping Ma (1999) in her research (see also Appendix A). The four questions were:

1. Let's spend some time thinking about one particular topic that you may work with when you teach, subtraction with regrouping. Look at these questions: $52 - 25$, $91 - 79$, etc.). How would you approach these problems if you were teaching second grade? What would you say pupils would need to understand or be able to do before they could start learning subtraction with regrouping?

2. Some sixth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate:

$$\begin{array}{r} 123 \\ \underline{\times\ 645} \end{array}$$

the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:

$$\begin{array}{r} 123 \\ \underline{\times\ 645} \\ 615 \\ 492 \\ \underline{738} \\ 1845 \end{array}$$

Instead of this:

```
   123
x  645
   615
  492
 738
 79335
```

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching sixth grade and you noticed that several of your students were doing this?

3. People seem to have different approaches to solving problems involving division with fractions. How do you solve a problem like this one?

$$1\frac{3}{4} \div \frac{1}{2} =$$

Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2}$?

4. Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she

has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove what she is doing:

Perimeter = 16 cm | Perimeter = 24 cm

Area = 16 square cm | Area = 32 square cm

What would you respond to this student? How would you engage with her around this idea?

During the interviews, I asked questions to probe teacher meaning and sometimes offered a conjecture or context that was intended to provoke a deeper response. I then asked one final questions of participants: What has been the greatest influence on how you currently think about and teach math?

All but one of the interviews were conducted in the teachers' classrooms, with the remaining interview taking place in a local coffee shop. Interviews took place either before or after school or, in the case of those conducted after the school year was complete, around scheduled end-of-year teacher meetings. They ranged in length from 25 to 60 minutes, with an average length of 38 minutes, and were both audio- and video-recorded, with the video camera focused on the written work the teacher was doing as opposed to the participant's face. Both video and audio files were transferred to a portable hard drive and to a secure, cloud-based storage platform. Each participant was given a piece of paper with each mathematical problem written out and was told to feel free to write or draw on it in order to capture their thinking. Those documents were scanned and uploaded to the portable hard drive and cloud-based storage. After each interview, I dictated a memo, which was also transcribed and archived, to capture my impressions of the participant, the classroom, and the flow and content of the interview. This gave me three sources of data: the survey responses, the video/audio files, and the written work done by each participant. IRB approval was granted for all recruitment and data collection procedures.

In the next section, I describe how I analyzed the data.

3.4 Data Analysis

Before analyzing the data, I had all interviews transcribed. Transcripts and document scans were then loaded into nVivo, a qualitative data analysis computer software. I created topic files by separating each interview into the four topic areas and adding in the images from the document scans that went with each participant's responses.

3.4.1 Coding for Specialized Content Knowledge

Most efforts to measure SCK have relied on a score from the multiple-choice test (LMT) designed by Ball and her colleagues (Hill et al., 2004; Hill et al., 2008). As I was evaluating interview data, I had to develop a strategy for identifying indicators of SCK for each content area. I first considered the elements of SCK as described by Lin et al. (2011), explanation, justification, and representation, and referred to three sources for elements that I could look for in those categories. The first source I consulted was the Common Core State Standards for Mathematics (CCSSM) content standards and Standards for Mathematical Practice (SMP) (CCSSI, 2010), and I identified those relevant to the questions I was asking my participants. As most topics are developed across grade levels, I looked at the standards for all grade levels up until I found a standard that implied mastery of that topic. Because multiple operations can be included in a single strand in the CCSSM, the standards I chose are not necessarily sequential. For example, for subtraction with regrouping, I include standards 2.NBT.1 and 2.NBT.5, but exclude the standards between them, which were not related to the topic of subtraction with regrouping, such as 2.NBT.3 (CCSSI, 2010, pg. 19), which calls for students to "Read and write numbers to 1000 using base-ten numerals, number names, and expanded form." Abbreviated versions of the chosen standards are shown in Table 3.3.

Table 3.3. Relevant CCSSM Standards

Topic	CCSSM Standards
Subtraction with regrouping	*K.NBT.1*: Compose and decompose numbers from 11 to 19 into ten ones and some further ones. *1.OA.4*: Understand subtraction as an unknown-addend problem. *1.OA.5:* [R]elate counting to addition and subtraction (e.g., by counting on 2 to add 2). *1.OA.6*: Add and subtract within 20, demonstrating fluency for addition and subtraction within 10. *2.NBT.1*: Understand that the three digits of a three-digit number represent amounts of hundreds, tens, and ones. *2.NBT.5*: Fluently add and subtract within 100 using strategies based on place value, properties of operations, and/or the relationship between addition and subtraction.
Multi-digit multiplication	*3.OA.7*. Fluently multiply and divide within 100, using strategies such as the relationship between multiplication and division or properties of operations. By the end of Grade 3, know from memory all products of two one-digit numbers.

	3.NBT.3. Multiply one-digit whole numbers by multiples of 10 in the range 10–90 using strategies based on place value and properties of operations. *4.NBT.5*. Multiply a whole number of up to four digits by a one-digit whole number, and multiply two two-digit numbers, using strategies based on place value and the properties of operations. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. *5.NBT.1.* Recognize that in a multi-digit number, a digit in one place represents 10 times as much as it represents in the place to its right and 1/10 of what it represents in the place to its left. *5.NBT.2.* Explain patterns in the number of zeros of the product when multiplying a number by powers of 10. *5.NBT.5.* Fluently multiply multi-digit whole numbers using the standard algorithm.
Division with fractions	*4.NF.4.* Apply and extend previous understandings of multiplication to multiply a fraction by a whole number. a. Understand a fraction a/b as a multiple of 1/b. For example, use a visual fraction model to represent 5/4 as the product 5 × (1/4), recording the conclusion by the equation 5/4 = 5 × (1/4). *5.NF.6*. Solve real world problems involving multiplication of fractions and mixed numbers. *5.NF.7*. Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole numbers by unit fractions. *6.NS.1*. Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions.

Area, perimeter, and proof	*3.MD.5*. Recognize area as an attribute of plane figures and understand concepts of area measurement. *3.MD.7.* Relate area to the operations of multiplication and addition. *3.MD. 8*. Solve real world and mathematical problems involving perimeters of polygons, including finding the perimeter given the side lengths, finding an unknown side length, and exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters. *4.MD.3.* Apply the area and perimeter formulas for rectangles in real world and mathematical problems. For example, find the width of a rectangular room given the area of the flooring and the length, by viewing the area formula as a multiplication equation with an unknown factor, *SMP 3.*Construct viable arguments and critique the reasoning of others.

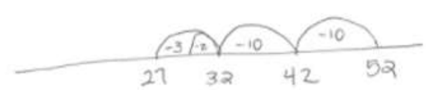
I then looked at Ma's (1999) findings from her work on Profound Understanding of Fundamental Mathematics, which I describe in the literature review, noting which of those elements I expected to see in the responses from my participants, based on the phrasing of the questions. I also looked at the best practices noted by Van de Walle et al. (2013), as mentioned in Chapter 2, and noted which of those were key to showing strong Specialized Content Knowledge. Drawing from these sources, I created a list of criteria for each topic that would indicate to me that a teacher had strong Specialized Content Knowledge in that area.

While looking through the interviews, it became clear to me that I could not just give a rating of "yes, this teacher has SCK", or "no, this teacher does not have SCK."

There seemed to be a range of SCK demonstrated for each of set of responses, with some teachers showing very strong SCK, some showing low SCK, and a group that fell between those two groups. Therefore, I decided to use three levels of SCK to describe the teachers' responses, and adjusted my coding manual to include criteria for all three levels. For subtraction with regrouping, I identified six criteria for Strong SCK. As the interviews were semi-structured and time limited, I could not expect the teachers to address both base ten and number line representations in each explanation, so I decided that exhibiting four out of the six criteria would demonstrate Strong SCK.

After using the manual to categorize each response, I recruited a colleague to help me determine the reliability of my coding scheme. We went through three participants' explanations for each topic, which I had selected to represent the various levels of SCK, and I confirmed with her that the SCK levels matched the criteria. She then coded six randomly selected explanations per topic and brought clarification questions to me. We worked together to modify the codebook to address her questions and re-examined those six cases per topic, and then compared the SCK levels we had assigned each explanation. For the topics of subtraction with regrouping, multi-digit multiplication, and division with fractions, we achieved agreement on 89% of the ratings. We then discussed those we disagreed on until we reached 100% agreement. I then reviewed and adjusted my coding for the remaining cases, based on the revised codebook and our collaborative coding. For the topic of area, perimeter, and proof, we had a lower level of agreement (67%) after the re-examination of our codes, and so we collaboratively coded all 18 of the cases until we reached 100% agreement. The final coding manual is shown in Table 3.4– Table 3.7.

Table 3.4. SCK Coding Manual for Subtraction with Regrouping

Level of SCK	Criteria	Examples
Strong (4 or more of the criteria)	Flexibly breaks up numbers.	So to set it up this way as well, that there are multiple ways to set up a number. 79 isn't just 7 10s and 9 1s, always. I can make it anyway I want. Right? What if make 5 10s and 29 1s.. We already know that we can make numbers infinite numbers of ways. You know? (P12)
	Uses counting up and/or down as a strategy.	There were number lines with counting up but then also counting back. (P15)
	Notes relationship between addition and subtraction.	[W]e try to teach it like, "Here's addition, and now the opposite of addition is subtraction." (P4)
	Provides at least one visual representation (number lines, manipulatives, etc.) or model.	 (P9)
	Mentions usefulness or difficulty of moving around and over decades.	Being able to comfortably go back 30 or go back 20 (P17)
	Talks about place value, such as 40 = 4 tens, 10 = ten ones, 52 = 4 tens and 12 ones.	…be able to describe it as five tens and two tens rather than just 52 (P11)
Moderate	Generally focused on the algorithm while noting that "borrowing one is really borrowing ten".	Someone might say, "One." I'll say no, we're not borrowing one at all, we're borrowing 10. If we're going to borrow 10, this doesn't become 3, it becomes 12. I talk about the fact that, when we're doing math, we're following procedures so that when problems are harder, we have a routine. Then we'll just go through the steps of solving that. (P5)
	Talks about place value, such as 40 = 4 tens, 10 = ten	…be able to describe it as five tens and two tens rather than just 52 (P11)

	ones, 52 = 4 tens and 12 ones.	
Low	Focus is only on the algorithm with no mention of "borrowing one is really borrowing 10"	[There were no examples for this level]

Table 3.5. SCK Coding Manual for Multi-digit Multiplication

Level of SCK	Criteria	Examples
Strong (all three criteria required)	Use of area model/box model/array/partial products that ties strongly back to the algorithm.	But knowing that this line [of the algorithm] represents 123 times five and that that's where the algorithm is more efficient. Here [in the area model] we have nine pieces, here we're going to have only three pieces. So helping them connect and see where that comes from…actually giving them the chance to see if they can figure out where the 615 is on their area model and make that connection themselves. (P3)
	Understanding of multiplication by ten.	'[W]hat's happening when you're [putting] a zero on the end?' And I really try to get them at least to verbalize 'I'm multiplying it by a power of ten.' (P1)
	Zero represents a value, not just a place holder.	But you can only put a digit in there. They are, maybe they are placeholders, but you have to put a digit. There's no value, then you can put zero. That's why zero's so important. I a place holder. It's the only place holder that you're allowed to use. (P2)
Moderate (at least two of the criteria)	Focusing on the procedure of multiplication.	I think really kind of breaking it down into what are our steps, why do we have to do that? (P15)

	Provides area model or box model, but seen as "managing" the process instead of providing essential understanding. (It's just another procedure.)	So just to emphasize to them that this is a way to save time and effort and improve their understanding. We start with just talking about it that way, but then we use the area model as *a way for them to manage it*. (Mr. Fields)
	Zero as a value, not a placeholder.	But you can only put a digit in there. They are, maybe they are placeholders, but you have to put a digit. There's no value, then you can put zero. That's why zero's so important. It's a place holder. It's the only place holder that you're allowed to use. (P2)
Low (one of the following criteria)	Expressed not knowing how to approach the problem.	But I'm not 100% sure how I would approach this because I'm not as fully confident in it. (P15)
	Did not reference the meaning of place value at any point.	
	Described zero as a placeholder.	

Table 3.6. SCK Coding Manual for Division with Fractions

Level of SCK	Criteria	Examples
Strong (all three criteria required)	Participant could solve the problem without using the algorithm.	So another way would be on the number line to think about. So if I have one and three-fourths, so there's my zero. Okay. So there's my one and three-fourths. I know when I think about how many halves go into that and so there is a half, there's a half and there's a half. So then noticing that okay, so I know that four halves would go into two and it's taking two of those pieces to make a half. So I have only part of a half in that case. So if you map it onto here I can see that there's one half, there's one half and then if I'm noticing it's taking four of those pieces to make a half I can switch and do it this way. Here's a half and that I only have a half of a half left. So then noticing that I have three halves but then there's this little half of a half. So if I have a fourth leftover. So those are my pieces that I have. (P3)

	Correct context and/or representation.	I need 1 3/4 cups of flour, but I only have a 1/2 cup scoop. How many of them are am I going to need? I need 3 1/2 scoops full to get my whole 1 3/4 cups of flour. (P12)
	Talked about what the problem is asking. (What does it mean to divide?)	They've foundationally been taught that that's what dividing is. Which doesn't change here. Basically, what I tell them is, to understand what's happening here, if I have 7/4, how many groups of 1/2s can I make? They can't necessarily concept that in their head, but at least they know what we're doing. I'm trying to break up 7/4 into pieces that are a size of a 1/2. So that's why that dividing by fractions makes it bigger, because I can fit multiple 1/2s in there. (P12)
Moderate (both criteria required)	Could solve without the algorithm	But if I did it this way, I would just keep subtracting. So you would come up with, so you did it at one, two, three, the remainder, one fourth. Would that be right? Is that right? [We rarely write remainders with fractions.] Oh, it's half of the half. So it'd be three and a half. (P2)
	Could not provide a correct context or representation.	You have a whole sandwich and three quarters, and you want to divide that in half so two people can share them. (P18)
Low (all three criteria required)	Could not solve or could only solve with the algorithm.	This is a mixed number and so the mixed number has to be changed into a fraction. So that's four times one, add three is seven fourths and then reversing the operation and flipping the fraction, don't ask me why, and we're doing 14 fourths and then that can be reduced. (P18)
	No sense of how or why the algorithm works.	I really have no idea. I remember that you're supposed to switch, change, flip. But I have no idea why and I think that that is a testament to my own math education. (P15)

	Could not provide a correct context or representation.	You have a whole sandwich and three quarters, and you want to divide that in half so two people can share them. (P18)

Table 3.7. SCK Coding Manual for Area, Perimeter, and Proof

Level of SCK	Criteria	Examples
Strong (meets both criteria)	Suggested trying other combinations.	I would say, "Good on you. Let's check this out further. Right? Let's just extend this and see if it's always, always true." Because we talk about math laws, right? And then we explore a lot of different examples, but we don't want to get to the point where we go, "That is always true. It's a math law. We feel confident." So I would say like, "What do we think? Do we think it's a math law? Let's explore a lot of different cases." (Ms. Blake)
	Created representations that were a counterexample AND/OR Mentioned keeping either Area or Perimeter constant and showing the variety of combinations the other factor could be to illustrate a counterexample.	I might go for an area of say 100, because that's a nice friendly number and if this kid clearly knows their multiplication, then I can say, "Hmm I have a 100 plus ten times ten is 100 and yet the perimeter is only forty". So I think that's what I would have them look at other factors for the number, because I'm also thinking about you could have this, but you could have fifty times two. And so now your perimeter is 108. For exactly the same area. So why are the perimeters that much bigger? (P1)

Moderate (all three criteria required)	Suggested trying other combinations and played with some.	And so, then I might ask them, "Okay, so you're telling me with this problem and this problem, we've confirmed that what you said is true, but now we have to test your theory again." Because, the whole point of a theory is that we have to keep testing it, and if you find that one time that your theory doesn't hold up, it's not true. (P4)
	Could not think of a counterexample.	Absence of a counterexample
	Did not arrive at a conclusion about correctness	So I still don't think I have a definitive answer to theirs because they are just saying if the perimeter increases that the area also increases. (P3)
Low (at least one criteria met)	Could not or did not engage with the problem.	I think, yeah. I would obviously spend some time saying, "Oh, I'm so excited you see this." Other than that, I'm kind of at a loss. (P16)
	Focused only on formulas.	You have to define the terms for them first. Once we get perimeter down, and they talk about the formula for it, then we talk about what about the inside, the surface? Then here's the area of my hand, how do you measure that? Talking about labeling units and so forth. (Mr. Fields)
	Reply to student was good job or they thought the student's work looked sufficient.	Yeah, I don't know if this ever said that to me. I think it's true. Yes. It makes sense. So I say, " Great…." I don't know. It makes sense. It's a great observation. (Mr. Fields)

3.4.2 Selecting the Focal Cases

Throughout Chapters 4 and 5, I use examples from participants' explanations to describe my findings and, in addition, I select three participants as focal case studies to illuminate the status of SCK across content areas for individual teachers. To select the participants who best seemed to represent the different levels of SCK in this study, I first gave each SCK level a numerical value: Strong=3, Moderate=2, Low=1. This allowed me to compare average SCK levels of the participants. I then grouped the participants into groups by their average ratings, with Strong having an average of 2.5 or greater (n=6, 33%), Moderate having an average between 2 and 2.5 (n =8, 44%), and Low having an average of 1.75 or lower (n =4, 22%). Within each group, I considered the attempted thoroughness of the teacher responses, and the match of those responses to the elements in the criteria for determining SCK, in order to select participants who seemed to best illustrate a teacher who had an overall rating of Strong, one who had an overall rating of Moderate, and one who had an overall rating of Low SCK. The three selected participants, who are described fully in Chapter 4, gave very thoughtful responses and all expressed a goal of teaching math conceptually. Also, because some of my participants were math specialists or had experience teaching elementary math content at a college level, I chose focal participants who were representative of the general classroom teacher. The selected participants were Participants 3, 5, and 8. I have given them the pseudonyms of Ms. Sutton, Mr. Fields, and Ms. Blake, respectively, for the remainder of this study.

3.4.3 Coding for Themes in the Explanations

To code for the second research question, focusing on themes I identified that reflected what teachers themselves saw as important, I employed thematic analysis (Braun & Clarke, 2006), which they describe (pg. 79) as “ a method for identifying, analyzing and reporting patterns (themes) within data. It minimally organizes and describes your data set in (rich) detail.” This was a theoretical analysis, as opposed to inductive, as I was coding for elements that answered the specific research question, and I had preconceived ideas as to the type of codes I would use, given the focus on explanation, representation, and justification (Lin et al.,), though I was also open to finding unanticipated codes and themes. I was also looking at the data on a semantic level, taking the teachers comments at face value as opposed to trying to read into the teachers' comments and interpret any deeper meaning. Braun and Clarke (2006, pg. 87) lay out six phases of thematic analysis:

1. Familiarizing yourself with your data
2. Generating initial codes
3. Searching for themes
4. Reviewing themes
5. Defining and naming themes
6. Producing the report.

I followed the six phases of thematic analysis by (1) familiarizing myself with the data, then (2) generating initial codes. These initial codes noted if the work was correct, and what elements of explanation, representation, or justification I was seeing in the interview transcripts. I kept a running list of elements of the features of the explanations, such as "number lines", "manipulatives", or "place value" mentioned by participants, and kept a tally of the number of participants per topic who included those elements. I then (3) grouped the elements into categories, such as "representations", "justifications", and "context", in order to search for themes, while also creating categories for unexpected themes. For example, with the topic Subtraction with Regrouping, I put mentions of base 10 blocks, Cuisenaire rods, and number lines into the category of representations. I noticed that some of those representations (base 10 blocks and Cuisenaire rods) require regrouping a power of ten, while others (number lines) do not, so identified “Multiple ways of regrouping” as one of my themes, which was not an anticipated theme . There was not a firm threshold of comments needed for something to rise to the level of a theme, but generally it was an element seen in at least one-third (i.e., at least 6) of the responses.

I (4) reviewed the themes, then (5) defined and named them. The final step was (6) producing this report, offering excerpts from the interviews to illustrate the various themes and to provide the “vivid, compelling extract examples” suggested by Braun and Clarke (2006, pg. 87) to illustrate the findings.

3.5 Delimitations

In any study, choices must be made that can impact the nature and generalizability of the findings. For this study, I chose to do one-on-one interviews, which led to a smaller sample, which does mean the data is illustrative but not generalizable to the population. It was important to me, however, to get a fuller picture of the math understandings held by teachers than could be obtained through a multiple-choice test or survey.

Also, while I asked my participants how they would teach or explain a topic, I did not watch them teach to see if their responses were confirmed in practice. Given that I was trying to evaluate SCK for four different topics that would be addressed at four different grade levels, it would have been impractical to see all the topics from any one teacher, and it was important for the study that I was able to get information about a broader set of math concepts from each participant.

There were also situations when, in reading through the transcripts, I see times when I could have asked better follow-up questions when time wasn't as much an issue. As the interviews progressed, the quality and nature of those questions improved.

In Chapter 4, I describe my findings for the first research questions, examining SCK demonstrated by the teachers' explanations. Chapter 5 addresses the second research question, exploring the themes I found in those explanations.

CHAPTER 4

FINDINGS –SPECIALIZED CONTENT KNOWLEDGE

4.1 Introduction

The purpose of this first research question was to determine if current teachers are offering strictly algorithmic explanations or more conceptual explanations that require a higher level of Specialized Content Knowledge (SCK), in alignment with the expectations new state and national math standards have placed on math instruction. The methodology used to analyze the level of SCK was presented in Chapter 3. This chapter consists of results that answer research question 1 by exploring the SCK levels found for each topic. Illustrations are offered through the explanations given by three focal participants.

The research questions examined for this section are:

How do the explanations given by the participants demonstrate Specialized Content Knowledge (SCK) for the following topics?

a. Subtraction with regrouping?

b. Multiplying multi-digit numbers?

c. Division with fractions?

d. The relationship between area and perimeter?

4.2 Overview of Findings: Specialized Content Knowledge

Table 4.1 shows the level of Specialized Content Knowledge (SCK) shown by each participant for each of the four mathematical topic areas. As described in Chapter 3, I developed criteria for the three levels of SCK for each topic, using the Common Core State Standards for Mathematics (2010), Ma's (1999) findings on profound understanding of fundamental mathematics, and best practices as described by Van de Walle et al. (2013). I then compared each participant's explanation to the criteria and assigned an SCK level for each topic.

As Table 4.1indicates, SCK varied greatly by participant and by topic. Table 4.2. shows the number of participants who were in each SCK level by topic. While 13 (72%) of participants showed Strong SCK in Subtraction with Regrouping, only four (22%) showed Strong SCK in Area, Perimeter, and Proof. Division with Fractions had the largest number categorized as having Low SCK (*n*=11, 61%), while only two participants (11%) showed Low SCK in Multi-digit Multiplication. This indicates that SCK can vary greatly by topic for a given participant. There was no clear relationship between grade level taught or number of years of teaching experience and SCK for any of the topic areas.

Table 4.1. Level of Specialized Content Knowledge for each topic area.

Participant	Subtraction with Regrouping	Multi-digit Multiplication	Division with Fractions	Area, Perimeter, and Proof	Participant Average rating
1	Strong	Strong	Strong	Strong	3
2	Strong	Strong	Moderate	Moderate	2.5
Ms. Sutton	Strong	Strong	Strong	Moderate	2.75
4	Strong	Strong	Low	Moderate	2.25
Mr. Fields	Moderate	Moderate	Low	Low	1.5
6	Moderate	Moderate	Moderate	Moderate	2
7	Strong	Moderate	Low	Moderate	2
Ms. Blake	Strong	Moderate	Low	Strong	2.25
9	Strong	Moderate	Low	Low	1.75
10	Strong	Strong	Strong	Moderate	2.75
11	Strong	Strong	Low	Strong	2.5
12	Moderate	Strong	Strong	Moderate	2.5
13	Strong	Strong	Low	Low	2
14	Moderate	Moderate	Low	Low	1.5
15	Strong	Low	Low	Strong	2
16	Moderate	Strong	Strong	Low	2.25
17	Strong	Low	Low	Low	1.5
18	Strong	Strong	Low	Low	2
Topic Average Rating	2.7	2.4	1.7	1.8	

Table 4.2. Number of Participants in Each SCK Level by Topic

Topic	Strong	Moderate	Low
Subtraction with Regrouping	13 (72%)	5 (28%)	0 (0%)
Multi-digit Multiplication	10 (56%)	6 (33%)	2 (11%)
Division with Fractions	5 (28%)	2 (11%)	11 (61%)
Area, Perimeter, and Proof	4 (22%)	7 (39%)	7 (39%)

Below are descriptions of the three teachers selected as focal cases for the remainder of the finding sections. Then I address the specific findings for each topic area, illustrated by excerpts from the interviews with the three focal teachers.

4.3 Selection of Focal Cases

Using the selection process described in Chapter 3, the following three cases were chosen:

4.3.1 Focal Case 1: Ms. Sutton

Ms. Sutton has been teaching for 9 years with experience in grades 5 and 6. She holds an undergraduate degree from a liberal arts college in a field other than education, and has two Master's degrees, one in Teaching. For professional development, she has participated in a university-based math fundamentals course, along with three courses on math remediation offered to district teachers. Two of the courses focused on whole numbers and operations, while the third dealt with fractions. Ms. Sutton noted that she finds the Everyday Mathematics (The University of Chicago Schools Mathematics Project, McGraw-Hill Education) curriculum very algorithm focused, and so tries to supplement with conceptual explanations of the algorithms. She has not been a member of any math-related organization, such as NCTM. Regarding the CCSSM, she reports having general familiarity with the standards, but she "can't quote specifics." She has received some professional development around standards-based grading, but not around the standards themselves.

Ms. Sutton and I met before school, so there was a definite time limit to the interview, but it did not feel rushed. She seemed to enjoy having someone to talk with about math and the professional development she had been taking. Student desks were arranged in clusters, and it looked like she had a relatively small class size, on the order of 16 students.

The recent professional development on both number sense and fractions significantly influence how she currently thinks about and teaches math. She was excited to show me the books that the trainings were based on, as well as the notebooks created for the professional development sessions. She also mentioned the program several times when explaining concepts, noting that the technique she was using was based on recent learning from the PD.

Ms. Sutton had an average SCK rating of 2.75, placing her in the highest group. She received ratings of Strong in subtraction with regrouping, multi-digit multiplication, and division with fractions. Her SCK in area, perimeter, and proof was evaluated as Moderate.

4.3.2 Focal Case 2: Ms. Blake

Ms. Blake has been teaching for 30 years, with experience in grades 3 through 7 and as a math interventionist, with the bulk of her teaching at grade 4 (24 years). She holds a bachelor's degree in Elementary Education, with a Master's in Education. For professional development in mathematics, she indicated that she participated in a three-year cycle of courses at a local community college, with each semester focusing on a different topic. She also participated in three courses on math remediation that were offered to district teachers. Two of the courses focused on whole numbers and operations, while the third dealt with fractions. Ms. Blake felt that the courses on whole numbers and operations were very worthwhile, but that the resources for the fraction course were not as well-developed or usable. In the past, she has been a member of NCTM and a regional math organization, but is not currently involved with either. She reports being somewhat familiar with the CCSSM, but has not received and professional development relative to the standards.

I met with Ms. Blake in her classroom one morning before school. Her classroom was set up with desks in pairs, all facing the same direction. While we had a strict time limit before the students arrived, the interview did not feel rushed in any way, and Ms. Blake seemed to engage with each question, giving them a lot of thought.

When I asked her about the greatest influences on how she currently thinks about and teaches math, she started by saying, "…not my teaching degree. That was worthless." She went on to mention that, earlier in her career, she was in a school that used the Investigations curriculum. "…We didn't have a lot of great training on it, so I was really like fumbling in the dark. I went through the motions for an entire year not knowing what I was doing or why I was doing it. But I think that second year was like, 'Oh, that's...This makes so much sense' You know what I mean? Things began to sort of fall into place gradually. Then, we did get some more training, which helped a lot, and then once you have drunk the Kool-Aid, I mean, you're all in."

She then sought out more training, including those at a local community college, mentioned above. “They were kind of brutal because it was all day Saturday, but they were so good, it was worth it. You know that feeling where you feel like, ‘I feel so bad for all the kids that I taught before, because I didn't know what the hell I was doing.’” Ms. Blake felt that the two district courses on whole numbers and operations were “great,” and that they had given her many tools for determining student understanding of structuring, numeracy, and ordering, and also activities for remediating misunderstandings or bringing students up to grade level. She did not find the fractions unit as helpful, noting that the materials felt poorly written and hard to follow. She follows the general expectations of the adopted curriculum, but finds her own resources for delivering the material, as she finds Everyday Mathematics (The University of Chicago Schools Mathematics Project, McGraw-Hill Education) too focused on procedure and thinks that the spiraling does not give students adequate time to engage with the mathematical ideas.

At the end of the interview, Ms. Blake commented that she likes math “a lot,” and that “It's really interesting to me and there's a big payoff for me, because it's fun, and when you see kids being really excited about math and making connections, this is great.” She tries to share her excitement with both her students and her fellow teachers.

Ms. Blake had an average SCK rating of 2.25, placing her in the middle group. She received ratings of Strong in subtraction with regrouping and area, perimeter, and proof, Moderate in multi-digit multiplication, and Low in division with fractions.

4.3.3 Focal Case 3: Mr. Fields

Mr. Fields has been teaching for 19 years, but only 9 of those have been as a classroom teacher. He taught 3rd grade for one year and has taught 4th grade for 8 years. His undergraduate major was in Elementary Education, and he has earned both a Master's degree and an EdD. He has not participated in any math professional development, nor joined any math-related organizations, such as NCTM. In reference to the CCSSM, he rated himself as not very familiar with them, and said that he has not received any professional development relative to the standards. He adheres faithfully to the Everyday Mathematics (The University of Chicago Schools Mathematics Project, McGraw-Hill Education) curriculum adopted by the district.

Our conversation took place after school in Mr. Fields' classroom. The desks were clustered in groups facing the white board, there was a large carpeted area in one corner and a well-stocked reading area separate from that. The classroom felt quite spacious and comfortable. While Mr. Fields was welcoming, he also seemed tired at the end of a busy school day near the end of the school year, and he didn't linger over any of the questions.

When I asked him about the greatest influences on how he currently teaches math, he started by mentioning the curriculum the school uses. "You read the lessons and think about the approaches they are using now." He noted that he has been using the current curriculum for seven years and, "I think that as time goes on, you recognize the key aspects of the concepts and emphasize those more." He mentioned that the curriculum materials do not call for using manipulatives or white boards, but that he has found those tools important for student learning of the concepts. His average SCK rating was 1.5, which put him in the lowest group. He earned a rating of Moderate in subtraction with regrouping and multi-digit multiplication, and Low in division with fractions and area, perimeter, and proof.

4.4 Specialized Content Knowledge by Topic

In this section, I report on the level of Specialized Content Knowledge (SCK) evidenced by the explanations for each topic and provide illustrations from the focal participants.

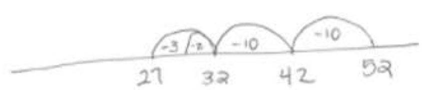
4.4.1 Subtraction with Regrouping

The question leading to responses for this topic, as described in Chapter 3, was:

> Let's spend some time thinking about one particular topic that you may work with when you teach, subtraction with regrouping. Look at these questions: $52 - 25$, $91 - 79$, etc.). How would you approach these problems if you were teaching second grade? What would you say pupils would need to understand or be able to do before they could start learning subtraction with regrouping?

As described in Chapter 3, these responses were coded as showing Strong Specialized Content Knowledge (SCK), Moderate SCK, or Low SCK. Criteria for SCK levels for Subtraction with regrouping are shown in Table 4.3.

Table 4.3. Coding manual for Specialized Content Knowledge for Subtraction with Regrouping

Level of SCK	Criteria	Examples
Strong (4 or more of the criteria)	Flexibly breaks up numbers.	So to set it up this way as well, that there are multiple ways to set up a number. 79 isn't just 7 10s and 9 1s, always. I can make it anyway I want. Right? What if make 5 10s and 29 1s.. We already know that we can make numbers infinite numbers of ways. You know? (P12)
	Uses counting up and/or down as a strategy.	There were number lines with counting up but then also counting back. (P15)
	Notes relationship between addition and subtraction.	[W]e try to teach it like, "Here's addition, and now the opposite of addition is subtraction." (P4)
	Provides at least one visual representation (number lines, manipulatives, etc.) or model.	 (P9)
	Mentions usefulness or difficulty of moving around and over decades.	Being able to comfortably go back 30 or go back 20 (P17)
	Talks about place value, such as 40 = 4 tens, 10 = ten ones, 52 = 4 tens and 12 ones.	…be able to describe it as five tens and two tens rather than just 52 (P11)
Moderate (meets both criteria)	Generally focused on the algorithm while noting that "borrowing one is really borrowing ten".	Someone might say, "One." I'll say no, we're not borrowing one at all, we're borrowing 10. If we're going to borrow 10, this doesn't become 3, it becomes 12. I talk about the fact that, when we're doing math, we're following procedures so that when problems are harder, we have a routine. Then we'll just go through the steps of solving that. (Ms. Fields)

	Talks about place value, such as 40 = 4 tens, 10 = ten ones, 52 = 4 tens and 12 ones.	…be able to describe it as five tens and two tens rather than just 52 (P11)
Low	Focus is only on the algorithm with no mention of "borrowing one is really borrowing 10"	[There were no examples for this level]

Thirteen participants (72%) were judged to show Strong SCK for Subtraction with Regrouping. Features of Strong SCK included the use of a visual model that related to the regrouping used. Those who used base 10 models, for example, would focus on breaking up a ten, and those who used number lines would talk about flexibility in breaking up numbers. Other features included a focus on counting up and/or down (especially noting issues with the decades) and noting the relationship between addition and subtraction.

Five (28%) showed Moderate SCK (see Table 4.4). Features indicating Moderate SCK were a focus on the algorithm without a provided representation, while noting the importance of understanding that a ten is 10 ones, or that 40 is 4 tens.

No participant was placed in the Low SCK category. Recall that the criterion for Low SCK was that a participant focused only on the subtraction algorithm without reference to place value or other features of Moderate or Strong SCK. Contrasting with the decades-old finding of Ma (1999), this finding that no teacher had Low SCK indicates that my participants had a more conceptual grasp of subtraction and how to teach that content than Ma saw in her study. Note that this result on its own does not speak to participants' SCK in other content areas.

Table 4.4. Level of SCK for each participant for subtraction with regrouping

Participant	Level of SCK
1	Strong
2	Strong
Ms. Sutton	Strong
4	Moderate
Mr. Fields	Moderate
6	Moderate
7	Strong
Ms. Blake	Strong
9	Strong
10	Strong
11	Strong
12	Strong
13	Strong
14	Moderate
15	Strong
16	Moderate
17	Strong
18	Strong

All of the participants who showed the elements of Strong SCK used number lines in their explanations, and several mentioned a second form of representation or visualization, such as base ten blocks or Cuisenaire rods.

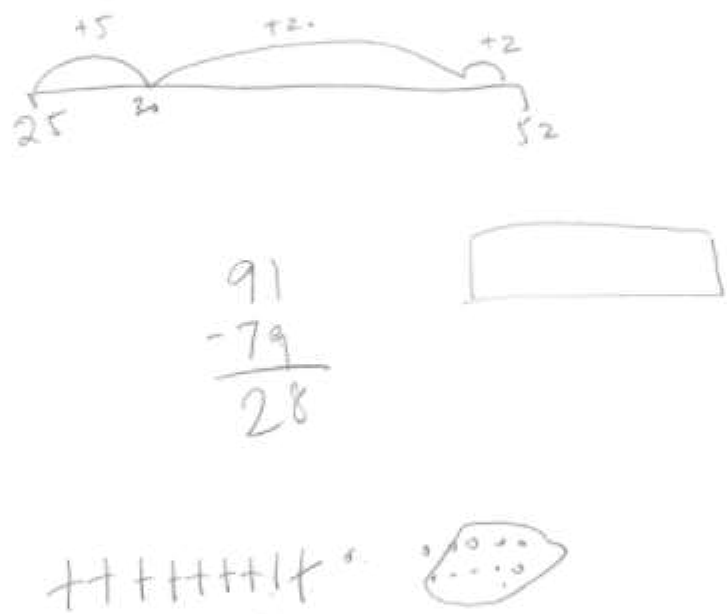
4.4.1.1 Multiple Case Study Analysis for Subtraction with Regrouping

I present the cases in order of SCK, from higher SCK to lower. Both Ms. Blake and Ms. Sutton were determined to show Strong SCK through their explanations of subtraction with regrouping, and Mr. Fields showed Moderate SCK.

Ms. Blake, whose response indicated a Strong SCK, displayed evidence of 5 of the 6 features in Table 4.3: flexibly breaking up numbers, counting up or down, visual representation, difficulty of decades, and place value. Ms. Blake approached the problem differently, noting that she would likely introduce this in a math talk, which is a short discussion usually featuring a problem that students have been asked to solve mentally, and in which they share their different solution strategies. She would expect students to offer solutions that involved counting up, doing the standard algorithm mentally, or with a model involving base ten blocks. She saw her role as faithfully recording their responses. Ms. Blake's response indicated Strong SCK. She created several visual representations, including number lines and base 10 blocks, which was one of the criteria for that SCK level.

$52 - 25$

$91 - 79$



For the number line offering, she would expect at student to say:

"Well, I started at 25, and I went up 5," then I would be doing, "Oh, so you started at 25, and you went up five. And then what did you do?" You'll answer it, "30. What was your next move?" Oh, and then I did whatever. Maybe I did... Where

> am I going, 52? So maybe I did 20, and then maybe I did 2. Right? And then I went up a total of 27.

Ms. Blakes statements reflected flexibly breaking up numbers (splitting the jump into a 5, a 20, and a 2) and counting up (moving from 25 to 52), each of which are criteria for Strong SCK. When asked about what students need to know or be able to do in order to tackle subtraction with regrouping, Ms. Blake, addressed counting up and moving across decades, both indicators of Strong SCK.

> What is it they really need to know? …it's that ability to count up, which it seems to be pretty intact, except when they're going over the decades, and sometimes, you'll go over the hundreds. Does that interfere? I'm not totally sure.

She then went on to talk about place value, which is another criterion for Strong SCK.

> I wonder if the structure of numbers is partly what is tripping them up? Right? If I'm not understanding that 91 is nine 10s and one 1, and that 79 is seven 10s and nine 1s.

While Ms. Blake did not address the relationship of addition and subtraction, she did talk about the difficulty her students have with subtraction.

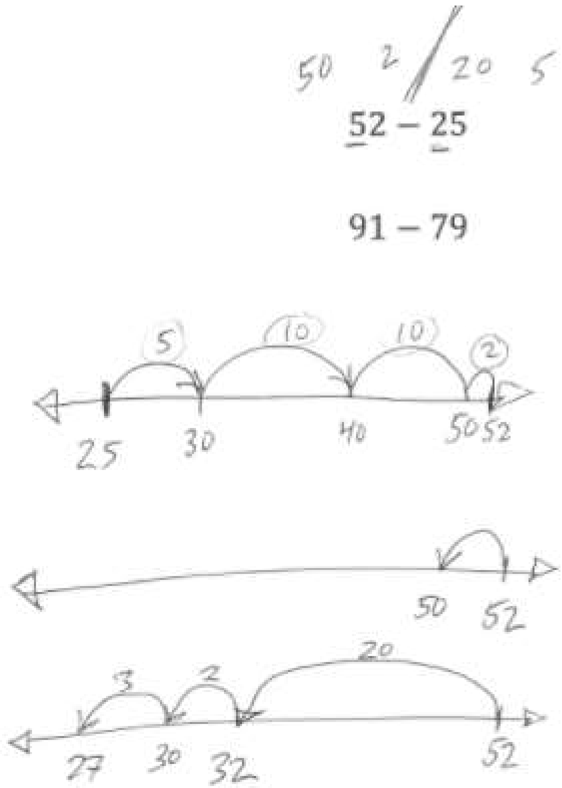
> [B]ut it seems something about that operation that is most problematic for them. There's something about that taking away, and maybe it's because it can be represented… it can be a difference, right? It can be a taking away. It can be a comparison. Maybe that's part of what messes them up is, it presents itself in different ways that look like something other than their default.

Ms. Blake's explanation was typical of those with Strong SCK, and similar to Ms. Sutton's reply.

Ms. Sutton, also judged to indicate Strong SCK in subtraction with regrouping, provided a response reflecting 4 of the 6 criteria: flexibly breaking up numbers, counting up or down, creating a visual representation, and working around and with decades. Similar to Ms. Blake, she also used number lines in modeling subtraction with regrouping, but she didn't specify if she would present the material or have students come up with the "jumps" themselves. She starts by talking about jumping decades, counting up, and flexibly breaking up numbers, which address three of the criteria.

> So, for example, if we've been practicing jumping to the decade then we might have practiced with a number line and so if we're starting with our 25 and we're basically going to add up to trying to get to our 52. So I might jump first up five to get to the decade of 30 and then noticing here that I might jump up 10 more to get to 40 and then 10 more to get to 50 and then I have my two left. So that's my distance because subtraction we can think about it as taking away, but it's also thinking of the distance between those two numbers. So I can think about I have 10, 20, 25 [pointing to the five], 26, 27 [pointing to the 2] as my distance between 25 and 52.

She also presents a visual representation of the number lines, meeting a fourth criteria, as shown below.



When asked what she thinks students need to know or be able to do before they can take on subtraction with regrouping, Ms. Sutton pointed to flexibility with grouping numbers, further addressing this criterion for Strong SCK, as she describes below:

> So they have to be able to do problems that involve regrouping. They have to be able to partition numbers and think of them in different ways. So we traditionally think of a very strict place value that that's a 50 and that's a two. That that's a 20 and that's a five. But there's a lot of other ways that you can do that.
>
> So in this case it was convenient for me to break apart 25 into a 20 and a five or rather 10 and 10 and five. But then thinking about that there might be other ways that are just as useful.

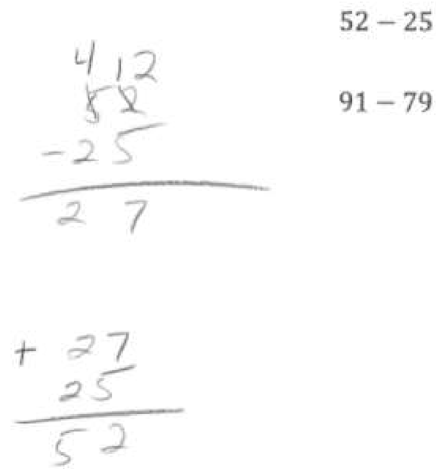
Ms. Sutton's use of number lines and flexibly breaking up numbers was very typical of the participants who showed Strong SCK in subtraction with regrouping.

Mr. Fields was one of the teachers who focused extensively on the procedure of the subtraction, and his response, shown below, was determined to indicate Moderate SCK for subtraction with regrouping, meaning that Mr. Fields did not create a representation for the operation, use counting up or down for a strategy, flexibly break up numbers, or mention issues students have with decades. He did, however, talk about place value. In responding to the initial prompt, Mr. Fields replied:

> I would say that, write it vertically and emphasize the place value obviously. I'd say, at the ones column, we can't do that. What steps do we take?

This was determined to show ‘Generally focused on the algorithm while noting that "borrowing one is really borrowing ten",’ which is consistent with a Moderate SCK level (see Table 4.3). He also focuses on place value, noting that the “one” that is borrowed is actually a ten, which is worth ten ones, as he describes below.

> We usually call it...I haven't used regroup. We'd say borrowing. So I'd say, " We're going to need to borrow from the tens column because we don't have enough in the ones. Let's turn this into 4. How much are we borrowing?" Someone might say, "One." I'll say no, we're not borrowing one at all, we're borrowing 10. If we're going to borrow 10, this doesn't become 3, it becomes 12. So, I have everyone do that before we go. Some of the kids are like, "I already know the answer." I talk about the fact that, when we're doing math, we're following procedures so that when problems are harder, we have a routine. Then we'll just go through the steps of solving that.



While he talks about wanting students to have conceptual understanding of the topic, his response indicated that he wants them to be able to justify the algorithm, but he does not seem open to other strategies or to exploring student thinking.

When asked what he thought students needed to know or be able to understand in order to before they could learn subtraction with regrouping, he noted a need to understand place value and the meaning of subtraction, but he did not elaborate on what that meaning was. Note that place value is a critical aspect to the subtraction algorithm but, without being accompanied by representations or ideas about flexibly breaking apart numbers, it is not enough on its own to be considered Strong SCK.

The majority of participants demonstrated a Strong SCK for subtraction with regrouping, and all showed at least a moderate level of SCK for subtraction with regrouping in the ways that they said they would explain the topic to students. Two-thirds showed a high level of SCK by providing strong representations, focusing on the importance of flexibility in regrouping numbers, and noting the multiple meanings of subtraction and why that is challenging for students, consistent with expectations from current standards and best practices (CCSSO, 2010; Van de Walle et al., 2013).

4.4.2 Multi-Digit Multiplication

The question leading to responses for this topic, as described in Chapter 3, was:

> Some sixth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate:
>
> $$\begin{array}{r} 123 \\ \underline{\times\ 645} \end{array}$$
>
> the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:
>
> $$\begin{array}{r} 123 \\ \underline{\times\ 645} \\ 615 \\ 492 \\ \underline{738} \\ 1845 \end{array}$$
>
> Instead of this:
>
> $$\begin{array}{r} 123 \\ \underline{\times\ 645} \\ 615 \\ 492 \\ \underline{738} \\ 79335 \end{array}$$

> While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching sixth grade and you noticed that several of your students were doing this? What would you say pupils would need to understand or be able to do before they could start learning multi-digit multiplication?

As described in Chapter 3, these responses were coded as showing Strong Specialized Content Knowledge (SCK), Moderate SCK, or Low SCK. Criteria for SCK levels for Multi-digit Multiplication are shown in Table 4.5.

Table 4.5. Coding manual for Specialized Content Knowledge for Multi-digit Multiplication

Level of SCK	Criteria	Examples
Strong (all three criteria required)	Use of area model/box model/array/partial products that ties strongly back to the algorithm.	But knowing that this line [of the algorithm] represents 123 times five and that that's where the algorithm is more efficient. Here [in the area model] we have nine pieces, here we're going to have only three pieces. So helping them connect and see where that comes from…actually giving them the chance to see if they can figure out where the 615 is on their area model and make that connection themselves. (Ms. Sutton)
	Understanding of multiplication by ten.	‘[W]hat's happening when you're [putting] a zero on the end?’ And I really try to get them at least to verbalize ‘I'm multiplying it by a power of ten.’ (P1)
	Zero represents a value, not just a place holder.	But you can only put a digit in there. They are, maybe they are placeholders, but you have to put a digit. There's no value, then you can put zero. That's why zero's so important. I a place holder. It's the only place holder that you're allowed to use. (P2)

Moderate (at least two of the criteria)	Focusing on the procedure of multiplication.	I think really kind of breaking it down into what are our steps, why do we have to do that? (P15)
	Provides area model or box model, but seen as "managing" the process instead of providing essential understanding. (It's just another procedure.)	So just to emphasize to them that this is a way to save time and effort and improve their understanding. We start with just talking about it that way, but then we use the area model as *a way for them to manage it*. (Ms. Fields)
	Zero as a value, not a placeholder.	But you can only put a digit in there. They are, maybe they are placeholders, but you have to put a digit. There's no value, then you can put zero. That's why zero's so important. It's a place holder. It's the only place holder that you're allowed to use. (P2)
Low (one of the following criteria)	Expressed not knowing how to approach the problem.	But I'm not 100% sure how I would approach this because I'm not as fully confident in it. (P15)
	Did not reference the meaning of place value at any point.	
	Described zero as a placeholder.	

Ten of the explanations were judged to indicate Strong SCK for multi-digit multiplication, as they created representations (area models, arrays, partial products) that tied strongly back to an understanding of the algorithm and showed an understanding of the value of zero as a result of multiplication by a power of ten. Six explanations showed Moderate SCK as they focused on the procedure of multiplication and, if they provided a model, it was presented as another procedure rather than a tool for understanding the algorithm. Those explanations also featured an understanding of the role of zero as a value and not simply a placeholder. Two of the explanations were evaluated as showing

Low SCK, reflecting a level that did not emerge in data for subtraction with regrouping. Those participants coded as showing Low SCK either did not know how to approach the problem or saw zero as strictly a placeholder with no understanding of multiplying by powers of ten. A summary of SCK level by participant is shown in Table 4.6. Illustrations of the features of the explanations are then offered for the three focal participants.

Table 4.6. Level of SCK for each participant for multi-digit multiplication

Participant	Level of SCK
1	Strong
2	Strong
Ms. Sutton	Strong
4	Strong
Mr. Fields	Moderate
6	Moderate
7	Moderate
Ms. Blake	Moderate
9	Moderate
10	Strong
11	Strong
12	Strong
13	Strong
14	Moderate
15	Low
16	Strong
17	Low
18	Strong

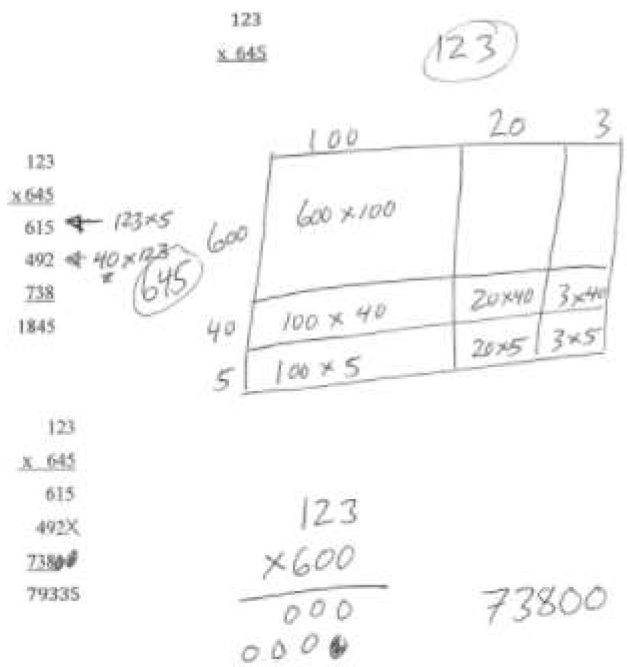
4.4.2.1 Multiple Case Study Analysis for Multi-digit Multiplication

I present the cases in order of SCK, from higher SCK to lower. For multi-digit multiplication, Ms. Sutton was determined to show Strong SCK, and Ms. Blake and Mr. Fields showed Moderate SCK through their explanations.

Ms. Sutton addressed all three of the criteria for Strong SCK for multi-digit multiplication: using a model that ties to the algorithm, understanding of multiplication by ten, and addressing properly the role of zeros. She used an area model, which she directly connected to the algorithm, addressing the first criterion.

Our curriculum is very algorithm centric but really wanting to make sure students understand the numbers and what's going on. So often times I go back and look at the area model since that's a way that we've taught. So if I'm breaking apart 123, what makes the most sense is by place value. 100, 20 and 3.

She then created the area model representation and talked about the factors that would be multiplied to fill each box:



She pointed to the upper left box of the model, saying:

> Making sure that they understand that this is the 600 times 100 . Helping them see that here's my 100 times my five, my 20 times my five and my three times five which all line up to be this line here. [K]nowing that this line represents 123 times five and that that's where the algorithm is more efficient. Here we have nine pieces; here [with the algorithm]we're going to have only three pieces. So, helping them connect and see where that comes from.

She noted that they would not start with three-by-three digit numbers:

> We would start with two-digit numbers or a much more simplified problem. But helping them actually do both versions of [area model and algorithm] and actually giving them the chance to see if they can figure out where the 615 is on their [calculation] or on their area model and make that connection themselves. So that they're noticing what was actually being multiplied here and while that's kind of the trick of the algorithm, I always tell them, "You're not thinking of the four as a 40, you're thinking of it as a four times a three, a four times a two. A four times a one," that's the trick of the algorithm. And why people really like it is because you're keeping the digits and numbers small. But really you have to always have that conceptual understanding and know what you're doing for the algorithms to really be effective of knowing that that four is actually a 40. So knowing then that this line is not four times 123, but that it's 40 times 123 and then what different people do to account for that.

In the interview, the participant discussed what should be placed in the "empty" spots in the presented problem. Through that discussion, Ms. Sutton addressed both multiplication by ten and zero as a value, both criteria for Strong SCK. Ms. Sutton noted that:

> So some students have learned to place an X [instead of a zero] and this is often I inherit them when they come to fifth or sixth grade where they already have a strategy where some of them are placing X's and some are doing these funky filled in zeroes. So they have a wide variety of things. Some students just put a traditional zero there.

> Mathematically what is the most accurate would be a zero. But I think what some of the other teachers might be trying to accommodate for is if there's a zero would be the next digit.

In this she showed understanding of the role of zero, then went on to talk about multiplication by ten, focusing on the language some teachers and students use when describing the placement of zeros in multi-digit multiplication problems.

> I was like all right, well then if you take a number and you add zero what do you get? So talking about our language and making sure when we're saying like oh, you're meaning are we multiplying by 10 or are you saying we're moving a place value over? We're not adding a zero. So rather than say you add two zeroes because it's not a six it's 600. Then you're saying well actually you're multiplying by 10 and then you're multiplying by 10 again or you're multiplying by 100 so you placed two zeroes at the end of the number or something like that. It's not adding.

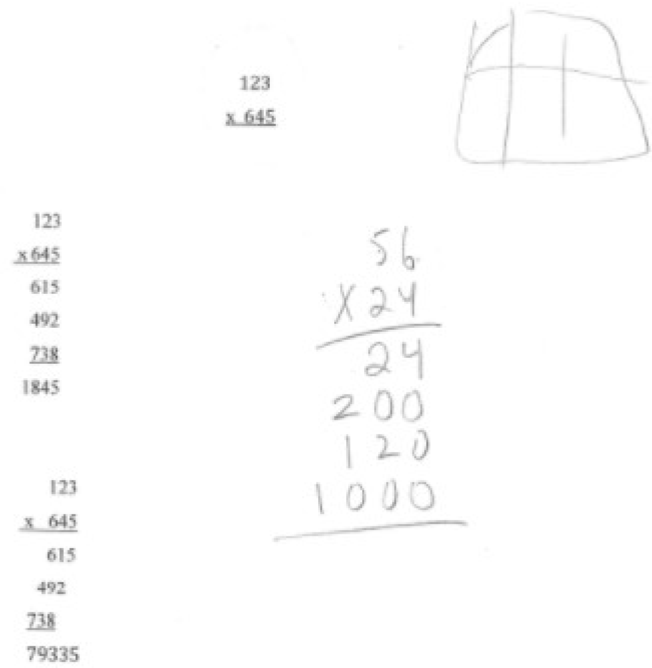
When I asked her what students need to know or be able to do before they can successfully engage in multi-digit multiplication, she said:

> Being able to work with numbers more fluently with partitioning and being able to break apart numbers. So while with multiplication it's most common to break apart based on place value, there are times where that doesn't make sense. Having familiarity with strategies about doubling and having familiarity with multiples of 10 so they're not using the algorithm to multiply 123 times 600. It's really much more efficient when they know their basic multiplication facts as well.

Ms. Sutton connected her representation to the algorithm, focused on multiplication by ten, and recognized that zero has value and is the only appropriate placeholder in the multiplication problem. These facets of her explanation place her solidly in the Strong SCK category.

Ms. Blake offered a response that indicated Moderate SCK in the area of multi-digit multiplication, meeting two of the three criteria: providing partial products that are not connected to the algorithm and using zero to represent the place values. She focused on the use of partial products to calculate the answer, noting that in the fourth grade they do not address the algorithm at all. While she mentions an area model, she did not demonstrate its connection to the partial products or the algorithm. She justifies the focus on partial products, saying:

> We do the partial products, and just leave it there. And so that does address that problem to some extent. Because this idea that, is that a 4 [in 645]? Not really. It's a 40. So I think that does help keep that place value intact. They have an efficient strategy. It demonstrates place value much more explicitly than the standard algorithm does. I'm going to use some smaller numbers.



She included all of her zeros in the partial products, to show the effects of multiplying by ten (“is that a 4? [referring to the 4 in 645] Not really. It's a 40.”), and indicating that as the only proper value.

When I asked what she felt students needed to know or be able to do before engaging with multi-digit multiplication, she mentioned skip counting, being able to move fluently around the number line by different numbers, and opportunities to play around with creating equal groups, both physically (like 8 bowls each holding 7 balls) and through drawn representations of those scenarios.

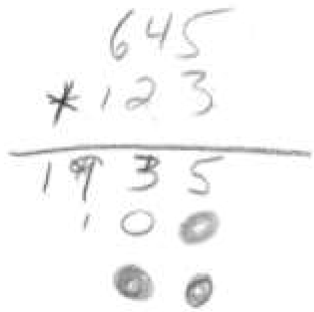
Ms. Blake seems to have foundational knowledge for building multiplication skills, but did not address the heart of the student error shown in the problem. Because she did not connect the partial product model to the algorithm shown, her explanation met the criteria for Moderate SCK.

Mr. Fields, whose response also indicated Moderate SCK, met all of the criteria for that level: focusing on the procedure, providing a model that was seen as managing the procedure, and recognizing zero as a value. He focused primarily on the traditional algorithm for multi-digit multiplication. He also offered an area model, but presented it in an algorithmic way instead of as a tool for understanding the procedure.

> Again, it's about place value. This is just the six or two until you teach them what the coding is essentially. So, I would definitely want to do that first. Generally, the approach we've taken is, I would have them put the larger number on top, especially in this particular problem because of the fact these [digits in 123] are much easier numbers to manage.

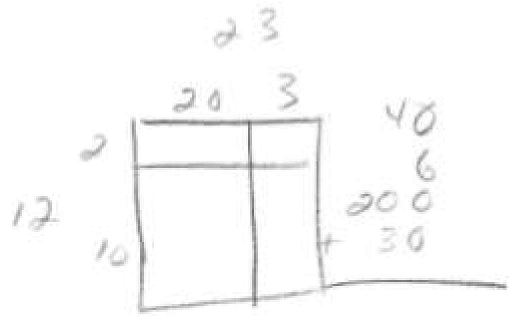
He then went through the procedure for the multiplication. When he got to multiplying by the 20, he said:

> Now we're in the 10's column. How are we going to account for that? Sometimes I'll let them make the mistake in saying, " All right 2 times 5 is 10. No, but that's not really 2" So I talk about what's called a place holder. We fill [the zero] in as a way to impress in their minds about what you're doing rather than just put a zero. Okay, now we are in the hundreds column we need two places so just so we can start there [placing two filled in zeros].



He then brought in the area model, but noted that it was used to manage the multiplication, never tying it in to the algorithm.

> We also, obviously, for smaller problems we use an area model. So, something like 12 times 23 or something. I wait to talk about how [23] there and [12] goes here, and then, let's break it apart, so it's more manageable.



> So just to emphasize to them that [multiplication] is a way to save time and effort [over repeated addition] and improve their understanding. We start with just talking about it that way, but then we use the area model as a way for them to manage it. Then eventually move on to the algorithm itself.

I mentioned the use of the filled in circles (the zeroes) and asked him if it matters what is used in those spaces. He stated that it should be some form of a zero, as that addresses the need to fill in that place value. The use of asterisks or elephants would just distract the students from what they are doing and from conceptual thinking. By noting that zero is a value, he fulfilled the final criteria for Moderate SCK.

When I asked him what students need to know or be able to do before learning multi-digit multiplication, he said place value, and noted its importance in understanding the algorithm.

> Lots of things that can happen if they don't understand how the columns are lined up and what their purpose is. All you're doing is teaching them a formula. You're not teaching them a concept. I know how to do this because Mr. Fields said how. Instead we want to have them thinking, I'm starting on the tens column. I can start with the ones. That's to me like the key. Most of them get it.

While he rails against just "teaching them a formula," Mr. Fields seems to concentrate heavily on the steps of the algorithm. And even though he is seeking to justify the steps of the algorithm, he does not connect a representation to the standard algorithm, even the partial products model he drew. Because his focus tended to be on managing the algorithm, Mr. Fields showed Moderate SCK for multi-digit multiplication.

For multi-digit multiplication, all but one of the participants indicated what conceptual misunderstanding the students in the problem were having, and all but two of the participants showed at least a moderate level of SCK. Ten of the teachers were able to offer representations that tied strongly to the algorithm and could explain the placement of zeroes into the partial products, and six others offered some strategies for addressing the place value issues, even though they didn't create linked representations. The two participants who were evaluated as having low SCK for this topic could not provide any strategy for addressing the misconception with students.

4.4.3 Division with Fractions

The question leading to responses for this topic, as described in Chapter 3, was:

> People seem to have different approaches to solving problems involving division with fractions. How do you solve a problem like this one?
>
> $$1\frac{3}{4} \div \frac{1}{2} =$$
>
> Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2}$? What would you say pupils would need to understand or be able to do before they could start learning division with fractions?

As described in Chapter 3, these responses were coded as showing Strong Specialized Content Knowledge (SCK), Moderate SCK, or Low SCK. Criteria for SCK levels for Division with Fractions are shown in Table 4.7.

Table 4.7. Coding manual for Specialized Content Knowledge for division with fractions

Level of SCK	Criteria	Examples
Strong (all three criteria required)	Participant could solve the problem without using the algorithm.	So another way would be on the number line to think about. So if I have one and three-fourths, so there's my zero. Okay. So there's my one and three-fourths. I know when I think about how many halves go into that and so there is a half, there's a half and there's a half. So then noticing that okay, so I know that four halves would go into two and it's taking two of those pieces to make a half. So I have only part of a half in that case. So if you map it onto here I can see that there's one half, there's one half and then if I'm noticing it's taking four of those pieces to make a half I can switch and do it this way. Here's a half and that I only have a half of a half left. So then noticing that I have three halves but then there's this little half of a half. So if I have a fourth leftover. So those are my pieces that I have. (Ms. Sutton)
	Correct context and/or representation.	I need 1 3/4 cups of flour, but I only have a 1/2 cup scoop. How many of them are am I going to need? I need 3 1/2 scoops full to get my whole 1 3/4 cups of flour. (P12)
	Talked about what the problem is asking. (What does it mean to divide?)	They've foundationally been taught that that's what dividing is. Which doesn't change here. Basically, what I tell them is, to understand what's happening here, if I have 7/4, how many groups of 1/2s can I make? They can't necessarily concept that in their head, but at least they know what we're doing. I'm trying to break up 7/4 into pieces that are a size of a 1/2. So that's why that dividing by fractions makes it bigger, because I can fit multiple 1/2s in there. (P12)
Moderate (both criteria required)	Could solve without the algorithm	But if I did it this way, I would just keep subtracting. So you would come up with, so you did it at one, two, three, the remainder, one fourth. Would that be right? Is that right? [We rarely write remainders with fractions.] Oh, it's half of the half. So it'd be three and a half. (P2)

	Could not provide a correct context or representation.	You have a whole sandwich and three quarters, and you want to divide that in half so two people can share them. (P18)
Low (all three criteria required)	Could not solve or could only solve with the algorithm.	This is a mixed number and so the mixed number has to be changed into a fraction. So that's four times one, add three is seven fourths and then reversing the operation and flipping the fraction, don't ask me why, and we're doing 14 fourths and then that can be reduced. (P18)
	No sense of how or why the algorithm works.	I really have no idea. I remember that you're supposed to switch, change, flip. But I have no idea why and I think that that is a testament to my own math education. (P15)
	Could not provide a correct context or representation.	You have a whole sandwich and three quarters, and you want to divide that in half so two people can share them. (P18)

Five participants (28%) were judged to show Strong SCK for Division with Fractions (see Table 4.8), meeting the criteria of solving the problem without using the algorithm, creating a context or representation, and talking about the meaning of division. Two participants (11%) showed Moderate SCK, meaning they could solve the problem without using the algorithm but could not create a correct context or representation. Eleven participants (61%) were placed in the Low SCK category, indicating that they could not solve the problem or could only solve it with the algorithm, that they did not know how the algorithm worked, and they could not provide a correct context or representation. Focal participant responses will show the aspects of these features.

Table 4.8. Level of SCK for each participant for division with fractions

Participant	Level of SCK
1	Strong
2	Moderate
Ms. Sutton	Strong
4	Low
Mr. Fields	Low
6	Moderate
7	Low
Ms. Blake	Low
9	Low
10	Strong
11	Low
12	Strong
13	Low
14	Low
15	Low
16	Strong
17	Low
18	Low

4.4.3.1 Multiple Case Study Analysis for Division with Fractions

I again present the cases in order of strongest SCK to lowest SCK. For the topic of division with fractions, Ms. Sutton was in the Strong category, and both Ms. Blake and Mr. Fields were in the Low category.

Ms. Sutton showed Strong SCK for Division with Fractions, meeting all three criteria: solving the problem without using the algorithm, providing a correct context, and offering a strong understanding of what division with fractions means. In solving the problem, she started by talking about what division means, meeting the third criterion. She led into her explanation by focusing on what 3 divided by 1 would mean.

> [O]ne strategy that I'm teaching my students is to change the numbers and make sure they understand what's going on. So literally, I'm going to do a three divided by a one. All right, so what does that mean? If I'm thinking about if I have three of something and I'm dividing that among one. So I'm thinking about how many

> times does the one and even using words if need be. How many times, so then we talk about the connection, …and both multiplication and division are thinking in groups. So how many times does the one fit into the three? Or how many groups of one fit into the three? That language.

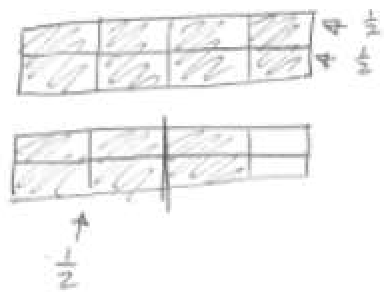
She then talked about how that would relate to 3 divided by ½.

> So then if I'm like all right if you have that sense with whole numbers, default to one-half right. So try it with where you have a whole number and then you have a half. Half is the most familiar of the fractions. Students have been working with it maybe from second grade. So thinking about now how many times or how many groups of one-half fit into three wholes. So then I'm going to think about I'm going to actually draw a picture for this one. Now I'm going to think about if each of those are my wholes that I'm making halves and now I can see my halves in there and I can actually, literally, count my halves.

After setting up the idea of what it means to divide, and to divide by a fraction, which meets the third criterion, she then focuses on the problem given.

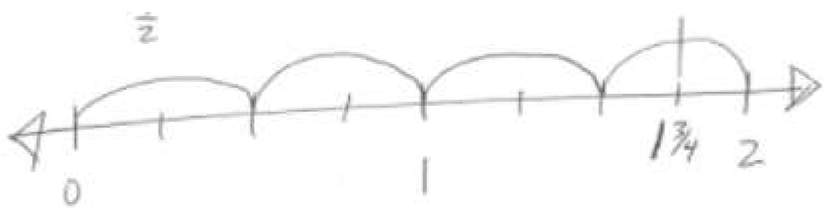
> So now I'm going to think about and go back to my original numbers. So making sure they really conceptually have an idea, going all the way back to whole numbers if need be, and then introducing a fraction piece by piece into that idea. So now I have a fraction of something. I have a whole and a fraction of something. So if I have my one whole and I have my three-fourths now I have to think about how many halves are in there which is really tricky to think about.

She then drew a representation of the given problem, showing the halves located in the $1\frac{3}{4}$.



Extending her thinking further, she proposed a number line model for determining the answer:

> So another way would be on the number line. So there's my one and three-fourths. I know when I think about how many halves go into that and so there is a half, there's a half and there's a half. So then noticing, I can see that there's one half, there's one half and here's a half and that I only have a half of a half left. So then noticing that I have three halves but then there's this little half of a half. So if I have a fourth leftover. So those are my pieces that I have.



She was able then to determine that the answer was 3½ halves. Using this number line model to calculate the answer met the first criterion for Strong SCK.

Ms. Sutton then noted that sixth graders learn the algorithm for solving division with fractions problems:

> But that again is a magic trick where they sometimes learn what keep change flip. It's just a trick. It's like a magic trick and there's really not this conceptual understanding of what it means. I find both with division problems and subtraction problems those are the ones where conceptually understanding what's going on is probably the hardest part.

This led directly into her providing a context for the problem, meeting the second criterion for Strong SCK. She didn’t hesitate at all in being able to produce it.

> So if I have a cup and three-fourths and then I want to take half cup measurements out of that how many portions would that be and building a granola sort of problem or something like that.

When I asked her what students need to understand or be able to do in order to work with division of fractions, she was able to offer many background skills and knowledge.

> Fractions are really not elementary at all. Students have to know how to exhaust the whole so that when they're creating their fractional pieces they're using the entire whole. They have to know that they have to make equal parts within that. They have to understand that there's always this reference to the whole. Students will sometimes figure out a problem of well it's a half of a fourth and I'm like okay well we don't usually go to the store and say I need a half of a fourth of a pound of flour. We always go back and refer to what one whole is. So they have to be able to hold those parts. So they have to be able to coordinate quite a few units.

She commented more on the coordination of units in working with fractions, showing deep understanding of the issues students face when learning these concepts.

> So in this case they have to be able to hold the whole. They have to be able to hold the fractional part and then they also have to be able to hold in this case, they have to be able to hold the, only I don't even know what to call it. The other fractional unit. So if this fractional unit is thinking in fourths but then they also have to be able to hold the concept of one-half at the same time. But when a student isn't yet able to coordinate three levels of unit, it's like how I'm expecting you to memorize the magic trick and it's not meaningful and they get tripped up really easily.

Ms. Sutton then mentioned a key idea – that fractions are numbers in their own right and not just parts of a whole.

> So there's all of this work that is within just understanding parts to a whole, but then there's all this part extending beyond that and seeing individual fractions as numbers themselves. Then being able to work with that. Then the magic trick part has this whole idea of reciprocals in there and inverse relationships and the connection between multiplication and division and that's a third layer of understand fractions. So if a student isn't yet coordinating that with whole numbers and then now I'm expecting them to do it with yet even a third layer and to do it with fractional numbers they're not ready yet.

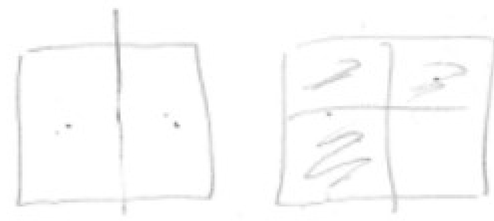
Ms. Sutton's firm understanding of representations of division with fractions, her ability to develop a context for that operation, and her strong understanding of the foundational concepts that lead to the operation met all three criteria for strong SCK for the topic, and her description of what students need to understand showed a deep understanding of the concepts underlying this topic.

Ms. Blake was placed in the Low SCK category for division with fractions, as she could only solve with the algorithm, did not know how the algorithm worked, and was unable to generate a correct context for the problem. In offering her solution, she mentioned a mnemonic she learned for the algorithm:

> Oh God, this is like beyond my fourth-grade abilities here. Right? Well, I think I'm going to do it like "Yours is not the reason why. Yours is just to invert and multiply." I'm going to do it that way, because that's the way, how I was taught. All right, so that'd be what? 7 fourths, and then it would be... I don't even know if I'm right. Am I right? It'll go 14 over 4, which would be like 3 and one half?
>
> $$\frac{7}{4} \times \frac{2}{1} = \frac{14}{4} \quad 3\tfrac{1}{2}$$
>
> Okay. But if I were really doing it right, it'd be like I got one whole, I have 3 fourths of another whole and I'm dividing that by half. What the heck does that mean? Right? Honestly, what on earth does that mean? Right? To divide by ½?

Given her question and that, as a fourth grade teacher, she does not encounter division with fractions, I hoped to further understand the character of her SCK for division with fractions by first describing division by 2, something that comes up in Grade 3 and sometimes Grade 2. I asked her what it mean to divide by two, which we agreed could be thought about as finding how many groups of two are in a number. So dividing by ½ would indicate how many groups of ½ are in a number.

She then drew:



And said:

Oh, so I have 1, 2, 3 halves, but I got 3 and a half.

I pointed out that she still had the ¼ to account for, which she then reasoned was half of a half. She then noted, “[W]hen you see 3 and a half, you think three wholes and a half. But it's referring to 3 and a half halves.”

This highlights the coordination of units struggle that Ms. Sutton mentioned in her description of what students need to know or understand in order to learn division with fractions.

The context she provided was problematic as she started with $1\frac{3}{4}$ pizzas and wanted to give half a slice of pizza to every person. She expressed that it was challenging to figure out the relationship of the part to the whole, again struggling with coordinating the units, which she mentioned when I asked her what students needed to know in order to tackle division with fractions.

With fractions, it feels like that ability to hold on to those layers of units. Right? I can hold on to what the whole is, I can hold on to what the units are that make up that whole, and like here, I don't know how many layers there are, but there's a lot of layers in units. Right? I mean, we have the wholes, and then we're defining basically the half as a whole when we're dividing. So that becomes whole, right?

She also noted:

> I did a lot more number line work with fractions this year, which I felt was like a way to think about them that I hadn't done. It's not just area with fractions. Right? It's also number line, and I feel like that helped a lot too. It keeps kind of widening that idea about what a fraction is.

Ms. Blake's SCK for Division with Fractions was judged to be low. Her only strategy for solving the problem was the standard algorithm, which she expressed she did not understand conceptually, and her context was problematic.

Mr. Fields was one of the participants who did not attempt to solve the problem, nor to provide a context, indicating Low SCK for division with fractions. As he said, "I may not be able to solve it myself, [but] I can talk about working with fractions."

His expression indicated a level of fear, so I did not push him initially to work with the problem given, but instead listened to how he works with fractions in his classroom. He spoke of adding and subtracting fractions and the need for both a common denominator and the ability to compare the fractions. He continued by saying:

> I would teach that using what we call the super one. Have you heard of that?

$$\frac{1}{2} \times \boxed{\frac{2}{2}} = \frac{2}{4}$$

We’ll start the concept thinking by saying, "What do you get when you multiply that number by one, 784 times 1 is 1 [sic]. "What do you get when you multiply a triangle by 1, you get 1 [sic]." The same thing with fractions. When we multiply them by 1, we get the fraction but in a different form which is what's cool. One half …We can multiply it by one over one. We'll talk about how 2 over 2 is a form of 1. So, what's 1 times 2? It's 2. What's 2 times 2 is 4.

He then focused on converting mixed numbers into improper fractions, saying:

> We talk about ways that they need to think about the fractions, so they can decide which one is going to be changed. Can we turn three fourths into halves? Well, so they'll start thinking like that. Let's try it. They really wouldn't know division with fractions. I wouldn't say, “Oh, we can divide it by a super one." I'd say, “Well, let's do change this one because we know we can do it.” Most of them could handle this right now, except as a multiplication or division.

He converted the $1\frac{3}{4}$ into 7/4 and the ½ into 2/4, focusing again on the fact that his students had been working with common denominators.

$$\frac{7}{4} \div \frac{2}{4}$$

Hoping to determine if he could use the common denominator method of solving the problem, I posed the following question to him:

> So if we got the seven fourths and the two fourths, I'm just wondering if they could do it like a repeated subtraction problem and think about it that way. It's almost like doing seven somethings divided by two somethings,

He did not take up the suggestion and offered that his students were just diving into division and working with partial quotients, so to add fractions into the mix would be daunting.

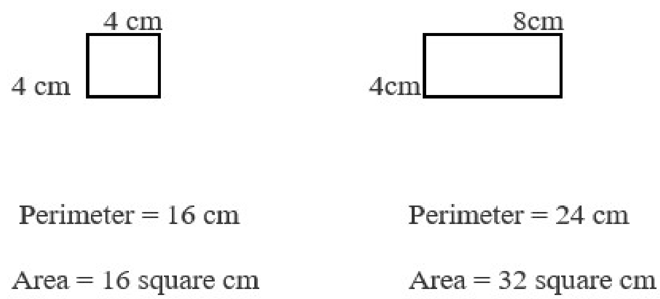
When I asked him what students need to know or be able to do in order to work with fraction division, he noted that they would need to understand what division is doing, and suggested modeling using things they could hold on to, noting that he prefers Cuisenaire rods, but he did not offer sufficient elaboration to know exactly what this means to him. He then mentioned using the idea that division is the opposite of multiplication. When I suggested that perhaps that strategy could be used on the fraction question I had asked, asking what times a half is $1\frac{3}{4}$, he said that he didn't think they could think that way with multiplication of fractions. While he mentioned that understanding the meanings of division and multiplication were important, he didn't elaborate on what those meanings are. His responses indicated that he did not have an understanding of either multiplication or division with fractions, and, though he mentioned representing multiplication with a manipulative, he did not offer any contexts for division, leading to an evaluation of Low Specialized Content Knowledge for this topic.

4.4.4 Area, Perimeter, and Proof

The question leading to responses for this topic, as described in Chapter 3, was:

> Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure

increases, the area also increases. She shows you this picture to prove what she is doing:



What would you respond to this student?

As described in Chapter 3, these responses were coded as showing Strong Specialized Content Knowledge (SCK), Moderate SCK, or Low SCK. Criteria for SCK levels for Area, Perimeter, and Proof are shown in Table 4.9.

Table 4.9. Coding manual for Specialized Content Knowledge for area, perimeter, and proof

Level of SCK	Criteria	Examples
Strong (meets both criteria)	Suggested trying other combinations.	I would say, "Good on you. Let's check this out further. Right? Let's just extend this and see if it's always, always true." Because we talk about math laws, right? And then we explore a lot of different examples, but we don't want to get to the point where we go, "That is always true. It's a math law. We feel confident." So I would say like, "What do we think? Do we think it's a math law? Let's explore a lot of different cases." (Ms. Blake)
	Created representations that were a counterexample AND/OR Mentioned keeping either Area or Perimeter constant and showing the variety of combinations the other factor could be to illustrate a counterexample.	I might go for an area of say 100, because that's a nice friendly number and if this kid clearly knows their multiplication, then I can say, "Hmm I have a 100 plus ten times ten is 100 and yet the perimeter is only forty". So I think that's what I would have them look at other factors for the number, because I'm also thinking about you could have this, but you could have fifty times two. And so now your perimeter is 108. For exactly the same area. So why are the perimeters that much bigger? (P1)
Moderate (all three criteria required)	Suggested trying other combinations and played with some.	And so, then I might ask them, "Okay, so you're telling me with this problem and this problem, we've confirmed that what you said is true, but now we have to test your theory again." Because, the whole point of a theory is that we have to keep testing it, and if you find that one time that your theory doesn't hold up, it's not true. (P4)
	Could not think of a counterexample.	Absence of a noted counterexample
	Did not arrive at a conclusion about correctness	So I still don't think I have a definitive answer to theirs because they are just saying if the perimeter increases that the area also increases. (Ms. Sutton)

Low (at least one criteria met)	Could not or did not engage with the problem.	I think, yeah. I would obviously spend some time saying, "Oh, I'm so excited you see this." Other than that, I'm kind of at a loss. (P16)
	Focused only on formulas.	You have to define the terms for them first. Once we get perimeter down, and they talk about the formula for it, then we talk about what about the inside, the surface? Then here's the area of my hand, how do you measure that? Talking about labeling units and so forth. (Mr. Fields)
	Reply to student was good job or they thought the student's work looked sufficient.	Yeah, I don't know if this ever said that to me. I think it's true. Yes. It makes sense. So I say, " Great…." I don't know. It makes sense. It's a great observation. (Mr. Fields)

Four of the explanations (22%) were judged to indicate Strong SCK for area, perimeter, and proof, as they suggested trying other combinations to test the theory set forth by the student and then created either a single counterexample or looked at rectangles with either constant areas or perimeters to show that the theory was false. Seven explanations (39%) showed Moderate SCK as they suggested testing other combinations, but either focused primarily on the formula or failed to correctly determine the validity of the claim. Seven of the explanations (39%) were evaluated as showing Low SCK, as those participants either did not engage with the problem at all, or focused on the formulas to determine that the theory was correct. A summary of SCK level by participant is shown in Table 4.10. Illustrations of the features of the explanations are then offered for the three focal participants.

Table 4.10. Level of SCK for each participant for area, perimeter, and proof

Participant	Level of SCK
1	Strong
2	Moderate
Ms. Sutton	Moderate
4	Moderate
Mr. Fields	Low
6	Moderate
7	Moderate
Ms. Blake	Strong
9	Low
10	Moderate
11	Strong
12	Moderate
13	Low
14	Low
15	Strong
16	Low
17	Low
18	Low

4.4.4.1 Multiple Case Study Analysis for Area, Perimeter, and Proof

I present the cases in order of SCK, from Strong to Low. Ms. Blake's response was determined to show Strong SCK, Ms. Sutton's to show Moderate SCK, and Mr. Fields' to show Low SCK.

Ms. Blake's response met both criteria for Strong SCK for this topic, as she both suggested trying other combinations and created a counterexample. She was one of the few participants who went beyond the counterexample and suggested finding conditions under which the claim was true.

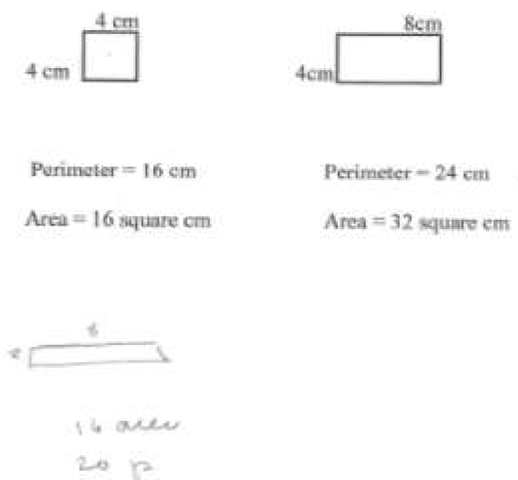
She first addressed the issue of proof, noting that a single example does not prove a math law:

> Good on you. Let's check this out further. Right? Let's just extend this and see if it's always, always true." Because we talk about math laws, right? The ones that

are always, always true. So I would say like, "What do we think? Do we think it's a math law? Let's explore a lot of different cases." Right?

So what if it were... So you're saying 4 by 4 is 16, area was 16. Increase the perimeter to 24, the area increased. So let's try, I don't know, let's try a 2 by 8, let's say. Right? That would be a 16 area, and it would be, what, 8 and 8 is 16, and 4, that'd be a 20 perimeter.

This created a clear counterexample to the theory by finding a rectangle with a larger perimeter (20) than the one on the left (16), but having the same area (16). Since the perimeter increased and the area did not, the student’s conjecture does not hold.



Ms. Blake went on to suggest a whole class exploration of the topic to try to determine the conditions under which the conjecture could be true:

And, then, can we throw this up on a chart, can we begin to explore this? Let's pretend there's other people in the class. What do they think? Maybe this would be a great morning math. Right? Like, "So and so thinks they have a math law. What do we think?"

> And could we make it a math law with some revisions?
>
> Could we say, "Well this is true if, right, or only if..." or whatever?
>
> Could we revise it in a way that it is, we do feel confident that it's a math law?

When I asked her what types of knowledge, skills, and understandings students need in order to work with perimeter and area, she said:

> We start small, we build stuff, but certainly the idea of adding, and the idea of multiplying become huge. That array model becomes a huge part of it, and that ability to connect it to all the other things. Right? If you're already figuring out multiplication using array model, well, you're already doing area. Right? You got it. It's there. It might be presented to you in a different way, but it is what you're already doing.
>
> So, yeah, definitely connecting to multiplication, I think, is really big, and then kids always get confused. "Is it the inside, and which one's the perimeter?"

Ms. Blake's suggestion to try other combinations and her creation of a clear counterexample, combined with her sense of what it would mean to justify a theory or create conditions under which a theory was true, were indications of Strong SCK for this topic.

Ms. Sutton was found to have a Moderate level of SCK for this topic, meeting the criteria of suggesting trying other combinations, absence of a noted counterexample, and not arriving at a conclusion about correctness. She said that she would reply:

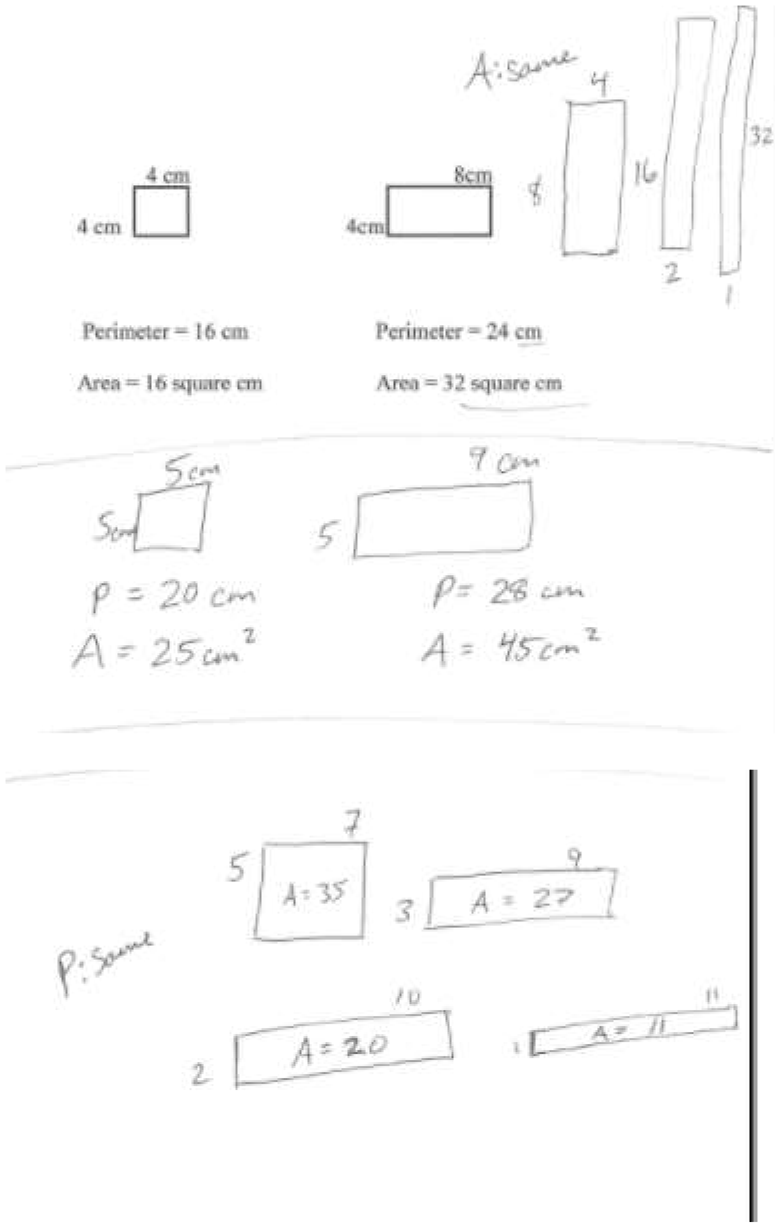
> So then I would say that there are instances where something seems to be a pattern that was established, but that if we just have to make sure it really is a pattern. So sometimes students can see a pattern and they run with it. Then it's

> like it's not always a pattern. So I guess I would encourage them to try then another example of that.

She went on to suggest a strategy for testing other combinations,

> So their theory was that when the perimeter increases that the area also increases. So then I'm trying to think. It would be cool to somehow come up with a systematic chart that has every combination in between. But then there's also thinking about how they can change ... you can have the same area but your perimeter can change. So all of the options of this one. So if I'm keeping my area the same that I could also ... well do I want to keep my area the same or do I want to keep my perimeter the same?
>
> So yeah, I guess I would just encourage them to continue to play with it as I clearly do not have an exact no you're right, yes you're wrong solution.

During this time, she drew the following representation, creating rectangles with the same area but different perimeters, and then a set with the same perimeter but different areas.



While these examples of changing perimeters with a constant area provide a counterexample to the claim that increasing perimeter also increases area, she was unable to state an opinion of the claim. I asked her to reflect further on the rectangles she created, and she said:

> So here I even played with the idea of, I'm going to keep the area the same and see what is happening and then over here I had to say that I will keep the

> perimeter the same and see what happened. So when I did that at first I was like okay, those are getting larger. Nope, now my perimeters are getting smaller. So here I have it showing those different connections there. So I still don't think I have a definitive answer to theirs because they are just saying if the perimeter increases that the area also increases.

When I asked her what skills or knowledge she thought students needed in order to engage in this exploration or type of learning, she answered:

> Well I think there's noticing patterns. There's clearly multiplication skills and understanding the different between area and perimeter and what each of them means and how to calculate them. There's understanding of units, although they could have those all wrong and still have the correct numbers for their area and their perimeter. But I think it's noticing pattern and using that and then also noticing that you can't necessarily make ... I mean you can make a theory based on part of a pattern but that it doesn't necessarily ... you have to continue the pattern farther sometimes to notice.

While Ms. Sutton was able to engage with the problem and test various areas and perimeters, she was unable to use the evidence to come to a conclusion. For that reason, her SCK for area, perimeter, and proof was judged to be at the Moderate level.

Mr. Fields was one of the participants who was rated as having Low SCK for area, perimeter, and proof, as he did not engage with the problem, focused on the formulas, and thought the student had reached a good conclusion. When asked how he would respond to the student he said:

> Yeah, I don't know if this was ever said that to me. I think it's true. Yes. It makes sense. So I say, “ Great. Let's try it out, show me some examples.” I don't know. It makes sense. It's a great observation.

This lack of engagement and agreement with the claim were two of the criteria for Low SCK. When I asked what students need to know or be able to understand as they start working with perimeter and area, his reply indicated a focus on the formulas, the final criteria for Low SCK:

> You have to define the terms first. Once we get perimeter down, and then talk about the formula for it, then we talk about what about the inside the surface? So here's the perimeter of my hand. And how do you measure that? …Then here's the area of my hand, how do you measure that? Talking about labeling units and so forth, to make sure that they understand. Then with area, obviously they're multiplying, so we teach them that formula too once we get perimeter solid.

Mr. Fields was focused almost entirely on the formulas used to calculate the area and perimeter, and in making certain his students understood the vocabulary for both. While he mentioned “show me some examples,” he indicated that he thought the theory was correct and that those examples were only for further illustration. At no time did he question the correctness of the theory or mention that further examination for a counterexample would be needed. These were all indicators of Low SCK for the topic.

4.5 Specialized Content Knowledge Across Participants

SCK varies not only by participant over the set of topics, but also by topic for the participants. In this section, I look at how SCK varies by topic for different participants, specifically those chosen as focal cases.

As described in Chapter 3, I assigned each SCK level a numerical rating: 3 for Strong, 2 for Moderate, and 1 for Low. I used the average SCK rating for each participant to separate the participants into three categories: Strong overall, Moderate overall, and Low overall. I then chose one participant from each of those categories who was representative of the teachers in that group. As Table 4.1shows, only two of the participants (11%) had the same SCK level for all four topics. The other 16 participants (89%) had SCK that varied over the topics. The three focal participants all had varied SCK levels, as shown in Figure 4.1.

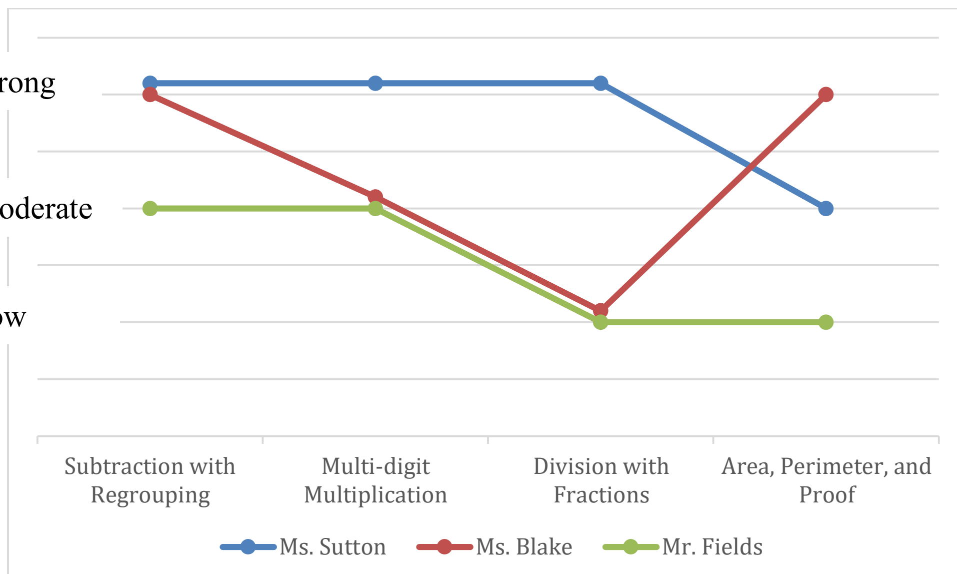


Figure 4.1. SCK levels for the focal participants across the four topics.

Ms. Sutton was able to provide conceptual explanations, representations, and justifications for the first three topics (subtraction, multiplication, and fraction division), and she also explored the topic of area, perimeter, and proof using a mathematically appropriate strategy. She was, however, unable to use that representation to conclude that the student's conjecture was not correct. Her Strong SCK in the first three areas did not ensure Strong SCK in the fourth. A variation in SCK was true for 5 of the 6 participants in the Strong SCK group, indicating that Strong SCK in one or more topics areas does not necessarily lead to Strong SCK in others.

Ms. Blake was typical of the Moderate group, as 7 out of the 8 (88%) of those participants also showed a variation in SCK. Her SCK was Strong in subtraction and area, perimeter, and proof, Moderate in multiplication, and Low in division with fractions, where she struggled to conceptualize what $1\frac{3}{4} \div \frac{1}{2}$ means. It would be challenging to predict Ms. Blake's SCK for a new topic based on the variation she demonstrated in these four areas.

Mr. Fields' explanations tended to be thin and procedural, despite his stated desire for students to understand the math conceptually. He was typical of the 4 (22%) participants in the Low group, who all had variations in SCK over the four topics. While these participants all showed Low SCK for at least two of the topics, they each had areas in which they could demonstrate some level of understanding of the explanations, representations, and justifications of the mathematics they encountered. Their struggle came in knowing how to connect those elements. Two of those participants (11%) were rated as showing Strong SCK in subtraction with regrouping, further illustrating that an SCK level in one topic is not necessarily a predictor of SCK in another area.

4.6 Summary of Findings: Specialized Content Knowledge

A level of Specialized Content Knowledge (SCK) was determined for each participant for each topic. These SCK levels varied greatly from topic to topic and from participant to participant, as indicated in Table 4.11. Participants showed the greatest SCK for subtraction with regrouping and multi-digit multiplication, but overall lower levels of SCK for division with fractions and the area/perimeter/proof problem. Fewer than one-third of participants showed Strong SCK in division with fractions and the area problem, indicating that many teachers are not adequately prepared to teach those topics at the elementary level. Subtraction with regrouping is expected to be mastered by the end of second grade, multi-digit multiplication by the end of fifth grade, division with fractions by the end of sixth, and area and perimeter relationships by the end of third grade. There did not appear to be a relationship between years of math teaching experience and level of SCK on these topics, except perhaps for division with fractions. The sample size is too small to make a conclusion as to the significance of that relationship. SCK also varied for almost all of the participants over the four topics. A participant could show Strong SCK in one or more areas and Moderate or Low SCK in others, as seen in Figure 4.1.

In Chapter 5, I describe the themes I found in the participants' responses, looking at the features that indicate explanation, representation, and justification. In Chapter 6, I will discuss the findings of these chapters, in conjunction with the relevant literature, and suggest some implications of the findings. I will also suggest areas for further research as we continue to work on improving teacher SCK.

CHAPTER 5

FINDINGS – THEMES IN TEACHER EXPLANATIONS

5.1 Introduction

The purpose of the second research question was to explore the themes in the features and representations teachers highlight when explaining math concepts and procedures to students. In particular, it focused on topics that have often been taught in a strictly algorithmic manner (subtraction with regrouping; multi-digit multiplication; division with fractions; and proof involving area and perimeter) to see if the explanations have features that can be considered more conceptual. Conceptual explanations would be in alignment with the expectations new state and national math standards have placed on math instruction, and would require a higher level of Specialized Content Knowledge.

The methodology used to analyze this question was presented in Chapter 3. This chapter consists of results that answer research question 2 by first offering an overview of the findings, then noting the themes found in the explanations given by participants for each topic, especially in light of the variation in SCK for each topic. Illustrations for the themes are offered throughout each section. The following chapter will discuss the findings from this chapter and the previous chapter and offer implications and areas of future research.

The research questions examined for this section are:

What themes are found in teachers' explanations for the following topics?

a. Subtraction with regrouping?

b. Multiplying multi-digit numbers?

c. Division with fractions?

d. The relationship between area and perimeter?

5.1.1 Overview of Findings

Themes found in the teacher explanations revolved around representing and justifying the algorithms for each type of calculation. In this overview, I consider each of the four content areas. Similar to the way SCK varied across the four topics, themes were not necessarily consistent across content areas. As I further describe in this chapter, some themes were not necessarily associated with Strong, Moderate, or Low SCK, while other themes were.

For subtraction with regrouping, most teachers were able to provide a conceptual foundation for their strategies, and several teachers used more than one strategy, which was coded as "multiple ways of regrouping." While some of the explanations focused on representing and justifying the standard algorithm of borrowing ten using base ten models, others bypassed the algorithm entirely by using number line models and flexible regrouping to introduce subtraction with regrouping. Participants also noted that "subtraction is challenging," because it has multiple contexts and meanings, and because the language we use for it has common meanings that can make it confusing for students.

For multi-digit multiplication, similar results were found. Teachers tended to choose a representation that justified the algorithm, such as an area model or use of partial products, and some teachers suggested that learning the algorithm was not important in light of those other strategies. This led to the theme of "Representations explain the algorithm." The participants also focused on place value misunderstandings as the source of student error ("the nature of the error"), and were nearly unanimous in declaring that zeros are the only appropriate values to act as placeholders in the traditional multiplication algorithm ("zero has meaning").

Teachers tended to struggle with division with fractions. Though most of the participants could solve the problem ("long live the algorithm"), only about one-third could do so without using the standard algorithm ("what does it mean to divide?). Only two of the 18 participants (11%) attempted to justify the algorithm. The teachers struggled to develop contexts for the given division problem, and many confused dividing by half with dividing in half ("context was an issue"). When discussing important fraction knowledge for approaching the topic of division with fractions, most of the participants focused on "features of fractions" and vocabulary, along with the part-whole definitions of fractions.

For the topic of area, perimeter, and proof, most teachers suggested that a student test their theory to look for counterexamples, thus showing an understanding of the rigorous nature of proof. Few teachers could suggest a strategy for testing the theory, many did not arrive at a conclusion about the truth of the claim, and some teachers thought the claim was correct. A link was seen between a teacher's conclusion about the claim and the suggestions they made about exploring the claim, leading to the sole theme "teacher knowledge affected the response.".

Each of the topic areas has a theme that touches on representation, whether through relating it to the algorithm (subtraction with regrouping and multi-digit multiplication), creating a model or context (division with fractions), or creating a counterexample (area, perimeter, and proof). Other themes are more topic-specific, such as the role of zero in multi-digit multiplication.

5.2 Subtraction with regrouping

The question leading to responses for this topic, as described in Chapter 3, was:

> Let's spend some time thinking about one particular topic that you may work with when you teach, subtraction with regrouping. Look at these questions: $52 - 25$, $91 - 79$, etc.). How would you approach these problems if you were teaching second grade? What would you say pupils would need to understand or be able to do before they could start learning subtraction with regrouping?

As described in Chapter 3, I employed a thematic analysis to identify relevant mathematical aspects of participants' responses to the subtraction with regrouping prompt. Recall that I grouped features of responses that seemed to be related, such as "representation" or "vocabulary" or "context." and then examined those groups to look for commonalities or differences that seemed to highlight a mathematical idea. I did not see the boundaries of the themes as sharply delineated, but as aspects of the participants' explanations. For subtraction with regrouping, I identified two themes: multiple ways of regrouping, and subtraction is challenging. I first present the major elements of strategies and responses noted for the topic and describe the themes that I defined based on those elements. I then provide excerpts from the interviews that illustrated each theme.

The first theme identified was "Multiple ways of regrouping," referring to the fact that participants presented different strategies to address the need to regroup. Descriptions that fit this theme were provided by 14 of 18 teachers (77%). One strategy for teaching subtraction with regrouping is to use the standard algorithm which has students "borrowing" a ten to gain more ones, sometimes accompanies by a base ten representation, such as base 10 blocks. While 3 participants (17%) in this study did suggest using a base-ten representation only, 5 (28%) instead only used flexible regrouping and number lines to perform the subtraction calculations, and 6 (33%) teachers presented both of those strategies.

The second theme identified for this topic was "Subtraction is challenging." Eight participants (44%) noted that subtraction is more challenging for students to learn than is addition, as there are more contexts and meanings for subtraction. The use of number lines with both counting up and counting down strategies was seen as a way to more closely match the context of a subtraction problem to its representation.

Data and quotes to illustrate both themes are presented in the next sections.

5.2.1 Multiple ways of regrouping

There was variety in the ways that teachers described regrouping, with some providing strong mathematical justification for the standard algorithm and others using techniques that did not address the mathematical underpinnings of the algorithm. The two main strategies participants used to address the topic of subtraction with regrouping were number lines and base ten representations. Five of the teachers (28%) mentioned only number lines, three (17%) suggested they would use only base ten representations to justify the algorithm, and six (33%) used both number lines and base ten representations in their explanations. Among the remaining participants, two (11%) would explain how to use the algorithm, one (6%) mentioned moving from number line strategies to the algorithm without mentioning a base ten representation, and one (6%) mentioned counting objects to work with the subtraction problems. The use of the number line strategies with flexible regrouping and the justification of the algorithm using base ten representations were both characteristics of Strong Specialized Content Knowledge as described in Chapter 4.

Base ten representations included base 10 blocks (n=10, 55%) and place value counters (*n*=1, 6%). The ten participants who included this representation as part of how they would explain the topic of subtraction with regrouping to students used drawings of base 10 blocks to indicate the tens and ones in the problem 52-25, as shown in Figure 5.1. The dots indicate unit-cubes from a set of base 10 blocks, which represent ones, and the vertical lines reflects "rods," which each represent ten (see Chapter 2). In the actual physical manipulatives, each rod is exactly the length of ten unit-cubes placed end to end. For example, Participant 17 (Strong SCK in subtraction with regrouping) initially drew five ten-rods and two one cubes, as illustrated in Figure 5.1.

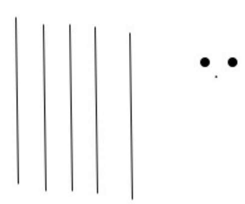


Figure 5.1. Illustration of 52 with base 10 representation.

She explained the technique as follows:

> [W]e have to take away five ones, can we? And we talk about how, no, there's only two ones. So we talk about trading in a 10. And what I do is we put a T on it to show that we're trading it. (See Figure 5.2.) (P17)

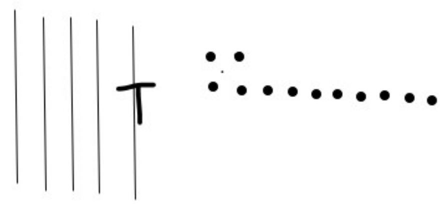


Figure 5.2. Trading in one of the ten-rods for 10 unit cubes.

[She draws in the ten dots to represent the ten ones]… so we've traded that in, and I emphasize when I'm first teaching this. And then we can take away the five ones, and then how many tens do we have to take away? And we talk a lot at the beginning how this [points to the ten with a T drawn on it] is not [still there]... We traded that. We have to take away, and we do Xs for taking away.

She marks Xs on two of the ten-rods and slashes through five of the unit-cubes. (See Figure 5.3)

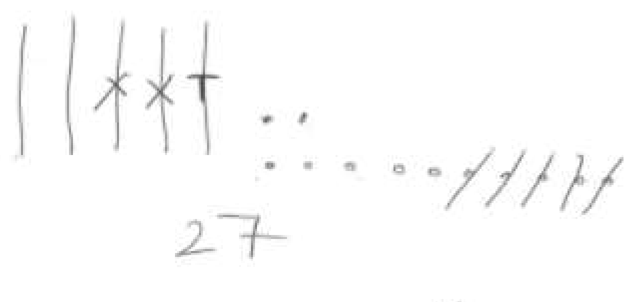


Figure 5.3. Taking away 25 by crossing out the rods and cubes.

And then we count what's left, which would be 10, 20, 21, 22, 23, 24, 25, 26, 27. And that's the answer. So that's like the most concrete way. And this is the first way I go about it, but most kids don't stay with this very long because it's cumbersome. (P17)

She then wrote the problem to show how the representation represented the algorithm in expanded form, as shown in Figure 5.4. She rewrote the 52 as 50 + 2 and the 25 as 20 + 5. In order to subtract the five ones, she needed more than two ones in the minuend, so traded in one of the tens for ten ones, leaving her with four tens, or 40, and 12 ones. She then subtracted by place value, subtracting the five ones from the 12 ones and the 20 from the 40, leaving a difference of 27.

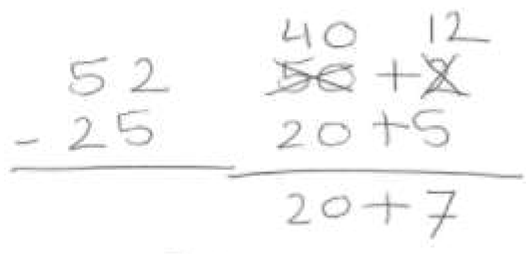


Figure 5.4. Representation of the expanded form algorithm that matched the base ten representation in Figure 5.3 (P17).

Like Participant 17, other participants noted that using this type of representation makes it clear that there is a ten, and not a one, that is being "borrowed" from the minuend. There was not a common vocabulary used in the trading in of a ten for ten ones, with participants using borrow, trade, and exchange for the process, and sometimes using more than one of the terms in the same explanation.

The number line strategy, on the other hand, bypasses the need to exchange a ten for ten ones, and instead requires a more flexible type of regrouping. While there was only one type of base ten representation offered for each subtraction problem, there are three types of number line representations possible: working down from the minuend to the subtrahend and counting the total of the jumps working down from the minuend by the subtrahend and noting the location after the final jump, and working up from the subtrahend to the minuend and counting the total of the jumps. Two of these were offered as representations by the participants: counting down by the subtrahend and counting up to the minuend. One participant described her counting down strategy as follows, illustrated in Figure 5.5:

> [If] I wanted to do a taking away from my 52, I could subtract away 20 first. So then students would get more facile at being able to just immediately take away the 20 and get to a 32 and then think through ‘now I need to partition my five because I'm going across a decade’. So I might want to take away a two first and then I might want to take away a three next in order to get to my answer. (Ms. Sutton)

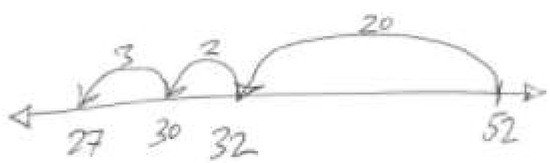


Figure 5.5. Number line representation counting down from the minuend to the difference by the subtrahend.

Using a counting up strategy, as shown in Figure 5.6, Ms. Sutton (Strong SCK in subtraction with regrouping) said:

> But one of the things that they have in there is practicing with jumping different moves that they have. So for example, if we've been practicing jumping to the decibel then we might have practiced with a number line and so if we're starting with our 25 and we're basically going to add up to trying to get to our 52. So I might jump first up five to get to the [decade] of 30 and then noticing here that I might jump up 10 more to get to 40 and then 10 more to get to 50 and then I have my two left. So that's my distance, because subtraction we can think about it as

taking away, but it's also thinking of the distance between those two numbers. So I can think about I have 10, 20, 25, 26, 27 as my distance between 25 and 52. (Ms. Sutton)

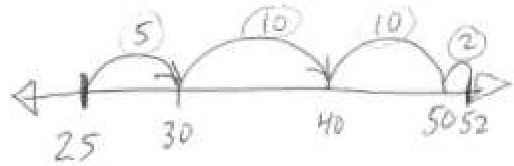


Figure 5.6. Number line representation counting up from the subtrahend to the minuend.

For the number line strategies, there are not prescribed jumps a student has to take. Participants mentioned that they expected different students to take different sizes and orders of jumps to reach the answer, depending on the students' comfort level with moving around and over decades. I note that concern for students' own entry points, consistent with the attention to comfort level here, is consistent with best practices for teaching mathematics (see Van de Walle et al., 2013). The participants expected most students would add or subtract to or by a decade or group of decades, then work with the remaining numbers to arrive at the difference. A student who was counting up from 25 to 52 might go by a five (to 30), then by ten (to 40), another ten (to 50), and two (to arrive at 52). They could then add the jumps to conclude the difference was 27. They could also start at the 25, add 20 (to 45), and know that seven more would get them to 52, also arriving at an answer of 27. The teachers noted that using a number line strategy requires students to be able to flexibly break apart numbers. They need to see 25 as two tens and five ones, but also as a ten, another ten, and a five; or two tens, a three, and a two. Flexible regrouping was also a criteria of Strong SCK in Chapter 4.

Participant 10 (Strong SCK in subtraction with regrouping), along with presenting a number line model, used a strategy that employed Cuisenaire rods. These are colored rods, with each color representing a different value as indicated by its length. Unlike base 10 blocks, Cuisenaire rods do not reflect base ten properties in their design. She proposed using them to demonstrate the need to increase the ones in order to subtract, but in a way that bypasses exchanging the ten for ten ones. Through this role play, one child represents the ones place and another represents the tens place, with a third child representing the number being subtracted.

She described it in the following way:

> I would have one child holding five orange rods [with a value of ten each] and one child holding a red [which has a value of two]. And then I have a person who says, 'excuse me. Do you have five that you can give me.' And the ones place will say, 'no, just a moment. I'm going to get some help.' And so this person turns to the next door neighbor and says, 'can you please help?' And the next-door neighbor is this whole thing. Because it's tens and the one. And the next-door neighbor is only ever allowed to give one of what they have. And so then they would say, 'sure. I'm happy to help you.' And then they just slide a ten over.

Now the child representing the ones place is holding a ten and a two, instead of twelve ones as would be standard when using base ten blocks. She goes on to say:

> So it doesn't actually involve regrouping. It doesn't involve trading anything, you just move that over. It's quite elegant. So that this then becomes, when it slides

over, now I have my 12 and now I have 40 and then I always say to them, 'wait. Did you take anything away yet? I see you have 40 and 12. Have you given anything away?' 'no, no. 40 plus 12 is 52. I haven't done anything, I've just renamed my number.' And the interviewer comes, 'can I have five?' And the 12 says, 'sure, you can have five.' And then at that point it gets pretty easy question to figure out what the difference is between five and 12.

So then they say, 'sure, I can give you [five].' So then they'll give him the [five] and then they'll say, 'okay, and now I still have my 20 and my 20 has to be taken away from the four tens. So can I do that? Yes.' And so there we go. (P10)

She did not explain how the giving away of the five would be represented by the students, nor how the remaining seven would be shown.

5.2.3 Subtraction is challenging

Several participants (*n* =8 , 44%) commented on the challenge that subtraction poses for their students, first because it has so many more contexts and meanings than does addition, and second because of the vocabulary we use. Addressing the first point, Ms. Blake (Strong SCK in subtraction with regrouping) said:

> But it seems something about that operation that is most problematic for them. There's something about that taking away, and maybe it's because it can be represented. It can be a difference, right? It can be a taking away. It can be a comparison. Maybe that's part of what messes them up is, it presents itself in different ways that look like something other than their default. (Ms. Blake)

Where addition is seen as joining or combining, subtraction could be, according to the participants, comparison, taking away, difference, distance, or a missing addend.

These different contexts can lead to different choices in representation: counting up to the total, counting down to the subtrahend, or counting down by the subtrahend. In considering these contexts, number line representations are seen as bringing meaning to the various contexts in ways the standard algorithm cannot, even if at the same time the number line does not well reflect the regrouping that is part and parcel of multidigit subtraction.

Participant 18 (Strong SCK for this topic) highlighted a different challenging aspect of subtraction, the vocabulary we use, which has everyday meanings as well as mathematical meanings. The multiple meanings can be challenging for students, especially those who are just learning English.

> And then just understanding the difference, which I find especially with ELL students, that's a challenging concept in math because when we talk about difference in math, it's very different. Like what's the difference between 91 and 79 and they'll say, well, 91 is bigger. It has nine tens or one you know? And it's like, what's the distance? I try to encourage that. What's the distance? How many spaces in between those numbers is the difference? (P18)

While subtraction can be seen as a very basic operation, the participants brought forth these ideas about why subtraction is challenging for students, and how they try to encourage understanding through multiple strategies and clarifying vocabulary. As noted in Chapter 4, the majority of teachers had Strong SCK in this topic area, and the rest had Moderate SCK. Their Specialized Content Knowledge level is corroborated by their descriptions of why subtraction is challenging, and what would be supportive pedagogically to help students overcome any challenges.

5.3 Multi-digit multiplication

The question leading to responses for this topic, as described in Chapter 3, was:

> Some sixth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate:
>
> $$\begin{array}{r} 123 \\ \underline{\times\ 645} \end{array}$$
>
> the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:
>
> $$\begin{array}{r} 123 \\ \underline{\times\ 645} \\ 615 \\ 492 \\ \underline{738} \\ 1845 \end{array}$$
>
> Instead of this:
>
> $$\begin{array}{r} 123 \\ \underline{\times\ \ 645} \\ 615 \\ 492 \\ \underline{738} \\ 79335 \end{array}$$

> While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching sixth grade and you noticed that several of your students were doing this? What would you say pupils would need to understand or be able to do before they could start learning multi-digit multiplication?

For multi-digit multiplication, I identified three themes: the nature of the error, representations explain the algorithm, and zero has meaning. Since themes are interwoven within a teacher's explanations, they are not always presented with the phrasing I have chosen to represent the themes. Therefore, theme boundaries are not sharply delineated.

Multi-digit multiplication has traditionally been taught as a procedure that involves moving partial products "over" as one proceeds through the place values of the multiplier, and often needs "carrying" or regrouping in the process. The first theme for this topic is "The nature of the error," as 17 of the 18 participants (94%) identified that the error was in place value understanding and not in misunderstanding the procedure.

The second theme is "Representations explain the algorithm," which notes the ways in which participants justified or bypassed the algorithm. While three of the participants (17%) addressed the error by simply restating the procedure, the rest (*n*=15, 83%) focused instead on a representation that justified the procedure or could even be used to bypass it. The representations and justifications were characteristics of Strong Specialized Content Knowledge as discussed in Chapter 4.

Participants were also nearly unanimous (*n*=15, 83%) in the call for including all zeros in the partial products, which were not present in the given problem, to indicate that there were no ones or tens in those places, leading to the third theme of "Zero has meaning."

Data and quotes to support these themes are found in the following sections.

5.3.1 The nature of the error

A lack of understanding of place value was the reason given by almost all (*n* = 17, 94%) of the participants as the cause of the error in the students' work. In referring to place value, the teachers seemed not to be implying that the students had forgotten which place the product should start in, but rather the teachers recognized that the students were seeing the four in 645 as a single digit of four, rather than understanding that it represented 40. As Participant 12 noted, "So this 4 doesn't mean I'm multiplying everything by 4, I'm multiplying everything by 40."   Not surprisingly, then, the representations they chose to address the mistake were intended to illustrate the place values of the digits in the multiplicand and/or the multiplier, as described below.

5.3.2 Representations explain the algorithm

Participants generally used array models or partial product models to justify the standard algorithm and to make it make sense to students (*n*=15, 83%). Several noted that the algorithm is not just a set of steps, but a shorthand way of working through the separate multiplications that comprise it. The array model, also called the box model or area model, was used by ten (56%) of the participants. In this model, the factors are presented as sides of a rectangle that has been divided into regions. Students find the area of each region, then add the areas together to find the total area or product. As Participant 1 (Strong SCK for multi-digit multiplication) explained, as illustrated in Figure 5.7:

> I'm going with the open array. I guess I need three [sections], so that you would have your 100, your 20 and your 3. Your 600, your 40 and your 5. I look at this and say, "Well I know I have to multiply each of these parts". Because it looks like somebody's like okay, so I know I'm supposed to multiply that part and I know I'm supposed to multiply that part, but I'm forgetting that it's really 40 times three.
>
> But now I can do all of these pieces. [She found the product of each set of factors.] There's your 500, your 100 and your 15. So there's my 615 right there. [Pointing at the row headed by the 5.] This is a 615, but this [pointing to the row headed by the 40] is not 492 and this [pointing to the row headed by the 600] is not 738, has to be 7,3800. (P1)

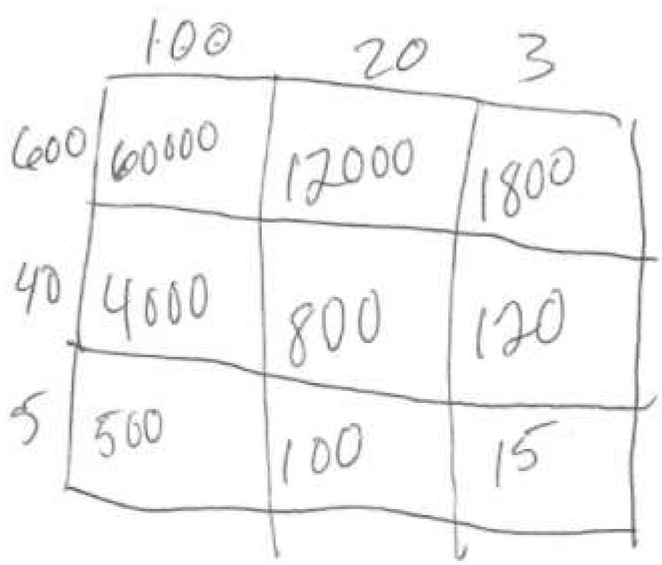


Figure 5.7. An array model of multi-digit multiplication.

With an array model, each element of the product is shown separately, and the partial products are added to find the total product. Some participants tried to scale the regions to represent the lengths assigned, but most did not, in keeping with creating a reasonably-sized model. Some found the array model to be cumbersome beyond multiplying two-digit numbers, as expressed by Participant 2 (Strong SCK for this topic), who said:

> I don't mind if kids are doing partial product. And you know what I mean by partial product? Instead of doing the algorithm. And it's, it looks much nicer when it's two by two. Three by starts to get really junky. (P2)

By "junky" this participant seemed to indicate that the number of cells in the model or elements in the partial products begin to get unwieldy for adding once the factors expand beyond two-by-two digit numbers.

Partial products, mentioned by eight (44%) of the participants can also be found without the array, in a way that generates fewer elements to be added together, as shown in Figure 5.8. For this problem, one participant broke the original problem down into three multiplication problems that can be calculated by multiplying by a single digit and then applying the appropriate place value. The three partial products align with the three product lines from the standard algorithm and can be helpful in understanding why the values need to be "moved over."

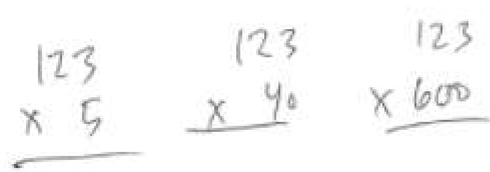


Figure 5.8. Partial products method for multi-digit multiplication.

While performing a similar calculation on a smaller, two-by-two, multiplication problem, as shown in Figure 5.9, Ms. Blake (Moderate SCK for this topic) described the process as:

> Because we had 56 times 24, so it'd be, okay, 4 x 6, that's 24. 4 times 50, that's 200. 20 times 6, that's 120, and 20 times 50 would be 1,000, and then I have my four partial products, and then add them up. (Ms. Blake)

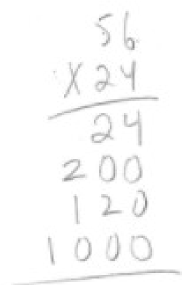


Figure 5.9. Illustrating the partial products method of multi-digit multiplication.

She even challenged the use of the standard algorithm, stating :

> Well, in fourth grade, we honestly never get to the standard algorithm. We do the partial products, and just leave it there. And so that does address that problem to some extent. Because, this idea that, is that a 4? Not really. It's a 40. So I think that does help keep that place value intact. Where you're just doing the partial products, and again, we decided like let's not rush to the standard algorithm. Why? They have an efficient strategy. It demonstrates place value much more explicitly than the standard algorithm does. (Ms. Blake)

Connecting representations and their related algorithms is key for creating strong mathematical understanding of a topic. That so many participants worked to create those connections is further illustration of their Strong SCK for this topic.

5.3.3. Zero has meaning

Fifteen of the sixteen participants (94%, or 83% of the whole sample) who addressed the question of what should be in the empty spots in the given problem noted that the only appropriate placeholders for the given problems were zeros, and several of the participants were very uncomfortable that the zeros were left out of the presented problem. The teachers voiced that the missing values were zeros, and should be included to indicate that there were no ones or no tens. Zero is not "nothing, nor merely a placeholder, but it is a value that needs to be shown for clarity. I categorized this separately from "the nature of the error," as one could know that the student error was with place value and still believe that it was acceptable to omit the "placeholder" zeros or to use another symbol to represent the placeholder.

As Participant 12 (Strong SCK for multi-digit multiplication) noted,

> The 0 is really important, because it's ... Yes, it's a placeholder, but you need to know what place value you're working in. Because that changes everything. 10 is hugely different from 10,000. So you need to have those 0s to help structure that. And I don't think that telling kids that it doesn't matter, trying to put that in their head is helping that.
>
> Mathematically what is the most accurate would be a zero. (P12)

When I asked if another symbol could be used to be the placeholder, Participant 2 (Strong SCK for this topic) said,

> No. No. But you can only put a digit in there. They are, maybe they are placeholders, but you have to put a digit. There's no value, then you can put zero.

> That's why zero's so important. It's the only place holder that you're allowed to use. My Hero. Zero. (P2)

Some of the participants mentioned the phrasing that is often used when teachers talk about the zeros that result from multiplying by powers of ten. As Participant 1 (Strong SCK for this topic) mentioned,

> That's like oh you just add a zero at the end and I'm like, "No, you don't just add a zero at the end", and I know we have a fifth grade teacher who said she slapped a zero on the end and I'm like, we're not slapping numbers around here. It means I have zero, and I really try to use that language of saying I have zero ones. There's nothing there. Slapping zero on the end. No, and actually what I say to them is 'what's happening when you're slapping a zero on the end?' And I really try to get them at least to verbalize 'I'm multiplying it by a power of ten.' (P1)

Two participants (11%) noted that they recommend students use filled-in zeros as placeholders, which seemed odd to me, as this is not an approach I have seen mentioned in resources like Van de Walle et al. (2013), and there does not seem to be a mathematical meaning communicated through the shading of a numeral. Ms. Sutton (Strong SCK in multi-digit multiplication) teacher, however, explained the practice.

> Mathematically what is the most accurate would be a zero. But I think what some of the other teachers might be trying to accommodate for is if there's a zero would be the next digit. (Ms. Sutton)

That is to say, if we were multiplying 23 by 5, there would be a zero from the power of ten in 20 times 5, but also another zero created by the 5 times 2. By using a filled-in zero, students could potentially recognize the difference between the zero required from place value considerations and a zero that was the result of a non-zero digit calculation.

Participants mentioned the importance of understanding multiplication by powers of ten, saying,

> I had to understand place value. I had to understand multiplying by [multiples of ten] is more than tacking on zeros on the end, or adding on zeros on the end. (P9)

and, when asked about foundational knowledge,

> having familiarity with multiples of 10 so they're not using the algorithm to multiply 123 times 600. (Ms. Sutton)

Participants in this study were able to identify the mathematical nature of the error in the given problem, and generally created representations that would encourage conceptual understanding of the traditional algorithm and the place value concepts inherent in it. They also recognized the importance of using zeros in the "placeholder" positions that were left blank in the problem that was presented, and that the zeros were representative of value.

5.4 Division with fractions

The question leading to responses for this topic, as described in Chapter 3, was:

> People seem to have different approaches to solving problems involving division with fractions. How do you solve a problem like this one?

$$1\frac{3}{4} \div \frac{1}{2}$$

> Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2}$? What would you say pupils would need to understand or be able to do before they could start learning division with fractions?

I identified four themes for the topic of division with fractions: long live the algorithm, what does it mean to divide?, context was an issue, and part-whole definitions of fractions.

Fifteen of the 18 participants (83%) were able to correctly solve the given problem, one (6%) solved it incorrectly, and two (11%) did not attempt a solution. Twelve (67%) of the participants used the traditional algorithm, noting either "invert and multiply" or "keep-change-flip," and seven (39%) were able to illustrate the solution by determining how many halves were in 1¾. One (6%) of the participants used algebraic properties to justify the algorithm, and one (6%) cited the definition of division (that dividing by a number is the same as multiplying by its reciprocal) in justifying multiplying by the reciprocal. The focus on the algorithm led me to the first theme of "Long live the algorithm." The ways in which some of the participants addressed the problem without using the algorithm is the focus of the second theme, "What does it mean to divide?".

My third theme, “Context was an issue,” addresses the errors made when teachers tried to develop contexts. Confusing dividing by half and dividing in half was the most common error when teachers tried to generate a context or representation, leading me to wonder about the language we use in describing those operations.

Participants tended to talk about fractions in terms of vocabulary, procedures, and part-whole definitions, which could complicate their understanding of both dividing by fractions and creating contexts for division. I address this in the theme “Features of fractions.”

Data and quotes to support these themes are presented in sections below.

5.4.1 Long live the algorithm

Ms. Blake, who was rated as Low in SCK for division with fractions, was representative of participants who relied on the traditional algorithm to solve the problem, mentioning different mnemonics they used to remember the procedure. She said:

> Well, I think I'm going to do it like "yours is not to reason why. Yours is just to invert and multiply." I'm going to do it that way, because that's the way, how I was taught. All right, so that'd be what? 7 fourths, and then it would be... I don't even know if I'm right. Am I right? It'll go 14 over 4, which would be like 3 and one half? Okay. But if I were really doing it right, it'd be like I got one whole, I have 3 fourths of another whole and I'm dividing that by half. What the heck does that mean? Right? (Ms. Blake)

An illustration of that calculation is shown in Figure 5.10.

$$\frac{7}{4}\times\frac{2}{1}=\frac{14}{4}\quad\frac{7}{2}=3\frac{1}{2}$$

Figure 5.10. Solving the division with fractions problem using the standard algorithm.

Several participants also noted that they had no idea why or how the algorithm worked. As Participant 15 (Low SCK for this topic) said,

> I really have no idea. I remember that you're supposed to switch, change, flip. But I have no idea why and I think that that is a testament to my own math education. I was not the most confident math student growing up and so I don't even remember this and I'm like ugh right now. (P15)

And Participant 18 (Low SCK for division with fractions) noted,

> This is a mixed number and so the mixed number has to be changed into a fraction. So that's four times one, add three, is seven fourths and then reversing the operation and flipping the fraction, don't ask me why, but that said, then we're doing 14 fourths and then that can be reduced. Like [dividing] that by two and that by two and then turning that into a mixed fraction is saying three and one half, which I had no idea if that would be right or not, or why we do it. (P18)

Several of the teachers talked about the challenge of teaching the topic, and the tendency to revert to teaching it strictly algorithmically.

> But in sixth grade they then learn the algorithm…But that again is a magic trick where they sometimes learn keep change flip. It's just a trick. It's like a magic trick and there's really not this conceptual understanding of what it means. I find

both with division problems and subtraction problems, those are the ones where conceptually understanding what's going on is probably the hardest part. (Ms. Sutton – Strong SCK)

When I suggested some ideas for contexts and activities to Participant 14 (Low SCK), he replied,

That's too much work. But this is one of the most difficult things to teach…You want them to be able to do it with something in their hands, but it's just like... it's so difficult to teach. Like I've tried to do it this way. I had to prep myself a lot. And then expect a million frustrated kids in the classroom. So it just usually ends up being just, "Keep change, flip". We try this [hands-on or conceptual work] for a couple of days and after a while I was just like, "Yeah, you're probably not going to use this very often, so let's just give you the formula". (P14)

Only two (11%) of the participants justified the algorithm, Participant 16 (strong SCK) using algebraic properties, as described below and shown in Figure 5.11. While her use of the word “side” is incorrect in this context, the horizontal nature of her equation likely led to the misnomer.

But then I also teach them, for the algorithm, the shortcut that you are allowed to use. By this time they've done algebra, so we can say whatever I do to this side, I can do to this side. So if I multiply with the reciprocal to this, I'm essentially turning it into one, and I can multiply the same thing that I did to this side, I can do to this side. I've just created the shortcut. (P16)

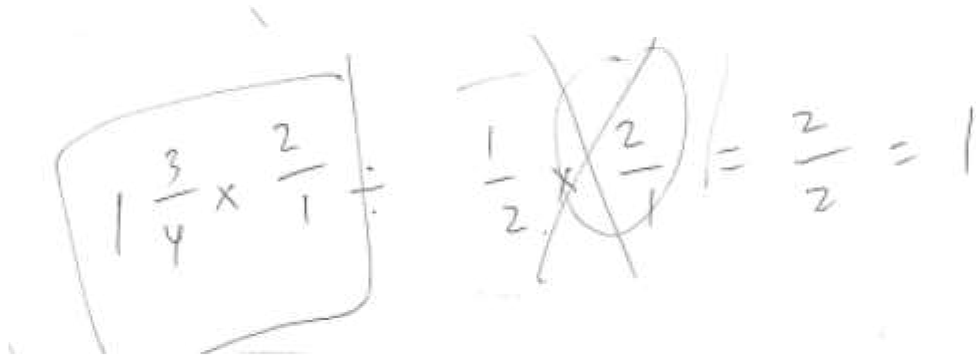


Figure 5.11. Justifying the algorithm using algebraic properties.

Participant 11 (Low SCK) used the definition of division to justify multiplying by the reciprocal,

> Yeah. Okay. You know, and so ... So, one is equivalent to, like ... four and four, and so ... so I might have, like, seven, four ... and by ... one half, and I know when we are dividing by a half, that would be equivalent of multiplying by two, and so twice as much is seven over four is 14 fours, and then I would just simplify that, and so maybe ... seven halves, and then if I wanted the mixed number again ... three, and ... one half. (P11)

Just over one-third of the participants (*n*=7, 39%) had a strategy other than the algorithm they could use to approach solving this problem. The reliance on the algorithm, largely without a way to justify its use, was common among the teachers in this study, and was also a factor in so few teachers showing Strong specialized content knowledge for the topic, as seen in Chapter 4.

5.4.2 What does it mean to divide?

Those participants who had a non-algorithmic strategy approached the problem by asking, “what does it mean to divide by ½?” They then noted that it was asking how many halves were in the 1¾ given in the problem. From there, they either used repeated subtraction or a drawing to determine the answer. Participant 16 (Strong SCK) said,

> We talk about how division is a quick way to subtract. We say if we have one and three fourths, how many halves can we take out of it? Because kids need pictures, if I have a whole, and I have another three fourths, then I'm wondering how many halves I can get out of that? I can get out a half, another half, so there's one. There's two, there's three, and then there's a half of a half, so there's three and a half. (P16)

She drew one full circle and one three-quarters of a circle, then shaded in the three half circles, as shown in Figure 5.12. The one-quarter of a circle left represented the "half of a half" mentioned above.

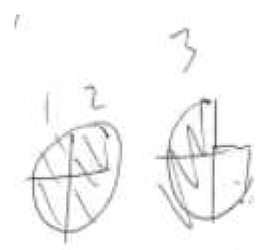


Figure 5.12. Representing division of fractions as finding groups of ½.

Participant 2 (Moderate SCK for this topic), after struggling to try to justify or understand the algorithm for several minutes, noted,

> Well what is division? It's repeated subtraction. So you can just keep taking, subtracting. That's probably where I honestly would start (see Figure 5.13). But if I did it this way, I would just keep subtracting. And this is a really long way to do it, …So you would come up with, so you did it at one, two, three, the remainder, one fourth. (P2)

When I pointed out that we generally don't write the answer to a fraction division problem with a remainder, he worked to figure out what the fourth represented, finally arriving at,

> Oh, it's half of the half. So it'd be three and a half. (P2)

$$1\frac{3}{4} - \frac{1}{2} = 1\frac{1}{4}$$

$$1\frac{1}{4} - \frac{1}{2} = \frac{3}{4}$$

$$\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$$

Figure 5.13. Using subtraction to answer the division with fractions problem.

Those participants who could work through the problem in a way other than strictly the algorithm, which was a criterion for Strong SCK, were generally able to create a context for the problem, as described below.

5.4.3. Context was an issue

When asked about creating a context for the division problem, six out of the 18 (33%) created a context that correctly aligned with the problem, four (22%) created incorrect contexts, and eight (44%) were unable to create any context at all. Eight of the participants (44%) struggled with the difference between divided BY one half and divided IN half, and their solutions and contexts reflected that confusion. Creation of a correct context was a criterion for strong SCK.

Correct contexts were generally aligned with the measurement, or quotitive, model, such as the two shown below,

> I love to bake, so immediately that comes to my mind. I need 1 3/4 cups of flour, but I only have a 1/2 cup scoop. How many of them are am I going to need? I

need 3 1/2 scoops full to get my whole 1 3/4 cups of flour. And we actually use baking as a way to solidify this. One day I baked cupcakes with them. (P12)

I have one and three fourths feet of rope and I need a half of a foot. How many halves of a foot will I be able to get? I need halves of a foot for a project. So how many pieces of rope will I get? (P10)

Incorrect contexts generally aligned with fair-shares, or partitive, thinking about fractions, and involved dividing the numerator IN half, rather than BY half, as shown in the following examples.

"So, you ordered one and 3/4 of a pizza, yeah, because you're weird. You decided you wanted 3/4 of a pizza," And it gets them laughing, totally engaged, I'm like, "... and you decided you were going to eat half now and half later. How much do you eat?" (P4)

You have a whole sandwich and three quarters, and you want to divide that in half so two people can share them. (P18)

No one created a multiplication problem, such as “If half of the distance around the lake is 1 ¾ miles, how far is the full distance around the lake?” which could be converted it into a quotitive division problem, to create a context.

5.4.4 Features of fractions

In talking about what students need to know or understand about fractions before learning division with fractions, participants tended to focus on vocabulary and procedures such as converting improper fractions to mixed numbers, finding equivalent fractions, and finding reciprocals (*n*=8, 44%). Three of the teachers (17%) mentioned that being able to coordinate units was a key idea. As one said,

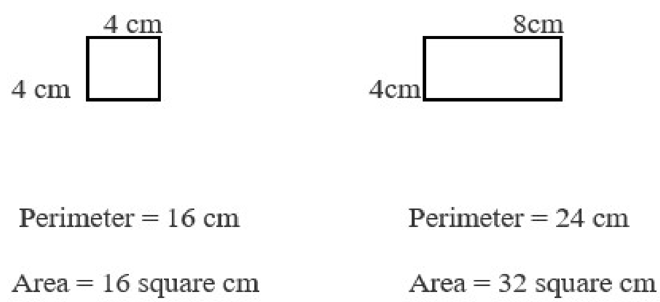
> So students have to know how to exhaust the whole so that when they're creating their fractional pieces that they're using the entire whole. They have to know that they have to make equal parts within that. They have to understand that there's always this reference to the whole. Students will sometimes figure out a problem of well it's a half of a fourth and I'm like okay well we don't usually go to the store and say I need a half of a fourth of a pound of flour. We always go back and refer to what one whole is. So they have to be able to hold those parts. So they have to be able to coordinate quite a few units.
>
> So in this case they have to be able to hold the whole. They have to be able to hold the factional part and then they also have to be able to hold in this case, they have to be able to hold the only don't even know what to call it. The other fractional unit. So if this fractional unit is thinking in fourths but then they also have to be able to hold the concept of one-half at the same time. (Ms. Sutton)

Many participants (*n* = 9, 50 %) mentioned the importance of knowing that a fraction represents part of a whole. This definition is limiting when considering division of fractions. No one mentioned 7/4 as 7 * ¼, which can be useful when thinking through the division problem, as 7 ¼ s divided by 2 ¼ s per group is similar to 7 apples divided by 2 apples per group. Only two participants (*n*=2, 11%) mentioned that students needed to understand that fractions were numbers in their own right and not just part of a whole.

5.5 Area, perimeter, and proof

The question leading to responses for this topic, as described in Chapter 3, was:

> Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove what she is doing:



> What would you respond to this student?

When looking at how participants said they would address the claim with a student, I noticed that there seemed to be a relationship between their approach and the conclusion they had reached about the correctness of the claim. This led me to identify a theme of "Teacher knowledge affected the response."

Nine of the participants (50%) did not state an opinion or conclusion as to the correctness of the claim, three (17%) said that the claim was correct, and six (33%) concluded that the claim was incorrect. One (6%) participant was able to indicate under which conditions the claim would be true. Four of the participants (22%) focused on the formulas used and verified that the student had performed the calculations correctly, while twelve (67%) noted that the student was putting forth a theory that needed further testing to see if it always held true.

Data and quotes to illustrate that theme are found below.

5.5.1 Teacher knowledge affected the response

There were four main responses teachers said they would offer to the student: 1) that it was a theory that needed further testing, but with no strategy for exploration; 2) great thinking!, with no further exploration; 3) a strategy for exploration that would lead to a counterexample; and 4) showing a counterexample that would disprove the theory. The determination participants made about the correctness of the theory seemed to have a strong impact on how they would respond to the student. Eight of the nine participants who were uncertain as to the correctness of the claim made by the student - that as the perimeter of a figure increases, its area also increases – said that they would respond to the student that the claim needed more testing. As Participant 4 (Moderate SCK for this topic) said,

> And so, then I might ask them, "Okay, so you're telling me with this problem and this problem, we've confirmed that what you said is true, but now we have to test your theory again." Because, the whole point of a theory is that we have to keep testing it, and if you find that one time that your theory doesn't hold up, it's not true. (P4)

And Participant 6 (Moderate SCK) said similarly,

> If you're going to have a theory, you've got to do many, many, many examples of [that] theory. And I would actually bring this up during class, because I really like

> it. I always say, "You [students] think differently than I do often. So this is someone's theory, let's prove it or disprove it." (P6)

While these participants did not have a clear strategy for the further testing and exploration of the concept, they knew that two examples were not enough to prove a theory and that one counterexample was enough to disprove a theory.

Six of the participants (33%) identified that the student's proposed relationship between area and perimeter was not true in all cases. Of those participants, four said they would suggest a strategy of exploration to the student that would lead to a counterexample. These strategies generally involved creating multiple rectangles that had the same area but were created by different side lengths, as shown in Figure 5.14, leading to differing perimeters. By having the student discover that a range of perimeters could all have the same area, they could help her realize that her theory was not correct as stated. This exploration is one of the third grade standards in the Common Core State Standards for Mathematics (CCSSO, 2010), and one of the criteria for Strong Specialized Content Knowledge as seen in Chapter 4.

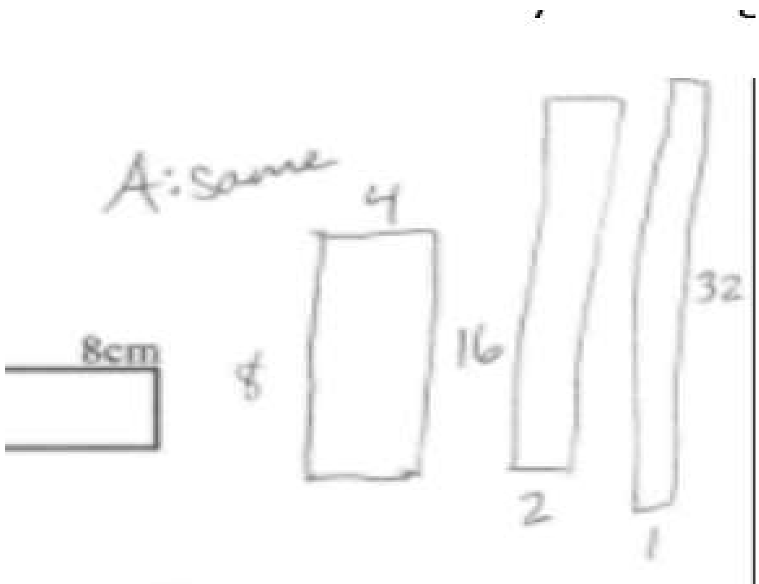


Figure 5.14. Exploring figures with the same area, but different perimeters. (Ms. Sutton)

Ms. Blake (Strong SCK) would take it further and ask the class to decide under which conditions the claim could be true, saying,

> And could we make it a math law with some revisions? Could we say, "Well this is true if, or only if..." (Ms. Blake)

The other two participants who determined the theory to be incorrect would tell the student that, and would show a counterexample to disprove the claim, rather than presenting it as an exploration. As Participant 11 (Strong SCK) described,

> But, I might also introduce another shape where I knew that maybe that wasn't the case. And so, if we had maybe a shape that was just one unit wide, and so let's say it was like one by eight ... and so I'd ask what the perimeter is in this case. The perimeter would be greater than the area. (P11)

Of the three participants (17%) who thought the student was correct in the relationship of area and perimeter, two offered no further work on the topic beyond a “well done,” while Participant 18 (Low SCK) said that she would ask the student to explain why it is true,

> Could she predict what the next ones would be? I would be interested to know if she could explain why that's happening. (P18)

Participants who knew that the theory was false tended to have strategies they could suggest to the student, while those who did not have as clear a picture on the correctness of the theory did not, even though they knew the student needed to further test the proposition. This seems to indicate a link between a teacher’s own knowledge and their ability to teach to a topic. Those with Strong SCK in this topic could lead students through explorations that could address misconceptions, but those with Low SCK could not.

5.6 Summary

The teachers in this study were generally able to offer conceptual explanations for subtraction with regrouping, and highlighted multiple ways of regrouping. They recognized the challenges students face with the topic of subtraction, noting its multiple contexts and confusing vocabulary. Their explanations of multi-digit multiplication also showed use of multiple representations, with both partial products and array models explaining the algorithm. Most identified the problem students were having as misunderstandings about place value and not confusion about the algorithm, and they strongly supported including the “placeholder” zeros as those indicate important values.

For division with fractions, though most participants could solve the problem, few even attempted to justify the algorithm, and most could not develop a context for the problem. Those who could solve the problem without the algorithm called on the definition of division to create representations, and they were generally able to create a context.

The participants struggled to suggest strategies for investigating a claim about area and perimeter, though most did show understanding of the nature of proof. Teacher responses to the hypothetical student seemed closely linked to whether or not they believed the claim to be true and whether or not they had a strategy for examining the claim.

In the next chapter I will discuss the findings from this chapter and the chapter 4, noting how they are different from or similar to findings from previous studies, especially Ma's study. I will also highlight the implications of these findings and offer suggestions for areas of further research.

CHAPTER 6

DISCUSSION AND IMPLICATIONS

6.1 Overview of the Chapter

In this chapter, I will first summarize the major findings from Chapters 4 and 5. I will then discuss how those findings relate to those of other studies, especially the work of Liping Ma (1999). Following the discussion, I will set forth implications of this research and suggest further topics of study to build on my findings. The findings were in response to the two research questions:

1. How do teachers' explanations of mathematics content demonstrate Specialized Content Knowledge (SCK) for the following topics?
    a. Subtraction with regrouping?
    b. Multiplying multi-digit numbers?
    c. Division with fractions?
    d. The relationship between area and perimeter?
2. What themes are found in teachers' explanations for the following topics?
    a. Subtraction with regrouping?
    b. Multiplying multi-digit numbers?
    c. Division with fractions?
    d. The relationship between area and perimeter?

6.2 Summary of Findings

6.2.1 Specialized Content Knowledge

My findings indicate that a teacher's Specialized Content Knowledge (SCK) can vary greatly by topic. While some participants, such as Ms. Sutton, showed fairly consistent levels of SCK, others, such as Ms. Blake, showed a wide range of SCK levels. Most participants showed stronger SCK in subtraction with regrouping and multi-digit multiplication than in division with fractions and area, perimeter, and proof. In fact, no one showed a higher level of SCK in division with fractions that they did in multi-digit multiplication, though one participant was stronger in division with fractions than in subtraction with regrouping.

That only 28% of participants were Strong in SCK in division with fractions is concerning, as is the small number of teachers (n = 4, 22%) who showed Strong SCK in area, perimeter, and proof. There did not seem to be a relationship between the grade level taught and the SCK in each topic area, indicating that being required to teach a topic does not necessarily mean one attains the necessary level of content knowledge to do so conceptually.

6.2.2 Themes in Explanations

Themes in the explanations were clustered around the use of representation, justifying algorithms, and the factors participants noted as important in the teaching and learning of each topic. For the topics of subtraction with regrouping and multi-digit multiplication, the participants generally gave conceptual explanations that included representations and justifications. Their explanations were more procedural for the topics of division with fractions and area, perimeter, and proof, with few teachers providing representations and justifications for their work.

The teachers in this study were generally able to offer conceptual explanations for subtraction with regrouping, and highlighted multiple ways of regrouping. Some of the explanations used base ten representations to justify and explain the standard algorithm, while others employed number line models that used flexible regrouping which avoided the need for the “borrowing” found in the standard algorithm. Participants recognized the challenges students face with the topic of subtraction, noting its multiple contexts and confusing vocabulary.

There did not seem to be common vocabulary for the “borrowing” action in the standard algorithm, with participants using trading, exchanging, and borrowing as the most common terms. Participants were clear, however, that there was a decomposition of a ten into ten ones to provide enough ones to do the subtraction.

In the explanations of multi-digit multiplication, participants also used multiple representations. Array models and partial products were both used to justify the algorithm, but those strategies could also be used to bypass the need for the algorithm. Participants used the array and partial product strategies to keep students from losing the place values of the digits being multiplied. They identified mathematical misunderstandings about place value, and not just a misunderstanding of the procedure, as the source of the student error in the given problem. Participants were nearly unanimous in their assertion that the zeros should be included in the partial products of the multiplication problem, as they indicate a value for the digit in that spot and are not just placeholders.

Participants struggled with the topic of division with fractions. Though many were able to solve the problem using a standard algorithm, resulting in the theme “long live the algorithm,” no one created a representation to justify the algorithm, and only two participants had other strategies for justifying the algorithm. The teachers also struggled to create a context for the given problem, with many participants not providing one and many others giving a context that would generate a dividing “in half” instead of dividing “by half.” This type of mathematical misunderstanding indicates a lack of connection of division with whole numbers and division with fractions among my participants. The use of an “in half” context in a classroom would likely lead to significant confusion for students as they are trying to make sense of division with fractions. When describing what students needed to know or be able to do in order to be successful with division with fractions, most of the participants focused on features of fractions such as numerators, denominators, and converting mixed numbers to improper fractions, instead of concepts such as how multiplication and division are related or what the meaning of division is. This procedural focus is a further indicator that the participants did not have strong conceptual understanding of the operation themselves, so could not identify the key ideas students would need to hold.

The area/perimeter/proof problem was also challenging for many participants. The few participants who recognized or determined that the conjecture was false were able to suggest a strategy for a student to arrive at a counterexample. Those who were not able to draw a conclusion about the claim tended to suggest that the student keep testing to ensure there was not a counterexample, but they could not provide a strategy for that testing. Participants who thought the claim was correct did not suggest further testing or discuss the nature of proving a conjecture true or false.

6.3 Discussion

6.3.1 SCK is Topic-Dependent

Using this multiple case-study approach (Yin, 2018), I was able to look at each participant across the set of topics and find that a teacher's Specialized Content Knowledge can vary greatly by topic. Recall that Ms. Blake's responses on subtraction with regrouping and area, perimeter, and proof showed Strong SCK, while her explanation of multi-digit multiplication showed Moderate SCK, and division of fractions showed Low SCK. This provides a possible lens for understanding the findings of Hill et al. (2008), who struggled to correlate the teaching quality of those who scored in the middle 50% on the multiple-choice assessment. Teachers who earned high MKT scores in that study had strong SCK across topics, and those who earned low MKT scores likely had low SCK across topics, and that likely played out in observations of teaching quality. Those in the middle 50% likely had a mix of SCK across topics and could have been observed teaching a topic they were either very strong, creating higher quality scores than expected, or a topic they were not strong in, creating lower quality scores than expected. One can imagine that Ms. Blake would appear to have high quality math instruction if we were to observe her teaching subtraction with regrouping, where she was able to address representations of the algorithm, but low quality math instruction teaching division with fractions, where she struggled to explain what the problem was asking or to create a context.

I developed my criteria for Strong SCK to reflect the expectations of the Common Core State Standards for Mathematics (CCSSI, 2010) as well as the best practices described by Van de Walle et al. (2013), which both encourage conceptual understanding of a topic. While most of the participants in my study seemed ready to teach subtraction with regrouping or multi-digit multiplication to children in a way that addresses the new standards, few show an understanding of division with fractions or area, perimeter, and proof that indicate the same level of readiness. When teachers do not hold strong Specialized Content Knowledge in a topic they are teaching, it is their students who are shortchanged.

6.3.2 Changes from Ma's Findings

My findings from the cross-case comparison (Yin, 1981) suggest that current elementary teachers are understanding and teaching both subtraction with regrouping and multi-digit multiplication in more conceptual ways than have been seen in the past. Where Ma (1999) found that 77% of her US participants limited their explanation to the procedural steps of the algorithm, only two of my participants (11%) focused primarily on the procedure of subtracting using the standard algorithm. My participants who did feature the standard algorithm focused on the decomposition of a ten into ten ones when they talked about borrowing, and half (*n* =9) linked their explanations to base ten representations. Eleven (61%) of my participants used strategies that did not require the standard algorithm at all, and instead focused on flexible regrouping to count up or count down, generally using a number line, to determine the difference. These strategies help students to develop mental representations of subtraction and can be helpful for mental calculations. If teachers Specialized Content Knowledge includes this flexibility, they can draw upon flexible approaches in their teaching, so that their students will have the opportunity to develop flexible thinking about these topics. Ma did not report any use of number line representations by her Chinese participants, but one-third did use non-standard regrouping to talk about another way of thinking through the subtraction.

Almost half of my teachers (n = 8, 44%) commented that students tend to find subtraction significantly more challenging than addition, suggesting that the large number of contexts is the most significant factor in that challenge. Van de Walle et al. (2013) agree with that conjecture and suggest that teachers start first with student-invented strategies for subtraction, use manipulatives to model contexts, work with number line representations, and support flexible regrouping. They note that counting up, a strategy they call "think addition", is often the most logical strategy for students in modeling certain situations. This strategy was mentioned by 11 (61%) of my participants in their explanations, showing that they are familiar with current best practices.

My participants generally used representations such as area models (n=10, 56%) and partial products (n =8, 44%) to justify the standard algorithm for multi-digit multiplication. In some cases, they advocated skipping the algorithm and only focusing on other representations, at least for multiplying two-digit by two-digit numbers, a position advanced by Lampert in 1986. The majority 77% of Ma's (1999) participants mentioned a lack of understanding of the procedure for the algorithm as the source of the error in the given multi-digit multiplication problem. This was in stark contrast to my findings that all but one of my participants (94%) identified the error as mathematical, that is, a lack of understanding that the 4 in 645 represented 40. The area model and partial products strategies are designed to explicitly focus on the place value each digit represents, so there is less likelihood a student will forget that their multiplication involves powers of ten. That so many of my participants were able to link the algorithm to either partial products or an array model shows growth in conceptual understanding when compared to earlier studies (Harkness & Thames, 2008; Lo et al., 2008; Southwell & Penglase, 2005). As my participants have come from different preservice education programs, have different years of teaching experience, and have participated in different amounts and types of professional development, this improvement in conceptual knowledge suggests a collective improvement in the way math educators teach multi-digit multiplication.

The participants in this study also recognized that zeros, which were not shown in the given problem, are the only appropriate "placeholder" for multi-digit multiplication, and that students will be less likely to have place value errors if those zeros are included when writing the partial products. This is dramatically different from the US teachers in Ma's (1999) study, who focused on the role of zeros only in the way that they help children properly line up the partial products. In fact, two of Ma's participants suggested that one could use another symbol, such as an asterisk, as a placeholder in lieu of a zero, showing a lack of understanding of the role of the zero in the problem, and focusing only on the procedure of lining up the digits. While two of my participants, including Mr. Fields, suggested the use of a filled in zero, it was to distinguish it from a zero that might result from multiplying the significant digits and not simply a procedural placeholder.

These improvements in teacher explanations of subtraction with grouping and multi-digit multiplications, when compared with Ma's (1999) findings, indicate that we have made progress in teacher conceptual understanding of whole number operations. Many of my participants have developed a repertoire of representations for subtraction and multiplication and can justify the algorithms using representations, which are criteria for Strong SCK (Ball et al, 2008; Lin et al., 2011). While teachers did not have the same preservice backgrounds, had not participated in the same professional development, and were not necessarily using the same curriculum, most had developed at least Moderate SCK and many had Strong SCK in both topics. Somewhere along their teaching journey, they had acquired that conceptual knowledge. The same does not seem true of division with fractions and area, perimeter, and proof, but there were some ways in which my participants showed improvement over Ma's findings for those two areas.

For the topic of division with fractions, a higher percentage of my participants (*n* =15, 83%) were able to solve the problem than Ma (1999) found in her study (39%), but there was still a lack of conceptual understanding of the algorithm, and most of my participants (*n* =11, 61%) had no other way to solve or represent the problem. My participants struggled to provide a context for the division problem, with eight (44%) confusing dividing by half with dividing in half, similar to the participants in Ma's study. Only one-third of my participants (*n* =6) developed correct contexts for the division problem, which is an improvement over Ma's single participant, but the low number is concerning as it is a sixth-grade standard (CCSSI, 2010).

When I asked participants how they would respond to the student's claim about area and perimeter, twelve (67%) focused on the fact that one example does not make a proof, and they said they would have the student explore the topic further. Four (22%) were able to provide strategies for that exploration, and the rest were not. Ma's teachers, on the other hand, tended to suggest they would look for an answer in a textbook and did not try to explore the question.

Representation and visualization were mentioned frequently by my participants. Number lines and base 10 representations were featured in explanations on subtraction with regrouping. Area models were created for multi-digit multiplication. Pizzas and cookies were drawn to explore division with fractions. And many rectangles were sketched in exploring area and perimeter. Ma did not feature non-numerical representations in her work, nor did she seem to indicate that they were part of the profound understanding of fundamental mathematics (PUFM) knowledge packages. The ability to choose, create, and compare representations is one of the key elements of Specialized Content Knowledge, and is embedded in the current standards (Ball et al., 2008; CCSSI, 2010).

6.3.3 Similarities to Ma's Findings

My findings showed some improvements in division with fractions, when compared with Ma's (1999) study, but some of the issues she found persisted. While only 39% of Ma's US participants were able to solve the division problem correctly, 83% of my participants could. And while all of her participants used the algorithm, 39% of my participants had a more conceptual way of solving the problem. Two (11%) of my teachers also justified the algorithm, one using algebraic properties and one using the definition of division, but most of the Chinese teachers in Ma's study were able to provide justification using the definition of division.

When it came to developing a context for the division problem, only one (4%) of Ma's US participants arrived at a context, and it was problematic in terms of the units. In contrast, one-third of my participants (33%) created a correct context. This still falls far short of the 90% of Chinese teachers who developed correct contexts in Ma's study, and is a result seen in many prior studies (Unlu & Ertekin, 2012; Alenazi, 2016; Nillas, 2003; Jansen & Hohensee, 2016; Işik & Kar, 2012; Lo & Luo, 2012). One of the common confusions seen in both studies was participants creating contexts that confused dividing by half and dividing in half, as Ms. Blake did. Dividing in half is to divide by two or to multiply by one-half, which is the inverse of the operation I was asking the teachers to perform. This seems to be an enduring misconception that cuts across countries (Nillas, 2003; Işik & Kar, 2012). It makes me wonder if there are linguistic underpinnings to the error, and if the same confusion would come if we were dividing by a number like 2/3, as we never say, "I am going to divide it into 2/3."

There was little evidence that the participants defined fractions in ways beyond the part/whole relationship. Only two teachers (11%) mentioned that fractions are numbers in and of themselves, and no one mentioned that 1¾ was 7 * ¼ or seven groups of ¼. Seeing fractions as their own quantities or as iterations of unit fractions can make it easier to understand the action of division with fractions (Van Steenbrugge, et al., 2014). It especially makes the common denominator method of division more conceptually clear (Van de Walle et al., 2013). If we took the initial problem, $1\frac{3}{4} \div \frac{1}{2}$, and transformed the divisor and dividend to have common denominators, $\frac{7}{4} \div \frac{2}{4}$, we could then see the problem as 7 groups of ¼ divided by 2 groups of ¼, which should be accessible to students who have experience with whole number division. This was the strategy I tried to engage in with Mr. Fields, but he was resistant to considering the topic of division at all. While this does not offer insight into the more traditional "keep, change, flip" algorithm, it does present an algorithm that can be taught with conceptual underpinnings.

One of my participants related the problem to multiplication with fractions, which is another avenue to approaching this topic more conceptually. An explanation of this type would show understanding beyond the algorithm and indicate strong SCK. If teachers and students could relate the division problem $1\frac{3}{4} \div \frac{1}{2}$ to the multiplication question, ½ of what is equal to $1\frac{3}{4}$, they could potentially generate contexts for the situation and create models that would help them understand the traditional algorithm, as highlighted in Ma's (1999) findings on this topic. In writing the equation $\frac{1}{2}x = 1\frac{3}{4}$, it can be shown that we could solve the equation by multiplying both sides by 2, which is the reciprocal of ½. This is, in fact the calculation required by the traditional algorithm. If a teacher could think of the problem as "half the length of a rope is $1\frac{3}{4}$ feet, how long is the rope?", mathematical connections between multiplication and division would indicate that doubling would give the whole length.

For the topic of area, perimeter, and proof, most participants would encourage students to explore further, but many did not have the core knowledge of the lack of a linear relationship between area and perimeter. While twelve of my eighteen participants (67%) suggested that the student should continue to test their theory, as finding a counterexample would prove it incorrect, only six (33%) actually investigated the claim themselves. While this is an improvement of the 13% found in Ma's (1999) study, it falls far short of the 92% of Chinese teachers who investigated the claim. Only three of my participants said the claim was correct (17%), which is more than the 9% Ma found, but much better than the 72% found by Livy et al. (2012). In that study the misconception persisted in spite of activities designed to prove it false.

The exploration that would lead to finding counterexamples for the claim that as perimeter increased area also increases is a third-grade standard in the CCSSM content standards (CCSSI, 2010). In that standard, students are expected to be able to solve problems involving “exhibiting rectangles with the same perimeter and different areas or with the same area and different perimeters” (CCSSI, 2010, p. 25). If our teachers have not encountered activities to explore the claim and engaged in discussions that explicitly call out the mathematical relationships found through those explorations, they will be unable to help students dispel the misconceptions around the relationship of area and perimeter (Fuller, 1996; Kow & Yeo, 2008). A case in point is Ms. Sutton, who could create counterexamples but did not recognize them as such, indicating a lack of deep understanding of the relationships she was seeing.

Fewer than 30% of my participants showed Strong SCK for division with fractions or area, perimeter, and proof, indicating that they do not have knowledge of the topic sufficient to teach to the current standards. While they might be able to teach the algorithm for division that allows a student to answer a calculation problem on a standardized test, they will not be able to give students an opportunity to learn the topic in a way that would allow them to understand, represent, and justify the algorithm. Similarly, few participants showed that they could address the standard involving the comparison of perimeters of rectangles with a constant area, or the comparison of areas of rectangles with a common perimeter.

6.4 Major implications

In this study, I found that only 33% of participants could be considered to have overall Strong SCK, and only one participant (6%) had Strong SCK on all four topics.

These findings suggest that, while strong in some topic areas, many current elementary teachers are not equipped to teach all math topics to the expectations of current standards and best practices. It is perhaps telling that 17 of my participants (94%) reported no professional development on CCSSM, and few reported taking part in much math-oriented professional development (PD). Many of the participants did mention individual content standards they were expected to address, or knew in which grade mastery of a topic was expected, but they did not indicate that they had received any PD related to their own understanding of the mathematics involved or related to teaching to those standards.

While this study is not focused on the ways professional development on the CCSSM might support SCK growth, my findings do indicate that teachers need opportunities to develop their conceptual understanding of mathematical topics and PD is one avenue for those experiences. In their interviews, most of the teachers reported being taught in a very procedural way, so it is not surprising that, for topics they have not had more conceptual experiences with, they teach procedurally. These PD opportunities need to go beyond common content knowledge and explore the concepts and connections embedded in each topic.

While seven of the teachers (39%) told me about the AVMR training the district was providing, only two of the classroom teachers (11%), Ms. Blake and Ms. Sutton, noted that they were taking part in those modules. (One participant was leading those trainings.) Two of the modules focused on number and whole number operations, and the other on fractions. Ms. Sutton mentioned in her interview that those workshops had greatly improved her understanding of how to best teach number and operations, and also her understanding of fractions and operations with fractions, and she demonstrated Strong SCK in the three related topic areas. Ms. Blake was enthusiastic about the number and operation courses, but did not find the fractions workshop to be as helpful to her. This sense that she had not embraced the material in that course was reflected in the Low SCK she showed for the topic of division with fractions.

The finding that only 28% of participants had Strong SCK in division with fractions indicates a need to focus more attention on that topic. To increase understanding of division with fractions, and fractions in general, we need to reconsider how we introduce the topic and the focus of the activities we choose. Chinese texts focus on division as the inverse of multiplication, and they include activities that allow students to get to the algorithm through that lens (Li, 2008). US texts, on the other hand, have been criticized as being focused on the algorithm and on providing exercises that draw primarily on use of the algorithm (Son & Senk, 2010). This leads to PSTs and teachers who remember procedures but don't understand how or why to apply them, nor how and why they work (Jones Newton, 2008; Van Steenbrugge et al., 2014). In preservice education, we should consider designing courses that encourage conceptual work on fractions and that tie representations and contexts to the algorithms explicitly (Whitehead & Walkowiak, 2017; Alenazi, 2016). If we do not work on developing these concepts, we will continue to create generations of teachers who believe that their fraction skills and knowledge are stronger than they actually are, and generations of students who never develop strong fraction knowledge (Li & Kulm, 2008).

The multiple case study methodology for this study (Yin, 2018) allowed me to look at the SCK of a teacher across several topics. The findings, in which only two teachers had consistent SCK levels for all four topics, strongly suggest that there is variation in a teacher's SCK depending on the topic that is studied. This variation is not captured in a test like the LMT, which does have some reporting by strand, but is not intended to report on the MKT of a single participant or for any specific topic (Selling et al., 2016). While case studies offer rich indications of individual participants' knowledge, they are not efficient for large-scale studies of teacher SCK. This implies that there is a need for topical tests of SCK that are intended for specific content areas. If teachers or PSTs are showing Strong SCK in subtraction with regrouping, on a group level, we could focus professional development efforts or teacher education on other topics where participants have shown lower levels of SCK. For individual PSTs, we could use pre-tests of SCK in topic areas to craft individualized course requirements that would address the needs and prior knowledge of each student.

We need to make certain that teacher education programs are not solely focused on number and operations, but address the other content strands our graduates are expected to teach. While number and operation are foundational to mathematics, they are not sufficient given the expectations of the CCSSM and associated high-stakes exams our students must take. We also need to make certain that professional development is available, and perhaps required, for teachers in topics that go beyond number and operation. Most of the participants in this study showed Strong SCK for subtraction with regrouping and multi-digit multiplication even without participating in the district-offered AVMR training on number and operation, leading me to question if that is the most needed PD for those teachers. Perhaps a cycle of professional development could be developed that focuses on the standards by strand, with all teachers learning the concepts and best practices along the K-6 continuum for each strand, or perhaps there could be grade-level focused PD to help teachers master their grade-level expectations.

We should acknowledge the strides that have been made in teacher education and teacher knowledge, as this study indicates that there have been improvements to teacher SCK since Ma's (1999) study was done. Policy makers who base recommendations on Ma's findings should seek more current data to help craft future requirements and policy recommendations.

Hill et. al recommended in 2008 that we consider having math taught strictly by math specialists in elementary schools, in the ways that subjects like physical education, art, and music are often taught by specialists. I have struggled to embrace that recommendation, as I have been concerned it will make children think that only some people can do math. In light of these findings, however, I am reconsidering my stance. While most teachers showed Strong SCK in subtraction with regrouping and multi-digit multiplication, there were still teachers who had Moderate or Low SCK, indicating that there is no guarantee of Strong SCK for the topics taught in the early grades. We continue to offer many children a suboptimal math education when they are placed with teachers who have low SCK. Until we have the teacher education and professional development in place that allow all teachers to develop the necessary SCK to teach math deeply and conceptually, it would be fairer to children to have math specialists who have demonstrated strong SCK for all of the topics and standards they are expected to teach. Again, this would require us to develop topic-specific measures of SCK by grade level.

That SCK varies so greatly by participant and by topic should also be considered by principals in their process of hiring teachers. Interview protocols might be changed to include questions, like those from this study, that assess topic-specific SCK for grade-level standards a prospective teacher would be expected to address in their classroom. If teachers are better matched to grade levels by their SCK, students would have a greater chance of learning the mathematics in a way that prioritizes justification and representation.

6.5 Limitations

This study is not generalizable as it is small-scale, focuses on only one school district, and in only one state. I was conducting the interviews near the end of the school year, when many teachers are feeling great stress to get everything done, and several teachers cited that as a reason to not participate. Even so, I was able to engage 18 participants, and those teachers represented a mix of teacher preparation, grade levels taught, years of experience, and professional development accessed. Replication of this study will be necessary to determine if these findings are generalizable to different settings and populations.

While I was asking the participants about their classroom practices, I did not observe them in their teaching to see if they used the strategies that they mentioned in the interviews, or if they had more strategies that they did not mention. Teacher responses may also have been constrained by the questions that were, or were not, asked directly. Further studies should include confirmation of practices, but those studies will need to be part of longer-term longitudinal research projects.

As is true in all research that uses qualitative methods, the quality of the results is dependent on the skills and resources held by the researcher. In this study, time was a resource that was often lacking. Many teachers were gracious enough to meet me for an interview before the start of the school day or at the end of a long school day, but that often led to a time crunch, which meant that some interviews felt rushed and that I sometimes didn't have time for the follow-up questions I wished I could ask. There were also times when I should have asked a follow-up question that I didn't, even when time was not a consideration.

6.6 Further Research

To support and build on this study, I propose several avenues of further research. The first is to replicate this study in some form on a larger scale, to see if my findings of improvement over the results of Ma's (1999) study hold true for an expanded population and other settings. The second is to develop and study a teacher education curriculum that would address the SCK needs PSTs have if they are to teach to current standards. The third is to develop and test measures of SCK that are topic-specific and perhaps include grade-level expectations. If we could pinpoint a teacher's SCK level for the different strands, we could better match teachers to grade levels they are well-prepared to teach, giving our students the strongest chance to be taught conceptually for deep understanding.

Outside of teacher education, we should be studying the professional development programs to see how they affect SCK for in-service teachers, and perhaps developing targeted PD for topics and strands that seem to be generally under-addressed. We should also study how teachers with strong SCK have developed their knowledge of explanation, representation, and justification, looking at family, school, teacher education, and professional development influences. If we know how Strong SCK teachers gain their strength of understanding, we could perhaps develop programs to help other teachers become more knowledgeable about mathematics.

6.7 Conclusion

This study provides evidence that there have been improvements in teacher conceptual understanding and SCK related to the expectations brought forth by the NCTM Standards and Practices (NCTM, 2000) and the Common Core State Standards (CCSSI, 2010). Teachers in my study showed strong progress in conceptual understanding of subtraction with regrouping and multi-digit multiplication compared with the findings of Ma (1999). They also showed increased knowledge of the procedure of division with fractions and the concept of proof for the area, perimeter, and proof problem. Findings also indicate that there is still significant work to be done in order to provide every student with a teacher who has strong Specialized Content Knowledge in the topics they teach. With focused educational experiences, at all levels of schooling, I am confident that teachers can learn math conceptually and learn to teach it conceptually. It is up to us as teacher educators to develop and deliver those opportunities, and our elementary students should not have to wait any longer for more progress.

APPENDIX A

INTERVIEW QUESTIONS

The first four questions are taken from Ma (1999).

1. Let's spend some time thinking about one particular topic that you may work with when you teach, subtraction with regrouping. Look at these questions: $52 - 25$, $91 - 79$, etc.). How would you approach these problems if you were teaching second grade? What would you say pupils would need to understand or be able to do before they could start learning subtraction with regrouping?

2. Some sixth-grade teachers noticed that several of their students were making the same mistake in multiplying large numbers. In trying to calculate:

$$\begin{array}{r} 123 \\ \underline{\times\ 645} \end{array}$$

the students seemed to be forgetting to "move the numbers" (i.e., the partial products) over on each line. They were doing this:

$$\begin{array}{r} 123 \\ \underline{\times 645} \\ 615 \\ 492 \\ \underline{738} \\ 1845 \end{array}$$

Instead of this:

$$
\begin{array}{r}
123 \\
\underline{\times\ \ 645} \\
615 \\
492 \\
\underline{738} \\
79335
\end{array}
$$

While these teachers agreed that this was a problem, they did not agree on what to do about it. What would you do if you were teaching sixth grade and you noticed that several of your students were doing this?

3. People seem to have different approaches to solving problems involving division with fractions. How do you solve a problem like this one?

$$1\frac{3}{4} \div \frac{1}{2} =$$

Imagine that you are teaching division with fractions. To make this meaningful for kids, something that many teachers try to do is relate mathematics to other things. Sometimes they try to come up with real-world situations or story-problems to show the application of some particular piece of content. What would you say would be a good story or model for $1\frac{3}{4} \div \frac{1}{2}$?

4. Imagine that one of your students comes to class very excited. She tells you that she has figured out a theory that you never told the class. She explains that she has discovered that as the perimeter of a closed figure increases, the area also increases. She shows you this picture to prove what she is doing:

Perimeter = 16 cm	Perimeter = 24 cm
Area = 16 square cm	Area = 32 square cm

What would you respond to this student? How would you engage with her around this idea?

5. What has been the greatest influence on how you currently think about and teach math?

APPENDIX B

QUESTIONNAIRE ITEMS

1. What teaching licenses do you hold?
2. How long have you been teaching?
3. What grade level(s) do you teach/have you taught? How long at each?
4. Where and when did you complete your undergraduate education?
5. What was your major?
6. If you have an advanced degree, where and when did you complete that?
7. What professional development have you participated in for math?
8. Were these school-based, district-based, or from another organization, such as NCTM?
9. Are you a member of NCTM or similar math-education organization?
10. How familiar are you with the current standards for math teaching and learning?
11. Have you gotten specific PD or training on these standards?

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