COMBINATORIAL NUMBER THEORY THROUGH DIAGRAMMING AND GESTURE

BY

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DISSERTATION

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Abstract

Within combinatorial number theory, we study a variety of problems about whole numbers that include enumerative, diagrammatic, or computational elements. We present results motivated by two different areas within combinatorial number theory: the study of partitions and the study of digital representations of integers. We take the perspective that mathematics research is mathematics learning; existing research from mathematics education on mathematics learning and problem solving can be applied to mathematics research. We illustrate this by focusing on the concept of diagramming and gesture as mathematical practice. The mathematics presented is viewed through this lens throughout the document.

Joint with H. E. Burson and A. Straub, motivated by recent results working toward classifying (s, t)-core partitions into distinct parts, we present results on certain abaci diagrams. We give a recurrence (on s) for generating polynomials for s-core abaci diagrams with spacing d and maximum position strictly less than $ms - r$ for positive integers s, d, m, and r. In the case $r = 1$, this implies a recurrence for $(s, ms - 1)$-core partitions into d-distinct parts, generalizing several recent results.

We introduce the sets $Q(b; \{d_1, d_2, \ldots, d_k\})$ to be integers that can be represented as quotients of integers that can be written in base b using only digits from the set $\{d_1, \ldots, d_k\}$. We explore in detail the sets $Q(b; \{d_1, d_2, \ldots, d_k\})$ where $d_1 = 0$ and the remaining digits form proper subsets of the set $\{1, 2, \ldots, b - 1\}$ for the cases $b = 3$, $b = 4$ and $b = 5$. We introduce modified multiplication transducers as a computational tool for studying these sets. We conclude with discussion of $Q(b; \{d_1, \ldots d_k\})$ for general b and digit sets including $\{-1, 0, 1\}$.

Sections of this dissertation are written for a nontraditional audience (outside of the academic mathematics research community).

To my students, past, present, and future, who inspire me to keep learning and growing as a mathematician.

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Notation

$\mathbb{N}$	The natural numbers.
$\mathbb{Z}$	The integers.
λ	A partition.
$\lvert\lambda\rvert$	The size of the partition λ.
$\nu(\lambda)$	The number of parts of the partition λ.
$\mathcal{C}^d_{s,m,r}$	The set of $(s, ms+r)$-core partitions into d-distinct parts.
$\boldsymbol{C}^d_{s,m,r}(q)$	The generating polynomial for the number of parts in the set $\mathcal{C}^d_{s,m,r}$.
$\mathcal{A}^d_{s,m,r}$	The set of s-core abaci with largest position less than $ms+r$ and spacing d.
$\boldsymbol{A}^d_{s,m,r}(q)$	The generating polynomial for the number of parts in the set $\mathcal{A}^d_{s,m,r}$.
$S(b;\{d_1,d_2,\ldots,d_k\})$	The set of integers that can be represented in base b using the digits $\{d_1,d_2,\ldots d_k\}$.
$Q(b;\{d_1,d_2,\ldots,d_k\})$	The set of integers that can be represented as quotients of elements of $S(b;\{d_1,d_2,\ldots,d_k\})$.
$\mathcal{T}_{m,b}$	The transducer that multiplies by m, reading and writing the digits $\{0,1,\ldots,b-1\}$ in base b.

Chapter 1

Introduction

This dissertation contains three parts. The first part is a discussion of diagramming and gesture in mathematical practice. The second and third parts are results from projects in combinatorial number theory, in the broad areas of partitions, and digital representations of integers, respectively. I use diagramming and gesture as a lens to examine the mathematical content.

1.1 Overview of the Chapters

The overview below includes formal definitions of the mathematical objects presented. Individual chapters present more informal definitions and examples. Each section of mathematical content was written to be accessible to a specific audience, inspired by people in my life who have a wide variety of mathematical backgrounds, most of whom do not identify as research mathematicians. More information about the specific audiences, and the choices I made regarding this nontraditional writing process are in Appendix B. A result of this is that the next few pages are the most formal and dense in the document; for more accessible definitions and examples, please refer to the individual chapters.

1.1.1 Diagramming and Gesture

Mathematics research is mathematics learning. As such, how we practice research is impacted by how we learn math. Mathematics education researchers, in [34], [36], and [38], have explored how academic mathematicians think about mathematics in the context of explaining existing mathematics. In this dissertation, I extend this discussion to the practice of generating new mathematical ideas in research.

I take the perspective of R. Hersh [20] that "mathematics has a front and a back": mathematics presented in public differs from mathematics practiced in private (like the separation between theater auditorium and backstage). Furthermore, the "back" is personal, in that it is different for each of us. I chose to focus

on diagramming and gesture as a practice that is particularly salient to my personal interaction with this content (the "back" of this dissertation). I do not claim this practice is universal. I expect that some aspects of this may resonate with some mathematicians, and I hope to encourage further discussion about other practices and methods through which mathematicians engage with mathematics research.

Chapter 2 includes a review of the literature on the use of diagram and of gesture, both of which have been studied in relation to problem solving and mathematics. In [13], Elizabeth de Freitas and Nathalie Sinclair define diagramming and gesture as inherently related. Diagrams inspire gesture (e.g. one might gesture to move or rearrange elements in a diagram), and gesture inspires diagrams (e.g. we often draw diagrams to capture a gesture on paper). Throughout my work on the problems described below, I have reflected on this combined process of *diagramming and gesture* as a core component of my mathematical practice. I use diagramming and gesture as a lens for exploring the mathematics in this dissertation. Toward the end of Chapter 2, on page 21, I list specific diagramming and gesture practices that I employ in my mathematics research. Throughout Chapters 3-8, I highlight these practices using offset Diagramming and Gesture boxes.

Diagramming and Gesture

Within each box, I will describe diagramming and gesture techniques used in the immediately preceding mathematical content, with the **specific practices** highlighted in bold.

In addition to showing the "back" of my research process, this approach provides evidence that drawing upon knowledge from mathematics education research can enrich our conversations about how we practice mathematics.

1.1.2 Simultaneous Core Partitions and Abaci

Chapters 3 and 4 contain a subset of the material in [9], which is the result of collaboration with Hannah Elizabeth Burson and Armin Straub. A complementary subset is presented in [8].

A *partition* λ of an integer $n \in \mathbb{N}$ is a finite, non-increasing sequence of positive integers $(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r)$ such that $\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_r = n$. We say a partition of n is a partition into *d-distinct parts* if the difference $\lambda_i - \lambda_{i-1} \geq d$ for all i, and when $d = 1$ we say *distinct* (instead of 1-distinct). Each partition has a *Ferrers diagram*, which represents the partition as a left-justified array of rows of cells, where the length of the first row corresponds to λ_1, the length of the second row corresponds to λ_2, etc.. For each cell u in a

Ferrers diagram, we can assign a hook length, which counts the cell itself, all of the cells to its right, and all the cells below.

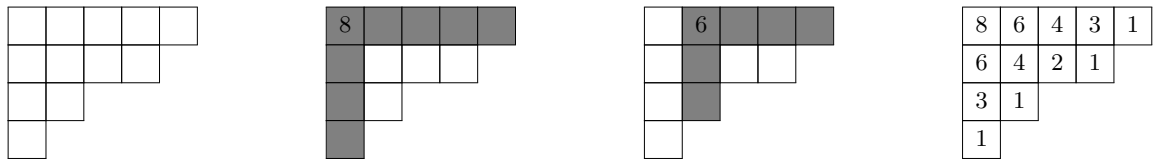


Figure 1.1: Hook lengths in the Ferrers diagram of the partition $(5, 4, 2, 1)$.

A partition is *t-core* if the integer t does not appear as a hook length for any of the cells in the partition's Ferrers diagram. The example in Figure 1.1 is simultaneously 5-core and 7-core, and we would call the partition $(5, 7)$-core. Questions related to counting certain simultaneous core partitions have been of interest since Anderson [4] showed that, for s and t coprime, the number of (s, t)-core partitions is

$$\frac{1}{s+t}\binom{s+t}{s}.$$

When $gcd(s, t) > 1$, there are infinitely many (s, t)-core partitions. Enumerating (s, t)-core partitions into distinct parts remains an open problem. Following this, in the last several years, there has been a surge of interest in simultaneous core partitions, including [6], [23], [27], [31], [33], and [41].

Let $\mathcal{C}^d_{s,m,r}$ denote the set of $(s, ms + r)$-core partitions into d-distinct parts. We use $\nu(\lambda)$ to denote the number of parts of a partition λ and define

$$\boldsymbol{C}^d_{s,m,r}(q) = \sum_{\lambda \in \mathcal{C}^d_{s,m,r}} q^{\nu(\lambda)},$$

to be the generating polynomial for the number of parts of the partitions in $\mathcal{C}^d_{s,m,r}$.

In Chapter 3, we introduce s-core abaci as an alternative diagram for representing core partitions (as defined in [27]). An *s-abacus* is an array consisting of s columns, labelled 0, 1, 2, ..., $s - 1$, and some number of rows, labelled 0, 1, 2, ..., where each entry is either occupied by a bead or unoccupied (by a spacer). The entry in row i and column j is said to be in position $is + j$. The s-abacus corresponding to a partition λ is obtained by placing the set of first-column hook lengths on an s-abacus. We say an s-abacus is an *s-core abacus* if the first column is empty, and there are no spacers below any bead.

The partition $\lambda = (5, 4, 2, 1)$ has first column hook lengths $\{1, 3, 6, 8\}$, as illustrated in Figure 1.1. Figure 1.2 shows the partition $(5, 4, 2, 1)$ as a 5-core abacus diagram, and a 7-core abacus diagram,

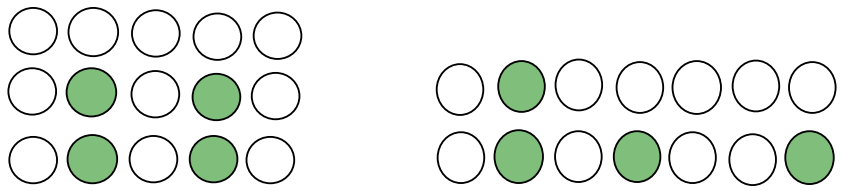


Figure 1.2: 5-core and 7-core abacus diagrams for the partition $(5,\ 4,\ 2,\ 1)$.

The number of parts of λ is equal to the number of beads in a corresponding abacus A, and we write $n(A)$ to represent the number of beads in A. We say an abacus has *spacing* d if there are at least d spacers between the positions of any two beads. Both diagrams in Figure 1.2 have spacing 1.

Let $\boldsymbol{A}^d_{s,m,r}(q)$ be the generating polynomial for the number of beads for the set $\mathcal{A}^d_{s,m,r}$ of s-core abacus diagrams that have spacing d and maximum position less than $ms+r$. If $r=-1$, we show that,

$$\boldsymbol{C}^d_{s,m,-1}(q) = \boldsymbol{A}^d_{s,m,-1}(q),$$

and give a bijection between the set $\mathcal{C}^d_{s,m,-1}$ of $(s, ms-1)$-core partitions into d-distinct parts and the set $\mathcal{A}^d_{s,m,-1}$ of s-core abaci that have spacing d and maximum position less than $ms-1$. There is a similar bijection when $1 \leq r \leq d$ as shown in [8].

In [8], Hannah Burson proves the following theorem about abaci.

Theorem. *Let $d, m \geq 1$, and $r \geq 0$. If $s > d+1$ and $s > r$, then*

$$\boldsymbol{A}^d_{s,m,r}(q) = \boldsymbol{A}^d_{s-1,m,r}(q) + (q+q^2+\ldots+q^m)\boldsymbol{A}^d_{s-d-1,m,r}(q),$$

In the case $1 \leq r \leq d$, this gives a recurrence for $\boldsymbol{C}^d_{s,m,r}(q)$.

In Chapter 4 (Theorem 32), we show that the same recurrence holds for negative values of r, with modified conditions on the variables.

Theorem. *Let $d, m \geq 1$, $r > 0$, and write $f_s(q) = \boldsymbol{A}^d_{s,m,-r}(q)$. If $s > 2d+r$, then*

$$\boldsymbol{A}^d_{s,m,-r}(q) = \boldsymbol{A}^d_{s-1,m,-r}(q) + (q+q^2+\ldots+q^m)\boldsymbol{A}^d_{s-d-1,m,r}(q).$$

In the special case $r=-1$, this gives the following statement (Theorem 37) about core partitions.

Theorem. *Let $m, d \geq 1$ be integers. Then, for $s > 2d+1$ (or $s > 2$, if $d = 1$), we have*

$$\boldsymbol{C}^d_{s,m,-1}(q) = \boldsymbol{C}^d_{s-1,m,-1}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{C}^d_{s-d-1,m,-1}(q).$$

This gives a generalization of the special case $(d, r, q) = (1, 1, -1)$, which was proved independently by Nath and Sellers [27] and Straub [37].

1.1.3 Integer Quotients from Restricted Digit Sets

Every natural number can be written in base b with digits from the set $\{0, 1, \ldots, b-1\}$. Suppose we choose a restricted set of digits, $\{d_1, \ldots, d_k\}$, where $k < b$. We consider the following set of integers with nonstandard representations using these digits,

$$S(b; \{d_1, \ldots, d_k\}) := \left\{ s \in \mathbb{N} : s = \sum_{i=0}^{\infty} \alpha_i b^i, \qquad \text{with } \alpha_i \in \{d_1, \ldots, d_k\} \text{ for all } i \right\}.$$

Sets of integers with "nonstandard" representations have been studied extensively, as summarized in [14]. Less is known about integers that can be represented as quotients of these sets,

$$Q(b; \{d_1, \ldots d_k\}) := \{x \in \mathbb{Z} : x = s/s' \text{ for some } s, s' \in S(b; \{d_1, \ldots, d_k\})\}.$$

Chapters 5 through 8 contain results about $Q(b; \{d_1, \ldots d_k\})$ for different bases b and restricted digit sets $\{d_1, \ldots, d_k\}$, with $d_1 = 0$, using tools from number theory, theoretical computer science, and graph theory.

Modified Multiplication Transducers

A *finite automaton* is a theoretical computing machine with a finite number of states, some subset of which are "accepting" states. The automaton reads a finite string of inputs, and transitions between states based on the input read. The string is *accepted* if the automaton is in an "accepting" state at the end of the process. A finite state automaton is *deterministic* if it always produces the same result for a given input string. It is useful to represent an automaton as a labeled, directed graph.

Transducers are automata which read an input string, and then write an output string that depends on the input string. A transducer can be thought of as a "translating machine"; it translates the input string into an output string. To describe a transducer, we require three finite alphabets: an input alphabet, an output alphabet, and a set of states.

In Chapter 5, we define multiplication transducers, transducers that perform base b multiplication, as described in [7]. For each positive integer m, we create a transducer in base b that computes $mr = s$ for an input string r (essentially, "translating" the input r to the output $mr = s$). The transducer $\mathcal{T}_{m,b}$ which multiplies by m in base b has the set $\{0, 1, \ldots b-1\}$ as both the input and output alphabet, and $\{0, 1, \ldots, m-1\}$ as the set of states. The movement between states is constructed in the following way: in state c, reading input a, the machine writes the digit $k \in \{0, \ldots, b-1\}$ that satisfies $ma + c \equiv k \pmod{b}$ and goes to state $c' = \left[\frac{ma+c}{b}\right]$, where $[x]$ represents the greatest integer less than or equal to x.

The transducer that multiplies by m in base b allows us to compute multiplication $m \cdot r$ for any positive integer r. To compute multiplication by integers with restricted digit sets in base b, we remove the paths that read and write the forbidden digits. This produces a transducer with the restricted digit set $\{d_1, \ldots d_k\}$ as both the input and output alphabets. The computation always begins in state 0.

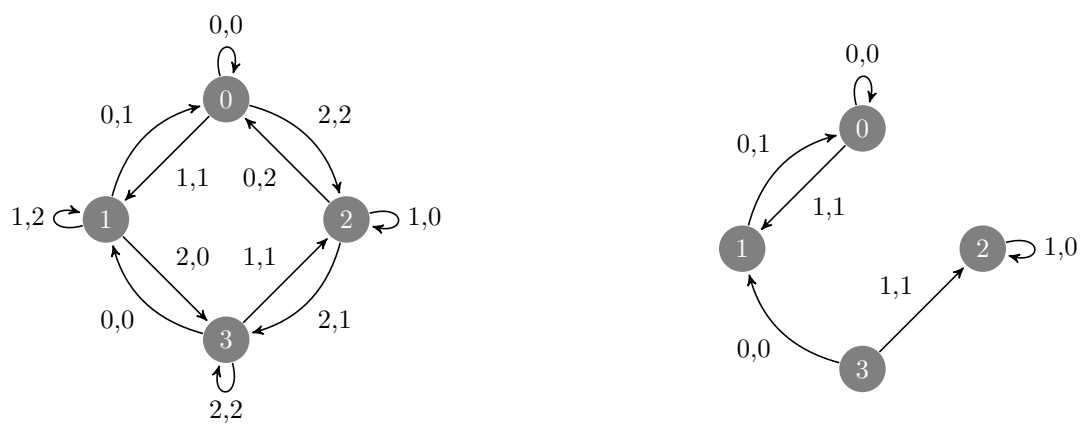


Figure 1.3: $\mathcal{T}_{4,3}$ (left) and modified $\mathcal{T}_{4,3}$ with digit 2 forbidden (right)

The left image in Figure 1.3 is the complete transducer $\mathcal{T}_{4,3}$ that multiplies by 4 in base 3. This transducer can read any nonnegative integer r (written in base 3) and write the product $4r$ in base 3. The labels on each edge correspond to the digits read and written, respectively, when following that path. For example, to multiply 7 by 4 (in base 3 using $\mathcal{T}_{4,3}$), we would look at the base 3 representation of 7, $[21]_3$, and follow the edges that read these digits from right to left. So, the first step would read 1, write 1, and move to state 1 (following the path below left of state 0). The next step would read 2, write 0 and move to state 3. Then, we would read 0 for as many steps as we need to return to the initial state, 0, moving back to state 1 and then state 0. The complete string written would be $[1001]_3 = 27 + 1 = 28$, which is indeed the product $4 \cdot 7$.

The right image in Figure 1.3 is the modified transducer when we restrict to only permit the digits $\{0, 1\}$ in

base 3, deleting all edges that either read or write the digit 2.

Quotients in Base Three

In Chapter 6, we focus on the base three case. In base three, restricting to the digits $\{0,1\}$, $S(3;\{0,1\})$ is the set of sums of distinct powers of 3. The set $Q(3;\{0,1\})$, integers that have representations as quotients of elements of $S(3;\{0,1\})$, have a natural connection to the Cantor Set. The results in this chapter use modified multiplication transducers to extend recent work on division in the Cantor Set [5].

Quotients of sums of distinct powers of three are exactly quotients of nonzero left-hand endpoints of the Cantor Set (defined to be the left-hand endpoint of any interval created in the "middle-thirds" definition of the Cantor Set). It is shown in [5] that the set of quotients of elements of the Cantor Set is contained in $\bigcup_{j\in\mathbb{Z}}\left[\frac{2}{3}\cdot 3^j, \frac{3}{2}\cdot 3^j\right]$. Since elements of $S(3;\{0,1\})$ are congruent to 0 or 1 $\pmod 3$, elements of $Q(3;\{0,1\})$ must also be congruent to 0 or 1 $\pmod 3$. If $x \in Q(3;\{0,1\})$ and $x \equiv 0 \pmod 3$, then $x = 3^i k$ for some $k \equiv 1 \pmod 3$. Thus, the problem reduces to integers $m \equiv 1 \pmod 3$, with $m \in \bigcup_{j\in\mathbb{Z}}\left[\frac{2}{3}\cdot 3^j, \frac{3}{2}\cdot 3^j\right]$.

The conditions above are necessary but not sufficient. Let A be the set of $m \equiv 1 \pmod 3$, with $m \in \bigcup_{j\in\mathbb{Z}}\left[\frac{2}{3}\cdot 3^j, \frac{3}{2}\cdot 3^j\right]$. Using SAGE computations that generate transducers for each such m, I have computationally checked that, up to 6200000, the only $m \in A$ with $m \notin Q(3;\{0,1\})$ are: 529, 592, 601, 616, 5368, 50281, 4072741, 4074361, 4088941, and 4245688. This was independently shown by Sajed Haque [17], who also found 37884151 to be the only other element of A not in S/S between 6200000 and 107883526. These computational results, led to the following conjecture.

Conjecture (Haque)**.** *Ternary representations of the form* $211((2222)^k)021$ *are not represented in* Q*.*

In addition to the computational results described above, Chapter 6 contains commentary on the structure of the transducers as graphs.

Quotients in Base Four

In Chapter 7, we apply transducers to the sets $Q(4;\{0,d_1,d_2\})$, where $d_1, d_2 \in \{1,2,3\}$, proving the following theorems (respectively, Theorem 73 and Theorem 75).

Theorem. $Q(4;\{0,2,3\}) \subseteq \mathbb{N} \setminus \{2^{2k+1} : k \in \mathbb{N}\}$.

Theorem. $Q(4;\{0,1,3\}) \subseteq \mathbb{N} \setminus \{2^{2j+1}(2\ell-1) : \ell, j \in \mathbb{N}\}$.

We conjecture that these statements are actually equalities, and the conjectures (respectively Conjecture 74, Conjecture 78, and Conjecture 79) below are supported by computational data up to 20,000, 35,000 and

35,000.

Conjecture. $Q(4;\{0,2,3\}) = \mathbb{N} \setminus \ \{2^{2k+1} : k \in \mathbb{N}\}$.

Conjecture. $Q(4;\{0,1,3\}) = \mathbb{N} \setminus \ \{2^{2j+1}(2\ell - 1) : \ell, j \in \mathbb{N}\}$.

Conjecture. $Q(4;\{0,1,2\}) = \mathbb{N}$.

In 1987, J. H. Loxton and A. J. van der Poorten published a paper entitled "An awful problem about integers in base four," in which they proved that every odd integer can be written as a quotient of numbers which can be represented using the digits 0, 1, and -1 in base four; that is, $2\mathbb{Z} + 1 \subseteq Q(4;\{-1,0,1\})$ [25]. This is the extent of references to this problem in the mathematics literature that we know of, and has been referenced periodically in relationship to related problems in theoretical computer science. Both [15] and [32] give examples of using such theoretical computing machines for problems related to the "awful problem," [25]. In [32], Rowland and Shallit introduce a theory of k-automatic sets of rational numbers using deterministic finite state automata that read pairs of integer inputs to determine if a quotient is accepted or rejected. In [15], Gonzalez defines a k-free set, and proves a condition for sets of size k chosen from $\mathbb{N}$ to be k-free, generalizing the result of [25].

Conjecture 78 implies that $Q(4;\{0,1,3\}) = Q(4;\{-1,0,1\})$, as the latter is described in [25]. This connection to the "awful problem" is rather surprising, since the transducers associated to the same element in each set are not isomorphic.

Quotients in Base 5 and Beyond

In Chapter 8, we begin by quotients in base 5 with three digits (including 0), developing the following conjectures (Conjectures 90 – 93) from computational data.

Conjecture. $Q(5;\{0,1,2\}) = \mathbb{N} \setminus \{n \in \mathbb{N} : n \equiv 4 \pmod 5\}$.

Conjecture. $Q(5;\{0,1,3\}) \subset Q(5;\{0,1,2\})$.

Conjecture. $Q(5;\{0,3,4\}) \subset Q(5;\{0,1,2\})$.

Conjecture. $Q(5;\{0,1,4\}) \subset Q(5;\{0,2,3\})$.

We then present the following conjecture for general $b > 4$ (Conjecture 96), which was inspired by the initial study of $b = 5$.

Conjecture. *For all bases $b > 4$,*

$$Q(b;\{0,d_1,d_2,\ldots,d_{b-2}\}) = \mathbb{N}$$

for any choice of $d_1, d_2, \ldots, d_{b-2}$ in $\{1,2,\ldots,(b-1)\}$

This conjecture is supported by computational data for bases $5 \leq b \leq 9$ up to $m = 2000$ (higher for the smaller bases), and if true would imply that, for example, every natural number is a quotient of two integers that do not include the digit "9". For example, even though 2019 uses a "9", we have $2019 = \frac{4038}{2}$, and neither 4038 or 2 uses the digit 9.

Continuing our exploration of general $b > 4$, we prove the following result (Theorem 98).

Theorem. *If an integer m is in $Q(b; \{-1, 0, 1\})$, then*

$$b^k \cdot \frac{b-2}{b} < m < b^k \cdot \frac{b}{b-2},$$

for some positive k.

This has implications for analogues of the "awful problem" in bases $b > 4$. Chapter 8 concludes with connections to polynomials, other approaches, and anticipated future work.

1.2 Positioning Myself

Following recent conversations within mathematics education research communities about the benefits of positioning oneself within research [12], I have chosen to include in this introduction some commentary on my goals for this document. The necessity of examining the role of the researcher's position is founded in the acknowledgement that all research is political and subjective. We make choices about what questions to explore, what results are important, and what information is worthy of sharing. These choices are shaped by our personal perspectives, as well as external influences including the histories of our disciplines within academic communities.

I came to mathematics research as a teacher first. I decided I wanted to be a math teacher after my junior year in high school, and chose to attend college at the University of Maine at Farmington as a secondary education major with a concentration in mathematics. Inspired by my upper-division college mathematics courses, I later chose to complete a second degree in mathematics. My first research project, [22], was in mathematics education, with my first mathematics research experience coming the following summer at the Mathematical Sciences Research Institute's Summer 2013 Undergraduate Program. Throughout my undergraduate and graduate education, I have been both a teacher and a student of mathematics.

My goals are rooted in a desire to break down barriers and make mathematics more inclusive. The things that make me a mathematician: what I like to think about and how I think about it, are shaped by my

identity and values. My hope in writing this document is that it creates windows and mirrors[1]. By writing sections without the presumption of graduate-level coursework in mathematics, I aim to create windows into mathematics research for folks who might not identify as mathematicians or as part of the formal mathematics research community, and mirrors to show how the formal mathematics research process is similar to other mathematics learning experiences outside of academic research environments. By including commentary on diagramming and gesture as an element of mathematical practice, I hope to offer a window into my process and an opportunity for readers to see where their process might mirror mine. I hope this will lead to further conversations between mathematicians about how we think about mathematics similarly or differently. This is not necessarily an either/or; I invite you to think about places where you might identify with part, but not all of a process.

I enjoy working on problems at the intersection of combinatorics and number theory that have significant diagrammatic, enumerative, and computational components. A motivating factor in the research I do, and why it interests me, is accessibility. I love working with positive integers, because they are mathematical objects that are familiar to people outside of the mathematics community. I chose to study combinatorial number theory explicitly because the problems have entry points at all levels of mathematical experience. To illustrate the accessibility of the mathematical content in this document, I have written sections to be accessible for a variety of audiences.

The first sections of Chapter 3 were written with my nephew in mind (age 10). The final section of Chapter 3 and the entirety of Chapter 4 were written for students I've had the opportunity to work with as an Education Justice Project member, most of whom have had formal instruction through Calculus II, supplemented with the informal conversations of a vibrant academic community. Sections of Chapter 5 were written for my nieces (ages 9 and 12). Chapters 6-8 were written for an advanced undergraduate audience, where students might have some familiarity with reading and writing proofs, equivalence classes, and modular arithmetic; there are a variety of people in my life who fit this description and who I had in mind when writing these chapters.

[1] I use the metaphor of windows and mirrors as in [16], "quality curricula [includes] a window and a mirror – a mirror in the sense of offering students a chance to see oneself; a window in the sense of being able to see a different view onto the world," (p. 44)

Chapter 2

Diagramming and Gesture

In this chapter, we provide context for the choice of diagramming and gesture as a lens, followed by reviews of the literature on embodied cognition, diagram, and gesture. We conclude with a description of diagramming and gesture, examples of specific diagramming and gesture practices, and describe how this lens is applied in the remaining chapters.

2.1 The Front and the Back of Mathematics

"Mathematics has a front and a back." – Reuben Hersh.

This statement, from his 1991 article, conveys the idea that mathematics presented in public differs from mathematics practiced in private [20]. The idea is borrowed from sociologist Erving Goffman, who described separations between public and private regions of various social institutions, and extended by Hersh to the social institution of mathematics. The "front" and "back" in mathematics can be thought of as analogous to the separation between the theater auditorium and backstage. A consequence of this separation in mathematics is that, for many outside of the academic mathematics community, the process of mathematics - messy, subjective, and the result of extensive practice - is obscured by the polished product.

The "front" of mathematics, including most presentations, textbooks, papers, and PhD dissertations, often emphasizes brevity, abstraction, and the appearance of objectivity. Part of what we are trained to do in graduate mathematics classrooms and formal research experiences is to tease apart the language, fill in the gaps, and understand the work that went into it. We also practice producing public mathematics, through written assignments, exams, and presentations. We learn what work we are meant to show, when instructed to "show work." If you tried something and it did not work, it does not go in the finished product.

For about five years, I was involved in preparing and staging a wide variety of theater productions. I was most often the stage manager, though I occasionally worked with light design or moving sets. As a stage

manager, I attended all of the rehearsals, watching and recording the blocking (where actors were meant to stand during a scene). I took all of these notes in pencil, so they could be erased and replaced as the play evolved. Once we got closer to performance time, we worked with other technical folks to hang and focus lights, tested sound effects, and planned how to move around set and prop pieces during the show. Often, we would spend an hour hanging and focusing one light to be used during one scene.

Since I spent so much time involved in various aspects of this process, when I see a theater production, I cannot help but notice technical elements that I probably would not have noticed before. Even when I am in the audience, my experience of being backstage influences my perspective on the new theater experience. I was a stage manager, so there are many aspects of my experience that I would expect to resonate with other stage managers. Just as each person involved in those productions had a different experience, and a different perspective, people involved in the production of mathematics have a wide range of experiences, that may share some common threads. As an audience for mathematics (for example, in a seminar talk), we can sometimes get glimpses of these perspectives in the way math is presented (e.g. the choice of definitions and examples made by the presenter), or in the different types of questions that are asked. However, these nuances are often only visible to those familiar with the subject, who might know that a choice is being made.

One goal of this document is to highlight a small corner of my experience of the "back" of my mathematical process. The mathematics presented in this document has been shaped by the people I have worked with and learned from, the books and articles I have read, and the formal coursework I have taken. It is also influenced by my personal history, values, and lived experiences. In the next section, we will talk more formally about this personal relationship to mathematics and how it interacts with the "front" and the "back."

Ideational and Conceptual Mathematics

In [35], Schiralli and Sinclair introduce a distinction in the use of the word mathematics. When referring to mathematics as a subject matter or discipline, they use the term *conceptual mathematics* (CM). They state that (CM) is "in its core purposes a public activity, an ongoing game in progress, whose rules are continuously negotiated as shared (even if not perfectly shared) meanings among the participants," (p. 81). Conceptual mathematics contrasts with *ideational mathematics* (IM), a term which includes the internal representations that individuals make for themselves about mathematical concepts.

Definition 1. *Conceptual Mathematics (CM): Mathematics as a subject matter or discipline, including mathematical concepts presented publicly that conform to logical patterns and rules established by the math-*

ematical research community.

Definition 2. *Ideational Mathematics (IM): Individuals' internal representations of mathematical concepts that are grounded in the individual's experience.*

This framework makes a distinction between mathematical concepts as "publicly accessible tool[s]," (CM), with the mathematical ideas that individual mathematicians may form of them (IM). In the example below, I compare the CM definition with my own IM ideas about the concept of evenness of integers.

Example 3. *CM: An integer m is even if there exists some integer k such that $m = 2k$.*

IM: An internal image of breaking a set of objects into two even groups, counting by twos, or thinking about 'every other' integer on a number line.

The internal representation I have for evenness may be consistent with, or different from the representations you have (e.g. maybe I imagine two columns, and you picture two rows). Many of us were encouraged to develop IM related to even integers, to play and practice with manipulatives. Different concrete representations of what evenness represents are included as part of CM in elementary texts, curricula, etc..

An example of IM for a more abstract math concept can be seen in [34]. The authors interviewed six mathematicians, and analyzed their response to the question: "What do you think of what I say the word 'eigenvector'?", followed by the formal definition, which is stated in abstract, symbolic terms. This concept is taught in introductory linear algebra courses. The authors chose this question in particular because they conjectured that it would highlight the ways in which mathematicians' ways of thinking about the concept would be different than how it is presented in the textbook. One mathematician described vectors as arrows (hands) on a clock, another used ellipses as a representation, and another talked about stresses on plates. All of them used gestures to help illustrate their respective interpretations.

This attempt to highlight the mathematicians' IM, in contrast to the CM of the formal definition, was particularly powerful because the mathematicians surveyed had distinctly different perspectives. As someone who had a great deal of difficulty understanding the CM concept of eigenvector, I realized in reading this paper that I was not encouraged to develop my own IM understanding of eigenvector, and was not shown examples of others' IM interpretations of eigenvectors as a student. I had merely memorized the definition and learned to apply it when necessary.

IM and CM in the "Front" and the "Back"

How does this connect to the "front" and the "back" of mathematics? While there are explicit ways that CM lives in the "front," through mathematics performed publicly (written and oral), IM does not encompass the entirety of the "back". Viewing the "front" and "back" of mathematics as regions of the social institution of mathematics, I see the negotiation, development, and publication of CM as a process that lives primarily in the "back." All individuals, not just academic research mathematicians, develop IM for themselves. The distinction between CM and IM happens at the level of the individual; whether the individual has access to the "back," and how much access, influences the development of their IM and their relationship to CM.

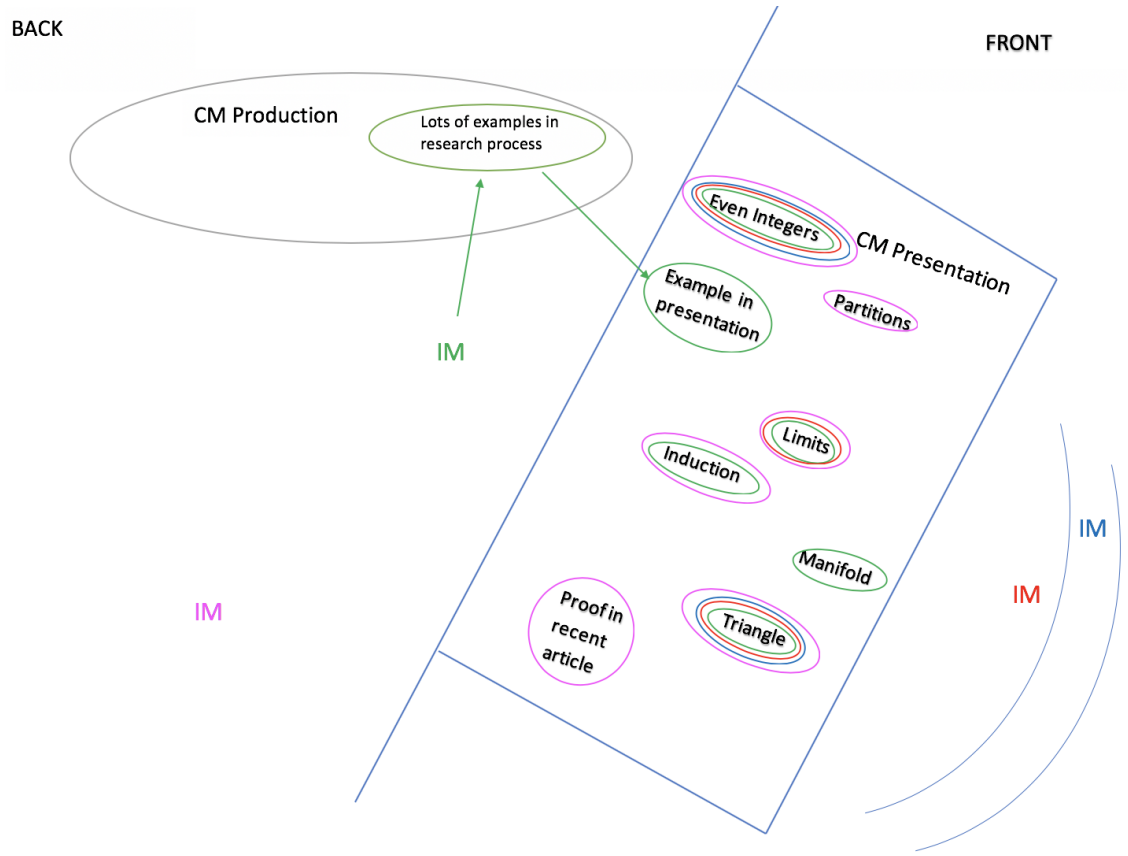


Figure 2.1: CM and IM in the "Front" and "Back"

In Figure 2.1, we illustrate mathematics through the metaphor of an auditorium, where CM ideas are presented on the stage. Depending on their mathematical experience, and the areas they have access to, different individuals may or may not develop IM representations for different CM ideas. In the image, each color represents an individual, some of whom may have access to the "back" and some of whom may not. For example, the individual represented in blue has developed IM regarding the concepts of even integers

and triangles, but not the other examples on the stage.

One thing that we learn with experience in formal math learning environments (especially in graduate mathematics) is to navigate between IM and CM, to develop IM that works for us in navigating the CM of academic papers, textbooks, and presentations. Spending time in the "back" helps us differentiate when a proof is "rigorous," meaning that it would be accepted as CM, and when there are gaps, or aspects that need more justification. A caveat here is that there are subtle changes within the research mathematics community about what is "accepted", and differences between the amount of justification offered by individual mathematicians. This is part of the ongoing negotiation of CM that happens in the "back".

My IM informs how I think about and make sense of the mathematics in this document. Within my own interactions with mathematics, my internal representations of mathematical concepts and their relationships have embodied roots. In the next section, we will introduce some perspectives on embodied cognition in mathematics.

2.2 Embodied Cognition, Diagram, and Gesture

2.2.1 Embodied Roots of Conceptual Mathematics

Embodied cognition is a branch of cognitive study that focuses on the ways that cognitive processes are grounded in physical experiences and interactions with the world. The connection between embodied cognition and mathematics was expanded on by George Lakoff and Rafael E. Nuñez in their book, *Where mathematics comes from* [24]. Lakoff and Nuñez introduce conceptual mechanisms that exist outside of formal mathematics and are also "central to mathematics – especially advanced mathematics – as it is embodied in human beings," (p. 30). We will explore this through an example:

Example 4. *Containment: The concept of containment is grounded in the physical world. We think of an apple in a fruit bowl as being contained in the bowl. In mathematics, we think of 5 as being contained in the set of prime numbers.*

In this example, our understanding of what containment means outside of formal mathematics influences how we think of the set of prime numbers. The idea of set containment is grounded in an embodied notion of what containment means outside of mathematics.

Conceptual mechanisms like this form a foundation for mathematical thinking. Lakoff and Nuñez illustrate how mathematical concepts, from arithmetic to calculus, are grounded in embodied experiences. Some of

these can be thought of as shared metaphors, which form embodied roots for CM.

Perspectives on Embodied Cognition

In [40], Margaret Wilson promotes the idea that "the mind must be understood in the context of its relationship to a physical body that interacts with the world," (p. 625). Wilson provides a review of support for embodied cognition (primarily from cognitive psychology perspectives), and describes several perspectives on embodied cognition. I will use two of these perspectives: that off-line cognition is based in the physical body, and that we off-load cognitive work onto the environment.

We distinguish cognitive function between on-line cognition, which happens in immediate interaction with the environment, and off-line cognition, which happens without direct interaction with the environment. Wilson describes off-line cognition being body based in the following way: "Even when decoupled from the environment, the activity of the mind is grounded in mechanisms that evolved for interaction with the environment – that is, mechanisms of sensory processing and motor control," (p. 626). What this means for me is that when I am sitting and thinking about mathematics, the way I engage with the mathematical concepts has roots in how my body moves through the world – I am drawing on sensory resources. If I am lying in my bed, not moving, the process of thinking is using my body and my lived experience in my body. One example of body-based off-line cognition is counting.

Example 5. *Learning to count: I begin by counting physical objects, then counting on my fingers, then eventually being able to count "in my head." The progression can be see as going from large movements, to smaller movements, to eventually internalized movement.*

While I do not actually remember learning how to count, I imagine that my experience was similar to what I have observed with the kids in my life. If I had developed IM related to counting that was grounded in physical movement, that would have developed my off-line cognition related to counting. Even if I count in my head today, the counting would still be drawing on the same sensory and motor-control resources that I initially used to count.

Another perspective on embodied cognition, as described in [40] is cognitive off-loading, which is a way of altering the environment to reduce cognitive work. When trying to solve a problem, rather than trying to hold all of the details mentally, we off-load some of the information onto the environment around us. One common form of cognitive off-loading is through using physical gestures, for example gesturing when trying to figure out how to pack boxes into the trunk of a car, or fit tupperware containers into a fridge. Another form of off-loading is writing notes or diagrams to keep track of information, for example drawing out a

calendar when making a plan that involves several peoples' schedules.

2.2.2 Diagram and Gesture

I analyze my mathematics research through my process of diagramming and gesture. There is one body of literature that focuses on diagrams, and another that focuses on gesture. A third line argues that gesture and diagramming are inherently linked. In this section, I will address each individually and then discuss the theoretical background that considers diagramming and gesture as an intertwined process.

Diagrams

Diagrams, as defined by Chu et. al. in [11], are schematic visual representations that express information via spatial relationships. Spatial diagrams can be used both in algebraic story problems and in symbolic equations. In a story problem, diagrams are used to illustrate a physical process, or a story, for example to show the distance between cars that are driving away from one another. Chu et. al. show that providing diagrams can also have positive effects on students' solving of symbolic equations (that do not have a clear physical representation) [11]. While many studies focus on the diagrams themselves, analyzing their structure ([1] and [29]), or focusing on the effectiveness of providing diagrams ([11]), there are fewer that look at spontaneous use of diagram in problem solving.

Abstract diagrams can also be used as tools for organizing information and problem solving. Novick has studied three interrelated spatial diagrams: hierarchies, matrices, and networks, extensively, noting specific ways they are able to convey information about relationships between objects [29]. For example, if the relationships are directional, a hierarchical diagram would convey this aspect of the relationship better than a matrix.

Diagrams have also been studied extensively in the context of student learning of geometry. Alshwaikh studied the role of diagrams in proof in Euclidan geometry, borrowing from a framework from social semiotics to classify geometric diagrams as either narrative or conceptual [1]. Narrative diagrams employ a temporal component, illustrating a story that happens over time. The author notes that narrative diagrams tend to have the following structures: arrowed, dotted, shaded, a sequence of diagrams, or construction. Conceptual diagrams use their structure to highlight or illustrate concepts, by using identifying, spatial (portion/size), and classificational structures. Their analysis explores diagrams in textbooks, and discuss the role of the human in looking at geometrical diagrams.

In addition to looking at a diagram in a book, there are other ways humans interact with a given diagram. Herbst introduces several different modes of human interaction with diagrams, again using examples from geometry, [19]. In these examples, the diagrams represent known geometric objects. One mode of interaction that we will consider is the generative mode, which involves engaging with the diagram by manipulating, adding, or hypothesizing beyond what is given (e.g. "if I move this, then...").

I want to differentiate here between the word diagram as a noun (an object to be used), and diagramming as a verb. We will define diagramming to be the act of engaging with a diagram, either through observation, or by creating or manipulating a diagram.

There are certain limitations to studying diagrams. The process of diagramming is time-consuming; there is a lag, or delay before a diagram is complete. The diagram comes into physical being through deliberate action, which means that study of diagrams may fail to capture spontaneous thinking before, during, and after the process of putting pen to paper. Though it has been said that a picture is worth a thousand words, diagrams often miss capturing depth and motion. In [1], Alshwaikh shows how some diagrams highlight elements of motion, however the diagrams themselves cannot account for what [19] calls generative interaction.

Gesture

In the English language we often use the word gesture to describe physical movements made to express or emphasize information during interpersonal communication. While some math education researchers consider gestures to be "hand and arm movements that accompany speech," (p. 249, [39]), in other parts of the literature, the definition of gesture can be much more broad.

While we often think of gesture as serving communication with others, it also has self-oriented functions, which serve the individual performing the gesture. Kita, Alibali, and Chu provide a survey that analyzes evidence for self-oriented functions of gesture in [21]. Most of their evidence comes from studies that experimentally manipulate gesture production (e.g. by creating situations in which gesture is prohibited or restricted, and comparing to a group without the restrictions). They found that gesture helps us activate, manipulate, package, and explore spatio-motoric information.

One function of gesture is to off-load information onto the environment, especially when the information is complex. One example of this is that people tend to use more gesture when describing something from memory than when there is visible related stimulus. People also tend to use more gesture when working with motoric information. It has been shown that experience manipulating objects in the real world can

influence gesture about objects that we cannot physically manipulate (for example, tectonic plates). Gesture can schematize this information, which helps us generalize information to new contexts [21].

Nemirovsky and Ferrara include gesture within the definition of utterances (a broader category that includes bodily activity such as facial expression, sound production and eye motion), and make the distinction between "overt" and "covert" aspects of utterance [28]. Since we are focusing on gesture, we will use the term overt gesture to describe physically noticeable bodily movement, while covert gesture is the internal component. If we think about pantomiming throwing a ball, we activate the part of the central nervous system that would initialize that overt gesture. This distinction between overt and covert gesture shows how we can consider gesture as encompassing both physical movement and some forms of body-based cognition.

Like in [39] and [28], most of the literature on gesture focuses on overt gesture, which is the aspect that is observable and recordable, and uses problems that have a strong geometric foundation (e.g. proofs involving triangles). Gesture is difficult to analyze in part because, with the exception of video recording, it is inherently impermanent; as such, the role of gesture in mathematical reasoning is a fairly recent area of study in mathematics education. Combining this with theories of covert gesture and the fact that gesture is rarely included in written communication, we see some of the limitations of analyzing gesture.

The role that gesture plays in problem solving can be significant as shown in [39]. In [26], the authors examine gestures and speech of graduate students working collaboratively to prove a theorem on a chalkboard. In both of these cases, the gestures examined are linked to concepts that have CM visual representations (triangles, real-valued functions). The gestures provide different information about the mathematical thinking being analyzed than diagrams. The concept of gesture addresses some of the limitations we see in the study of diagram, and studying diagrams can likewise enhance our conversations about gesture.

2.3 Diagramming and Gesture

My perspective on the process of diagramming and gesture aligns with the perspective of de Freitas and Sinclair, as described in [13]. They use the work of philosopher Giles Châtelet, [10], to highlight the relationship between diagramming and gesture in mathematical practice. In the field of education, we often talk about "teaching and learning" as a complete phrase; it is difficult to talk about teaching without considering learning, and vice versa. Even in independent learning we can think about texts, or other aspects of the environment, as performing the teaching role. In much the same way, de Freitas and Sinclair view diagramming and gesture as connected and complementary.

Both diagrams and gestures have mobility and potentiality. Mobility refers to motion, or virtual motion, and potentiality is the inherent capacity for growth, development, or coming into being. The mobility in a diagram can include ways that the diagram encodes motion (for example through dotted lines or arrows to indicate movement or the passage of time), as well as the potential for movement in changing or restructuring the diagram. The mobility in a gesture is the movement of the gesture. Diagrams and gestures hold within them a potential to grow and develop through human interaction, to be reimagined and remade anew. In the words of de Freitas and Sinclair in [13], "gestures give rise to the very possibility of diagramming, and diagrams give rise to new possibilities for gesturing," (p. 137).

The relationship between gestures and diagrams is complementary; Châtelet [10] describes a diagram as capturing a gesture "mid-flight" (p. 10). In this way, diagramming provides a snapshot that shows some aspects of what is encoded in a gesture. In addition to preserving these aspects of a gesture, it can act as cognitive off-loading of information, creating space for engaging with other aspects of the concept. Similarly, gesture offers a way to interact with diagrams: imagining new ways to move, change, add to and subtract from what is there.

Due to this complementary nature, when we are engaged in the process of problem solving, we may bounce back and forth between diagramming and gesturing; it is not clear where one ends and the other begins. In my own work, I notice myself starting with a gesture, moving to start drawing a diagram, abandoning it midway through, using gesture with the incomplete diagram and beginning a new diagram, moving back and forth between the two. Thus, following the tradition of Châtelet, de Freitas, and Sinclair I choose to use the phrase "diagramming and gesture" to represent this process.

Definition 6. *Diagramming and gesture: The process of engaging with an embodied concept through the complementary practices of diagramming and gesture.*

One instance of this relationship plays out in mathematicians' engagement with mathematical concepts highlighted in [34], the study on mathematicians' IM understanding of eigenvectors. They found that the mathematicians employed gesture and diagram to explain the concept, and used several different embodied ideas and metaphors to explain the concept. These authors, again using Châtelet as a base, describe gesture as a "core mediating link between the body and the mathematical diagram," (p. 236).

To analyze diagrams in this framework of diagramming and gesture, de Freitas and Sinclair define five techniques to communicate temporal and mobile dimensions of a concept through diagram: successive framing, dotted lines, perspective, arrows, and shading [13]. These are similar to the characteristics of narrative diagrams proposed by [1], but their lens includes the ideational perspective and internal gestural

representations of the different mathematicians. This analysis was done in the context of having a group of students watch a stop-motion film and then draw their interpretation of it.

2.3.1 Diagramming and Gesture Techniques

In this document, I build on the work of de Freitas and Sinclair, adding two more techniques, use of color and nested diagrams, to their five techniques. The use of color adds another dimension to a diagram and can be used to differentiate elements. I define nested diagrams to mean diagrams in which smaller diagrams are themselves elements. These nested diagrams come from engaging with diagrams through gesture by moving the diagrams around (either physically or theoretically) to highlight relationships between them. This results in dynamic diagrams similar to the abstract diagrams analyzed by [29], but whose elements are also dynamic diagrams. I also highlight several diagramming and gesture practices that may involve interaction with the diagrams, but are not visible in the diagrams themselves.

Tables 2.1 and 2.3 list examples of practices that fall under my definition of diagramming and gesture.

Technique	Description
successive framing*	Similar to taking snapshots of a process at different points in time, successive framing includes multiple images, often oriented to display progression over time.
dotted lines*	Dotted lines can be used to display passage of time within a single diagram, flexibility/continuation within the diagram (as ellipses might in written language), or to differentiate between real elements of the diagram from potential elements.
perspective*	Perspective can be used to highlight various parts of the diagram, by rearranging their physical representation on the page.
arrows*	Arrows encapsulate movement within a diagram; they can be used to indicate spatiomotoric or temporal movement, orientation, or the potential for movement.
shading*	Shading can be used to display dimension, construction or removal of elements within the diagram, or to create perspective.
color	Similar to shading, color is used to create contrast between different parts of a diagram, to highlight relationships, or create emphasis.
nested diagrams	Nested diagrams are diagrams whose elements are diagrams in their own right, created by building or arranging the smaller diagrams.

*de Freitas & Sinclair (2012, p. 149)

Table 2.1: **Diagramming and Gesture Techniques Visible in Diagrams**

Some of the practices are from the studies of diagramming, some are from studies of gesture, and some from the studies of diagramming and gesture; they are all linked through the discussion above. In Table 2.1, we list practices that can be seen through analysis of the diagrams. Table 2.2 gives examples of each of these different techniques.

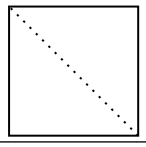
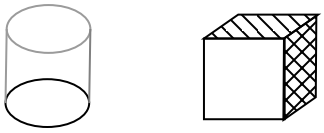
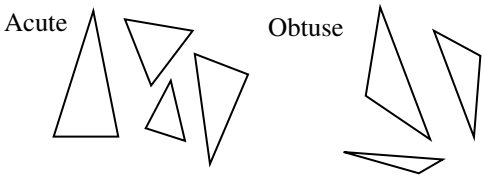
Technique	Example	
successive framing	When illustrating how to approach the line tangent to a curve using secant lines, I might draw successive frames of secant lines that get closer and closer to the tangent.	
dotted lines	If we were to talk about folding a square along a diagonal, dotted lines could be used to illustrate this potential element.	
perspective	Depending on the problem, one might draw a torus (doughnut shape) using different perspective.	
arrows	The first example of an arrow shows change or movement from the open circle to the solid circle. In the second example, the arrow is used to illustrate orientation of the 60° angle.	
shading	To construct a cylinder from the base of a circle, I might draw the added dimension with a lighter shade of line to highlight the difference between the initial circle and the new lines. To illustrate depth in a cube I might shade different faces using hatch marks.	
color	To pair angles in a triangle with their corresponding opposite edges, I might use the same color for an angle and its opposite edge.	
nested diagrams	Individual triangles are diagrams in their own right. When we group them into categories, we have a diagram composed of the smaller triangle diagrams. This is especially relevant in a context where we might be thinking about individual triangles, and then 'zoom out' to see how they relate to each other.	

Table 2.2: **Examples of Diagramming and Gesture Techniques in Diagrams**

Technique	Description
intentional self-oriented gesture	Intentional gesture that is motivated by and serves the individual gesturing (as opposed to serving communication with others), often used to off-load information onto the environment.
spontaneous overt/covert gesture	Overt (visible) or covert (internal, unobservable) gesture that results as spontaneous reaction, often in problem-solving. Spontaneous gesture can be overt or covert, but is unplanned.
exploration of temporal (mobile) dimensions	Using diagramming and gesture to ask and answer questions related to progression of time, such as construction of elements or moving from a designated 'start'.
exploration of spatio-motoric information	Using diagramming and gesture to ask and answer questions about relationships and information that is stored spatially by exploring and manipulating those relationships and information.
organizing information	Using diagramming and gesture to sift and represent information in a way that serves the problem-solving process.

Table 2.3: **Diagramming and Gesture Techniques Not Visible in Diagrams**

Table 2.3 focuses on practices that are not visible in a diagram but are still a part of the diagramming and gesture process; these practices relate to motivation and generative interaction with the diagramming and gesture process. The first two, self-oriented gesture and spontaneous overt or covert gesture, are practices that describe the function and process of gesture. The following three can describe motivation for other diagramming and gesture practices, focusing on how one might choose to interact with a concept through diagramming and gesture to serve a specific purpose.

These techniques are a mixture of ideas from across the literature on diagramming and gesture. I differentiate between intentional self-oriented gesture (planned gesture, where I make a conscious decision to gesture) and spontaneous overt or covert gesture (which is not planned). The concepts of self-oriented gesture and overt/covert utterance (a broader term that includes gesture) are treated without emphasis on intentionality in [21] and [28], respectively.

The examples below show how these techniques might be used. While they have implicit connections to diagrams, the diagrams are not shown because the techniques involve interacting with the ideas of the diagram, not necessarily the physical diagrams themselves.

Technique	Example
intentional self-oriented gesture	"Drawing" the graph of a function with my finger when thinking about the end behavior of that function. This self-oriented gesture is intentional, which
spontaneous overt/covert gesture	When thinking about a sphere expanding or contracting, I might make a ball with my hands and move them in and out to model this (overt). If instead of using my hands, I just think about that same motion it would be a covert gesture.
exploration of temporal (mobile) dimensions	Building a cube by taking depiction of its surface area in two dimensions and folding it into three dimensions, or "unrolling" the cube into the 2-dimensional surface area encodes mobile dimensions of the concept with respect to time. Starting with one representation and then moving toward the other representation.
exploration of spatio-motoric information	Taking a point on a line and thinking about how dragging it along the line might affect the relationships in the diagram.
organizing information	Taking a set of numbers and thinking about physically moving them to group odds and evens together.

Table 2.4: **Examples of Diagramming and Gesture Techniques Not Visible in Diagrams**

These practices are fluid, and often overlap. For example, successive framing in a diagram may be a way of illustrating exploration of a temporal or mobile dimension. I also often find myself using several practices from Table 2.1 within the same diagram, for example using arrows, perspective, and shading to highlight different aspects of the same diagram. Similarly, self-oriented gesture might be motivated by an impulse to organize information.

As these practices appear in the context of the mathematics in this document, I will highlight them using "Diagramming and Gesture" boxes, as shown below.

Diagramming and Gesture

Within each box, I will describe diagramming and gesture techniques used in the immediately preceding mathematical content, with the **specific practices** highlighted in bold.

Table 2.5 indicates the page numbers where these "Diagramming and Gesture" boxes appear, grouped by technique.

Technique	Page Numbers
successive framing	48, 102
dotted lines	40, 164
perspective	97, 109, 122, 127, 155
arrows	92, 97, 120, 134
shading	48, 74, 107,
color	58, 107, 125
nested diagrams	58, 64, 153
intentional self-oriented gesture	33, 92, 164
spontaneous overt/covert gesture	58, 130, 161
exploration of temporal (mobile) dimensions	33, 102, 120
exploration of spatio-motoric information	40, 48, 92, 109, 127
organizing information	58, 64, 97, 134, 161

Table 2.5: Diagramming and Gesture Techniques by Page Number

2.4 Applications to Research Mathematics

Previous research has been focused on individuals' interactions with established CM, through teaching and/or learning. In this document, I extend this work to examine the role of these ideas within the creative process of research. The practice of diagramming and gesture is a core component of my own mathematical sense-making, and part of how I build IM models for abstract concepts. As in the example of eigenvectors, there is power in sharing parts of one's IM publicly alongside the CM when teaching. I hope to demonstrate that this value extends to sharing IM models about new research ideas (which become CM through the research, publication, and presentation process).

One of the things that makes collaborations and discussions in the "back" of mathematics fruitful is that we gain depth by comparing perspectives about the content, because we all experience mathematics differently. Talking formally about using diagramming and gesture in my research is intended to encourage discussions about how this is or is not consistent with others' IM. This is meaningful to me because there is power in naming a thing; it provides a vocabulary for us to engage about how we think about mathematics, which enriches conversations by helping us notice commonalities and appreciate differences.

In Figure 2.2, I layer the information discussed in Sections 2.2 and 2.3 onto our mathematics auditorium. Diagramming and gesture, as described above, is a primary method through which I develop my IM related

to mathematics, and in particular to the research in this document. I engage with my mathematics research (CM production) through diagramming and gesture, much of which happens in the "back". In addition to presenting the original results which are a product of the research process, I also endeavor to show some of my process in the formal CM presentation that is this dissertation.

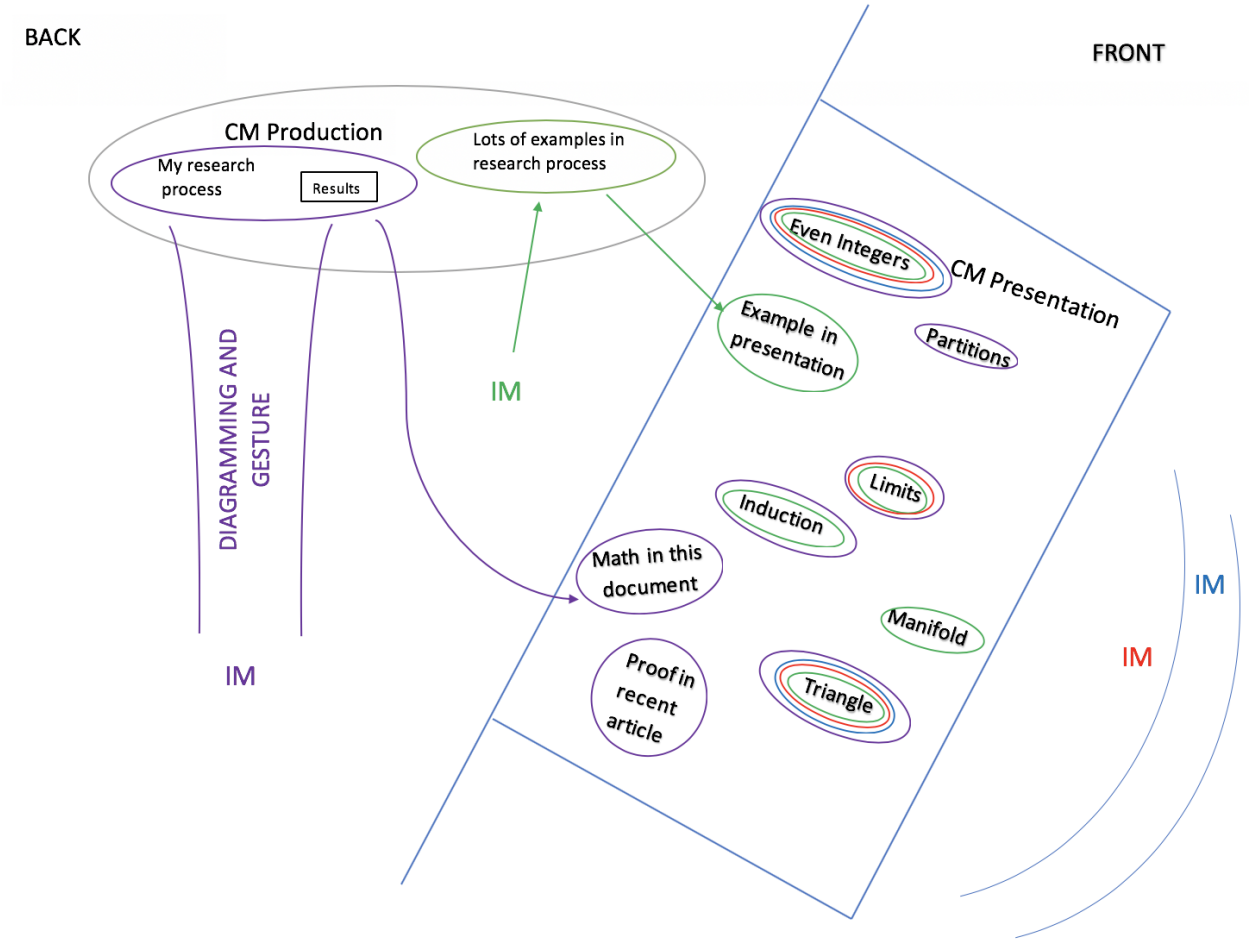


Figure 2.2: Diagramming and Gesture in the "Back"

In the following chapters, I will illustrate ways I used diagramming and gesture in the production of the mathematics being presented. This means that sometimes I will show a couple of different ways I thought of representing an idea, instead of just showing the diagram that eventually made the most sense to me. I see diagramming and gesture as forming a port of entry for me to engage with the mathematics, and a framework for thinking about the ideas. Ultimately, this process is a path, grounded in my embodied experience, that eventually culminates in the formal results. As I present the CM ideas, I will attempt to highlight ways diagramming and gesture contributed to how I developed my own IM about the same ideas.

I offer this document as a proof of concept for illustrating the use of diagramming and gesture as a process to engage with mathematics. I do not claim that this is the only way of seeing the research process; it undoubtedly is not, but rather one of many lenses through which we might approach talking about research. This is intended to be a model for reimagining the performance of mathematics to show a little bit of the "back."

Chapter 3

Introduction to Core Partitions and Abaci

This chapter introduces in more detail definitions that will be used in Chapter 4. Section 3.1 was written for, and read by my nephew (age 9) between his 4th and 5th grade years. Section 3.2 provides more notation and definitions, and was written to provide background for a workshop with Education Justice Project students, most of whom have taken formal courses through Calculus II.

3.1 Definitions

A *partition* is a way of breaking up a whole number into whole number parts.

Example 7. *There are five different partitions of the number four:*

$$4 \qquad 3+1, \qquad 2+2, \qquad 2+1+1, \quad \textit{and} \;\; 1+1+1+1$$

We often like to think about partitions like arranging blocks in rows of a diagram. The rows are lined up so that they match along the left side and the largest row is on top.

Example 8. *The five partitions of four, as diagrams*

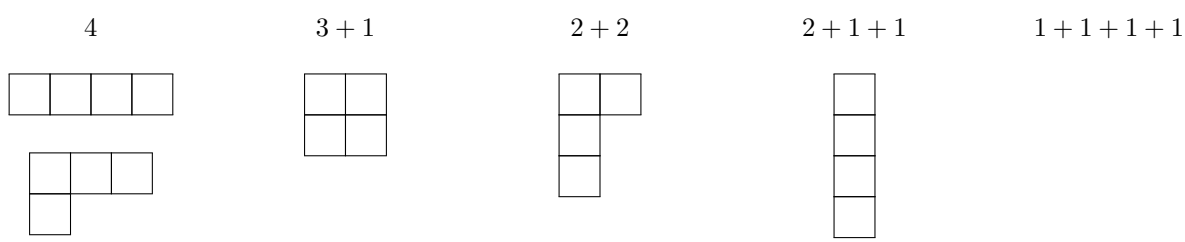


Figure 3.1: Ferrers diagrams for the five partitions of four.

TRY: Can you find the three partitions of the number three and draw their diagrams?

We will focus on partitions into *distinct* parts. This means that there are no repeated parts.

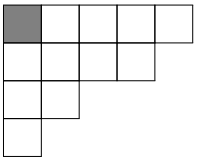
Example 9. *There are only two partitions of four into distinct parts:* 4 *and* $3+1$.

TRY: Which of your partitions of three have distinct parts?

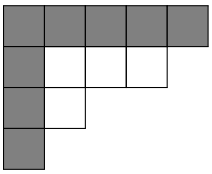
For each diagram, we can write a number in each box that represents the size of the *hook* that has that box in its upper left corner.

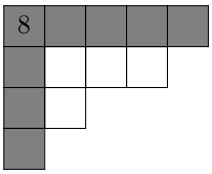
Example 10. *The partition* $5 + 4 + 2 + 1$ *breaks up the number 12. We will find the hooks for each of the boxes in this diagram.*

We will start by finding the hook length for the highlighted box:

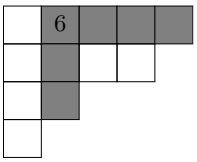


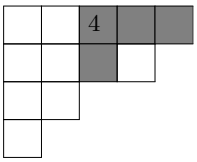
Count the number of boxes in the hook, which includes our original box, plus all of the boxes to the right and below our original box. That number goes in the box.

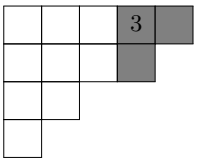


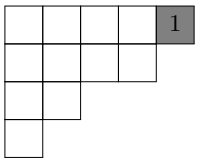


We then do this for each of the boxes in the diagram.

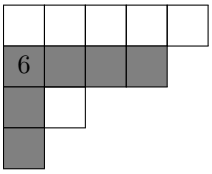


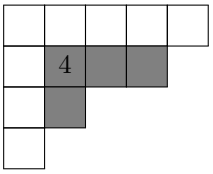


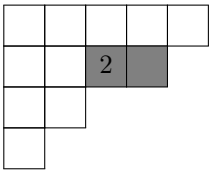


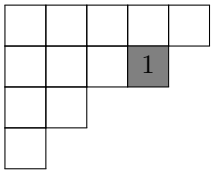


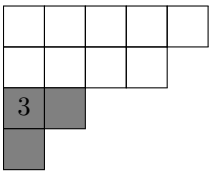
Sometimes there are no boxes below or to the right. That is okay.

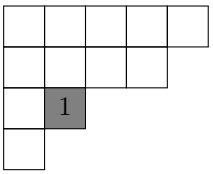


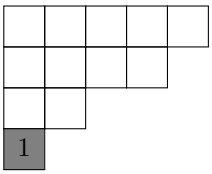










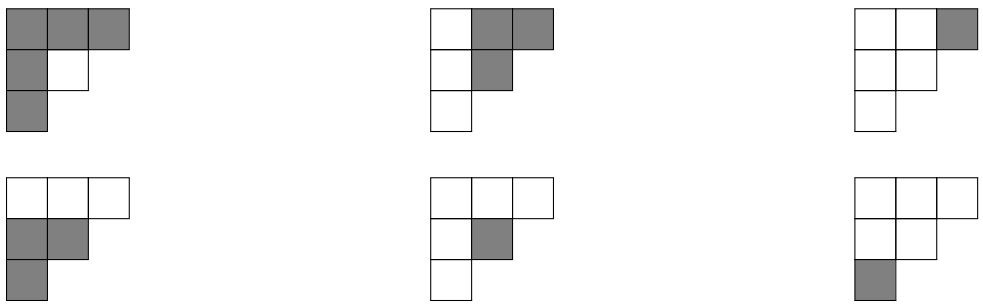


Once we have found the size of each of the hooks, we can put them all together and fill in the diagram.

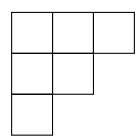
8	6	4	3	1
6	4	2	1	
3	1			
1				

Figure 3.2: Hook lengths in the Ferrers diagram for the partition $5 + 4 + 2 + 1$.

TRY: Find the hook lengths for each box in the partition $3 + 2 + 1$



Now, fill in all the boxes in the diagram.



Once we have filled in all of the boxes, if the number 5 is not in any of the boxes, we call that partition 5-core.

Example 11. *Our partition* $5+4+2+1$ *from before is* 5*-core, because there are no 5's in any of the boxes.*

8	6	4	3	1
6	4	2	1	
3	1			
1				

This partition is also 7-core, but it is not 2-core because there is a box that has a hook length of 2.

If a partition is two different types of core at the same time, to save space we write them together. So we would say the partition above is a $(5, 7)$-*core* partition, because it is both 5-core and 7-core. [1]

[1] It can be shown that if a partition is a 5-core, it is also 10-core, and 15-core, and 20-core etc.. This is because 10 = 2*5, 15= 3*5, 20 = 4*5.... So while this partition is both 2-core and 4-core, we usually would not write (2,4)-core.

TRY: What numbers do NOT appear in this diagram?

5	3	1
3	1	
1		

When we study examples of core-partitions, it can take a lot of work to fill in all of those boxes every single time. Luckily, there is an easier way! While it is not obvious, as long as we know what goes in the first column of boxes, we can use a different type of diagram that makes it easier to see the cores. We will now go through an example of how to construct these diagrams.[2]

Example 12. *Let us look at our partition* $5 + 4 + 2 + 1$ *from before:*

8				
6				
3				
1				

Figure 3.3: First column hook lengths in the Ferrers diagram for the partition $5 + 4 + 2 + 1$.

We can think about the numbers in the first column of boxes like beads on a string.

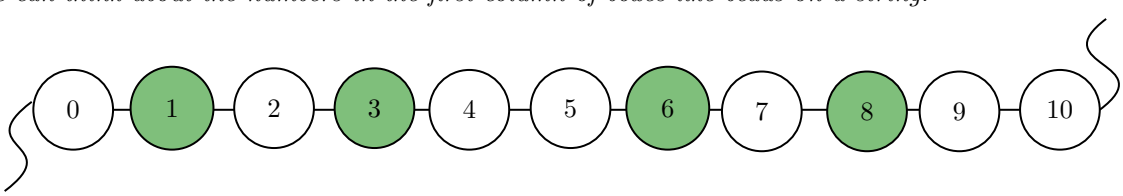


Figure 3.4: First column hook lengths on a string of beads.

To figure out if the partition is 5-core, we line up the beads in blocks of 5, keeping them in order. If there are no green beads in the first column, and all of the green beads have only green beads beneath them, then the partition is 5-core.

Because our example had distinct parts, the towers of green beads are separated from each other by columns of white beads.

[2] A partition is uniquely determined by its first-column hook lengths. To see this, we use the example of constructing the partition associated to the first-column hook lengths (8, 6, 3, 1). Reading right to left. the smallest part of the partition should be 1, the second smallest part should be $3 - 1 = 2$, the third smallest part should be $6 - 2 = 4$, and the largest part should be $8 - 3 = 5$. In general the first column hook lengths $(h_k, h_{k-1}, \ldots, h_1)$ give us a partition into k parts with the smallest part $= h_1$, and the jth smallest part is hook length $h_j - (j - 1)$.

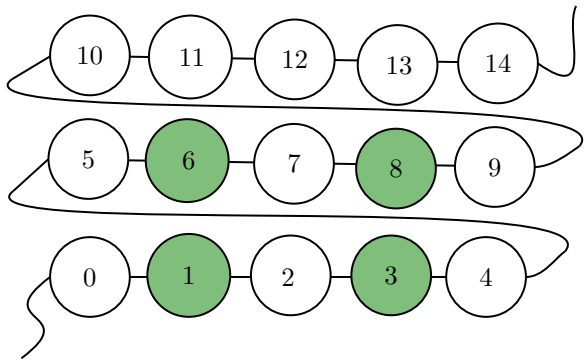


Figure 3.5: 5-column bead diagram for the partition $5 + 4 + 2 + 1$.

In this same example, we can use the beads to show that the partition is also 7-core.

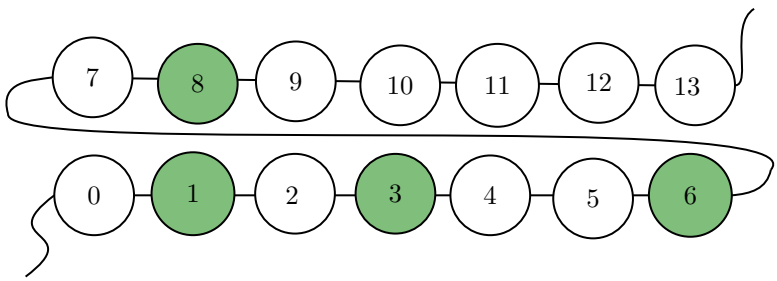


Figure 3.6: 7-column bead diagram for the partition $5 + 4 + 2 + 1$.

Example 13. *To see that our example* $5 + 4 + 2 + 1$ *is NOT 4-core, we can follow the same process.*

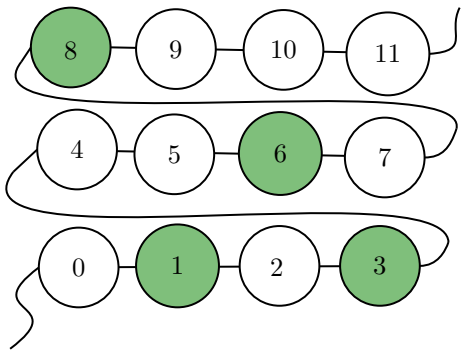


Figure 3.7: 4-column bead diagram for the partition $5 + 4 + 2 + 1$.

This shows that the partition is NOT 4-core because there are green beads (beads 6 and 8) that have white beads underneath them. I like to think about it like the white beads are 'empty'. The green beads 6 and 8 would fall down because they are not supported by other green beads.

TRY: Use the beads to figure out if this partition is 6-core.

Most of the time, to save time when drawing the pictures, we do not draw the string or the numbers, because we know that they would always work the same way. So, the image on the right represents the same partition as the image on the left.

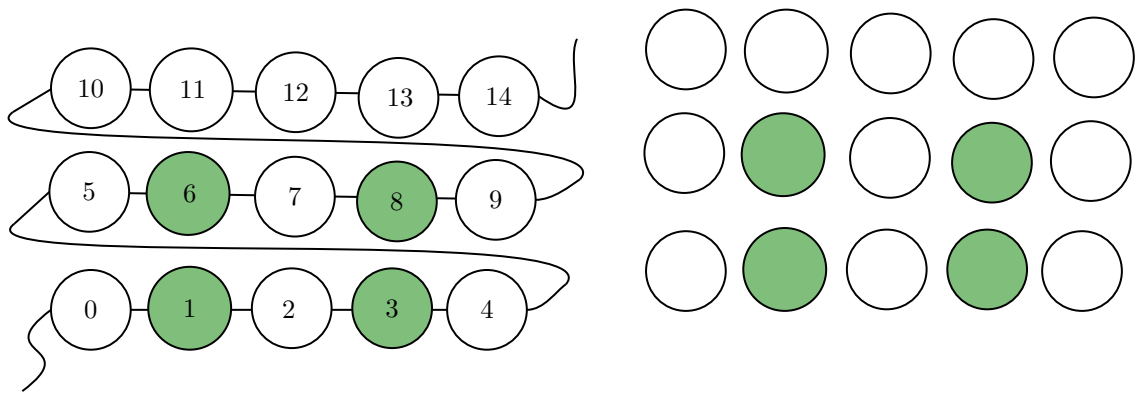


Figure 3.8: A more abstract representation of a bead diagram.

Diagramming and Gesture

The idea of representing the abaci as first beads on a string came from trying to think of a way to explain my **intentional self-oriented gesture** of thinking about creating different abacus diagrams through stacking beads. The string represents the continuity I see in the ordering of the numbers of beads, and makes it easier to connect, for example a 5-abacus and a 7-abacus with the same bead structure. Another thing it does is illustrate the **exploration of a temporal dimension**. In this case, I see the object as if there is an initial diagram of beads on a string, that can be manipulated to create different s-abaci. The temporal aspect sees the beads on a string as the initial state, and the stacking of beads into an s-abacus happens as we move forward in time, unstacking is moving back to the original state.

3.1.1 Families of Core Partitions

We study partitions that are core partitions for two values that are related to each other. For example, we might study the family of partitions that includes $(2, 5)$-core, $(3, 8)$-core partitions and $(4, 11)$-core partitions. There is a relationship between the first and second number in each of these examples. We get the second number by:

1. multiplying the first number by three, and then 2. subtracting one.

Example 14. *Look at (**2**,5)-cores because* $3 \times \mathbf{2} - 1 = 5$.

*Look at (**3**,8)-cores because* $3 \times \mathbf{3} - 1 = 8$.

*Look at (**4**,11)-cores because* $3 \times \mathbf{4} - 1 = 11$.

TRY: Can you come up with another pair of cores that would fit in this family? (Hint: pick a first number, then come up with the second number that goes with it).

m is for Multiply

This particular family, which we will call $(s, 3s-1)$-core partitions had actually already been studied by several people, including Armin Straub [37], Rishi Nath and James Sellers [27]. We write it this way because the s is a placeholder for a changing value.

We get $(\underline{\mathbf{2}}, 5)$ when <u>s is **2**</u>, and $(\underline{\mathbf{3}}, 8)$ when <u>s is **3**</u>, etc..

In fact, they studied a lot of families like this one, including $(s, 4s-1)$, $(s, 5s-1)$, $(s, 6s-1)$. For these, instead of multiplying the first number by 3, we multiply by 4, or 5, or 6, and so on. Later we will want to talk about this number that we multiply by in general, and we will call it m.

r is for Remove

Instead of subtracting 1, we could subtract 2, or 3, or 4 (or whichever positive number you like). When we are talking about the number we subtract, we will call it r (for 'remove', because we already used s for the first number in the pair).

Example 15. *If instead of subtracting 1, we subtracted 2, we would be looking at relationships between partitions in the family* $(s, 3s-2)$. *So, we could look at*

*(**2**,4)-cores because* $3 \times \mathbf{2} - 2 = 4$, [3]

*(**3**,7)-cores because* $3 \times \mathbf{3} - 2 = 7$,

*(**4**,10)-cores because* $3 \times \mathbf{4} - 2 = 10$,

*(**5**,13)-cores because* $3 \times \mathbf{5} - 2 = 13$.

[3] This is a little more complicated when the two numbers are not relatively prime. There are actually infinitely many (2,4)-cores! To look at these partitions, we will consider an additional restriction, the maximum number of parts, which we will talk more about in the next chapter.

TRY: Pick another number (besides 1 or 2) to subtract, and list three or four different pairs of cores that would belong to that family.

d is for Distinct

At the very beginning, we said we would be looking at partitions into distinct parts (any two columns of green beads are separated by columns of white beads). Another thing our work does is looks at how the partitions are related if we make a rule that the green beads have to be separated by at least two columns of white beads, or if they have to be separated by at least three columns.

Example 16. *The partition* $8+4+1$ *has distinct parts, and if we make a bead diagram to show that it is 7-core, we can see that the green columns are separated by each other by at least 2 white columns. We call this 2-distinct.*

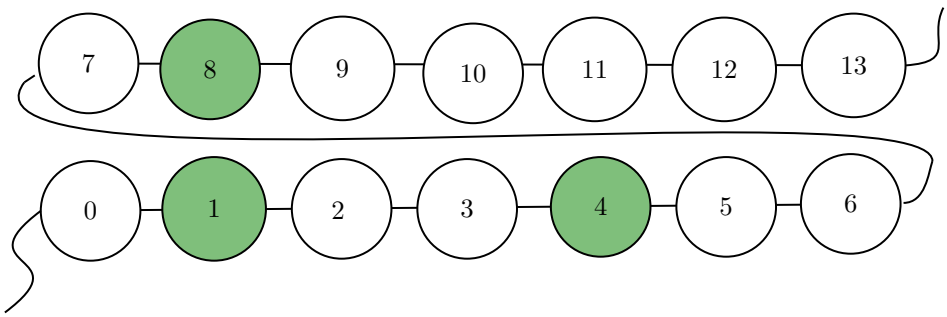


Figure 3.9: The 7-column bead diagram for the 2-distinct partition $8+4+1$.

Our initial example, $5+4+2+1$ *is not 2-distinct because we can see two columns of green beads are separated by only one column of white beads.*

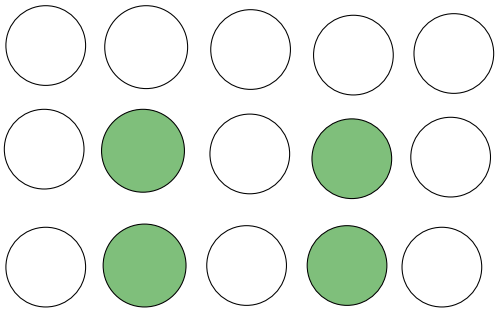


Figure 3.10: The 5-column bead diagram for the partition $5+4+2+1$ that is not 2-distinct.

Other mathematicians have studied this type of distinctness for core partitions with different properties,

including Murat Sahin in [33] and Noah Kravitz in [23]. When we are talking about the number of columns between green columns, we use the letter d.

What the next section will cover is how we use the bead diagrams to describe relationships over all of these changing values. One benefit of being able to describe these relationships is that we can count how many there are, and know something about what the pictures would look like, without having to actually draw all the pictures.

By putting all of our variables together, we can describe families of

$$(s, \mathbf{m}s - \mathbf{r})\text{-core partitions into } \mathbf{d}\text{-distinct parts,}$$

for any positive whole number values of m and d. We looked a little bit at

$(s, \mathbf{3}s - \mathbf{1})$-core partitions into $\mathbf{1}$-distinct parts,

But we could have just as easily done:

$(s, \mathbf{5}s - \mathbf{1})$-core partitions into $\mathbf{3}$-distinct parts, or

$(s, \mathbf{2}s - \mathbf{3})$-core partitions into $\mathbf{2}$-distinct parts.

TRY: Choose your favorite three positive whole numbers and fill in the blanks below.

$$(s, \_\_\_s - \_\_\_) - \text{core partitions into } \_\_\_ - \text{distinct parts.}$$

Our results describe how partitions are related in these families, in particular, using smaller s-values to build the larger s values.

3.2 Abacus Diagrams and Notation

In order to describe relationships in the families of $(s, ms - r)$-core partitions into d-distinct parts, it will help to introduce some notation. We saw earlier that we can draw s-core partitions using s-abacus diagrams. We will introduce some notation to help us explore relationships between sets of diagrams, which can help us understand the relationships in the families of partitions.

Before, we introduced two different types of diagrams to represent partitions, block diagrams (also called Ferrers diagrams) and bead diagrams. A bead diagram looks a lot like an abacus, and so we call these diagrams *abacus diagrams*.

In the example from before, we moved the beads into 5 columns to illustrate that a partition is (or is not) 5-core. If a partition is 5-core, we call the diagram a 5-core abacus diagram. Similarly, for any s, we can think of the set of *s-core abacus diagrams*.

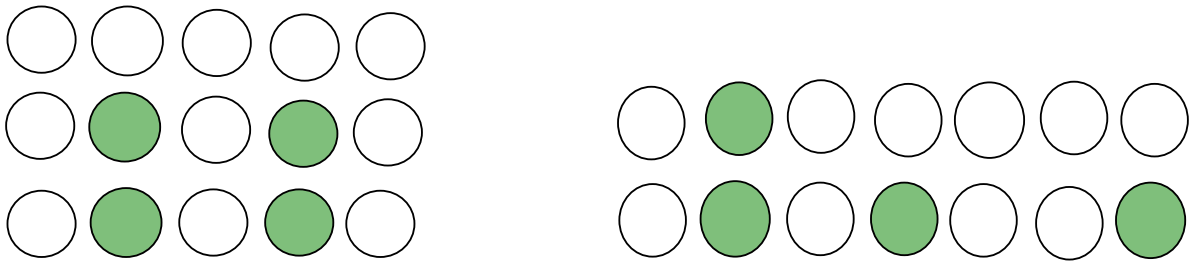


Figure 3.11: A 5-core abacus diagram and a 7-core abacus diagram

One example we have looked at is $(s, 3s-1)$-core partitions into distinct parts.

Example 17. *When $s = 4$, these are $(4, 3(4)-1)$-core, or $(4, 11)$-core partitions. The complete set of $(4, 11)$-core partitions is:*

$$\{0,\ 1,\ 2,\ 3, 2+1, 4+1, 5+2, 6+3, 3+2+1, 5+2+1, 7+4+1, 8+5+2, 4+3+2+1,\ \textit{and}\ 5+4+3+2+1\}.$$

In Figure 3.12 we list the 4-core abacus diagrams that correspond to these partitions. We use 0 to represent the 'empty partiton.' Note: these are 4-core abacus diagrams that correspond to $(4, 3(4)-1)$-core, or $(4, 11)$-core partitions into distinct parts. The position of the beads corresponds to the first column hook lengths, and not the parts of the partition.

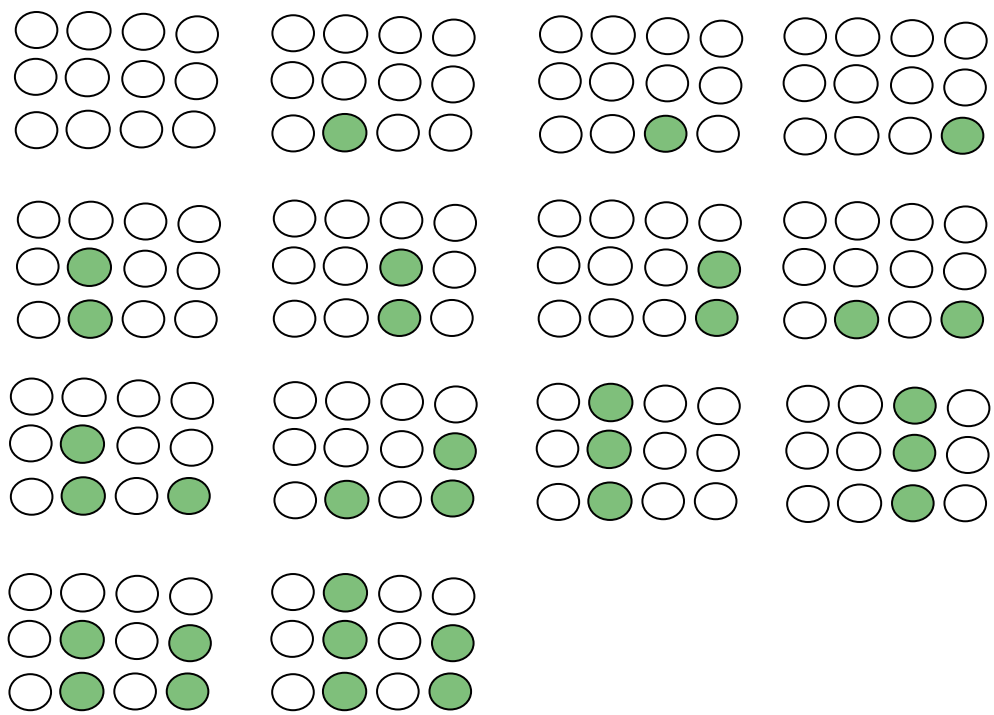


Figure 3.12: 4-core abacus diagrams for the set of $(4, 11)$-core partitions.

When looking at this set of 4-core abacus diagrams into distinct parts, we can notice that all of them have at most 3 rows with green beads, there is at least one column of empty beads between columns of green beads, and the highest placed green bead is below position 11.

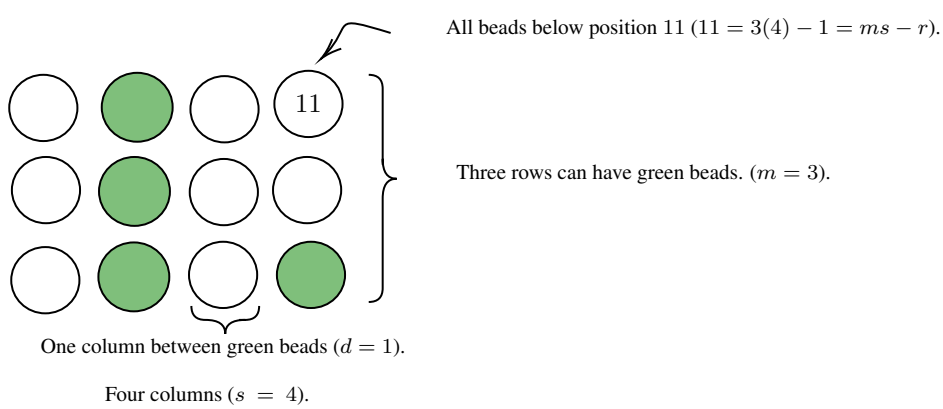


Figure 3.13: General sketch of 4-core abaci corresponding to $(4, 11)$-core partitions.

We are interested in analogous sets of beads for different choices of the variables m, d, and r. We will be talking about the set of s-core abaci with (at most) m rows, that have spacing of d empty columns between

columns of colored beads, and that have a maximum position of strictly less than $ms - r$.

When we are talking about these sets, it can be helpful to use a shorthand name. We will use the notation $\mathcal{A}^d_{s,m,-r}$ to describe the set of s-core abaci with (at most) m rows, spacing d and maximum position strictly less than $ms - r$. I tend to think of the set $A^d_{s,m,-r}$ as the diagrams that have these characteristics, and fit into the diagram outlined below in Figure 3.14.

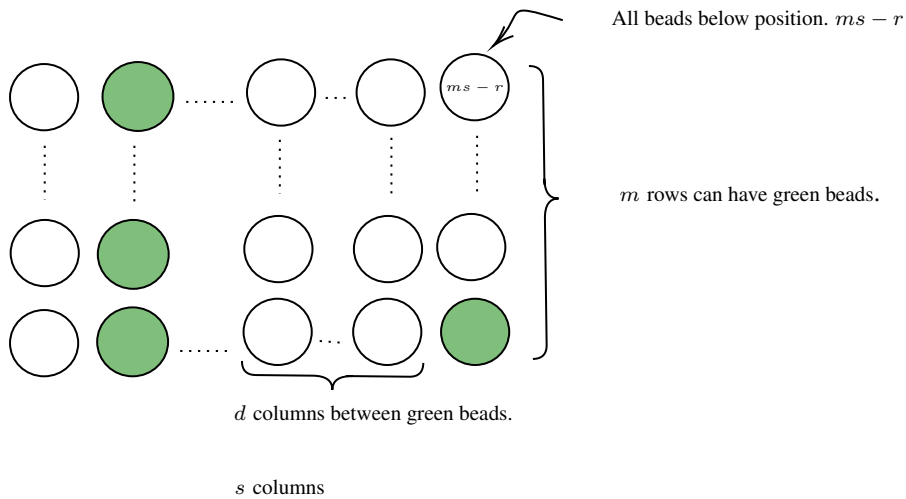
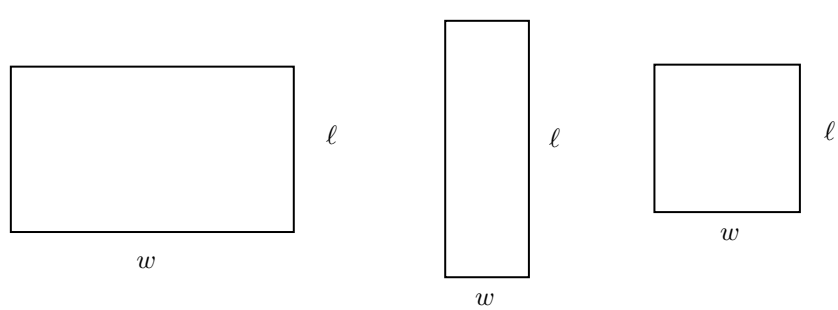


Figure 3.14: General diagram for s-core abaci with (at most) m rows, spacing d and maximum position strictly less than $ms - r$.

We have a bunch of variables (d, s, m, r). It is a little bit like trying to solve an optimization problem for area enclosed by a fence. If I drew the rectangle labelled with length and width, I have some idea that depending on the actual values of ℓ and w, it might look different than my picture. Having the diagram can help me solve the problem, by helping me see how the variables might relate to each other.



Diagramming and Gesture

The figure represents a generalization, my attempt to illustrate the intuition I developed about these diagrams. The **dotted lines** represent flexibility in that there are dimensions that can be expanded and contracted based on the values of the variables. This internal model that encodes relevant information about the definition of an abacus, facilitates **manipulation and exploration of the spatio-motoric information** that these mathematical objects have. It enables me to ask questions about a 'general abacus', thinking about what might happen 'in general' to an 'arbitrary diagram' if we change a variable.

I developed this sense of a 'general abacus' after drawing many examples of specific abaci, and throughout the project the model developed and changed based on context. In order to do this work, it was helpful that I could transition back and forth between individual examples and the 'general abacus.' While working on the project, and especially when trying to explain or formalize an argument about why something worked the way it did, I would draw 'general' diagrams such as this, labelling different elements and thinking of movement of the general diagram that echoes movement (e.g. removal/insertion of beads/columns) in the specific diagrams. We will see more examples of this later.

Generating Polynomials

Sometimes we use polynomials to help us keep track of partitions, and following a convention of this particular branch of mathematics, we will use the variable q in our polynomials. Before, we looked at the five partitions of the number four. We will use the *generating polynomial* to count this set, while keeping track of the number of parts in the different partitions.

Example 18. *We count the number of parts in each partition of 4, and label each as powers of q.*

$$4, \qquad 3+1, \qquad 2+2, \qquad 2+1+1, \quad \textit{and} \quad 1+1+1+1$$

$$q, \qquad q^2, \qquad q^2, \qquad q^3, \qquad \textit{and} \qquad q^4$$

So, the generating polynomial for the partitions of four is $q + 2q^2 + q^3 + q^4$, because there is one partition that has one part, two that have two parts, and one each with three and four parts.

One thing that is useful about these generating polynomials, is that if we substitute 1 for q, we get exactly

the number of things in the set (there are $5 = 1 + 2(1)^2 + 1(1)^3 + 1(1)^4$ partitions of four). So the polynomial stores that information (how many total) in addition to storing the additional information (how many have 1 part, how many have 2 parts, etc.).

TRY: Write the generating polynomial that keeps track fo the number of parts for the seven partitions of 5.

$$5 \qquad 4+1 \qquad 3+2 \qquad 3+1+1$$

$$2+2+1 \qquad 2+1+1+1 \qquad 1+1+1+1+1$$

We will use these generating polynomials to be able to write algebraically about the sets of abaci $\mathcal{A}^d_{s,m,-r}$ and their relationships to each other.

For each abacus, A, define $n(A)$ to be the function that gives the number of green beads in the diagram. This is the same as the number of parts in the corresponding partition.

Example 19. *Using the same set of abaci, $\mathcal{A}^1_{4,3,-1}$, we used earlier, we can calculate the value of $n(A)$ for each abacus, A.*

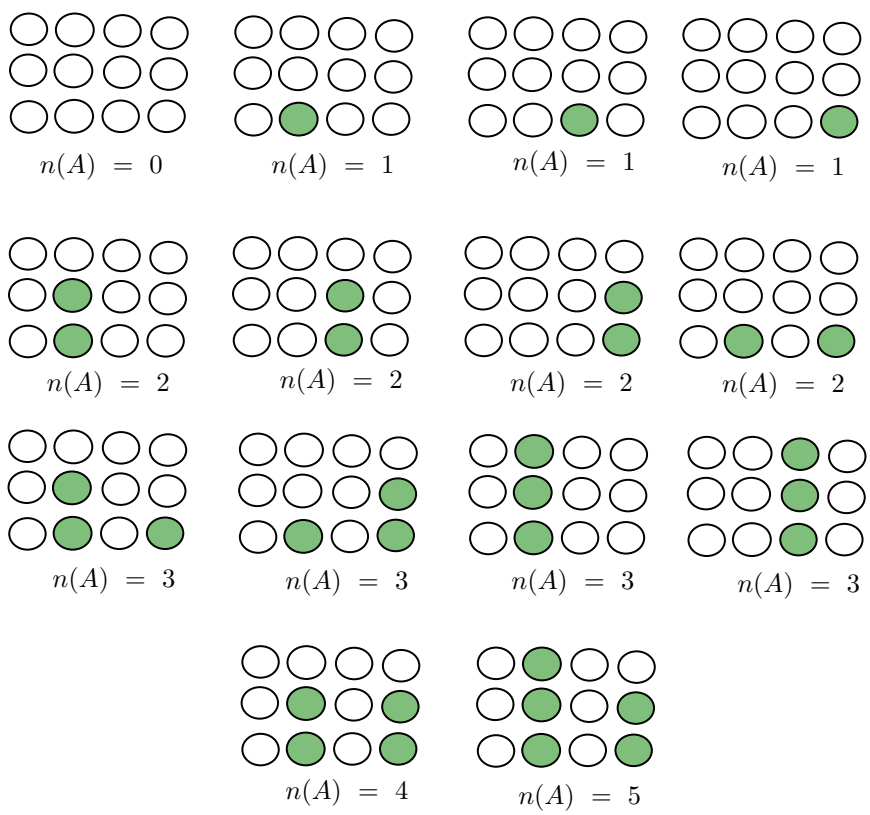


Figure 3.15: Number of beads for abaci corresponding to $(4, 11)$-core partitions.

This is the set of 4*-core abaci with (at most)* 3 *rows, spacing* 1 *and maximum position strictly less than* $3(4)-1=11$*. We saw earlier how these correspond to* $(4, 3(4)-1)$*-core partitions.*

From each abacus in the example, we can think of counting that diagram by $q^{n(A)}$. So we can write the generating polynomial for the number of beads in the 4-core abaci with (at most) 3 rows, spacing 1 and maximum position strictly less than $3(4)-1=11$ (the set $\mathcal{A}^1_{4,3,-1}$) by summing up all of these terms. We use the notation $\mathbf{A}^1_{4,3,-1}(q)$ to represent the generating polynomial for the number of beads in the abaci in the set $\mathcal{A}^1_{4,3,-1}$, using the variable q.

$$q^0+q^1+q^1+q^1+q^2+q^2+q^2+q^2+q^3+q^3+q^3+q^3+q^4+q^5$$

$$=q^5+q^4+4q^3+4q^2+3q+1=\mathbf{A}^1_{4,3,-1}(q)$$

We use summation notation as a shorthand to describe these generating polynomials. When we were adding up the different terms, each abacus was counted by one term. In the past (e.g. in Calc II), we have used summation notation where the sum iterates over integers, like: $\sum_{i=1}^{4} i^2 = 1^2+2^2+3^2+4^2=30$

Here, we will sum over a set of diagrams, but the idea is still the same. We have a set of abaci, and each abacus in the set contributes a term. We showed this in the earlier example, for $(4,11)$-core partitions, and the way we write this formally is

$$\mathbf{A}^1_{4,3,-1}(q)=\sum_{A\in\mathcal{A}^1_{4,3,-1}} q^{n(A)}=q^5+q^4+4q^3+4q^2+3q+1$$

The equation above was an example, with given values $d=1$, $s=4$, $m=3$ and $r=1$. The general definition is:

$$\mathbf{A}^d_{s,m,-r}(q)=\sum_{A\in\mathcal{A}^d_{s,m,-r}} q^{n(A)}.$$

We say that $\mathbf{A}^d_{0,m,-r}(q)=q^0=1$, which represents the empty abacus (no green beads), since if $s=0$, there are no columns, and therefore no spaces which may hold green beads. When $s<0$, we set $\mathbf{A}^d_{s,m,-r}(q)=0$ (this makes some intuitive sense, since there is no way to have negative columns, so there would be no abaci with a negative number of columns).

Chapter 4

Counting Abaci

The results presented in this chapter come from joint work with Hannah Elizabeth Burson and Armin Straub [9]. Analogous theorems for the cases $r \geq 0$ are proved in [8]. In Section 4.1, we give some preliminary lemmas and context for the work and the types of arguments we use. In Section 4.2, we enumerate s-core abaci with spacing d of certain height.

4.1 Preliminaries

4.1.1 Some 'Small' Proofs.

To get to know these diagrams better, before we move into proving the main results, we'll work through proofs of smaller facts. When we are proving a 'small' fact, we call it a Lemma. Lemmas can be used in proofs of other statements, as a way to shorten the proof by saying "we already showed that this is true, so we can just state it again here without any more justification."

For each of the Lemmas in this section, we'll go through an example that illustrates why the Lemma is true, and then give the proof of the Lemma in more formal, academic language.

We continue to use the phrase s-core abacus to refer to an s-abacus corresponding to an s-core partition (as we defined them in Chapter 3). This definition comes from existing literature, and in [27, Lemma 7], which is equivalent to [41, Lemma 2.1], the following statement appears with proof.

Lemma 20. *[27, Lemma 7] An s-abacus corresponds to an s-core partition if and only if the first column is empty and no spacers occur below a bead.*

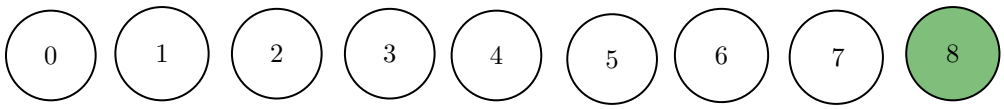
Lemma 21. *If $1 \leq r \leq d$, then every $(s, s+r)$-core partition into d-distinct parts has maximum hook length $< s + r$.*

Lemma 21 Restated: As long as r is positive and less than or equal to d, then we know that the maximum hook length (the largest hook in the upper left box of the Ferrers diagram) has to be less than $s + r$, no

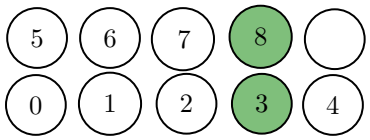
matter what s is. We will illustrate how the proof works with an example first, and then write it with more formal "academic math"-y language below.

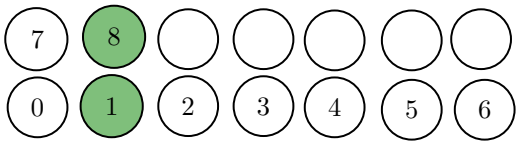
We will use the examples $r = 2$, $d = 2$, and $s = 5$. The proof uses the technique of contradiction, which means we suppose there is some example that contradicts the statement, and then show that cannot possibly be true.

Assume that there is some $(5, 7)$-core partition ($s = 5$ and $r = 2$) into 2-distinct parts that has a maximum hook length of 8 (larger than 7). In the abacus diagrams, that means there would be a green bead in position 8.

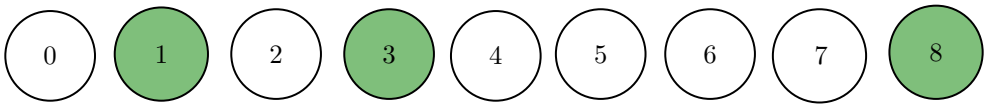


Then, since we said that this partition is 5-core and 7-core, it must have valid 5-core and 7-core abacus diagrams. In order for this to be true, there must be beads in both position 3 (to support position 8 in the 5-core abacus) and position 1 (to support position 8 in the 7-core abacus).





Since both things must be true at once, the partition diagram must have at least beads in positions 1, 3, and 8.



But, we also said the partition should be 2-distinct, which means any two green beads must be separated by two white beads. This cannot possibly be true at the same time as the assumption we made: that there is some $(5, 7)$-core partition into 2-distinct parts that has maximum hook length of 8. So we, have reached a contradiction, and the assumption must have been false.

Note: I chose 8 as the example here, but we would run into the same problem for any integer greater than 7. The issue was that the positions $8 - 7 = 1$ and $8 - 5 = 3$ only have one bead between them.

TRY: Convince yourself that the same issue comes up for a different maximum hook length (e.g. 9, or 10, or 34...)

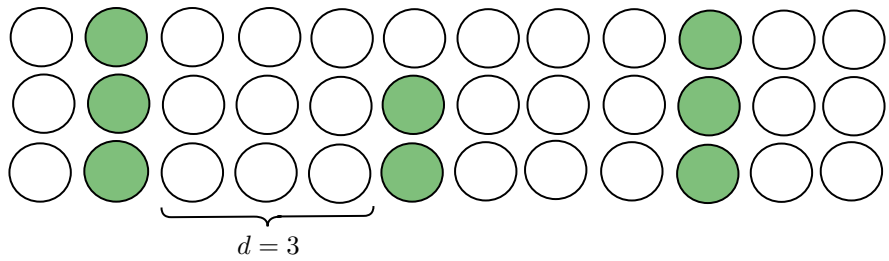


Figure 4.1: An example of a 12-core abacus diagram with spacing 3.

The proof below explains the process we did in our example, using general variables to show that this works for all $(s, s+r)$-core partitions into d-distinct parts, for any values of r, d, and s, provided $1 \leq r \leq d$.

Lemma 20 refers to white beads in an abacus as 'spacers'. From here on, we will use this convention, so an abacus has 'beads' (colorful beads) and 'spacers' (white beads).

General Proof of Lemma 21. Let A be the s-abacus of an $(s, s+r)$-core partition λ into d-distinct parts. We need to show that all beads of A are in positions with labels $< s+r$. We remember that the beads of A correspond to the first column hook lengths of the box diagram of the partition λ. Assume, for the sake of contradiction, that A has a bead in position $x \geq s+r$. By Lemma 20, because λ is an s-core partition, there must also be a bead in position $x \geq s+r$. By Lemma 20, because λ is an s-core partition, there must also be a bead in position $x - s$. Likewise, because λ is an $(s+r)$-core partition, there must be a bead in position $x - (s+r)$. However, $|(x-s) - (x-(s+r))| = r \leq d$, which contradicts the requirement that positions of beads in A differ by more than d. $\square$

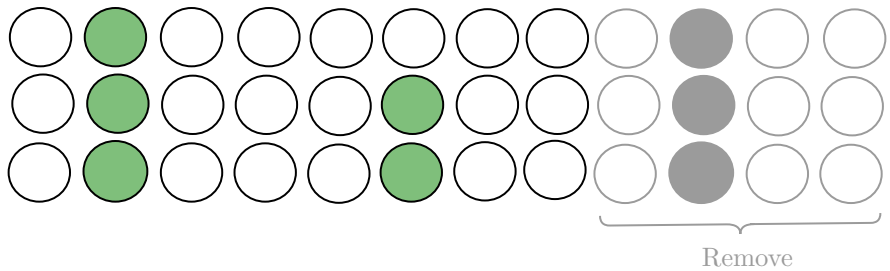
The next lemma supports the main argument of most of our results.

Lemma 22. *Let $s > d+1$. Removing the last $d+1$ columns, not all empty, from an s-core abacus with spacing d results in an $(s-d-1)$-core abacus with spacing d.*

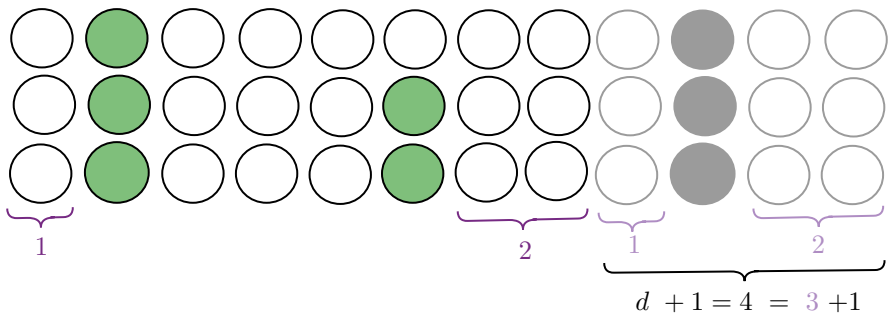
Lemma 22 Restated: If we have an s-core abacus with spacing d and we remove the last $d+1$ columns, the abacus diagram that we create by removing those columns also has spacing d.

For example, if we start with the diagram below, we have a 12-core abacus with spacing 3.

If we remove the last 4 columns ($d+1 = 4$ because $d = 3$), we will get a new diagram, which we can see below. Since we are removing columns from the right side, we have not changed the empty first column and have not moved any of the remaining beads, so our new diagram will definitely be an 8-core abacus.



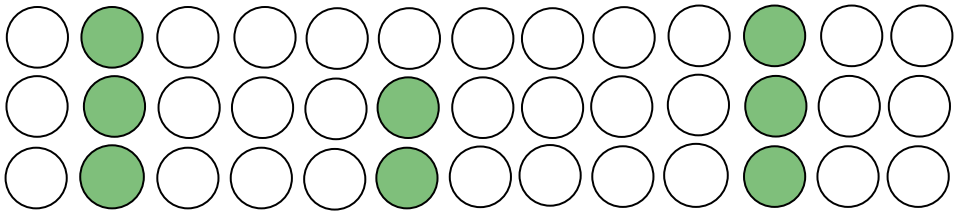
What is left to show is to prove that the diagram that is left is also 3-distinct (has spacing 3). In order to do this, we show that there is a gap of at least 3 beads between the rightmost bead in the first row and the leftmost bead in the second row. If it is true for those beads, then it will similarly be true for all of the other beads in those columns.



In the diagram above, we see that when we remove the $4 = d + 1$ columns, we can think of it as removing 1 column of spacers, 1 column of beads, and a set of 2 columns of spacers. The remaining abacus has 2 columns of spacers after the rightmost bead in the first row, and one column before the leftmost bead in the second row, so the diagram is 3-distinct.

Our example had exactly 3 columns of spacers between each column of beads, however the process would still hold if there were more than 3 columns of spacers.

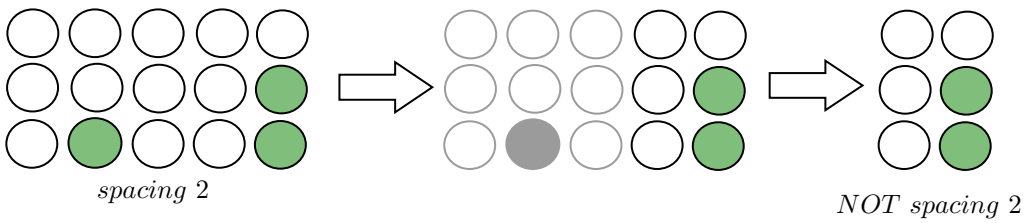
TRY: Convince yourself that if we started with this slightly different 3-distinct abacus and removed the last $d + 1 = 4$ columns, the remaining abacus would still be 3-distinct.



The proof below explains the process from this example in more general terms.

General Proof of Lemma 22. From our definition of s-core abaci, the result is clearly an $(s-d-1)$-core abacus. It remains to observe that the reduced abacus has spacing d. For that, it suffices to check that, in the reduced abacus, each bead in the first row is followed by d empty positions. Since the initial abacus had spacing d, we need only consider the last such bead. In the initial abacus, this bead was followed by $d_1 \geq d$ spacers, then a bead (in the last $d+1$ columns), followed by another $d_2 \geq d$ spacers. After removal of the last $d+1$ columns, it is therefore followed by $d_1 + d_2 - d \geq d$ spacers. $\square$

Example 23. *We note that this result is not true for removing the first $d+1$ columns. This is illustrated, for instance, with $d = 2$, by the 5-core abacus with beads in positions $1, 4, 9$. The resulting abacus does not have spacing 2, because there are beads in positions 1 and 3, with only one spacer between them.*



Diagramming and Gesture

In Lemma 3, we discuss the implications of removing columns from an abacus diagram. The idea of removing columns from a diagram is a way of interacting with the diagram. My interaction with the abaci is through **manipulation of spatio-motoric information in the diagrams**, and the gestures that are associated to this manipulation. I think of each bead (spacer) in the diagram as independently movable, as well as grouped by columns and rows. By removing columns from an abacus, we get a new abacus, but one that shares some of the characteristics of the initial diagram. Being able to relate these abaci means being comfortable with holding some information steady, while changing other dimensions. The abacus diagrams encode the information for me more easily than the abstract notation.

The figures accompanying the example above illustrate the use of **successive framing** and **shading** to illustrate movement within a diagram. When I create examples for myself, I often only physically draw a subset of the three diagrams shown here. However, whether I would draw the first two, or the second two, or just the middle one would depend on what I was trying to figure out. I often use shading to illustrate columns that are being removed or added, to highlight where the changes happen. These diagrams represent a process, and I would use drawings to highlight part of that process.

4.1.2 Abacus Diagrams and Core Partitions

At this point, it is important to talk about work that has already been done on counting simultaneous-core partitions, as this was the primary motivator for our embarking on this particular mathematical journey. This line of inquiry has received increasing interest since Anderson [4] proved that the number of (s,t)-core partitions is

$$\frac{1}{s+t}\binom{s+t}{s}$$

if s and t are coprime (otherwise, there are infinitely many such partitions). When we say s and t are coprime, that means they do not share any factors. For example, 5 and 6 are coprime, but 4 and 6 are not (because they both have a factor of 2).

Note that $\binom{s+t}{s} = \frac{(s+t)!}{s!t!}$ is the notation for the binomial coefficient, which counts the number of ways to choose s objects from a set of $s+t$ objects. You may remember this from conversations about the triangle

of binomial coefficients, (sometimes referred to as Pascal's triangle). If not, no worries, we will not use this definition again.

It remains an open problem to come up with a formula that counts (s,t)-core partitions into distinct parts. Towards that problem, it was shown in [37] that $(s, ms-1)$-core partitions into distinct parts are counted by Fibonacci-like numbers (below). This count was generalized by Nath and Sellers [27] to also include $(s, ms+1)$-core partitions.

We use the notation introduced in [27], where $(s, ms \pm 1)$-core partitions refers to "$(s, ms-1)$-core or $(s, ms+1)$-core partitions" and the statements will be true for '$+$' and '$-$', respectively.

Theorem 24 below is our first example of a recurrence. An example of a recurrence that may be familiar is the Fibonacci sequence $\{1,\ 1,\ 2,\ 3,\ 5,\ 8, \dots\}$, where each term is generated by summing previous terms. This theorem says that the number of $(s, ms \pm 1)$-core partitions into distinct parts (represented using the notation $C^{\pm}_{s,m}$, where $C^{+}_{s,m}$ is the number of $(s, ms+1)$-core partitions into distinct parts and $C^{-}_{s,m}$ is the number of $(s, ms-1)$-core partitions into distinct parts) is generated by a recurrence.

Theorem 24. *Let $m, s \geq 1$. The number $C^{\pm}_{s,m}$ of $(s, ms \pm 1)$-core partitions into distinct parts is characterized by the following recurrence,*

$$C^{\pm}_{s,m} = C^{\pm}_{s-1,m} + mC^{\pm}_{s-2,m},$$

for $s \geq 3$, with the initial conditions $C^{\pm}_{1,m} = 1$, $C^{-}_{2,m} = m$ and $C^{+}_{2,m} = m+1$.

In the case $m = 1$, the numbers $C^{-}_{s,1} = F_s$ are the Fibonacci numbers $\{1,\ 1,\ 2,\ 3,\ 5,\ 8, ...\}$. This special case was conjectured (guessed) by Amdeberhan [2] and also independently proved by Xiong [41]. Nath and Sellers [27] proved (the case of $(s, ms+1)$-core partitions of) Theorem 24 combinatorially using properties of abacus diagrams.

At this point we have been referring to the connection between s-core partitions and s-core abacus diagrams. In the following Lemma, we formalize this relationship in the case of $(s, ms-1)$-core partitions.

Lemma 25. *The abaci in $\mathcal{A}^{d}_{s,m,-1}$ are in one-to-one correspondence with $(s, ms-1)$-core partitions into d-distinct parts.*

Lemma 25 Restated: Each diagram in the set $\mathcal{A}^{d}_{s,m,-1}$ of s-core abacus diagrams with spacing d and maximum position less than $ms-1$ corresponds to exactly one $(s, ms-1)$-core partition into d-distinct parts.

We know that each s-core abacus diagram has a corresponding $(s, ms-1)$-core partition by the way we have defined them. However, in order to show they pair up one-to-one, we need to show that all of the $(s, ms-1)$-core partitions also satisfy the maximum hook length condition of the definition of the abacus

diagrams.

We first illustrate this correspondence with an example. Suppose $s = 5$ and $m = 4$, then we are looking at $(5, 4(5) - 1)$-core partitions, or $(5, 19)$-core partitions. What we want to show is that all of these have largest hook smaller than 19.

Since every 5-core partition is also 10-core, and 15-core, etc., we can use Lemma 21 to help us prove this. Any $(5, 19)$-core partition is also 20-core, so we can say it is $(20, 19)$-core, and rewrite this as $(19, 20)$-core to keep with convention.

Then, Lemma 21 says that any $(19, 19 + 1)$-core partition has maximum hook length $< 19 + 1 = 20$. Since all of these partitions are also 19-core, they cannot have maximum hook $= 19$, so we can say that they have maximum hook < 19. Thus, the corresponding abacus will have maximum position < 19.

The proof below generalizes from this example.

Proof. Suppose $r < 0$. Note that every $(s, ms - r)$-core partition is $(ms - r, ms)$-core. Hence, if $d \geq |r|$, then Lemma 21 shows that every $(s, ms - r)$-core partition into d-distinct parts has maximum hook length $< ms$. If $r = -1$, it follows that the maximum hook length is $< ms - 1$. □

We started by looking at $(s, ms - r)$-core partitions, which are a motivation for these and have shown that, when $r = 1$, the abaci have this one-to-one correspondence with abaci. However, the correspondence is not exactly one-to-one when we look at values of $r > 1$. One way to see why this does not quite work is that there are times when, in $(s, ms - r)$ the larger value is actually s.

Example 26. *Consider the case $d = 2$ $r = 2$, $s = 6$ and $m = 1$. Then, there is a partition $4 + 1$ which is $(s, ms - r) = (6, 4)$-core and 2-distinct. However its maximum hook length is $5 \not< ms - r = 4$.*

5	3	2	1
1			

This example shows that the one-to-one correspondence between partitions and abaci does not hold when we look at $(s, ms - r)$-core partitions for larger values of r. However, we can show it does hold when we look at $(s, ms + r)$-core partitions for some larger values of r. In fact, Lemma 25 has the following analog.

Lemma 27. *[8] If $1 \leq r \leq d$, then the abaci in $\mathcal{A}^d_{s,m,r}$ are in one-to-one correspondence with $(s, ms+r)$-core partitions into d-distinct parts.*

We recall that the notation $\mathcal{A}^d_{s,m,r}$ refers to s-core abacus diagrams with spacing d and maximum position less than $ms + r$ and $\boldsymbol{A}^d_{s,m,r}(q)$ is the generating polynomial for the number of beads.

We define $\mathcal{C}^d_{s,m,r}$ to be the set of $(s, ms+r)$-core partitions into d-distinct parts. and $\boldsymbol{C}^d_{s,m,r}(q)$ as the generating polynomial for the number of parts in $\mathcal{C}^d_{s,m,r}$. In [8], Burson presents the following recurrence for generating polynomials of core partitions. Since there are several variables, we point out that only the variable s is changing, so we can say that this is a recurrence "on s."

Theorem 28. *[8] Let $d, m, r \geq 1$. If $s > d+1$ and $r \leq d$, then*

$$\boldsymbol{C}^d_{s,m,r}(q) = \boldsymbol{C}^d_{s-1,m,r}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{C}^d_{s-d-1,m,r}(q).$$

For $r \geq 1$, Theorem 28 follows directly from Theorem 29 due to the correspondence between s-core partitions and s-core abaci. A complete proof of Theorem 29 is given in [8]. This theorem says that, given slightly different initial conditions, the same recurrence holds for s-abaci with spacing d and maximum part less than $(ms+r)$.

Theorem 29. *Let $d, m \geq 1$, and $r \geq 0$. If $s > d+1$ and $s > r$, then*

$$\boldsymbol{A}^d_{s,m,r}(q) = \boldsymbol{A}^d_{s-1,m,r}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}^d_{s-d-1,m,r}(q). \tag{4.1}$$

While we originally were motivated by the connection to core partitions, we can still study the sets of abacus diagrams; $\mathcal{A}^d_{s,m,-r}$ as mathematical objects in their own right. Many of the patterns we noticed when studying the abacus diagrams that correspond with $(s, ms+r)$-core partitions extend to diagrams in $\mathcal{A}^d_{s,m,-r}$. In the following section, we will focus entirely on statements involving the abaci.

4.1.3 Building with Beads

Most of the Lemmas and Theorems we prove have statements that are equations, and we prove our results through combinatorial arguments. This means that the arguments we use to prove them show that what is counted by the expression on the left side of the equation is the same as what is counted by the expression on the right side.

Furthermore, many of the statements are recurrences, and I think of our arguments as building larger abacus diagrams from smaller ones. Our first example of a recurrence, the Fibonacci numbers $\{1,\ 1,\ 2,\ 3,\ 5,\ 8, \ldots\}$, each term is generated by summing previous terms, in other words, we make the later entries in the sequence by adding previous ones. When we talk about recurrences involving abaci, we will build s-abacus diagrams from smaller (previous) abacus diagrams. For example, we might build a 5-core abacus diagram by adding

beads to a 4-core abacus diagram. We will see more examples of this as we go on.

In [27], the authors share a connection between $(s, ms-1)$-core partitions and $(s, ms+1)$-core partitions into distinct parts, which translates to an equivalent connection between the sets $\mathcal{A}^1_{s,m,-1}$ and $\mathcal{A}^1_{s,m,1}$. Since there can never be a bead in position ms of an s-core abacus, we know the sets $\mathcal{A}^1_{s,m,1}$ and $\mathcal{A}^1_{s,m,0}$ are equivalent. We will use the notation $\boldsymbol{A}_{s,m}(q)$ to refer to the generating polynomial for s-core abaci with spacing 1 and maximum position less than ms (leaving off the variables $d = 1$, $r = 0$ from the original $\boldsymbol{A}^1_{s,m,0}(q)$). We follow this convention and let $\boldsymbol{A}_{s,m}(q) = \boldsymbol{A}^{(}_{s,m,0}q)$. The result from [27] gives us

$$\boldsymbol{A}_{s,m,-1}(q) = \boldsymbol{A}_{s-1,m}(q) + (q + q^2 + \ldots + q^{m-1})\boldsymbol{A}_{s-2,m}(q) \tag{4.2}$$

using an algebraic argument based on generating polynomials. We will do a little algebra manipulation to come up with an equivalent statement, which will be what we use in the following proof.

We first take Equation (4.1) from Theorem 29 (using the conditions $d = 1$ and $r = 0$) which gives us the following equation. (When we refer to an equation that appeared previously, we refer to the number that is to the right of that equation.)

$$\boldsymbol{A}^1_{s,m,0}(q) = \boldsymbol{A}^1_{s-1,m,0}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}^1_{s-2,m,0}(q).$$

By subtracting the second term from the right side, we have

$$\boldsymbol{A}_{s,m}(q) - [(q + q^2 + \ldots + q^m)\boldsymbol{A}_{s-2,m}(q)] = \boldsymbol{A}_{s-1,m}(q).$$

Then, substituting the left side above for $\boldsymbol{A}_{s-1,m}(q)$, in Equation (4.2) we have that Equation (4.2) is equivalent to

$$\boldsymbol{A}_{s,m,-1}(q) = \boldsymbol{A}_{s,m}(q) - [(q + q^2 + \ldots + q^m)\boldsymbol{A}_{s-2,m}(q)] + (q + q^2 + \ldots + q^{m-1})\boldsymbol{A}_{s-2,m}(q).$$

And by combining like terms, it follows that Equation (4.2) is equivalent to

$$\boldsymbol{A}_{s,m,-1}(q) = \boldsymbol{A}_{s,m}(q) - q^m\boldsymbol{A}_{s-2,m}(q),$$

which we can rewrite as

$$\boldsymbol{A}_{s,m}(q) = \boldsymbol{A}_{s,m,-1}(q) + q^m\boldsymbol{A}_{s-2,m}(q). \tag{4.3}$$

To prove this, we actually prove a more general statement in Lemma 30. Equation (4.3) is the special case of Equation (4.4) in Lemma 30 for the value $r = 1$. We can prove this statement using a combinatorial argument (building with abacus diagrams), on the generating polynomials $\boldsymbol{A}_{s,m,-r}(q)$ for any $r \geq 1$.

Lemma 30. *Let $m, r \geq 1$. If $s \geq r$, then*

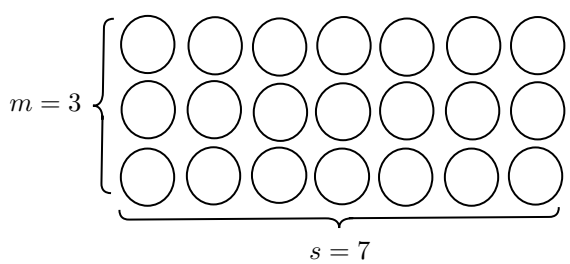
$$\boldsymbol{A}_{s,m}(q) = \boldsymbol{A}_{s,m,-r}(q) + q^m \sum_{k=1}^{r} \boldsymbol{A}_{s-k-1,m}(q)\boldsymbol{A}_{k-1,m-1}(q). \tag{4.4}$$

Lemma 30 Restated: The set of s-core abaci with spacing 1 and maximum position strictly less than ms includes the entire set $\mathcal{A}_{s,m,-r}$, of s-core abaci with spacing 1 and maximum position strictly less than $ms - r$, plus those that can be counted by the generating polynomial

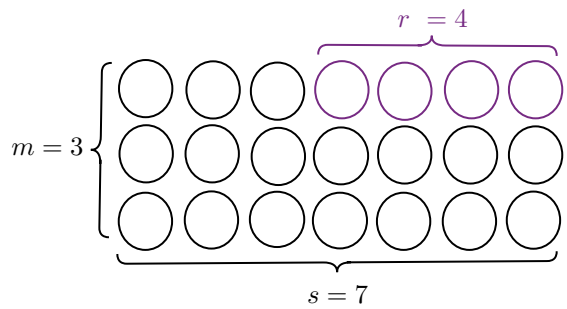
$$q^m \sum_{k=1}^{r} \boldsymbol{A}_{s-k-1,m}(q)\boldsymbol{A}_{k-1,m-1}(q),$$

where q keeps track of the number of parts. These two sets of abaci sum exactly to give the set of s-core abaci with spacing 1 and maximum positions ms.

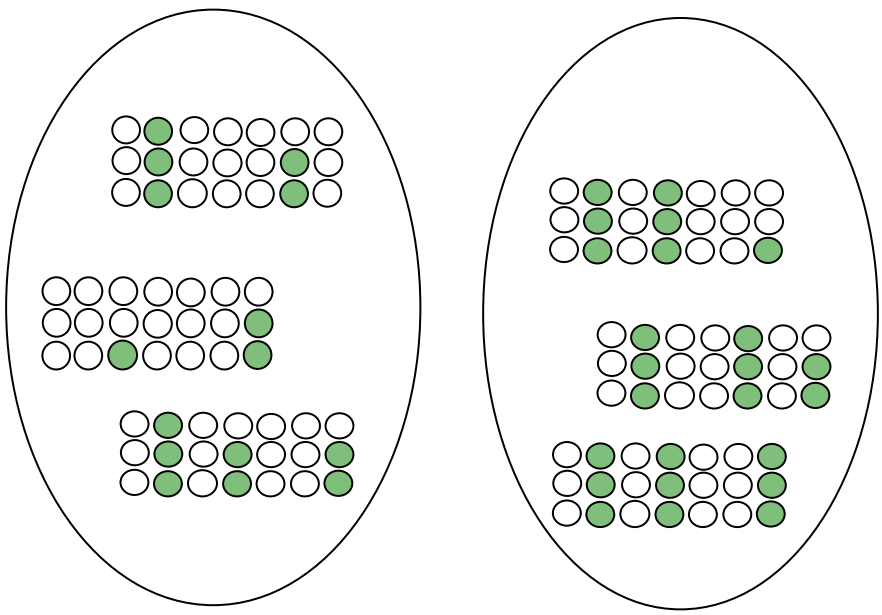
We will work through this proof first using the example of $s = 7$, $m = 3$, and $r = 4$. The first thing to note is that $\boldsymbol{A}_{7,3}(q)$ is the generating polynomial for the set of 7-core abaci with spacing 1 and maximum position strictly less than 21. This means, abaci with spacing 1 (at least 1 column of spacers between columns of beads) that have dimensions matching the diagram below.



When we look at each diagram that fits this description, we can ask: Does it have at least one bead in the rightmost four spaces of the top row?



Asking this question gives us a way to split up the set of abaci into two groups. If we do this for all of the abaci in the set, we will get two groups: those that do not have a bead in the rightmost four spaces of the top row and those that do. The image below shows three examples from each set (though there are many more not pictured).



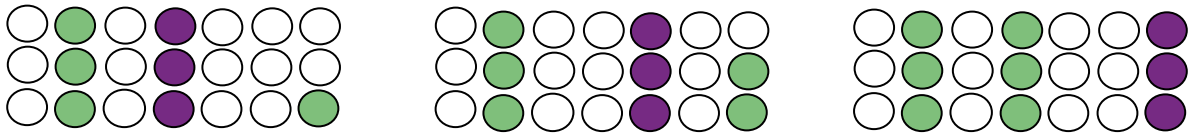
The set on the left includes all 7-core abaci with spacing 1 and maximum position strictly less than 21 that also do not have beads in positions 17, 18, 19, and 20. These are exactly the 7-core abaci with spacing 1 and maximum position strictly less than 17, which is the set $\mathcal{A}_{7,3,-4}$.

We now move to the set on the right, and our goal is to show that the abaci in that set can be counted by the generating polynomial

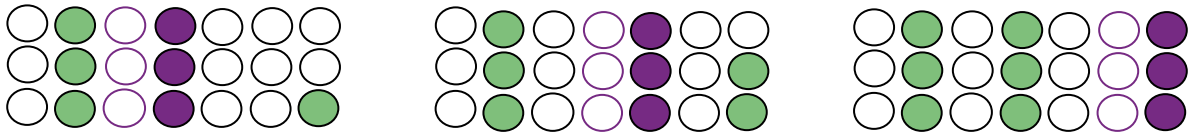
$$q^3 \sum_{k=1}^{4} \boldsymbol{A}_{7-k-1,3}(q)\boldsymbol{A}_{k-1,2}(q),$$

where q keeps track of the number of beads in the diagram. We will do it with pictures first, and then talk about how the pictures match the mathematical expression.

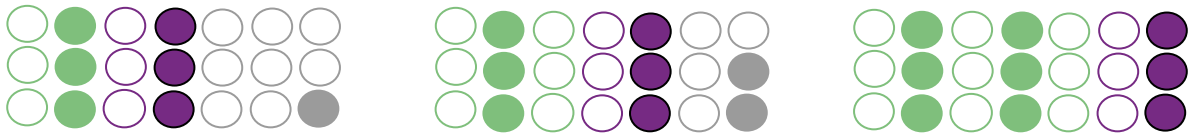
Since each element of this set has at least one bead in the last four columns of the top row, that must mean there is a 'full' column in the last four columns. We highlight the rightmost 'full' column of our three examples below.



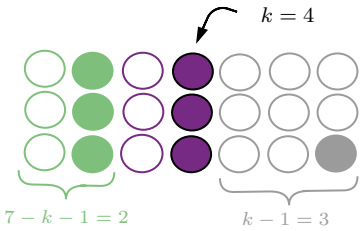
This 'breaks up' the diagram into two separate parts, and the size of those parts depends on where our 'full' column is. Since our original abaci are 1-distinct, there are empty columns of spacers on either side of the full column. The one on the right will get included in the abacus, and we will hold on to the one on the left, tying it to the full column.



Then, for each of our examples, we have an abacus on the left, and an abacus on the right. Note that in the rightmost example the abacus on the right is the 'empty abacus.'

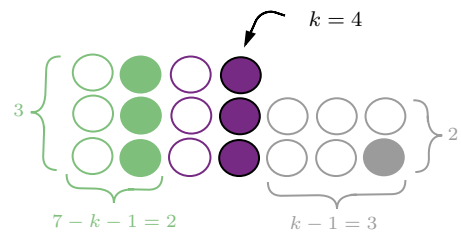


Our highest purple bead (the highest placed bead in the rightmost full column) must have been in position 17, 18, 19, or 20. We will think about these as $21-4$, $21-3$, $21-2$, and $21-1$, respectively (generally, $21-k$, counting as the kth space from the right side in the top row). We can describe the other abaci in terms of k, working through each of the three examples separately.

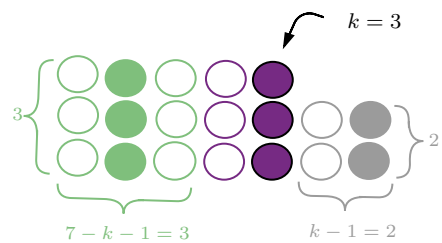


In the first example, the highest placed bead is in position 17, or $(21-4)$. The abacus to the left (green) is a 2-core abacus (we note here that $7-4-1=2$, and will come back to this later). The abacus to the right (gray) is a 3-core abacus (note $4-1=3$).

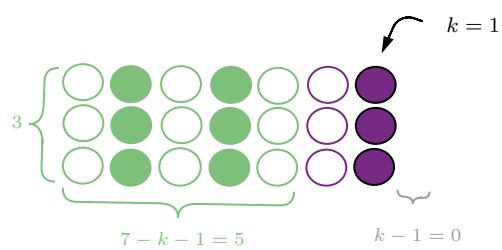
We will also note that the maximum position of the left 2-core abacus is strictly less than $6 = 3(2)$. The maximum position is counted only using positions in the green abacus, and the 3 comes from our initial m-value. The maximum position in the gray abacus is less than $6 = 2(3)$. Since our highest placed bead is counted to the left of the gray abacus, there will not be any beads in the top row of the gray abacus, so we can think about the 2 as one less than our initial m-value ($2 = m - 1$).



We now take the next example, and note that similarly, the highest placed bead is in position 18 ($21 - 3$), so we have $k = 3$. The abacus to the left (green) is a 3-core abacus (note, $7 - 3 - 1 = 3$) with maximum position less than $9 = 3(3)$. The abacus to the right (gray) is a 2-core abacus (note, $4 - 1 = 3$) with maximum position less than $4 = 2(2)$.



For our last example, the the highest placed bead is in position 20 ($21 - 1$), so we have $k = 1$. The abacus to the left (green) is a 5-abacus (note, $7 - 1 - 1 = 5$) with maximum position less than $15 = 3(5)$. There is no abacus to the right, so we think of it as the empty abacus is to the right.



Now we will talk about how these examples relate to the notation $q^3 \sum_{k=1}^{4} \boldsymbol{A}_{7-k-1,3}(q)\boldsymbol{A}_{k-1,2}(q)$. The functions inside the sum are generating polynomials:

- $\boldsymbol{A}_{7-k-1,3}(q)$ is the generating polynomial for $(7 - k - 1)$-core abaci with maximum position less than $3(7 - k - 1)$ into distinct parts, where q counts the number of parts.

- $\boldsymbol{A}_{k-1,2}(q)$ is the generating polynomial for $(k-1)$-core abaci with maximum position less than $2(k-1)$ into distinct parts, where q counts the number of parts.

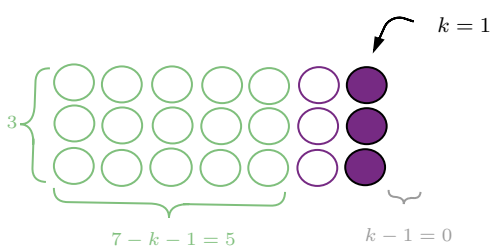
If we expand the expression

$$q^3 \sum_{k=1}^{4} \boldsymbol{A}_{7-k-1,3}(q)\boldsymbol{A}_{k-1,2}(q) = q^3\boldsymbol{A}_{5,3}(q)\boldsymbol{A}_{0,2} + q^3\boldsymbol{A}_{4,3}(q)\boldsymbol{A}_{1,2} + q^3\boldsymbol{A}_{3,3}(q)\boldsymbol{A}_{2,2} + q^3\boldsymbol{A}_{2,3}(q)\boldsymbol{A}_{3,2},$$

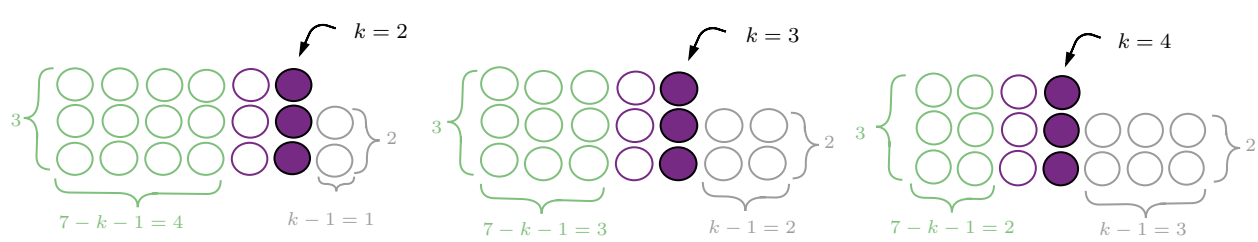
each term has a q^3 multiplied by a term from the generating polynomial $\boldsymbol{A}_{7-k-1,3}(q)$ multiplied by a term from the generating polynomial $\boldsymbol{A}_{k-1,2}(q)$.

In each example the q^3 corresponds to our full column of beads (purple), the green abacus is a member of the set corresponding to $\boldsymbol{A}_{7-k-1,3}(q)$, and the gray abacus is a member of the set corresponding to $\boldsymbol{A}_{k-1,2}(q)$, for the given values of k.

We count all of the abaci in our original set by summing over k values between 1 and 4. So, the sum would first count the abaci that fit into the following form. They are counted by the term $q^3\boldsymbol{A}_{5,3}(q)\boldsymbol{A}_{0,2}(q)$, and we note that $\boldsymbol{A}_{0,2}(q) = 1$ (there is only one way to have the empty abacus), so this is equivalent to saying $q^3\boldsymbol{A}_{5,3}(q)$. I think about this as adding the purple columns to every abacus in the set $\mathcal{A}_{5,3}$, abaci whose beads would fit in the green outlined dimensions below.



So, we have accounted for the first term in the sum. Similarly, each of the next terms represent abaci whose beads would fit in the dimensions outlined in the three remaining cases below.



Each of these cases corresponds to a term in the sum

$$q^3 \sum_{k=1}^{4} \boldsymbol{A}_{7-k-1,3}(q)\boldsymbol{A}_{k-1,2}(q) = q^3\boldsymbol{A}_{5,3}(q)\boldsymbol{A}_{0,2} + q^3\boldsymbol{A}_{4,3}(q)\boldsymbol{A}_{1,2} + q^3\boldsymbol{A}_{3,3}(q)\boldsymbol{A}_{2,2} + q^3\boldsymbol{A}_{2,3}(q)\boldsymbol{A}_{3,2},$$

Each 7-core abacus in the remaining set is counted in exactly one of the 4 cases, so the expression does count what we claimed it does. When we put it all together, we can say that the Lemma is true for this example.

TRY: Determine which of the terms count each of the three examples we used. Come up with a different example that would be counted in each of the terms.

General Proof of Lemma 30. Suppose $s > r$. Observe that $\boldsymbol{A}_{s,m}(q) - \boldsymbol{A}_{s,m,-r}(q)$ consists of those s-core abaci with spacing $d = 1$, which have m rows and last bead in position $ms - k$ for $k = 1, 2, \ldots, r$. Fix one of these values for k, corresponding to one of the last r columns. That column always contains m beads, contributing q^m to Equation (4.4). The $k-1$ columns after that column form an abacus in $\mathcal{A}_{k-1,m-1}$, while the first $s-k-1$ columns form an abacus in $\mathcal{A}_{s-k-1,m}$. The one remaining column is the empty column preceding the column with m beads.

If $s = r$, the same argument still applies but $k = r$ is not possible. Since $\boldsymbol{A}_{-1,m}(q) = 0$, the summand corresponding to $k = r$ is zero, so that Equation (4.4) still holds. □

Diagramming and Gesture

In the example for Lemma 30, I use **nested diagrams** to **organize information through spatial diagrams**. By thinking of the set of abaci as a group of objects, we then organize the objects according to a given characteristic. This is the first example of using the abaci as elements of a larger diagram, keeping each individual abacus fixed and moving them around in relation to one another. However, the abaci themselves are still dynamic diagrams (especially when we think about adding or removing beads). We used **color** to illustrate some of these dynamic components.

When I was first drawing these examples, I would draw the set all together. The impulse to move them around into groups is a **spontaneous covert gesture**. I would not physically gesture to move the diagrams, but rather think about where and how I might move them, and then redraw the examples to illustrate. The gesture is covert because there is not an external or visible component (even if it isn't visible, thinking about moving activates the same parts of the brain).

4.2 Enumerating s-core Abaci of Bounded Height

4.2.1 Primary Results

We recall the statement about abacus diagrams proved in [8], restated from Theorem 29.

Theorem. *Let $d, m \geq 1$, and $r \geq 0$. If $s > d+1$ and $s > r$, then*

$$\boldsymbol{A}^d_{s,m,r}(q) = \boldsymbol{A}^d_{s-1,m,r}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}^d_{s-d-1,m,r}(q). \tag{4.5}$$

In this section, we will show that the same recurrence holds for $\boldsymbol{A}^d_{s,m,-r}(q)$, with slightly different restrictions on initial conditions for the variables. We recall that $\boldsymbol{A}^d_{s,m,-r}(q)$ is the generating polynomial that counts the set $\mathcal{A}^d_{s,m,-r}$ of s-core abacus diagrams that have spacing d and maximum position less than $ms - r$. Many of the results in this section use arguments involving finding bijections between sets of abaci, while some others use algebraic manipulation. For each statement (Lemma or Theorem), we give an outline of the proof argument, with the complete proof below. We will prove first Lemma 31, which we use in the complete proof of Theorem 32. Then, we will discuss cases in which the initial conditions on the recurrence may be relaxed, in particular when r is small, and when $d = 1$.

Lemma 31. *Let $d, m \geq 1$.*

$$\boldsymbol{A}^d_{2d+2,m,-1}(q) = \boldsymbol{A}^d_{2d+1,m,-1}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}^d_{d+1,m,-1}(q). \tag{4.6}$$

Lemma 31 Restated: The recurrence holds for any $d, m \geq 1$ when $r = 1$, and $s = 2d + 2$.

Proof Sketch: Our objective is to prove the statement

$$\boldsymbol{A}^d_{2d+2,m,-1}(q) = \boldsymbol{A}^d_{2d+1,m,-1}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}^d_{d+1,m,-1}(q).$$

We actually prove the equivalent statement

$$\boldsymbol{A}^d_{2d+2,m,-1}(q) - \boldsymbol{A}^d_{2d+1,m,-1}(q) = (q + q^2 + \ldots + q^m)\boldsymbol{A}^d_{d+1,m,-1}(q).$$

We will take the set of abaci counted by the left side of the second equation and show that it is equal to the set of abaci counted by the right side of the equation. The proof follows the steps below:

1) We get the expanded generating polynomial $\boldsymbol{A}^d_{d+1,m,-1}(q)$ by explicitly counting the $(d+1)$-core abaci in

$\mathcal{A}^d_{d+1,m,-1}$, so we know that the right hand side of the equation is

$$(q+q^2+\ldots+q^m)\boldsymbol{A}^d_{d+1,m,-1}(q) = (q+q^2+\ldots+q^m)(1+(d-1)q)(1+q+\ldots+q^{m-1}).$$

2) We note that we can build some of the abaci in $\mathcal{A}^d_{2d+2,m,-1}$ by appending an empty column to abaci in $\mathcal{A}^d_{2d+1,m,-1}$.

3) The abaci that are in the set $\mathcal{A}^d_{2d+2,m,-1}$ and cannot be obtained by appending an empty abacus to an abacus in the set $\mathcal{A}^d_{2d+1,m,-1}$ are counted by the generating polynomial $\boldsymbol{A}^d_{2d+2,m,-1}(q) - \boldsymbol{A}^d_{2d+1,m,-1}(q)$. [Note we are saying this set is counted using the generating polynomial on the left side of the equation.] Each of these fall into one of three categories:

(i) Abaci in $\mathcal{A}^d_{2d+2,m,-1}$ that have beads in the last column (between 1 and $m-1$ beads, since there cannot be a bead in position $ms-1$).

(ii) Abaci in $\mathcal{A}^d_{2d+2,m,-1}$ that do not have beads in the last column (empty last column) with the gap g between the last bead in the first row and the first bead in the last row exactly equal to d.

(iii) Abaci in $\mathcal{A}^d_{2d+2,m,-1}$ for which the second to last column contains m beads and, to avoid double counting, there are more than d spaces following the last bead in the first row. These are the only abaci that could have an empty last column, gap $g > d$, and not have been formed by appending an empty column to abaci in $\mathcal{A}^d_{2d+1,m,-1}$ (in other words, the abaci that are left over).

We find the generating polynomial for each of these cases, add them up, and show that the sum is equal to the generating polynomial $(q+q^2+\ldots+q^m)\boldsymbol{A}^d_{d+1,m,-1}(q)$. So, we are saying that the same set of abaci is counted by both the left hand side and the right hand side, thus proving the two sides of the equation are equal.

The proof below gives the argument sketched above in more formal detail, with images to illustrate that represent the case $d = 4$ and $m = 3$.

Proof. We need to show that

$$\boldsymbol{A}^d_{2d+2,m,-1}(q) = \boldsymbol{A}^d_{2d+1,m,-1}(q) + (q+q^2+\ldots+q^m)\boldsymbol{A}^d_{d+1,m,-1}(q). \tag{4.7}$$

Because each abacus in $\mathcal{A}^d_{d+1,m,-1}$ has $d+1$ columns and spacing d, there can be at most one column of beads, we get:

$$\boldsymbol{A}^d_{d+1,m,-1}(q) = \text{ the empty abacus } + \text{ abaci where the last column is the one with beads}$$

$$+(d-1)(\text{abaci where any other column has beads}) =$$

$$q^0 + (q^1 + q^2 + \ldots q^{m-1}) + (d-1)(q^1 + \cdots + q^m)$$

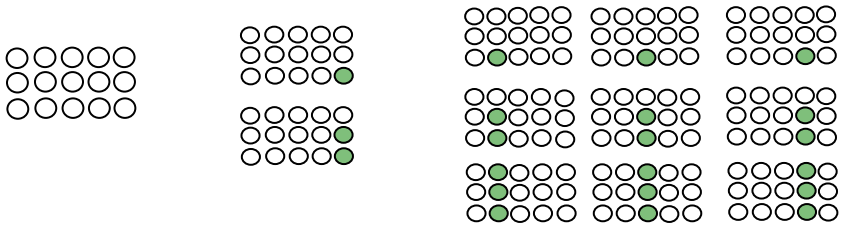


Figure 4.2: Abaci in the set $\mathcal{A}^4_{4+1,3,-1}$ organized to illustrate Equation (4.8).

$$\begin{aligned} \boldsymbol{A}^d_{d+1,m,-1}(q) &= 1 + (q + q^2 + \ldots + q^{m-1}) + (d-1)(q + q^2 + \ldots + q^m) \\ &= (1 + (d-1)q)(1 + q + \ldots + q^{m-1}). \end{aligned} \tag{4.8}$$

On the other hand, abaci in $\mathcal{A}^d_{2d+2,m,-1}$ and $\mathcal{A}^d_{2d+1,m,-1}$ have at most two columns of beads. Appending an empty column to each abacus in $\mathcal{A}^d_{2d+1,m,-1}$ yields an abacus in $\mathcal{A}^d_{2d+2,m,-1}$. The abaci in $\mathcal{A}^d_{2d+2,m,-1}$ not obtained in this fashion fall into three groups:

(i) Those for which the last column is not empty (and hence contains $1, 2, \ldots, m-1$ many beads). Since these abaci have exactly $2d+2$ columns, there can be at most two columns with beads in them (to satisfy spacing d. One of these is the last column by definition, and then there are either no other beads, beads in column $d+1$ (up to m of them), or beads in one of the columns $1, \ldots, d$ (only one bead is possible in these to maintain spacing d. We can think of the generating polynomial as a product (choices of beads in the last column) $\times$ (choices for other beads). The generating polynomial for these abaci is

$$(q + q^2 + \ldots + q^{m-1})(1 + (q + q^2 + \ldots + q^m) + (d-1)q),$$

based on whether there is no second column of beads (contributing the 1 in the second factor), there is a column of beads including position d (contributing $q + q^2 + \ldots + q^m$), or there is a single bead in positions $1, 2, \ldots, d-1$ (contributing $(d-1)q$).

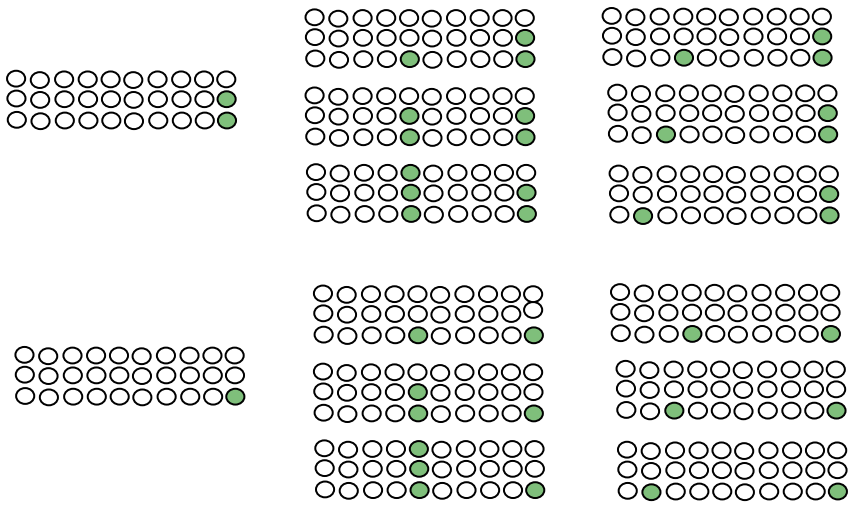


Figure 4.3: Abaci in $\mathcal{A}^4_{2(4)+2,3,-1}$ that have nonempty last column.

Each possible abacus is created as a product of one term from the first factor times one term from the second factor.

ii Those for which the last column is empty and there is a gap of exactly d spaces between the last bead in the first row and the first bead in the second row. Since there are $d-1$ possibilities for the location of the first column of beads (containing $2, 3, \ldots, m$ many beads), each of which determines the location of the second column (containing $1, 2, \ldots, m$ beads), the generating polynomial for these abaci is

$$(d-1)(q^2+q^3+\ldots+q^m)(q+q^2+\ldots+q^m).$$

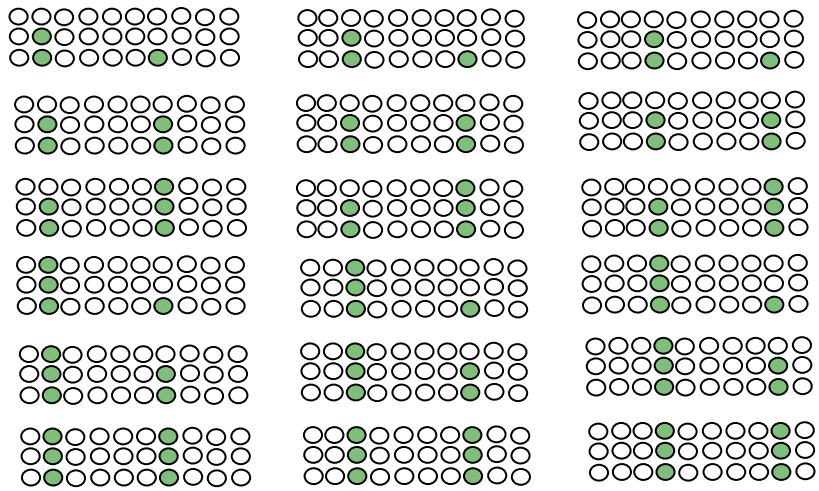


Figure 4.4: Abaci in $\mathcal{A}^4_{2(4)+2,3,-1}$ with empty last column and exactly 4 spacers between the last bead in the first row and first bead in the second row.

iii Those for which the second to last column contains m beads and, to avoid double counting, there are more than d spaces following the last bead in the first row. The generating polynomial for these abaci is

$$q^m(1 + (d-1)q),$$

because, in addition to the column of m beads, there can only be a single bead in positions $1, 2, \ldots, d-1$.

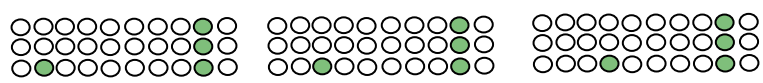


Figure 4.5: Abaci in $\mathcal{A}^4_{2(4)+2,3,-1}$ with 3 beads in the second-to-last column and gap between the last bead in the first column and first bead in the second column > 4.

Adding these three generating polynomials, we have shown that

$$\boldsymbol{A}^d_{2d+2,m,-1}(q) - \boldsymbol{A}^d_{2d+1,m,-1}(q) = q(1 + (d-1)q)(1 + q + \ldots + q^{m-1})^2.$$

Together with Equation (4.8), this proves Equation (4.7). □

Diagramming and Gesture

Similar to Lemma 30, to prove this theorem we did a lot of examples **organizing information** using **nested diagrams**. For example, I would think about the set $\mathcal{A}^d_{d+1,m,-1}$ as a group of diagrams that can be moved around in relation to each other. Then, to look at the generating polynomial for this, I would try to organize the diagrams based on characteristics to make sure everything was getting counted appropriately.

I attempted to illustrate this in the proof by writing out

$$\boldsymbol{A}^d_{d+1,m,-1}(q) = \text{ the empty abacus } + \text{ abaci where the last column is the one with beads}$$

$$+(d-1)(\text{abaci where any other column has beads}) =$$

$$q^0 + (q^1 + q^2 + \ldots q^{m-1}) + (d-1)(q^1 + \cdots + q^m).$$

This is meant to illustrate the process of starting with the set of abaci counted by the generating polynomial, moving the diagrams around into relevant groups, and then translating back to generating functions. When I became more comfortable with the generating polynomial notation, I would often not need to draw all of the pictures multiple times, instead having 'general' diagrams stand in for each term of the sum in my notes.

Since s is the variable that is changing (we have a recurrence on s), we can get the generating polynomial for a value of s by combining generating polynomials for smaller values (in particular $s-1$ and $s-d-1$). We note that the other variables m, d, and r remain fixed in the recurrence. We will use the notation $f_s(q)$ to represent $\boldsymbol{A}^d_{s,m,-r}(q)$ in the following Theorem, and throughout the rest of the section. This can be helpful because it lets us focus on s, the variable that changes and how it changes (since m, d, and $-r$ stay the same in the recurrence).

Theorem 32. *Let $d, m \geq 1$, $r > 0$, and write $f_s(q) = \boldsymbol{A}^d_{s,m,-r}(q)$. If $s > 2d + r$, then*

$$f_s(q) = f_{s-1}(q) + (q + q^2 + \ldots + q^m) f_{s-d-1}(q). \tag{4.9}$$

Theorem 32 Restated: The same recurrence on s holds for $\boldsymbol{A}^d_{s,m,-r}(q)$ when $s > 2d + r$ (for any choice of d, m, and r where $d, m \geq 1$, $r > 0$).

Proof Sketch: The proof below relies on rewriting the desired equations to be able to use Theorem 29 and the

process of proof by induction to show show the equivalence between the left and right sides of the equation. We do this in two parts, first the case $r = 1$ and $s > 2d+2$, and then the rest (following a similar argument). In each of the two parts:

1) We use the definition of the diagrams and Theorem 29 to expand the left side of the equation into a sum of four different generating functions.

2) Then we show that each part of that sum satisfies the same recurrence, so the left side of the equation must also satisfy this recurrence.

By showing the statement is true for $r = 1$ and $s > 2d + 2$, and taking that along with Lemma 31 (which proves the recurrence for the case $r = 1$ and $s = 2d + 2$), we get that the statement is true for $r = 1$, when $s > 2d + 1 = 2d + r$.

Once we know this, we use it as the base case of our induction. The general idea of the induction part of the argument is that if we know the statement is true for $r = 1$, we can use that fact to show $r = 2$. Then, if we know it is true for $r = 1$ and $r = 2$, we could use this to show that it is true for $r = 3$, and so on. Instead of showing those steps one at a time, and then saying 'and so on,' we assume the statement is true for all $r' = 1, \ldots r - 1$, and use that assumption in our argument for a general value of $r > 1$. The argument would hold if we substitute $r = 2$, or $r = 3$, etc., so we can then say that it is true for all $r > 1$. The part where we 'assume the statement is true for $r' = 1, \ldots r - 1$' is called the induction hypothesis.

We separate the proof into two components below, first proving the case $r = 1$ for $s > 2d + 2$, and then separately proving the case $r > 1$, for $s > 2d + r$

Proof of Theorem 32 (Case $r = 1$, $s > 2d + 2$). Let $d, m \geq 1$. We first prove the claim that Equation (4.9) holds when $r = 1$ and $s > 2d + 2$. Let t be an integer such that $t > d + 1$ (note: $t > s - d - 1$). By the definition of the abacus diagrams, we have

$$\boldsymbol{A}^d_{t,m,-r}(q) = \boldsymbol{A}^d_{t,m-1,t-r}(q). \tag{4.10}$$

The left side of the equation is the generating polynomial that counts t-core abacus diagrams with spacing d and maximum part less than $mt - r$. The right side of the equation is the generating polynomial that counts t-core abaci with spacing d and maximum part less than $(m - 1)t + (t - r) = mt - r$.

In the current case, we note that Equation (4.10) for our given $r = 1$, and $t > d + 1$ is

$$\boldsymbol{A}^d_{t,m,-1}(q) = \boldsymbol{A}^d_{t,m-1,t-1}(q). \tag{4.11}$$

We recall Theorem 29: *Let $d, m \geq 1$, and $r \geq 0$. If $s > d + 1$ and $s > r$, then*

$$\boldsymbol{A}^d_{s,m,r}(q) = \boldsymbol{A}^d_{s-1,m,r}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}^d_{s-d-1,m,r}(q).$$

Since our $t > d + 1$ by definition, and $t > 1 = r$, and $t - 1 \geq 0$, we can apply Theorem 29 to the right side of Equation (4.11) to get:

$$\boldsymbol{A}^d_{t,m,-1}(q) = \boldsymbol{A}^d_{t-1,m-1,t-1}(q) + (q + q^2 + \ldots + q^{m-1})\boldsymbol{A}^d_{t-d-1,m-1,t-1}(q) \tag{4.12}$$

Then, applying Equation (4.10) to each of the generating function terms of the sum on the right side, we have

$$\boldsymbol{A}^d_{t-1,m-1,t-1}(q) = \boldsymbol{A}_{(t-1),m,(t-1)-0} = \boldsymbol{A}^d_{t-1,m,0}(q), \quad \text{and}$$

$$\boldsymbol{A}^d_{t-d-1,m-1,t-1}(q) = \boldsymbol{A}^d_{(t-d-1),m-1,(t-d-1)-(-d)}(q) = \boldsymbol{A}^d_{t-d-1,m,d}(q).$$

Where instead of the t and r in the first equation we have $t - 1$ and 0, and in the second equation, we have $t-d-1$ and $-d$. We note that for $r = 0$ we typically leave off that index and write $\boldsymbol{A}^d_{t-1,m,0}(q) = \boldsymbol{A}^d_{t-1,m}(q)$. Substituting these both in to Equation (4.12), we have

$$\boldsymbol{A}^d_{t,m,-1}(q) = \boldsymbol{A}^d_{t-1,m}(q) + (q + q^2 + \ldots + q^{m-1})\boldsymbol{A}^d_{t-d-1,m,d}(q). \tag{4.13}$$

This actually looks fairly close to the recurrence that we want. We want to use the tools we have, and one of the tools we have is Theorem 29. A similar statement to Equation (4.13) that is true by Theorem 29 is:

$$\boldsymbol{A}^d_{t,m,d}(q) = \boldsymbol{A}^d_{t-1,m,d}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}^d_{t-d-1,m,d}(q).$$

Then, we note that we can split up the rightmost term to get

$$\boldsymbol{A}^d_{t,m,d}(q) = \boldsymbol{A}^d_{t-1,m,d}(q) + (q + q^2 + \ldots + q^{m-1})\boldsymbol{A}^d_{t-d-1,m,d}(q) + q^m\boldsymbol{A}^d_{t-d-1,m,d}(q)$$

and, subtracting over to the other side, we get

$$\boldsymbol{A}^d_{t,m,d}(q) - \boldsymbol{A}^d_{t-1,m,d}(q) - q^m \boldsymbol{A}^d_{t-d-1,m,d}(q) = (q + q^2 + \ldots + q^{m-1})\boldsymbol{A}^d_{t-d-1,m,d}(q).$$

The point of doing this is so that the right-hand side is exactly a term of Equation (4.13), so we can make the following substitution, and we remember that $t \geq s - d - 1$.

$$\boldsymbol{A}^d_{t,m,-1}(q) = \boldsymbol{A}^d_{t-1,m}(q) + \boldsymbol{A}^d_{t,m,d}(q) - \boldsymbol{A}^d_{t-1,m,d}(q) - q^m \boldsymbol{A}^d_{t-d-1,m,d}(q) \tag{4.14}$$

Now, since this is true for $t \geq s - d - 1$, it is true in particular for $t = s$. Our last step is to show that the generating polynomial in each of the four terms on the right individually satisfy the recurrence in the case $t = s$. Since we are working in the case $s > 2d + 2$, we will show that when $t = s$, for each of these terms, the conditions of Theorem 29 are satisfied. We note that Theorem 29 applies to $\boldsymbol{A}^d_{s,m,r}(q)$ when $m, d \geq 1$, $r \geq 0$ and $s \geq d + 1$, and recall that we already have $m, d \geq 1$, so what is left to show is only $r \geq 0$ and $s \geq d + 1$. We will also assign each of these a shortened name to make the next step a little clearer.

$$A^1_s = \boldsymbol{A}^d_{s-1,m}(q), \quad [r = 0,\ s - 1 > 2d + 1 > d + 1]$$

$$A^2_s = \boldsymbol{A}^d_{s,m,d}(q) \quad [r = d > 0,\ s > 2d + 2 > d + 1]$$

$$A^3_s = \boldsymbol{A}^d_{s-1,m,d}(q) \quad [r = d > 0,\ s - 1 > 2d + 1 > d + 1]$$

$$A^4_s = \boldsymbol{A}^d_{s-d-1,m,d}(q) \quad [r = d > 0,\ s - d - 1 > 2d + 2 - d - 1 = d + 1]$$

Since the conditions of Theorem 29 are satisfied, we have:

$$A^1_s = A^1_{s-1} + (q + q^2 + \ldots + q^m)A^1_{s-d-1} \qquad A^3_s = A^3_{s-1} + (q + q^2 + \ldots + q^m)A^3_{s-d-1}$$

$$A^2_s = A^2_{s-1} + (q + q^2 + \ldots + q^m)A^2_{s-d-1} \qquad A^4_s = A^4_{s-1} + (q + q^2 + \ldots + q^m)A^4_{s-d-1}$$

Then since each of these were terms of Equation (4.14), we substitute them back in to get:

$$\boldsymbol{A}^d_{s,m,-1}(q) = A^1_s + A^2_s - A^3_s - q^m A^4_s$$

$$= A^1_{s-1} + (q + q^2 + \ldots + q^m)A^1_{s-d-1} + A^2_{s-1} + (q + q^2 + \ldots + q^m)A^2_{s-d-1}$$

$$- \left[A^3_{s-1} + (q + q^2 + \ldots + q^m)A^3_{s-d-1}\right] - q^m \left[A^4_{s-1} + (q + q^2 + \ldots + q^m)A^4_{s-d-1}\right]$$

and with a little rearranging, this becomes

$$\boldsymbol{A}^d_{s,m,-1}(q) = A^1_{s-1} + A^2_{s-1} - A^3_{s-1} - q^m A^4_{s-1}$$

$$+(q + q^2 + \ldots + q^m)\left[A^1_{s-d-1} + A^2_{s-d-1} - A^3_{s-d-1} - q^m A^4_{s-d-1}\right].$$

By showing the recurrence is true for each of the terms of the sum, we have shown that

$$\boldsymbol{A}^d_{s,m,-1}(q) = \boldsymbol{A}^d_{s-1,m,-1}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}^d_{s-d-1,m,-1}(q) \tag{4.15}$$

Thus the recurrence holds when $r = 1$, for $s > 2d + 2$. Combining this with Lemma 31, we have that the recurrence holds when $r = 1$ for $s > 2d + 1$. □

We now proceed to show that the statement is true for $r > 1$ when $s > 2d + r$.

Proof of Theorem 32 (Case $r > 1$, $s > 2d + r$). Let us prove the claim that Equation (4.9) holds for all $s > 2d + r$. Fix $r > 1$ and suppose, for the purposes of induction, that this claim is true when r is replaced with $1, 2, \ldots, r - 1$. In the following, we are going to show that the claim holds for r as well, using the assumption that it is true for each of $1, 2, \ldots, r - 1$. This assumption is called the induction hypothesis.

Suppose that $s > 2d + r$. In the following, let t be such that $t \geq s - d - 1$ (we intend to rewrite each of the terms in Equation (4.9) in terms of t later on). Note that the condition $s > 2d + r$ translates into $t > d + r - 1$.

We recall Equation (4.10), which is a consequence of the definition of abacus diagrams and states that, for any integer $t \geq 0$, we have

$$\boldsymbol{A}^d_{t,m,-r}(q) = \boldsymbol{A}^d_{t,m-1,t-r}(q).$$

Since $t \geq r$ and $t > d + 1$, we may apply Theorem 29 to the right-hand side of Equation (4.10) to obtain

$$\boldsymbol{A}^d_{t,m,-r}(q) = \boldsymbol{A}^d_{t-1,m-1,t-r}(q) + (q + q^2 + \ldots + q^{m-1})\boldsymbol{A}^d_{t-d-1,m-1,t-r}(q) \tag{4.16}$$

Then, we rewrite each of the generating functions on the right side so that the notation matches the right side of Equation (4.10), and then apply Equation (4.10) to each, to get

$$\boldsymbol{A}^d_{t-1,m-1,t-r}(q) = \boldsymbol{A}_{(t-1),m-1,(t-1)-(r-1)} = \boldsymbol{A}^d_{t-1,m,1-r}(q), \quad \text{and}$$

$$\boldsymbol{A}^d_{t-d-1,m-1,t-r}(q) = \boldsymbol{A}^d_{(t-d-1),m-1,(t-d-1)-(r-d-1)}(q) = \boldsymbol{A}^d_{t-d-1,m,d+1-r}(q).$$

Putting these back in Equation (4.16), we get

$$\boldsymbol{A}^d_{t,m,-r}(q) = \boldsymbol{A}^d_{t-1,m,1-r}(q) + (q + q^2 + \ldots + q^{m-1})\boldsymbol{A}^d_{t-d-1,m,d+1-r}(q). \tag{4.17}$$

Again, this looks fairly close to the recurrence that we want. We want to use the tools we have, and one of the tools we have is Theorem 29. A similar statement to Equation (4.13) that is true by Theorem 29 is:

$$\boldsymbol{A}^d_{t,m,d+1-r}(q) = \boldsymbol{A}^d_{t-1,m,d+1-r}(q) + (q + q^2 + \ldots + q^{m})\boldsymbol{A}^d_{t-d-1,m,d+1-r}(q), \tag{4.18}$$

We know that this equation satisfies the initial conditions of Theorem 29 because either $d + 1 - r \geq 0$, or $d + 1 - r < 0$ and the condition $t > 2d + (r - d - 1) = d + r - 1$ is satisfied, so we can use the induction hypothesis. Rewriting Equation (4.18), we have

$$\boldsymbol{A}^d_{t,m,d+1-r}(q) = \boldsymbol{A}^d_{t-1,m,d+1-r}(q) + (q + q^2 + \ldots + q^{m-1})\boldsymbol{A}^d_{t-d-1,m,d+1-r}(q) + q^m\boldsymbol{A}^d_{t-d-1,m,d+1-r}(q),$$

and then by rearranging we get,

$$\boldsymbol{A}^d_{t,m,d+1-r}(q) - \boldsymbol{A}^d_{t-1,m,d+1-r}(q) - q^m\boldsymbol{A}^d_{t-d-1,m,d+1-r}(q) = (q+q^2+\ldots+q^{m-1})\boldsymbol{A}^d_{t-d-1,m,d+1-r}(q). \tag{4.19}$$

Then, making a substitution on the left side of Equation (4.19) for $(q + q^2 + \ldots + q^{m-1})\boldsymbol{A}^d_{t-d-1,m,d+1-r}(q)$ in Equation (4.17), we conclude that, for $t \geq s - d - 1$,

$$\begin{aligned} \boldsymbol{A}^d_{t,m,-r}(q) &= \boldsymbol{A}^d_{t-1,m,1-r}(q) - q^m\boldsymbol{A}^d_{t-d-1,m,d+1-r}(q) \\ &+ \boldsymbol{A}^d_{t,m,d+1-r}(q) - \boldsymbol{A}^d_{t-1,m,d+1-r}(q). \end{aligned} \tag{4.20}$$

Now, since this is true for $t \geq s - d - 1$, it is true in particular for $t = s$. Our last step is to show that the generating polynomial in each of the four terms on the right individually satisfy the recurrence in the case $t = s$. Since we are working in the case $s > 2d + r$, we will show that when $t = s$, for each of these terms, the recurrence is satisfied either because the initial conditions of Theorem 29 are satisfied, or due to the induction hypothesis . We note that Theorem 29 applies to $\boldsymbol{A}^d_{s,m,r}(q)$ when $m, d \geq 1$, $r \geq 0$ and $s \geq d + 1$, and recall that we already have $m, d \geq 1$, so what would need to be shown is $r \geq 0$ and $s \geq d + 1$. At this

point, it is helpful to restate our induction hypothesis. We use r_1 in place of r, and the induction hypothesis (what we assume to be true) is that the recurrence below is true for for all r_1 in the set $1, \ldots, r-1$

$$\boldsymbol{A}^d_{s,m,-r_1}(q) = \boldsymbol{A}^d_{s-1,m,-r_1}(q) + (q + q^2 + \ldots + q^m)A^d_{s-d-1,m,-r_1}(q).$$

Below we indicate why each generating polynomial term of the right side of Equation (4.20) either is covered by the induction hypothesis, or satisfies the initial conditions of Theorem 29.

$\boldsymbol{A}^d_{s-1,m,1-r}(q)$ is the case $-r_1 = 1 - r$, so $0 < r_1 < r$ and $s_1 = s - 1 > 2d + r - 1 > 2d + r_1$, satisfying the induction hypothesis. For each of the remaining three generating polynomial terms, $\boldsymbol{A}^d_{s-d-1,m,d+1-r}(q)$, $\boldsymbol{A}^d_{s,m,d+1-r}(q)$, and $\boldsymbol{A}^d_{s-1,m,d+1-r}(q)$, there are two options. If $d+1-r \geq 0$, then this, along with $s > 2d+r$, implies s, $s-1$, and $s-d-1$ are all $> r$ and $> d+1$, satisfying the initial conditions of Theorem 29. In the other case, if $d+1-r < 0$, then $-r_1 = d+1-r < 0$, which implies $1 < r_1 < r$, and s, $s-1$, and $s-d-1$ are all greater than $2d+r_1$, so we can use the induction hypothesis. Each of these has been shown to satisfy the recurrence, and we rewrite them in a simplified notation to make the last step a little bit clearer:

$$\boldsymbol{A}^d_{s-1,m,1-r}(q) = A^1_s = A^1_{s-1} + (q + q^2 + \ldots + q^m)A^1_{s-d-1},$$

$$\boldsymbol{A}^d_{s-d-1,m,d+1-r}(q) = A^2_s = A^2_{s-1} + (q + q^2 + \ldots + q^m)A^2_{s-d-1},$$

$$\boldsymbol{A}^d_{s,m,d+1-r}(q) = A^3_s = A^3_{s-1} + (q + q^2 + \ldots + q^m)A^3_{s-d-1},$$

$$\boldsymbol{A}^d_{s-1,m,d+1-r}(q) = A^4_s = A^4_{s-1} + (q + q^2 + \ldots + q^m)A^4_{s-d-1}.$$

Then since each of these were terms of Equation (4.20), we substitute them back in to get:

$$\boldsymbol{A}^d_{s,m,-r}(q) = A^1_s + A^2_s - A^3_s - q^m A^4_s = A^1_{s-1} + (q+q^2+\ldots+q^m)A^1_{s-d-1} + A^2_{s-1} + (q+q^2+\ldots+q^m)A^2_{s-d-1}$$

$$- \left[A^3_{s-1} + (q + q^2 + \ldots + q^m)A^3_{s-d-1}\right] - q^m \left[A^4_{s-1} + (q + q^2 + \ldots + q^m)A^4_{s-d-1}\right]$$

and with a little rearranging, this becomes

$$\boldsymbol{A}^d_{s,m,-r}(q) = A^1_{s-1} + A^2_{s-1} - A^3_{s-1} - q^m A^4_{s-1} + (q+q^2+\ldots+q^m)\left[A^1_{s-d-1} + A^2_{s-d-1} - A^3_{s-d-1} - q^m A^4_{s-d-1}\right].$$

Since the recurrence is true for each of the terms of the sum, we have shown that

$$\boldsymbol{A}^d_{s,m,-r}(q) = \boldsymbol{A}^d_{s-1,m,-r}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}^d_{s-d-1,m,-r}(q) \tag{4.21}$$

for the conditions $d, m > 0$, $r > 1$, $s > 2d + r$. Combining this with the result above ($r = 1, s > 2d + 2$) and Lemma 31, we have proved Theorem 32.

□

The proof above is very long and algebra heavy. When we wrote this up for the paper, the proof was much shorter, without all of the details worked out explicitly. That is fairly common in a paper, and when I am reading a paper, I typically fill in those details for myself in the margins. The idea for proving Theorem 32 in this way was due to my collaborator, Armin Straub. I tend to gravitate towards trying to use combinatorial arguments (working with the diagrams) than manipulating the generating functions.

The bound $s > 2d + r$ in Theorem 32 can be improved to $s > 2d + 1$ in the case of small r, specifically $r \leq d + 1$. When we discuss improving the bound, this means that we would need fewer initial steps in order to perform the recurrence. In our Fibonacci sequence example, we need the first two values (F_1 and F_2) in order to be able to perform the recurrence (after that every Fibonacci number can be built from the previous ones).

Here we will show through a bijective proof that if r is small ($r \leq d + 1$), then for every $s > 2d + 1$ we can perform the recurrence; we would only need to know the generating function for s-values from 1 through $2d + 1$ as initial conditions in order to build all the abaci. This is the best possible bound for these small r-values, which we know from having found examples that show the recurrence does not hold for lower values of s.

Theorem 33. *Let $m \geq 1$, $1 \leq r \leq d + 1$, and write $f_s(q) = \boldsymbol{A}^d_{s,m,-r}(q)$. If $s > 2d + 1$, then*

$$f_s(q) = f_{s-1}(q) + (q + q^2 + \ldots + q^m) f_{s-d-1}(q). \tag{4.22}$$

Proof Sketch: The main idea of the proof is to take the set $\mathcal{A}^d_{s,m,-r}$ of s-core abaci with maximum position $< ms - r$ and spacing d, and break them up into two groups, based on if they

1. have a gap of size $> d$ between the last bead in the first row and first bead in the second row, an empty last column, and no bead in the maximum allowed position, or
2. fail to meet one of these three conditions.

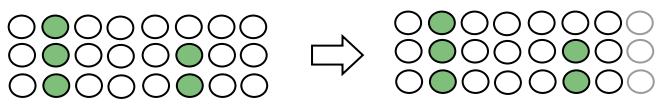
We then show the set of abaci satisfying case (a) is in bijection with the set $\mathcal{A}^d_{s-1,m,-r}$ of $(s-1)$-core abaci with maximum position $< m(s-1) + r$ and spacing d, giving the first term in the sum. For the set of abaci satisfying case (b), we show that there is a map from that set to the set $\mathcal{A}^d_{s-d-1,m,-r}$ of $(s-d-1)$-core

abaci with maximum position $< m(s-d-1)-r$ and spacing d. This map is m-to-one, which means there are m abaci in the first set that map to one abacus in the second set.

In the proof below, we provide one example to illustrate each case, using the values $m = 3$, $r = 3$, $d = 2$ and $s = 8$.

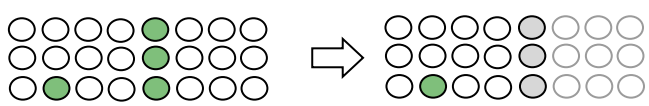
Proof. By definition, $f_s(q)$ is the generating polynomial for s-core abaci with spacing at least d between beads, which consist of m rows such that the top row has no beads in the last r positions. As in the proof of Theorem 29, let g be the size of the gap between the position of the last bead in the first row and the position of the first bead in the second row. There are two disjoint possibilities for such abaci.

1. Suppose that $g > d$ with the last column empty and no bead in the final allowed position (that is, no bead in position $ms - r - 1$). By removing the last column, we see that these abaci are in bijective correspondence with abaci in $\mathcal{A}^d_{s-1,m,r}$.

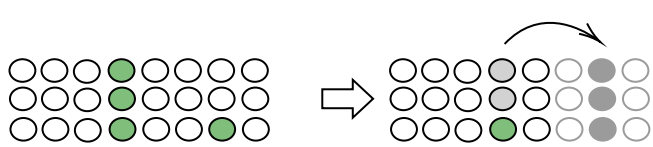


2. Suppose the conditions of 1 are not satisfied. We distinguish the following cases.

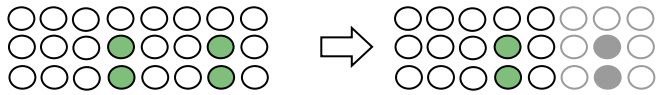
   i Suppose that the last $d+1$ columns are empty. If this is true and not all of 1 is true, then $r = d+1$ and there is a bead in the highest allowed position (that is, column $s-r-1$ has m beads). We then remove the last $d+1$ columns, as well as the m beads from column $s-r-1$, and obtain an abacus in $\mathcal{A}^d_{s-d-1,m,-r}$.



   ii Otherwise, there is precisely one non-empty column among the last $d+1$ columns. Suppose (additionally) that one of the r columns $s-d-r, \ldots, s-d-2, s-d-1$, say column j, contains m beads (equivalently, removing the last $d+1$ columns does not result in an abacus in $\mathcal{A}^d_{s-d-1,m,-r}$). In that case, the one non-empty column among the last $d+1$ must be one of the last r and, as such, contains $b < m$ beads. We then remove the last $d+1$ columns and reduce the number of beads in column j by $m-b$. The resulting abacus is in $\mathcal{A}^d_{s-d-1,m,-r}$ and has precisely m fewer beads.



iii Otherwise, we remove the last $d+1$ columns and, by Lemma 22, obtain an abacus in $\mathcal{A}^d_{s-d-1,m,-r}$.



Fix an abacus $A \in \mathcal{A}^d_{s-d-1,m,-r}$. To complete the proof, we need to show that A is obtained through the described process from m Case 2 abaci $B \in \mathcal{A}^d_{s,m,-r}$, with $1, 2, \ldots, m$ additional beads. We show this by building each of the corresponding B from A (instead of taking away from A to make B). Let $h \geq 1$ be the number of empty positions at the beginning of the second row of A. If $h \leq d$, then let $k = s - (d - h)$. Otherwise, let $k = s$. We can construct B from A by adding $d + 1$ empty columns, and then filling column k with b beads, where $1 \leq b < m$. (Note that, if $h \leq d$, then $g = d$ for B, while otherwise B has a non-empty last column.) It remains to similarly construct an abacus B from A by adding $d + 1$ columns and m beads. For that, we distinguish three cases:

- If column k is not among the last r columns of B, then it can also be filled with m beads.
- Otherwise, if one of the last r columns of A contains $c \geq 1$ beads, then we add $m - c$ beads to the corresponding column of B and c beads to column k of B.
- Otherwise, we add m beads to column $s - r - 1$ of B.

In each case, we check that abacus B corresponds to Case 2 and results in abacus A by the process described for Case 2.

□

While the proof describes the direction of the maps created by removing columns, when I was working on this I thought about it in the other direction – building the larger s-abaci from the smaller $(s-1)$-abaci and $(s - d - 1)$-abaci. When we went to write it up, we decided it was clearer to write in the other direction (removing columns instead of adding).

Diagramming and Gesture

I use **shading** to use one diagram as shorthand for a group of diagrams that share most if not all of the same characteristics. I learned this strategy by working alongside Hannah Burson on this problem. We would both draw examples, and I noticed that she was using this shorthand to save time while still getting a sense of drawing 'all' of the diagrams for a given set $\mathcal{A}^d_{s,m,-r}$. For example, when we have two abaci that share a characteristic, we would represent both in the same diagram (the half-shaded bead in the bottom image) indicates that we are considering both the diagram where this bead is colored, and the one where it is not.

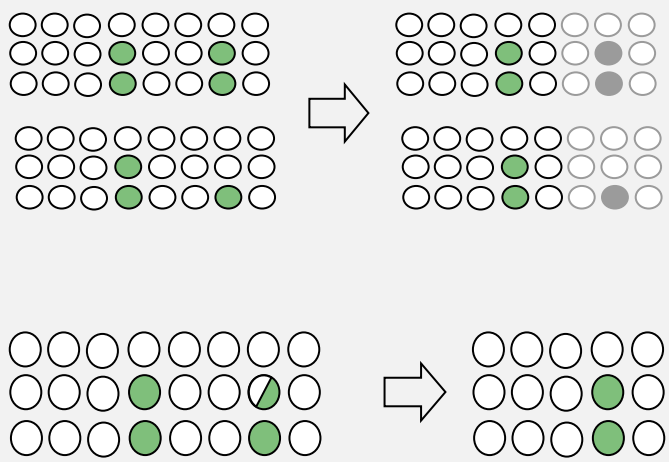


Figure 4.6: Example of shading to represent multiple abaci in one diagram.

We round out this discussion with a quick proof that Theorem 32, in the case $d = 1$, continues to hold for all $s > r + 1$. This is the best possible lower bound for s in the recurrence, which we know because we have found examples showing that the recurrence does not hold when $s \leq r$. We note that if $s \leq r$, when we subtract $ms - r$, this would be removing an entire row of beads (or more), and we could instead take r' such that $r' < s$ and $r = is + r'$, and rewrite this as $(m - i)s - r'$.

Lemma 34. *Let $m, r \geq 1$, and write $f_s(q) = \boldsymbol{A}_{s,m,-r}(q)$. If $s > r + 1$, then Equation (4.9) holds.*

Lemma 34 Restated: When $d = 1$, the recurrence holds for all s-values greater than $r+1$ (for any $m, r \geq 1$).

Proof Sketch: We use the fact that the set $\mathcal{A}_{s,m,-r}$ of s-core abaci with maximum position $< ms - r$ and spacing 1, are a subset of the set $\mathcal{A}_{s,m}$ of s-core abaci with maximum position $< ms$ and spacing 1, and we can create the set $\mathcal{A}_{s,m,-r}$ by subtracting from $\mathcal{A}_{s,m}$ all those abaci with beads in position j for some

$ms - r \leq j < ms$. Then, we use the previous results to show that the recurrence holds for both $\mathcal{A}_{s,m}$, and the set of abaci that are removed. Then, it must still be true of their difference.

Proof. By Lemma 30, for $s \geq r$,

$$\boldsymbol{A}_{s,m,-r}(q) = \boldsymbol{A}_{s,m}(q) - q^m \sum_{k=1}^{r} \boldsymbol{A}_{s-k-1,m}(q)\boldsymbol{A}_{k-1,m-1}(q).$$

We consider the terms $\boldsymbol{A}_{s,m}(q)$ and $\boldsymbol{A}_{s-k-1,m}(q)$, for $k = 1, 2, \ldots, r$, on the right-hand side. To prove our claim, it suffices to show that, for $s \geq r + 2$, each of these terms satisfies the recurrence. The recurrence

$$\boldsymbol{A}_{s,m}(q) = \boldsymbol{A}_{s-1,m}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}_{s-2,m}(q),$$

follows from Theorem 29 with $r = 0$ (because with $s \geq r+2 = 2$ the conditions of the theorem are satisfied). Furthermore, since $k = 1, 2, \ldots, r$, and $s \geq r + 2$, we have $s - k - 1 > 0$ for all k, and so

$$\boldsymbol{A}_{s-k-1,m}(q) = \boldsymbol{A}_{s-k-2,m}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{A}_{s-k-3,m}(q),$$

again applying Theorem 29 with $r = 0$, and noting that for $d = 1$, the condition $s > d + 1$ may be dropped because the recurrence would still hold for $s = 1$ and $s = 2$ under the assumption that $\boldsymbol{A}_{s,m}(q) = 0$ when $s < 0$.

□

4.2.2 Initial Conditions and Observations

The Theorems we proved in the previous section depend on our values of s being large enough to support the recurrence. When s is small, the set of abaci $\mathcal{A}^d_{s,m,-r}$ cannot be built using the recurrence. However, we can still define these sets separately, which is what we will do in this section. We begin with values of $s \leq d + 1$.

The following proof breaks this into two cases: $s \leq d$ and $s = d + 1$.

Lemma 35. *Let $d, m \geq 1$ and $r \geq 0$. For $s \in \{1, 2, \ldots, d\}$,*

$$\boldsymbol{A}^d_{s,m,-r}(q) = 1 + \min(s - 1, ms - r - 1)^+ q.$$

For the case $s = d+1$, write $-r = sr' + r_0$ with $0 \le r_0 \le d$. If $m + r' < 0$, then $\boldsymbol{A}^d_{d+1,m,-r}(q) = 1$. If $m + r' \ge 0$, then

$$\boldsymbol{A}^d_{d+1,m,-r}(q) = 1 + d(q + q^2 + \ldots + q^{m+r'}) + (r_0 - 1)^+ q^{m+r'+1}.$$

Lemma 35 Restated: For small values of s (when $1 \le s \le d$) all s-core abaci with maximum position $< ms - r$ and spacing d are either the empty abacus, or have exactly one bead. The number of such abaci with only 1 bead is the minimum of $(s-1, ms-r-1)$ (or zero if one of those values is negative).

When $s = d+1$, there can be at most one column of beads and there are d such columns that can have beads. If we rewrite $-r = sr' + r_0$ for some $0 \le r_0 \le d$ (example: $-5 = 3(-2) + 1$ for $r = 5$ and $s = 3$). Any column that has beads either has $m + r'$ beads (for columns $s - r_0$, through $s - 1$) or $m + r' + 1$ beads (for columns 1 through $s - r_0 - 1$).

Proof Sketch: When $1 \le s \le d$, if a column had more than one bead, the gap between those beads would be $\le d$, which cannot happen. So, there can be at most one bead in the abacus. Then, we look at the possible spaces for beads in the first row to find the exact number.

When $s = d+1$, rewrite $-r$ as $sr' + r_0$ (so we have a negative r' and a positive $\le r_0 \le d$). For example, if $s = 3$ and $-r = -5$, we could rewrite as $-5 = 3(-2) + 1$. Since $s = d+1$, there can be at most one column of beads, and that column can have at least $m + r'$ beads (note: since r' is negative, the value $m + r' < m$). This counts the number of beads that can fit in the rectangle s by $m + r'$. Then, there are $r_0 - 1$ columns that could have one more bead in them.

Proof. In the case where $s \le d$ the abaci have no more than d columns. Thus, there could be at most $d - 1$ spacers between any two beads, which is not allowed. So there can be at most one bead.

The bead cannot be in the first column (by the definition the first column must be empty). Thus there can be at most $s - 1$ possible positions for a single bead. Any such abacus is an s-core abacus with maximum position less than $ms - r$. Thus, it is also an $ms - r$-core abacus. In the case where $ms - r < s$, it there will be at most $ms - r - 1 < s - 1$ possible positions for a single bead. So, we have that the set $\mathcal{A}^d_{s,m,-r}$ contains the empty abacus, plus those $\min(s-1, ms+r-1)$ abaci that have exactly one bead.

In the case $s = d+1$, we can have a full column of beads in any of the $s - 1 = d$ columns after the first (and there can be no bead in a second column). Note that $\mathcal{A}^d_{d+1,m,r} = \mathcal{A}^d_{d+1,m+r',r_0}$. Since the case $m + r' < 0$ is trivial, suppose $m + r' \ge 0$. Each of the last d columns can accomodate $m + r'$ many beads. Additionally, any of the $r_0 - 1$ columns $1, 2, 3, \ldots, r_0 - 1$ can accomodate an additional bead in the $(m + r' + 1)$st row. $\square$

Below is an example of Lemma 35.

Example 36. *In the case $d = 1$, we find that $\boldsymbol{A}_{1,m,r}(q) = 1$ and*

$$\boldsymbol{A}_{2,m,r}(q) = 1 + q + q^2 + \ldots + q^{m+\lfloor r/2 \rfloor}. \tag{4.23}$$

In the cases $r = \pm 1$, $q = 1$, these specialize to the initial conditions in Theorem 24.

A Return to Core Partitions

Let us now consider s-core partitions with maximum hook length $ms - r$ for all $m, r \in \mathbb{Z}$.

We recall that $\mathcal{C}^d_{s,m,-1}$ is the set of $(s, ms-1)$-core partitions into d-distinct parts and

$$\boldsymbol{C}^d_{s,m,-1}(q) = \sum_{\lambda \in \mathcal{C}^d_{s,m,-1}} q^{n(\lambda)} \tag{4.24}$$

is the generating polynomial for the number of parts of the partitions in $\mathcal{C}^d_{s,m,-1}$. If d is omitted in the notation, it is implicit that $d = 1$.

As a consequence of Lemma 25,

$$\boldsymbol{C}^d_{s,m,-1}(q) = \boldsymbol{A}^d_{s,m,-1}(q), \tag{4.25}$$

where $\boldsymbol{C}^d_{s,m,-1}(q)$, defined in Equation (4.24), is the generating polynomial for the number of parts of $(s, ms-1)$-core partitions into d-distinct parts.

By combining Equation (4.24) with Theorem 33 and Lemma 34, we get the following statement on abaci.

Theorem 37. *Let $m, d \geq 1$ be integers. Then, for $s > 2d + 1$ (or $s > 2$, if $d = 1$), we have*

$$\boldsymbol{C}^d_{s,m,-1}(q) = \boldsymbol{C}^d_{s-1,m,-1}(q) + (q + q^2 + \ldots + q^m)\boldsymbol{C}^d_{s-d-1,m,-1}(q).$$

Example 38. *In the case $(d, r) = (1, -1)$, we have $\boldsymbol{A}_{s,m,r}(q) = \boldsymbol{C}_{s,m,r}(q)$. For instance, for $(s, m) = (4, 3)$ we have $ms - 1 = 11$ and the $(4, 11)$-core partitions into distinct parts are*

$$\emptyset, \quad (1), \quad (2), \quad (3), \quad (2,1), \quad (4,1), \quad (5,2), \quad (6,3), \quad (3,2,1),$$
$$(5,2,1), \quad (7,4,1), \quad (8,5,2), \quad (4,3,2,1), \quad (6,3,2,1), \quad (5,4,3,2,1).$$

There are $\boldsymbol{C}_{4,3,-1}(1) = 15$ such partitions and the generating polynomial for the number of parts in these

partitions is

$$\boldsymbol{C}_{4,3,-1}(q) = 1 + 3q + 4q^2 + 4q^3 + 2q^4 + q^5 = (1 + q + q^2)(1 + 2q + q^2 + q^3).$$

We note that such a factorization always exists for the polynomials $\boldsymbol{C}_{s,m,-1}(q)$. Indeed, for $s \geq 2$, we claim that $\boldsymbol{C}_{s,m,-1}(q)$ is divisible by $1 + q + q^2 + \ldots + q^{m-1}$. This is true for $\boldsymbol{C}_{2,m,-1}(q)$ by Equation (4.23) and the claim follows inductively from Theorem 37.

Example 39. *Continuing the previous example in the special case $m = 2$, we conclude from Theorem 37 that*

$$\boldsymbol{C}_{s,2,-1}(q) = (1 + q)^{s-1}. \tag{4.26}$$

In particular, the number of $(s, 2s - 1)$-core partitions into k distinct parts is $\binom{s-1}{k}$. Less generally, as observed in [37], there are 2^{s-1} many $(s, 2s - 1)$-core partitions into distinct parts.

We also make a brief note that in the case $s > 2$, we can use the structure of abaci to note the maximum number of parts in $(s, ms - 1)$-core partitions into d-distinct parts

Lemma 40. *Let $M^d_{s,m,-1}$ be the maximum number of parts of $(s, ms - 1)$-core partitions into d-distinct parts. Let $d, m \geq 1$. If $s > 2$, then*

$$M^d_{s,m,-1} = \left\lfloor \frac{s}{d+1} \right\rfloor m + \begin{cases} -1, & \text{if } d = 1 \text{ and } s \equiv 0 \pmod 2, \\ 0, & \text{otherwise, if } s \equiv 0, 1, 2 \pmod{d+1}, \\ 1, & \text{otherwise.} \end{cases}$$

Proof. The case $r = -1$, follows from using $M^d_{s,m,-1} = M^d_{s,m-1,s-1}$. We can think about an s-core abacus with spacing d and maximum position $< ms - 1$ instead as an s-core abacus with spacing d and maximum position $< m(s - 1) + (s - 1)$. In this case, we can apply Lemma 41 using the value $r = s - 1$ (stated below). □

The following is presented in [8].

Lemma 41. *Let $M^d_{s,m,r}$ be the maximum number of beads of an abacus in $\mathcal{A}^d_{s,m,r}$. Let $d, m \geq 1$. If $s \geq r > 1$, then*

$$M^d_{s,m,r} = \left\lfloor \frac{s}{d+1} \right\rfloor m + \left\lfloor \frac{r-2}{d+1} \right\rfloor + 1.$$

The maximal initial gap is the spacing between the first and second part of a partition (size of the largest part − size of the second largest part. We spent some time looking partitions grouped by initial gap, and

found the following.

Lemma 42. *Let $s, m \geq 1$. If $r = 1$, then there is a natural one-to-one correspondence between $(s, ms-r)$-core partitions into d-distinct parts with maximal initial gap and $(s, (m-1)s - r)$-core partitions into d-distinct parts.*

Proof. The claimed bijective correspondence $\mathcal{G}^d_{s,m,r} \to \mathcal{C}^d_{s,m-1,r}$ is given by the map

$$(\lambda_1, \lambda_2, \lambda_3, \ldots) \mapsto (\lambda_2, \lambda_3, \ldots),$$

with the understanding that the empty partition is sent to itself. This map removes the largest part of the partition. □

Corollary 43. *In particular, if $r = -1$, we get an enumeration of $(s, ms + r)$-core partitions into d-distinct parts with maximal initial gap from Theorem 28.*

We end with an observation about an interesting open problem. In Example 39, we note that there are 2^{s-1} many $(s, 2s-1)$-core partitions into distinct parts. Yan, Qin, Jin, Zhou [42], Zaleski, Zeilberger [43], Baek, Nam, Yu [6], and Paramonov [31] have shown that, for odd s, the number of $(s, s+2)$-core partitions into distinct parts is 2^{s-1} as well.

In our notation,

$$\boldsymbol{C}_{s,1,2}(1) = 2^{s-1},$$

It would be interesting to be able to count these more generally, as we did for the other sets above, and determine a formula

$$\boldsymbol{C}_{s,1,2}(q) = \sum_{n=0}^{\infty} c_n(s) q^n,$$

that keeps track of the number of parts. Additionally, since the sets of $(s, 2s-1)$-core partitions into distinct parts and $(s, s+2)$-core partitions have the same size for odd s, we would very much like to show a bijection between these sets.

Chapter 5

Quotients from Restricted Digit Sets and Multiplication Transducers

This chapter provides background information on integer quotients of elements of the set of integers with nonstandard digital representations in base b, which is the context for Chapters 6-8. In Section 5.1 we introduce definitions and notation. In Sections 5.2 and 5.3 we present modified multiplication transducers as mathematical objects that can be used as a tool for exploring these sets of quotients. Section 5.1.1 was written for my niece, who is currently in the third grade. Sections 5.2 and 5.3 were written for another niece who is currently in seventh grade.

5.1 Definitions and Notation

5.1.1 Base b Numeration Systems

The number system most of us know best uses ten digits: $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$.

We build the rest of the numbers using these digits and grouping by tens, then hundreds ($100 = 10 \times 10$), then thousands ($1000 = 10 \times 10 \times 10$). We call this number system *base 10*, because we get each place by multiplying by ten again. For example, the number 136 we can think of as

$$1\ (100) + 3\ (10) + 6\ (1),$$

or, one '100' + three '10s' + six '1s'. We call each of the place values *powers of ten* and write them with exponents. So $10 = 10^1$, which reads as "10 to the first power", $100 = 10 \times 10 = 10^2$, which reads as "10 to the second power", etc..

TRY: What would you call $100000 = 10 \times 10 \times 10 \times 10 \times 10 = 10^5$?

In our base 10 number system, every number has a digit from $(0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$ in each place value.

For the number thirteen, we could think of it as having three 1s, one group of 10 and zero for the bigger

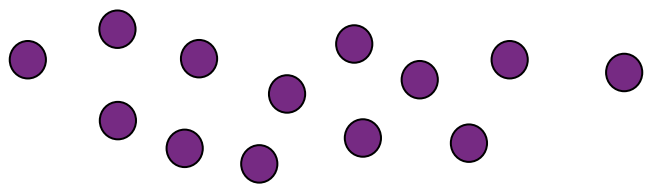
powers of 10. Usually we do not write the zeros in front.

$10000 = 10^4$	$1000 = 10^3$	$100 = 10^2$	$10 = 10^1$	$1 = 10^0$
0	0	0	1	3

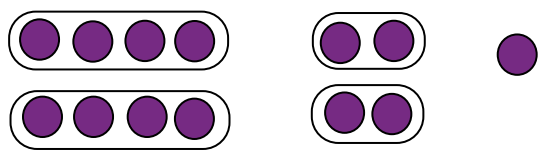
The 'ones' place we call 10^0, because it is what is left over that cannot be made into a group of ten.

We usually use base 10, but there are lots of other numbers we could use as a base instead! Let's try using 2 as a base. We want to build numbers out of groups of ones, twos, then fours ($4 = 2 \times 2$), then eights ($8 = 2 \times 2 \times 2$), and so on.

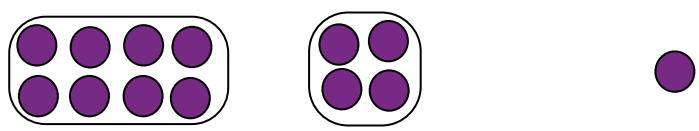
Example 44. *If I wanted to make the number thirteen, I can think of having 13 dots.*



Then, I want to group them using the powers of 2. The powers of two are $1, 2, 4, 8, 16...$ *The rule is that if we can move dots into a larger group, we do. So, I might try first to group:*



But that is not quite right, because my two groups of 2 could also be a group of 4, and my two groups of 4 could be a group of 8. So we try again.



Now that we cannot combine groups any more, we can count the number of groups in each place value and fill in a base 2 table.

$16 = 2 \times 2 \times 2 \times 2 = 2^4$	$8 = 2 \times 2 \times 2 = 2^3$	$4 = 2 \times 2 = 2^2$	$2 = 2^1$	$1 = 2^0$
	1	*1*	*0*	*1*

So, the number thirteen looks like 13 when we write it in base 10, but looks like 1101 when we write it in base 2! To keep track of which base we are using, we put brackets around the numbers, with a little number to show the base at the bottom.

$$[13]_{10} = [1101]_2$$

TRY: Try writing the number twenty three in base 2.

When we write numbers in base 2, we only need to use the digits 1 and 0, because if we had two ones, we would combine them to make one group of two, if we had two fours, we could combine them to make one group of 8.

In the base 10 system, we have ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. We could also choose to write numbers using base 3 (called the ternary system), or base 4, or base 5, or with any positive whole number you like as a base. In base 2, we saw the rightmost digit counts how many ones (2^0) there are, the next digit counts how many 2s, then how many 2^2s, etc.. In base 3, the rightmost digit still counts how many ones (3^0), the next digit counts 3s, then 9s (since $9 = 3^2$), etc..

Each number has a representation using each of the bases. For example, the number three in base 2 is written as $[11]_2$ because we think about it as one group of two plus one left over. In base 3, the number three is written as $[10]_3$ because we think about it as one group of three plus no ones left over. The table below shows the ways to write the numbers one through eight in base 2 and in base 3. In base 3 we have digit 0, 1, or 2 in each place. We cannot have the digit 3; if we had 3 ones we would have a group of three and move that group to the next place.

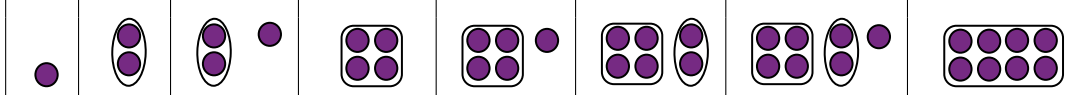
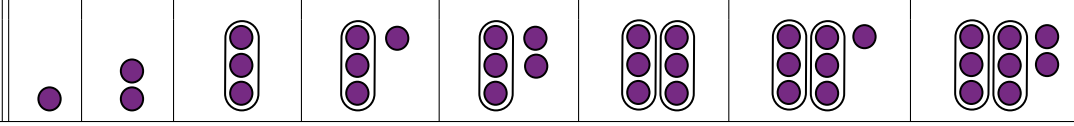
Base 10	1	2	3	4	5	6	7	8
Base 2	$[1]_2$	$[10]_2$	$[11]_2$	$[100]_2$	$[101]_2$	$[110]_2$	$[111]_2$	$[1000]$
	1	$2+0$	$2+1$	$4+0+0$	$4+0+1$	$4+2+0$	$4+2+1$	$8+0+0+0$
								
Base 3	$[1]_3$	$[2]_3$	$[10]_3$	$[11]_3$	$[12]_3$	$[20]_3$	$[21]_3$	$[22]_3$
	1	2	$3+0$	$3+1$	$3+2$	$2\times 3+0$	$(2\times 3)+1$	$(2\times 3)+2$
								

Table 5.1: The Numbers 1 through 8 in Base 2 and Base 3

In base 2, we use the digits 0 and 1. In base 3, we use the digits 0, 1, and 2. One way to think about it is the digits are the options for how many ones we can have before making a group in that base.

TRY: Which digits would we use if we wanted to write numbers in base 4? Write the first eight numbers using base 4.

Look at the statements below to explore the pattern.

- There is exactly one way to write each positive whole number using the digits 0 and 1 in base 2.
- There is exactly one way to write each positive whole number using the digits 0, 1, and 2 in base 3.
- There is exactly one way to write each positive whole number using the digits 0 through 9 in base 10.

This is true for any base you like. We often use the letter b to represent the base. Then we can say: **There is exactly one way to write each positive whole number in base b using the digits 0 through $b-1$.** It is helpful say it using the letters, because the bold sentence is true no matter what positive whole number you choose to put in for b.

TRY: Rewrite the bold sentence from above using the number 7 in place of b.

Nonstandard base b numeration systems

What if we do not allow all of the digits in a given base? For example, we could try to study the positive whole numbers that have no nines. Then, we would have a set of numbers that looks like this:

$$1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 20, 21, \ldots, 86, 87, 88, 100, 101, \ldots$$

The problems I study focus on these special sets of numbers, for different bases and different choices for which digits are allowed. For example, the positive whole numbers that have no digit 2 in base 3:

$$[1]_3, [10]_3, [11]_3, [100]_3, [101]_3, [110]_3, [111]_3, [1000]_3, [1001]_3, [1010]_3, [1011]_3, [1100]_3, [1101]_3, \ldots$$

$$= 1, \quad 3, \quad 4, \quad 9, \quad 10, \quad 12, \quad 13, \quad 27, \quad 28, \quad 30, \quad 31, \quad 36, \quad 37, \quad \ldots$$

In particular, I look at the different combinations and divide them by each other. So, I can make the number 7 by dividing 28 by 4, and both 28 and 4 are in the special set above. This is cool because 7 is not in the special set, but I can still make it by dividing two of the numbers that are in the special set.

5.1.2 More formal notation

The positive, whole numbers we have been talking about are often called the *natural numbers*. We know that we can represent every natural number in base b if we allow the digits $\{0, 1, \ldots, b-1\}$. Suppose we

pick a natural number and call it s. When we talk about writing s in base b, what we mean is that we can make s by adding up of powers of b. For example, in base 10, the number 24 ($s = 24$) can be written as $2(10) + 4(1) = [24]_{10}$. In base 3, it is $2(9) + 2(3) + 0(1) = [220]_3$.

In mathematical notation, saying you can represent every natural number in base 3 is the same as saying, for any number s, we can write

$$s = \alpha_0 3^0 + \alpha_1 3^1 + \alpha_2 3^2 + \alpha_3 3^3 + \dots$$

where each α_i comes from the set of digits $\{0, 1, 2\}$. In our example,

$$24 = 0(1) + 2(3) + 2(9) + 0(27) + 0(81) + \cdots = \alpha_0 3^0 + \alpha_1 3^1 + \alpha_2 3^2 + \alpha_3 3^3 + \dots.$$

The expanded form above is equivalent to writing $[00220]_3$. Sometimes we write the smallest term first in a sum, which is reverse order from the base 3 notation with the brackets. The most general way of writing this notation is saying

$$s = \sum_{i=0}^{\infty} \alpha_i 3^i, \text{with } \alpha_i \in \{0, 1, 2\} \text{ for all } i.$$

The symbol $\sum_{i=0}^{\infty}$ means add up all of the terms, starting when $i = 0$, ($\alpha_0 3^0$), then when $i = 1$ (adding $\alpha_1 3^1$), and then when $i = 2$, and so on. The coefficients α_i that multiply each power of 3 have to come from the set of numbers $\{0, 1, 2\}$.

What happens if we can only have some of the digits? We can ask questions about sets of numbers that can be represented with restricted digit sets.

Example 45. *The set $S(3; \{0, 1\})$ is the set of numbers whose ternary representation has no twos – set of natural numbers that can be written in base 3 using only digits from the two digit set $\{0, 1\}$.*

$$S(3; \{0, 1\}) = \{1, 3, 4, 9, 10, \dots\} \textit{ because } [1]_3 = 1, [10]_3 = 3 + 0 = 3, [11]_3 = 3 + 1 = 4, \dots$$

We can use this notation for any base b and set of digits.

Example 46. *The set $S(10; \{0, 1, 2, 3, 4, 5, 6, 7, 8\})$ is the set of numbers whose standard base 10 representation has no 9s.*

TRY: Use the same notation to describe the set of numbers whose base 5 representation has only even digits. (Hint: Think about what digits would be in a typical base 5 representation first.)

The most general way to write these sets uses the following notation.

Definition 47.

$$S(b;\{d_1,\ldots,d_k\}) := \left\{ s \in \mathbb{N} : s = \sum_{i=0}^{\infty} \alpha_i b^i, \qquad \textit{with } \alpha_i \in \{d_1,\ldots,d_k\} \textit{ for all } i \right\}.$$

The set $S(b;\{d_1,\ldots,d_k\})$ represents all of the numbers that can be represented in base b using only the digits $\{d_1,\ldots d_k\}$. The set of digits has k digits in it, and the subscript (little number on the bottom right) keeps track of how many digits there are. In my work, k must be less than b; we are looking at sets that have fewer than b digits in base b. So, in our example $S(3;\{0,1\})$, the base is $b = 3$ and the set of digits is $\{0,1\} = \{d_1, d_2\}$.

We are actually interested in studying the set of positive whole numbers that we can make by dividing the numbers that are in these special sets. The word "quotient" means the number resulting from dividing one number by another, so we will use Q as our symbol for these sets.

Example 48. *The number* 7 *is in* $Q(3;\{0,1\})$*, since* $7 = 28/4$*, and* $28 = [1001]_3$*,* $4 = [11]_3$ *are both elements of* $S(3;\{0,1\})$*.*

The general way to write these sets of quotients is using the following notation.

Definition 49.

$$Q(b;\{d_1,\ldots d_k\}) := \{x \in \mathbb{Z} : x = s/s' \textit{ for some } s, s' \in S(b;\{d_1,\ldots,d_k\})\}.$$

The set $Q(b;\{d_1,\ldots d_k\})$ includes only the integers that can be represented as quotients of elements of $S(b;\{d_1,\ldots,d_k\})$. We write $x = s/s'$ instead of $s'x = s$, because we want to highlight that $s' \neq 0$.

5.2 Automata and Multiplication Transducers

To study these sets of numbers, we use pictures that represent simple computers. There are algorithms, or processes, which we use to multiply in base b effectively. Just like we develop systems for multiplying numbers in base 10, there are similar algorithms that multiply in base b.

TRY: Use any method you know to multiply 13×56. Would you do something similar if you instead wanted to multiply 13×68?

What were the steps you did to perform the multiplication above? The machines we will be using follow

specific steps. They are simple; they can read strings of numbers (one digit at a time), store a very small amount of information, and write strings of numbers (one digit at a time). The action they perform is multiplication, so they 'read' a number, do the action of multiplying it by a given number, and 'write' the product.

The steps we will use to do multiplication with the machines may be different than the steps you used above. We will talk through these steps to multiply 56 by 13 first (using our standard base 10 system), and then show how to follow the same steps in other bases.

Some important ideas:

- The way the machine stores information is by 'carrying' over to the next place value. We start by 'carrying' zero.
- The machine reads the input right to left (smallest to largest place value), so for the number 56, we will 'read' 6 first, and then 5.
- The machine writes the output from right to left as well.

Example 50. *The 'Place' column refers to the place value of the digit we are reading and writing. For each place value, we multiply the digit we are reading by 13, figure out what digit to write, and how many tens to carry to the next place value. For the first step, we multiply 6 by 13, to get 78. Since we are in the* 10^0 *(or ones) place, we write* 8 *and then carry 7 tens to the next step.*

Step 1:

Place	*Carry*	*Read*	*Total*	*Write*	*Carry*
10^0	0	6	$0 + 13(6) = 78$ $= 8 + 10(7)$	8	7

We carry 7 tens to the next step, and read 5. So we will be adding 7 tens to our 13 *times* 5 *tens in the next step. This gives us* 72 *tens, so we write* 2*, and carry over 7 tens into the* 10^2 *place.*

Step 2:

Place	*Carry*	*Read*	*Total*	*Write*	*Carry*
10^1	7	5	$7 + 13(5) = 72$ $= 2 + 10(7)$	2	7

Now, since we were multiplying 56, there are no more digits left to read. However, because we carried over 7, we are not done yet. We continue on, by 'reading' 0 for as many steps as we need to until we no longer have anything to carry.

Step 2:

Place	*Carry*	*Read*	*Total*	*Write*	*Carry*
10^2	7	0	$7 + 13(0) = 7$ $= 7 + 10(0)$	7	0

Now that we have nothing left to carry, we are done. If we look at what we wrote (the purple numbers), we wrote an 8 in the one's place, a 2 in the ten's place and a 7 in the hundred's place. Putting it all together gives us 728 which is 13 times 56.

This system may seem overly complicated, especially since we have other methods we are more familiar with when multiplying in base 10. However, it can help when we are multiplying in bases we are less familiar with. We will give an example using base 3, and then talk about how to create the multiplication machines that can perform these computations.

In the following example, we will multiply 25 by 4 in base 3. So this is the action performed by the machine that 'multiplies by 4 in base 3'.

Reminders:

- The way the machine stores information is by 'carrying' over to the next place value. We start by 'carrying' zero.
- The machine reads the input right to left (smallest to largest place value), and will read inputs in base 3. In our example, $25 = [221]_3$, so the first digit read is '1',
- The machine writes the output right to left as well.

Example 51. *Input* $25 = [221]_3$*, multiplying by* 4 *in base* 3 *Step 1:*

Place	*Carry*	*Read*	*Total*	*Write*	*Carry*
3^0	0	1	$0 + 4(1) = 4$ $= 1 + 1(3)$	1	1

We start by carrying 0, and read the digit 1*. Then our action is 'multiply by 4', so we multiply 4(1), and add it to what we were carrying (0). Because we need to write in base* 3*, we look at how many ones are in our total (in base* 3*, the number* 4 *has one 1), and save all the* 3*s. So, we write* 1*, and carry* 1 *three to the next line.*

Step 2:

Place	*Carry*	*Read*	*Total*	*Write*	*Carry*
3^1	1	*2*	$1 + 4(2) = 9$ $= 0 + 3(3)$	0	3

Here, we have 1 *that we have carried from the previous line, and read the digit* 2 *(which was the second rightmost digit in our original input). Then our action is 'multiply by 4', so we multiply 4(2), and add it to what we were carrying (1). Because we need to write in base* 3*, we look at how many ones are in our total (in base* 3*, the number* 9 *has no ones), and save all the* 3*s. So, we write* 0*, and carry three* 3*s to the next line.*

Step 3:

Place	*Carry*	*Read*	*Total*	*Write*	*Carry*
3^2	3	2	$3 + 4(2) = 11$ $= 2 + 3(3)$	2	3

Here, we have 3 *that we have carried from the previous line and read the digit* 2 *(which was the third rightmost digit in our original input). Then our action is 'multiply by 4', so we multiply 4(2), and add it to the* 3 *we were carrying. Because we need to write in base* 3*, we look at how many ones are in our total (in base* 3*, the number* 11 *has* 2 *ones), and save all the* 3*s. So, we write* 2*, and carry three* 3*s to the next line.*

So far, we have read from right to left our initial input $[221]_3$*, and wrote as an output (also right to left)* $[201]_3$*. We are at the end of our inputs, but we are not done yet! We still have a 3 that needs to be carried over from the most recent step. Here, we'll keep 'reading' 0s until we have exhausted our carrying.*

Step 4

Place	*Carry*	*Read*	*Total*	*Write*	*Carry*
3^3	3	0	$3 + 4(0) = 3$ $= 0 + 1(3)$	*0*	*1*

Here, we have 3 *that we carried from the previous line, and 'read' the digit* 0*. Then our action is 'multiply by 4', so we multiply 4(0), and add it to the* 3 *we were carrying. Because we need to write in base* 3*, we look at how many ones are in our total (in base* 3*, the number* 3 *has* 0 *ones), and save all the* 3*s. So, we write* 0*, and carry one* 3 *to the next line.*

We are not done yet, because we have more to carry over.

Step 5:

Place	*Carry*	*Read*	*Total*	*Write*	*Carry*
3^4	1	0	$1 + 4(0) = 1$ $= 1 + 0(3)$	1	0

Here, we have 1 *that we carried from the previous line, and 'read' the digit* 0*. Then our action is 'multiply by 4', so we multiply 4(0), and add it to what we were carrying (1). Because we need to write in base* 3*, we look at how many ones are in our total (in base* 3*, the number* 1 *has* 1 *one), and save all the* 3*s. So, we write* 1*, and carry no* 3*s to the next line.*

Now that we have nothing left over to carry, we are finally done. The result is that we read $[00221]_3$*, and wrote* $[10201]_3$*. In base* 3*, we have*

$$[10201]_3 = 1(3^4) + 0(3^3) + 2(3^2) + 0(3^1) + 1(3^0) = 1(81) + 2(9) + 1(1) = 100.$$

Our original goal was to perform multiplication by 4 *in base* 3 *on the input 25. Indeed, our result* $[10201]_3 = 100 = 4(25)$.

We worked through each step of the process, and now we will talk about a visual way of showing how our multiplication algorithm works. The picture that matches our example is a machine that multiplies by 4 in base 3. We call this type of multiplication machine a *transducer*. The word transducer is related to the word translate – this is a machine that translates by taking in a number in base 3 and translating to that number multiplied by four. The image below represents the transducer, $\mathcal{T}_{4,3}$ that multiplies by 4 in base 3.

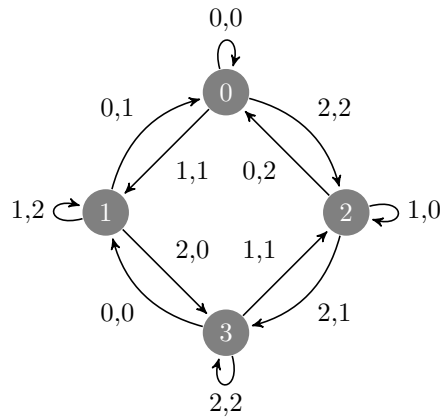
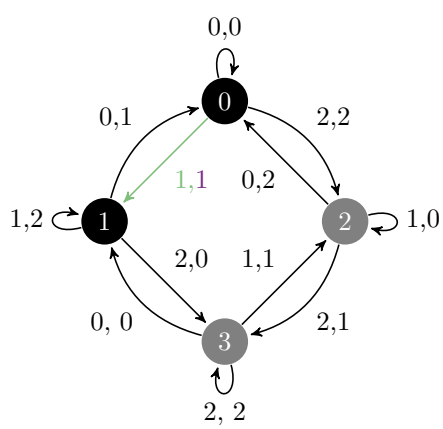


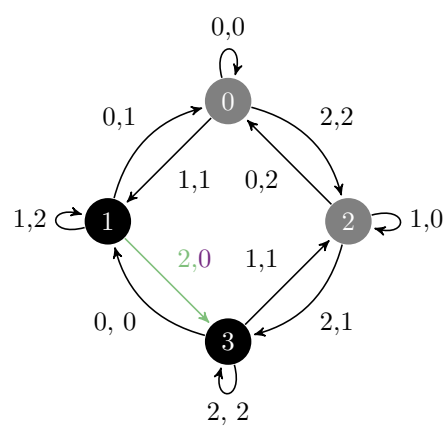
Figure 5.1: The transducer $\mathcal{T}_{4,3}$ that multiplies by 4 in base three.

We will now talk about how this picture contains the multiplication algorithm we described in the example. In our first step of the example computation, we started by carrying 0, and read an input of 1, to write an output of 1 and carry 1.

Place	Carry	Read	Total	Write	Carry
3^0	0	1	$0 + 4(1) = 4$	1	1
			$= 1 + 1(3)$		

In the picture, we will always start at the state marked 0, because the state is what keeps track of the carries. The input, what we read, is the first number on the path. We will follow the path labelled 1, and then write the number (1) that follows it.



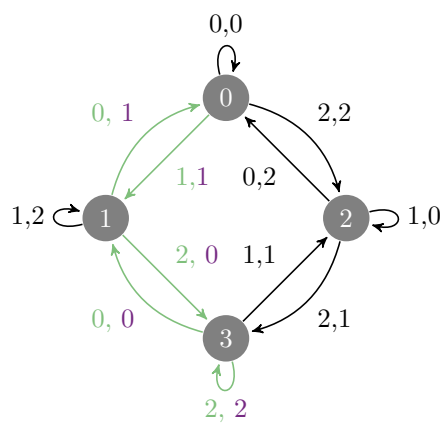


Then, we had carried a 1, so we moved to state 1, and proceeded to read the next digit, which was 2. So,

our next step would be to follow the path that starts with 2, leading us to write 0, the number that follows it.

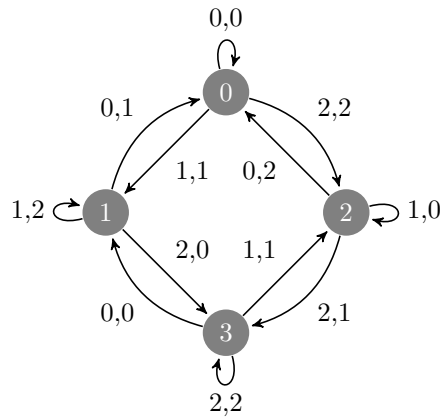
We continue in this manner, and see that performing all of the steps of the algorithm (listed below), correspond to following the green highlighted path in the diagram.

Place	Carry	Read	Total	Write	Carry
3^0	0	1	4	1	1
3^1	1	2	9	0	3
3^2	3	2	11	2	3
3^3	3	0	3	0	1
3^4	1	0	1	1	0



Our example was taking an input of 25 and when we wrote it in base 3 and followed the digits from right to left, the machine performs the computation (writing the output $[10201]_3$ which is 100 in base 3). The machine can actually have as an input any positive number in base 3, and will perform the computation of multiplying that number by 4.

TRY: Follow the paths in the transducer below to find the product of 43 and 4 in base 3. (Hint: In base 3, $43 = [1121]_3$.)



Diagramming and Gesture

The transducers are diagrams that employ **arrows** to illustrate movement from state to state. My interactions with these diagrams help me make sense of the multiplication algorithm through **exploration of the spatio-motoric information** that is encoded by the transducer. I often physically trace the paths with my finger, as I note the digits that are read and written. This decision to physically trace the path is an **intentional self-oriented gesture**. I think about the information of what is being carried differently by being able to see how it interacts with the other states in the diagram. For example, if we are already carrying a '2' in the example above, our next carry cannot be a '1', which is information that we might not see as readily through the algorithm.

Constructing Multiplication Transducers

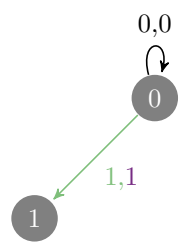
How would we go about making a diagram like this one? Well, we can think about what would happen for every possible input digit, for every possible carry. If we stay with our original example, we could start with state 0 (where we are carrying nothing). Then, because we are in base 3, there are only three possible digits that we could have as the input:

If our input is 0, we get the path that goes from 0 to itself (also writing 0).

Place	Carry	Read	Total	Write	Carry
3^0	0	0	$0 + 4(0) = 0$ $= 0 + 0(3)$	0	0

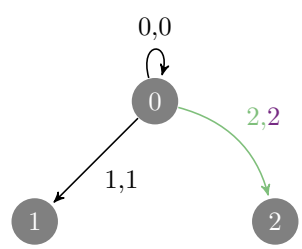
If our input was 1, we would add another path:

Place	Carry	Read	Total	Write	Carry
3^0	0	1	$0 + 4(1) = 4$	1	1
			$= 1 + 1(3)$		



The possibility of an input of 2 would give us yet another path (still starting from 0).

Place	Carry	Read	Total	Write	Carry
3^0	0	2	$0 + 4(2) = 8$	2	2
			$= 2 + 2(3)$		

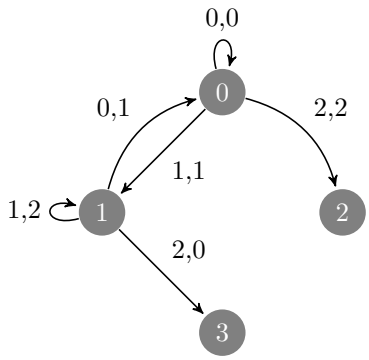


Now, those the three possible inputs that could be read at the start. Another way to think about this is: every whole number is in one of the following three groups

A multiple of three:	1+ a multiple of three:	2+ a multiple of three:
$\{0, 3, 6, 9, \dots\}$	$\{1, 4, 7, 10 \dots\}$	$\{2, 5, 8, 11 \dots\}$
$\{[0]_3, [10]_3, [20]_3, \dots\}$	$\{[1]_3, [11]_3, [21]_3, \dots\}$	$\{[2]_3, [12]_3, [22]_3, \dots\}$

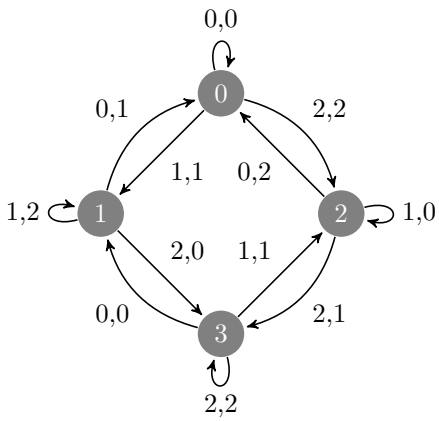
So, each of those numbers, written in base 3, has rightmost digit either 0, 1, or 2.

We have worked out exactly what the options are for the very first step. We could do the same thing for each state (carry). For example, if we next thought about our three options when we carry 1, we would get:



TRY: Figure out the path that describes what would happen if we read the digit 0 while carrying 2 (the path leaving state 2), and add it to the diagram above.

We can repeat this process to construct the whole diagram, and once we have filled out all of the possible options, we would get the complete diagram.



The formal, general way of describing this process, is that, in a state ℓ, reading an input $0 \leq j \leq b-1$, the transducer has a path to the state $\left[\frac{mj+\ell}{b}\right] = k$, and writes the remainder (where $\left[\frac{mj+\ell}{b}\right]$ represents the largest whole number less than or equal to $\frac{mj+\ell}{b}$).

In our example above, if we start in state 1, then $\ell = 1$ and we read a $j = 2$, the transducer (which multiplies by $m = 4$ in base $b = 3$) has a path to state $k = 3$, because

$$\left[\frac{mj+\ell}{b}\right] = \left[\frac{4(2)+1}{3}\right] = \left[\frac{9}{3}\right] = 3 = k.$$

Properties of Multiplication Transducers

As I got to know these pictures, and drew a lot of them, I started to figure out some things about how they work. When I started drawing these pictures, I did that computation for each arrow that I needed to draw in the diagram. After doing many of them, I realized that there are some patterns that can make constructing the diagrams easier. I would start by working through each state one at a time. To keep track of what my edges will be, I first write out the paths in the order

$$\text{start carry} \xrightarrow{\text{input digit, output digit}} \text{end carry.}$$

So, for the diagram above, my notes would look like:

$$\begin{array}{ccc} & \xrightarrow{0,0} & 0 \\ 0 & \xrightarrow{1,1} & 1 \\ & \xrightarrow{2,2} & 2 \end{array} \qquad \begin{array}{ccc} & \xrightarrow{0,1} & 0 \\ 1 & \xrightarrow{1,2} & 1 \\ & \xrightarrow{2,0} & 3 \end{array} \qquad \begin{array}{ccc} & \xrightarrow{0,2} & 0 \\ 2 & \xrightarrow{1,0} & 2 \\ & \xrightarrow{2,1} & 3 \end{array} \qquad \begin{array}{ccc} & \xrightarrow{0,0} & 1 \\ 3 & \xrightarrow{1,1} & 2 \\ & \xrightarrow{2,2} & 3 \end{array}$$

Then, I use the notes to help me draw the whole diagram. I would do the same for the transducer that multiplies by 5 in base 3.

Example 52. *Constructing the transducer that multiplies by* 5 *in base* 3*. Looking at the steps I could take from* 0 *first, I would think of the outcomes by multiplying the input digit by* 5*, and adding* 0*.*

$$\begin{array}{ccccl} & \xrightarrow{0,0} & 0 & & \textit{because } 5(0)+0=0(3)+0 \\ 0 & \xrightarrow{1,2} & 1 & & \textit{because } 5(1)+0=1(3)+2 \\ & \xrightarrow{2,1} & 3 & & \textit{because } 5(2)+0=3(3)+1 \end{array}$$

Continuing the same way, I would make the notes for the other four possible states (carries) to get

$$\begin{array}{ccc} & \xrightarrow{0,0} & 0 \\ 0 & \xrightarrow{1,2} & 1 \\ & \xrightarrow{2,1} & 3 \end{array} \qquad \begin{array}{ccc} & \xrightarrow{0,1} & 0 \\ 1 & \xrightarrow{1,0} & 2 \\ & \xrightarrow{2,2} & 3 \end{array} \qquad \begin{array}{ccc} & \xrightarrow{0,2} & 0 \\ 2 & \xrightarrow{1,1} & 2 \\ & \xrightarrow{2,0} & 4 \end{array} \qquad \begin{array}{ccc} & \xrightarrow{0,0} & 1 \\ 3 & \xrightarrow{1,2} & 2 \\ & \xrightarrow{2,1} & 4 \end{array} \qquad \begin{array}{ccc} & \xrightarrow{0,1} & 1 \\ 4 & \xrightarrow{1,0} & 3 \\ & \xrightarrow{2,2} & 4. \end{array}$$

Then, I take this information and put it together into the transducer picture.

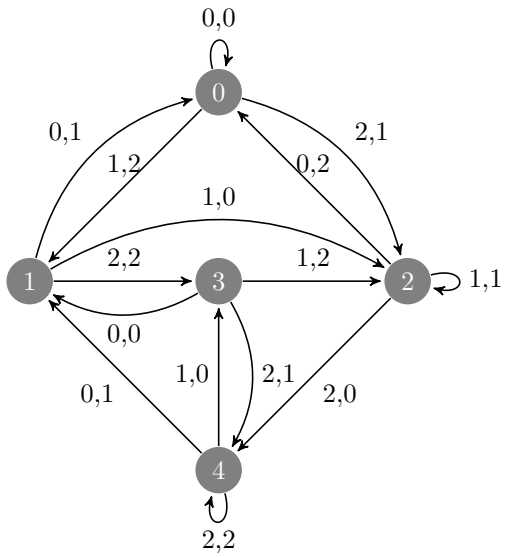
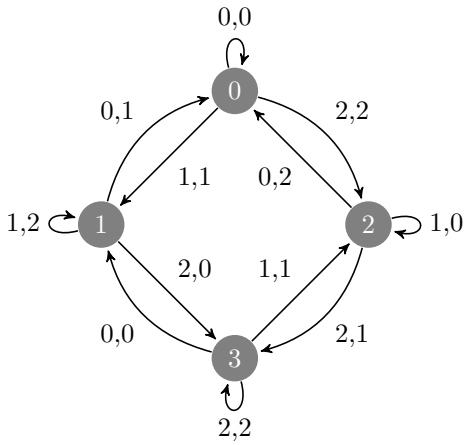
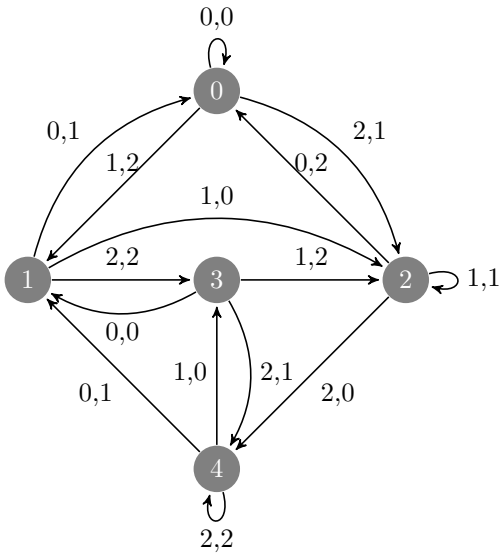


Figure 5.2: The transducer $\mathcal{T}_{5,3}$ that multiplies by 5 in base 3.

In drawing a lot of these pictures, there are things we can notice that are consistent between the different examples. For example, one thing I notice about the diagrams above is that in the transducer that multiplies by 4, there are 4 states: 0, 1, 2, and 3. When we look at the transducer that multiplies by 5, it has 5 states: 0, 1, 2, 3, and 4.

TRY: Look at the two pictures of transducers we have worked with so far. List two more similarities that you notice between the pictures:





Diagramming and Gesture

Through drawing more examples of transducers, I developed a shorthand notation as part of the process of drawing the diagrams. I employed **arrows** to first map out which edges would be present in the transducer. This step was motivated by a desire to **organize information** to more effectively and efficiently draw the diagrams. I utilize **perspective** in two ways. First, the shorthand notation is an example of perspective by choosing to highlight the diagram differently (focusing on individual paths), where it is easier to read (from right to left) the possible paths leaving each state. Then, I chose to draw the transducers in a way that made the graphs planar (no edges crossing), which helps me see cycles more clearly. I make these choices in perspective based on what I think will make things easier to see, but they are definitely not the only choices one could make. I think about the elements of the diagrams as being flexible (easy to move around, with edges moving like string rather than like rigid poles). I could have chosen another perspective (for example, changing the position of states 3 and 4 in the example $\mathcal{T}_{5,3}$).

It has been proven that every transducer that multiplies by m in base 3 (and in fact, any transducer that multiplies by m in any base b) has exactly the m states: 0, 1, 2, ..., $m-1$. So, what I noted about the examples $\mathcal{T}_{4,3}$ and $\mathcal{T}_{5,3}$ is true in general. This is a consequence of statements that were proved by Blanchard, Dumont, and Thomas in their 1992 paper [7]. We first will talk through why it is true for the examples we have seen, and then give the formal, general statement.

In our examples above, we see that starting from state $m-1$ and reading 2 (the largest digit we could read), keeps us coming back to state $m-1$. So, in $\mathcal{T}_{4,3}$, from state 3 we move to

$$\left[\frac{mj+\ell}{b}\right] = \left[\frac{4(2)+3}{3}\right] = \left[\frac{11}{3}\right] = 3 = k,$$

writing a 2. In $\mathcal{T}_{5,3}$, in state 4 we move to

$$\left[\frac{mj+\ell}{b}\right] = \left[\frac{5(2)+3}{3}\right] = \left[\frac{14}{3}\right] = 4 = k,$$

writing a 2. Then, if we were in any state d smaller than the state $m-1$, we would have a smaller number in the numerator of the fraction $\left[\frac{mj+\ell}{b}\right]$, and so would get a value for k that is less than or equal to $m-1$.

Proposition 53. *[7] The multiplication transducer that multiplies by m in base b, where the set of states is the set of all carries 0 to $m-1$, is always irreducible and synchronizing for input.*

Proposition Restated: This proposition includes two statements:

1. The number of states corresponds to the possible options for carrying to the next place value in the algorithm. This lemma is the general way of saying that if we are multiplying by m in base b, there are steps that carry each value from 0 to $m-1$, and the largest value we could carry is $m-1$. This is what the authors mean by "irreducible."

2. The transducer can read every string, or number, that is composed of digits from the set $\{0, 1, 2\}$ in base 3, and it will give the same result for a number every time it reads that number. This is what they mean when they say "synchronizing for input."

In addition to noticing patterns about the what the completed transducer looks like, I also find patterns when I am in the middle of constructing transducers. Some things are easier to see when written one way as opposed to another. When I was writing out my notes before drawing transducer, I noticed that the paths that end in a given state are in groups of three. For example, in the transducer that multiplies by 5 in base 3, all the paths that enter state 0 come from a group in order 0, 1, and 2.

$$\begin{array}{ccccccccccccccc}
 & \xrightarrow{0,0} & 0 & & & \xrightarrow{0,1} & 0 & & & \xrightarrow{0,2} & 0 & & & \xrightarrow{0,0} & 1 & & & \xrightarrow{0,1} & 1 \\
0 & \xrightarrow{1,2} & 1 & & 1 & \xrightarrow{1,0} & 2 & & 2 & \xrightarrow{1,1} & 2 & & 3 & \xrightarrow{1,2} & 2 & & 4 & \xrightarrow{1,0} & 3 \\
 & \xrightarrow{2,1} & 3 & & & \xrightarrow{2,2} & 3 & & & \xrightarrow{2,0} & 4 & & & \xrightarrow{2,1} & 4 & & & \xrightarrow{2,2} & 4.
\end{array}$$

The paths that enter state 3 may not look like they are in order at first glance: 4, 0, and 1, However, if we think of them as part of set of remainders from dividing by 5, they are in order in this way.

Number	1	2	3	4	5	6	7	8	9	10	11	...
Remainder /5	1	2	3	4	0	1	2	3	4	0	1	...

When we talk about numbers being in order of their remainder after dividing by 5, we say that they are *consecutive* (mod 5). The following lemma is the formal, general way of saying that what we noticed about the arrows going in to a given state coming from states that are consecutive (mod 5) is actually true in general (so for the transducer $\mathcal{T}_{m,b}$, the states are consecutive (mod m)).

Lemma 54. *The paths in to any given state in the multiplication transducer $\mathcal{T}_{m,b}$ are from a set of exactly b states that are consecutive* (mod m).

Lemma Restated: There are exactly b states that have direct paths into any given state in a transducer that

multiplies by m in base b, and these b states are consecutive $\pmod{m}$. Consecutive $\pmod{m}$ means the remainder wraps around in order, i.e. $\{5, 6, 0, 1\}$ when multiplying by 7.

Proof. By definition, the transducer $\mathcal{T}_{m,b}$ in a state ℓ, reading an input $0 \leq j \leq b-1$, moves to the state $\left[\frac{mj+\ell}{b}\right] = a$.

It is equivalent to say that a state a, with $0 \leq a \leq m-1$ is the end of paths from the states

$$\ell_i = m \cdot \text{frac}\left(\frac{ba+i}{m}\right) \text{ for each } 0 \leq i < b,$$

where the symbol $\text{frac}\left(\frac{x}{y}\right) = \frac{x}{y} - \left[\frac{x}{y}\right]$ represents the fractional part (or part after the decimal point) in $\frac{x}{y}$ (for example, $\text{frac}\left(\frac{6}{5}\right) = \frac{1}{5}$). We note that these states are consecutive $\pmod{m}$, since $\ell_i = m \cdot \frac{k}{m}$ for some $0 \leq k \leq m-1$. Since the i come from the range $0 \leq i < b$, there are exactly b of them. $\square$

Some facts related to the above Lemma:

- We can also see the k paths into each state by the construction of the transducer, which has mk paths distributed equally among m states before removing paths.
- All of these paths read the same input except those that are directed into the states

$$\left\{\left[\frac{m}{k}\right], \left[\frac{2m}{k}\right], \ldots, \left[\frac{(k-1)m}{k}\right]\right\}$$

  (these are the same ones that come from wrapping around).
- When $m \equiv 0 \pmod{k}$, there is no wrapping around.

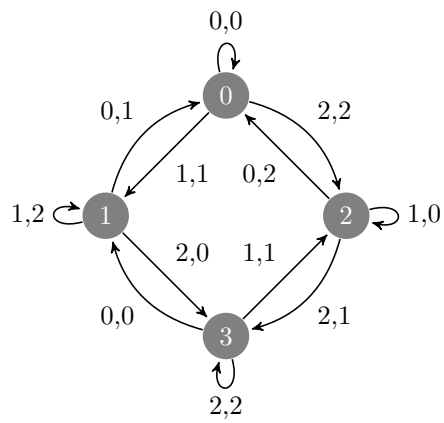
5.3 Modified Multiplication Transducers

We now return to the original question about restricting digits. We asked in the first section, which numbers are in the sets of numbers that use no 9s in base 10, or which are in the set that have only digits 0 and 1 in base 3? We then want to study the quotients of these numbers (which numbers can be made by dividing numbers in the set).

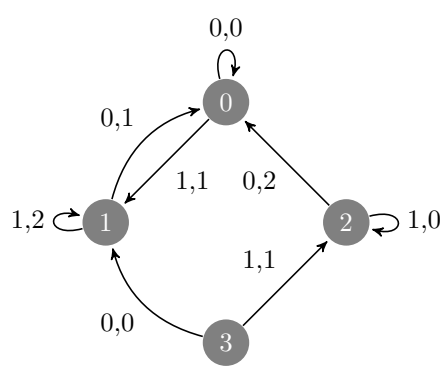
Multiplication and Division

Multiplication and division are inverse operations. If I have the statement $4 \times 25 = 100$, I can also write $4 = \frac{100}{25}$, and we say 4 is the *quotient* (outcome of the division). When we write this in base 3, we have $4 = \frac{[10201]_3}{[221]_3}$, so we have written 4 as a quotient of numbers in base 3 using the transducers.

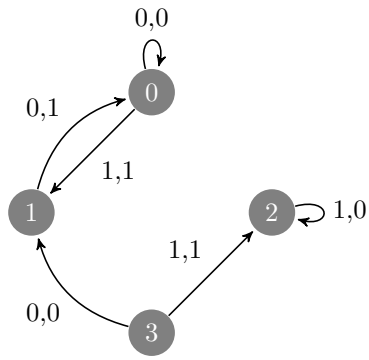
Restricting Digits is Removing Edges



We are interested in whether we can write 4 as a quotient of two numbers that do not have any 2s in their base 3 representation. We do not know which number we would want to multiply by, but we do know that it should not have any 2s. So, we can modify the diagram to only show the paths that do not read any 2s (no 2s in the number we are multiplying by, which becomes the denominator of the fraction).



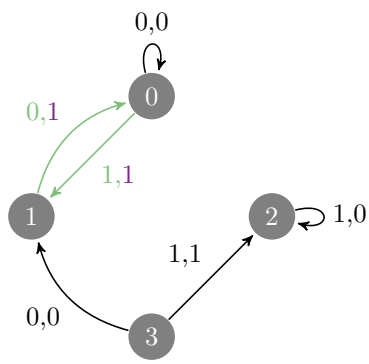
We also do not want to have the product (output) have any digits that have 2, so we similarly remove the paths that 'write' a digit 2 as an output.



What is left is a modification of the original transducer that only 'reads' and 'writes' the digits 0 and 1. This machine will no longer multiply by any number in base 3. What it can do is tell us if we take some number that only uses 0s and 1s in base 3, and we multiply it by 4, whether the product also has only the digits 0 and 1. Another way of thinking about this, is the machine tells us if the number 4 can be represented as a quotient of numbers with only digits 0 and 1 in their base 3 representations.

It turns out it does, there is a loop starting and ending at state 0 that shows

$$4 = \frac{4}{1} = \frac{[11]_3}{[01]_3}.$$



The punchline is that we can construct a machine that multiplies by any number m in base 3, and then remove edges to get a modified transducer that tells us if m can be represented as a quotient of two integers with base 3 representation using only digits 0 and 1. If there is a loop that starts and ends at the state 0 in the machine, then m has such a representation. Most of the time, we ignore the loop that is just one step from 0 to itself, because it represents the equation $0 \cdot m = 0$, but this does not give us a valid quotient (since we cannot divide by 0).

Diagramming and Gesture

When initially drawing the modified multiplication transducers, I used the strategy of **successive framing** to show, side by side, the complete transducer and the modified transducer with edges removed. This allowed me to see the effect of removing edges by being able to compare with the original (as opposed to erasing, or only making the modified transducer). By drawing the transducer in this way, I also **explore a temporal dimension** of the diagram. Thinking of the complete transducer as the 'original' that is built, and then removing edges gives a timeline to the process.

At this point we can remember that we gave the "set of numbers that can be represented as a quotient of two integers with base 3 representation using only digits 0 and 1" a name, $Q(3; \{0, 1\})$. So, we have shown that 4 is in the set $Q(3; \{0, 1\})$ by looking for a loop in the machine.

We could similarly construct a machine that multiplies by 7 in base 3. Once we delete all the edges that read and write the digit 2, we are left with the following diagram.

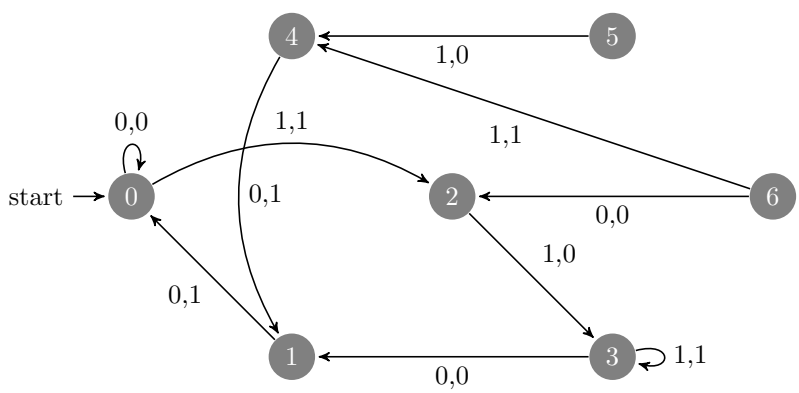
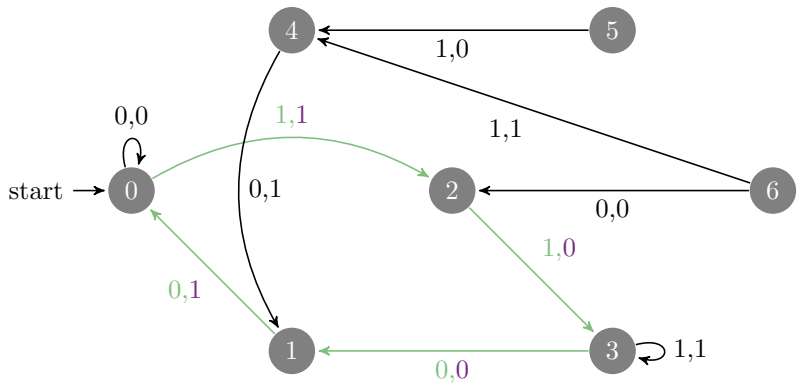


Figure 5.3: $\mathcal{T}_{7,3}$ with edges that read or write the digit 2 removed.

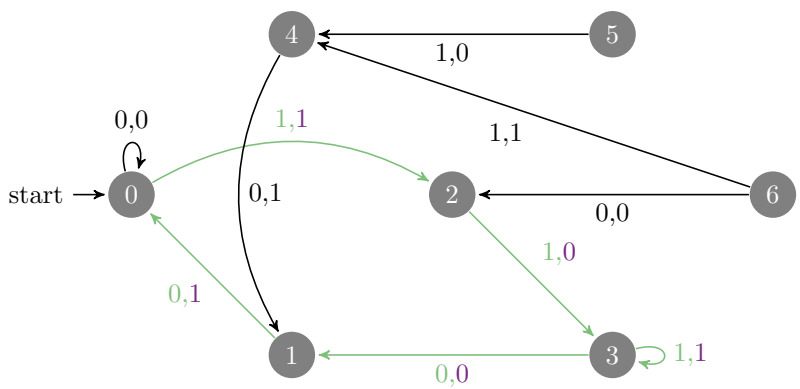
There is a loop starting and ending at 0 that shows the example we saw previously:

$$7 = \frac{28}{4} = \frac{[1001]_3}{[11]_3}.$$



So, 7 is in the set $Q(3;\{0,1\})$. In fact, it has several representations. For example, the loop below shows

$$7 = \frac{91}{13} = \frac{[10101]_3}{[111]_3}.$$



Indeed, there are infinitely many distinct representations depending on how many 1s we insert into the middle.

$$7 = \frac{[1001]_3}{[11]_3} = \frac{[10101]_3}{[111]_3} = \frac{[101101]_3}{[1111]_3} = \frac{[1011101]_3}{[11111]_3} = \frac{[101\ \overbrace{1\ldots1}^{\text{any number}}\ 101]_3}{[11\ \underbrace{1\ldots1}_{\text{same number}}\ 11]_3}$$

Lemma 55. *A cycle that starts and ends at* 0 *corresponds to a distinct representation as a quotient.*

These machines are especially helpful for telling us if an integer is definitely NOT in $Q(3;\{0,1\}$. For example, when $m = 6$ we get the following graph.

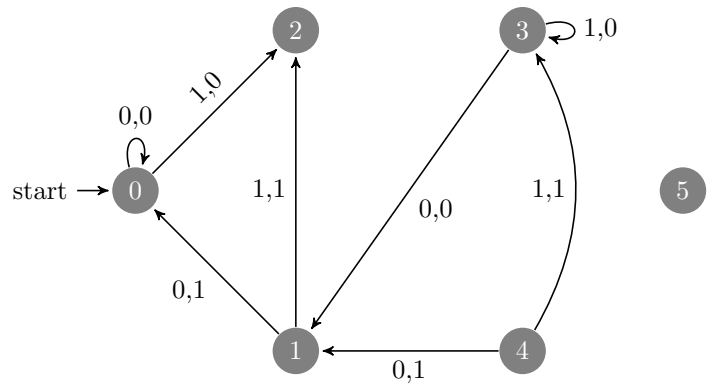


Figure 5.4: $\mathcal{T}_{6,3}$ with edges that read or write the digit 2 removed.

Here, if we look closely, we will see that the only nontrivial path out of 0 goes to state 2 (nontrivial means not the one step from 0 to itself). But there are no paths that leave state 2! So, we would not be able to get back to state 0. This means there is no way to write 6 as a quotient of integers with representations using digits 0 and 1 in base 3 (or, 6 is not in $Q(3; \{0, 1\})$).

5.3.1 Ways to Draw Modified Multiplication Transducers

In working on these I also played around with different ways to draw them. This is part of seeing the diagrams as dynamic, as well as efforts to explain them better, and understand them better myself.

As part of an effort to explain more formally the relationships I was seeing to my advisor, Bruce, I wrote a document in July 2018 with an example that used color and shading instead of labels on the edges.

Example 56. *The transducer below is the machine that multiplies by 7 in base 4. The color of the arrow indicates the digit read (the digit from the denominator, which is being multiplied by* 7*). The colors are:*

Gray reads 0 *Green reads 1* *Purple reads 2* *Black reads 3.*

The size of the dotted line indicates which digit is written (the digit in the numerator, which would be the digit written in the corresponding place in the multiplication algorithm).The lines are:

Solid writes 0 *Dense Dots write 1* *Loose Dots write 2* *Dashes write 3*

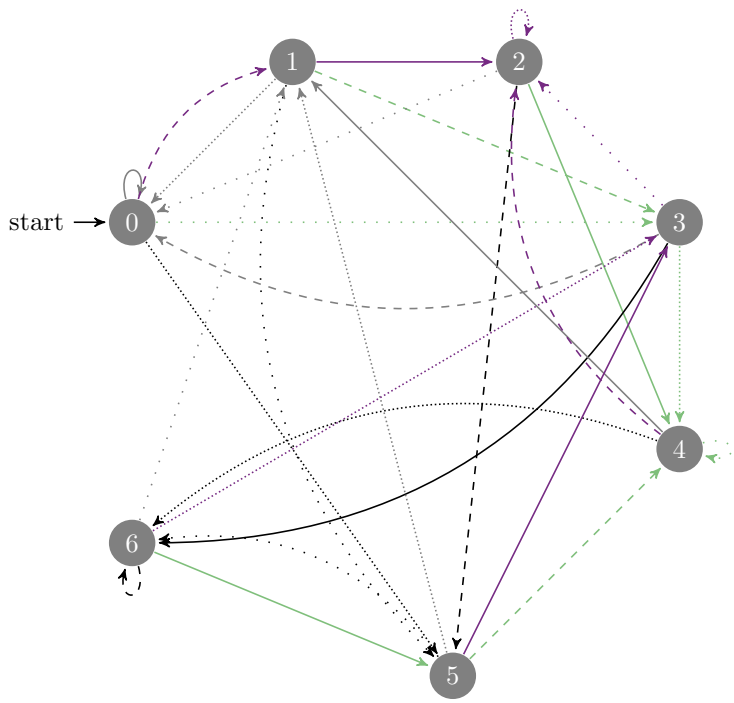
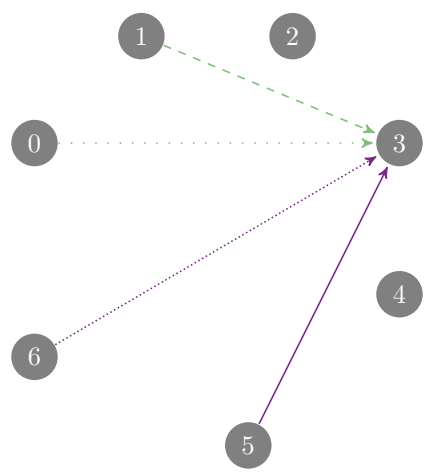
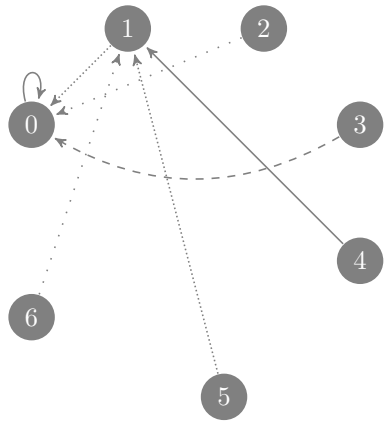


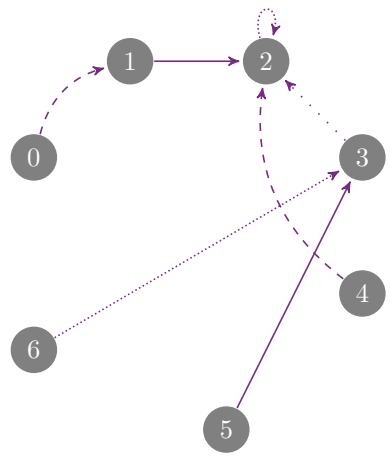
Figure 5.5: The transducer $\mathcal{T}_{7,4}$ that multiplies by 7 in base 4.

This is an extension of the way I had been drawing the graphs in my notes, using colors to represent at different points the digits that are being read and written. I often draw the diagrams with the states roughly in a circle in increasing order, again working off of the shorthand notes to figure out where to draw the paths. This has become the fastest way for me to do it, and helped to illustrate some of the properties below. What we saw before in Lemma 54 wrapping around, we can see by looking at the paths into the state 3. We can also notice that the four paths into any given state are of all four different densities, which means they each write a different value.



By looking at the the different colors, we can see that there is a pattern to all of the paths that read specific inputs. For example, if we highlight all of the paths that read 0 (gray colored edges), or all of the edges that read 1 (purple colored edges) we get:





I think about this almost as though the edges are "filling in" the possible incoming edges, starting from the smallest possible carry. There are 7 possible gray edges, because the edges 'read' 0 from each of the seven possible states. Then, there are four possible ways to enter state 0 (corresponding to the remainder, or the digit that is written), so the gray edges from the first four states fill in those slots (remainder 0, 1, 2, and 3). Then, there are four possible ways to enter state 1, and those will be filled by the gray edges from the next four states. But there are only three more states! So those three gray edges enter state 1, and there is one slot remaining as we move to the next input digit. then we have one path that reads a 1 that can enter state 1. The remaining 6 paths that read 1 are split with the next four of them entering state 2, and then the remaining two entering state 3.

$$0 \quad \begin{array}{ll} \xrightarrow{0,0} & 0 \\ \xrightarrow{1,3} & 1 \\ \xrightarrow{2,2} & 3 \\ \xrightarrow{3,1} & 5 \end{array} \qquad 1 \quad \begin{array}{ll} \xrightarrow{0,1} & 0 \\ \xrightarrow{1,0} & 2 \\ \xrightarrow{2,3} & 3 \\ \xrightarrow{3,2} & 5 \end{array} \qquad 2 \quad \begin{array}{ll} \xrightarrow{0,2} & 0 \\ \xrightarrow{1,1} & 2 \\ \xrightarrow{2,0} & 4 \\ \xrightarrow{3,3} & 5 \end{array} \qquad 3 \quad \begin{array}{ll} \xrightarrow{0,3} & 0 \\ \xrightarrow{1,2} & 2 \\ \xrightarrow{2,1} & 4 \\ \xrightarrow{3,0} & 6 \end{array}$$

$$4 \quad \begin{array}{ll} \xrightarrow{0,0} & 1 \\ \xrightarrow{1,3} & 2 \\ \xrightarrow{2,2} & 4 \\ \xrightarrow{3,1} & 6 \end{array} \qquad 5 \quad \begin{array}{ll} \xrightarrow{0,1} & 1 \\ \xrightarrow{1,0} & 3 \\ \xrightarrow{2,3} & 4 \\ \xrightarrow{3,2} & 6 \end{array} \qquad 6 \quad \begin{array}{ll} \xrightarrow{0,2} & 1 \\ \xrightarrow{1,1} & 3 \\ \xrightarrow{2,0} & 5 \\ \xrightarrow{3,3} & 6 \end{array}$$

We formalize this phenomenon in the following lemma.

Lemma 57. *The paths that read* $0 \leq i \leq b-1$*, enter the states* $\left[\left[\frac{mi}{b}\right], \ldots \left[\frac{m(i+1)-1}{b}\right]\right]$.

Proof. The transducer $\mathcal{T}_{m,b}$ in a state ℓ, reading an input i, moves to the state $\left[\frac{mi+\ell}{b}\right] = a$. When we look at the set of states $\{0, \ldots, m-1\}$, we see that reading an input i moves to the set of states $\left\{\left[\frac{mi+\ell}{b}\right] : \ell \in \{0, \ldots, m-1\}\right\}$. This is exactly the set of states $\left[\left[\frac{mi}{b}\right], \ldots \left[\frac{m(i+1)-1}{b}\right]\right]$. □

Diagramming and Gesture

In this example, I used **color** and **shading** to highlight the digits being read/written. I see this as using the strategy of shading, rather than dotted lines, because I use loose dots, dense dots, and dashes to shade the edges, exploring differences in the edges. In this way, color and shading complement each other, because I needed to show both the digit read and the digit written on the same edge. When choosing which colors to put where, I tried to alternate between the lighter and darker colors (so that it would still be clear when printed in black and white). When choosing the shading of the lines, I tried to place the shading so that the 'darker' shadings (dense dots and dashes), bracket the solid line. This helped me illustrate the cyclical nature of the digits written (solid, dense dots, loose dots, dashed, solid, dense dots, ...) in the diagram.

This diagram was written explicitly to try to highlight things that I had internalized based on drawing lots and lots of pictures by hand. Many of the diagrams I drew by hand had some but not all of these characteristics; for example, I would use colors to highlight the digits read in one diagram, and then redraw the same diagram using the same colors to highlight the digits written. When writing it up to share with Bruce Reznick, I tried to include as much information as possible in the same diagram.

Drawing Reduced Diagrams

If we were interested in eliminating a digit (for example the digit 1), we would first eliminate all purple arrows from our original diagram (because the denominator would not be able to have any 1s, so we would take no steps that read a 1), giving us the diagram on the left. Then, we would also want to eliminate all paths that would write a 1 (these are the densely dotted lines). We are left with the diagram on the right of Figure 5.3.1.

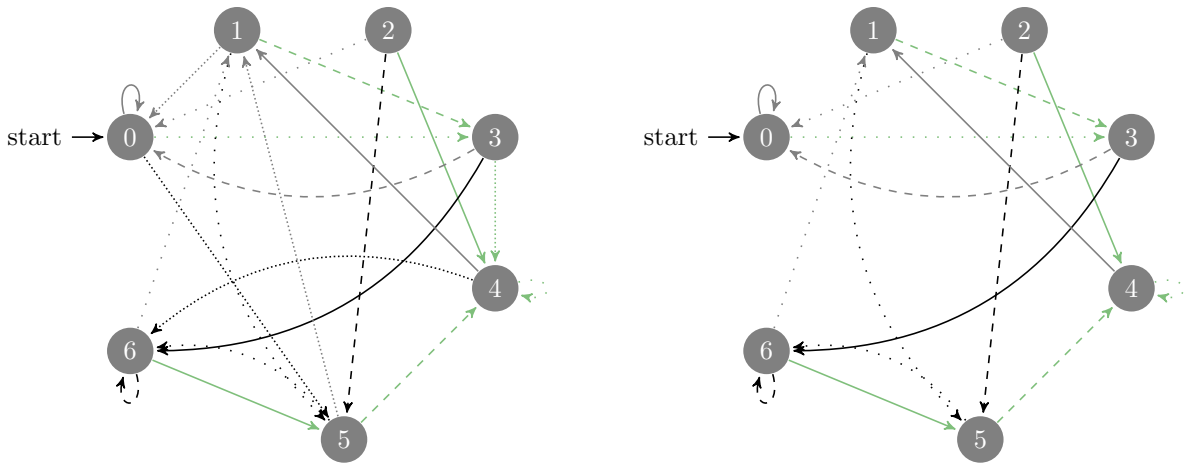


Figure 5.6: The transducer that multiplies by 7 in base 4 with edges that read 1 deleted (left), and edges that either read or write 1 deleted (right).

Now, we can see that there are no paths that lead to state 2 in the resulting diagram. Since there are no paths into the state 2, every cycle containing 0 has to come through state 3. If we are interested in looking at cycles containing 0, it can be helpful to rearrange this transducer (and eliminate state 2).

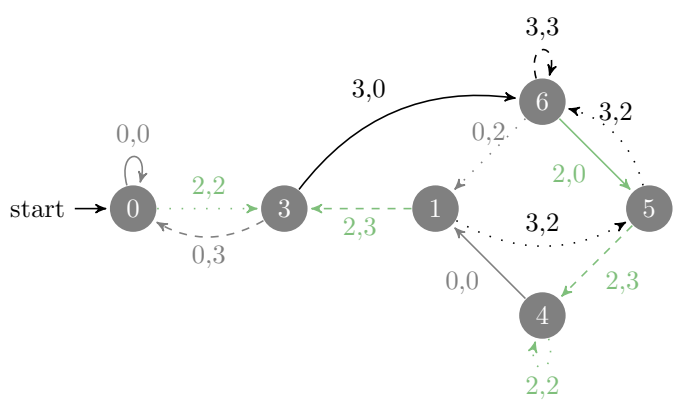


Figure 5.7: The modified transducer $\mathcal{T}_{7,4}$ with no edges that read or write 1, rearranged to be planar.

I often draw only the parts of the diagram that are relevant to the question I am currently trying to answer. We will see more of this in the next chapter.

Diagramming and Gesture

Most often, I start drawing the diagrams with states in a circle (almost like a clock), with the state 0 in the upper left corner. I chose this **perspective** once I had begun drawing many diagrams at a time and realized that it would be easier to compare them if I was consistent in my approach. However, after deleting some edges, or otherwise modifying a diagram, I would try to change perspective to see if I could get a better sense of a particular attribute I am looking at. In the example above, I redrew the diagram to focus on only the cycles containing 0.

These changes in perspective facilitate **exploration of spatio-motoric information**. The diagrams contain information, like the cycles containing a certain state, and I can understand that information better by playing around with the spatial relationships. I think of the relationships between the states as fixed, but the diagram itself as having the potential for motion, so I can redraw the diagram from many different perspectives, depending on what I would like to focus on.

Chapter 6

Quotients in Base 3

This chapter addresses progress in classifying the set $Q(3;\{0,1\})$ of quotients of sums of distinct powers of three. In Section 6.1 we discuss relationships between $Q(3;\{0,1\})$ and the Cantor Set as a motivation for exploring this problem. In Section 6.2 we provide an algorithm which gives another perspective on determining whether an integer m is in $Q(3;\{0,1\})$, and if so, giving an explicit construction for a representation of m as a quotient of sums of distinct powers of three. We conclude with computational results towards classifying $Q(3;\{0,1\})$.

6.1 Quotients of Sums of Distinct Powers of Three

6.1.1 The Cantor Set

The Cantor Set (which we will refer to as $\mathcal{C}$) is a mathematical object that has connections to a variety of mathematical areas, including topology, analysis, and fractals.

It is a set of numbers that lie in the interval $[0,1]$, and can be constructed by taking the interval $[0,1]$ and repeatedly removing middle thirds. Begin with the interval $[0,1]$, and remove the middle third to get $\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right]$. With the remaining two intervals (in the second level), remove the middle third of each to get the third level, to get $\left[0,\frac{1}{9}\right]\cup\left[\frac{2}{9},\frac{1}{3}\right]\cup\left[\frac{2}{3},\frac{7}{9}\right]\cup\left[\frac{8}{9},1\right]$. Repeating this process forever, we then take the intersection of all of the levels (note that level k has 2^{k-1} intervals of size $\frac{1}{3}^{k-1}$). The intersection of the levels is the set of points that are in every level. For example, $\frac{2}{3}\in\mathcal{C}$ because the point $\frac{2}{3}$ remains even as we continue removing middle thirds. The point $\frac{1}{8}$ is not in $\mathcal{C}$, because after the second level it is removed, and so it is not in the intersection of the sets of intervals.

This procedure of removing middle thirds continues forever. Every element in $\mathcal{C}$ is an endpoint of one of these intervals at some level. Previous work by Jayadev Athreya, Jeremy Tyson, and Bruce Reznick describes properties of arithmetic between Cantor Set elements, [5] .

Figure 6.1: The first 4 levels of the 'middle-thirds' definition of the Cantor Set.

They show that the set of quotients of Cantor Set elements is

$$\left\{\frac{x}{y} : x, y \in \mathcal{C}, y \neq 0\right\} = \bigcup_{k=-\infty}^{\infty} \left[\frac{2}{3} \cdot 3^k, \frac{3}{2} \cdot 3^k\right].$$

The right side of this equation represents the set of quotients of Cantor Set elements, and they prove that set is equal to the union of intervals,

$$\ldots \cup \left[\frac{2}{9}, \frac{1}{2}\right] \cup \left[\frac{2}{3}, \frac{3}{2}\right] \cup \left[2, \frac{9}{2}\right] \cup \left[6, \frac{27}{2}\right] \cup \ldots$$

However, their proof does not provide an algorithm for expressing a given element $u \in \bigcup_{k=-\infty}^{\infty} \left[\frac{2}{3} \cdot 3^k, \frac{3}{2} \cdot 3^k\right]$ as a quotient of elements of $\mathcal{C}$. What this means is that we know that every number in the union of those intervals can be written as a quotient of Cantor Set elements, but we do not know how to come up with the particular quotient for a given number.

We consider integers u in the set

$$\bigcup_{k=1}^{\infty} \left[\frac{2}{3} \cdot 3^k, \frac{3}{2} \cdot 3^k\right] = [2, 4.5] \cup [6, 13.5] \cup [18, 40.5] \cup \ldots.$$

The remainder of this section will be focused on concretely describing how to come up with the quotient for some of these integers.

We define left-hand endpoints of $\mathcal{C}$ to be elements of $\mathcal{C}$ that are the leftmost points of closed intervals in the 'middle-thirds' definition of the Cantor Set.

These left-hand endpoints are the points which may be expressed as $\sum_{k=1}^{r} \frac{2}{3^{m_k}}$ for some finite sequence

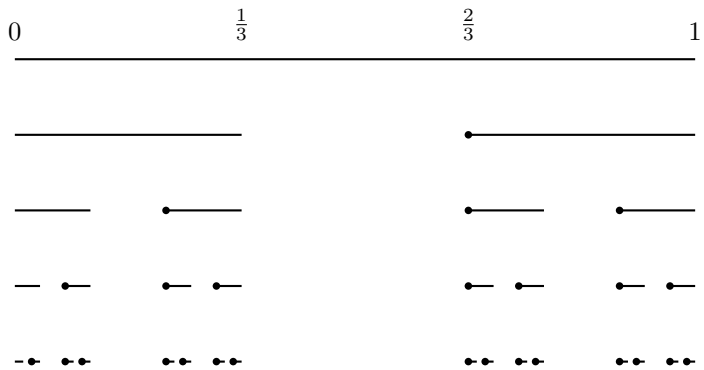


Figure 6.2: Nonzero left hand endpoints in the first 5 levels.

$\{m_1 < m_2 < \cdots < m_r\}$ of natural numbers. For example $\frac{2}{3} + \frac{2}{9}$ is a left-hand endpoint of the fourth interval in the third level, and $\frac{2}{9} + \frac{2}{27} + \frac{2}{81}$ is a left-hand endpoint of the eighth interval in the fifth level. Given a quotient of nonzero left-hand endpoints of $\mathcal{C}$, we can perform a bit of manipulation to rewrite the quotient as a quotient of sums of distinct powers of three.

$$\frac{\sum_{k=1}^{r} \frac{2}{3^{m_k}}}{\sum_{l=1}^{t} \frac{2}{3^{n_l}}} = \frac{\sum_{k=1}^{r} \frac{1}{3^{m_k}}}{\sum_{l=1}^{t} \frac{1}{3^{n_l}}} = \frac{\frac{3^{m_r}}{3^{m_r}} \sum_{k=1}^{r} \frac{1}{3^{m_k}}}{\frac{3^{n_t}}{3^{n_t}} \sum_{l=1}^{t} \frac{1}{3^{n_l}}} = \frac{\frac{1}{3^{m_r}} \sum_{k=1}^{r} 3^{m_r - m_k}}{\frac{1}{3^{n_t}} \sum_{l=1}^{t} 3^{n_t - n_l}} = \frac{\sum_{k=1}^{r} 3^{m_r - m_k + n_t}}{\sum_{l=1}^{t} 3^{n_t - n_l + m_r}}.$$

We discuss progress on classifying which positive integers can be expressed as the quotients of these left-hand endpoints and how multiplication transducers defined in Chapter 5 can be used as tools for studying quotients of sums of distinct powers of three.

We recall that the notation $Q(3; \{0, 1\})$ represents the set of integer quotients of sums of distinct powers of three and that $S(3; \{0, 1\})$ represents the set of integers that are sums of distinct powers of three.

6.1.2 Narrowing the Question

The set $Q(3; \{0, 1\})$ is equal to set of integer quotients of left endpoint Cantor Set elements, and so is contained in the union of the intervals of real numbers $\left[\frac{2}{3} \cdot 3^j, \frac{3}{2} \cdot 3^j\right]$ for every integer j. When $j = 1$, we have the interval $[2, 9/2]$, when $j = 2$, we have the interval $[6, 27/2]$, and so on. So, in particular, we know that $\{2, 3, 4, 6, 7, 8, 9, 10, 11, 12, 13\}$ are in these intervals, but 5 is not, since $4.5 < 5 < 6$.

An initial question we might ask is: Are all of the positive integers in these intervals in $Q(3; \{0, 1\})$?

To answer this question, we consider the positive integers in the set of intervals as being grouped $\pmod 3$

and look at one group at a time:

$\{2, 8, 11, \dots\}$	$\{3, 6, 9, 12, \dots\}$	$\{4, 7, 10, 13, \dots\}$
$\equiv 2 \pmod 3\}$	$\equiv 0 \pmod 3$	$\equiv 1 \pmod 3$

Consider the numbers in the first group, those that are in the union of intervals (quotients of Cantor elements) and are equivalent to 2 (mod 3). Proposition 58 shows that these integers cannot be in $Q(3; \{0, 1\})$.

Proposition 58. *If $m \in \{n \in \mathbb{N} : n \equiv 2 \pmod 3\}$, then $m \notin Q(3; \{0, 1\})$.*

Proof. Let $m \in \{n \in \mathbb{N} : n \equiv 2 \pmod 3\}$. By definition, $m = \alpha_k 3^k + \alpha_{k-1} 3^{k-1} + \cdots + \alpha_1 3^1 + 2$ for some $k \in \mathbb{N}$, $\alpha_i \in \{0, 1, 2\}$. We know that elements of $Q(3; \{0, 1\})$ are quotients of elements of $S(3; \{0, 1\})$. Each of these elements of $S(3; \{0, 1\})$ are equivalent to 0 or 1 (mod 3). We proceed by contradiction. Suppose $m = \frac{x}{y}$ for some $x, y \in S(3; \{0, 1\})$. Since $m \equiv 2 \pmod 3$, m does not have 3 as a factor, and since $my = x$, there is some y', $x' \in S(3; \{0, 1\})$ such that $m = \frac{x'}{y'}$, where 3 does not divide y' (by factoring out any powers of three in the original y, and since m is not a multiple of three, there must be at least that many powers of three in the original x). Then, by definition of $S(3; \{0, 1\})$,

$$y' = \beta_j 3^j + \beta_{j-1} 3^{j-1} + \cdots + \beta_1 3^1 + 1, \text{ for some } j, \text{ where } \beta_i \in \{0, 1\}.$$

Then, if we multiply the expansions for m and y', combining like terms, we get

$$my' = \alpha_k \beta_j 3^{j+k} + (\alpha_k \beta_{j-1} + \alpha_{k-1} \beta_j) 3^{j+k-1} + \cdots + (\alpha_1 + 2\beta_1) 3^1 + 2$$

However, $my' = x'$, and this gives us a contradiction, since $x' \in S(3; \{0, 1\})$ and thus $x' \not\equiv 2 \pmod 3$. Therefore, our original assumption must have been false, and m cannot be written as a quotient of sums of distinct powers of three. $\square$

Next, consider the numbers in the second group, those that are in the union of intervals (quotients of Cantor elements) and are equivalent to 0 (mod 3). Suppose we have a positive integer u that belongs to this set. By definition, u is divisible by 3. So $u = 3^i w$ for some positive integer i and some $w \equiv 1$ or $2 \pmod 3$ (by factoring out the highest power of 3 that divides u). Notice that k would still be in our union of intervals, because if $u \in \left[\frac{2}{3} \cdot 3^j, \frac{3}{2} \cdot 3^j\right]$ for some integer j, then $w \in \left[\frac{2}{3} \cdot 3^{j-i}, \frac{3}{2} \cdot 3^{j-i}\right]$.

Since our goal is to find a representation of u as a quotient in base 3, this is equivalent to finding a representation of w as a quotient in base 3 (to get our quotient for u, we would multiply the numerator by

3^i). So, the case when u is divisible by 3 reduces to the case of $w \equiv 1$ or $2 \pmod 3$. This means that the case $u \equiv 0 \pmod 3$ depends on the two possible cases for w below.

$$\{2, 8, 11, \dots\} \qquad\qquad \{4, 7, 10, 13, \dots\}$$

$$\equiv 2 \pmod 3\} \qquad\qquad \equiv 1 \pmod 3$$

We have already shown that numbers in the first set, $\{n \in \mathbb{N} : n \equiv 2 \pmod 3\}$ cannot be in $Q(3; \{0, 1\})$. This implies that also elements of the set $\{u \in \mathbb{N} : u = 3^i w, \text{ for some integers } i \geq 0, w \equiv 2 \pmod 3\}$ cannot be in $Q(3; \{0, 1\})$.

At this point, we will recap what we have done in this section. We took the original problem looking at positive integers in the set $\bigcup_{k=1}^{\infty} \left[\frac{2}{3} \cdot 3^k, \frac{3}{2} \cdot 3^k\right]$. We then showed that some of these integers, in particular the set $\{u \in \mathbb{N} : u = 3^i w, \text{ for some integers } i \geq 0, \ w \equiv 2 \pmod 3\}$, are not in $Q(3; \{0, 1\})$. What remains is the set $\{v \in \mathbb{N} : v = 3^i w, \text{ for some integers } i \geq 0, \ w \equiv 1 \pmod 3\}$. It is sufficient to look at only the integers that are equivalent to $1 \pmod 3$, because whether $w \in Q(3; \{0, 1\})$ directly determines whether $v = 3^i w \in Q(3; \{0, 1\})$ for any i.

Since integers that are $\equiv 2 \pmod 3$ are not in $Q(3; \{0, 1\})$ and $3^k \frac{3}{2}$ is not an integer for any k, the endpoints of the interval $\left[\frac{2}{3} \cdot 3^k, \frac{3}{2} \cdot 3^k\right]$ will not be in $Q(3; \{0, 1\})$. So, we can consider the open intervals $\left(\frac{2}{3} \cdot 3^k, \frac{3}{2} \cdot 3^k\right)$.

Thus, we have reduced the question of which positive integers can be expressed as the quotients of the left-hand endpoint Cantor Set elements to the question:

Which integers $m \in \bigcup_{k=1}^{\infty} \left(\frac{2}{3} \cdot 3^k, \frac{3}{2} \cdot 3^k\right)$, with $m \equiv 1 \pmod 3$, are quotients of sums of distinct powers of three?

6.2 Testing with Directed Graphs

In order to answer the question above, we want to be able to test whether a given integer m is in $Q(3; \{0, 1\})$.

From above, we see that the question may be reduced to the question of whether a given integer $m \in \mathcal{A}$, where we define

$$\mathcal{A} = \left\{ m : m \equiv 1 \pmod 3, m \in \bigcup_{k=1}^{\infty} \left(\frac{2}{3} \cdot 3^k, \frac{3}{2} \cdot 3^k\right) \right\},$$

is in $Q(3; \{0, 1\})$. These conditions are necessary but not sufficient. For any integer m that satisfies these conditions, we may determine whether m can be written as a quotient of sums of distinct powers of 3

using the following algorithm. The algorithm is illustrated with an example ($m = 55$, below), and can be represented using a directed graph.

Definition 59. *A directed graph is a set of vertices connected by edges, where the edges have a direction associated with them.*

The vertices in our directed graphs are integers, and the edges are arrows which correspond to steps in the algorithm below. We will see an example of a directed graph when we go through the algorithm for $m = 55$.

This algorithm is originally how I was introduced to the problem by my advisor, Bruce Reznick, and how I became acquainted with the problems.

Algorithm: This algorithm iterates over a series of steps for a given integer $m \in \mathcal{A}$. If m can be written as a quotient of sums of distinct powers of three, the algorithm constructs integers r and p such that $mr = p$ and both r and p are sums of distinct powers of three. If m is not a quotient of sums of distinct powers of three, the algorithm determines that fact.

Given $m \in \mathcal{A}$, in each step we write

$$r_n \cdot m = 3^n a_n + b_n,$$

where r_n and b_n are sums of distinct powers of 3, less than 3^n. We start with the initial step ($n = 1$) that takes $r_1 = 1$, $a_1 = \frac{m-1}{3}$, and $b_1 = 1$ always, since $m \equiv 1 \pmod 3$.

We may then construct each step from the previous step, choosing from the appropriate case below depending on the congruence class of $a_n \pmod 3$. This is an iterative process, starting with $n = 1$ and ending if for some n, a_n is a sum of distinct powers of 3.

(a) If $a_n \equiv 1 \pmod 3$, take $r_{n+1} = r_n$ and note that $r_n \cdot m = 3^{n+1}\left(\dfrac{a_n - 1 + 1}{3}\right) + b_n$, which gives us $a_{n+1} = \frac{a_n - 1}{3}$, and $b_{n+1} = b_n + 3^n$.

(b) If $a_n \equiv 2 \pmod 3$, take $r_{n+1} = r_n + 3^n$ and note that $(r_n + 3^n) \cdot m = 3^n + b_n + 3^{n+1}\left(\dfrac{m}{3}\right)$, which gives us $b_{n+1} = b_n$, and $a_{n+1} = \frac{a_n + m}{3}$

(c) If $a_n \equiv 0 \pmod 3$, we have two choices:

  (i) Take $r_{n+1} = r_n$, and get $a_{n+1} = \frac{a_n}{3}$, $b_{n+1} = b_n$.

  (ii) Take $r_{n+1} = r_n + 3^n$, and get $a_{n+1} = \frac{a_n + m - 1}{3}$, $b_{n+1} = b_n + 3^n$.

In each step, we either keep $r_{n+1} = r_n$, or take $r_{n+1} = r_n + 3^n$. Both options are allowed when $a_n \equiv 0 \pmod 3$. In case (a), when $a_n \equiv 1 \pmod n$, we get a distinct factor of 3^n from the term $3^n a_n$. If we were

to take $r_{n+1} = r_n + 3^n$, we would have two factors of 3^n, leading to a digit of 2, which is forbidden.

In case (b), if $a_n \equiv 2 \pmod{n}$, then we get two factors of 3^n from the term $3^n a_n$. If we were to take $r_{n+1} = r_n$, we would have exactly two factors of 3^n, which is forbidden. So, in this case we must take $r_{n+1} = r_n + 3^n$, resulting in three factors of 3^n, leading to a digit 0 for that place.

If at some step n, a_n is a sum of distinct powers of 3, then $m = \dfrac{3^n a_n + b_n}{r_n}$ is a quotient of sums of distinct powers of 3. If we never arrive at such an a_n, then m cannot be written as a quotient of sums of distinct powers of 3.

Example: We now illustrate the algorithm using a specific numerical example. Take $m = 55$. Because $55 \equiv 1 \pmod{3}$, we have our initial step (always case (a)):

Step 1:

$$55 = 54 + 1$$
$$55 = 3(18) + 1$$

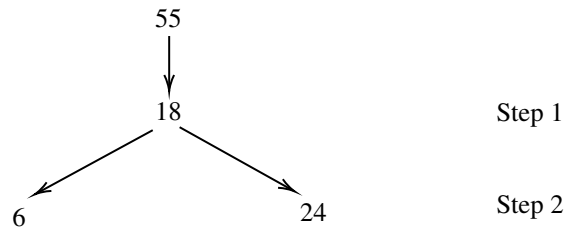
where the second equation is $r_1 \cdot m = 3^1(a_1) + b_1$ for the values $r_1 = 1$, $a_1 = 18 = \frac{54}{3} = \frac{55-1}{3}$ and $b_1 = 1$. We will keep track of the a_n values and have $m \to a_1$, or $55 \to 18$.

Since $18 \equiv 0 \pmod{3}$, we are in case (c). That means there are two possible options for step 2. Then we have:

Step 2:

$$55 = 3(18) + 1 \qquad\qquad 55 = 3(18) + 1$$
$$55 = 3^2(6) + 1 \qquad\qquad (3+1)55 = 3^2(24) + 3 + 1$$

Where the equation on the left comes from (c) (i) and the equation on the right comes from (c) (ii). Taking the appropriate values in the expression $r_2 \cdot m = 3^2 a_2 + b_2$, which are $[r_2 = r_1 = 1,\ a_2 = \frac{a_1}{3} = \frac{18}{3} = 6$, and $b_2 = b_1 = 1]$ and $[r_2 = r_1 + 3^1 = 1 + 3,\ a_2 = \frac{a_1+m-1}{3} = \frac{18+55-1}{3} = \frac{72}{3} = 24$, and $b_2 = b_1 + 3^1 = 3 + 1]$, respectively. Keeping track of the a_ns, we have $18 \to 6$ and $18 \to 24$. To put these together with the first step, we have



To move to the next step, we will now have two branches, and will perform Step 3 in two parts (once for $a_2 = 6$ and once for $a_2 = 24$).

Step 3:

When $a_2 = 6$, because 6 is a multiple of 3, we again have two possible options:

$$55 = 3^2(6) + 1 \qquad\qquad 55 = 3^2(6) + 1$$

$$55 = 3^3(2) + 1 \qquad\qquad (3^2 + 1)55 = 3^3(20) + 3^2 + 1$$

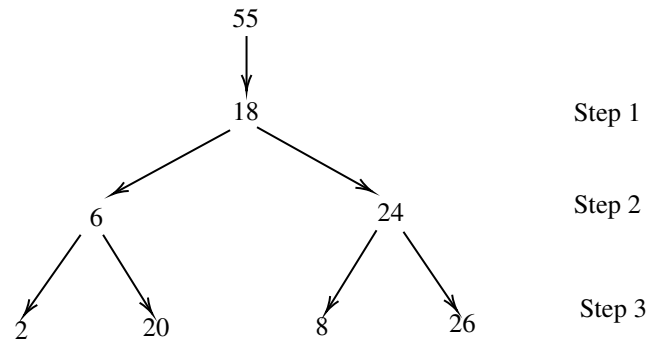
The equation on the left comes from following (c) (i) and the equation on the right comes from (c) (ii), with the step 2 equation $55 = 3^2(6) + 1$ [$r_2 = 1$, $a_2 = 6$, and $b_2 = 1$]. Taking the appropriate values in the expression $r_3 \cdot m = 3^3 a_3 + b_3$, which are [$r_3 = r_2 = 1$, $a_3 = \frac{a_2}{3} = \frac{6}{3} = 2$, and $b_3 = b_2 = 1$] and [$r_3 = r_2 + 3^2 = 1 + 3^2$, $a_3 = \frac{a_2+m-1}{3} = \frac{6+55-1}{3} = \frac{60}{3} = 20$, and $b_3 = b_2 + 3^2 = 3^2 + 1$], respectively. These show that the steps from $a_2 = 6$ are $6 \to 2$ and $6 \to 20$. We need also perform Step 3 on the second Step 2 output.

When $a_2 = 24$, because 24 is a multiple of 3, we have two possible options:

$$(3 + 1)55 = 3^2(24) + 3 + 1 \qquad\qquad (3 + 1)55 = 3^2(24) + 3 + 1$$

$$(3 + 1)55 = 3^3(8) + 3 + 1 \qquad\qquad (3^2 + 3 + 1)55 = 3^3(26) + 3^2 + 3 + 1$$

The equation on the left comes from following (c) (i) and the equation on the right comes from (c) (ii), with the step 2 equation $(3+1)55 = 3^2(24) + 3 + 1$ [$r_2 = 3 + 1$, $a_2 = 24$, and $b_2 = 3 + 1$]. Taking the appropriate values in the expression $r_3 \cdot m = 3^3 a_3 + b_3$, which are [$r_3 = r_2 = 3+1$, $a_3 = \frac{a_2}{3} = \frac{24}{3} = 8$, and $b_3 = b_2 = 3+1$] and [$r_3 = r_2 + 3^2 = 1+3+3^2$, $a_3 = \frac{a_2+m-1}{3} = \frac{24+55-1}{3} = \frac{78}{3} = 26$, and $b_3 = b_2 + 3^2 = 3^2+3+1$], respectively. These show that the steps from $a_2 = 24$ are $24 \to 8$ and $24 \to 26$. Combining all of Step 3 into our directed graph, we have,



We now move on to the next step.

Step 4:

None of the values for a_3 are divisible by 3, so each case produces exactly one value of a_4, listed below:

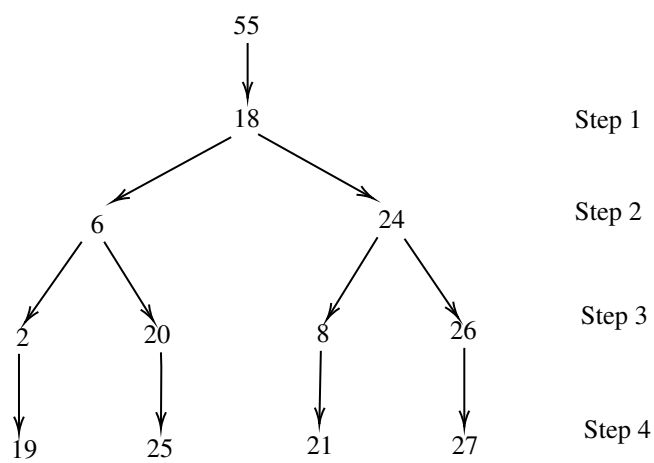
$$55 = 3^3(2) + 1 \qquad (3+1)55 = 3^3(8) + 3 + 1$$

$$(3^3 + 1)55 = 3^4(19) + 1 \qquad (3^3 + 3 + 1)55 = 3^4(21) + 3 + 1$$

$$(3^2 + 1)55 = 3^3(20) + 3^2 + 1 \qquad (3^2 + 3 + 1)55 = 3^3(26) + 3^2 + 3 + 1$$

$$(3^3 + 3^2 + 1)55 = 3^4(25) + 3^2 + 1 \qquad (3^3 + 3^2 + 3 + 1)55 = 3^4(27) + 3^2 + 3 + 1$$

Where each pair of equations moves from $r_3 = 3^3(a_3) + b_3$ to $r_4 = 3^4(a_4) + b_4$, following the algorithm. Then, if we put these into our graph, we have:



We note that in the final path of the example, we get $a_4 = 27 = 3^3$, and so we have

$$(3^3 + 3^2 + 3 + 1)55 = 3^4(27) + 3^2 + 3 + 1 \qquad \Rightarrow \qquad 55 = \frac{3^4(3^3) + 3^2 + 3^1 + 3^0}{3^3 + 3^2 + 3^1 + 3^0}.$$

In this example, a_4 is a power of three, however as long as a_n is a sum of powers of 3, then $3^n a_n + b_n$ is a sum of distinct powers of 3.

We stop at 27, because it is a power of three, but we would get a valid ratio for any a_n that is a sum of distinct powers of three. If we were to keep going past the fifth level, we would see that path actually continues to terminate at the vertex 0.

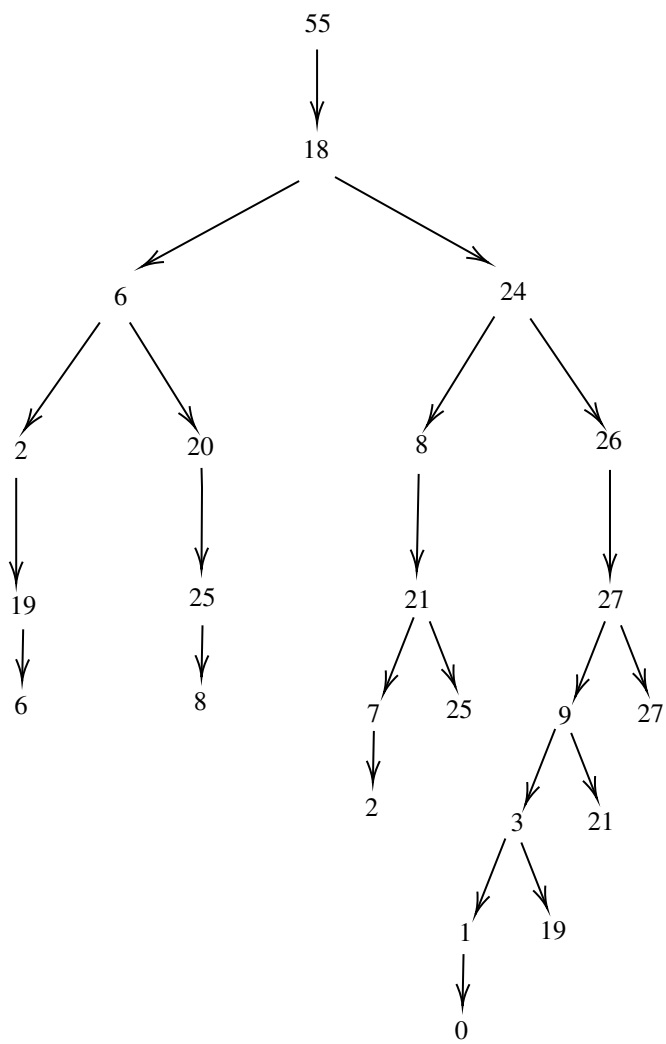
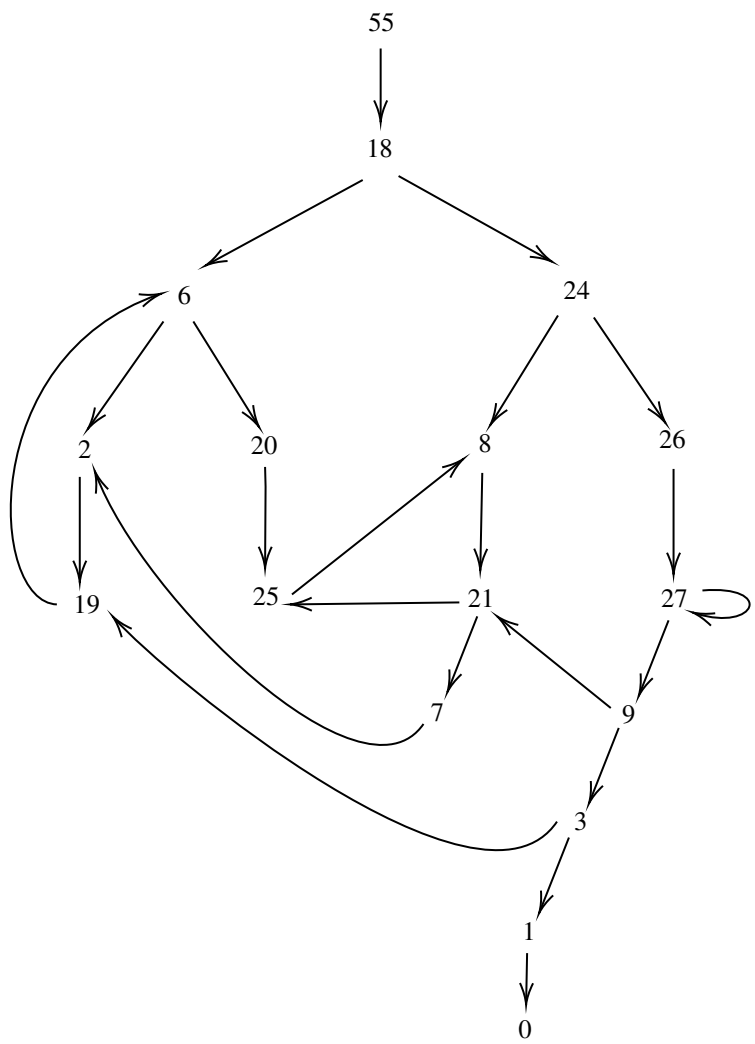


Figure 6.3: Tree created by completing all steps of the algorithm for $m = 55$ in base 3.

Diagramming and Gesture

We use **arrows** to illustrate the steps in the algorithm. Similar to the arrows in the transducers described in Chapter 5, these arrows represent movement through the algorithm as a path in a directed graph. The process of drawing the steps of the algorithm as levels in a directed graph (each level of the graph represents one step of the diagram) is an **exploration of the temporal and mobile dimensions** of the algorithm and of the graph. The graph can be read top-to-bottom, and that orientation represents the order of time in the algorithm.

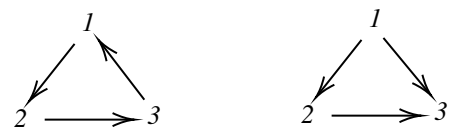
We notice that some numbers show up more than once. Here, I stopped each path once we reached a number that had already appeared. We can redraw the graph to have the arrows go 'back' to the same vertex, as shown below.



We now introduce another definition that will help us talk about these directed graphs that have edges which loop back to previous steps.

Definition 60. *A cycle is a path of edges that begins and ends with the same vertex in a directed graph.*

Example 61 (Nonexample)**.** *The directed graph on the left is a cycle. The graph on the right is not a cycle, because the orientation of the directed edges causes a problem. In a cycle, we can start at a given vertex and follow the arrows all the way around to get back to where we started.*



Suppose we look at the nodes in the directed graph we constructed above and take them $\pmod{55}$. The node 0 and the node 55 would be equivalent, and the path that ends at 0 actually becomes a cycle. This makes sense, because if we think about the edge from m as $m \to \frac{m-1}{3}$, and one of the edges from $0 \equiv 0 \pmod 3$ as $0 \to \frac{0+m-1}{3}$, we see that they end in the same vertex. The second edge from 0 would be $0 \to \frac{0}{3}$, or a loop from 0 to itself. We typically do not draw this loop because it does not impact the question we are trying to answer.

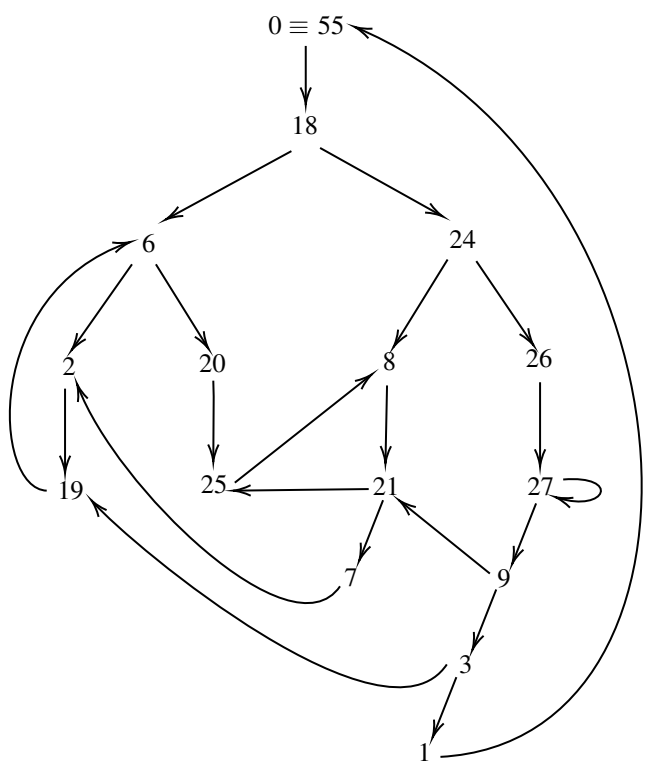


Figure 6.4: Directed graph for $m = 55$.

Diagramming and Gesture

By rearranging the edges out of the strict 'tree' formation of the original algorithm, we are changing **perspective** to highlight different aspects of the relationships between vertices. When we set $0 \equiv 55$ to be the same vertex in the diagram, we are aligning 'start' and 'end', which deemphasizes the linear progression of time in the algorithm to focus on 'returning'. Redrawing the diagrams in this way was important for me as I shifted from thinking about performing the algorithm to studying properties of the graphs themselves.

If we think about it this way, it is a subgraph of a particular directed graph on $\lceil \frac{m}{2} \rceil - 1$ vertices, labelled from 0 to $\lceil \frac{m}{2} \rceil - 1$ that can be constructed by the following procedure. For each vertex k, create an edge :

- from k to $\frac{k-1}{3}$ if $k \equiv 1 \pmod 3$
- from k to $\frac{k+m}{3}$ if $k \equiv 2 \pmod 3$
- from k to $\frac{k}{3}$ if $k \equiv 0 \pmod 3$
- from k to $\frac{k+m-1}{3}$ if $k \equiv 0 \pmod 3$.

This will give us a graph with $\lceil \frac{m}{2} \rceil - 1$ vertices, where each vertex that is not divisible by 3 ($\equiv 1 \pmod 3$ and $\equiv 2 \pmod 3$) has exactly one edge leaving it and each vertex that is divisible by three has two vertices leaving it. [1]

If we construct a smaller example ($m = 19$) following the original algorithm, we would do it in 3 steps, as shown on the left below. If we construct the larger directed graph using the procedure above, we would get the graph on the right.

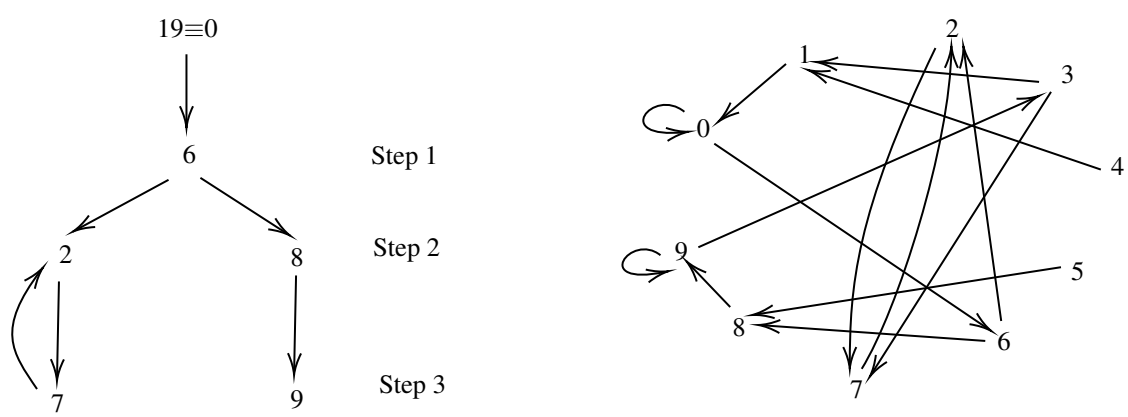


Figure 6.5: The tree created by the algorithm (left) is a subset of the larger directed graph (right) for $m = 19$.

[1] The graph that is constructed in this way is equivalent to the transducer $\mathcal{T}_{m,3}$ modified to remove edges that read and write the digit 2, as defined in Chapter 5.

While it seems like we will get the same information, it can make a difference in the speed of the computation, especially if we ask actual computers to help us draw the graphs.

Using the method on the left, the computer has to add vertices to the graph in each step, as well as check to make sure those vertices do not already exist elsewhere. For example, instead of adding a new 2 to receive the edge from 7, the computer would have to check all of the previous vertices (stored in order by step), to see if we already have a 2. When the graphs get quite large, this begins to be a very slow process. Additionally, the computer would need to either check if each new vertex is a sum of distinct powers of three, or keep going until it has exhausted all paths from all vertices.

In contrast, the method on the right needs to iterate over each vertex, creating the paths for each. The program begins with all possible vertices, and so the amount of time it takes to generate each graph is much more consistent. While it is true that the graph contains more information than we may need (e.g. this example includes the vertices 4 and 5, even though there is no path into them), it does not require the same amount of checking and storing of information, which can make the computation faster. Additionally, these graphs are interesting in their own right, and we will spend the remainder of this section using this second method of drawing the graphs.

I introduce both methods (which in many senses are equivalent), because the first method was the first method I learned, and shows clearly how we get the corresponding quotient from the algorithm. When I became more comfortable with that algorithm (after drawing lots and lots of pictures), it was much easier to see how to generalize it and how it connects to existing algorithms for base b multiplication (the transducers introduced in Chapter 5). The way I drew the pictures evolved based on what simplified the procedure for me, and then later by trying to figure out how to streamline the process for the computer (while still being able to store and read off all the information I wanted).

The directed graph constructed using the second procedure encodes all possible valid outcomes of the algorithm as cycles including the vertex labelled 0. For a cycle starting and ending at the vertex 0, reading the path starting at 0, we have a list of vertices $\{0 = v_0, v_1, \ldots, v_j = 0\}$. If we take 3^i to be a part of our sum for each i where $v_i < v_{i+1}$, we get a sum of distinct powers of 3. Let this sum of distinct powers of three be r_j. Then, $r_j m$ is also a sum of distinct powers of three, and so we have a representation of m as a quotient of sums of distinct powers of three. We note that if v_k is some sum of powers of 3, then there must be a sequence $v_k > v_{k+1} > \cdots > v_j = 0$.

We return now to our initial example graph, $m = 55$. I think about each edge $v_i \to v_{i+1}$ as being an

"up-step" if $v_i < v_{i+1}$ and a *"down-step"* if $v_i > v_{i+1}$. More specifically, the edges

$$k \to \frac{k+m-1}{3}, \quad k \to \frac{k+m}{3} \qquad \text{when } k \equiv 0 \text{ and } k \equiv 2 \pmod 3, \text{ respectively,}$$

are up-steps and the edges,

$$k \to \frac{k}{3}, \quad k \to \frac{k-1}{3} \qquad \text{when } k \equiv 0 \text{ and } k \equiv 1 \pmod 3, \text{ respectively,}$$

are down-steps. When I started drawing the pictures, I would attempt to color-code the down and up steps to better understand the patterns. In the image below, the up-steps are colored green.

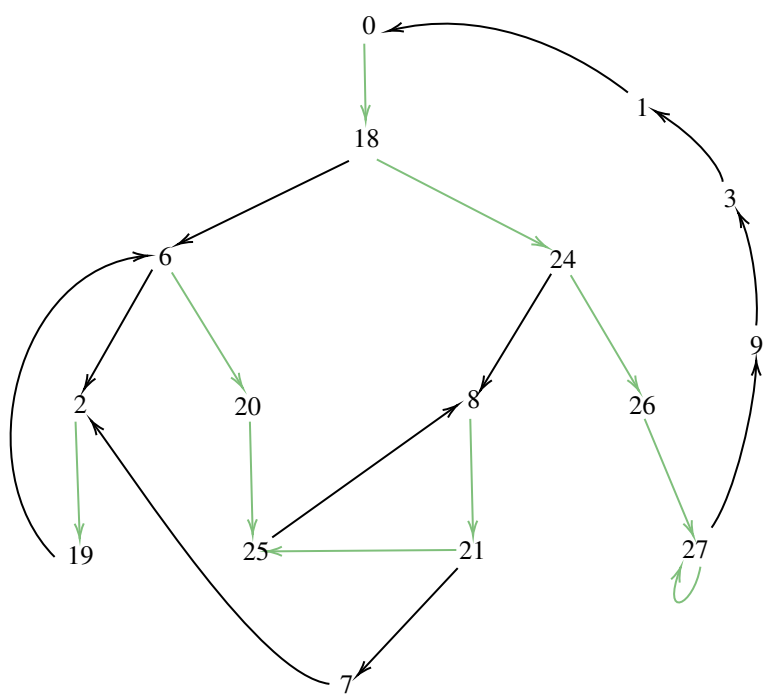


Figure 6.6: The directed graph for $m = 55$, color coded for "up-steps" and "down-steps."

If we look at the up-steps in our cycle, we see that the steps $0 \to 18 \to 24 \to 26 \to 27$ are the up-steps, and they are consecutive starting from the initial vertex $v_0 = 0$. So the highlighted portion of the graph would correspond to the value of $r_4 = 3^3 + 3^2 + 3^1 + 3^0$, which was the denominator that resulted from our original algorithm.

The green edges correspond to the digit 1 in the base 3 representation of the multiplier. Starting from 0,

each up-step contributes a 1 (moving from right to left). So, we have

$$55 = \frac{[10000111]_3}{[1111]_3},$$

following the path $0 \to 18 \to 24 \to 26 \to 27 \to 9 \to 3 \to 1 \to 0$. If we go around the loop at 27, we would get,

$$55 = \frac{[100000111]_3}{[11111]_3}.$$

The directed graphs described above perform the same computation as the modified multiplication transducers we discussed in the previous section (this particular example would be a subgraph of $\mathcal{T}_{55,3}$, and more specifically the modified transducer created by removing edges that read and write the digit 2 from $\mathcal{T}_{55,3}$, as well as states that have no in-edges).

Diagramming and Gesture

When I first started exploring these graphs, I noticed differences between "up-steps" and "down-steps". By using **color** to encode these differences in the diagram, it was easier to see these patterns, and compare them.

6.2.1 Observations about the Graphs

In these graphs, we include only vertices up to $\lceil \frac{m}{2} \rceil - 1$, since for every $k > \lceil \frac{m}{2} \rceil - 1$, all edges that map into k originate from a vertex $j > \lceil \frac{m}{2} \rceil - 1$. So there is no possible way to reach k through a path that starts at vertex 0, and as a result, it is impossible for any such k to be a part of a cycle in this directed graph.

Down-steps do not depend on m, and up-steps depend on m. A result of this is that down steps will be the same across the set of graphs. For example, if we look at our example $m = 19$ from before, and compare the down steps to the example $m = 22$, we see that the down steps are the same.

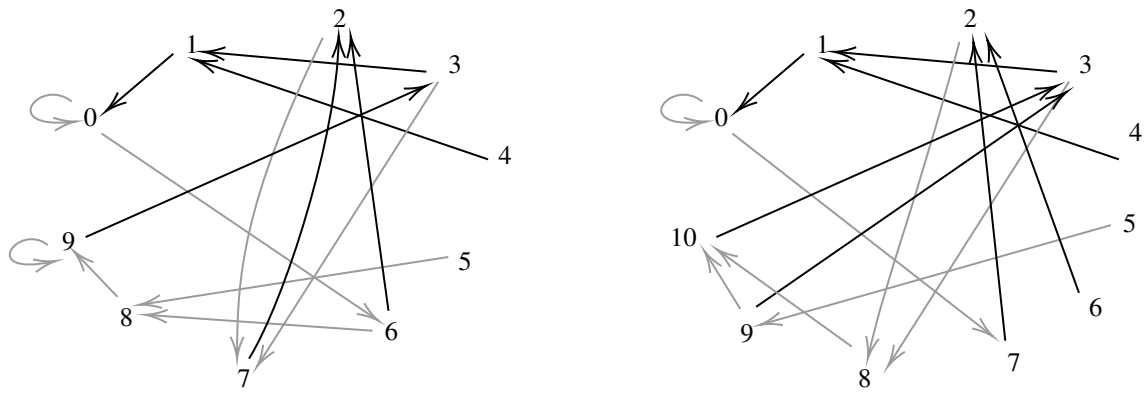


Figure 6.7: Down steps in the graphs for $m = 19$ and $m = 22$.

We can also notice that paths going to 0 must go through the down step $1 \to 0$, and we can follow that pattern to see that the paths of down steps in any graph generated by m that terminate at 0 are a subset of the infinite tree:

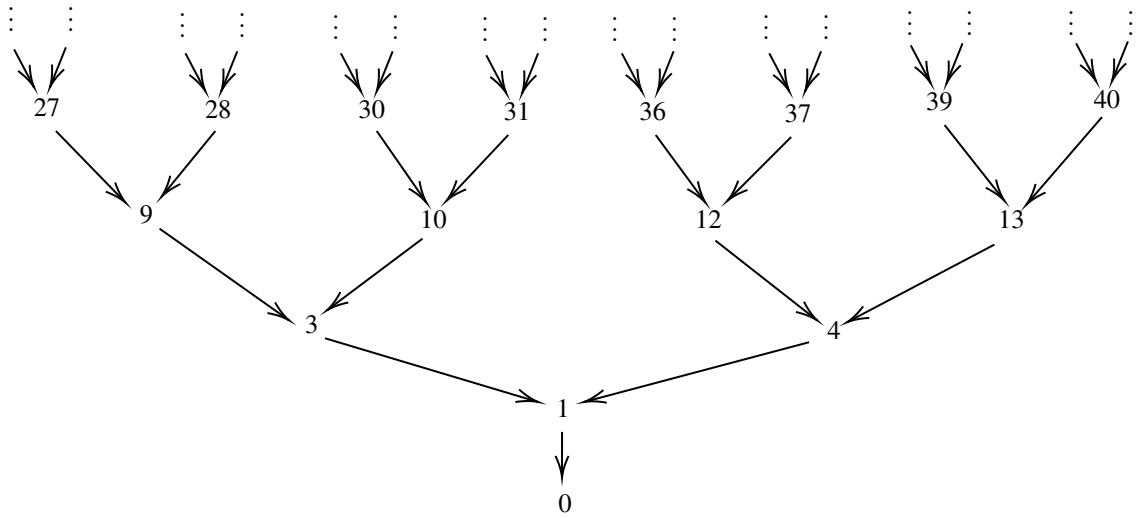


Figure 6.8: The tree of possible down-steps that lead to 0.

This infinite graph does not include all of the down steps, for example $7 \to 2$ and $6 \to 2$, but all cycles containing 0 must at some point join a path in this graph. We also notice that the vertices in this graph are exactly sums of distinct powers of three. They have a direct path to 0, and so this shows how the argument of having a cycle containing 0 in the graph generated by m is consistent with the original algorithm terminating at a sum of distinct powers of three. If the original algorithm terminates, then there is a path from 0 to one

of the vertices in this graph, and then the cycle is completed by taking down steps back to 0.

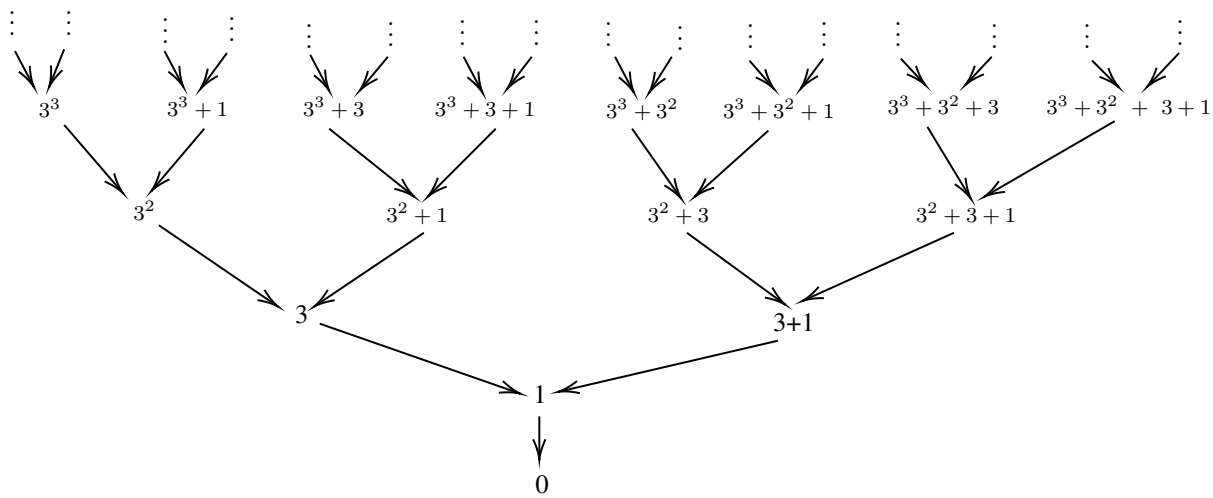


Figure 6.9: Sums of distinct powers of three in the tree of down-steps that lead to 0.

Diagramming and Gesture

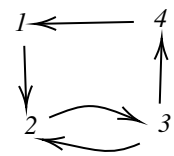
We shift **perspective** again, this time zooming out to think about the edges that are shared between graphs. Because the edges represented are "down-steps" I draw the edges pointing "down" to 0 instead of pointing up. Taking this perspective, we can see that a cycle containing 0 must include one of these sums of distinct powers of three. Instead of looking at a specific transducer, this graph shows some edges that different transducers have in common.

The diagram is focused on the **spatiomotoric information** that is in a section of the cycle that is the path to 0 that reads only 0's. By drawing this graph, we can think about which integers are in it, and which are not, then comparing those relationships to other relationships in other diagrams.

Looking at up-steps and down-steps can also tell us about relationships between intervals of vertices. The smallest number that is the result of an up-step is $\frac{m-1}{3}$, which is an up-step from 0. The largest number that is a result of a down-step is $\lceil \frac{m}{6} \rceil$ is the down step from the largest number in the graph that is $\equiv 1 \pmod 3$ or $\equiv 0 \pmod 3$. So, the vertices in $\left(\lceil \frac{m}{6} \rceil, \frac{m-1}{3}\right)$ have in-degree 0. In-degree is a term that refers to the number of edges entering a particular vertex.

Definition 62. *A k-cycle in a directed graph is a cycle containing exactly k different vertices.*

Example 63. *The directed graph on the left contains one two-cycle,* $2-3-2$ *and one four-cycle* $1-2-3-4-1$*. We note that doing a 2-cycle twice would not be considered a 4-cycle because there are not 4 different vertices.*



In the discussion below, we use the word "graph" to represent an element of the set of directed graphs as constructed above for a given $m \in A$.

Proposition 64. *Each graph has at least one one-cycle, and at most* 2 *one-cycles.*

Proof. In every graph, $0 \to 0$, since $0 \equiv 0 \pmod 3$ and $\frac{0}{3} = 0$. So, each graph contains a one-cycle at vertex 0. We now show that there exists at most one other one-cycle.

Consider the graph for some $m \equiv 1 \pmod 3$ in the appropriate set of intervals. If there is a nonzero vertex k that would map to itself, it would be in one of the following cases:

(a) $k \equiv 1 \pmod 3$ and $k = \frac{k-1}{3}$.

(b) $k \equiv 2 \pmod 3$ and $k = \frac{k+m}{3}$.

(c) $k \equiv 0 \pmod 3$ and either $k = \frac{k}{3}$ or $k = \frac{k+m-1}{3}$.

We approach each case individually.

(a) By solving for k we get $k = \frac{-1}{2}$. This is not a valid vertex, so this case is impossible.

(b) By solving for k in terms of m, we get $k = \frac{m}{2}$. However since we include only vertices up to $\lceil \frac{m}{2} \rceil - 1$, there is no such value of k in our graph.

(c) Since k is nonzero, $k \neq \frac{k}{3}$. In the second case, by solving for k in terms of m, we get $k = \frac{m-1}{2}$.

Thus, there is exactly one additional one-cycle at a vertex k if and only if there is an integer value $k = \frac{m-1}{2}$ when $k \equiv 0 \pmod 3$ (note, m must be odd). Thus, we can have at most two one-cycles in a graph generated by m. □

We note that the one-cycle $0 \to 0$ is of the form $k \to \frac{k}{3}$ and as such would be classified as a down-step, while if there is a second one-cycle, it is of the form $k \to \frac{k+m-1}{3}$ and would be classified as an up-step.

Example 65. *The graph for* $m = 19$ *has exactly two one-cycles, one at vertex* 0 *and one at vertex* $9 = \frac{19-1}{2}$.

Proposition 66. *Each graph can have at most one two-cycle.*

Proof. Any two-cycle could be seen oriented as

$$\text{a smaller number } \equiv 0, \text{ or } 2 \pmod 3 \quad \rightleftarrows \quad \text{a larger number } \equiv 0, \text{ or } 1 \pmod 3.$$

The uniqueness of two-cycles can be shown by cases. Suppose $k_1 \equiv 0 \pmod 3$ is the smaller number in some two-cycle in the graph associated with $m \equiv 1 \pmod 3$. Then, either

$$\begin{aligned} 3k_1 &= \tfrac{k_1+m-1}{3} \\ 9k_1 &= k_1 + m - 1 \\ 8k_1 + 1 &= m, \end{aligned} \qquad \text{or} \qquad \begin{aligned} 3k_1 + 1 &= \tfrac{k_1+m-1}{3} \\ 9k_1 + 3 &= k_1 + m - 1 \\ 8k_1 + 4 &= m. \end{aligned}$$

Suppose towards a contradiction that there were some other $k_2 \equiv 0 \pmod 3$, $k_1 \neq k_2$, that was the smaller number in some two-cycle. Then, with identical arithmetic to above, we have $8k_2 + 1 = m$ or $8k_2 + 4 = m$. Going through all of the combinations,

$$8k_1 + 1 = m \text{ and } 8k_2 + 1 = m \qquad 8k_1 + 1 = m \text{ and } 8k_2 + 4 = m$$

$$8k_1 + 4 = m \text{ and } 8k_2 + 4 = m \qquad 8k_1 + 4 = m \text{ and } 8k_2 + 1 = m$$

All four possibilities lead to a contradiction ($k_1 = k_2$ in the left two cases, $8k_1 = 8k_2 \pm 3$ in the right two cases). Thus, if there is some other two-cycle, the smaller number in that two-cycle is $k_2 \equiv 2 \pmod 3$. If this is true, then

$$\begin{aligned} 3k_2 &= \tfrac{k_2+m}{3} \\ 9k_2 &= k_2 + m \\ 8k_2 &= m, \end{aligned} \qquad \text{or} \qquad \begin{aligned} 3k_2 + 1 &= \tfrac{k_2+m}{3} \\ 9k_2 + 3 &= k_2 + m \\ 8k_2 + 3 &= m. \end{aligned}$$

Then, we have either, $8k_2 = m$ or $8k_2 + 3 = m$, and also $8k_1 + 1 = m$ or $8k_1 + 4 = m$. This leads to a contradiction, since m cannot be $\equiv 0 \pmod 8$ or $\equiv 3 \pmod 8$ while simultaneously being $\equiv 1 \pmod 8$ or $\equiv 2 \pmod 8$.

The remaining case is, if $k_1,\ k_2 \equiv 2 \pmod 3$ with $k_1 \neq k_2$ are the smaller numbers of two different two-cycles in the graph. This also leads to a contradiction, and so there is at most one two-cycle in any such graph. $\square$

Example 67. *The graph for $m = 19$ has exactly one 2-cycle, which comes from the case $k_1 = 2$. Then $8(k_1) + 3 = 19$, and the two cycle is $2 \to 7 \to 2$.*

Diagramming and Gesture

When I started to think about two-cycles, I realized that each two-cycle must contain two vertices, and so one must be larger than the other. My internal response was to think of a two-cycle as an 'up-step' and a 'down-step.' The **spontaneous covert gesture** associated to this realization is that I think about stepping up (like going up one step on a set of stairs), and then stepping down. Since my automatic reaction was to think of stepping 'up' first, I approached the proof by starting with the smaller number and looking at the step up to a larger number.

The following statement and proof are similar to the one above, but we see that with a larger cycle, there are more options to chase through when we are counting.

Proposition 68. *Each graph can have at most two three-cycles.*

Proof. Any given three-cycle must be either of form

$$A: \quad k \quad \nearrow \quad k_1 \quad \searrow \quad k_2 \quad \nearrow \quad k \qquad \text{or} \qquad B: \quad k \quad \searrow \quad k_1 \quad \nearrow \quad k_2 \quad \searrow \quad k,$$

where the arrows $\nearrow$ and $\searrow$ correspond to up steps and down steps, respectively. Note, the beginning and end of the cycle must be the same number. Given a cycle we can choose k to force the cycle into form A or B. There is no way to have either three up steps or three down steps in a cycle (because if we only step up or down we could not possibly get back to where we started), so we pick k to be the vertex in between the two up steps (form A) or the vertex between the two down steps (form B).

We know that we have two different possibilities for up-steps, $k \to \frac{k+m}{3}$ or $k \to \frac{k+m-1}{3}$, and two different possibilities for down steps: $k \to \frac{k}{3}$ or $k \to \frac{k-1}{3}$. We will look at the two cases separately.

Case A:

We break this case into two cases, based on whether the first up step is $k \to \frac{k+m}{3}$ or $k \to \frac{k+m-1}{3}$.

(i) Suppose the first up step is $k \to \frac{k+m}{3}$. Then, our second vertex is $k_1 = \frac{k+m}{3}$. There are two choices for the second step ($k_1 \to k_2$, which is a down step): $k_1 \to \frac{k_1}{3}$ or $k_1 \to \frac{k_1-1}{3}$. Then our third vertex is $k_2 = \frac{k_1}{3}$ or $k_2 = \frac{k_1-1}{3}$, and the last step is an up step, where there are again two choices for each k_2 (either $k_2 \to \frac{k_2+m}{3}$ or $k_2 \to \frac{k_2+m-1}{3}$). So putting this all together we have four options in case (i):

$$k \quad \to \quad \frac{k+m}{3} \quad \to \quad \frac{k+m}{3^2} \quad \to \quad \frac{\frac{k+m}{3^2}+m}{3}$$

$$k \quad \to \quad \frac{k+m}{3} \quad \to \quad \frac{k+m}{3^2} \quad \to \quad \frac{\frac{k+m}{3^2}+m-1}{3}$$

$$k \quad \to \quad \frac{k+m}{3} \quad \to \quad \frac{\frac{k+m}{3}-1}{3} \quad \to \quad \frac{\frac{\frac{k+m}{3}-1}{3}+m}{3}$$

$$k \quad \to \quad \frac{k+m}{3} \quad \to \quad \frac{\frac{k+m}{3}-1}{3} \quad \to \quad \frac{\frac{\frac{k+m}{3}-1}{3}+m-1}{3}$$

In each of these options, the ending state must be equal to k in order to form a cycle. So, in the first option, we have

$$k = \frac{\frac{k+m}{3^2}+m}{3} \quad \Rightarrow \quad 3k = \frac{k+m}{3^2}+m \quad \Rightarrow \quad 3k-m = \frac{k+m}{3^2}$$

$$\Rightarrow \quad 9(3k-m) = k+m \quad \Rightarrow \quad 27k-9m = k+m \quad \Rightarrow \quad k = \frac{10m}{26}.$$

If we do this for each of the four options we have, respectively,

$$k = \frac{10m}{26}, \qquad k = \frac{10m-9}{26}, \qquad k = \frac{10m-3}{26}, \quad \text{and} \quad k = \frac{10m-12}{26}.$$

Now, since we also need k to be an integer, the second and third options are ruled out, since the numerators $10m-9$ and $10m-3$ are both of the form even - odd, and thus are odd and cannot possibly be divisible by 26 (even). So, case (i) gives us two potential relationships between k and m: $k = \frac{10m}{26}$ and $k = \frac{10m-12}{26}$. This is possible when $m \equiv 0$ or $m \equiv 9 \pmod{13}$.

(ii) We now instead suppose the first up step is $k \to \frac{k+m-1}{3}$. Then, our second vertex is $k_1 = \frac{k+m-1}{3}$. There are two choices for the second step (which is a down step): $k_1 \to \frac{k_1}{3}$ or $k_1 \to \frac{k_1-1}{3}$. Then our third vertex is $k_2 = \frac{k_1}{3}$ or $k_2 = \frac{k_1-1}{3}$, and the last step is an up step, where there are again two choices for each k_2 (either $k_2 \to \frac{k_2+m}{3}$ or $k_2 \to \frac{k_2+m-1}{3}$). So putting this all together we have four options in case (ii):

$$k \quad \to \quad \frac{k+m-1}{3} \quad \to \quad \frac{k+m-1}{3^2} \quad \to \quad \frac{\frac{k+m-1}{3^2}+m}{3},$$

$$k \quad \to \quad \frac{k+m-1}{3} \quad \to \quad \frac{k+m-1}{3^2} \quad \to \quad \frac{\frac{k+m-1}{3^2}+m-1}{3},$$

$$k \quad \to \quad \frac{k+m-1}{3} \quad \to \quad \frac{\frac{k+m-1}{3}-1}{3} \quad \to \quad \frac{\frac{\frac{k+m-1}{3}-1}{3}+m}{3},$$

$$k \quad \to \quad \frac{k+m-1}{3} \quad \to \quad \frac{\frac{k+m-1}{3}-1}{3} \quad \to \quad \frac{\frac{\frac{k+m-1}{3}-1}{3}+m-1}{3}.$$

We then set each of the ending states equal to k for each of the four options, and have, respectively,

$$k = \frac{10m-1}{26}, \qquad k = \frac{10m-10}{26}, \qquad k = \frac{10m-4}{26}, \quad \text{and} \quad k = \frac{10m-13}{26}.$$

Since we also need k to be an integer, the first and fourth options are ruled out, since the numerators $10m-1$ and $10m-13$ are both of the form even - odd, and thus are odd and so cannot possibly be divisible by 26 (even). Thus case (ii) gives us two potential relationships between k and m: $k = \frac{10m-10}{26}$ and $k = \frac{10m-4}{26}$. This is possible when $m \equiv 1$ or $m \equiv 3 \pmod{13}$.

From Case A we have four possibilities for k in terms of m, and we note that at most one of these can be true at once (depending on the congruence class of $m \pmod{13}$). We note the possible congruence classes for $m \pmod{26}$, that we will reference later. In Case A, the possible congruence classes for m are:

$$0,\ 1,\ 3,\ 9,\ 13,\ 14,\ 16,\ 22 \pmod{26}.$$

Case B:

In this case, we begin with a down step, and break into cases based on whether the first down step is $k \to \frac{k}{3}$ or $k \to \frac{k-1}{3}$.

(i) Suppose the first down step is $k \to \frac{k}{3}$. Then our second vertex is $k_1 = \frac{k}{3}$. There are two choices for the second step (which is an up step): $k_1 \to \frac{k_1+m}{3}$ or $k_1 \to \frac{k_1+m-1}{3}$. Then our third vertex is either $k_2 = \frac{k_1+m}{3}$ or $k_2 = \frac{k_1+m-1}{3}$, and the last step is a down step, where there are two choices for each k_2. Putting this all together:

$$k \quad \to \quad \frac{k}{3} \quad \to \quad \frac{\frac{k}{3}+m}{3} \quad \to \quad \frac{\frac{k}{3}+m}{3^2}$$

$$k \quad \to \quad \frac{k}{3} \quad \to \quad \frac{\frac{k}{3}+m}{3} \quad \to \quad \frac{\frac{\frac{k}{3}+m}{3}-1}{3}$$

$$k \quad \to \quad \frac{k}{3} \quad \to \quad \frac{\frac{k}{3}+m-1}{3} \quad \to \quad \frac{\frac{k}{3}+m-1}{3^2}$$

$$k \quad \to \quad \frac{k}{3} \quad \to \quad \frac{\frac{k}{3}+m-1}{3} \quad \to \quad \frac{\frac{\frac{k}{3}+m-1}{3}-1}{3}$$

Setting the ending states equal to k and solving for k in terms of m we have, respectively,

$$k = \frac{3m}{26}, \qquad k = \frac{3m-9}{26}, \qquad k = \frac{3m-3}{26}, \quad \text{and} \quad k = \frac{3m-12}{26}.$$

(ii) Suppose the first down step is $k \to \frac{k-1}{3}$. Then our second vertex is $k_1 = \frac{k-1}{3}$. There are two choices for the second step (which is an up step): $k_1 \to \frac{k_1+m}{3}$ or $k_1 \to \frac{k_1+m-1}{3}$. Then our third vertex is either $k_2 = \frac{k_1+m}{3}$ or $k_2 = \frac{k_1+m-1}{3}$, and the last step is a down step, where there are two choices for each k_2. Putting this all together:

$$k \quad \to \quad \frac{k-1}{3} \quad \to \quad \frac{\frac{k-1}{3}+m}{3} \quad \to \quad \frac{\frac{k-1}{3}+m}{3^2}$$

$$k \quad \to \quad \frac{k-1}{3} \quad \to \quad \frac{\frac{k-1}{3}+m}{3} \quad \to \quad \frac{\frac{\frac{k-1}{3}+m}{3}-1}{3}$$

$$k \quad \to \quad \frac{k-1}{3} \quad \to \quad \frac{\frac{k-1}{3}+m-1}{3} \quad \to \quad \frac{\frac{k-1}{3}+m-1}{3^2}$$

$$k \quad \to \quad \frac{k-1}{3} \quad \to \quad \frac{\frac{k-1}{3}+m-1}{3} \quad \to \quad \frac{\frac{\frac{k-1}{3}+m-1}{3}-1}{3}$$

Setting the ending states equal to k and solving for k in terms of m we have, respectively,

$$k = \frac{3m-1}{26}, \qquad k = \frac{3m-10}{26}, \qquad k = \frac{3m-4}{26}, \quad \text{and} \quad k = \frac{3m-13}{26}.$$

From Case B we have eight possibilities for k in terms of m, and we note that at most one of these can be true at once (depending on the equivalence class of $3m \pmod{26}$).

Taken all together, the possible congruence classes for m are:

$$0,\ 1,\ 3,\ 4,\ 9,\ 10,\ 12,\ 13 \pmod{26}.$$

Thus, for any m, there is at most one k that satisfies Case A and at most one k that satisfies case B, so there are at most two three-cycles. There are exactly two three-cycles in the case where $m \equiv 0,\ 1,\ 3,\ 9$, or $13 \pmod{26}$. □

Example 69. *When $m = 55$, we are in the case $m \equiv 3 \pmod{26}$, and have two three-cycles. Corresponding to the cases in the proof above, we have a Case A three-cycle: $21 \to 25 \to 8 \to 21$ and a Case B three cycle $2 \to 19 \to 6 \to 2$. These can be seen in Figure 6.4.*

Diagramming and Gesture

When working on a large number of cases, like in the proof above, I use diagramming and gesture to help **organize information**. The explanation above has several options in each case of k. When I was working on this, I actually wrote them all out as part of one big tree (over several sheets of paper), with **arrows** branching for each of the different cases. I worked through the algebra for each case one at a time. When I came back to write it, I used that diagram to help guide how to split the cases in the proof.

We could continue to count the number of n-cycles in a given graph, but as n increases, the number of cases to consider would get large quickly. We instead consider other information we could get from looking at patterns in up-steps and down-steps.

Proposition 70. *The maximum possible number of consecutive down-steps in the graph generated by m is bounded by the smallest k with $\frac{m}{3^k} < 1$.*

We note that this is equivalent to saying the smallest k with $m < 3^k$, or $\log_3 m < k$, so we could calculate k as $k = \lfloor \log_3 m \rfloor$. The way it is stated above is the way I first thought of it, and makes it easier for me to see in the proof.

Proof. Let $m \equiv 1 \pmod 3$ be a positive integer in the appropriate union of intervals. Let k be the smallest k such that $\frac{m}{3^k} < 1$ (note, this also implies $3^{k-1} \leq m < 3^k$). Assume towards a contradiction that there is a path of $k+1$ down steps contained in the graph generated by m. Let n be the starting vertex of this path. The first step must take $n \to n_1$ where $n_1 = \frac{n}{3}$ or $n_1 = \frac{n-1}{3}$ (we can say $n_1 \in [\frac{n-1}{3}, \frac{n}{3}]$). Then, the next step would take $n_1 \to n_2$, where $n_2 = \frac{n_1}{3}$ or $n_2 = \frac{n_1-1}{3}$ (note, if we follow the options for n_1, this implies that $n_2 \in \left[\frac{\frac{n-1}{3}-1}{3}, \frac{n}{3^2}\right]$). Let $n_2^- = \frac{\frac{n-1}{3}-1}{3}$, the lower bound of the interval. We continue in this manner, letting $n_3^- = \frac{\frac{\frac{n-1}{3}-1}{3}-1}{3}$, and n_k^- represent k such iterations. Since by our assumption there are $k+1$ consecutive down steps, we have a path from n to n_{k+1}, where

$$n_{k+1} \text{ lies in the interval } \left[n_{k+1}^-, \frac{n}{3^{k+1}}\right]$$

However, we also know that n is a vertex in the graph, and so $n < m$. Thus, $\frac{n}{3^{k+1}} < \frac{m}{3^{k+1}} < \frac{m}{3^k} < 1$. This is a contradiction, because $\left[n_{k+1}^-, \frac{n}{3^{k+1}}\right] \subset (0,1)$, and so n_{k+1} cannot be a positive integer. Thus, a path of $k+1$ down steps cannot exist in the graph, and so the length of a path of consecutive down steps is bounded by the smallest k with $\frac{m}{3^k} < 1$. $\square$

Proposition 71. *The maximum possible number of consecutive up-steps is bounded by the smallest k with*

$$\frac{(m-1)\sum_{i=0}^{k} 3^i}{3^{k+1}} \geq \left\lceil \frac{m}{2} \right\rceil - 1.$$

Proof. Let $m \equiv 1 \pmod 3$ be a positive integer in the appropriate union of intervals. Let k be the smallest k such that $\frac{(m-1)\sum_{i=0}^{k} 3^i}{3^{k+1}} \geq \left\lceil \frac{m}{2} \right\rceil - 1$. Assume towards a contradiction that there is a path of $k+1$ up steps contained in the graph generated by m. Let n be the starting vertex of this path. The first step must take $n \to n_1$ where $n_1 = \frac{n+m}{3}$ or $n_1 = \frac{n+m-1}{3}$ (we can say $n_1 \in [\frac{n+m-1}{3}, \frac{n+m}{3}]$). Then, the next step would take $n_1 \to n_2$, where $n_2 = \frac{n_1+m}{3}$ or $n_2 = \frac{n_1+m-1}{3}$ (if we follow the options for n_1, this implies that $n_2 \in \left[\frac{\frac{n+m-1}{3}+m-1}{3}, \frac{\frac{n+m}{3}+m}{3}\right]$). Let $n_1^- = \frac{n+m-1}{3}$, $n_2^- = \frac{\frac{n+m-1}{3}+m-1}{3}$ and so on, and $n_1^+ = \frac{n+m}{3}$, $n_2^+ = \frac{\frac{n+m}{3}+m}{3}$ and so on. Then $n_{k+1} \in \left[n_{k+1}^-, n_{k+1}^+\right]$. We further note that, since n is the value of a vertex in the graph, $n \geq 0$, so

$$n_{k+1}^- = \frac{\frac{\frac{\frac{n+m-1}{3}+m-1}{3}+m-1}{3} \cdots}{3} \geq \frac{\frac{\frac{\frac{m-1}{3}+m-1}{3}+m-1}{3} \cdots}{3} = \frac{\frac{\frac{\frac{m-1}{3}+m-1}{3}+m-1}{3} \cdots}{3} = \frac{(m-1)\sum_{i=0}^{k} 3^i}{3^{k+1}},$$

where the '$\ldots$' represents the continuation to $k+1$ iterations.

To illustrate the arithmetic above, we show for $k = 2$.

$$n_3^- = \frac{\frac{\frac{n+m-1}{3}+m-1}{3} + m - 1}{3} \geq \frac{\frac{\frac{m-1}{3}+m-1}{3} + m - 1}{3} = \frac{\frac{m-1+3(m-1)}{3^2} + m - 1}{3}$$

$$= \frac{m - 1 + 3(m-1) + 3^2(m-1)}{3^3} = \frac{(m-1)\sum_{i=0}^{2} 3^i}{3^3}$$

Going back to general k, we have that $n_{k+1} \in [n_{k+1}^-, n_{k+1}^+]$, and since n_{k+1}^- is bounded below by $\frac{(m-1)\sum_{i=0}^{k} 3^i}{3^{k+1}}$, so is n_{k+1}. Thus, we have

$$n_{k+1} \geq \frac{(m-1)\sum_{i=0}^{k} 3^i}{3^{k+1}} \geq \left\lceil \frac{m}{2} \right\rceil - 1.$$

This is a contradiction, because all vertices in the graph will be less than $\left\lceil \frac{m}{2} \right\rceil - 1$ by definition. Thus, our assumption must have been false and there could not be a path of $k+1$ up steps contained in the graph generated by m. $\square$

6.3 Computational Results

Now that we have familiarity with the graphs, we can test individual numbers that belong to the set

$$\mathcal{A} = \left\{ m : m \equiv 1 \pmod 3, m \in \bigcup_{k=1}^{\infty} \left(\frac{2}{3} \cdot 3^k, \frac{3}{2} \cdot 3^k \right) \right\},$$

to see which are in $Q(3; \{0, 1\})$. Working by hand, it seems at first like every element of the first set is an element of the second set. However, when we get to larger numbers we find some counterexamples. The first integer that is in $\mathcal{A}$ that is not in $Q(3; \{0, 1\})$ is 529.

We can test this by drawing the graph for 529 (Figure 6.10), using the original algorithm, to discover that all nontrivial paths originating from the state 0 will not loop back to create a cycle containing 0. We say nontrivial because the 'trivial' one-cycle from 0 to itself would be present in every graph, but does not tell us anything about representations in $Q(3; \{0, 1\})$ [this is essentially saying $0 \cdot m = 0$ always, which is true, but $\frac{0}{0}$ would not be a valid representation as a quotient].

However, when testing each m value in $\mathcal{A}$, we do not draw all of the graphs by hand. Instead, we use a computer program to generate the graphs, and then test to see if there is a nontrivial cycle containing the vertex 0. I started doing this with computer programs in SAGE. Using SAGE computations that generate graphs for each such m, I have computationally checked that, up to 6200000, the only $m \in A$ with $m \notin Q(3; \{0, 1\})$ are:

$$529,\ 592,\ 601,\ 616,\ 5368,\ 50281,\ 4072741,\ 4074361,\ 4088941,\ \text{and}\ \ 4245688.$$

Of these, it is possible to draw easily by hand the first four using the original algorithm (the example 529 is below). The first six had been found by Bruce and Sakulbuth Ekvittayaniphon (another student of Bruce, one of my academic siblings who graduated in 2018). I had verified these, and then had a program running to keep testing larger numbers while I worked on other things. When the program was not returning any more counterexamples, we thought that the six (relatively small) integers might be the only ones.

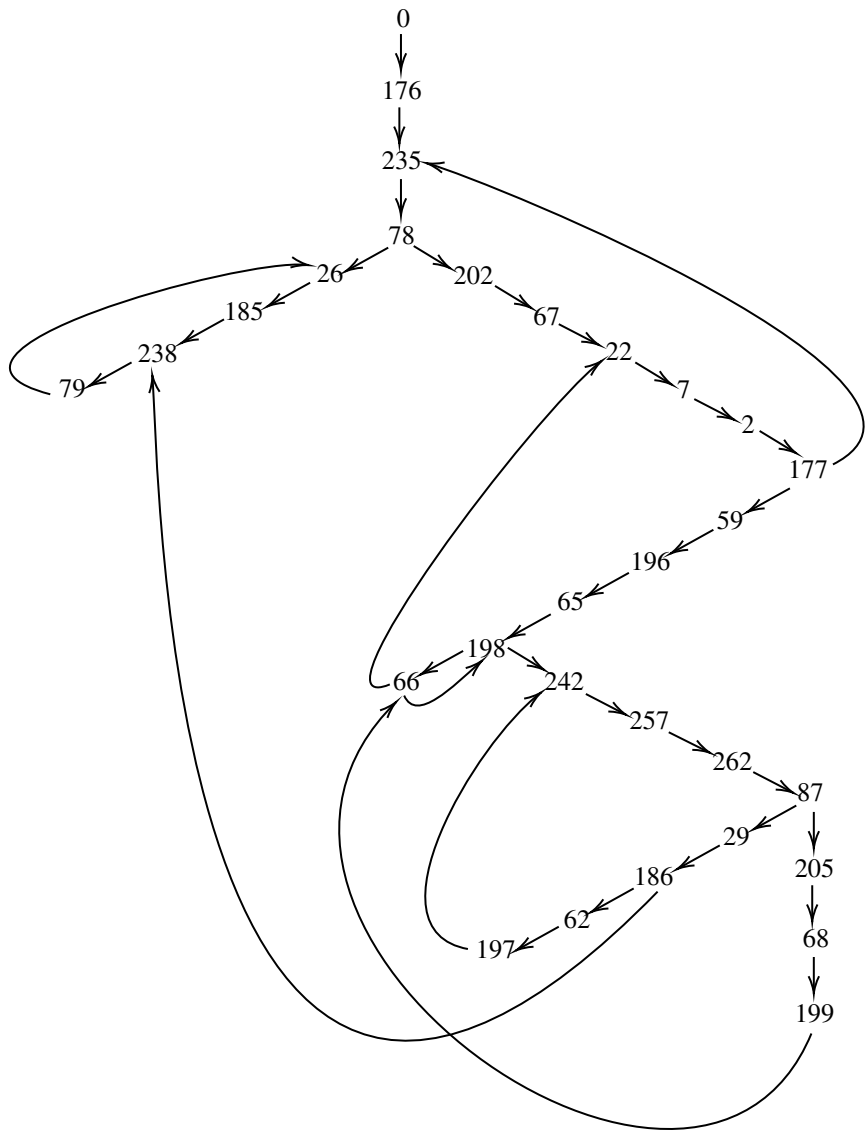


Figure 6.10: The directed graph for $m = 529$.

However, once the program returned four counterexamples in the four millions, this led us to think instead that there might be more as we tested larger numbers. Shortly after I found these next four counterexamples, I put them in an OEIS (Online Encyclopedia of Integer Sequences) search. The OEIS is an online wiki and

search engine that stores information about integer sequences. The information is crowd-sourced, much like Wikipedia, and mathematicians regularly add new sequences and comments about existing sequences as they discover them. Through a comment in the page associated to $Q(3;\{0,1\})$, we discovered that Jeffrey Shallit (a computer science professor at the University of Waterloo) had also worked on this problem [30]. Through communications with him, we connected with one of his students, Sajed Haque, who verified the results that I had, and reported that he found 37884151 to be the only other element of $\mathcal{A}$ not in $Q(3;\{0,1\})$ between 6200000 and 107883526 [17].

Since these integers are relatively few and far between, we were curious to see if there is something that makes them special. In particular, since our goal is to classify elements of $Q(3;\{0,1\})$, and we reduced that problem to determining which elements of $\mathcal{A}$ are in $Q(3;\{0,1\})$, it would be great to be able to have a statement along the lines of: **Elements of $\mathcal{A}$ are in $Q(3;\{0,1\})$ except the ones that are** *special*. Where *special* is something that we could describe formally: i.e. they have some property in common, or belong to a specific set.

I had looked for patterns in the integers that I have. One thing we tried was to see whether there was a pattern in the factorization of these special numbers.

$529 = 23^2$	$50281 = 7 \cdot 11 \cdot 653$
$592 = 2^4 \cdot 37$	$4072741 = 17 \cdot 107 \cdot 2239$
$601 = 601$	$4074361 = 31 \cdot 131431$
$616 = 2^3 \cdot 7 \cdot 11$	$4088941 = 4088941$
$5368 = 2^3 \cdot 11 \cdot 61$	$4245688 = 2^3 \cdot 530711$

I did not find a pattern this way. However Sajed was able to find a pattern by looking at the ternary representations of these integers.

Conjecture 72 (Haque)**.** *Ternary representations of the form* $211((2222)^k)021$ *(or* $211(2^{4k})021$*) are not represented in* $Q(3;\{0,1\})$*.*

The cases $0 \leq k \leq 2$, $\{601, 50281, 4074361\}$ that are in A but not in $Q(3;\{0,1\})$ appeared in both of our computations:

$$601 = [211021]_3 \qquad 50281 = [2112222021]_3 \qquad 4074361 = [21122222222021]_3$$

The fourth one, $330024841 = [211222222222222021]_3$ appeared in Sajed's computations, and I tested it independently after talking with him. The $k = 4$ case was too large of a computation for my machine to handle. I tried to prove Haque's conjecture by analyzing the structure of the transducers associated to these values, however they are so big, that only the first one is reasonable to look at the entire graph, and I was not able to find any patterns.

This conjecture implies that there would be infinitely many integers in the set $\mathcal{A}$ that cannot be written as quotients of sums of distinct powers of three.

Chapter 7

Quotients in Base 4

This chapter focuses on results towards classifying integer quotients from sets of integers with restricted digits in base four. We explore the sets $Q(4;\{0,2,3\})$, $Q(4;\{0,1,3\})$, and $Q(4;\{0,1,2\})$. In Section 7.2, we note connections to $Q(4;\{-1,0,1\})$ which is described by [25], and is the only prior appearance of quotients in base 4 in the literature.

7.1 Quotients in Base 4 with Three Digits

My work in base four came directly out of the work in base three. After looking at $Q(3;\{0,1\})$, I thought of that set as integers representable as 'quotients with one digit forbidden.' When I moved to looking at base four, I approached it from the same lens, thinking of which integers are quotients of integers that have one digit forbidden in base four. This section discusses the sets of integers $Q(4;\{d_1,d_2,d_3\})$, where $d_1 = 0$, $d_2, d_3 \in \{1,2,3\}$. In other words, the sets of integers that can be represented as quotients of integers whose base 4 representation have at most three digits (including 0). We will use the following shorthand notation, where Q_i is quotients with the digit i forbidden.

$Q_i := Q(4;\{\{0,1,2,3\}\backslash\{i\}\})$ = integers that can be represented as quotients of integers whose base four representations contain no i's. For each of $Q_1 = Q(4;\{0,2,3\})$, $Q_2 = Q(4;\{0,1,3\})$, and $Q_3 = Q(4;\{0,2,3\})$, I have conjectures for how to describe the set and have proven one direction of the containment.

Working with these sets, I wrote computer programs that draw transducers for integers m to test whether $m \in Q_i$ for each i. I used this data to formulate conjectures about each of the sets, and then attempted to understand why they might be true by exploring the transducers for specific integers. The proofs in this section include my original proofs using the transducers.

We use an argument about the structure of the transducers to show that if m is an odd power of 2 (e.g.

$2, 8, 32, \ldots$), then m cannot be an element of Q_1. We first sketch the proof using a picture, and then give the formal proof below.

Theorem 73. *Let* $Q_1 = Q(4; \{0, 2, 3\})$. $Q_1 \subseteq \mathbb{N} \setminus \{2^{2k+1} : k \in \mathbb{N}\}$.

Proof Sketch: In the modified transducer that multiplies by 2^{2k+1} in base 4, with digit 1 forbidden, every path that terminates at state 0 originates in the interval $[2^{2k-1}, 2^{2k})$. Since the states in this interval have in-degree 0 (the edges entering them would have read 1, which is forbidden), there can be no nontrivial cycle containing the state 0.

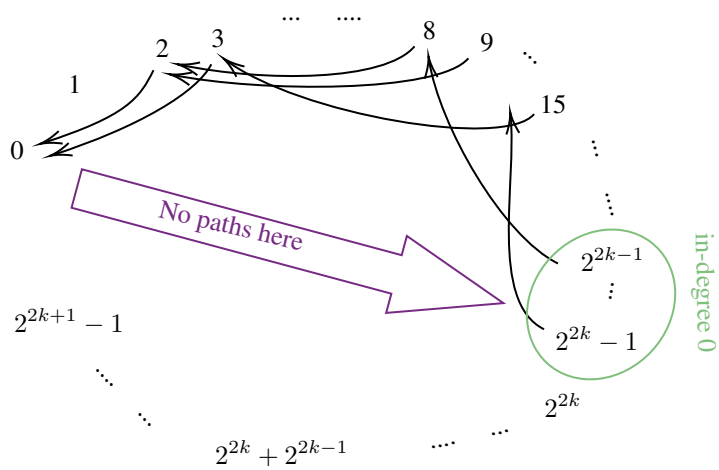


Figure 7.1: General sketch of the modified transducer for $m = 2^{2k+1}$ with digit 1 forbidden in base 4.

Proof. We will show that the integers 2^{2k+1}, where $k \in \mathbb{N}$, cannot be written as elements of $Q(4; \{0, 2, 3\})$. We first note that if $k \in \mathbb{N}$, $2^{2k+1} \equiv 0 \pmod 4$. The multiplication transducer for 2^{2k+1} in base 4 has exactly 2^{2k+1} states, labelled $\{0, \ldots, 2^{2k+1} - 1\}$.

In state ℓ, if the transducer reads an input of 1, the transducer moves to state $\left[\frac{2^{2k+1}+\ell}{4}\right]$. So, the paths that read an input of 1 map into the states in the interval $\left[\frac{2^{2k+1}}{4}, \frac{2 \cdot 2^{2k+1}-1}{4}\right] = [2^{2k-1}, 2^{2k})$. (We note that these would be the only paths entering these states, since by construction, reading an input of 0 in any state maps into the states $\left[0, \frac{2^{2k+1}-1}{4}\right]$ and reading an input of 2 in any state maps into the states $\left[\frac{2 \cdot 2^{2k+1}}{4}, \frac{3 \cdot 2^{2k+1}-1}{4}\right]$.) Since paths that read an input of 1 are forbidden in the modified transducer that restricts the digits to $\{0, 2, 3\}$, here are no paths that originate from the initial state and map into the states $[2^{2k-1}, 2^{2k})$.

We now show that all paths mapping in to the initial state must originate from one of the states in $[2^{2k-1}, 2^{2k})$. Since we have eliminated paths that read or write the digit one, we have that the states 0, 2, and 3 are the only ones that map into the initial state. The states that map into the interval $[2, 3]$ are the states j with $\left[\frac{j}{4}\right] = 2$ and $\left[\frac{j}{4}\right] = 3$, that is, the states in $[2^3, 2^4)$. Similarly, the states that map into the interval

$[2^3, 2^4)$ are the states $[2^5, 2^6)$, and in general, the states that map into the interval $[2^{2i-1}, 2^{2i})$ are the states $[2^{2i+1}, 2^{2(i+1)})$. Thus, all nontrivial paths to the inital state come through the states 2 and 3, and so must originate in the interval $[2^{2k-1}, 2^{2k})$.

Since these states have no incoming edges, there is no path from the intial state into these states. Thus, there can be no nontrivial path containing the initial state. It follows that the strongly connected component (set of states that all have paths to reach each other) containing the initial state contains only the initial state. So, $2^{2k+1} \notin Q_1$ for any $k \in \mathbb{N}$. $\square$

Theorem 73 is also a special case more general Theorem 105 in the following chapter.

I conjecture that these are the only m that are not in Q_1. I have tested this conjecture up to $m = 200000$ using a SAGE program that draws the transducer $\mathcal{T}_{m,4}$ for an integer m and deletes the edges labeled 1 before searching for a cycle.

Conjecture 74. $Q_1 = \mathbb{N} \setminus \{2^{2k+1} : k \in \mathbb{N}\}$.

In order to prove this conjecture, we would need to show that $Q_1 \subseteq \mathbb{N} \setminus \{2^{2k+1} : k \in \mathbb{N}\}$ and $Q_1 \supseteq \mathbb{N} \setminus \{2^{2k+1} : k \in \mathbb{N}\}$. We have shown the former, but the latter presents some challenges. It can be much easier to show that something specific (i.e. odd powers of 2) has a certain behavior, than to show that everything else would have a cycle. I have tried many different approaches, working with the graphs and with the numbers themselves, but have not yet been able to prove it. We can show that different families are definitely representable, for example, integers with representations using only the digits 0 and 1 are representable by multiplying by 2 in the following way:

$$17 = [101]_4 = \frac{[101]_4}{[1]_4} = \frac{[202]_4}{[2]_4}.$$

However, working with specific patterns and restrictions in the digits of m is not going to help us show that everything except odd powers of two is representable, unless we could exactly partition the set $\mathbb{N} \setminus \{2^{2k+1} : k \in \mathbb{N}\}$ into groups, and then show that each subset is representable. I have tried this by partitioning into equivalence classes $\pmod 4$, and looking for patterns in the transducers, but I have not been successful at proving the theorem. We will discuss more about this process in the next section.

After looking at the set Q_1, I moved on to the set Q_2, integers that have representations as quotients in base four with the digit 2 forbidden.

We show below that the numbers $m = 2^{2j+1}(2\ell - 1)$ (numbers m that have an odd power of 2 as a factor)

are not elements of Q_2. This result is related to the argument presented in [25] that the integers $k \equiv \pm 4^a \pmod{4^b}$ for $a < b$ are representable in base four using just the digits 0, 1, or -1. We include more about this result in Section 7.2. In fact, Theorem 75 is equivalent to $Q(4; \{0, 1, 3\}) \subseteq Q(4; \{-1, 0, 1\})$, as described in [25].

We provide two proofs of Theorem 75 below. The first uses the structure of the transducers, and the second uses more conventional techniques from number theory.

Theorem 75. *Let* $Q_2 = Q(4; \{0, 1, 3\})$. $Q_2 \subseteq \mathbb{N} \setminus \{2^{2j+1}(2\ell - 1) : \ell, j \in \mathbb{N}\}$.

Proof. We first show that $Q_2 \subseteq \mathbb{N} \setminus \{x = 2(2\ell - 1) : \ell \in \mathbb{N}\}$, in other words that $m \equiv 2 \pmod 4$ is not representable. Let $m \equiv 2 \pmod 4$. Then, $m = 4\ell + 2$. Consider the first step in the transducer originating from the state 0. Since our allowed digits are $\{0, 1, 3\}$, we look at each of these in order to show that there are no productive steps starting from 0.

- If we read a 0, we write a 0. This is the trivial step and in order to have a representation we must be able to create a nontrivial cycle, thus taking some nontrivial step.
- If we read a 1, we would write a two and move to state ℓ. However, we cannot have a 2 in our numerator, so this is not a permitted step.
- If we read 3 we would write two and move to state $3\ell+1$ (because $0+3(4\ell+2) = 12\ell+6 = 4(3\ell+1)+2$). However, we cannot have a 2 in our numerator, so this is not a permitted step.

Since we also would never write a 2, there is no possible initial step that will lead us into a nontrivial cycle containing 0. Thus $m \notin Q_2$.

We now show that if $2(2\ell - 1)$ is not representable, then $2^{2k+1}(2\ell - 1)$ is not representable for any $k \in \mathbb{N}$. Suppose $m = 2(2\ell - 1)$ is not representable and $2^{2k+1}(2\ell - 1)$ is. Then, $2^{2k+1}(2\ell - 1) = \frac{p}{q}$ for some p and q whose base 4 representations use only the digits $\{0, 1, 3\}$ (and $q \neq 0$). Then, we have $2^{2k}m = \frac{p}{q}$, and it follows that $m = \frac{p}{4^k q}$. However, $4^k q$ would also only use the digits $\{0, 1, 3\}$ (since multiplying by 4^k is equivalent to appending k 0's to the right of q). This contradicts the statement that m is not representable.

Thus, since every $2(2\ell - 1)$ is not representable, and this implies that $2^{2k+1}(2\ell - 1)$ is not representable, we have proved the statement.

□

My initial interaction with this result came through my understanding of the transducers, and the proof above was my original proof. The proof below is due to an observation by Bruce Reznick, and makes use of

the following definition.

Definition 76. *For any prime p, the function $v_p(n)$ (sometimes called a discrete logarithm) is the power of p that divides n.*

Example 77. *For the prime $p = 2$, $v_2(8) = 3$, because $8 = 2^3$. We also have $v_2(6) = 1$, since $6 = 3^1 2^1$. We note that $v_2(48) = v_2(8 \cdot 6) = v_2(8) + v_2(6) = 3 + 1 = 4$, echoing properties of logarithms that may be familiar. Likewise, $v_2(6) = v_2(48) - v_2(8)$.*

We will use the property illustrated above, that $v_p(r) = v_p(s) - v_p(t)$ for any $r = \frac{s}{t}$ in the proof below.

Proof. We first show that every element n of $S(4; \{0, 1, 3\})$ has $v_2(n)$ an even integer. Suppose n is in $S(4; \{0, 1, 3\})$. Then, $n = \sum_{j=0}^{\infty} a_j 4^j$, where the $a_j \in \{0, 1, 3\}$. Take k to be the smallest index for which $a_j \neq 0$. Then

$$n = a_k 4^k + a_{k+1} 4^{k+1} + ... = 4^k (a_k + 4a_{k+1} + ...),$$

and since a_k is in $\{1, 3\}$, it follows that $v_2(n) = 2k$ is even.

We now show by contradiction that if $m = 2^{2i+1}(2\ell - 1)$, then $m \notin Q(4; \{0, 1, 3\})$. Suppose there is some $m = 2^{2i+1}(2\ell - 1)$ and $m \in Q(4; \{0, 1, 3\})$. Then $m = \frac{s}{t}$ for some s, $t \in S(4; \{0, 1, 3\})$. By the construction above, $v_2(s) = 2k_1$ and $v_2(t) = 2k_2$ for some natural numbers k_1 and k_2. We know that $v_2(m) = v_2(s) - v_2(t)$, so $v_2(m)$ is a difference of two even numbers, and so must be even. But, since $m = 2^{2i+1}(2\ell - 1)$, then $v_2(m) = 2i + 1$ is odd. Since $v_2(m)$ cannot be both odd and even, we have a contradiction and every $m = 2^{2i+1}(2\ell - 1)$ for some i, ℓ is not in $Q(4; \{0, 1, 3\})$. □

I have verified the following conjecture computationally up to $n = 35000$ using a SAGE program that generates $\mathcal{T}_{m,4}$ and deletes the edges that have digit 2 as an input or output, for each m.

Conjecture 78. $Q(4; \{0, 1, 3\}) = \mathbb{N} \setminus \{2^{2j+1}(2\ell - 1) : \ell, j \in \mathbb{N}\}$

The right side of the equation has been proven to be the set $Q(4; \{-1, 0, 1\})$ by Loxton and van der Poorten in [25], and so Conjecture 78 is equivalent to $Q(4 : \{0, 1, 3\}) = Q(4; \{-1, 0, 1\})$.

On the surface, this conjecture may not seem that surprising since $3 \equiv -1 \pmod 4$. However, when I looked into proving these statements, my first instinct was to try to compare the transducers. Though it appears that the sets are the same, the transducers are not in any way isomorphic. We can see this in the example for $m = 7$ below. The integer 7 is an element of both $Q(4; \{0, 1, 3\})$ and $Q(4; \{-1, 0, 1\})$, which can be shown

via the cycles in transducers below (left is the transducer constructed using digits $\{0, 1, 3\}$, and on the right digits $\{-1, 0, 1\}$).

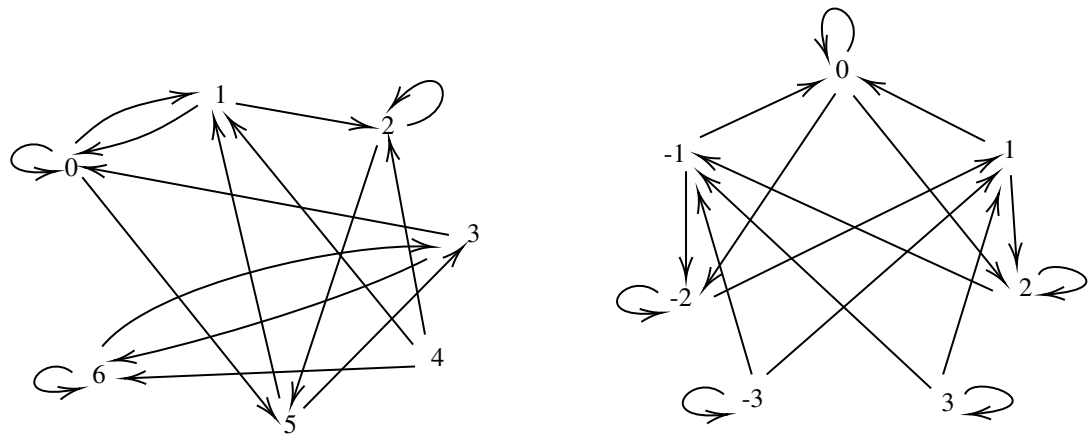


Figure 7.2: Transducers that multiply by 7 in base 4 with digit 2 forbidden using the digits $\{0, 1, 3\}$ (left) and $\{-1, 0, 1\}$ (right).

We provide more detail about the construction of graphs using negative digits and their properties in Appendix A.

Originally, my thought process was that, if these sets are equal, there might be some relationship between the representation of an integer m in $Q(4; \{0, 1, 3\})$ and its representation in $Q(4; \{-1, 0, 1\})$. I had hoped to formally describe this relationship, and develop a system (bijection) to show it explicitly for any given m. However, we have not been able to find such a relationship, which itself is an interesting result. This remains an open problem.

Finally, I went to look at Q_3, the set of integers who have representations as quotients in base 4 with digit 3 forbidden. The following conjecture has been verified computationally up to $m = 35000$.

Conjecture 79. $Q_3 = Q(4; \{0, 1, 2\}) = \mathbb{N}$.

For Q_1 and Q_2, I was able to prove the $\subseteq$ direction. Here, the $\subseteq$ direction is trivial. This means that it follows directly from the definition - we know that elements of Q_3 are already natural numbers and so there is nothing to prove. We know how to test a natural number and get its representation as a quotient using the transducers. However, this does not guarantee that every natural number you could think of has such a representation (we would need to test it to check).

7.1.1 Observations about the Graphs.

For $Q(4;\{0,2,3\})$. Look $\pmod 4$. We will take four examples to represent the four equivalence classes $\pmod 4$, and compare what we notice for each set Q_i. We will employ the strategy I use to draw the graphs efficiently, keeping track of the start and end states (carries) and input/output digits, I use shorthand notation

$$\text{start carry} \xrightarrow{\text{input digit, output digit}} \text{end carry.}$$

We will do this for each of the possibilities for the m values in $\{8,9,10,11\}$, to have the values for the full multiplication transducer that multiplies by m in base 4, and talk through the modified transducers related to each Q_i for each equivalence class of $m \pmod 4$.

Example 80. *To multiply* 9 *by* $m = 8$ *in base* 4*, we would read the base* 4 *representation of* $9 = [21]_4$ *from right to left. We would start at state* 0*, read a '1', then a '2', and then as many '0's as we need to get back to state* 0*. This path would look like*

$$0 \xrightarrow{1,0} 2 \xrightarrow{2,2} 4 \xrightarrow{0,0} 1 \xrightarrow{0,1} 0.$$

Since we also write the product from right to left, we would end up writing $[1020]_4 = 72 = 9 \cdot 8$*, and we would have moved from state 0 to state 2 to state 4 to state 1 to state 0. Because neither the input nor output includes the digit* 3 *we can see this appear as a cycle in the third graph of Figure 7.1.1.*

$\mathbf{m = 8}$:

$$0 \begin{array}{l} \xrightarrow{0,0} 0 \\ \xrightarrow{1,0} 2 \\ \xrightarrow{2,0} 4 \\ \xrightarrow{3,0} 6 \end{array} \qquad 1 \begin{array}{l} \xrightarrow{0,1} 0 \\ \xrightarrow{1,1} 2 \\ \xrightarrow{2,1} 4 \\ \xrightarrow{3,1} 6 \end{array} \qquad 2 \begin{array}{l} \xrightarrow{0,2} 0 \\ \xrightarrow{1,2} 2 \\ \xrightarrow{2,2} 4 \\ \xrightarrow{3,2} 6 \end{array} \qquad 3 \begin{array}{l} \xrightarrow{0,3} 0 \\ \xrightarrow{1,3} 2 \\ \xrightarrow{2,3} 4 \\ \xrightarrow{3,3} 6 \end{array}$$

$$4 \begin{array}{l} \xrightarrow{0,0} 1 \\ \xrightarrow{1,0} 3 \\ \xrightarrow{2,0} 5 \\ \xrightarrow{3,0} 7 \end{array} \qquad 5 \begin{array}{l} \xrightarrow{0,1} 1 \\ \xrightarrow{1,1} 3 \\ \xrightarrow{2,1} 5 \\ \xrightarrow{3,1} 7 \end{array} \qquad 6 \begin{array}{l} \xrightarrow{0,2} 1 \\ \xrightarrow{1,2} 3 \\ \xrightarrow{2,2} 5 \\ \xrightarrow{3,2} 7 \end{array} \qquad 7 \begin{array}{l} \xrightarrow{0,3} 1 \\ \xrightarrow{1,3} 3 \\ \xrightarrow{2,3} 5 \\ \xrightarrow{3,3} 7 \end{array}$$

In the case $m \equiv 0 \pmod 4$, we see that every edge leaving a given state writes the same digit. A consequence of this is that these graphs have many more *sinks*, or states that have edges coming in and no edges coming

out. This particular example is an odd power of two ($8 = 2^3$), so the graphs below help illustrate that $8 \notin Q_1$ or Q_2. Figure 7.3 shows the modified multiplication transducers that omit edges reading and writing 1s (top left), 2s (top right), and 3s (bottom center).

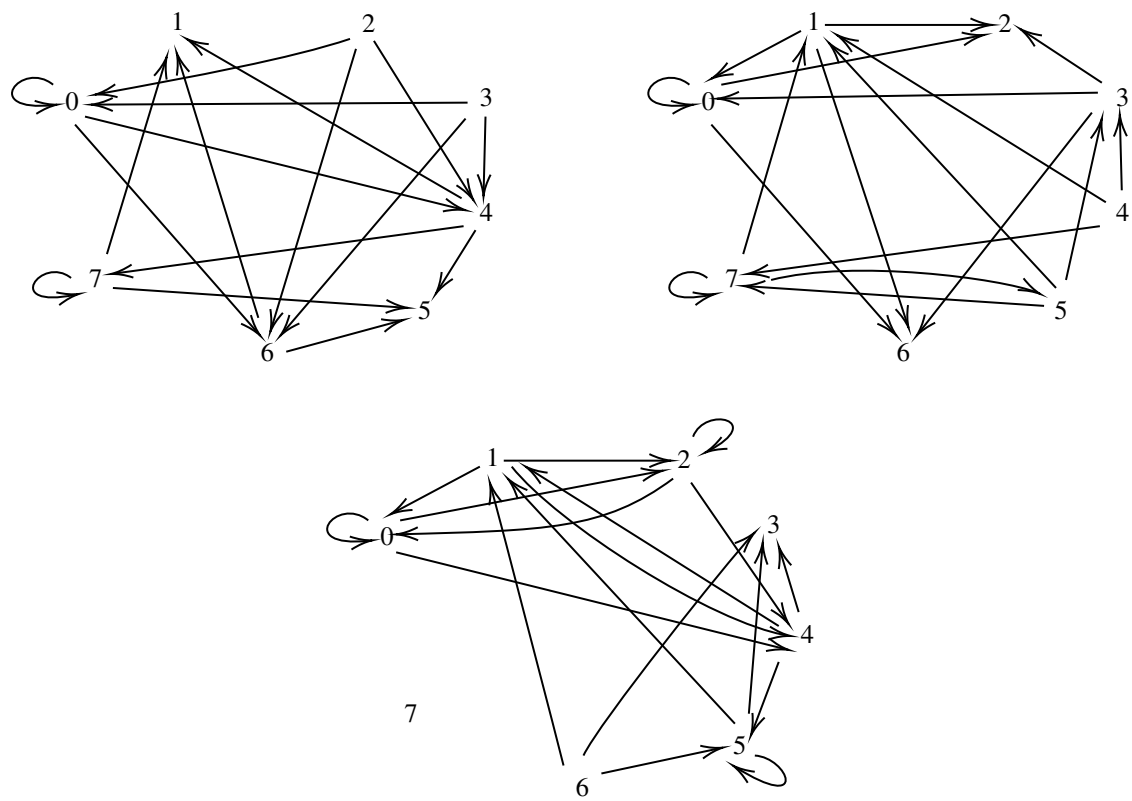


Figure 7.3: Modified transducers for $m = 8$ in base 4 that forbid digit 1, digit 2, and digit 3.

One thing we can recall about these transducers is that, when we forbid the digit i, the states in the interval $\left[\lceil \frac{im}{4} \rceil, \lfloor \frac{(i+1)m-1}{4} \rfloor\right]$ will have in-degree 0 (no edges entering them). This is true no matter what equivalence class m is in $\pmod 4$. In the example above, the states with in-degree 0 are those in $\left[\lceil \frac{8i}{4} \rceil, \lfloor \frac{8(i+1)-1}{4} \rfloor\right]$, or $[2, 3]$, $[4]$, and $[6, 7]$, for $i \in \{1, 2, 3\}$. Another thing we might notice is that, since every edge leaving a given state writes the same digit, the edges leaving states $\{k, k+1, k+2, k+3\}$ that read the same digit all end up in the same state (for any $k \equiv 0 \pmod 4$). An example is that edges leaving the states $\{0, 1, 2, 3\}$ in the original (unmodified) transducer that read the digit 0 end in state 0, those starting in this set that read the digit 1, end in state 2, etc.. This is not true for other equivalence classes of $m \pmod 4$, which we will see below.

We now move to looking at $m \equiv 1 \pmod 4$, through the example of $m = 9$.

m = 9:

$$0 \begin{array}{ll} \xrightarrow{0,0} & 0 \\ \xrightarrow{1,1} & 2 \\ \xrightarrow{2,2} & 4 \\ \xrightarrow{3,3} & 6 \end{array} \qquad 1 \begin{array}{ll} \xrightarrow{0,1} & 0 \\ \xrightarrow{1,2} & 2 \\ \xrightarrow{2,3} & 4 \\ \xrightarrow{3,0} & 7 \end{array} \qquad 2 \begin{array}{ll} \xrightarrow{0,2} & 0 \\ \xrightarrow{1,3} & 2 \\ \xrightarrow{2,0} & 5 \\ \xrightarrow{3,1} & 7 \end{array} \qquad 3 \begin{array}{ll} \xrightarrow{0,3} & 0 \\ \xrightarrow{1,0} & 3 \\ \xrightarrow{2,1} & 5 \\ \xrightarrow{3,2} & 7 \end{array} \qquad 4 \begin{array}{ll} \xrightarrow{0,0} & 1 \\ \xrightarrow{1,1} & 3 \\ \xrightarrow{2,2} & 5 \\ \xrightarrow{3,3} & 7 \end{array}$$

$$5 \begin{array}{ll} \xrightarrow{0,1} & 1 \\ \xrightarrow{1,2} & 3 \\ \xrightarrow{2,3} & 5 \\ \xrightarrow{3,0} & 8 \end{array} \qquad 6 \begin{array}{ll} \xrightarrow{0,2} & 1 \\ \xrightarrow{1,3} & 3 \\ \xrightarrow{2,0} & 6 \\ \xrightarrow{3,1} & 8 \end{array} \qquad 7 \begin{array}{ll} \xrightarrow{0,3} & 1 \\ \xrightarrow{1,0} & 4 \\ \xrightarrow{2,1} & 6 \\ \xrightarrow{3,2} & 8 \end{array} \qquad 8 \begin{array}{ll} \xrightarrow{0,0} & 2 \\ \xrightarrow{1,1} & 4 \\ \xrightarrow{2,2} & 6 \\ \xrightarrow{3,3} & 8 \end{array}$$

Here, we can notice that each state has exactly one edge leaving, and one edge entering corresponding to each digit. A result of this is that, once we have eliminated edges that read/write the digit i, each state will have at least two edges leaving it. There will still be states that have in-degree 0 as described above.

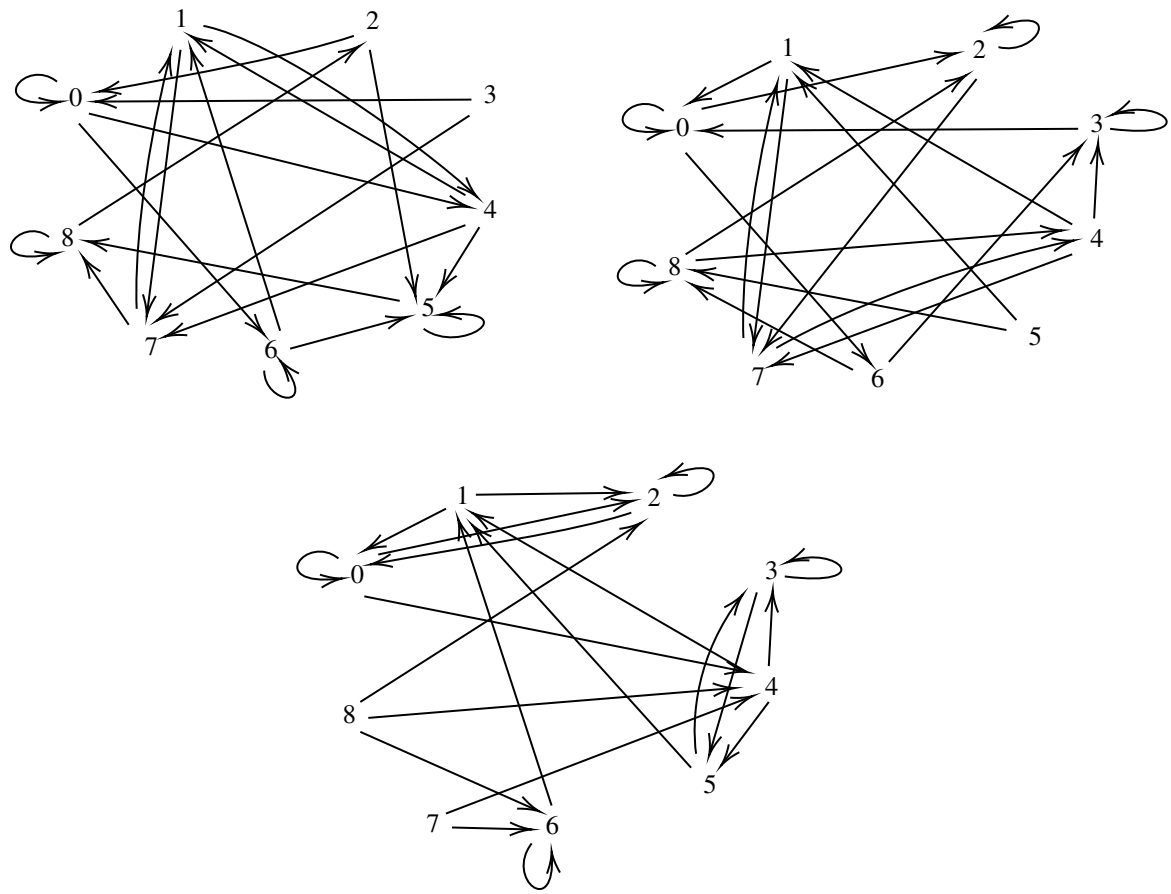


Figure 7.4: Modified transducers for $m = 9$ in base 4 that forbid digit 1, digit 2, and digit 3.

Since every state has at least two edges leaving it, and the interval $\left[\lceil \frac{im}{4} \rceil, \lfloor \frac{(i+1)m-1}{4} \rfloor\right]$ is shorter (since both m and i are not multiples of 4), most states have edges both entering and leaving. We note that $m = 9$ is in each of Q_1, Q_2, and Q_3.

We proceed to draw the modified transducers for $m = 10$.

m = 10:

	$\xrightarrow{0,0}$	0		$\xrightarrow{0,1}$	0		$\xrightarrow{0,2}$	0		$\xrightarrow{0,3}$	0		$\xrightarrow{0,0}$	1
0	$\xrightarrow{1,2}$	2	1	$\xrightarrow{1,3}$	2	2	$\xrightarrow{1,0}$	3	3	$\xrightarrow{1,1}$	3	4	$\xrightarrow{1,2}$	3
	$\xrightarrow{2,0}$	5		$\xrightarrow{2,1}$	5		$\xrightarrow{2,2}$	5		$\xrightarrow{2,3}$	5		$\xrightarrow{2,0}$	6
	$\xrightarrow{3,2}$	7		$\xrightarrow{3,3}$	7		$\xrightarrow{3,0}$	8		$\xrightarrow{3,1}$	8		$\xrightarrow{3,2}$	8

	$\xrightarrow{0,1}$	1		$\xrightarrow{0,2}$	1		$\xrightarrow{0,3}$	1		$\xrightarrow{0,0}$	2		$\xrightarrow{0,1}$	2
5	$\xrightarrow{1,3}$	3	6	$\xrightarrow{1,0}$	4	7	$\xrightarrow{1,1}$	4	8	$\xrightarrow{1,2}$	4	9	$\xrightarrow{1,3}$	4
	$\xrightarrow{2,1}$	6		$\xrightarrow{2,2}$	6		$\xrightarrow{2,3}$	6		$\xrightarrow{2,0}$	7		$\xrightarrow{2,1}$	7
	$\xrightarrow{3,3}$	8		$\xrightarrow{3,0}$	9		$\xrightarrow{3,1}$	9		$\xrightarrow{3,2}$	9		$\xrightarrow{3,3}$	9

For each state, there are only two possible digits that can be written while leaving this state. This causes problems, particularly when forbidding the digit 2, which stops us right at the start (this was the main idea of our proof that $Q_2 \subseteq \mathbb{N} \setminus \{2^{2j+1}(2\ell-1) : \ell, j \in \mathbb{N}\}$. Indeed, $10 = 2^1(5)$ falls into the set of integers that must not be in Q_2. For the other forbidden digits (no 1s or no 3s), we can see that we do not have any issues taking the first step.

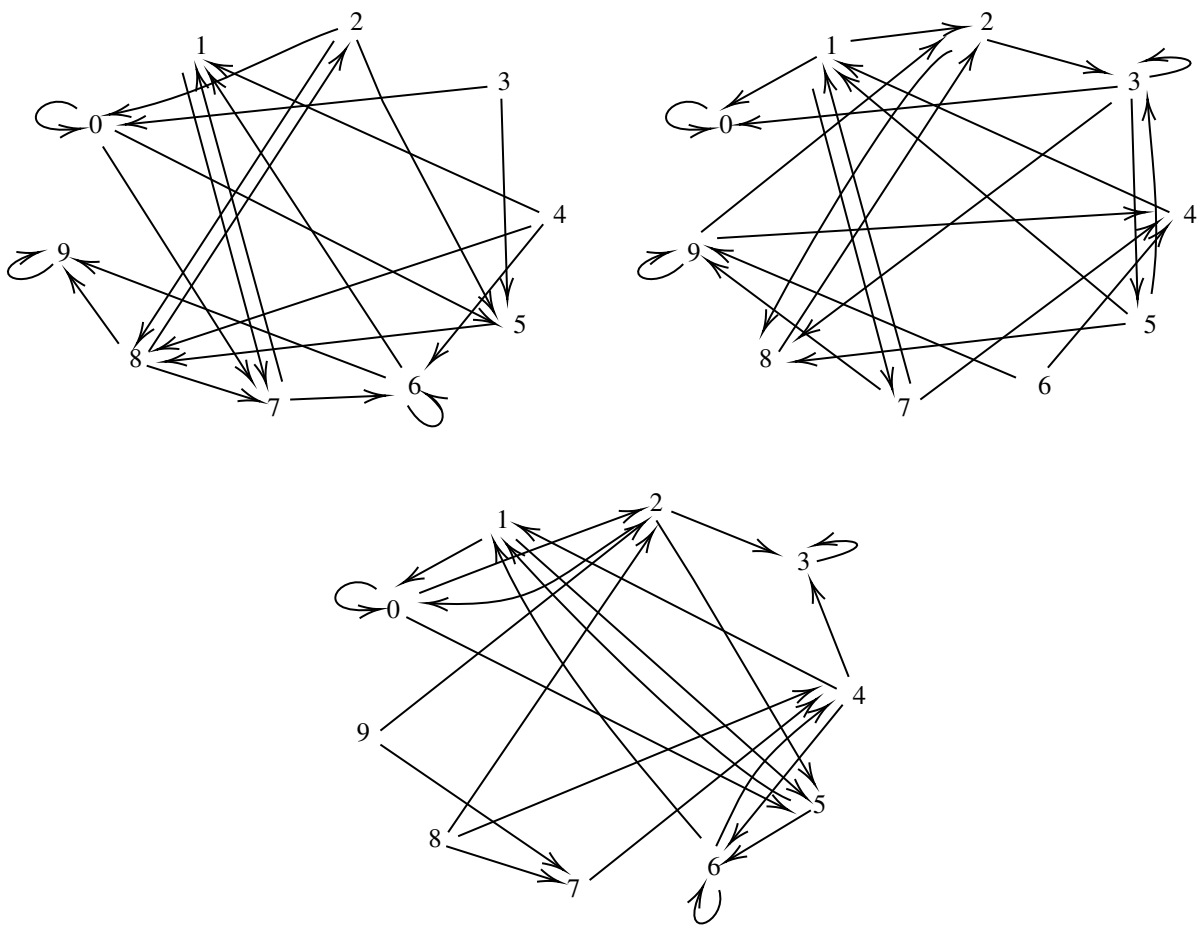


Figure 7.5: Modified transducers for $m = 10$ in base 4 that forbid digit 1, digit 2, and digit 3.

Finally, we arrive at $m \equiv 3 \pmod 4$, with the example of $m = 11$. Similarly to the case $m = 9$, each state has exactly one edge leaving and one edge entering corresponding to each digit.

m = 11:

$$0 \begin{array}{ll} \xrightarrow{0,0} & 0 \\ \xrightarrow{1,3} & 2 \\ \xrightarrow{2,2} & 5 \\ \xrightarrow{3,1} & 8 \end{array} \qquad 1 \begin{array}{ll} \xrightarrow{0,1} & 0 \\ \xrightarrow{1,0} & 3 \\ \xrightarrow{2,3} & 5 \\ \xrightarrow{3,2} & 8 \end{array} \qquad 2 \begin{array}{ll} \xrightarrow{0,2} & 0 \\ \xrightarrow{1,1} & 3 \\ \xrightarrow{2,0} & 6 \\ \xrightarrow{3,3} & 8 \end{array} \qquad 3 \begin{array}{ll} \xrightarrow{0,3} & 0 \\ \xrightarrow{1,2} & 3 \\ \xrightarrow{2,1} & 6 \\ \xrightarrow{3,0} & 9 \end{array}$$

$$4 \begin{array}{ll} \xrightarrow{0,0} & 1 \\ \xrightarrow{1,3} & 3 \\ \xrightarrow{2,2} & 6 \\ \xrightarrow{3,1} & 9 \end{array} \qquad 5 \begin{array}{ll} \xrightarrow{0,1} & 1 \\ \xrightarrow{1,0} & 4 \\ \xrightarrow{2,3} & 6 \\ \xrightarrow{3,2} & 9 \end{array} \qquad 6 \begin{array}{ll} \xrightarrow{0,2} & 1 \\ \xrightarrow{1,1} & 4 \\ \xrightarrow{2,0} & 7 \\ \xrightarrow{3,3} & 9 \end{array} \qquad 7 \begin{array}{ll} \xrightarrow{0,3} & 1 \\ \xrightarrow{1,2} & 4 \\ \xrightarrow{2,1} & 7 \\ \xrightarrow{3,0} & 10 \end{array}$$

$$8 \begin{array}{ll} \xrightarrow{0,0} & 2 \\ \xrightarrow{1,3} & 4 \\ \xrightarrow{2,2} & 7 \\ \xrightarrow{3,1} & 10 \end{array} \qquad 9 \begin{array}{ll} \xrightarrow{0,1} & 2 \\ \xrightarrow{1,0} & 5 \\ \xrightarrow{2,3} & 7 \\ \xrightarrow{3,2} & 10 \end{array} \qquad 10 \begin{array}{ll} \xrightarrow{0,2} & 2 \\ \xrightarrow{1,1} & 5 \\ \xrightarrow{2,0} & 8 \\ \xrightarrow{3,3} & 10 \end{array}$$

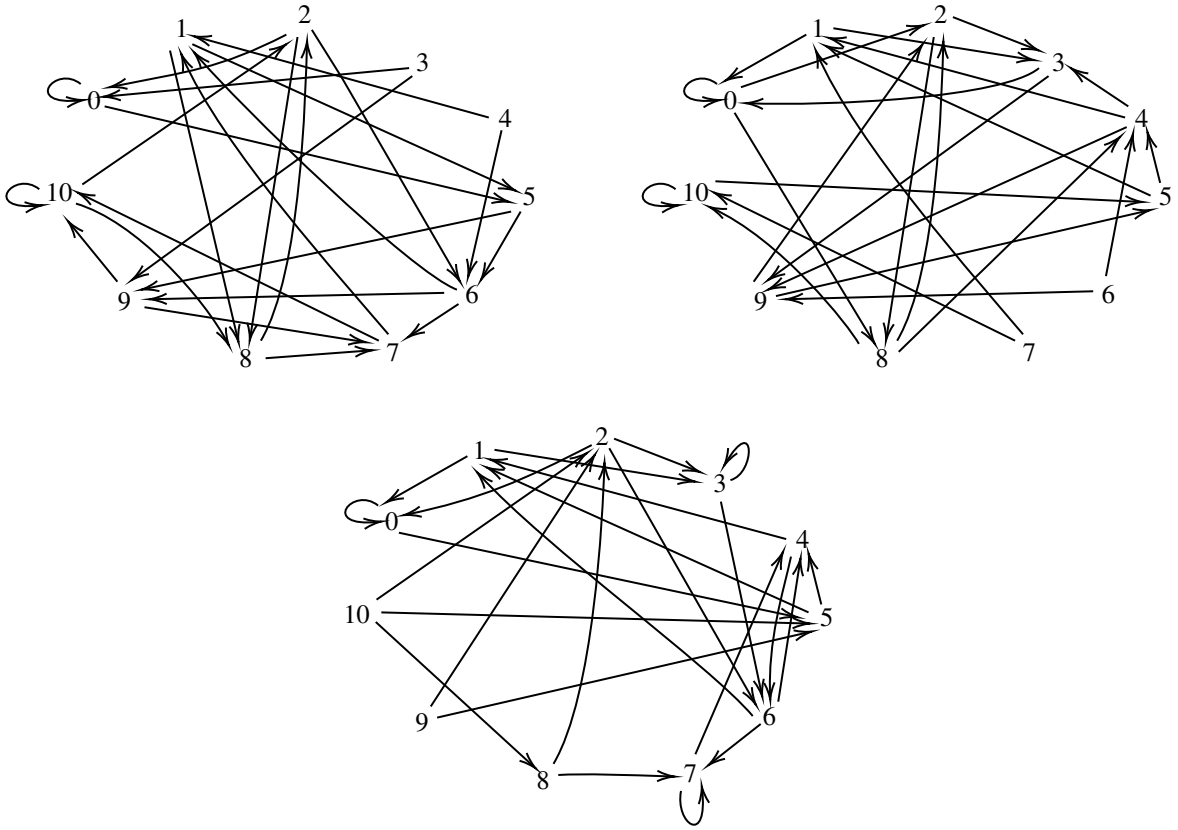


Figure 7.6: Modified transducers for $m = 11$ in base 4 that forbid digit 1, digit 2, and digit 3.

When looking at these transducers, and others like them, I more often grouped by forbidden digit (rather than m) to try to see patterns. For example, looking at the following four transducers which all have digit 1 forbidden.

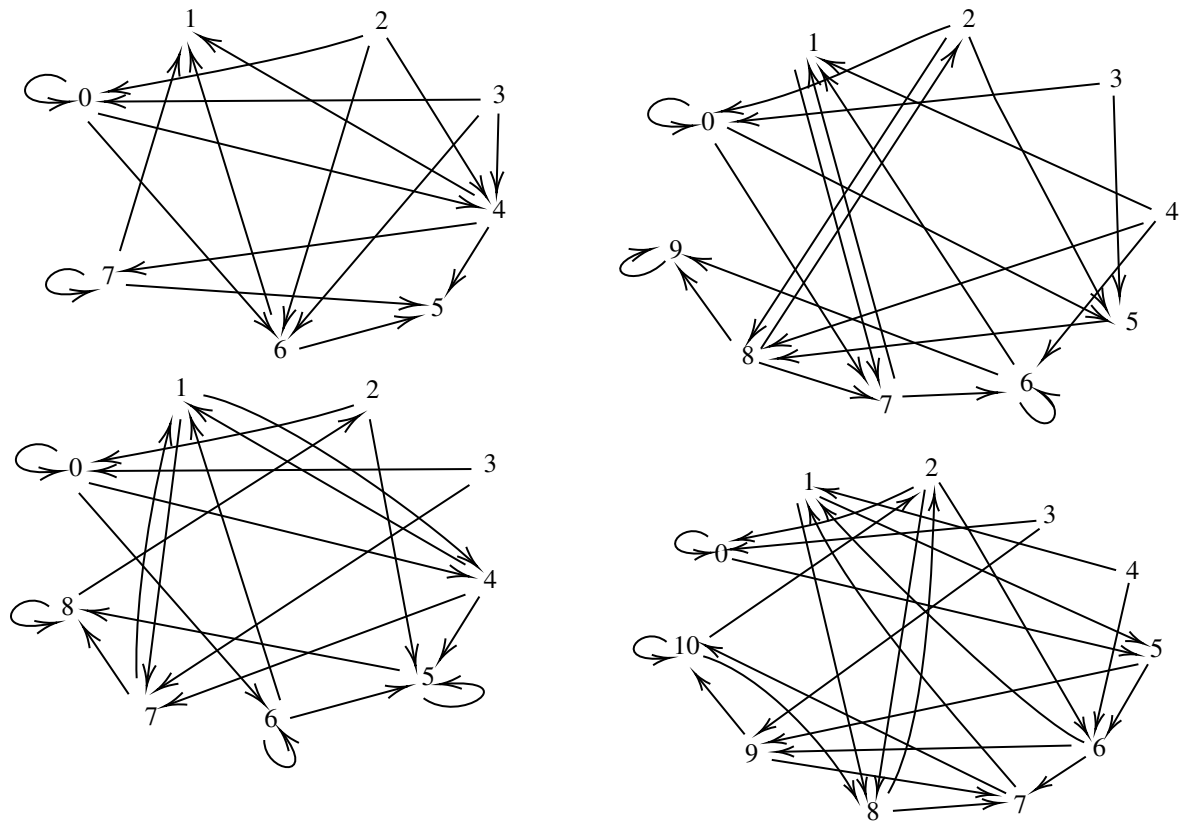


Figure 7.7: Modified transducers that forbid digit 1 in base 4 for $m = 8$ through 11.

Diagramming and Gesture

The way I interact with these examples is as **nested diagrams**. When drawing them, I would often draw several on the same page, and redraw them in different configurations to get a sense of how they relate to each other. I would also redraw the individual diagrams with different orientations, or subsets of edges, and look at them as a group together that shares those characteristics.

While I was unable to prove some of the conjectures in this section, drawing lots of these pictures helped me get a good start on developing intuition for why they might be true.

For the conjecture $Q_2 = \mathbb{N} \setminus \{2^{2j+1}(2\ell - 1) : \ell, j \in \mathbb{N}\}$, I looked at transducers grouped by equivalence class (mod 4) to try to see if I could make an argument on a case-by-case basis. For example, I would look at the transducers for $m = 9$, $m = 13$, $m = 17$,... together and look for patterns.

For the conjecture $Q_3 = \mathbb{N}$, looking at the images I notice that the states with in-degree 0 are the largest

states, which concentrates most of the edges in the transducer in the lower states. This is relevant because we eventually are hoping to get back to 0, so having more edges going "down" should make it easier to return to 0.

7.2 An Awful Problem

In 1987, J. H. Loxton and A. J. van der Poorten published a paper entitled "An awful problem about integers in base four," in which they proved that every odd integer can be written as a quotient of numbers which can be represented using the digits 0, 1, and -1 in base four. When we use the negative one as a digit, this indicates that we are multiplying that power of four by a negative one (essentially subtracting it). To make it easier to write, Loxton and van der Poorten write the 'negative' one with an overline.

Example 81. $[10\overline{1}\,]_4 = 1(4^2) + 0(4^1) + -1(4^0) = 16 + 0 - 1 = 15.$

While they do give a way of finding what the quotient would be that represents a given odd number, their method is not linked to the way they were able to prove the statement.

Their result is equivalent to saying the set of odd numbers is contained in the set $Q(4; \{-1, 0, 1\})$, and they go further to completely characterize the set $Q(4; \{-1, 0, 1\})$.

To prove the statement, they define the set $S = S(4; \{0, 1\})$, prove a statement about integers in the set $S + kS$, and then use that statement to show the main result. The set S represents the set of integers which can be written in base four using just the digits 0 or 1, and they let S_n represent the subset of S that contains elements with at most n digits in base 4.

Example 82.

$$S_1 = \{0, 1\}, \quad S_2 = \{0, 1, 4, 5\} = \{[0]_4, [1]_4, [10]_4, [11]_4\},$$

$$S_3 = \{0, 1, 4, 5, 16, 17, 20, 21\} = \{[0]_4, [1]_4, [10]_4, [11]_4, [100]_4, [101]_4, [110]_4, [111]_4\}$$

We note that S_1 has 2 elements, S_2 has 4 elements, and S_3 has 8 elements. This is not a coincidence; if S_n includes up to n digits, and digits are chosen from the set $\{0, 1\}$, then we have two choices for each of the n places, which results in exactly 2^n elements in the set S_n. Since S_n has 2^n elements, there are 2^n elements also in kS_n for an odd number k (kS_n is the set we get when we multiply each element of S_n by k).

Example 83. *We saw* $S_3 = \{0, 1, 4, 5, 16, 17, 20, 21\}$ *above, and so* $3S_3 = \{0, 3, 12, 15, 48, 52, 60, 63\}$.

We might expect then, that the set $S_3 + 3S_3$ *would have* 4^3 *elements (adding each of the* 2^3 *elements from*

S_3 to each of the 2^3 elements from $3S_3$. However, we see this is not necessarily the case.

	$3S_3$	0	3	12	15	48	51	60	63
S_3	0	0	3	12	15	48	51	60	63
	1	1	4	13	16	49	52	61	64
	4	4	7	16	19	52	55	64	67
	5	5	8	17	20	53	56	65	68
	16	16	19	28	31	64	67	76	79
	17	17	20	29	32	65	68	77	80
	20	20	23	32	35	68	71	80	83
	21	21	24	33	36	69	72	81	84

The set of integers on the inside of the table are the elements of $S_3 + 3S_3$. We can notice that there are several integers ($4, 16, 17, 19, 20, 32, 52, 64, 65, 67, 68, 80$), that appear more than once. We can rewrite the set without duplicates below:

$$\{0, 1, 3, 4, 5, 7, 8, 12, 13, 15, 16, 17, 19, 20, 21, 23, 24, 28, 29, 31, 32, 33, 35, 36, 48, 49, 51, 52, 53, 55, 56, 60,$$

$$61, 63, 64, 65, 67, 68, 69, 71, 72, 76, 77, 79, 80, 81, 83, 84\}$$

The size of the set $S_3 + 3S_3$, is 47, which is less than $64 = 4^3$.

Diagramming and Gesture

When trying to make sense of the sets $S_n + kS_n$ in [25], I made grids of examples like the one above. By creating this **perspective** through how the sets are constructed, I was able to better understand their arguments. They did not organize the information in this way in their paper, this was the "back" end of my process in trying to make sense of it.

Loxton and van der Poorten prove that the same thing will happen for any odd number k. That is, for any odd k, there is some n such that $S_n + kS_n$ must have fewer than 4^n distinct elements (note, n is not necessarily the same number as k). They use contradiction to prove the statement below.

Theorem 84. *Let S be the set of integers which can be written in base four using just the digits 0 or 1, and for $n = 0,\ 1,\ 2,\ \ldots$ denote by S_n the subset of numbers in S with at most n digits. Let k be an odd integer. Then, for all sufficiently large n, there is a number $p < 4$ such that the set*

$$S_n + kS_n = \{s + ks' : s, s' \in S_n\}$$

has size $O(p^n)$.

This means that, for any k, there is some n for which there are fewer than 4^n elements in $S_n + kS_n$. It follows that at least one element of the set $S_n + kS_n$ has more than one representation as a sum of one element of S_n and one element of kS_n.

The main idea in using their theorem to get a proof of the corollary below is noticing what happens when we take two elements, s and s' from the set $S = S(4; \{0, 1\})$ of integers with representations in base 4 using only the digits 0 and 1. Since s and s' both have only digits 0 and 1, if we subtract one from the other (say, $s - s'$) we get an integer with a representation using only digits -1, 0, and 1 in base four (an integer in the set $S(4; \{-1, 0, 1\})$). If we think about this, for any given place value, the options are

$$1 - 1 = 0, \qquad 1 - 0 = 1, \qquad 0 - 1 = -1, \qquad \text{or } 0 - 0 = 0.$$

By doing this for each place value independently (with no carrying or borrowing), the difference must have digits from the set $\{-1, 0, 1\}$.

We will illustrate first with an example, and then outline the formal proof below. In our example, we saw that there are two ways to build the number 65 as a sum of one element from S_3 and one element from $3S_3$.

$$5 + 60 = 65 = 17 + 48$$

Since $60 = 3(20)$ and $48 = 3(16) \in 3S_3$, we can rewrite this as

$$\begin{aligned} 5 + 3(20) &= 17 + 3(16) \\ 3(20 - 16) &= 17 - 5 \\ 3 &= \frac{17 - 5}{20 - 16} = \frac{12}{4} \end{aligned}$$

We recall that $17, 5, 20$, and 16 were all elements of S_3, with representations using the digits 0 and 1 in base 4. So, when we subtract them, we get elements with representations using the digits $0, 1, \overline{1}$.

$$17 - 5 = [101]_4 - [11]_4 = [1\overline{1}0]_4 \qquad 20 - 16 = [110]_4 - [100]_4 = [10]_4$$

So, we have shown that $3 = \frac{12}{4} = \frac{[1\overline{1}0]_4}{[10]_4}$, and so the odd number three is a quotient of elements that have representations using only the digits 0, 1, and -1 in base four.

Corollary 85. *Let L be the set of integers that can be written in base four using just the digits* 1, 0, *or* -1. *Every odd integer can be expressed as the quotient of two integers in L.*

Proof. Let k be an odd number. We note that, using the definition of S_n from Theorem 1, that $S_0 = \{0\}$, and so S_0 has $1 = 4^0$ elements, and $S_1 + kS_1 = \{0, 1, k, k+1\}$ has exactly four elements. By Theorem 1, there is some n such that the set $S_n + kS_n$ has fewer than 4^n distinct elements, and $S_{n-1} + kS_{n-1}$ has exactly 4^{n-1} elements.

We note that $S_n = 4 * S_{n-1} + S_1$, and so $kS_n = k(4 * S_{n-1} + S_1)$. Then

$$S_n + kS_n = 4(S_{n-1} + kS_{n-1}) + S_1 + kS_1$$

Since the set $S_n + kS_n$ has fewer than 4^n distinct elements, but the set $S_{n-1} + kS_{n-1}$ has exactly 4^{n-1} distinct elements, there is some $s \in S_n + kS_n$ that has two distinct representations. Thus, we have $s = s_1 + ks_1'$ and $s' = s_2 + ks_2'$ for some $s_1, s_2, s_1', s_2' \in S_n$. Rewriting this, we have

$$s_1 + ks_1' = s_2 + ks_2' \qquad \Rightarrow \qquad s_1 - s_2 = k(s_2' - s_1') \qquad \Rightarrow \qquad \frac{s_1 - s_2}{s_2' - s_1'} = k.$$

Then, since $s_1, s_2, s_1', s_2' \in S_n$, it follows that $s_1 - s_2$, $s_2' - s_1' \in L$, and so k can be expressed as the quotient of two integers in L. $\square$

This proof relies heavily on the symmetry that we see with the digits $-1, 0, 1$ in base 4. The authors note that a similar symmetry appears for any base that is a perfect square. They assert briefly that their argument could be copied in base 9 using the digit set $\{-2, -1, 0, 1, 2\}$, and more generally in base b^2 using digit set $\{-(b-1), -(b-2), \ldots, -1, 0, 1, \ldots, (b-2), (b-1)\}$. While they give a proof of the statement, it is not tied to the method the authors use to find the actual quotient for a given odd number k. This is a big part of why they find the problem "awful." Their method is not the same as the algorithm and transducers we described in previous chapters. Instead they work through the representation of k in base four using the digits $\{-1, 0, 1, 2\}$, choosing digits of the multiplier with a goal of eliminating twos, and the strategy is only described roughly.

We have not yet applied transducers to the set $Q(4; \{-1, 0, 1\})$, but are optimistic that this might be an avenue for offering an alternative proof of $Q(4; \{-1, 0, 1\}) \subseteq \mathbb{N} \setminus \{2^{2j+1}(2\ell - 1) : \ell, j \in \mathbb{N}\}$. When I was originally looking at the examples in base four, I had not yet become familiar enough with the transducers to think about constructing them using negative digits. We continue a discussion of $Q(b; \{-1, 0, 1\})$ more generally (without the transducers) in Chapter 8.

Chapter 8

Quotients in Base 5 and Beyond

This chapter focuses on quotients for bases $b > 4$. Section 8.1 includes results for $b = 5$, which motivated the study of general $b \geq 5$, which is treated in Section 8.2. The results in this chapter are preliminary and serve as a launchpad for future work. Section 8.3 provides an example of a family of sets $Q(b; \{d_1, \ldots d_k\})$ of a rather different flavor, and concludes with directions for future work.

8.1 Quotients in Base 5

After looking at quotients in base 4, my next step was to look at quotients in base 5. I started by looking at sets with three digits permitted (which can also be thought of as digit sets with two digits forbidden) and sets with four digits permitted (one forbidden). We look first at the sets $Q(5; \{d_1 = 0, d_2, d_3\})$ (where d_2, d_3 are chosen from the standard nonzero digits in base five, $\{1, 2, 3, 4\}$).

We begin with some restrictions on these sets that follow directly from the facts of multiplication $\pmod{b}$. We first consider the example $Q(5; \{0, 1, 2\})$.

Example 86. *Suppose we have an integer $r \in Q(5; \{0, 1, 2\})$. Then, we know that $r = \frac{q}{p}$ where q and p are integers with base 5 representations using only the digits $\{0, 1, 2\}$. We can divide out all factors of 5 to get $r' = \frac{q'}{p'}$, or $r'p' = q'$ where r', q', and p' are not multiples of 5. By definition, we have $p' = 5p_1 + a_0$ where $a_0 \in \{1, 2\}$, and $r' = 5r_1 + b_0$, where $b_0 \in \{1, 2, 3, 4\}$ (since the only restrictions on the digits of r' is $r' \not\equiv 0 \pmod{5}$). We note that $q' = 5q_1 + c_0$ where $c_0 \in \{1, 2\}$ as well. So,*

$$r'p' = (5p_1 + a_0)(5r_1 + b_0) = 25p_1r_1 + 5p_1b_0 + 5r_1a_0 + a_0b_0 = 5q_1 + c_0 = q'.$$

Looking at the both sides $\pmod{5}$, *we have*

$$a_0b_0 \equiv c_0 \pmod{5}.$$

We know that our possible choices for a_0 are ($\{1,2\}$) and for b_0 are ($\{1,2,3,4\}$). When we combine these options, we get (grouped by the possibilities for b_0):

$$1 \cdot 1 \equiv 1 \pmod 5 \qquad 1 \cdot 2 \equiv 2 \pmod 5 \qquad 1 \cdot 3 \equiv 3 \pmod 5 \qquad 1 \cdot 4 \equiv 4 \pmod 5$$

$$2 \cdot 1 \equiv 2 \pmod 5 \qquad 2 \cdot 2 \equiv 4 \pmod 5 \qquad 2 \cdot 3 \equiv 1 \pmod 5 \qquad 2 \cdot 4 \equiv 3 \pmod 5$$

Inspecting these possibilities, we note that $b_0 = 4$ would lead to a contradiction, because c_0 must be either 1 or 2. This process shows that $r' \not\equiv 4 \pmod 5$, and so no integer congruent to $4 \pmod 5$ may be in $Q(5;\{0,1,2\})$.

We generalize this statement in the Theorem 87.

Theorem 87. *Let m be an integer in $Q(b : \{0, d_2, d_3\})$, for some natural number $b \geq 1$, with distinct $d_2, d_3 \in \{1, 2, \ldots, b-1\}$. If r is the rightmost nonzero digit of m, then $rj \equiv k \pmod b$ for some choice of $j, k \in \{d_2, d_3\}$.*

Proof. Let $m \in Q(b;\{0,d_2,d_3\})$. Then, we know that $m = \frac{q}{p}$ where q and p are integers with base b representations using only the digits $\{0, d_2, d_3\}$. We can reduce all factors of b to get $m' = \frac{q'}{p'}$, or $m'p' = q'$ where m', q', and p' are not multiples of b. Then, by definition, we have $p' = bp_1 + j$ where $j \in \{d_2, d_3\}$, and $m' = bm_1 + r$, where $r \in \{1, 2, \ldots, b-1\}$ (since we have no restrictions on the digits of m', but we know that $m' \not\equiv 0 \pmod 5$). We note that $q' = bq_1 + k$ where $k \in \{d_2, d_3\}$, and r is by definition the rightmost nonzero digit of m. Putting this all together, we have,

$$m'p' = (bp_1 + r)(br_1 + j) = b^2 p_1 r_1 + bp_1 j + br_1 r + rj = bq_1 + k = q'.$$

It follows that $rj \equiv k \pmod b$, and by definition, $j, k \in \{d_2, d_3\}$. □

As a direct consequence of Theorem 87, in the base 5 case, we can note that in particular:

Corollary 88. *$Q(5;\{0,d_2,d_3\} \subseteq \mathbb{N} \setminus \{n \in \mathbb{N} : n \equiv 4 \pmod 5\}$, for the sets $\{d_2, d_3\} = \{1,2\}, \{1,3\}, \{2,4\}$, and $\{3,4\}$.*

Corollary 89. *$Q(5;\{0,d_2,d_3\}) \subseteq \mathbb{N} \setminus \{n \in \mathbb{N} : n \equiv 2 \pmod 5,$ or $n \equiv 3 \pmod 5\}$, for the sets $\{d_2, d_3\} = \{1,4\}$ and $\{2,3\}$.*

We conjecture that in the case of $Q(5;\{0,1,2\})$, this is actually an equality, and have checked up to $m = 10000$.

Conjecture 90.

$$Q(5;\{0,1,2\}) = \mathbb{N} \setminus \{n \in \mathbb{N} : n \equiv 4 \pmod 5\}.$$

I generated a list of the integers that are (and are not) in $Q(5;\{d_1 = 0, d_2, d_3\})$ for each choice of d_2, $d_3 \in \{1,2,3,4\}$, up to 300 and compared those lists. In exploring the computational data, I came up with additional conjectures about relationships between the sets. Then, for each of the conjectures below, I tested larger numbers, up to 3000, where instead of having the list printed out, I asked it to report any integers that did not match the conjecture (there were none).

Conjecture 91.

$$Q(5;\{0,1,3\}) \subsetneq Q(5;\{0,1,2\}).$$

Conjecture 92.

$$Q(5;\{0,3,4\}) \subsetneq Q(5;\{0,1,2\}).$$

Conjecture 93.

$$Q(5;\{0,2,3\}) \subsetneq Q(5;\{0,1,4\}).$$

This is somewhat surprising, since the permitted nonzero digits in the two sets do not overlap in Conjectures 92 and 93. We also note that the containments are strict.

Example 94. *The integer* $23 = [43]_5$ *is in both* $Q(5;\{0,1,3\})$ *and* $Q(5;\{0,2,4\})$*,*

$$23 = \frac{[31333]_5}{[331]_5} \text{ and } 23 = \frac{[1121]_5}{[12]_5}.$$

To illustrate that the containments are strict: 17 *is in* $Q(5;\{0,1,2\})$*, but not in* $Q(5;\{0,1,3\})$*,* 32 *is in* $Q(5;\{0,1,2\})$ *but not* $Q(5;\{0,3,4\})$ *and* 11 *is in* $Q(5;\{0,1,4\})$*, but not* $Q(5;\{0,2,3\})$*.*

Given the direction of the conjectured set containment, we might hope that there would be a map to take a representation as a quotient using the digits $\{0,1,3\}$ and transform it into a quotient using the digits $\{0,1,2\}$. However, similarly to the conjectured equality $Q(4;\{0,1,3\}) = Q(4;\{-1,0,1\})$ in the previous

chapter, this is still an open problem.

We can also show that $Q(5;\{0,1,2\}) = Q(5;\{0,2,4\})$ by multiplying every digit in numerator and denominator of a quotient on the left by 2. Aside from the conjectures stated above, and the equality between $Q(5;\{0,1,2\})$ and $Q(5;\{0,2,4\})$, there are provably no other possible inclusions. This means that looking at the sets of integers generated for small cases, no choice of digits i, j generates a set $Q(5;\{0,i,j\})$ that is contained in $Q(5;\{0,k,\ell\})$ for any choice of k, ℓ, other than those already conjectured above. So, we know that $Q(5;\{0,3,4\})$ contains an element that is not in $Q(5;\{0,1,3\})$ and vice versa.

There does not seem to be a clear relationship between the sets in Conjecture 91 and Conjecture 93. It appears that the sets in Conjecture 91 are more dense (both with density near .75 in the interval $[5, 2000]$), while the sets in Conjecture 93 are less dense ($Q(5;\{0,1,4\})$ has density .5 and $Q(5;\{0,2,3\})$ has density .34 in the interval $[5, 2000)$).

We close this discussion of $Q(5;\{0,i,j\})$ by discussing the set of integers which are in $Q(5;\{0,i,j\})$, for all i, j.

By Theorem 87, we know that this set cannot contain any integers $m \equiv 2$, 3, or 4 (mod 5). Of the integers in the interval $[1, 300]$, the following are in $Q(5;\{0,i,j\})$ for any choice of two i, j from $\{1,2,3,4\}$.

$$\{1,\ 5,\ 6,\ 21,\ 25,\ 26,\ 30,\ 31,\ 36,\ 41,\ 76,\ 81,\ 86,\ 91,\ 96,\ 101,\ 105,\ 106,\ 111,\ 116,\ 121,\ 125,\ 126,$$

$$130,\ 131,\ 136,\ 141,\ 146,\ 150,\ 151,\ 155,\ 156,\ 161,\ 166,\ 171,\ 180,\ 181,\ 186,\ 191,\ 196,\ 201,\ 206\}$$

It remains an open question to classify the set of all integers which are in $Q(5;\{0,i,j\})$ for any choice of two i, j from $\{1,2,3,4\}$.

Diagramming and Gesture

When looking at the sets $Q(5;\{0,i,j\})$, I started with text files that listed the elements (up to 300) of $Q(5;\{0,i,j\})$ for each choice of i, j. In an attempt to **organize the information**, I found myself using **spontaneous overt gesture** to manipulate the text boxes on the computer screen. I had all the text boxes open at once, and moved them around to be near sets that looked similar (had similar numbers). This made it easier to see which sets were contained in which others, and provides an example of how diagramming and gesture interacts with using information on the computer.

In looking at what happens when there is only one digit forbidden, I developed the following conjecture.

Conjecture 95. *(Base 5)*

$$Q(5; \{0, d_1, d_2, d_3\}) = \mathbb{N}, \text{ where } d_1,\ d_2,\ d_3 \text{ are distinct elements of } \{1,\ 2,\ 3,\ 4\}$$

This translates to the statement: "Every natural number can be written as a quotient of integers with base 5 representation using 0 and any three nonzero digits from $\{1,\ 2,\ 3,\ 4\}$." We note that the choices of d_1, d_2, and d_3 must be distinct; we cannot have $d_1 = 1$ and $d_2 = 1$.

I tested each choice for $d_1, d_2, d_3 \in \{1, 2, 3, 4\}$ for every integer up to $m = 30000$, again using a Python program that draws the transducer, and then checks for cycles containing the state 0. We will see a generalization of this conjecture in the next section.

8.2 General Bases $b \geq 5$

After looking at the relatively small bases $b = 3$, $b = 4$, and $b = 5$ individually, I began to wonder if there were patterns in the small bases that extend to the larger bases. We note that the standard base b representation includes b digits, and so as we look at larger and larger values of b, the pictures become increasingly difficult to draw by hand.

After Conjecture 95, I wondered if taking a relatively large digit set (size $b - 1$) in base $b > 5$ would also appear to include all natural numbers. I started by computing examples for $b = 6$ (up to $m = 30000$) and $b = 7$ (up to $m = 20000$), and then continued to run examples for some larger values of b to generalize further.

Conjecture 96. *For all bases $b > 4$,*

$$Q(b; \{0, d_1, d_2, \ldots, d_{b-2}\}) = \mathbb{N}$$

for any choice of distinct $d_1, d_2, \ldots, d_{b-2}$ from $\{1, 2, \ldots, (b-1)\}$

I verified this conjecture through base $b = 9$ up to $m = 20000$. When I think about drawing multiplication transducers, and then removing edges that read or write a single digit, as the base gets larger, the fraction of removed edges to total edges from the original graph gets smaller. This is not concrete enough to prove the conjecture, and whether it can be formalized to give a proof is still an open question.

Following this intuition, we might expect that for larger values of b, we would see that forbidding 2 digits, or three digits, would have the same effect. I have not explored this line of reasoning yet, but hope to in the future.

8.2.1 Quotients with Digits $\{-1, 0, 1\}$

In addition to looking at digit sets chosen from the standard base 5 digit set of $\{0, 1, 2, 3, 4\}$, inspired by [25] I looked at the set $Q(5; \{-1, 0, 1\})$. In my experimentation, I noticed that large intervals of integers were not in $Q(5; \{-1, 0, 1\})$, and further, that the intervals were those without leading digit 2.

Proposition 97. *No positive integer whose base 5 representation has leading digit 2 is in $Q(5; \{-1, 0, 1\})$.*

Proof. Let m be an integer with leading digit 2 in base 5, and take k such that m has exactly $k+1$ digits when written in base 5. Suppose towards a contradiction that there exists some integer q such that $m \cdot q = p$, where $p, q \in S(5 : \{-1, 0, 1\})$. Let i be the number of digits in the base 5 representation of q. Without loss of generality, we assume that both p and q have leading digit 1 (since the only permitted digits are 1 and $\overline{1}$, either $p = q = 1$ or $p = q = -1$ for a positive m). We note that q has i digits, and all digits in q come from the set $\{-1, 0, 1\}$.

Thus, we have $[1\underbrace{\overline{1}\overline{1}\cdots\overline{1}}_{(i-1)}]_5 \leq q \leq [\underbrace{11\cdots1}_{i}]_5$. [1] By assumption, $2 \cdot 5^k \leq m < 3 \cdot 5^k$.

Since $p = mq$, we have $[2\underbrace{00\cdots0}_{k}]_5[1\underbrace{\overline{1}\overline{1}\cdots\overline{1}}_{(i-1)}]_5 \leq p < [3\underbrace{00\cdots0}_{k}]_5[\underbrace{11\cdots1}_{i}]_5$.

Note that $[2\underbrace{00\cdots0}_{k}]_5[1\underbrace{\overline{1}\overline{1}\cdots\overline{1}}_{(i-1)}]_5 = [1\underbrace{2\ldots2}_{i-2}3\underbrace{0\cdots0}_{k}]_5$.

And, $[3\underbrace{00\cdots0}_{k}]_5[\underbrace{11\cdots1}_{i}]_5 = [\underbrace{3\cdots3}_{i}\underbrace{0\cdots00}_{k}]_5$.

So, for our assumed i-digit integer q, we have

$$[1\underbrace{2\ldots2}_{i-2}3\underbrace{0\cdots0}_{k}]_5 \leq p < [\underbrace{3\cdots3}_{i}\underbrace{0\cdots00}_{k}]_5$$

However, this contradicts the statement that $p \in S(5; \{-1, 0, 1\})$. Thus, there can be no such q, and m is not in $Q(5; \{-1, 0, 1\})$.

□

[1] We denote -1 by $\overline{1}$ when writing negative digits in base b representations.

While we restrict the proof to positive integers, it is worth noting that if a positive integer $m \in Q(5; \{-1, 0, 1\})$, then $m = \frac{p}{q}$ for p and q with digits using only -1, 0 and 1 in base 5, and so $-p$ also uses only digits -1, 0, and 1. Thus if $m \in Q(5; \{-1, 0, 1\})$, we know $-m \in\in Q(5; \{-1, 0, 1\})$, and similarly, if $m \notin Q(5; \{-1, 0, 1\})$, then $-m \notin Q(5; \{-1, 0, 1\})$.

Diagramming and Gesture

The ellipses in the previous proof are an example of **dotted lines**. When thinking about a general set of integers that have a particular form (i.e. have some number of 0's in the middle), I use my thumb and index finger to hold that information through **intentional self-oriented gesture**. Moving my thumb and index finger closer together and further apart helps me remember that dynamic part of the representation. We can see this modeled in the text with braces that are labeled with the variable. Though the text is not strictly a diagram, this highlights ways in which mathematical notation within equations can echo the strategies we see in diagrams.

We proceeded to generalize this theorem for larger values of b, and we increased the interval slightly in doing so. When looking at the data in base 5, the interval of integers that were not represented appeared to be slightly larger; some integers a little less than $2 \cdot 5^k$ are not represented, and some integers a little more than $3 \cdot 5^k$ are not represented.

From looking at the data, my first conjecture was that for $b > (2^a + 1)$, for all $k > a$, integers in the following intervals would not be in $Q(b; \{-1, 0, 1\})$,

$$\left[[1248 \ldots (2^{a-1})(2^a + 1) \underbrace{0 \ldots 0}_{k-a}]_b, [(b-2) \underbrace{00 \ldots 00}_{k}]_b. \right].$$

After talking this through with my advisor, Bruce Reznick, we shifted from looking at intervals of m that are not in $Q(b; \{-1, 0, 1\})$ and instead looked at the complement of those (the intervals that m must be in in order to also be in $Q(b; \{-1, 0, 1\})$). We found that the intervals $\left(b^k \cdot \frac{b-2}{b}, b^k \cdot \frac{b}{b-2}\right)$ give a more precise bound on which m may be in $Q(b; \{-1, 0, 1\})$. Note that Theorem 98 implies that there are large intervals of integers (those with leading digits $2, 3, \ldots, (b-3)$ in base b) that are not in $Q(b; \{-1, 0, 1\})$ for each $b > 4$.

Theorem 98. *If an integer m is in $Q(b; \{-1, 0, 1\})$, then*

$$b^k \cdot \frac{b-2}{b} < m < b^k \cdot \frac{b}{b-2},$$

for some positive k.

Proof. Let $m > 1$ be a positive integer in $Q(b; \{-1, 0, 1\})$. By definition, $m = \frac{p}{q}$, where $p, q \in S(b; \{-1, 0, 1\})$. Take i to be the number of digits in the base b representation of q. Since $p > q$, define k such that $k + i \geq i$ is the number of digits in the base b representation of p. Without loss of generality, assume the leftmost digit of p and q are both 1.Then,

$$p = b^{k+i} + \sum_{j=0}^{k+i-1} \alpha_j b^j \quad \text{and} \quad q = b^i + \sum_{j=0}^{i} \beta_j b^j, \quad \text{where } \alpha_j, \beta_j \in \{-1, 0, 1\}$$

Since $p, q \in S(b; \{-1, 0, 1\})$, we know

$$[1\underbrace{\overline{11}\cdots\overline{1}}_{(k+i-1)}]_b \leq p \leq [\underbrace{11\cdots1}_{k+i}]_b,$$

which is equivalent to the expanded statement,

$$b^{k+i} - \frac{b^{k+i}-1}{b-1} = b^{k+i} - \sum_{j=0}^{k+i-1} b^j \leq p \leq b^{k+i} + \sum_{j=0}^{k+i-1} b^j = b^{k+i} + \frac{b^{k+i}-1}{b-1}$$

Then, $b^j(1 - \frac{1}{b-1}) = b^j - \frac{b^j}{b-1} < b^j - \frac{b^j-1}{b-1}$ and $b^j + \frac{b^j-1}{b-1} < b^j + \frac{b^j}{b-1} = b^j(1 + \frac{1}{b-1})$, for $j \geq 1$ so it follows that

$$b^{k+i}(1 - \frac{1}{b-1}) < p < b^{k+i}(1 + \frac{1}{b-1}).$$

Similarly, since

$$[1\underbrace{\overline{11}\cdots\overline{1}}_{(i-1)}]_b \leq q \leq [\underbrace{11\cdots1}_{i}]_b,$$

we have that

$$b^i(1 - \frac{1}{b-1}) < q < b^i(1 + \frac{1}{b-1}).$$

Since $m = \frac{p}{q}$, we can put this all together to get

$$b^k \cdot \frac{b-2}{b} = \frac{b^{k+i}(1 - \frac{1}{b-1})}{b^i(1 + \frac{1}{b-1})} < m < \frac{b^{k+i}(1 + \frac{1}{b-1})}{b^i(1 - \frac{1}{b-1})} = b^k \cdot \frac{b}{b-2},$$

□

In base 5, we get the intervals:

$$([3]_5, [13]_5) \cup ([30]_5, [131]_5) \cup ([300]_5, [1313]_5) \cup \ldots$$

In base 6 we get the intervals:

$$([4]_6, [13]_6) \cup ([40]_6, [130]_6) \cup ([400]_6, [1300]_6) \cup \ldots$$

In base 7 we get the intervals:

$$([5]_7, [12]_7) \cup ([50]_7, [125]_7) \cup ([500]_7, [1254]_7) \cup ([5000]_7, [12541]_7) \cup ([50000]_7, [125412]_7) \cup \ldots$$

The argument above comes from looking at the possible first digit 1, and upper/lower bounds (1/-1) for the second digits of p and q. We could do a similar process using the possibilities for the first two digits of p and q ([11...], [10...] and [$1\overline{1}$...], etc.) and would get that our p and q values must be in the intervals:

$$b^j \cdot \left[\left(1 - \frac{1}{b-1}, 1 - \frac{b-2}{b(b-1)}\right) \bigcup \left(1 - \frac{1}{b(b-1)}, 1 + \frac{1}{b(b-1)}\right) \bigcup \left(1 + \frac{b-2}{b(b-1)}, 1 + \frac{1}{b-1}\right)\right].$$

These intervals are a subset of the interval we showed above. We have shown that integers in $Q(b; \{-1, 0, 1\})$ must be in the interval $(b^k \frac{b-2}{b}, b^k \frac{b}{b-2})$, for some k, but that does not guarantee that every integer in those intervals is in $Q(b; \{-1, 0, 1\})$, and in fact we know this is not true.

8.2.2 The "Easy Way" and the "Hard Way."

A quick diversion into the land of polynomials. We note that saying $m \in Q(b; \{d_1, \ldots, d_k\})$ is equivalent to saying

$$m = \frac{p}{q}, \text{ for some } p,\ q,\ \in S(b; \{d_1, \ldots, d_k\}),$$

and recall that by the definition of $S(b; \{d_1, \ldots, d_k\})$, this means that

$$p = \sum_{i=0}^{\ell} a_i b^i \text{ and } q = \sum_{j=0}^{n} c_j b^j,$$

for some positive integers $\ell \geq n$, and coefficients a_i, c_j, chosen from the set $\{d_1, \ldots, d_k\}$. So, we can think of $p = f(b)$, and $q = g(b)$ as polynomials in the variable b.

Example 99. *It was shown in Chapter 5 that* $7 \in Q(3; \{0, 1\})$*, because* $7 = \frac{28}{4} = \frac{3^3+1}{3+1}$.

The polynomial $x + 1$ *is a factor of* $x^3 + 1$*, so we have*

$$\frac{x^3+1}{x+1} = x^2 - x + 1 \quad \Rightarrow \quad \frac{3^3+1}{3+1} = 3^2 - 3 + 1 = 7.$$

We can also notice that if we take $x = 4$*, we get*

$$\frac{x^3+1}{x+1} = x^2 - x + 1 \quad \Rightarrow \quad \frac{4^3+1}{4+1} = 4^2 - 4 + 1 = 13.$$

So, in some way, the fact that $7 = 3^2 - 3 + 1 \in Q(3; \{0, 1\})$ is related to that $13 = 4^2 - 4 + 1 \in Q(4; \{0, 1\})$, and $21 = 5^2 - 5 + 1 \in Q(5; \{0, 1\})$, and $b^2 - b + 1 \in Q(b; \{0, 1\})$. We call this the "easy way", because it is true for any base, so these are quotients that we can find relatively "easily" using facts about polynomials.

However, not every integer in $Q(3; \{0, 1\})$ is obtainable the "easy way".

Example 100. *We could use the transducers or the algorithm to show that*

$$22 = \frac{22 \cdot 37}{37} = \frac{3^6 + 3^4 + 3 + 1}{3^3 + 3^2 + 1}.$$

However, taken as polynomials, $x^3 + x^2 + 1$ *is not a factor of* $x^6 + x^4 + x + 1$*. We call this the 'hard way' because it is specific to the base, so* 22 *does not belong to a similar family of polynomials, and if we tried to take, for example,*

$$\frac{4^6 + 4^4 + 4 + 1}{4^3 + 4^2 + 1} = \frac{4357}{81},$$

we see that we will not get an integer quotient.

The 'easy way' and the 'hard way' will be considered in much greater detail in [3].

8.2.3 Families of Quotients

Most of the conjectures I have proved in this document have shown that certain integers are not in a given set $Q(b; \{d_1, \ldots, d_k\})$. We are also able to show that some families of integers are in a given set. This does not completely classify the set $Q(b; \{d_1, \ldots, d_k\})$, but being able to show some things definitely are not in the set, and other things definitely are gets us closer. Inspired by the "easy way", I began looking for families

that are related by the denominator of the quotient and can be found the easy way. I give below some general families of integers that are in $Q(b;\{-1,0,1\})$ for arbitrary b.

Proposition 101. *Let $m = [d_k d_{k-1} d_{k-2} \dots d_2 d_1 d_0]_b$ be an integer with digits $d_k = 1$, $d_0 \in \{-1,0,1\}$, and d_i with $d_i = d_{i-1} - 1$, $d_i = d_{i-1}$, or $d_i = d_{i-1} + 1$ for $1 \le i < k$. Then, m is in $Q(b;\{-1,0,1\})$.*

Proof. We will show that $n = m \cdot [1\overline{1}]_b \in S(b;\{-1,0,1\})$. $m \cdot [1\overline{1}]_b = m * b - m$, which we can think of as:

$$\begin{array}{cccccccc}
d_k & d_{k-1} & d_{k-2} & \dots & d_2 & d_1 & d_0 & \\
- & d_k & d_{k-1} & d_{k-2} & \dots & d_2 & d_1 & d_0 \\
\hline
n_{k+1} & n_k & n_{k-1} & \dots & n_3 & n_2 & n_1 & n_0
\end{array}$$

We note that $n_{k+1} = d_k = 1$, and $n_0 = -d_0 \in \{-1,0,1\}$. For $1 \le i \le k$, $n_i = d_{i-1} - d_i \in \{-1,0,1\}$. So we have $[n_{k+1} n_k n_{k-1} \dots n_3 n_2 n_1 n_0]_b \in S(b;\{-1,0,1\})$ and

$$m = \frac{[n_{k+1} n_k n_{k-1} \dots n_3 n_2 n_1 n_0]_b}{[1\overline{1}]_b} \in Q(b;\{-1,0,1\}).$$

□

This proof shows that any integer whose base b representation starts with the digit 1, ends with any digit from $\{-1,0,1\}$, and for which the difference between each digit and an adjacent digit is at most 1 is in $Q(b;\{-1,0,1\})$.

Example 102. *The integer $m = [123432101]_b$ is in $Q(b;\{-1,0,1\})$ for all b, because if we multiply this m by $[1\overline{1}]_b$ (the same as multiplying by b, and then subtracting m), we'll get a product in $S(b;\{-1,0,1\})$. We can show this using addition:*

$$\begin{array}{ccccccccc}
1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 1 \\
- & 1 & 2 & 3 & 4 & 3 & 2 & 1 & 0 & 1 \\
\hline
1 & 1 & 1 & 1 & \overline{1} & \overline{1} & \overline{1} & \overline{1} & 1 & \overline{1}
\end{array}$$

We note that the digits here are no longer restricted to $d_i \in \{0,1,\dots,b-1\}$, but rather could be any digits

that satisfy the conditions $d_i = d_{i-1} - 1$, $d_i = d_{i-1}$, or $d_i = d_{i-1} + 1$ for $1 \leq i < k$.

We can also think of Proposition 101 as a statement about polynomials. If we take $(x-1)$ and multiply any polynomial with the following form,

$$\sum_{i=0}^{k} d_i x^i \qquad \text{with } d_k = 1,\ d_0 \in \{-1,0,1\},\ \text{and } d_i = d_{i-1} - 1,\ d_i = d_{i-1},\ \text{or } d_i = d_{i-1} + 1 \text{ for } 0 < i < k,$$

we'll get a polynomial of the form

$$\sum_{i=0}^{k+1} c_i x^i \qquad \text{with } c_i \in \{-1,0,1\} \text{ for all } 0 < i < k+1.$$

The following two propositions create different families using the same process (using factors $(x+1)$ and (x^2-1), respectively).

Proposition 103. *Let $m = [d_k d_{k-1} d_{k-2} \ldots d_2 d_1 d_0]_b$ be an integer with digits $d_k = 1$, $d_0 \in \{-1,0,1\}$, and d_i with $d_i = -d_{i-1} + 1$, $d_i = -d_{i-1}$, or $d_i = -d_{i-1} - 1$, for $1 \leq i < k$. Then, m is in $Q(b; \{-1,0,1\})$.*

Proof. We will show that $n = m \cdot [11]_b \in S(b; \{-1,0,1\})$. $m \cdot [11]_b = m * b + m$, which we can think of as:

$$\begin{array}{cccccccc}
 & d_k & d_{k-1} & d_{k-2} & \ldots & d_2 & d_1 & d_0 \\
+ & & d_k & d_{k-1} & d_{k-2} & \ldots & d_2 & d_1 & d_0 \\
\hline
n_{k+1} & n_k & n_{k-1} & \ldots & n_3 & n_2 & n_1 & n_0
\end{array}$$

We note that $n_{k+1} = d_k = 1$, and $n_0 = -d_0 \in \{-1,0,1\}$. For $1 \leq i \leq k$, $n_i = d_{i-1} - d_i \in \{-1,0,1\}$. So we have $[n_{k+1} n_k n_{k-1} \ldots n_3 n_2 n_1 n_0]_b \in S(b; \{-1,0,1\})$ and

$$m = \frac{[n_{k+1} n_k n_{k-1} \ldots n_3 n_2 n_1 n_0]_b}{[11]_b} \in Q(b; \{-1,0,1\}).$$

□

Proposition 104. *Let $m = [d_k d_{k-1} d_{k-2} \ldots d_2 d_1 d_0]_b$ be an integer with digits $d_k = 1$, $d_{k-1}, d_1, d_0 \in \{-1,0,1\}$, and d_i with $d_i = d_{i-2} - 1$, $d_i = d_{i-2}$, or $d_i = d_{i-2} + 1$ for $2 \leq i < k-1$. Then, m is in $Q(b; \{-1,0,1\})$.*

Proof. We will show that $n = m \cdot [10\overline{1}]_b \in S(b; \{-1, 0, 1\})$. $m \cdot [10\overline{1}]_b = m * b^2 - m$, which we can think of as:

$$\begin{array}{ccccccccc}
d_k & d_{k-1} & d_{k-2} & \dots & d_2 & d_1 & d_0 & & \\
- & & d_k & d_{k-1} & d_{k-2} & \dots & d_2 & d_1 & d_0 \\
\hline
n_{k+2} & n_{k+1} & n_k & n_{k-1} & \dots & n_3 & n_2 & n_1 & n_0
\end{array}$$

We note that $n_{k+2} = d_k = 1$, $n_{k+1} = d_{k-1} \in \{-1, 0, 1\}$, $n_1 = -d_1 \in \{-1, 0, 1\}$ and $n_0 = -d_0 \in \{-1, 0, 1\}$. For $2 \leq i \leq k$, $n_i = d_{i-2} - d_i \in \{-1, 0, 1\}$. So we have $[n_{k+1}n_kn_{k-1} \dots n_3n_2n_1n_0]_b \in S(b; \{-1, 0, 1\})$ and

$$m = \frac{[n_{k+1}n_kn_{k-1} \dots n_3n_2n_1n_0]_b}{[10\overline{1}]_b} \in Q(b; \{-1, 0, 1\}).$$

□

We could keep going in this way, and note that if we chose a denominator with three digits, we would need to add or subtract three shifted copies of m.

8.3 A Multitude of Future Problems

We have looked a variety of specific cases and generalizations, but this has barely scratched the surface of possibilities. There are many variations of problems in this vein that have yet to be explored. We finish this section with a statement with a somewhat different flavor, taking a very specific digit set, that is true for all even bases. This was inspired by Theorem 73, which we proved using transducers in the previous chapter. In the case $m = 2$, Theorem 105 provides an alternate proof of Theorem 73.

Theorem 105. *Suppose $b = 2m$, and j is in $[2, m]$. Then, $j \cdot b^i$ is not in the set $Q(b; \{0, m, m+1, \dots, 2m-1\})$ for some positive i.*

Proof. We consider the set $S = S(b; \{0, m, m+1, \dots, 2m-1\})$ and for each $s \in S$, we have

$$s = \sum_{l=0}^{k} a_l b^l, \qquad \{a_l \in \{0, m, m+1, \dots, 2m-1\}, a_k \neq 0\}.$$

Then, since for each s, $a_k \in \{m, m+1, \dots, 2m-1\}$, we know that each $s \in S$ with $s \neq 0$, is in an interval $\left[\frac{1}{2}b^{k+1}, b^{k+1}\right]$.

Let $w = jb^i$, where $j \in [2, m]$. Suppose towards a contradiction that $w = \frac{s}{t}$, for some nonzero $s, t, \in S$. Then, $s = wt$, and since $t \in S$, we have

$$s = wt \in jb^i \cdot S = (jb^i)\left[\frac{1}{2}b^{k+1}, b^{k+1}\right) = \left[\frac{j}{2}b^{i+k+1}, \frac{j}{2}b^{i+k+1}\right).$$

By assumption, $2 \leq j \leq m$, so we have $b^{i+k+1} \leq s < m \cdot b^{i+k+1}$ by the above. Thus, the leading coefficient of b^{j+k+1} in the base b expansion of s is in the set $\{1, \ldots, m\}$

$\square$

This completes our discussion of the sets $Q(b; \{d_1, \ldots d_k\})$. What has been presented here only barely scratches the surface of study of $Q(b; \{d_1, \ldots d_k\})$. I look forward to continuing to explore variations of these sets in the future.

Appendices

A Constructing Transducers with Negative Digits

In this section, we show how to apply transducers in the case that includes negative digits. We first recall the notation

$$Q(b; \{d_1, \ldots d_k\}) := \{x \in \mathbb{Z} : x = s/s' \text{ for some } s, s' \in S(b; \{d_1, \ldots, d_k\})\}.$$

The set $Q(b; \{d_1, \ldots d_k\})$ includes only the integers that can be represented as quotients of elements of $S(b; \{d_1, \ldots, d_k\})$. In base 3 we chose a set of digits $\{0, 1\}$ which is a subset of the digit set of the standard base 3 representation $\{0, 1, 2\}$. This allowed us to modify transducers that perform multiplication in base 3 using that standard digit set.

However, our definition does not restrict the digit set $\{d_1, \ldots, d_k\}$ to be a subset of $\{0, 1, \ldots, (b-1)\}$ in base b. We can construct multiplication transducers for any set of digits we choose. For example, we earlier looked at a transducer that multiplies by 7 in base four.

The color of the arrow indicates the digit read (the digit from the denominator, which is being multiplied by 7). The colors are:

Gray reads 0 Purple reads 1 Green reads 2 Black reads 3.

The size of the dotted line indicates which digit is written (the digit in the numerator, which would be the digit written in the corresponding place in the multiplication algorithm).The lines are:

Solid writes 0 Dense Dots write 1 Loose Dots write 2 Dashes write 3

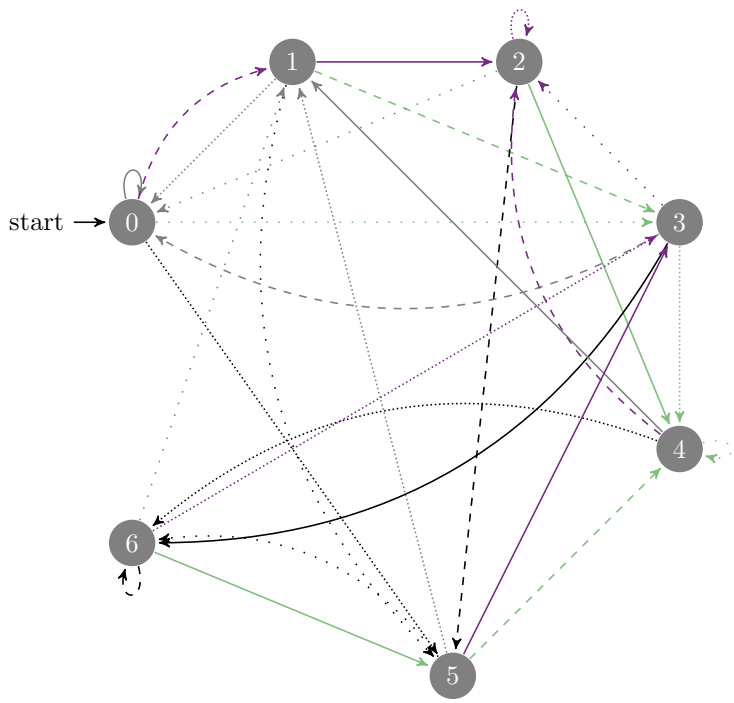


Figure A.1: The transducer that multiplies by 7 in base 4 using the digits $\{0,\ 1,\ 2,\ 3\}$.

To make this transducer, I made a shorthand version of all of the edges first. Each edge starting from a state k, reads a digit $i \in \{0,\ 1,\ 2,\ 3\}$ and moves to state ℓ, writing $j \in \{0,\ 1,\ 2,\ 3\}$, for ℓ, k, satisfying $7i + k = 4\ell + j$. In the shorthand notation, this would be $k \xrightarrow{i,j} \ell$.

To take an example, we start in state $k = 0$. If we read a digit $i = 1$, we have the equation $7(1)+0 = 4(1)+3$, so we would move to state 1 and write the digit 3. This step, $0 \xrightarrow{1,3} 1$ is represented by the second arrow in the first column below.

$m = 7$**, digits** $\{0,\ 1,\ 2,\ 3\}$**:**

$$0 \begin{array}{l} \xrightarrow{0,0} 0 \\ \xrightarrow{1,3} 1 \\ \xrightarrow{2,2} 3 \\ \xrightarrow{3,1} 5 \end{array} \qquad 1 \begin{array}{l} \xrightarrow{0,1} 0 \\ \xrightarrow{1,0} 2 \\ \xrightarrow{2,3} 3 \\ \xrightarrow{3,2} 5 \end{array} \qquad 2 \begin{array}{l} \xrightarrow{0,2} 0 \\ \xrightarrow{1,1} 2 \\ \xrightarrow{2,0} 4 \\ \xrightarrow{3,3} 5 \end{array} \qquad 3 \begin{array}{l} \xrightarrow{0,3} 0 \\ \xrightarrow{1,2} 2 \\ \xrightarrow{2,1} 4 \\ \xrightarrow{3,0} 6 \end{array}$$

$$4 \begin{array}{l} \xrightarrow{0,0} 1 \\ \xrightarrow{1,3} 2 \\ \xrightarrow{2,2} 4 \\ \xrightarrow{3,1} 6 \end{array} \qquad 5 \begin{array}{l} \xrightarrow{0,1} 1 \\ \xrightarrow{1,0} 3 \\ \xrightarrow{2,3} 4 \\ \xrightarrow{3,2} 6 \end{array} \qquad 6 \begin{array}{l} \xrightarrow{0,2} 1 \\ \xrightarrow{1,1} 3 \\ \xrightarrow{2,0} 5 \\ \xrightarrow{3,3} 6 \end{array}$$

We want to explore what happens when we take a digit set that contains negative numbers. Suppose we take the digit set $\{-1,\ 0,\ 1,\ 2\}$. We know that every integer has a unique representation in base 4 using digits from $\{-1,\ 0,\ 1,\ 2\}$, however, in this case, we would have $7 = [2\overline{1}]_4$. Since we no longer have the digit 3, we represent 7 as $2(4) + (-1)(1)$.

We can create a multiplication transducer that performs multiplication by 4 using the digits $\{-1,\ 0,\ 1,\ 2\}$ in the same way that we created the previous transducer. Each edge starting from a state k, reads a digit $i \in \{-1,\ 0,\ 1,\ 2\}$ and moves to state ℓ, writing $j \in \{-1,\ 0,\ 1,\ 2\}$, for ℓ, k, satisfying $7i + k = 4\ell + j$. In the shorthand notation, this would be $k \xrightarrow{i,j} \ell$.

Comparing the shorthand below with that above, some of the edges (which did not read/write a 3 or a -1) are the same. In the second example it is possible to 'carry' a negative number, or move to a negative state. For example, if we are in state 0 and read a -1, we have the equation $7(-1) + 0 = -2(4) + 1$, so we would write a 1 and move to state -2, which represents carrying -2 copies of 4 into our next step.

$m = 7$**, digits** $\{-1,\ 0,\ 1,\ 2\}$**:**

$$-2 \quad \begin{array}{ll} \xrightarrow{-1,-1} & -2 \\ \xrightarrow{0,2} & -1 \\ \xrightarrow{1,1} & 1 \\ \xrightarrow{2,0} & 3 \end{array} \qquad -1 \quad \begin{array}{ll} \xrightarrow{-1,0} & -2 \\ \xrightarrow{0,-1} & 0 \\ \xrightarrow{1,2} & 1 \\ \xrightarrow{2,1} & 3 \end{array} \qquad 0 \quad \begin{array}{ll} \xrightarrow{-1,1} & -2 \\ \xrightarrow{0,0} & 0 \\ \xrightarrow{1,-1} & 2 \\ \xrightarrow{2,2} & 3 \end{array} \qquad 1 \quad \begin{array}{ll} \xrightarrow{-1,2} & -2 \\ \xrightarrow{0,1} & 0 \\ \xrightarrow{1,0} & 2 \\ \xrightarrow{2,-1} & 4 \end{array}$$

$$2 \quad \begin{array}{ll} \xrightarrow{-1,-1} & -1 \\ \xrightarrow{0,2} & 0 \\ \xrightarrow{1,1} & 2 \\ \xrightarrow{2,0} & 4 \end{array} \qquad 3 \quad \begin{array}{ll} \xrightarrow{-1,0} & -1 \\ \xrightarrow{0,-1} & 1 \\ \xrightarrow{1,2} & 2 \\ \xrightarrow{2,1} & 4 \end{array} \qquad 4 \quad \begin{array}{ll} \xrightarrow{-1,1} & -1 \\ \xrightarrow{0,0} & 1 \\ \xrightarrow{1,-1} & 3 \\ \xrightarrow{2,2} & 4 \end{array}$$

When I draw these transducers, I start the shorthand from state 0, and continue in the positive direction until I have completed the paths for every positive state that has occurred. Starting from 0, I write the possibilities for each digit read in state 0, then state 1, then state 2, all the way to state 4. There is no need to write possibilities for states > 4 in this case, because we would not reach them starting from 0.

Then, I work my way in the negative direction, again continuing until I have completed the paths for each negative state that has occurred (here we only go to -2). Despite the order that I compute them, I tend to write them out in number-line (ascending) order. We can use these to draw the transducer that multiplies by 7 in base 4 using the digits -1, 0, 1, and 2.

We adapt the colors and shading from before as follows:

The color of the arrow indicates the digit read (the digit from the denominator, which is being multiplied by 7). The colors are:

Black reads -1 Gray reads 0 Purple reads 1 Green reads 2.

The size of the dotted line indicates which digit is written (the digit in the numerator, which would be the digit written in the corresponding place in the multiplication algorithm).The lines are:

Dashes write -1 Solid writes 0 Dense Dots write 1 Loose Dots write 2.

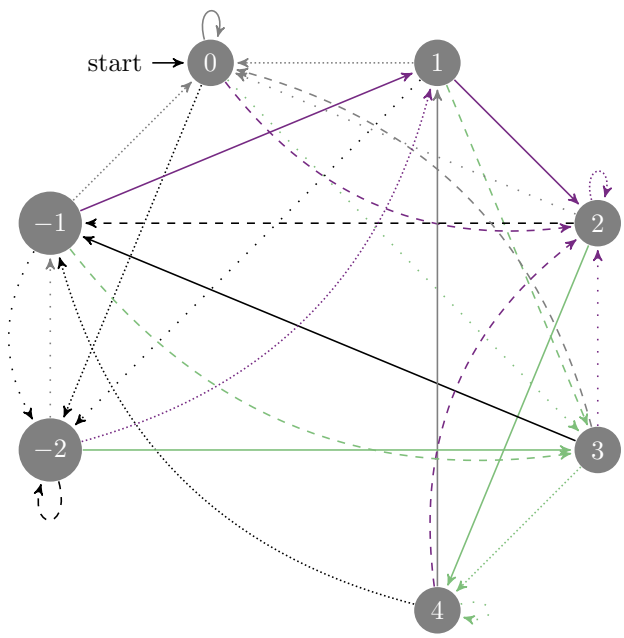


Figure A.2: The transducer that multiplies by 7 in base 4 using the digits $\{-1,\ 0,\ 1,\ 2\}$.

We can then delete edges to remove digits. For example, removing the digit 2 will create a transducer that tests whether $7 \in Q(4; \{-1, 0, 1\})$, which we know to be true due to the results in [25].

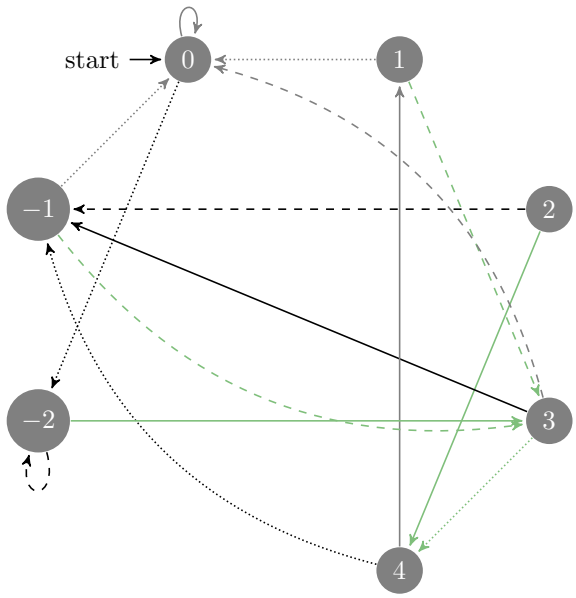


Figure A.3: The transducer for $m = 7$ in base 4 with digits $\{-1,\ 0,\ 1,\ 2\}$, no edges that read or write 2.

We note that if we removed the digit -1 from the transducer with negative digits, we would get almost the same transducer as if we removed the digit 3 from the original.

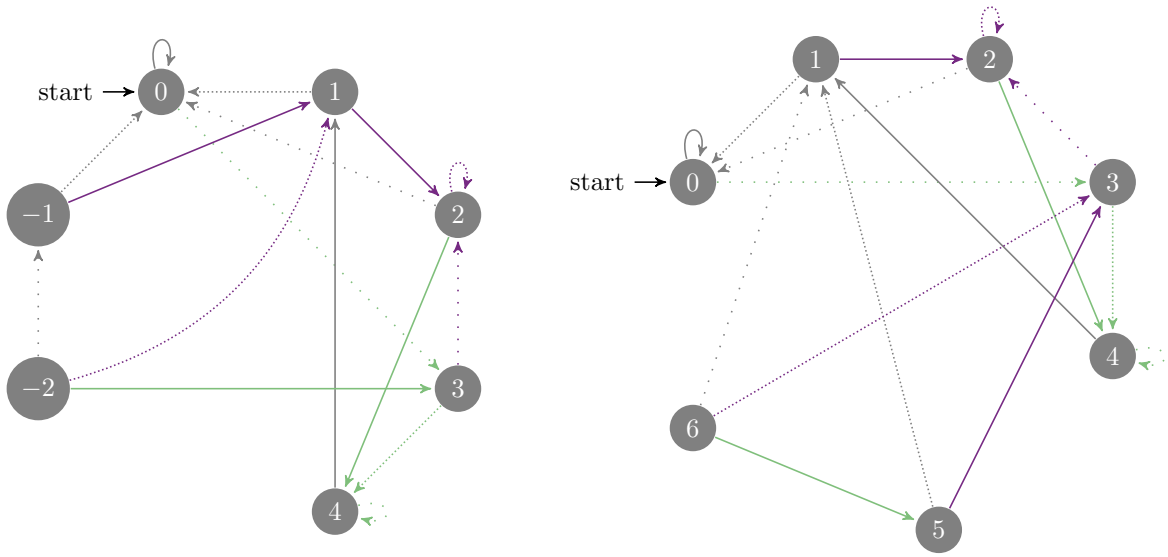


Figure A.4: Transducers for $m = 7$ in base 4 using digits $\{-1, 0, 1, 2\}$ and $\{0, 1, 2, 3\}$ with no digit -1 and 3 (respectively).

This is not a coincidence, but rather those computations would be exactly the same. In fact, if we remove the 'source' states (states that have no edges coming in, and so would not be included in the computation), the transducers are exactly the same.

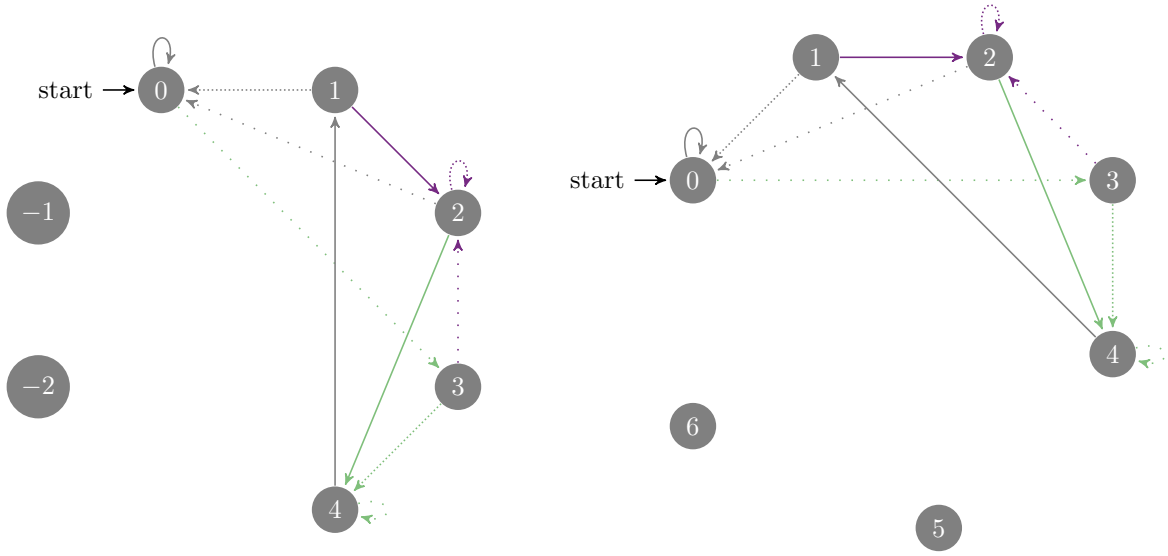


Figure A.5: The transducers that multiply by 7 in base 4 using the digits $\{-1,\ 0,\ 1,\ 2\}$ and $\{0,\ 1,\ 2,\ 3\}$ with digits -1 and 3 removed (respectively) and edges from source states removed.

However, if we remove the digit 2 from each of them, we get different transducers.

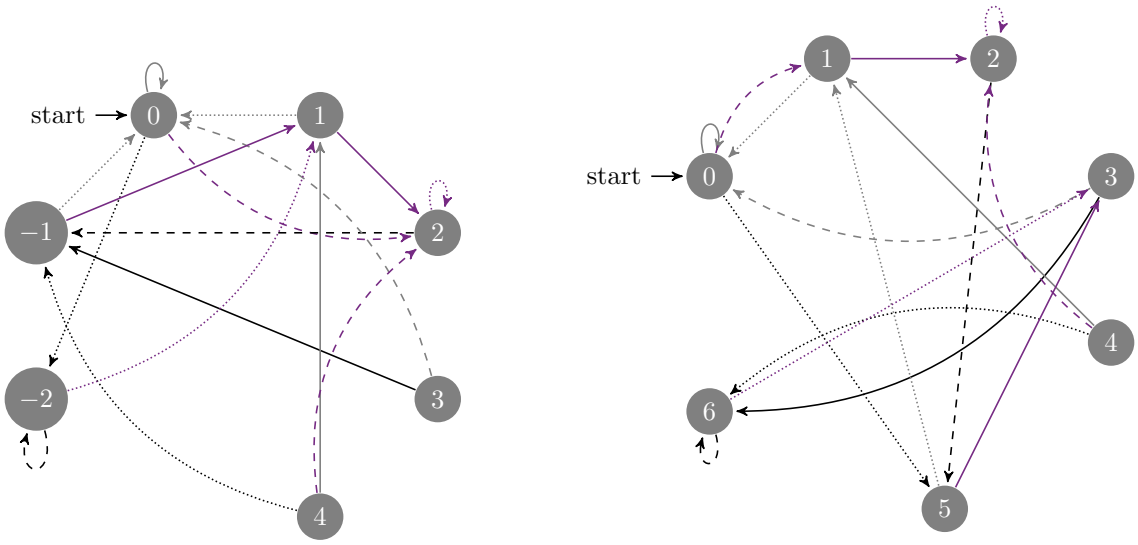


Figure A.6: The transducers that multiply by 7 in base 4 using the digits $\{-1,\ 0,\ 1,\ 2\}$ and $\{0,\ 1,\ 2,\ 3\}$, each with digit 2 removed.

While we can create transducers with negative digits, in order to have a functional multiplication transducer to start with, we need to choose one digit from each equivalence class $\pmod{b}$. So, we could create a transducer that uses the digits $\{-2,\ -1,\ 0,\ 1\}$, or $\{-2,\ 0,\ 1,\ 3\}$, or even $\{-2,\ 1,\ 3,\ 4\}$, but if we tried to choose $\{-1,\ 0,\ 1,\ 3\}$, we would run into issues. For example, if we tried to use the digits $\{-1,\ 0,\ 1,\ 3\}$, and in state 3 read 0, we wouldn't know whether to rewrite it as $0(7) + 3 = 0(4) + 3$ (write 3 and move to state 0), or $0(7) + 3 = 1(4) + -1$ (write -1 and move to state 1).

We hope to explore these transducers with digit sets that include negative digits more in the future.

B Commentary on a Nontraditional Dissertation

B.1 Descriptions of the Audiences

Below, I describe the audiences for particular sections in more detail. The sections presented in this document vary slightly from those that I have shared/will share with readers from the intended audiences. Most of this is cosmetic (figure captions, formatting, numbering). Occasionally I have added a few sentences to connect to previous sections of the document that I remove when sections are taken out of the context of the entire dissertation.

Chapter 3 Sections 3.1 and 3.2 were written with my nephew in mind. He read and provided feedback for these sections when he was 10 years old and in fourth and fifth grades. Section 3.3 was written for a subset of students who I have worked with in the Education Justice Project program, to be read alongside Sections 3.1 and 3.2 in a workshop session preparing to read Chapter 4.

Chapter 4 Chapter 4 was written for Education Justice Project students who were in a cohort Calculus sequence (Preparation for Calculus through Calculus II) from Spring 2017-Spring 2019, taught by myself and Joshua Wen, another mathematics PhD student.

Chapter 5 Section 5.1 was written for my niece (age 9, third grade). Sections 5.3 and 5.4 were written another niece (her sister, age 12, seventh grade). Section 5.2 and 5.5 are included to provide background and were written to be accessible to my sister, their mother.

Chapters 6-8 Chapters 6 through 8 were written for an advanced undergraduate audience. In particular, I thought of students who were in my Fall 2017 Math 347 (Fundamental Mathematics, an introduction to proof-writing), many of whom are graduating in Spring 2020.

B.2 Reflections on Writing for Nontraditional Audiences

I was inspired to write my thesis to be accessible beyond the traditional audience of research mathematicians when I happened upon Piper Harron's dissertation, [18], as a second-year graduate student, well before I began working on the mathematical content. The idea of writing a nontraditional thesis was in the back of my mind as I began working on problems. Throughout my research process, I was conscious of what background knowledge was truly required to understand the main ideas.

When it came time to write the dissertation, I had choices to make about how I wanted to approach it. In

[18], each chapter had a similar structure, with three tiers of accessibility. I chose instead to write different sections to be standalone, with the idea that individuals might read a section without having read the introduction, or even seeing the thesis in its complete form. In making my decisions, I reflected on my motivations, the audience, and how to approach writing for each specific audience. I have attempted to concentrate the things I considered during this process into the set of questions below. The questions were generated at the end of the writing process, to provide documentation of my process and possibly guide others who might want to attempt writing a similar type of document.

1. *Who is the audience?*

   When I first started thinking about writing, I knew that I wanted to write something accessible, but I wasn't sure what that would look like. I approached Matt Ando and, in a brainstorming meeting, he asked the pivotal question that really started this journey for me. "Who is the audience?" In other words, I was talking about thinking the mathematics was (or could be made) accessible, but to whom? Accessible to a middle-school student is different than accessible to an undergrad who has taken calculus, which is different than accessible to research mathematicians outside of my area, because each of them would have different levels of comfort with symbolic notation, jargon, and formal mathematical conventions. I ultimately chose different audiences for different sections, in service of my goals for the document.

2. *What are the goals?* As I progressed through my graduate career, I anticipated the thesis-writing process with dread. I find energy in teaching and collaborating with others, and the prospect of writing a thesis seemed isolating. I knew I would drag my feet spending months writing a long document that very few people outside of the committee would be interested in reading. Around the same time as my meeting with Matt, I decided to change my perspective; I tried to imagine a way of writing this thesis that would make me feel more connected to people that I care about, instead of isolated from them. My goal was to use this process as an opportunity to build connections, and to share what I've been working on with people who are not typically included in the formal research process.

3. *What is the context?* Originally, I thought I might turn my research into activities for a course or workshop to be taught at the Summer Illinois Mathematics Camp or through the Education Justice Project. However, as things progressed, I realized that it was important to me that the pieces of the actual written document be shared, not just the content. In order to do this, I had to think of the potential audience and the context in which they would engage with the written document, often by reading a printed copy. How are they used to engaging with mathematics? If the audience is

not accustomed to reading mathematics as text, I wanted to write those sections to be broken up by questions and examples, to feel more like play. I also thought about length of sections; I might expect an advanced undergraduate to read a much longer set of pages than a 9-year-old third grader.

4. *How can the content connect to the audience's prior knowledge?* Everyone has mathematical experiences. For each audience, I asked myself what I know about their experiences, and how I can connect this content to things they are already familiar with. In some cases, it means choosing examples that highlight multiplication, or reviewing notation that might be familiar from a calculus class. I also try to connect using similes or metaphors, to describe what something is *like*, to help create an image that resonates. I found myself using the phrase "I think about it like...," to transition into these less formal descriptions.

5. *What kinds of supports might the audience need?* Through my years of training as a mathematician, I have developed strategies for reading mathematics. I often stop after a definition is introduced to try to work out an example for myself, make notes in the margins, and ask myself lots of questions in between. It can take a long time to read a page of math, and I knew that many of the potential audiences have not practiced reading math very much. When writing, I tried to think about how I read math, and to build in scaffolding for that process. In some sections, there are explicit exercises, to prompt the audience to check their own understanding. In other sections this showed up as adding an extra line of an equation, or an extra frame to a set of diagrams to make it easier to follow.

6. *What are the main ideas, and where can formality/jargon be reduced?* I was lucky to attend a workshop entitled "What's the Story? Research Presentations for an Undergraduate Audience" at both the 2018 and 2019 Mathfest conferences, led by May Mei (professor at Denison University). One major takeaway from the workshop was the phrase: "introduce vocabulary like it costs money." While it may not cost actual money, there is a finite amount of space that an audience has for storing new ideas and definitions. When thinking about my audiences, I tried to focus on what vocabulary was truly necessary, and what vocabulary could be talked around, or left to a footnote without sacrificing the main ideas. For example, we often define a partition as a nonincreasing sequence of positive integers. Instead of using the word nonincreasing, in the introduction to Chapter 3, I build that element of the definition into the convention of the diagrams by putting the largest row first.

7. *How would I teach this in person?* I asked myself this question when I came to a section or idea that I did not know how to approach. By thinking about how I would approach teaching the content in a classroom, I could usually get unstuck. I would think about what questions I would ask, if I would

draw a picture, or where I might have students come up with their own examples, and then try to translate that into the text. Where I might hope to have more discussion when teaching in person, I would try to add more explanation or examples/nonexamples, to support the reader in taking a little more time to think about the topic.

8. *When and how should I include formal proofs?* My decision for when and how to include formal proof was dependent on the audience. For audiences who are less familiar with formal proof conventions, I tried to preface each proof with either a concrete example, or a rough sketch of the argument (Chapter 4). For audiences who have some experience with proof, I tried to remove excess notation, and wrote the proofs so that they could be followed by advanced undergraduates - adding details that might be skipped in an academic paper.

9. *How can I engage the potential audience in a way that is ethical and authentic?* Originally, I had hoped to have each section read by a representative (or group of representatives) from each potential audience. At the forefront of my mind when thinking of this was, how to invite potential readers while acknowledging the power imbalance in my relationship with them. My inspiration for audiences includes former students, incarcerated students, and children, all of whom are people I care deeply about, and I wanted to be careful to not exploit our relationships. I approached these conversations with transparency about these power dynamics, making clear that I had no expectation for them to participate, and acknowledging my appreciation for their time and intentions to take feedback seriously. I was most often met with enthusiasm for the idea. I had planned to meet with some of these readers towards the end of the spring semester 2020, and in those meetings to give them an opportunity to consult on how they are credited in the document. Some of these plans were disrupted by the COVID-19 pandemic, and as a result several sections have not yet been revised by the intended audience. I have every intention of connecting with these readers again once it is safe to do so.

C Sample Computer Programs

Below is an example SAGE program that uses a directed graph package to generate transducers, and creates modified multiplication transducers by deleting edges from a full transducer. The program can print a graphic of a given transducer. It was used early in the research process, and so are designed to only forbid one digit at a time.

```
def mult_transducer(k, p):
    #builds a transducer that multiplies by k in base p
    G = DiGraph(loops=True)#loops=True to show loops
    for i in range(k):
        G.add_vertex(i)
    for i in range(k):
        for j in range(p):
            s = mod(k*j+i, p)
            o = floor((k*j+i)/p)
            G.add_edge(i,o, label = [j,s])
    G.show(edge_labels=True)
    print G.edges()
    return G
def mult_transducer_noi(k,p,i):
    #builds a transducer that multiplies by k in base p and deletes edges labelled
with i
    G_initial = mult_transducer(k,p)
    G_noi = DiGraph(loops=True)
    listofedges= []
    for b in range(p):
        listofedges.append([i, b])
        listofedges.append([b,i])
    for a in range(k):
        G_noi.add_vertex(a)
    for e in G_initial.edges():
        if e[2] not in listofedges:
            G_noi.add_edge(e)
        else:
            continue
    G_noi.show(edge_labels=True)
    return G_noi
def cycles_cont_0(k,p):
    #gives a list of all of the cycles containing vertex 0 in the transducer that
multiply by k in base p
    G = mult_transducer(k,p)
    cycles = G.all_simple_cycles(starting_vertices=[0])
    print cycles
    return cycles
def cycles_to_m_noi(m,p,i):
    #prints the smallest cycle with length >1 in each transducer that multiplies by
values from p to m in base p without any labels "i"
    fout = file("cycles_base_4_digits_no_3"+str(m)+".txt","w")
    for l in range(m-p):
        G= mult_transducer_noi(p+l, p,i)
        cycles = G.all_simple_cycles(starting_vertices=[0])
        fout.write("cycles " + str(p+l) + " no 3 " + str(cycles[1]))
        fout.write("\n")
    fout.close()
```

```
def edge_labels_to_m_noi(m,p,i):
    #outputs a text file with the list of edges in each transducer that multiplies by
values from p to m in base p without any labels "i"
    fout = file("edge_lables_b4_digits_no_3"+str(m)+".txt","w")
    for l in range(m-p):
        G= mult_transducer_noi(p+l, p,i)
        cycles = G.all_simple_cycles(starting_vertices=[0])
        e = G.edges()
        fout.write("edges" + str(p+l) + " no 3 ")
        fout.write("\n")
        for j in range(len(e)):
            fout.write(str(e[j]))
            fout.write("\n")
    fout.close()
def quotients_to_m_noi(m,p,i):
    #writes the first 10 quotients with no i that give integers up to m. (all if <10)
    fout = file("quotients_base_"+str(p)+"_digits_no_"+str(i)+"_"+str(m)+".txt","w")
    for l in range(m-p+1):
        G= mult_transducer_noi(p+l, p,i)
        if mod(p+l, 10)==0:
            print p+l
        cycles = G.all_simple_cycles(starting_vertices=[0])
        fout.write("quotients " + str(k+l) + " no 3 ")
        fout.write("\n")
        for j in min(range(len(cycles)), range(10)):
            a = str()
            b = str()
            if len(cycles[j])>2:
                for e in range(len(cycles[j])-1):
                    m = G.edge_label(cycles[j][e], cycles[j][e+1])
                    a = str(m[0])+a
                    b= str(m[1]) +b
            else:
                continue
            fout.write(str(b)+ "/" +str(a)+", ")
        fout.write("\n")
    fout.close()
```

Below is an example of a Python program from later in the research process. This program constructs a modified transducer by initially drawing only the edges that are permitted (as opposed to drawing all possible edges and then deleting some of them.

```
import time
import copy
def modifiedtransducer(k,p,l_omit):
    #creates the transducer that multiplies by k in base p and has no paths labelled
with digits from l_omit, as a dictionary, removes first set of sinks and sources.
    gdict = {}
    to0list = []
    visited = {}
    sinkslist=[]
    sourceslist =[]
    for i in range(k):
        for l in l_omit:
            A = (l*k)//p +1
            B = (l*k + k-1)//p
            if i in range(A,B):
                sourceslist.append(i)
            ptest = 0
            for p1 in range(p):
                if (p1*k+i)%p==l:
                    ptest +=1
            if ptest == p:
                sinkslist.append(i)
    for i in range(k):
        if i not in sinkslist and i not in sourceslist:
            gdict[i]=[]
            visited[i] = False
    for i in gdict:
        for j in range(p):
            s = (k*j+i)%p
            o = (k*j+i)//p
            if o in gdict:
                sizel = len(l_omit)
                if j not in l_omit:
                    if s not in l_omit and i !=o:
                        js = str(j)
                        ss = str(s)
                        gdict[i].append(o)
                        if o == 0:
                            to0list.append(i)
                    else:
                        continue
            else:
                continue
    return gdict, to0list, visited
```

```
def ispathfromstart(graph,start,endlist,visited):
    #Determines if there is a path from start, not necessarily finding the path.
    stack, tested = [start], [start]
    temp = stack.copy()
    newlevel=[]
    for ends in endlist:
        if start == ends:
            return stack
    while len(temp)>=1:
        #print(temp)
        if temp[0]>=0:
            i = temp[0]
            leaves = []
            if i not in tested:
                tested.append(i)
            for j in graph[i]:
                if j in endlist:
                    tested.append(j)
                    return tested
                if j not in tested:
                    leaves.append(j)
                    newlevel.append(j)
                if j in tested:
                    continue
            stack.append(leaves)
            if len(temp)==1:
                temp = newlevel.copy()
                newlevel =[]
            else:
                temp.remove(i)
    return []
def unexpected_basep_text(j,k,p,l_omit):
    #Textfile that gives all of the integers that do not have a representation in base
p with no digits from l_omit
    start = time.time()
    fout = open("unexpected_transducer_from_"+
str(j)+"_to_"+str(k)+"_base_"+str(p)+"_no_"+str(l_omit)+"test.txt", "w")
    num_unexpected=0
    for i in range(j,k):
        if i %100 ==0:
            print(i)
            continue
        [graph,to0,v]= modifiedtransducer(i,p,l_omit)
        if len(ispathfromstart(graph,0,to0,v))==0:
            fout.write(str(i) +"\n")
            num_unexpected+=1
    if num_unexpected==0:
        fout.write("all expected"+"\n")
    fout.write("It took "+ str(time.time()-start)+ " seconds.")
    fout.close()
```

```
def list_loops_text(j,k,p,l_omit):
    #Textfile that gives loops for all of the integers between j and k that have a
representation in base p with no digits from l_omit
    start = time.time()
    fout = open("list_loops_from_"+
str(j)+"_to_"+str(k)+"_base_"+str(p)+"_no_"+str(l_omit)+"test.txt", "w")
    num_unexpected=0
    for i in range(j,k):
        if i %100 ==0:
            print(i)
            continue
        [graph,to0,v]= modifiedtransducer(i,p,l_omit)
        loop =ispathfromstart(graph,0,to0,v)
        if len(loop)>0:
            fout.write(str(i)+ " , "+ str(loop) +"\n")
    fout.write("It took "+ str(time.time()-start)+ " seconds.")
    fout.close()
```

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