LOW DILATATION PSEUDO-ANOSOVS ON PUNCTURED SURFACES AND VOLUME

BY

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DISSERTATION

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Abstract

For a pseudo-Anosov homeomorphism f on a closed surface of genus $g \geq 2$, for which the entropy is on the order $\frac{1}{g}$ (the lowest possible order), Farb-Leininger-Margalit showed that the volume of the mapping torus is bounded, independent of g. We show that the analogous result fails for a surface of fixed genus g with n punctures, by constructing pseudo-Anosov homeomorphism with entropy of the minimal order $\frac{\log n}{n}$, and volume tending to infinity.

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Chapter 1

Introduction

In this thesis we consider pseudo-Anosov homeomorphims $f\colon S \to S$ of a surface $S = S_{g,n}$ of genus g with n punctures; see section 2.1 for definitions. The dilatation $\lambda(f)$ measures the growth rate for lengths of curves on S under iteration of f. Let $l_{g,n} = \min\{\log(\lambda(f)) | f : S_{g,n} \to S_{g,n}\}$ denote the logarithm of the minimal dilatation of a pseudo-Anosov f on an orientable surface $S_{g,n}$ with genus g and n punctures, that is, the minimal topological entropy. The value of $l(g, n)$ is known in a few cases.

$$l_{1,1} = l_{0,4} = \log(\frac{3+\sqrt{5}}{2}).$$

$$l_{2,0} = l_{0,6} = \text{the largest root of } x^4 - x^3 - x^2 - x + 1.$$

See [HS07].

$$l_{1,2} = \text{largest root of } x^4 - 2x^3 - 2x + 1.$$

See [CH08].

When $n = 0$, Penner gives both upper bounds and lower bounds for $l_{g,0}$.

Theorem 1 (Penner)**.** *For any $g \geq 2$,*

$$\frac{\log 2}{12g - 12} < l_{g,0} < \frac{\log 11}{g}.$$

See [Pen91]. These bounds have been improved since Penner's original work. The upper bound is improved by Bauer [Bau92] to $\frac{\log 6}{g}$, by Hironaka and Kin [HK06a] to $\frac{\log(2+\sqrt{3})}{g}$. See also [Min06]. Aaber, Dunfield, Hironaka, Kin and Takasawa [AD10, KT13, Hir10] also improved the upper bound.

Theorem 2 (Aaber-Dunfield, Hironaka, Kin-Takasawa)**.**

$$\limsup_{g\to\infty} g l_{g,0} \leq \log(\frac{3+\sqrt{5}}{2})$$

To better understand where minimal dilatation pseudo-Anosov homeomorphism come from, in [FLM11], the authors consider the set

$$\Psi_L = \{f : S_{g,0} \to S_{g,0} | f \text{ is pseudo-Anosov, } \log(\lambda(f)) \leq \frac{L}{g}\}.$$

They show that for any $L > 0$ there exists finite number of hyperbolic 3-manifolds $M_1, \ldots, M_n$, such that for each $f \in \Psi_L$, the mapping torus M_f of f is obtained by Dehn fillings on some M_i. See [FLM11, Corollary 1.4]. As a consequence, the volume of M_f is bounded by a constant depending only on L; see [FLM11, Corollary 1.5]. See also [Ago11, KM18, BB16].

In [FLM11], for any $P \geq 1$, the set of small dilatation pseudo-Anosov homeomorphisms is defined as:

$$\Psi_P = \{f : S \to S | \chi(S) < 0, f \text{ is pseudo-Anosov, } \lambda(f) \leq P^{\frac{1}{|\chi(S)|}}\}.$$

Let $S^\circ \subset S$ be the surface obtained by removing all the singularities. Then let Ψ_P° be the set

$$\Psi_P^\circ = \{f|_{S^\circ} : S^\circ \to S^\circ | (f : S \to S) \in \Psi_P\}.$$

Theorem 3 (Farb-Leininger-Margalit)**.** *The set of homeomorphism classes of mapping tori of elements of Ψ_P° is finite.*

Corollary 1 (Farb-Leininger-Margalit)**.** For any $P > 1$ there exists finite number of hyperbolic 3-manifolds $M_1, \ldots, M_n$, such that for each $f \in \Psi_P$, the mapping torus M_f of f is obtained by Dehn fillings on some M_i.

For punctured surfaces of a fixed genus, Tsai [Tsa09] proved that $l_{g,n}$ has a different asymptotic behavior.

Theorem 4 (Tsai)**.** *For any fixed $g \geq 2$, for all $n \geq 3$, there is a constant $c_g \geq 1$ depending on g such that*

$$\frac{\log n}{c_g n} < l_{g,n} < \frac{c_g \log n}{n}.$$

Yazdi [Yaz18a, Yaz18b] improved the lower bound to $\frac{C(\alpha)}{g^{2+\alpha}} \frac{\log n}{n}$ for any positive real number α, where $C(\alpha)$ is a positive constant. Valdivia [Val12] showed that given any rational number r, the asymptotic behavior of $l_{g,n}$ along the ray defined by $g = rn$ is

$$\log(l_{g,n}) \asymp \frac{1}{|\chi(S_{g,n})|},$$

where $\chi(S_{g,n})$ is the Euler characteristic of $S_{g,n}$.

.

For fixed $g \geq 2, n \geq 0$, let

$$\Psi_{g,L} = \{f : S_{g,n} \to S_{g,n} | f \text{ is pseudo-Anosov, } \log(\lambda(f)) \leq L\frac{\log n}{n}\}.$$

We show that the analogue of the results of [FLM11] fail for $\Psi_{g,L}$. Specifically, we prove the following.

Main Theorem. *For any fixed $g \geq 2$, and $L \geq 162g$, there exists a sequence $\{M_{f_i}\}_{i=1}^{\infty}$, with $f_i \in \Psi_{g,L}$, so that $\lim_{n\to\infty} \mathrm{Vol}(M_{f_i}) \to \infty$.*

As a consequence, we have the following.

Corollary 2. For any $g \geq 2$, there exists L such that there is no finite set Ω of 3-manifolds so that for all M_f, $f \in \Psi_{g,L}$ are obtained by Dehn filling on some $M \in \Omega$.

The construction in the proof of the Main Theorem is based on the example in [Tsa09] of $f_{g,n} : S_{g,n} \to S_{g,n}$ with

$$\log(\lambda(f_{g,n})) < \frac{c_g \log n}{n}.$$

But for each g, one can show that $\{M_{f_{g,n}}\}_{n=1}^{\infty}$ are all obtained by Dehn fillings on a finite number of 3-manifolds, so we have to modify this construction. See also examples constructed by Kin-Takasawa [KT13]. The idea is to compose $f_{g,n}$ with homeomorphisms supported in a subsurface of $S_{g,n}$ that become more and more complicated as n gets larger. This has to be balanced with keeping the stretch factor bounded by a fixed multiple of $\frac{\log n}{n}$.

In Section 2 we recall some of the background we will need on fibered 3-manifold, hyperbolic geometry and Dehn surgery. In Section 3 we state Theorem 13, which is a version of the Main Theorem for punctured spheres based on a construction of [HK06b], then prove the Main Theorem based on that. In Section 4 we give the complete proof of Theorem 13 by giving the construction of the sequence $\{M_{f_i}\}_{i=1}^{\infty}$, which are obtained by cutting open and gluing in an increasing numbers of copies of a certain manifold with totally geodesic boundary, then applying Dehn fillings.

The motivation is to try to prove analogue of Theorem 3. According to Theorem 1, the set Ψ_L contains pseudo-Anosov on all closed surfaces of genus at least 2 when L is big enough. In particular, this set gives all the minimizers. Then Theorem 3 tells us the set is determined by a finite list of 3-manifolds. In particular, the minimizers are determined by the finite list. We hope the minimizers for punctured surfaces are also determined by a finite list of 3-manifolds, but Main Theorem says the direct analogue of Theorem 3 does not hold. We might hope that restricting to a smaller set might get rid of the problem. This lead us to the following question.

Question. *If we only consider the minimizers of the entropy, is the set determined by a finite list of 3-manifolds?*

Chapter 2

Background

2.1 Pseudo-Anosov homeomorphisms

A measured foliation on a closed surface S_c is a foliation $\mathcal{F}$ with singularities, together with a transverse measure that is invariant under holonomy. In the neighborhood of a nonsingular point, there exist a chart $u : U \to \mathbb{R}^2_{x,y}$, such that $u^{-1}(y = constant)$ consists of the leaves of $\mathcal{F}|_U$. If $U_i \cap U_j$ is nonempty, there exist transition funtions u_{ij} of the form

$$u_{ij}(x, y) = (h_{ij}(x, y), c_{ij} \pm y)$$

where c_{ij} is a constant. In these charts, the transverse measure is given by $|dy|$.

Let S be a closed surface S_c minus a finite number of points. We sometimes consider S as a compact surface with boundary components, and will confuse punctures with boundary components when convenient (the former obtained from the latter by removing the boundary). The following theorem is from [FLP12].

Theorem 5 (Thurston). *Any diffeomorphism f on S is isotopic to a map f' satisfying one of the following conditions:*

(i) f' has finite order.

(ii) f' preserves a disjoint union of essential simple curves.

(iii) There exists $\lambda > 1$ and two transverse measured foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, called the stable and unstable foliations, respectively, such that

$$f'(\mathcal{F}^s) = (1/\lambda)\mathcal{F}^s, f'(\mathcal{F}^u) = \lambda\mathcal{F}^u.$$

The three cases are called *periodic*, *reducible* and *pseudo-Anosov* respectively. The number $\lambda = \lambda(f)$ in case (iii) is called the *stretch factor* of f. The topological entropy of pseudo-Anosov homeomorphism $f : S \to S$ is $\log(\lambda(f))$

2.2 Fibered 3-manifolds

Let S be a compact surface properly embedded in a compact 3-manifold M. For any disk D embedded in M such that $D \cap S = \partial D$ and the intersection is transverse, if ∂D bounds a disk in S and S is not a 2-sphere, then S is incompressible. A 3-manifold M is atoroidal if it does not contain an embedded, non-boundary parallel, incompressible torus.

Let M be the interior of a compact, connected, orientable, irreducible, atoroidal 3-manifold that fibers over S^1 with fiber $S \subset M$ and monodromy f. That is, M is the mapping torus of f:

$$M = M_f = S \times [0,1]/(x,1) \sim (f(x),0).$$

Then S is a closed orientable surface with a finite number of punctures and negative Euler characteristic, and f is pseudo-Anosov with a unique expanding invariant foliation up to isotopy. Associated to (M,S) we also have

(i) $F \subset H^1(M,\mathbb{R})$, the open face of the unit ball in Thurston norm with $[S] \in (F \cdot \mathbb{R}^+)$. See [Thu86].

(ii) A suspension flow ψ on M, and a 2-dimensional foliation obtained by suspending the stable and unstable foliation of f. See [Fri79].

F is called a *fibred face* of the Thurston norm ball. The segments

$$x \times [0,1] \subset S \times [0,1]$$

glued together in M_f are leaves of the 1-dimensional foliation Ψ of M, the flow lines of ψ.

The Thurston norm measures the minimal complexity of an embedded surface in a given cohomology class. For an integral cohomology class ξ, the norm is given by:

$$||\xi||_T = \inf\{\chi(S_0) : (S,\partial S) \subset (S,\partial M) \text{ is dual to } \xi\}$$

where $S_0 \subset S$ excludes any S^2 or D^2 components of S. The unit ball of the Thurston norm is a polyhedron with rational vertices. Any fiber minimizes $|\chi(S)|$ in its cohomology class. Moreover, $[S]$ belongs to the cone $F \cdot \mathbb{R}$ over an open fibered face F of the unit ball in the Thurston norm.

The following theorem is from [Fri79] and [Fri82].

Theorem 6 (Fried)**.** *Let (M,S), F and Ψ be as above. Then any integral class in $F \cdot \mathbb{R}^+$ is represented by a fiber S' of a fibration of M over the circle which can be isotoped to be transverse to Ψ, and the first* *return map of ψ coincides with the pseudo-Anosov monodromy f', up to isotopy. Moreover, if $S' \subset M$ is any orientable surface with $S' \pitchfork \Psi$, then $[S'] \in \overline{F \cdot \mathbb{R}^+}$.*

If $f : S \to S$ is pseudo-Anosov on a surface with punctures, and $G \subset S$ is a spine, then we can homotope f to a map $g : S \to G$ so that $g|_G \colon G \to G$ a graph map; that is, g sends vertices to vertices and edges to edge paths. The growth rate of $g|_G$ is the largest absolute value of any eigenvalue of the Perron-Frobenious block of the transition matrix T induced by g, and is an upper bound for $\lambda(f)$, see [BH95].

The Perron-Frobenius Theorem tells that largest eigenvalue of a Perron-Frobenius matrix is bounded above by the largest row sum of the matrix. Recall that associated to a non-negative integral matrix $T = \{e_{ij}\}, 1 \leq i, j \leq n$ is a directed graph Γ, where $\{V_1, V_2, \ldots, V_n\}$ is the vertex set of Γ corresponding to the columns/rows of T, and e_{ij} represents the number of edges pointing from V_i to V_j. We have the following proposition. See [Gan59].

Proposition 1. Let Γ be the directed graph of an integral Perron-Frobenius matrix T with eigenvalue λ. Let $N(V_i, l)$ be the number of length-l paths emanating from vertex V_i in Γ. Then $\lambda^l \leq \max_i N(V_i, l)$.

The case when $f : S_{g,0} \to S_{g,0}$ is mentioned in introduction.

2.3 Hyperbolic geometry

Hyperbolic n-space is the maximally symmetric, simply connected, n-dimensional Riemannian manifold with a constant negative sectional curvature. A hyperbolic 3–manifold is a 3-manifold equipped with a hyperbolic metric, that is a Riemannian metric which has all its sectional curvatures equal to -1.

Theorem 7 (Thurston)**.** *The mapping torus of a surface automoprhism $f : S \to S$ is a hyperbolic 3-manifold if and only if f is isotopic to a pseudo-Anosov homeomorphism.*

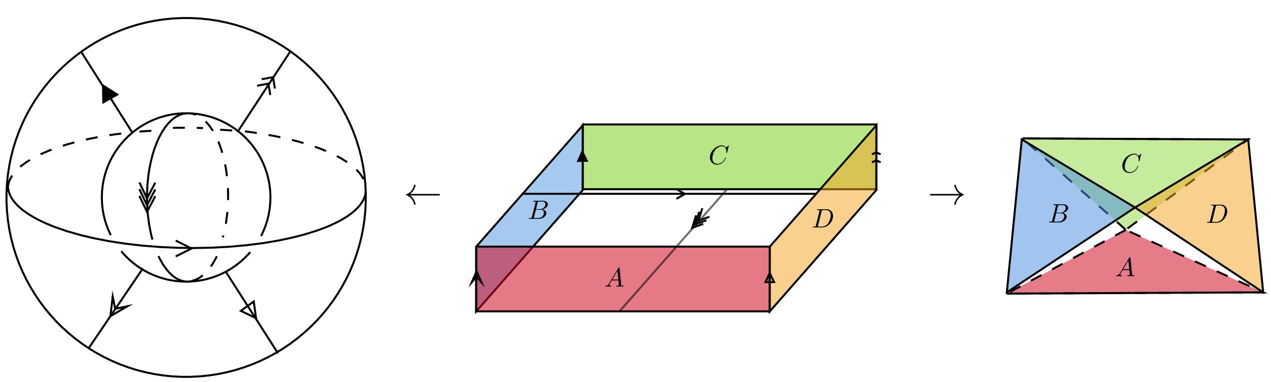

Figure 2.1: Left: A_0. Right: an ideal hyperbolic octahedron.

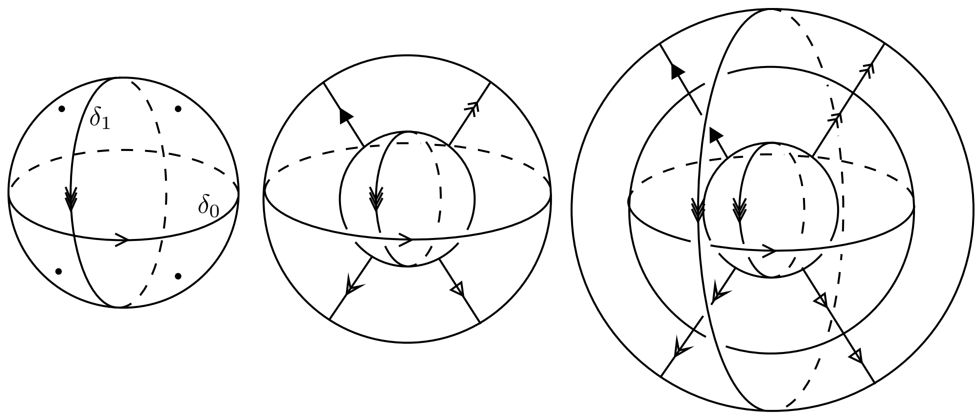

Figure 2.2: Left: Σ_4. Middle: A_0. Right: A.

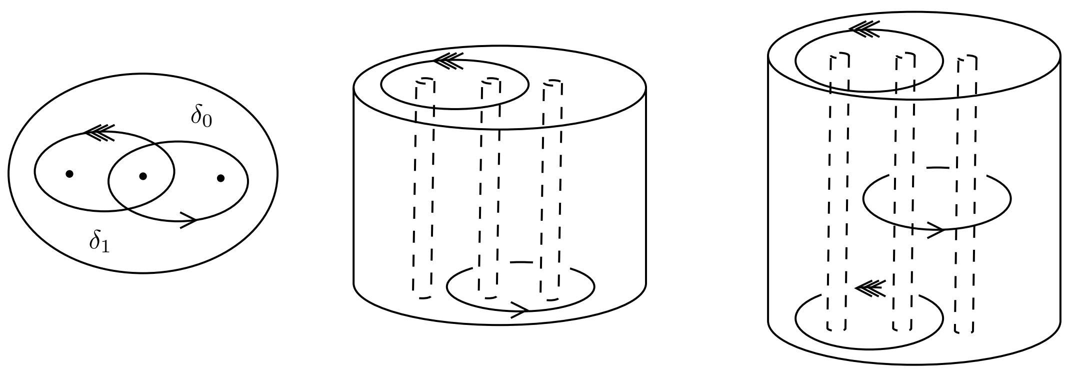

Figure 2.3: Left: Σ_4. Middle: A_0. Right: A.

The following construction is given by Agol in [Ago03]. Let Σ_4 denote the 4-puntured sphere, and let $\delta_0, \delta_1 \subset \Sigma_4$ be two circles on Σ_4 shown in Figure 2.2. Let A_0 be $\Sigma_4 \times [0,1] \backslash (\delta_0 \times \{0\} \cup \delta_1 \times \{1\})$. Let V_8 denote the volume of a regular, ideal, hyperbolic octahedron.

As shown in Figure 2.1, A_0 can be obtained by doubling the middle figure across the four faces A, B, C, and D, and removing the marked edges. If we crush all the marked edges in the middle figure, we get an octahedron on the right of Figure 2.1. Removing the vertices of the octahedron is the same as removing the marked edges, so we can obtain A_0 by doubling an ideal octahedron over the four faces A, B, C, and D. We view the octahedron as a regular ideal octahedron. Note that a regular ideal octahedron is an octahedron in $\overline{\mathbb{H}}^3$ with vertices at infinity, and the intersection with $\mathbb{H}^3$ is a convex polyhedron. All its dihedral angles are $\pi/2$. When we glue the corresponding 4 faces, the rest 4 faces, which are not glued, will be formed in the way that the diheral angle is π. Thus, the faces which are not identified form a totally geodesic boundary of A_0, which is a union of thrice-punctured spheres.

Proposition 2 (Agol)**.** A_0 has complete hyperbolic metric with totally geodesic boundary, with $\mathrm{Vol}(A_0) = 2V_8$.

For our purpose, it is more useful to draw the 4-punctured sphere as a 3-punctured disk, then A and A_0 are manifolds shown in Figure 2.2. Let A denote the manifold obtained by isometrically gluing two copies of A_0 along $\Sigma_4 \times \{0\} \backslash (\delta_0 \times \{0\})$, then we have

$$A \cong \Sigma_4 \times [0,1] \backslash (\delta_1 \times \{0,1\} \cup \delta_0 \times \{1/2\})$$

and A is a hyperbolic 3-manifold with totally geodesic boundary and

$$\mathrm{Vol}(A) = 4V_8$$

.

We will also need the following theorem, due to Adams [Ada85].

Theorem 8 (Adams). *Any properly embedded incompressible thrice-punctured sphere in a hyperbolic 3-manifold M is isotopic to a totally geodesic properly embedded thrice-punctured sphere in M.*

From this theorem one easily obtains the following.

Corollary 3. A disjoint union of pairwise non-isotopic properly embedded thrice-punctured spheres in a hyperbolic 3-manifold M are simultaneously isotopic to pairwise disjoint totally geodesic thrice-punctured spheres in M.

2.4 Dehn surgery

Let M be a compact 3-manifold with boundary $\partial M = \partial_1 M \sqcup \ldots \sqcup \partial_k M$ so that the interior of M is a complete hyperbolic manifold, where $\partial_i M$ is a torus for any $1 \leq i \leq k$. Choose a basis μ_i, ν_i for $H_1(\partial_i M) = \pi_1(\partial_i M)$. Then the isotopy class of any essential simple closed curve β_i on $\partial_i M$, called a *slope*, is represented by $p_i\mu_i + q_i\nu_i$ in $H_1(\partial_i M)$ for coprime integer p_i, q_i. Since we do not care about orientation of β_i, we use the notation $\beta_i = \frac{p_i}{q_i} \in \mathbb{Q} \cup \{\infty\}$. Given $\beta = (\beta_1, \ldots, \beta_k)$, where each β_i is a slope, let M_β denote the manifold obtained by gluing a solid torus to each $\partial_i M$, where β_i is the slope $\partial_i M$ identified with the meridian of the corresponding solid torus. We call $\beta = \{\beta_1, \ldots, \beta_k\}$ the Dehn surgery coefficients.

The following is from [Thu78, NZ85].

Theorem 9 (Thurston). *If the interior of M is a complete hyperbolic 3-manifold of finite volume and $\beta = \{\beta_1, \ldots, \beta_k\}$ are the Dehn surgery coefficients, then for all but finitely many slopes β_i, for each i, M_β*

is hyperbolic and $\mathrm{Vol}(M_\beta) < \mathrm{Vol}(M)$*. If* $\beta^n = (\beta_1^n, \ldots, \beta_k^n)$*, with* $\{\beta_i^n\}_{n=1}^\infty$ *an infinite sequence of distinct slopes on* $\partial_i M$ *for each* $1 \leq i \leq n$*, then* $\lim_{n\to\infty} \mathrm{Vol}(M_{\beta^n}) \to \mathrm{Vol}(M)$*.*

Theorem 10 (Thurston)**.** *If* M *is a compact irreducible atoroidal Haken manifold whose boundary has zero Euler characteristic, then the interior of* M *has a complete hyperbolic structure of finite volume.*

If $a \in H_k(M;\mathbb{R})$ is any homology class, then the Gromov norm of a is defined to be the infimum of the L_1-norms of cycles representing a.

$$||a||= \inf\{||z|| \mid z \text{ is a singular cycle representing } a\}$$

There's a natural extension of the norm on the chain group to the relative chain group. So there's a natual extension of the Gromov norm to the relative homology class for some $a \in H_k(M;A)$, where $A \subset M$ a submanifold.

$$||a||= \inf\{||z|| \mid z \text{ is a relative cycle representing } a\}$$

Let $||[M]||$ denote the *Gromov norm* of the fundamental class $[M] \in H_3(M;\partial M)$. Then we have the following two theorems. See [Thu78, Theorem 6.2, Proposition 6.5.2, Lemma 6.5.4]

Theorem 11 (Gromov)**.** *If the interior of* M *admits a complete hyperbolic metric of finite volume, then*

$$||[M]||= \frac{\mathrm{Vol}(M)}{v_3}.$$

Theorem 12 (Thurston)**.** *For any Dehn fillings with Dehn surgery coefficients* $\beta = \{\beta_1, \ldots, \beta_k\}$*,*

$$||[M_\beta]||\leq ||[M]||.$$

We will be interested in a special case of Dehn surgery in which M is obtained from a mapping torus

$$M_f = S \times [0,1]/(x,1) \sim (f(x),0)$$

by removing neighborhoods of disjoint curves $\alpha_1, \alpha_2, \ldots, \alpha_k$, $\alpha_i \subset S \times \{t_i\}$, for some

$$0 < t_1 < t_2 < \cdots < t_k < 1.$$

Then we can choose basis μ_i, ν_i of $H_1(\partial_i M)$, so that if $\beta_i = \frac{1}{r_i}$, then

$$M_\beta = M_{T_{\alpha_k}^{r_k} T_{\alpha_{k-1}}^{r_{k-1}} \dots T_{\alpha_1}^{r_1} f},$$

where T_{α_i} denotes the Dehn twist along α_i in S. See, for example, [Sta78].

Chapter 3

Reduction

Consider the sphere with $n+m+2$ puntures $S_{0,n+m+2}$. We can distribute the punctures as shown in Figure 3.1. Let x, y and z be the three of the punctures as shown. Let $X, Y \subset S_{0,n+m+2}$ be two embedded punctured disks centered at x and y as shown in Figure 3.1. There are n punctures in X arranged around x, m punctures in Y arranged around y, with one puncture shared in X and Y. Let p_n denote the homeomorphism which is supported inside X, fixes x and rotates the punctures around x by one counterclockwise. Let q_m denote the homeomorphism which is supported inside Y, fixes y and rotates the punctures around y by one clockwise. For any $n, m > 6$, let $f_{n,m} : S_{0,n+m+2} \to S_{0,n+m+2}$ be $f_{n,m} = q_m p_n$. These homeomorphisms $f_{n,m}$ were constructed by Hironaka and Kin in [HK06b] and were shown to be pseudo-Anosov.

Let $V_1, V_2, \ldots, V_n$ be the punctures in X, starting with V_1 in $X \cap Y$, ordered counter-clockwise, as shown in Figure 3.1. Let $\Sigma_0 \subset S_{0,n+m+2}$ be the subsurface containing 3 consecutive punctures $\{V_i, V_{i+1}, V_{i+2}\}$, with $\partial\Sigma_0 = \beta$ as shown in Figure 3.1. Let $\alpha, \gamma \subset \Sigma_0$ be the two essential closed curves shown.

We will consider the composition $hf_{n,m}^3$, where $h : S_{0,n+m+2} \to S_{0,n+m+2}$ is a homeomorphism supported in Σ_0. Note that if we replace h by $p_n^k h p_n^{-k}$ for $1 \le k \le n-(i+3)$, which is supported on $p_n^k(\Sigma_0)$, then q_m commutes with $p_n^j h p_n^{-j}$ for $1 \le j \le k$. So we have

$$\begin{aligned}
f_{n,m}^k h f_{n,m}^3 f_{n,m}^{-k} &= f_{n,m}^{k-1} q_m (p_n h p_n^{-1}) p_n f_{n,m}^{-k+3} \\
&= f_{n,m}^{k-1} (p_n h p_n^{-1}) q_m p_n f_{n,m}^{-k+3} \\
&= f_{n,m}^{k-1} (p_n h p_n^{-1}) f_{n,m}^{-k+4} \\
&= f_{n,m}^{k-2} q_m (p_n^2 h p_n^{-2}) p_n f_{n,m}^{-k+4} \\
&= \ldots \\
&= q_m (p_n^k h p_n^{-k}) p_n f_{n,m}^2 \\
&= (p_n^k h p_n^{-k}) f_{n,m}^3
\end{aligned}$$

That is, $hf_{n,m}^3$ is conjugate to $p_n^k h p_n^{-k} f_{n,m}^3$. In particular, we can assume Σ_0 surrounds V_i, V_{i+1}, V_{i+2} for any $2 \leq i \leq n-5$ at the expense of conjugation which does not affect stretch factor or the homeomorphism type of mapping torus. For this reason, in the following statements, Σ_0 is allowed to surround the punctures V_i, V_{i+1}, V_{i+2} for any $2 \leq i \leq n-5$.

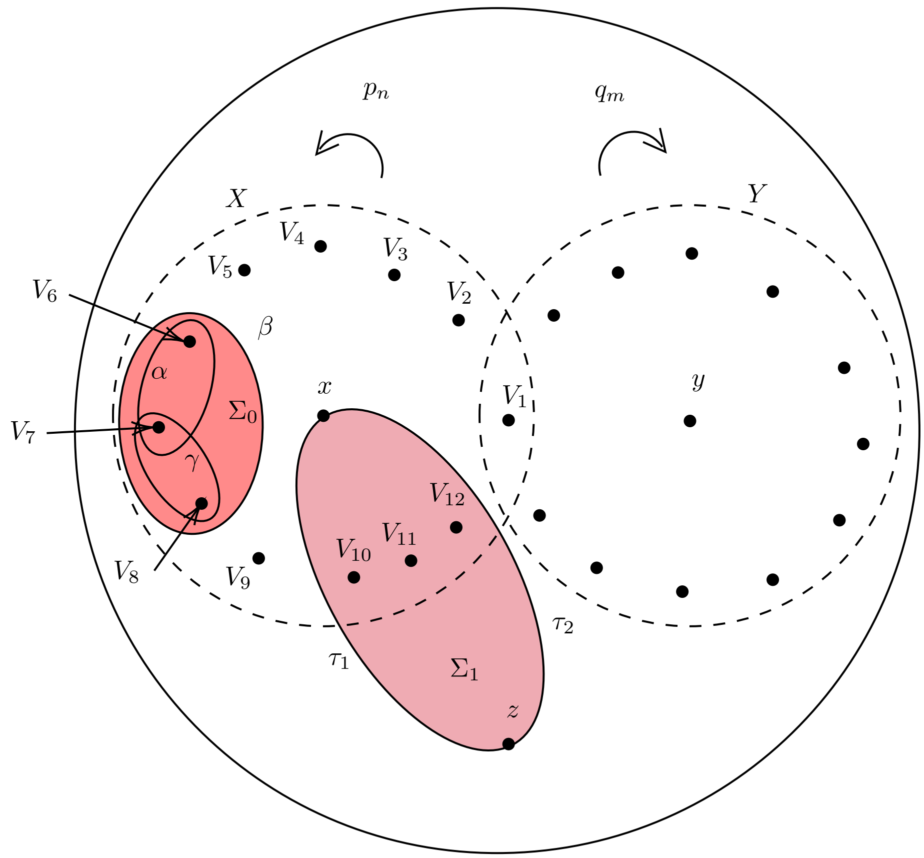

Figure 3.1: $S_{0,n+m+2}$ for $n = m = 12$

Theorem 13. *For any $k = 1, 2, 3, \ldots,$ there exists B_k such that if*

$$h_k = T_\alpha^{u_1} T_\gamma^{v_1} \ldots T_\alpha^{u_{k-1}} T_\gamma^{v_{k-1}} T_\alpha^{u_k} T_\beta^{v_k}$$

where $u_i, v_i \geq B_k$ for all i, then for $h_k f_{n,m} : S_{0,n+m+2} \to S_{0,n+m+2}$, we have

(1) $h_k f_{n,m}^3$ is pseudo-Anosov.

(2) $\mathrm{Vol}(M_{h_k f_{n,m}^3}) \geq 3kV_8$.

(3) there exists $N = N_k$, such that if $n = m > N$, then

$$\log \lambda(h_k f_{n,n}^3) \leq 54 \frac{\log(2n+2)}{2n+2}.$$

Assuming this theorem, we prove the Main Theorem from the introduction.

Main Theorem. *For any fixed $g \geq 2$, and $L \geq 162g$, there exists a sequence $\{M_{f_i}\}_{i=1}^{\infty}$, with $f_i \in \Psi_{g,L}$, so that* $\lim_{n\to\infty} \mathrm{Vol}(M_{f_i}) \to \infty$.

Proof. For any $g \geq 2$, [Tsa09] gives a construction of an appropriate cover $\pi : S_{g,s} \to S_{0,n+m+2}$ such that $s = (2g+1)(n+m+1)+1$ and

$$f_{n,m} : S_{0,n+m+2} \to S_{0,n+m+2},$$

lifts to $S_{g,s}$. Moreover, the construction of the cover is the cover corresponding to the kernel of a homomorphism from $\pi_1(S_{0,n+m+2})$ to a finite group so that all the peripheral loops around all punctures, except x, y and z, are trivial. Thus, each of α, β, γ lifts to $S_{g,s}$, so h_k lifts.

Let $\widetilde{f}_k : S_{g,s} \to S_{g,s}$ be the lift of $h_k \circ f_{n,m}^3$. Then $\log(\lambda(\widetilde{f}_k)) = \log(\lambda(h_k f_{n,m}^3))$. Also by Theorem 13, for $n = m > N_k$,

$$\log(\lambda(\widetilde{f}_k)) \leq 54\frac{\log(n+m+2)}{n+m+2} < 54\frac{\log(s)}{\frac{s-1}{2g+1}+1} < 162g\frac{\log s}{s}.$$

The first inequality comes from Theorem 13. The second and third hold for all $g \geq 2$ since $s > n+m+2$ and $\frac{s-1}{2g+1}+1 > \frac{s}{3g}$. Furthermore, $\mathrm{Vol}(M_{\widetilde{f_k}}) = deg(\pi)\mathrm{Vol}(M_{h_k f_{n,m}^3}) \geq 3kV_8 deg(\pi)$. Therefore, $\{M_{\widetilde{f_k}}\}_{k=1}^{\infty}$ is contained in the set for the theorem and $\mathrm{Vol}(M_{\widetilde{f_k}}) \to \infty$.

□

Corollary 1. For any $g \geq 2$, there exists L such that there is no finite set Ω of 3-manifolds so that all M_f, $f \in \Psi_{g,L}$, are obtained by Dehn filling on some $M \in \Omega$.

Proof. Let $L \geq 162g$. If the finite set Ω exist, then by Theorems 11 and 12,

$$\mathrm{Vol}(M_f) \leq v_3 \max_{M\in\Omega}\{|||[M]|||\} < \infty,$$

which contradicts the Main Theorem. □

Chapter 4

Proof of Theorem 13

Now fix some $n, m > 6$, let $f = f^3_{n,m}$. Let M_f be the mapping torus.

The proof of the following lemma is almost identical to the proof of [LM86, Theorem B].

Lemma 1. *$M_f \backslash ((\alpha \cup \beta) \times \{1/2\})$ is hyperbolic.*

Proof. Let

$$\Sigma = S_{0,n+m+2} \times \{1/2\}, \Sigma' = \Sigma \backslash ((\alpha \cup \beta) \times \{1/2\}) \subset M_f.$$

Let $T_0 \subset M_f$ be an embedded incompressible torus. By applying some isotopy, we can make every component of $T_0 \backslash \Sigma'$ be an annulus. Any annulus component should either miss no fiber or have boundary components parallel to α or β, and on opposite sides of some small neighborhood of α or β. Since α and β bound different number of punctures, a component parallel to α can never connect to a component parallel to β. Also, $f^{k_1}(\alpha)$ will never close up with $f^{k_2}(\alpha)$ if $k_1 \neq k_2$ since f is pseudo-Anosov. By Thurston's hyperbolization theorem (see [Thu82, Mor84, Ota96]), $M_f \backslash ((\alpha \cup \beta) \times \{1/2\})$ is hyperbolic. □

For any k, let $L_k \subset M_f$ be

$$L_k = \alpha \times \left\{ \frac{2}{4k}, \frac{4}{4k}, \dots, \frac{2k+2}{4k} \right\} \cup \gamma \times \left\{ \frac{3}{4k}, \frac{5}{4k}, \dots, \frac{2k+1}{4k} \right\} \cup \beta \times \left\{ \frac{1}{4k} \right\}.$$

Let $N(L_k)$ denote an tubular neighborhood of L_k and $M_k = M_f \backslash N(L_k)$. We can order the boundary components of M_k as

$$\partial M_k = \partial_1 M_k \sqcup \ldots \sqcup \partial_{2k+2} M_k,$$

where

$$\begin{cases} \partial_{2i} M_k = \alpha \times \{\frac{2i}{4k}\} & \text{for any } 1 \leq i \leq k+1 \\ \partial_{2i+1} M_k = \gamma \times \{\frac{2i+1}{4k}\} & \text{for any } 1 \leq i \leq k-1 \\ \partial_1 M_k = \beta \times \{\frac{1}{4k}\}. & \end{cases}$$

Lemma 2. *The interior of* $M_f \backslash N(L_k)$ *is hyperbolic and*

$$\mathrm{Vol}(int(M_f \backslash N(L_k))) \geq 4kV_8.$$

Proof. Glue k copies of A, top to bottom, to get

$$A_k \cong (S_{0,4} \times [0,1]) \backslash \left(\alpha \times \left\{ \frac{0}{2k}, \frac{2}{2k}, \dots, \frac{2k}{2k} \right\} \cup \gamma \times \left\{ \frac{1}{2k}, \frac{3}{2k}, \dots, \frac{2k-1}{2k} \right\} \right),$$

with the i-th copy identifying with

$$\left(S_{0,4} \times \left[\frac{2i-2}{2k}, \frac{2i}{2k} \right] \right) \backslash \left(\alpha \times \left\{ \frac{2i-2}{2k}, \frac{2i}{2k} \right\} \cup \gamma \times \left\{ \frac{2i-1}{2k} \right\} \right).$$

By Theorem 8, A_k has four totally geodesic thrice-punctured sphere boundary components, and $\mathrm{Vol}(A_k) = 4kV_8$.

Cut $M_f \backslash ((\alpha \cup \beta) \times \{1/2\})$ along the two thrice-punctured spheres, i.e. the two regions shown in Figure 4.1. The two thrice-punctured spheres can be assumed to be totally geodesic by Corollary 2. So the cut-open manifold has four totally geodesic thrice-punctured sphere boundary components. Now glue the top boundary of A_k to the top of the cut by an isometry, with the marked curves and colored faces glued correspondingly. Then apply the same to the bottom boundary. After applying an isotopy to adjust the height, we see that the result is homeomorphic to $M_f \backslash N(L_k)$. Moreover, A_k is isometrically embedded in $M_f \backslash N(L_k)$. Since $\mathrm{Vol}(A_k) \geq 4kV_8$, we have $\mathrm{Vol}(M_f \backslash N(L_k)) \geq 4kV_8$.

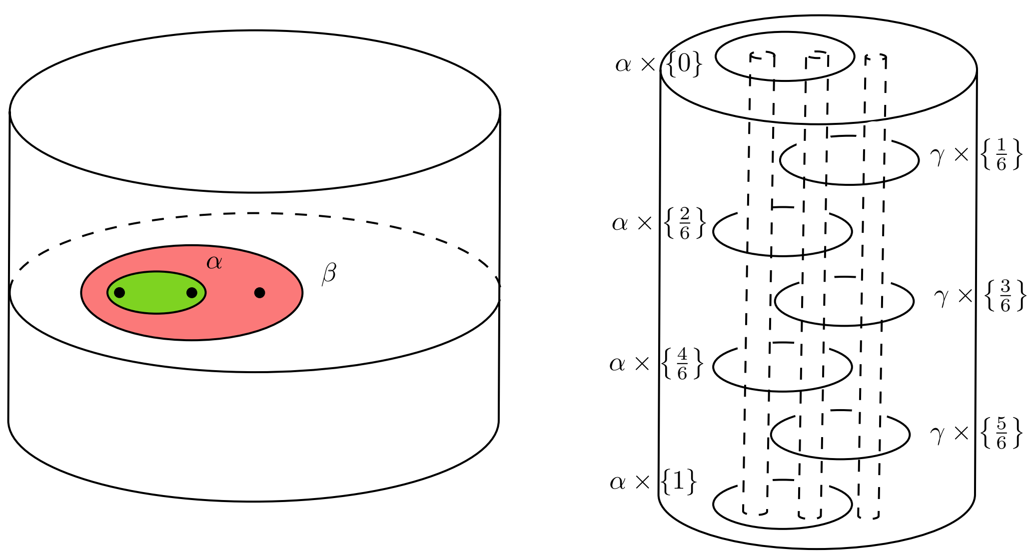

Figure 4.1: Cut and glue A_k to $M_f \backslash ((\alpha \cup \beta) \times \{1/2\})$ when $k = 3$

$\square$

Proposition 3. Given k, there exists B_k, such that if $u_i, v_i > B_k$, then $h_k f$ is pseudo-Anosov and $\mathrm{Vol}(M_{h_k f}) \geq 3kV_8$.

Proof. Let $M = M_f \backslash N(L_k)$. Let $\beta = \{\frac{1}{v_k}, \frac{1}{u_k}, \ldots, \frac{1}{v_1}, \frac{1}{u_1}\}$, then by Theorem 9, $M_{h_k f} = M_\beta$, and when u_i, v_i are big enough, the volume is approximatly equal to $\mathrm{Vol}(M_f \backslash N(L_k))$. In particular, if u_i, v_i are large enough,

$$\mathrm{Vol}(int(M_{h_k f})) \geq \mathrm{Vol}(int(M_f \backslash N(L_k))) - kV_8 \geq 3kV_8$$

by Lemma 2. $\square$

Lemma 3. *For $n, m > 3$, $M_{h_k f^3_{n,m}} \cong M_{h_k f^3_{n+3,m}} \cong M_{h_k f^3_{n,m+3}}$.*

Proof. By Proposition 1, $M^\circ = M_{h_k f} = M_{h_k f^3_{n,m}}$ is hyperbolic. Let Σ_1 be the subsurface in $S_{0,n+m+2}$ shown in Figure 3.1 containing 3 punctures, and let τ_1 and τ_2 denote the two components of $\partial\Sigma_1$, where τ_1 and τ_2 are two arcs connecting x and z, with $\tau_2 = f^3_{n,m}(\tau_1)$.

Construct a surface $\Sigma_2 \subset M$ as follows. First, define a map

$$\eta = (\eta_1, \eta_2) : \Sigma_1 \to S \times [0,1]$$

so that $\eta(\Sigma_1) \cap S \times \{0\} = \tau_2 \times \{0\}$, $\eta(\Sigma_1) \cap S \times \{1\} = \tau_1 \times \{1\}$ and η_1 is the inclusion of Σ_1 into S. Since $f(\tau_1) = \tau_2$, if we project $p : S \times [0,1] \to M_f$, η defines an embedding of $\Sigma_1/(\tau_1 \underset{f}{\sim} \tau_2)$, that is, Σ_1 with τ_1 glued to τ_2 by f. Since η_1 is the inclusion, $\Sigma_2 = p \circ \eta(\Sigma_1/\tau_1 \underset{f}{\sim} \tau_2)$ is transverse to the suspension flow. By Theorem 6, $[\Sigma_2] \in \overline{F \cdot \mathbb{R}^+}$.

We will define a surface S' such that $[S'] = [S] + [\Sigma_2]$ in $H^1(M_f)$ as follows. Let S_{τ_2} denote the surface obtained by cutting S along τ_2. Then S_{τ_2} has two boundary components, denote τ_2^+, τ_2^-. Since $\tau_2 = p \circ \eta(\Sigma_1)$, and $p \circ \eta(\tau_1) = p \circ \eta(\tau_2) = \tau_2 \subset S \subset M_f$, we can construct S' in M_f by gluing τ_2^+ to $\eta(\tau_2)$ and τ_2^- to $\eta(\tau_1)$, perturbed slightly to be embedded. Then $[S'] = [S] + [\Sigma_2]$ and $S' \pitchfork \Psi$. So S' is a fiber representing a class in $F \cdot \mathbb{R}^+ \subset H^1(M)$. By Theorem 6, the first return map of ψ is the monodromy $f' : S' \to S'$. This is given by

$$f'(x) = \begin{cases} \eta(x) & \text{if } x \in \Sigma_1 \\ f \circ \eta^{-1}(x) & \text{if } x \in \eta(\Sigma_1) \\ f(x) & \text{otherwise} \end{cases}$$

See Figure 4.2. As indicated by Figure 4.3, $S' \cong S_{0,n+m+5}$, and up to conjugation, $f' = f^3_{n+3,m}$. Therefore,

$M_{h_k f^3_{n,m}} \cong M_{h_k f^3_{n+3,m}}$. Similarly, if we pick another subsurface in Y homeomorphic to Σ_0, one can show $M_{h_k f^3_{n,m}} \cong M_{h_k f^3_{n,m+3}}$.

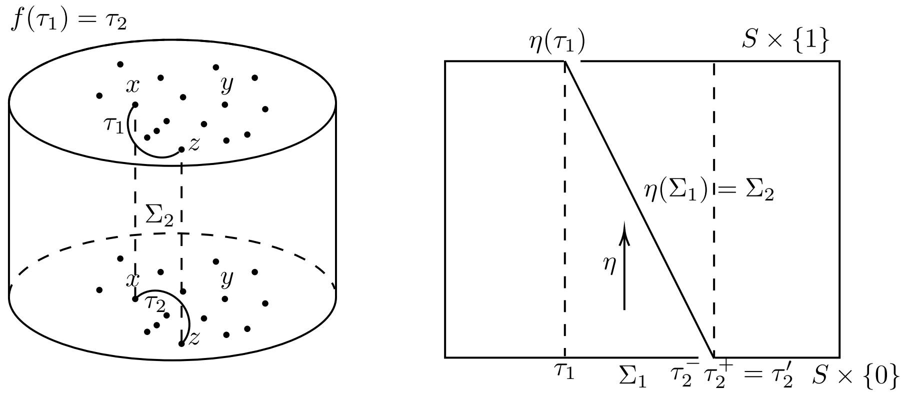

Figure 4.2: Obtain Σ_2 from $\eta : \Sigma_1 \to S \times [0,1]$ and S' from S and Σ_2 as shown.

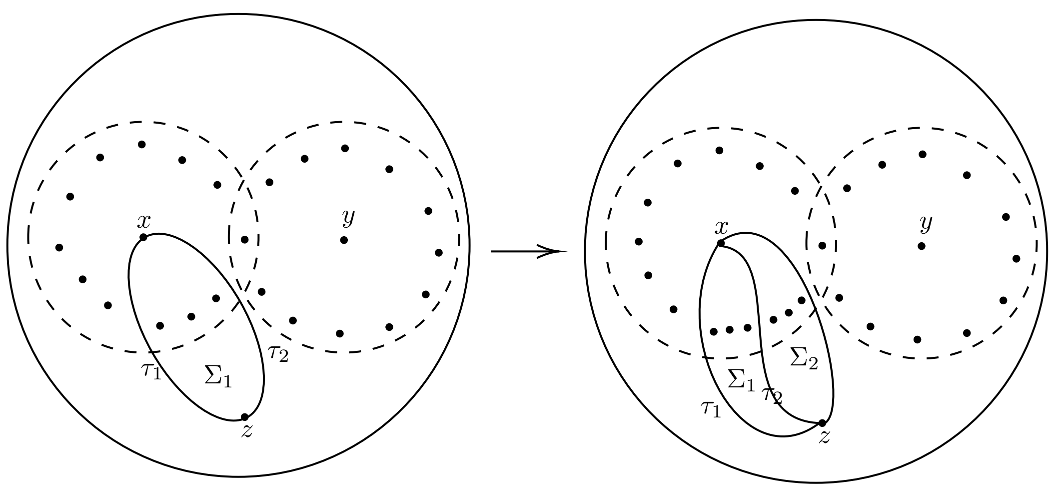

Figure 4.3: Left: S. Right: S'

$\square$

Lemma 4. *For fixed k, and fixed $u_i, v_i \geq B_k$ (the constant from Proposition 3), there exists $R > 0$ so that if $n = m \geq R$, then $h_k f^3_{n,n} : S_{0,2n+2} \to S_{0,2n+2}$ has $\log \lambda(h_k f^3_{n,n}) \leq 54 \frac{\log 2n+2}{2n+2}$.*

Proof. We can get the spine G as in Figure 4.4 on $S_{0,n+m+2}$. This is in fact a train track for $f_{n,m}$, as described in [HK06b], and hence also f. Then f induces a map $g : G \to G$.

The graph G contains the loop edges $a_1, a_2, \ldots, a_n$, and $a'_2, a'_3, \ldots, a'_m$, which g acts on as a permutation, and "peripheral" edges $b_1, b_2, \ldots, b_n$, and $b'_1, b'_2, \ldots, b'_m$, which g also acts on them as a permutation. The

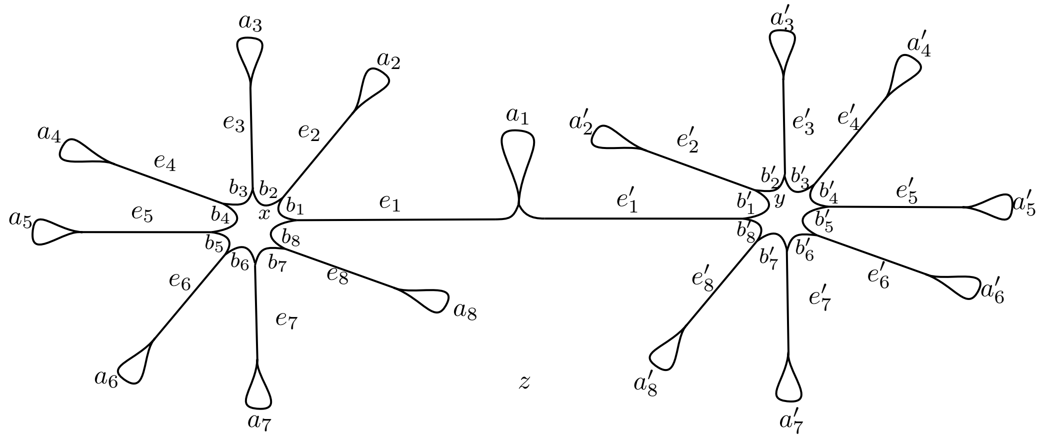

Figure 4.4: Spine of $S_{0,n+m+2}$ when $n = m = 8$

transition matrix has the following form:

$$T = \left[\begin{array}{c|c} A & * \\ \hline 0 & P \end{array}\right]$$

where P corresponds to $e_1, e_2, \ldots, e_n$, $e'_1, e'_2, \ldots, e'_m$. The matrix A is a permutation matrix corresponds to $a_1, a_2, \ldots, a_n$, $a'_1, a'_2, \ldots, a'_m$, $b_1, b_2, \ldots, b_n$, $b'_1, b'_2, \ldots, b'_m$. So the largest eigenvalue of T (in absolute value) will be the largest eigenvalue of P. If we remove all the non-contributing edges, we have

$$\begin{aligned}
e_i &\rightarrow e_{i+3} \;\text{ for } 1 \leq i \leq n-3 \\
e'_i &\rightarrow e'_{i+3} \;\text{ for } 1 < i \leq m-2 \\
e'_1 &\rightarrow e'_4 e'_4 e'_3 e'_3 e'_2 e'_2 e'_1 e_1 e_2 e_2 e_3 e_3 e_4 \\
e_n &\rightarrow e_3 e_3 e_2 e_2 e_1 e'_1 e'_2 e'_2 e'_3 e'_3 e'_4 \\
e'_m &\rightarrow e'_3 e'_3 e'_2 e'_2 e'_1 e_1 e_2 e_2 e_3 \\
e_{n-1} &\rightarrow e_2 e_2 e_1 e'_1 e'_2 e'_2 e'_3 \\
e'_{m-1} &\rightarrow e'_2 e'_2 e'_1 e_1 e_2 \\
e_{n-2} &\rightarrow e_1 e'_1 e'_2
\end{aligned}$$

Assume $n = m$, we get the directed graph Γ associated to f (or g) and T (with only the contributing edges) as shown in Figure 4.5. The graph is made of 6 big "loops" going clockwise, together with a subgraph D. The subgraph D is given by the relations determined by g above, as shown in Figure 4.6, containing one loop at e'_1. For simplicity, the graph of D in Figure 4.6 omits the arrows in between. All edges with omitted arrows implicitly point from left to right. The edges marked thick mean there are two edges connecting those vertices. Thus, a path with given length passing through D once will either

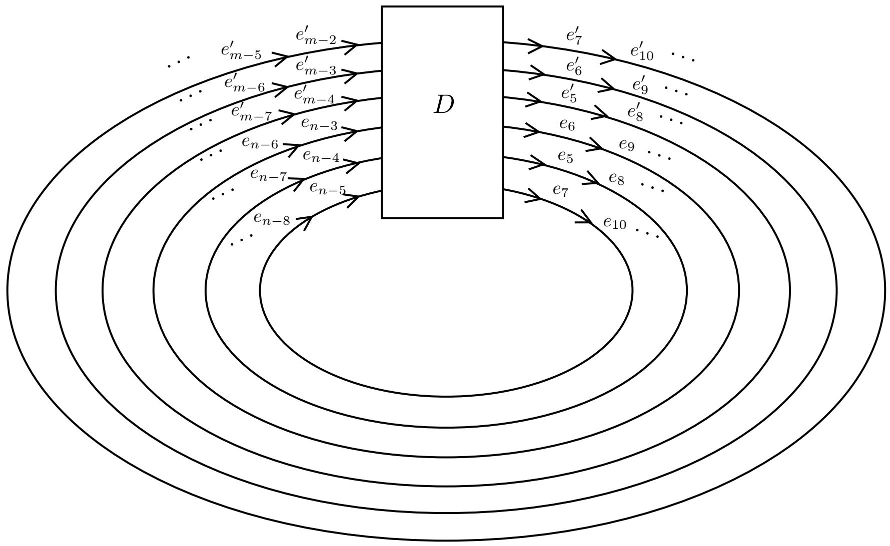

Figure 4.5: The directed graph Γ associated to f.

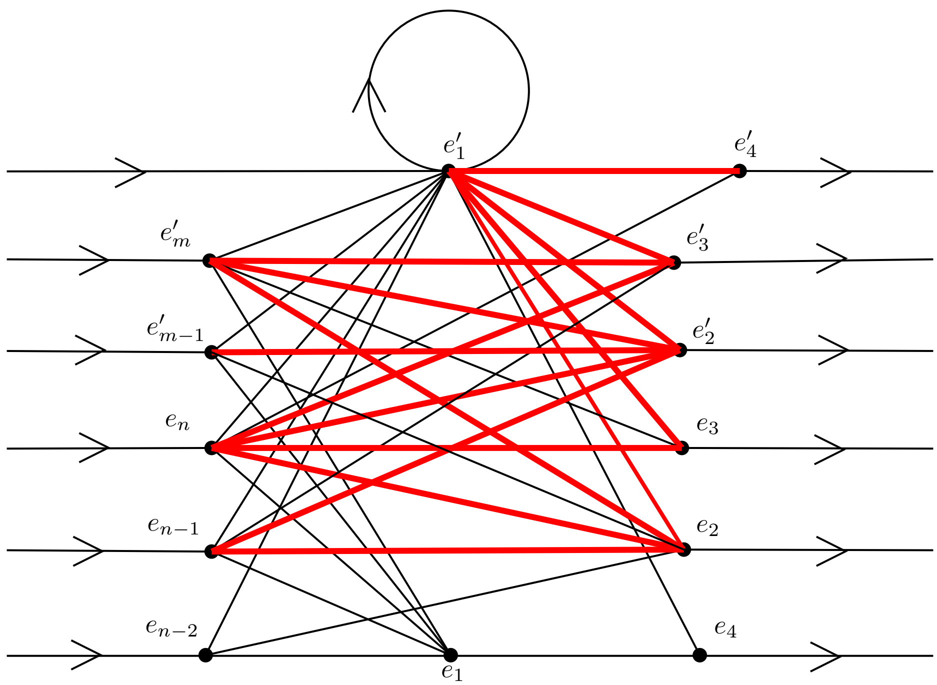

Figure 4.6: D: edges marked thick denote two directed edges between corresponding vertices

- directly go from left to right with length 1.
- go from left to e'_1, then wrap around the loop at e'_1 some number of times, then go to the right.
- pass e_1 and go to e_4.

Given two vertices, the number of paths of length $\frac{n}{13}$ between them which passes through D is therefore at most 2.

Now we let Σ_0 surround $V_{\lfloor \frac{n}{2} \rfloor -1}, V_{\lfloor \frac{n}{2} \rfloor}, V_{\lfloor \frac{n}{2} \rfloor +1}$, fix h_k and consider a graph map $g_k \simeq h_k f$ and its matrix T_k. Note that h_k is supported in a neighborhood of Σ_0. Let a_j, a_{j+1}, a_{j+2} denote the three loops wrapping around the three punctures in Σ_0. If we remove all the non-contributing edges, after homotopy, h_k sends e_j, e_{j+1}, e_{j+2} to a combination of e_j, e_{j+1}, e_{j+2} without acting on other edges. Thus $g_k \simeq h_k f$ sends $e_{j-3}, e_{j-2}, e_{j-1}$ to a combination of e_j, e_{j+1}, e_{j+2} and acts on the rest of the edges as $g \simeq f$ does.

Then we get the directed graph Γ_k associated to T_k and g_k as shown in Figure 4.7. The graph Γ_k is the same as Γ away from $e_{j-3}, e_{j-2}, e_{j-1}, e_j, e_{j+1}, e_{j+2}$, and has a subgraph D_k given by h_k. The subgraph D_k is a bipartite graph with 3 vertices in each set, $\{e_j, e_{j+1}, e_{j+2}\}$ and $\{e_{j-3}, e_{j-2}, e_{j-1}\}$. All edges of D_k point from right to left, from $\{e_{j-3}, e_{j-2}, e_{j-1}\}$ to $\{e_j, e_{j+1}, e_{j+2}\}$. The number of edges between any two vertices in different sets is bounded above by some $E_k > 0$ depending on h_k. See Figure 4.8.

Note that the distance between D and D_k is greater than $n/6 - 2$. Thus, when $n = m \geq 13$ any path of length $\frac{n}{13}$ can't intersect D and D_k simultaneously. Thus given any two vertices, the number of paths of length $\frac{n}{13}$ between those vertices is bounded above by $N_k = \max\{2, E_k\}$. Note that the total number of vertices in Γ_k is $2n$. Thus, the number of paths of length $\frac{n}{13}$ emanating from a given vertex is thus at most $2nN_k$. Then for λ_0, the leading eigenvalue of T_k, by Proposition 1, we have

$$\log \lambda_0 \leq \frac{\log 2nN_k}{\frac{n}{13}}.$$

When $n > N_k$ is big enough, we have

$$\log \lambda_0 \leq \frac{\log 2nN_k}{\frac{n}{13}} < \frac{2\log(2n+2)}{\frac{2n}{26}} < \frac{2\log(2n+2)}{\frac{2n+2}{27}} = 54\frac{\log(2n+2)}{2n+2}.$$

The third inequality holds when $n > 26$. Then the result follows since $\lambda(h_k f) \leq \lambda_0$. $\square$

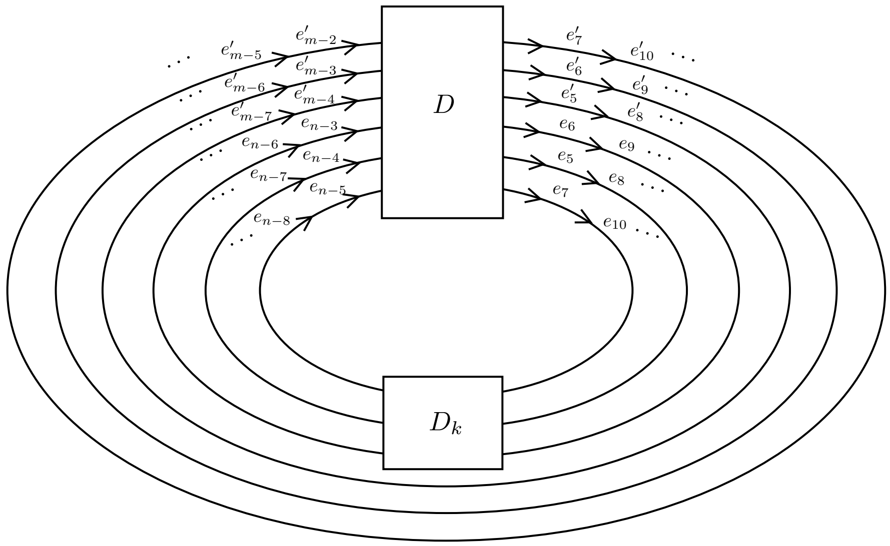

Figure 4.7: The directed graph Γ_k associated to $h_k f$.

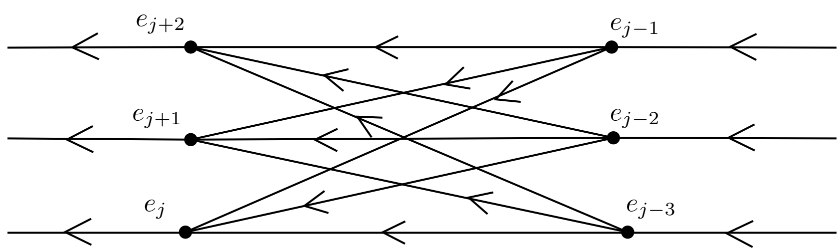

Figure 4.8: D_k: each directed edge in between represent $\leq E_k$ directed edge.

Now we finish Theorem 13. Part (1) is given by Lemma 2. Part (2) is given by Proposition 3. Part (3) is given by Lemma 4.

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