EXTREMAL PROBLEMS ON SPECIAL GRAPH COLORINGS

BY

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DISSERTATION

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Abstract

In this thesis, we study several extremal problems on graph colorings. In particular, we study monochromatic connected matchings, paths, and cycles in 2-edge colored graphs, packing colorings of subcubic graphs, and directed intersection number of digraphs.

In Chapter 2, we consider monochromatic structures in 2-edge colored graphs. A matching M in a graph G is *connected* if all the edges of M are in the same component of G. Following Łuczak, there are a number of results using the existence of large connected matchings in cluster graphs with respect to regular partitions of large graphs to show the existence of long paths and other structures in these graphs. We prove exact Ramsey-type bounds on the sizes of monochromatic connected matchings in 2-edge-colored multipartite graphs. In addition, we prove a stability theorem for such matchings, which is used to find necessary and sufficient conditions on the existence of monochromatic paths and cycles: for every fixed s and large n, we describe all values of $n_1, \dots, n_s$ such that for every 2-edge-coloring of the complete s-partite graph $K_{n_1,\dots,n_s}$ there exists a monochromatic (i) cycle C_{2n} with $2n$ vertices, (ii) cycle $C_{\geq 2n}$ with at least $2n$ vertices, (iii) path P_{2n} with $2n$ vertices, and (iv) path P_{2n+1} with $2n+1$ vertices. Our results also imply for large n of the conjecture by Gyárfás, Ruszinkó, Sárközy and Szemerédi that for every 2-edge-coloring of the complete 3-partite graph $K_{n,n,n}$ there is a monochromatic path P_{2n+1}.

Moreover, we prove that for every sufficiently large n, if $n = 3t + r$ where $r \in \{0, 1, 2\}$ and G is an n-vertex graph with $\delta(G) \geq (3n-1)/4$, then for every 2-edge-coloring of G, either there are cycles of every length $\{3, 4, 5, \dots, 2t + r\}$ of the same color, or there are cycles of every even length $\{4, 6, 8, \dots, 2t + 2\}$ of the same color. This result is tight and implies the conjecture of Schelp that for every sufficiently large n, every $(3n-1)$-vertex graph G with minimum degree larger than $3|V(G)|/4$, in each 2-edge-coloring of G there exists a monochromatic path P_{2n} with $2n$ vertices. It also implies for sufficiently large n the conjecture by Benevides, Łuczak, Scott, Skokan and White that for every positive integer n of the form $n = 3t + r$ where $r \in \{0, 1, 2\}$ and every n-vertex graph G with $\delta(G) \geq 3n/4$, in each 2-edge-coloring of G there exists a monochromatic cycle of length at least $2t + r$.

In Chapter 3, we consider a collection of special vertex colorings called *packing colorings*. For a sequence of non-decreasing positive integers $S = (s_1, \dots, s_k)$, a packing S-coloring is a partition of $V(G)$ into sets $V_1, \dots, V_k$ such that for each $1 \leq i \leq k$ the distance between any two distinct $x, y \in V_i$ is at least $s_i + 1$. The smallest k such that G has a packing $(1, 2, \dots, k)$-coloring is called the packing

chromatic number of G and is denoted by $\chi_p(G)$. The question whether the packing chromatic number of subcubic graphs is bounded appears in several papers. We show that for every fixed k and $g \geq 2k+2$, almost every n-vertex cubic graph of girth at least g has the packing chromatic number greater than k, which answers the previous question in the negative. Moreover, we work towards the conjecture of Brešar, Klavžar, Rall and Wash that the packing chromatic number of 1-subdivision of subcubic graphs are bounded above by 5. In particular, we show that every subcubic graph is $(1,1,2,2,3,3,k)$-colorable for every integer $k \geq 4$ via a coloring in which color k is used at most once, every 2-degenerate subcubic graph is $(1,1,2,2,3,3)$-colorable, and every subcubic graph with maximum average degree less than $\frac{30}{11}$ is packing $(1,1,2,2)$-colorable.

Furthermore, while proving the packing chromatic number of subcubic graphs is unbounded, we also consider improving upper bound on the independence ratio, $\alpha(G)/n$, of cubic n-vertex graphs of large girth. We show that "almost all" cubic labeled graphs of girth at least 16 have independence ratio at most 0.454.

In Chapter 4, we introduce and study the directed intersection representation of digraphs. A directed intersection representation is an assignment of a color set to each vertex in a digraph such that two vertices form an edge if and only if their color sets share at least one color and the tail vertex has a strictly smaller color set than the head. The smallest possible size of the union of the color sets is defined to be the directed intersection number (DIN). We show that the directed intersection representation is well-defined for all directed acyclic graphs and the maximum DIN among all n vertex acyclic digraphs is at most $\frac{5n^2}{8} + O(n)$ and at least $\frac{9n^2}{16} + O(n)$.

To my parents.

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LIST OF ABBREVIATIONS AND NOTATION

$\emptyset$	the empty set
$[n]$	for $n \in \mathbb{N}$, $[n] := \{1, \dots, n\}$
$\alpha(G)$	the independence number of a graph G
$\chi(G)$	the chromatic number of a graph G
$E(G)$	the edge set of a (hyper)graph G
$e(G)$	the size of $E(G)$
$\mathbb{N}$	the set of natural numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{Z}$	the set of integers
$V(G)$	the vertex set of a (hyper)graph G
$v(G)$	the size of $V(G)$
$c(G)$	the circumference of G, i.e., the length of a longest cycle in G
K_n	the complete graph on n vertices
C_n	the cycle on n vertices
P_n	the path on n vertices
$N(v)$	the neighborhood of a vertex, i.e., $\{u \in V(G) : uv \in E(G)\}$
$d(v)$	the degree of a vertex, i.e., $\lvert N(v)\rvert$

Chapter 1

Introduction

In this thesis, we study extremal problems on some specific graph colorings, including monochromatic structures on 2-edge-colored graphs, packing colorings, and directed intersection representation.

Ramsey theory is a branch of combinatorics that studies the conditions under which regularity must appear. It was first proved by Ramsey that the Ramsey number $R(s,t)$ exists: for any $s,t \geq 1$, there exists a smallest positive integer N ($R(s,t)$ is defined to be N) such that if we color the edges of K_N, the complete graph on N vertices, by red and blue, then the colored graph contains either a red K_s or a blue K_t. There have been many new results of similar type, and these evolved into the following Ramsey-type problem for graphs: under what conditions (of the host graph) does there exist a monochromatic copy of a collection of subgraphs when we color the edges of the host graph using r colors? In Chapter 2, we consider conditions on different types of host graphs such that respectively a monochromatic connected matching with n edges, a path with $2n$ vertices, a cycle with at least $2n$ vertices, or a cycle with exactly $2n$ vertices always exist.

The theory of coloring deals with the problem of partitioning objects into classes that avoid specific conflicts. It is one of the most important topics in discrete mathematics and has many applications in other fields of Mathematics, Computer Science, Information Theory and Electrical Engineering. In Chapter 3, we consider a specific type of coloring called packing colorings, which was motivated by a frequency assignment problem in broadcast networks: there are many broadcast stations in the world, and we would like to assign each station a frequency; stations assigned the same frequency are required to be at least a certain distance apart, and each frequency requires a different smallest distance; what is the minimum number of frequencies needed for such an assignment?

In the WWW network, a number of pages are devoted to topic or item disambiguation, such as Wikipedia pages. In these disambiguation pages, a number of identical names of designators are used to describe different entities which are further clarified and narrowed down in context via links to more specific pages. In Chapter 4, we introduce and study the notion of Directed Intersection Representation of digraphs: let D be a directed graph with vertex set V and arc set A, and assume that each vertex $v \in V$ is associated with a nonempty subset $\varphi(v)$ of a finite ground set $\mathcal{C}$, called the *color set*, such that $(u,v) \in A$ if and only if $|\varphi(u) \cap \varphi(v)| \geq 1$ and $|\varphi(u)| < |\varphi(v)|$; if such a representation is possible, we refer to it as a *directed intersection representation.*

1.1 Monochromatic connected matchings, paths and cycles in 2-edge-colored graphs

A *connected matching* in a graph G is a matching having edges only in one component of G. By M_n we will always denote a connected matching with n edges and by P_n, the path with n vertices. Also by C_n we denote the cycle with n vertices, and by $C_{\geq n}$, a cycle of length at least n. The circumference of a graph G is the length of a longest cycle in G. For graphs $G_0, \dots, G_k$ we write $G_0 \mapsto (G_1, \dots, G_k)$ if for every k-coloring of the edges of G_0, there is a copy of some G_i $(i \in [k])$ with all edges of color i. The *Ramsey number* $R(G_1, \dots, G_k)$ is the minimum N such that $K_N \mapsto (G_1, \dots, G_k)$, and we write $R_k(G) = R(G_1, \dots, G_k)$ when $G_1 = \ldots = G_k = G$.

The study of Ramsey-type problems of paths was initiated by Gerencsér and Gyárfás [45] in 1967. They proved that for positive integers k and ℓ with $k \geq \ell$, $R(P_k, P_\ell) = k - 1 + \lfloor \frac{\ell}{2} \rfloor$, which implies $R_2(P_n) = \lfloor \frac{3n-2}{2} \rfloor$. Many significant results bounding $R_k(P_n)$ for $k \geq 3$ and $R_k(C_n)$ for even n have been proved. In particular, the current best upper bounds, recently proven by Knierim and Su [64], are $R_k(C_n) \leq (k - \frac{1}{2} + o(1))n$ for even n and $R_k(P_n) \leq (k - \frac{1}{2} + o(1))n$. Many proofs used the Szemerédi Regularity Lemma [85]. A number of them used the idea of connected matchings in regular partitions due to Łuczak [73]. A flavor of it is illustrated by the following Lemma.

Lemma 1.1 (Figaj and Łuczak [39], 2007, Lemma 8 in [74])**.** *Let a positive real number c and a positive integer k be given. If it is true that for every $\epsilon > 0$ there exists a $\delta > 0$ and an n_0 such that for every even $n > n_0$, each graph G with $v(G) > (1 + \epsilon)cn$ and $e(G) \geq (1 - \delta)\binom{v(G)}{2}$ satisfy the property that for every k-edge-coloring of G there is a monochromatic connected matching $M_{n/2}$, then for large N, $R_k(C_N) \leq (c + o(1))N$ (and hence $R_k(P_N) \leq (c + o(1))N$).*

1.1.1 Monochromatic connected matchings, paths and cycles in 2-edge-colored multipartite graphs

Ramsey-type problems when the host graphs are not complete but complete bipartite were studied by Gyárfás and Lehel [47], Faudree and Schelp [42], DeBiasio, Gyárfás, Krueger, Ruszinkó and Sárközy [55], DeBiasio and Krueger [34], and Bucic, Letzter and Sudakov [24, 25]. When the host graphs are complete 3-partite, it was studied by Gyárfás, Ruszinkó, Sárközy and Szemerédi [54].

The main result of Gyárfás, Ruszinkó, Sárközy and Szemerédi [54] is the following theorem.

Theorem 1.2 (Gyárfás, Ruszinkó, Sárközy and Szemerédi [54], 2007)**.** *For every positive integer n, $K_{n,n,n} \mapsto (P_{2n-o(n)}, P_{2n-o(n)})$.*

The following exact bound was also conjectured:

Conjecture 1.3 (Gyárfás, Ruszinkó, Sárközy and Szemerédi [54], 2007)**.** *For every positive integer n, $K_{n,n,n} \mapsto (P_{2n+1}, P_{2n+1})$.*

We find in Section 2.1 exact bounds on the size of a maximum monochromatic connected matching in each 2-edge-colored complete multipartite graph $K_{n_1,\ldots,n_s}$. This generalizes (using the idea of connected matching in regular partitions), sharpens and extends the corresponding results in [54].

Theorem 1.4 (Balogh, Kostochka, Lavrov, Liu [6], 2020+). *Let $x_1 \geq x_2 \geq 1, s \geq 2$, and let G be a complete s-partite graph $K_{n_1,\ldots,n_s}$ with $N = n_1 + \ldots + n_s$ such that*

$$N \geq 2x_1 + x_2 - 1, \tag{1.1}$$

and

$$N - n_i \geq x_1 + x_2 - 1 \quad \text{for every } 1 \leq i \leq s. \tag{1.2}$$

Let $E(G) = E_1 \cup E_2$ be a partition of the edges of G, and let $G_i = G[E_i]$ for $i = 1, 2$. Then for some i, G_i has a connected matching of at least x_i edges.

We also consider in Section 2.2 necessary restrictions on $n_1 \geq n_2 \geq \ldots \geq n_s \geq 1$ providing that each 2-edge-coloring of $K_{n_1,n_2,\ldots,n_s}$ contains (a) a monochromatic path P_{2n} on $2n$ vertices, (b) a monochromatic path P_{2n+1} on $2n+1$ vertices, (c) a monochromatic cycle C_{2n} on $2n$ vertices or (d) a monochromatic cycle $C_{\geq 2n}$ on at least $2n$ vertices.

A different combination of the following seven conditions will be sufficient for each 2-edge-coloring of $K_{n_1,n_2,\ldots,n_s}$ to contain a monochromatic (a), (b), (c) and (d) respectively when n is sufficiently large.

Let $n_1 \geq \ldots \geq n_s$ and $N = n_1 + \ldots + n_s$. There are examples that demonstrating Conditions 1 and 2 are individually necessary for a monochromatic P_{2n}. It turns out that they are together also sufficient when n is sufficiently large.

Condition 1: $N \geq 3n - 1$.

Condition 2: $N - n_1 \geq 2n - 1$.

Theorem 1.5 (Balogh, Kostochka, Lavrov, Liu [7], 2020). *Let s and n be positive integers with $s \geq 2$ and n sufficiently large. Let $n_1 \geq \ldots \geq n_s$ and N satisfy Conditions 1 and 2. Then for each 2-edge-coloring of the edges of the complete s-partite graph $K_{n_1,\ldots,n_s}$, there exists a monochromatic path P_{2n}.*

To guarantee a monochromatic P_{2n+1}, we need to change Condition 1 to Condition 3, keep Condition 2, and add Condition 4 to deal with the bipartite case.

Condition 3: $N \geq 3n$.

Condition 4: If $n_3 = 0$ then $n_1 \geq 2n + 1$.

Theorem 1.6 (Balogh, Kostochka, Lavrov, Liu [7], 2020). *Let s and n be positive integers with $s \geq 2$ and n sufficiently large. Let $n_1 \geq \ldots \geq n_s$ and N satisfy Conditions 2,3 and 4. Then for each 2-edge-coloring of the edges of the complete s-partite graph $K_{n_1,\ldots,n_s}$, there exists a monochromatic path P_{2n+1}.*

Our result also implies the conjecture of Gyárfás, Ruszinkó, Sárkőzy and Szemerédi [54] for sufficiently large n.

Corollary 1.7. *If n be sufficiently large, then $K_{n,n,n} \mapsto (P_{2n+1}, P_{2n+1})$.*

For a cycle of length at least $2n$, we need Conditions 5 and 6 to handle the 'almost' bipartite case.

Condition 5: If $N - n_1 - n_2 \leq 2,$ then $n_1 \geq 2n - 1$.

Condition 6: If $N - n_1 - n_2 \leq 1,$ then $n_1 + N \geq 6n - 2$.

Theorem 1.8 (Balogh, Kostochka, Lavrov, Liu [7], 2020)**.** *Let s and n be positive integers with $s \geq 2$ and n sufficiently large. Let $n_1 \geq \ldots \geq n_s$ and N satisfy Conditions 1,2,5 and 6. Then for each 2-edge-coloring of the edges of the complete s-partite graph $K_{n_1,\ldots,n_s}$, there exists a monochromatic cycle $C_{\geq 2n}$.*

For a cycle of length exactly $2n$, we need Condition 7 to handle the 'almost' bipartite case.

Condition 7: If $N - n_1 - n_2 \leq 2,$ then $N \geq 4n - 1$.

Theorem 1.9 (Balogh, Kostochka, Lavrov, Liu [7], 2020)**.** *Let s and n be positive integers with $s \geq 2$ and n sufficiently large. Let $n_1 \geq \ldots \geq n_s$ and N satisfy Conditions 1,2 and 7. Then for each 2-edge-coloring of the edges of the complete s-partite graph $K_{n_1,\ldots,n_s}$, there exists a monochromatic cycle C_{2n}.*

Our main strategy to prove Theorem 1.9 (and also the other three theorems) is as follows: We first apply a 2-colored version of the Regularity Lemma to G to obtain a reduced graph G^r. Then, we apply our stability theorem (Theorem 2.11 in Section 2) that either G^r contains a large monochromatic connected matching or the 2-edge-coloring of G^r is restricted and has one of two particular forms (we call them 'bad' partitions). If G^r has a large monochromatic connected matching, then we find a long monochromatic cycle using Lemma 1.1. If G^r does not have a large monochromatic connected matching, then we obtain a bad partition of G^r. We then transfer the bad partition of G^r to a bad partition of G and apply theorems on Hamiltonian cycles to find a monochromatic cycle C_{2n} in G.

1.1.2 Long monochromatic paths and cycles in 2-edge-colored graphs with large minimum degree

There has been a series of papers showing that not only does K_{3n-1} arrow P_{2n}, but also some dense subgraphs of K_{3n-1} arrow P_{2n}. Li, Nikiforov, and Schelp [69], Benevides, Łuczak, Scott, Skokan, and White [11], Schelp [83], and Gyárfás and Sárközy [57] considered 2-edge-colorings of graphs with high minimum degree. In particular, Schelp [83] posed the following conjecture.

Conjecture 1.10 (Schelp [83], 2012)**.** *Suppose that n is large enough and G is a graph on $3n-1$ vertices with minimum degree larger than $3|V(G)|/4$. Then, every 2-edge-coloring of G contains a monochromatic P_{2n}.*

Gyárfás and Sárközy [57] and independently Benevides et al. [11] proved an asymptotic version of this conjecture. In fact, Benevides et al. [11] proved more:

Theorem 1.11 (Benevides, Łuczak, Scott, Skokan, and White [11], 2012)**.** *For every $0 < \delta \leq 1/180$, there exists an integer $n_0 = n_0(\delta)$ such that the following holds. Let G be a graph of order $n > n_0$ with $\delta(G) \geq 3n/4$. For every 2-edge-coloring of G with red graph R_G and blue graph B_G, either G has monochromatic circumference at least $(2/3 + \delta/2)n$ or one of R_G and B_G contains cycles of all lengths $\ell \in [3, (2/3 - \delta)n]$.*

This theorem provides not only monochromatic paths of length close to the one conjectured by Schelp [83], but also long monochromatic cycles. Benevides et al. [11] also conjectured the following.

Conjecture 1.12 (Benevides, Łuczak, Scott, Skokan, and White [11], 2012)**.** *Let G be a graph of order n with $\delta(G) \geq 3n/4$. Let $n = 3t + r$, where $r \in \{0, 1, 2\}$. Every 2-edge-coloring of G has monochromatic circumference at least $2t + r$.*

We prove in Section 2.3 the following theorem, which is tight and implies Conjectures 1.10 and 1.12.

Theorem 1.13 (Balogh, Kostochka, Lavrov, Liu [8], 2020+)**.** *There exists a positive integer n_0 with the following property. Let $n = 3t + r > n_0$, where $r \in \{0, 1, 2\}$. Let G be a graph of order n with $\delta(G) \geq (3n - 1)/4$. Then for every 2-edge-coloring of G, either there are cycles of every length $\{3, 4, 5, ..., 2t + r\}$ of the same color, or there are cycles of every even length $\{4, 6, 8, ..., 2t + 2\}$ of the same color.*

The proof of Theorem 1.13 uses the Szemerédi Regularity Lemma [85], the idea of connected matchings in regular partitions due to Łuczak [73], a stability theorem, and several classical theorems on existence of cycles in graphs.

1.2 Packing colorings of subcubic graphs

For a non-decreasing sequence $S = (s_1, s_2, ..., s_k)$ of positive integers, a *packing S-coloring* of a graph G is a partition of $V(G)$ into sets $V_1, ..., V_k$ such that for each $1 \leq i \leq k$ the distance between any two distinct $x, y \in V_i$ is at least $s_i + 1$. A *packing k-coloring* of a graph G is a packing $(1, 2, \ldots, k)$-coloring. The *packing chromatic number* (PCN), $\chi_p(G)$, of a graph G is the minimum k such that G has a packing k-coloring. The notion of PCN was first introduced by Goddard, Hedetniemi, Hedetniemi, Harris, and Rall [51] in 2008. The concept of packing S-colorings has attracted considerable attention recently: there are around 50 papers on the topic (see e.g. [1, 17, 19, 20, 21, 22, 23, 31, 38, 43, 44, 49, 84] and references in them). Fiala and Golovach [38] proved in 2010 that finding the PCN of a graph is NP-complete even in the class of trees. Sloper [84] showed that there are 4-regular graphs with arbitrarily large PCN and that any tree T with $\Delta(T) \leq 3$ has PCN at most 7.

The following question was first asked by Goddard et al. [51] in 2008 and has drawn the attention of many researchers.

Question 1.14. *Is it true that the PCN of all subcubic graphs is bounded by a constant?*

Gastineau and Togni [49] in 2016 gave a construction of a cubic graph G such that $\chi_p(G) = 13$, and another cubic graph with PCN 14 was recently found by Brešar, Klavžar, Rall, and Wash [21].

The 1-subdivision, $D(G)$, of a graph G is the graph obtained from G by replacing each edge with a path of length two. The following question was first asked by Gastineau and Togni [49] in 2016 and later conjectured by Brešar, Klavžar, Rall, and Wash [22].

Conjecture 1.15 (Brešar, Klavžar, Rall, and Wash [22], 2017)**.** *The PCN of the* 1*-subdivision of a subcubic graph is at most* 5*.*

We prove in Section 3.1 that 'many' cubic graphs have 'high' PCN and thus answer Question 1.14 in the negative.

Theorem 1.16 (Balogh, Kostochka, Liu [3], 2018)**.** *For each fixed integer $k \geq 12$ and $g \geq 2k + 2$, almost every n-vertex cubic graph of girth at least g has PCN greater than k.*

In contrast, we prove in Section 3.3 the first upper bound in the direction of Conjecture 1.15.

Theorem 1.17 (Balogh, Kostochka, Liu [5], 2019)**.** *If G is a subcubic graph, then $\chi_p(D(G)) \leq 8$.*

We also show in Section 3.3 that every subcubic graph is packing $(1, 1, 2, 2, 3, 3, k)$-colorable for every integer $k \geq 4$ via a coloring in which color k is used at most once, and every 2-degenerate subcubic graph is packing $(1, 1, 2, 2, 3, 3)$-colorable.

The following proposition of Gastineau and Togni [49] showed that if one can prove every subcubic graph except the Petersen graph is packing $(1, 1, 2, 2)$-colorable then $\chi_p(D(G)) \leq 5$ for every subcubic graph.

Proposition 1.18 (Gastineau and Togni [49], Proposition 1)**.** *Let G be a graph and $S = (s_1, ..., s_k)$ be a non-decreasing sequence of integers. If G is S-colorable then $D(G)$ is $(1, 2s_1 + 1, \ldots, 2s_k + 1)$-colorable.*

They also asked a stronger question that whether every subcubic graph except the Petersen graph is packing $(1, 1, 2, 3)$-colorable.

The maximum average degree, mad(G), is defined to be $\max\{\frac{2|E(H)|}{|V(H)|} : H \subset G\}$. We prove in Section 3.4 that every subcubic graph with maximum average degree less than $\frac{30}{11}$ is packing $(1, 1, 2, 2)$-colorable and thus confirmed Conjecture 1.15 for subcubic graph G with $mad(G) < \frac{30}{11}$.

Our idea of proving Theorem 1.16 is to first prove upper bounds on the sizes c_i of maximum i-independent sets in almost all cubic n-vertex graphs of large girth and an upper bound $c_{1,2,4}$ on the size of the union of an 1-independent, a 2-independent, and a 4-independent sets for almost all cubic n-vertex graphs of large girth, which is less than $c_1 + c_2 + c_4$. Then we show that for a fixed k and large n, the sum $c_{1,2,4} + c_3 + c_5 + \cdots + c_k < n$ for almost all cubic graphs with large girth, which forbids them to have a bounded PCN.

When proving Theorem 1.16, we also improve upper bound on the independence ratio, c_1/n, of cubic n-vertex graphs of large girth. We show that "almost all" cubic labeled graphs of girth at least 16 have independence ratio at most 0.454.

Let $i(r,g)$ denote the infimum of the ratio $\frac{\alpha(G)}{|V(G)|}$ over the r-regular graphs of girth at least g, where $\alpha(G)$ is the independence number of G, and let $i(r,\infty) = \lim_{g\to\infty} i(r,g)$.

Recently, several new lower bounds on $i(3,\infty)$ were obtained. In particular, Hoppen and Wormald [59] showed in 2015 that $i(3,\infty) \geq 0.4375$, and Csóka [29] improved it to $i(3,\infty) \geq 0.44533$ in 2016. Bollobás [13] proved the upper bound $i(3,\infty) < \frac{6}{13}$ in 1981, and McKay [76] improved it to $i(3,\infty) < 0.45537$ in 1987. There have been no improvements since then. In Section 3.2 we improve the upper bound to $i(3,\infty) \leq 0.454$.

Theorem 1.19 (Balogh, Kostochka, Liu [4], 2019)**.** $i(3,\infty) \leq 0.454$.

1.3 Directed intersection number and the information content of digraphs

Let D be a directed graph with vertex set V and arc set A. A *directed intersection representation* φ with *color set* C is an assignment of a nonempty subset of C to each vertex so that $(u,v) \in A$ if and only if $|\varphi(u) \cap \varphi(v)| \geq 1$ and $|\varphi(u)| < |\varphi(v)|$. We show that such a representation is always possible for directed acyclic graphs (DAGs) and refer to it as a *directed intersection representation.* The question of interest is to determine the smallest cardinality of the ground set $\mathcal{C}$ which allows for a directed intersection representation of a digraph D, henceforth termed the *directed intersection number (DIN)* of D.

The problem of finding directed intersection representations of digraphs is closely associated with the intersection representation problem for undirected graphs. Intersection representations are of interest in many applications such as keyword conflict resolution, traffic phasing, latent feature discovery and competition graph analysis. Formally, the vertices $v \in V$ of a graph $G(V,E)$ are associated with subsets $\varphi(v)$ of a ground set $\mathcal{C}$ so that $uv \in E$ if and only if $|\varphi(u) \cap \varphi(v)| \geq 1$. The intersection number of a graph G is the smallest size of the ground set $\mathcal{C}$ that allows for an intersection representation, and it is well-defined for all graphs. It was proved by Erdős, Goodman and Posa in [37] that finding the intersection number of a graph is equivalent to finding the edge clique cover number, and it was shown by Orlin [78] that determining the edge clique cover number is NP-hard. The intersection number of an undirected graph may differ vastly from the DIN of some of its directed counterparts, whenever the latter exists.

Theorem 1.20 (Liu, Machado, Milenkovic [71], 2020+)**.** *Every DAG D on n vertices admits a directed intersection representation. Moreover, $DIN(n) \leq \frac{5}{8}n^2 - \frac{1}{4}n$.*

An improved upper bound can be obtained using (nonconstructive) inductive arguments.

Theorem 1.21 (Liu, Machado, Milenkovic [71], 2020+). *Let D be an acyclic digraph on n vertices. If n is even, then $DIN(D) \leq \frac{5n^2}{8} - \frac{3n}{4} + 1$.*

We then introduce the notion of *DIN-extremal* DAGs, i.e., DAGs with largest DIN among DAGs with the same number of vertices, and find a constructive lower bound.

Theorem 1.22 (Liu, Machado, Milenkovic [71], 2020+). *There is an n-vertex DAG with DIN exactly*

$$\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

Chapter 2

Monochromatic connected matchings, paths, and cycles in 2-edge-colored graphs

Results in Chapter 2.1, 2.2, and 2.3 are joint work with Balogh, Kostochka, and Lavrov.

Recall that for graphs $G_0, \dots, G_k$ we write $G_0 \mapsto (G_1, \dots, G_k)$ if for every k-coloring of the edges of G_0, for some $i \in [k]$ there will be a copy of G_i with all edges of color i. The *Ramsey number* $R_k(G)$ is the minimum N such that $K_N \mapsto (G_1, \dots, G_k)$, where $G_1 = \dots = G_k = G$.

Gerencsér and Gyárfás [45] proved in 1967 that the n-vertex path P_n satisfies $R_2(P_n) = \left\lfloor \frac{3n-2}{2} \right\rfloor$. They actually proved a stronger result:

Theorem 2.1 ([45])**.** *For any two positive integers $k \geq \ell$, $R(P_k, P_\ell) = k - 1 + \left\lfloor \frac{\ell}{2} \right\rfloor$.*

A lot of progress in bounding $R_k(P_n)$ for $k \geq 3$ and $R_k(C_n)$ for even n was achieved after 2007 (see [11, 27, 34, 35, 41, 39, 40, 55, 56, 64, 73, 74, 82] and some references in them). All these proofs used the Szemerédi Regularity Lemma [85] and the idea of connected matchings in regular partitions due to Łuczak [73].

Recall that a matching M in a graph G is *connected* if all the edges of M are in the same component of G. We will denote a connected matching with n edges by M_n and the path with n vertices by P_n. Also by C_n we denote the cycle with n vertices, and by $C_{\geq n}$ – a cycle of length at least n. A vertex cover X in a graph G is a subset of $V(G)$ such that every edge in $E(G)$ has at least one endpoint in X. A Hamiltonian cycle of a graph G is a cycle that contains all of the vertices in G. The use of connected matchings is illustrated for example by the following version of a lemma by Figaj and Łuczak [39].

Lemma 2.2 (Lemma 8 in [74] and Lemma 1 in [64])**.** *Let a real number Let a positive real number c and a positive integer k be given. If it is true that for every $\epsilon > 0$ there exists a $\delta > 0$ and an n_0 such that for every even $n > n_0$, each graph G with $v(G) > (1+\epsilon)cn$ and $e(G) \geq (1-\delta)\binom{v(G)}{2}$ satisfy the property that for every k-edge-coloring of G there is a monochromatic connected matching $M_{n/2}$, then for large N, $R_k(C_N) \leq (c + o(1))N$ (and hence $R_k(P_N) \leq (c + o(1))N$).*

Ramsey-type problems when the host graphs are complete bipartite graphs were studied by Gyárfás and Lehel [47], Faudree and Schelp [42], DeBiasio, Gyárfás, Krueger, Ruszinkó, and Sárkőzy [55], DeBiasio and Krueger [34], and Bucic, Letzter and Sudakov [24, 25], and when the host graphs are complete 3-partite — by Gyárfás, Ruszinkó, Sárkőzy and Szemerédi [54]. The main result in [42] and [47] was

Theorem 2.3 ([42, 47])**.** *For every positive integer n, $K_{n,n} \mapsto (P_{2\lceil n/2\rceil}, P_{2\lceil n/2\rceil})$. Furthermore, $K_{n,n} \not\mapsto (P_{2\lceil n/2\rceil+1}, P_{2\lceil n/2\rceil+1})$.*

DeBiasio and Krueger [34] extended the result from paths $P_{2\lceil n/2\rceil}$ to cycles of length at least $2\lfloor n/2\rfloor$ for large n.

The main result in [54] was

Theorem 2.4 ([54])**.** *For every positive integer n, $K_{n,n,n} \mapsto (P_{2n-o(n)}, P_{2n-o(n)})$.*

The following exact bound was also conjectured:

Conjecture 2.5 ([54])**.** *For every positive integer n, $K_{n,n,n} \mapsto (P_{2n+1}, P_{2n+1})$.*

Since the papers [54, 24, 25] were proving asymptotic bounds, they used approximate bounds on maximum sizes of monochromatic connected matchings in edge-colored dense multipartite graphs. But for the exact bound [55, 56] (for large N) on long paths in 3-edge-colored K_N and for the exact bound by DeBiasio and Krueger [34] on long paths and cycles in 2-edge-colored bipartite graphs, one needs a stability theorem: *either the edge-colored graph has a large monochromatic connected matching, or the edge-coloring is very special.*

2.1 Monochromatic connected matchings in 2-edge-colored multipartite graphs

In Section 2.1, we find exact bounds on the size of a maximum monochromatic connected matching in each 2-edge-colored complete multipartite graph $K_{n_1,\ldots,n_k}$. This generalizes, sharpens and extends the corresponding results in [54] and can be considered as an extension of one of the results in [34]. We also prove a corresponding stability theorem in the spirit of [55] and [34]. In the follow-up section (Section 2.2) we use this stability theorem to prove among other results that for large n, Conjecture 2.5 and the relation $K_{n,n,n} \mapsto (C_{2n}, C_{2n})$ hold.

2.1.1 Notation and results

Let $\alpha'(G)$ denote the size of a largest matching in G and $\alpha'_*(G)$ denote the size of a largest connected matching in G. Let $\alpha(G)$ denote the independence number and $\beta(G)$ denote the size of a smallest vertex cover in G.

For a graph G and $W_1, W_2 \subseteq V(G)$, let $G[W_1, W_2]$ denote the subgraph of G consisting of edges with one endpoint in W_1 and the other endpoint in W_2.

We seek minimal restrictions on $n_1 \geq n_2 \geq \ldots \geq n_s$ guaranteeing that every 2-edge-coloring of $K_{n_1,n_2,\ldots,n_s}$ contains a monochromatic M_n. Let $N = n_1 + \ldots + n_s$. An obvious necessary condition is that

$$N \geq 3n - 1. \tag{2.1}$$

Indeed, even $K_{3n-2} \not\mapsto (M_n, M_n)$: for $G = K_{3n-2}$, partition $V(G)$ into sets U_1 and U_2 with $|U_1| = 2n-1$, $|U_2| = n-1$, and color the edges of $G[U_1, U_2]$ with red and the rest of the edges with blue. Then there is no monochromatic M_n; see Figure 2.1. The other natural requirement is that

$$N - n_1 = n_2 + \ldots + n_s \geq 2n - 1. \tag{2.2}$$

Indeed, for arbitrarily large n_1 and $N = n_1 + 2n - 2$, consider the graph H obtained from K_N by deleting the edges inside a vertex subset U_1 with $|U_1| = n_1$. Graph H contains every $K_{n_1,n_2,\ldots,n_s}$ with $n_2 + \ldots + n_s = 2n-2$. Partition $V(H) - U_1$ into sets U_2 and U_3 with $|U_2| = |U_3| = n-1$. Color all edges incident with U_2 red, and the remaining edges of H blue. Again, there is no monochromatic M_n; see Figure 2.2.

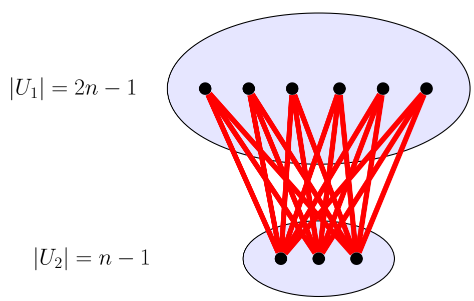


Figure 2.1: Example for condition (2.1).

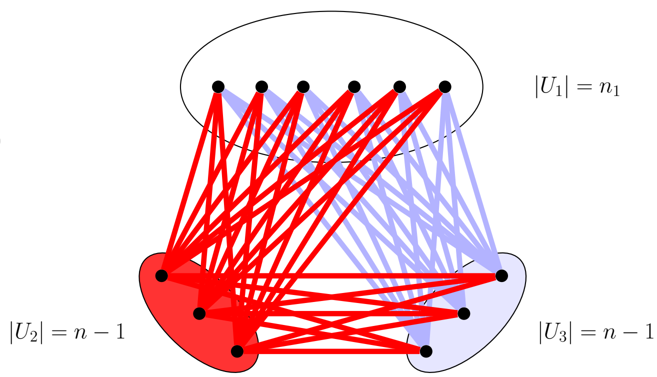


Figure 2.2: Example for condition (2.2).

Our first main result is that the necessary conditions (2.1) and (2.2) together are sufficient for $K_{n_1,n_2,\ldots,n_s} \mapsto (M_n, M_n)$. We prove it in the following more general form.

Theorem 2.6. *Let $x_1 \geq x_2 \geq 1, s \geq 2$, and let G be a complete s-partite graph $K_{n_1,\ldots,n_s}$ with $N = n_1 + \ldots + n_s$ such that*

$$N \geq 2x_1 + x_2 - 1, \tag{2.3}$$

and

$$N - n_i \geq x_1 + x_2 - 1 \quad \textit{for every } 1 \leq i \leq s. \tag{2.4}$$

Let $E(G) = E_1 \cup E_2$ be a partition of the edges of G, and let $G_i = G[E_i]$ for $i = 1, 2$. Then for some i, $\alpha'_*(G_i) \geq x_i$.

There are at least two types of 3-edge-colorings of K_{4n-3} with no monochromatic M_n. We use Theorem 2.6 to show the following generalization of the existence of a monochromatic connected matching M_n in each 3-edge-coloring of K_{4n-2}.

Theorem 2.7. *Let $1 \leq x_2, x_3 \leq x_1$, $N = 2x_1 + x_2 + x_3 - 2$, and $G = K_N$.*

Let $E(G) = E_1 \cup E_2 \cup E_3$ be a partition of the edges of G, and let $G_i = G[E_i]$ for $i = 1, 2, 3$. Then for some i, $\alpha'_*(G_i) \geq x_i$.

Finally, for the case $x_1 = x_2 = n$ of Theorem 2.6, we prove a stability result which will be used in [6] to prove Conjecture 2.5 for large N. This will require a few definitions to state.

Definition 2.8. *For $\epsilon > 0$ and $s \geq 2$, an N-vertex s-partite graph G with parts $V_1, \ldots, V_s$ of sizes $n_1 \geq n_2 \geq \ldots \geq n_s$, and a 2-edge-coloring $E = E_1 \cup E_2$, is (n, s, ϵ)-*suitable *if the following conditions hold:*

$$N = n_1 + \ldots + n_s \geq 3n - 1, \tag{2.5}$$

$$n_2 + n_3 + \ldots + n_s \geq 2n - 1, \tag{2.6}$$

and if $\widetilde{V}_i$ is the set of vertices in V_i of degree at most $N - \epsilon n - n_i$ and $\widetilde{V} = \bigcup_{i=1}^{s} \widetilde{V}_i$, then

$$|\widetilde{V}| = |\widetilde{V}_1| + \ldots + |\widetilde{V}_s| < \epsilon n. \tag{2.7}$$

We do not require $E_1 \cap E_2 = \emptyset$; an edge can have one or both colors. We write $G_i = G[E_i]$ for $i = 1, 2$.

Our stability result gives a partition of the vertices of near-extremal graphs called a (λ, i, j)-*bad partition*. There are two types of bad partitions.

Definition 2.9. *For $i \in \{1, 2\}$ and $\lambda > 0$, a partition $V(G) = W_1 \cup W_2$ of $V(G)$ is $(\lambda, i, 1)$-*bad *if the following holds:*

(i) $(1 - \lambda)n \leq |W_2| \leq (1 + \lambda)n_1$;

(ii) $|E(G_i[W_1, W_2])| \leq \lambda n^2$;

(iii) $|E(G_{3-i}[W_1])| \leq \lambda n^2$.

Definition 2.10. *For $i \in \{1, 2\}$ and $\lambda > 0$, a partition $V(G) = V_j \cup U_1 \cup U_2$, $j \in [s]$, of $V(G)$ is $(\lambda, i, 2)$-bad if the following holds:*

(i) $|E(G_i[V_j, U_1])| \leq \lambda n^2$;

(ii) $|E(G_{3-i}[V_j, U_2])| \leq \lambda n^2$;

(iii) $n_j = |V_j| \geq (1 - \lambda)n$;

(iv) $(1 - \lambda)n \leq |U_1| \leq (1 + \lambda)n$;

(v) $(1 - \lambda)n \leq |U_2| \leq (1 + \lambda)n$.

Our stability theorem is:

Theorem 2.11. *Let $n \geq s \geq 2$, $0 < \epsilon < 10^{-3}\gamma < 10^{-6}$ and $n > 100/\gamma$. Let G be an (n, s, ϵ)-suitable graph. If $\max\{\alpha'_*(G_1), \alpha'_*(G_2)\} \leq n(1 + \gamma)$, then for some $i \in [2]$ and $j \in [2]$, $V(G)$ has a $(68\gamma, i, j)$-bad partition.*

In Section 2.1.2, we remind the reader the notion and properties of the Gallai–Edmonds decomposition, and in each of the next three subsections (Section 2.1.3, Section 2.1.4, and Section 2.1.5) we prove one of the Theorems 2.6, 2.7 and 2.11.

2.1.2 Tools from graph theory

We make extensive use of the Gallai–Edmonds decomposition (called below *the GE-decomposition* for short) of a graph G, defined below.

Definition 2.12. *In a graph G, let B be the set of vertices that are covered by every maximum matching in G. Let A be the set of vertices in B having at least one neighbor outside B, let $C = B - A$, and let $D = V(G) - B$. The* GE-decomposition *of G is the partition of $V(G)$ into the three sets A, C, D.*

Edmonds and Gallai described important properties of this decomposition:

Theorem 2.13 (Gallai–Edmonds Theorem; Theorem 3.2.1 in [72]). *Let A, C, D be the GE-decomposition of a graph G. Let $G_1, \ldots, G_k$ be the components of $G[D]$. If M is a maximum matching in G, then the following properties hold:*

(a) M covers C and matches A into distinct components of $G[D]$.

(b) Each G_i is factor-critical and has a near-perfect matching in M.

(c) If $\emptyset \neq S \subseteq A$, then $N(S)$ intersects at least $|S| + 1$ of $G_1, \ldots, G_k$.

For bipartite graphs, we use the simpler König–Egerváry theorem, which we apply in two equivalent forms:

Theorem 2.14 (König–Egerváry Theorem; Theorem 1.1.1 in [72]). *In a bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.*

Equivalently, if H is a bipartite graph with bipartition (U, V), then

$$\alpha'(H) = \min_{U_1 \subseteq U} \{|U| - |U_1| + |N(U_1)|\}.$$

Finally, we also will use the following theorem on Hamiltonian cycles.

Theorem 2.15 (Las Vergnas [68], see also Theorem 11 on p. 214 in [12]). *Let H be a $2n$-vertex bipartite graph with vertices $u_1, u_2, \ldots, u_n$ on one side and $v_1, v_2, \ldots, v_n$ on the other, such that $d(u_1) \leq \ldots \leq d(u_n)$ and $d(v_1) \leq \ldots \leq d(v_n)$. Let q be an integer, $0 \leq q \leq n - 1$.*

If, whenever $u_iv_j \notin E(H)$, $d(u_i) \le i+q$, and $d(v_j) \le j+q$, we have

$$d(u_i) + d(v_j) \ge n + q + 1,$$

then each set of q edges that form vertex-disjoint paths is contained in a Hamiltonian cycle of G.

2.1.3 Connected matchings in 2-edge-colorings (Theorem 2.6)

Let G be a complete s-partite graph $K_{n_1,\dots,n_s}$ satisfying (2.3) and (2.4). Let $V_1,\dots,V_s$ be the parts of G with $|V_i| = n_i$ for $i = 1,\dots,s$.

We proceed by contradiction, assuming that there is a partition $E(G) = E_1 \cup E_2$ such that

$$\alpha'_*(G_1) < x_1 \text{ and } \alpha'_*(G_2) < x_2. \tag{2.8}$$

Among such edge partitions, we will find partitions with additional restrictions and study their properties. Eventually we will prove that such partitions do not exist.

2.1.3.1 Structure of G

Among all G and partitions $E(G) = E_1 \cup E_2$ satisfying (2.3), (2.4) and (2.8), choose one with the smallest N.

Claim 2.16. *If $n_1 \ge n_2 \ge \ldots \ge n_s$, then either $N = 2x_1+x_2-1$ or $n_1 = n_2$ and $N \le 2x_1+2x_2-s$.*

Proof. Suppose $N > 2x_1+x_2-1$ and $v \in V_1$. Let $G' = G-v$. Then (2.3) and (2.8) hold for G'. Hence by the minimality of G, (2.4) does not hold for G'. Since (2.4) does hold for G, we conclude that $n_1 = n_2$ and $N - n_1 = x_1+x_2-1$. The last equality implies that $n_2 = (x_1+x_2-1) - n_3 - \ldots - n_s \le x_1 + x_2 + 1 - s$. Hence

$$N = n_1 + (N - n_1) = n_2 + (x_1 + x_2 - 1) \le 2x_1 + 2x_2 - s,$$

as claimed. □

Claim 2.17. *G is not bipartite; that is, $s \ge 3$.*

Proof. Suppose $s = 2$. Then by (2.4), $n_1 = N - n_2 \ge x_1 + x_2 - 1$ and $n_2 = N - n_1 \ge x_1 + x_2 - 1$. It is sufficient to consider the situation that $n_1 = n_2 = x_1 + x_2 - 1$.

Suppose that for some $i \in \{1,2\}$, $\alpha'(G_i) = \alpha'_*(G_i)$ (and so by (2.8), $\alpha'(G_i) < x_i$). By Theorem 2.14, G_i has a vertex cover C with $|C| \le x_i - 1$. Hence all edges of G connecting $V_1 - C$ with $V_2 - C$ are in E_{3-i}. Thus G_{3-i} contains $K_{x_1+x_2-1-|C|,x_1+x_2-1-|C|}$, which in turn contains $K_{x_{3-i},x_{3-i}}$. Therefore $\alpha'_*(G_{3-i}) \ge x_{3-i}$, contradicting (2.8).

Therefore $\alpha'(G_i) > \alpha'_*(G_i)$ for both $i \in \{1,2\}$. This means that each of G_1 and G_2 has more than one nontrivial component. Let A be the vertex set of one nontrivial component in G_2 and $B = (V_1 \cup V_2) - A$. For each $i \in \{1,2\}$, let $A_i = V_i \cap A$, $B_i = V_i \cap B$, $a_i = |A_i|$, and $b_i = |B_i|$.

Then for both $i \in \{1,2\}$, $G_1[A_i \cup B_{3-i}] = K_{a_i,b_{3-i}}$. So if there is at least one edge connecting A_1 with A_2 or B_1 with B_2 in G_1, then G_1 is connected and so $\alpha'_*(G_1) = \alpha'(G_1)$, a contradiction. Thus, $G_2[A_1 \cup A_2] = K_{a_1,a_2}$ and $G_2[B_1 \cup B_2] = K_{b_1,b_2}$.

This means that $\min\{a_1,a_2\} < x_2$ and $\min\{b_1,b_2\} < x_2$. By the symmetry between a_1 and a_2, we may assume $a_1 < x_2$. Then $b_1 = (x_1 + x_2 - 1) - a_1 \geq x_1 \geq x_2$. Hence $b_2 < x_2$, and $a_2 = (x_1 + x_2 - 1) - b_2 \geq x_1$. But G_1 contains K_{b_1,a_2}, so it contains K_{x_1,x_1}, a contradiction to (2.8). $\square$

2.1.3.2 Components of G_i

Next, by analyzing the components of G_1 and G_2, we will reduce the problem to a case where G_1 and G_2 have no nontrivial components. Then it will be enough to find a large matching in either G_1 or G_2; the matching will automatically be connected, which will contradict assumption (2.8).

Claim 2.18. *For any $i \in \{1,2\}$, if G_i is disconnected, then $\alpha'_*(G_{3-i}) = \alpha'(G_{3-i})$.*

Proof. Suppose G_1 is disconnected. Let W_1 induce a component of G_1 and $W_2 = V(G) - W_1$. We consider three cases:

Case 1. For some $j \in [s]$, $W_1 \subseteq V_j$. Since V_j is independent, $W_1 = \{v\}$ for some $v \in V_j$. Then all vertices in $V(G_2) - V_j$ are adjacent to v. So, G_2 has a component D containing $V(G_2) - V_j + v$. Since V_j is independent, every edge in G_2 has a vertex in $V(G) - V_j$, and hence lies in D.

Case 2. For some distinct $j_1, j_2 \in [s]$, $W_1 \subseteq V_{j_1} \cup V_{j_2}$ and has a vertex $v_1 \in V_{j_1}$ and a vertex $v_2 \in V_{j_2}$. By Claim 2.17, $V(G) - V_{j_1} - V_{j_2} \neq \emptyset$, and by the case, each vertex in $V(G) - V_{j_1} - V_{j_2}$ is adjacent in G_2 to both, v_1 and v_2. Thus, a component D of G_2 contains $W_1 \cup (V(G) - V_{j_1} - V_{j_2})$. Furthermore, each vertex in $V_{j_1} - W_1$ is adjacent in G_2 to v_2, and each vertex in $V_{j_2} - W_2$ is adjacent in G_2 to v_1. It follows that G_2 is connected.

Case 3. For some distinct $j_1, j_2, j_3 \in [s]$, W_1 has a vertex $v_\ell \in V_{j_\ell}$ for all $\ell \in [3]$. Then each vertex in W_2 is adjacent in G_2 to at least two of v_1, v_2 and v_3. Thus, a component D of G_2 contains W_2. If each $v \in W_1$ has in G_2 a neighbor in W_2, then $D = V(G)$, i.e. G_2 is connected. Suppose there is $v \in W_1$ that has no neighbors in W_2 in G_2. We may assume $v \in V_{j_1}$. Then $W_2 \subset V_{j_1}$. This means all vertices in $V(G) - D$ are in V_{j_1}. Since V_{j_1} is independent, every edge in G_2 has a vertex in $V(G) - V_{j_1}$, and hence lies in D. $\square$

Claim 2.18 implies that $\alpha'_*(G_i) = \alpha'(G_i)$ holds for at least one i. This equality does not necessarily hold for both $i = 1$ and $i = 2$, but we show that it is enough to prove Theorem 2.6 in the case where it does.

Claim 2.19. *If there are partitions $E(G) = E_1 \cup E_2$ of $E(G)$ such that $G_1 = G[E_1]$ and $G_2 = G[E_2]$ satisfy* (2.8)*, then there is one satisfying all of the following:*

- $\alpha'_*(G_1) = \alpha'(G_1)$ *and* $\alpha'_*(G_2) = \alpha'(G_2)$*;*
- G_1 *has the GE-decomposition* (A, C, D) *such that if* $D_0 = C$ *and* $D_1, D_2, \dots, D_k$ *are the components of* $G_1[D]$ *with* $|D_1| \geq |D_2| \geq \dots \geq |D_k|$*, then* $G_1 - A$ *has at least three components, and* $G_2[D_j]$ *is empty for* $j = 0, 1, \dots, k$*.*

Proof. Suppose that $E(G) = E_1 \cup E_2$ is a partition of $E(G)$ such that $G_1 = G[E_1]$ and $G_2 = G[E_2]$ satisfy (2.8).

By Claim 2.18, there is some $i \in \{1, 2\}$ such that $\alpha'_*(G_i) = \alpha'(G_i)$. Pick such an i.

Let (A, C, D) be the GE-decomposition of G_i; let $D_0 = C$, $a = |A|$, and let $D_1, D_2, \dots, D_k$ be the components of $G_i[D]$.

We have $N = |V(G)| = |V(G_i)| \geq 2x_1 + x_2 - 1 \geq 2x_i$, and yet by assumption (2.8), $\alpha'(G_i) < x_i$. Therefore every maximum matching in G_i leaves at least two vertices uncovered; by Theorem 2.13, this means $k \geq 2$, since the number of uncovered vertices is $k - a$.

We want to show that $G_i - A$ actually has at least 3 components. Since $k \geq 2$, D_1 and D_2 are two of them. If $C = D_0 \neq \emptyset$, then it is a third component of $G_i - A$; if $A \neq \emptyset$, then $k \geq a + 2 \geq 3$. If $A = C = \emptyset$ and $k = 2$, then D_1 and D_2 are components of G_i as well. By assumption, $\alpha'_*(G_i) = \alpha'(G_i)$, so D_1 and D_2 cannot both be nontrivial components.

This leaves the possibility that D_2 is an isolated vertex of G_i and D_1 is the rest of $V(G)$, which we must also rule out. In this case, by Theorem 2.13, a maximum matching in G_i covers all vertices of D_1 except for one of them; we have

$$\alpha'_*(G_i) = \frac{N}{2} - 1 \geq \frac{2x_1 + x_2 - 1}{2} - 1 \geq x_i + \frac{x_{3-i} - 3}{2}.$$

But by assumption (2.8), $\alpha'_*(G_i) \leq x_i - 1$, which means $\frac{x_{3-i}-3}{2} \leq -1$, or $x_{3-i} \leq 1$. By (2.4), the degree of the single vertex in D_2 is at least $N - n_1 \geq x_1 + x_2 - 1 \geq 1$, and it is isolated in G_i; therefore $\alpha'_*(G_{3-i}) \geq 1 \geq x_{3-i}$, violating assumption (2.8). Therefore $G_i - A$ has at least three components.

Let Q be the set of edges in G_{3-i} that are either incident to A or else have both ends in the same D_i (including D_0). Modify the partition $E_1 \cup E_2$ by removing all edges of Q from E_{3-i} and adding them to E_i instead; let $E'_1 \cup E'_2$ be the resulting partition, with $G'_1 = G[E'_1]$ and $G'_2 = G[E'_2]$. The same GE-decomposition (A, C, D) witnesses that $\alpha'(G'_i) = \alpha'(G_i) = \alpha'_*(G_i) < x_i$; meanwhile, G'_{3-i} is a subgraph of G_{3-i}, so $\alpha'_*(G'_{3-i}) \leq \alpha'(G_{3-i}) < x_{3-i}$. Therefore the resulting partition still satisfies (2.8).

Next, we show that G'_{3-i} has at most one nontrivial component: equivalently, that $\alpha'_*(G_{3-i}) = \alpha'(G_{3-i})$. Suppose for the sake of contradiction that G'_{3-i} has at least two nontrivial components, say H_1 and H_2. Let $u_1u_2 \in E(H_1)$ and $v_1v_2 \in E(H_2)$.

We may rename the parts of G so that $u_1 \in V_1$ and $u_2 \in V_2$. Suppose $u_1 \in D_j$ and $u_2 \in D_{j'}$. By the definition of Q, $j' \neq j$. So, if $v_1 \notin V_1 \cup V_2$ or $v_1 \notin D_j \cup D_{j'}$, then $v_1u_1 \in E(G'_{3-i})$ or $v_1u_2 \in E(G'_{3-i})$, and hence $H_2 = H_1$. The same holds for v_2. Thus, since $v_1v_2 \in E(G'_{3-i})$, we may assume that $v_1 \in V_1 \cap D_{j'}$ and $v_2 \in V_2 \cap D_j$. We proved earlier that $G_i - A$ has at least three components; therefore we can choose $D_{j''} \neq D_j, D_{j'}$ with a vertex $w \in D_{j''}$. By the symmetry between V_1 and V_2, we may assume $w \notin V_1$. Then w is adjacent in G'_{3-i} with both u_1 and v_1, a contradiction.

The resulting partition $E'_1 \cup E'_2$ satisfies $\alpha'_*(G_1) = \alpha'(G_1)$ and $\alpha'_*(G_2) = \alpha'(G_2)$. The second condition of Claim 2.19 also holds if we had $i = 1$ in the proof above. If we had $i = 2$, then we may repeat this procedure with $i = 1$, finding a third partition $E''_1 \cup E''_2$. This still satisfies $\alpha'_*(G_1) = \alpha'(G_1)$ and $\alpha'_*(G_2) = \alpha'(G_2)$, but now the Gallai–Edmonds partition of G_1 has the properties we want, proving the claim. $\square$

2.1.3.3 Completing the proof of Theorem 2.6

From now on, we assume that the partition $E_1 \cup E_2$ satisfies the conditions guaranteed by Claim 2.19. Let (A, C, D) and $D_0, D_1, \ldots, D_k$ be as defined in the statement of Claim 2.19; let $a = |A|$.

Assumption (2.8) implies that $\alpha'(G_1) < x_1$ and $\alpha'(G_2) < x_2$. The following claim allows us to gradually grow a connected matching R.

Claim 2.20. *Let R be a matching in $G_2 - A$. Assume that $I \neq \emptyset$ is a set of isolated vertices in $G_1 - A$, with $I \cap V(R) = \emptyset$ and $A \cup I \cup V(R) \neq V(G)$. Suppose that R cannot be made larger by either of the following operations:*

- *Adding an edge of G_2 which has one endpoint in I and the other outside $A \cup I \cup V(R)$.*
- *Replacing an edge $e \in R$ with two edges $e', e'' \in E(G_2 - A)$ such that $e \subset e' \cup e''$ and $e' \cup e''$ has one vertex in I and one in $V(G) - A - R - I$.*

Then G violates assumption (2.8).

Proof. Let u be a vertex of G outside $A \cup I \cup V(R)$ and let $v \in I$. Since v is an isolated vertex in $G_1 - A$, uv cannot be an edge of G_1; by the maximality of R, uv cannot be an edge of G_2. Therefore there is some part V_i of G containing both u and v.

Next, we show that every edge of R has one endpoint in V_i. Suppose not; let $w_1w_2 \in R$ be an edge with $w_1, w_2 \notin V_i$. Note that uw_1, uw_2, vw_1, vw_2 are all edges of G. Since $w_1w_2 \in E_2$, w_1 and w_2 cannot be in the same component of $G_1 - A$. Therefore uw_1, uw_2 cannot both be in E_1; without loss of generality, $uw_1 \in E_2$. Since v is isolated in $G_1 - A$, the edge $w_1w_2 \in R$ can be replaced by the edges $uw_1, vw_2 \in E_2$, violating the maximality of R.

By (2.4), v has at least $x_1 + x_2 - 1$ neighbors in G, so it has at least $(x_1 + x_2 - 1) - a$ neighbors in $G - A$. Since v is an isolated vertex in $G_1 - A$, these are all neighbors of v in G_2; by the maximality of R, they all are in R, and by the argument in the previous paragraph, they are all in different edges of R.

Therefore $|R| \geq (x_1 + x_2 - 1) - a$. If $|R| \geq x_2$, then $\alpha'(G_2) \geq x_2$. By Claim 2.19, this violates assumption (2.8). If not, then $(x_1 + x_2 - 1) - a \leq x_2 - 1$, so $a \geq x_1$. By Theorem 2.13, there is a matching in G_1 saturating A; therefore $\alpha'(G_1) \geq x_1$, again violating assumption (2.8) by Claim 2.19. □

We consider two cases; in each, we construct the pair (I, R) of Claim 2.20 and arrive at a contradiction.

Case 1. $G_2 - A$ has no matching that covers all vertices which are not isolated in $G_1 - A$.

In this case, let $D_1, D_2, \dots, D_r$ be the components of $G_1[D]$ with at least 3 vertices. For each of these components, we pick a leaf vertex u_i of a spanning tree of $G_1[D_i]$. Since $G_1[D_i] - u_i$ is still connected, there is an edge $e_i \in G_1[D_i]$. At least one endpoint of e_i is a vertex v_i not in the same part of G as u_{i+1}, and is therefore adjacent to u_{i+1} in G_2.

To begin, let R_0 be the set of the $r - 1$ edges $u_{i+1}v_i$ found in this way, when $r > 0$, and the empty set otherwise. If I_0 is the set of all isolated vertices in $G_1[D]$, then $|I_0| = k - r$, and therefore $|I_0| + |R_0| \geq k - 1$.

Now build I and R by the following procedure. Start with $I = I_0$ and $R = R_0$. Whenever an edge (in G_2) connects I to $V(G) - (A \cup I \cup V(R))$, add it to R and remove its endpoint from I. Whenever we can replace an edge $e \in R$ with two other edges e', e'' such that $e \subset e' \cup e''$ and $e' \cup e''$ has exactly one vertex in I, do so, and remove from I the vertex contained in $e' \cup e''$. Once this process is complete, R satisfies the maximality conditions of Claim 2.20.

In this process, $|I| + |R|$ never changes. Therefore $|I| + |R| \geq k - 1$ at the end of this procedure.

By assumption (2.8), $|R| \leq \alpha'(G_2) \leq x_2 - 1$; therefore $|I| \geq k - 1 - |R| \geq k - x_2$.

Theorem 2.13 guarantees that $\alpha'(G_1) = \frac{N-(k-a)}{2} \geq \frac{N-k}{2}$. By assumption (2.8), $\alpha'(G_1) \leq x_1 - 1$, so we have

$$x_1 - 1 \geq \frac{N - k}{2} \geq \frac{(2x_1 + x_2 - 1) - k}{2} \implies 2x_1 - 2 \geq 2x_1 + x_2 - k - 1 \implies k - x_2 \geq 1.$$

Therefore $|I| \geq k - x_2 \geq 1$, so I is nonempty.

Moreover, $A \cup I \cup V(R) \neq V(G)$, since by the assumption in the case R cannot cover all the non-isolated vertices of $G_1 - A$. Therefore Claim 2.20 applies to the pair (I, R), contradicting assumption (2.8).

Case 2. $G_2 - A$ has a matching that covers all vertices which are not isolated in $G_1 - A$.

In this case, let R_0 be such a matching, and let R be a maximal matching in $G_2 - A$ that covers all vertices of $V(R_0)$. Let $I_0 = V(G) - V(R) - A$.

By assumption (2.8), $|V(R)| \leq 2\alpha'(G_2) \leq 2(x_2 - 1)$, so $|I_0| \geq N - 2(x_2 - 1) - a$. By (2.3),

$$|I_0| \geq (2x_1 + x_2 - 1) - 2(x_2 - 1) - a = (x_1 - a) + (x_1 - x_2) + 1 \geq x_1 - a + 1.$$

By Theorem 2.13, there is a matching in G_1 saturating A; therefore $a \leq \alpha'(G_1) \leq x_1 - 1$, and $x_1 - a \geq 1$. Therefore $|I_0| \geq 2$.

Choose any $u \in I_0$ and let $I = I_0 - \{u\}$. Then Claim 2.20 applies to the pair (I, R), with the maximality conditions holding because R is a maximum matching; once again, this contradicts assumption (2.8). $\square$

2.1.4 Connected matchings in 3-edge-colorings (Theorem 2.7)

2.1.4.1 Components of G_i

To prove Theorem 2.7, we begin by proving bounds on the sizes of components in G_2 and G_3. This is done by applying Theorem 2.6 to an appropriate subgraph of G.

Claim 2.21. *If there is an $i \in \{2, 3\}$ such that G_i has no component of size larger than $x_1 + x_i - 1$, then the conclusion of Theorem 2.7 holds.*

Proof. Without loss of generality, say $i = 3$. For each component of G_3, delete all edges in G between vertices of that component to create a graph G'. This graph has a 2-edge-coloring given by G_1 and G_2. It satisfies Condition (2.3) of Theorem 2.6 automatically, since $N \geq 2x_1 + x_2 - 1$. Also, no part is larger than $x_1 + x_3 - 1$, so

$$N - n_i \geq (2x_1 + x_2 + x_3 - 2) - (x_1 + x_3 - 1) = x_1 + x_2 - 1$$

and G' satisfies Condition (2.4). By Theorem 2.6, we have $\alpha'_*(G_i) \geq x_i$ for some $i \in \{1, 2\}$. $\square$

From now on, we assume that for each $i \in \{2, 3\}$, there is a component in color i on vertex set $S_i \subseteq V(G)$, with $|S_i| \geq x_1 + x_i$.

However, neither S_2 nor S_3 can be too large.

Claim 2.22. *If there is an $i \in \{2, 3\}$ such that $|S_i| \geq x_1 + x_2 + x_3 - 2$, then the conclusion of Theorem 2.7 holds.*

Proof. Without loss of generality, say $i = 3$. Let $B = V(G) - S_3$. If $G_3[S_3]$ contains a matching of size x_3, then we are done. If not, take the GE-decomposition (A, C, D) of $G_3[S_3]$.

We build a multipartite graph G', with the inherited 2-edge-coloring by

1. deleting the vertices of A from G, and
2. for each component of $G_3[V(G) - A]$, deleting all edges of G inside that component.

We must have $|A| \leq x_3 - 1$ because, by Theorem 2.13, a maximum matching in $G_3[S_3]$ matches each vertex of A to a vertex outside A. So G' contains at least $2x_1 + x_2 + x_3 - 2 - (x_3 - 1) = 2x_1 + x_2 - 1$ vertices, satisfying Condition (2.3) of Theorem 2.6.

If $C_1, \ldots, C_k$ are the components of $G_3[S_3 - A]$, then for each C_i we have $|A| + |C_i| \leq 2x_3 - 1$ because, by Theorem 2.13, $G_3[S_3]$ has a maximum matching that saturates the vertices in $A \cup C_i$. Therefore $G' - C_i$ contains at least

$$2x_1 + x_2 + x_3 - 2 - (2x_3 - 1) = 2x_1 + x_2 - x_3 - 1 \geq x_1 + x_2 - 1$$

vertices.

This verifies Condition (2.4) of Theorem 2.6 for the parts of G' that are contained in S_3. It remains to check this condition for parts of G' that are contained in B. Since all the vertices of $S_3 - A$ are vertices of G' outside such a part, the number of such vertices is at least

$$|S_3| - |A| \geq (x_1 + x_2 + x_3 - 2) - (x_3 - 1) = x_1 + x_2 - 1.$$

So Theorem 2.6 applies to G'. Therefore, for some $i \in \{1, 2\}$, $\alpha'_*(G_i) \geq \alpha'_*(G'_i) \geq x_i$, and the conclusion of Theorem 2.7 holds. □

2.1.4.2 Completing the proof of Theorem 2.7

From now on, we assume that the hypothesis of Claim 2.22 does not hold. Let $\overline{S_i} = V(G) - S_i$; our assumption implies that $|\overline{S_i}| \geq x_1 + 1$ for both $i \in \{2, 3\}$. We can use this to obtain a decomposition of $V(G)$ in which we know the colors of many edges.

Claim 2.23. *Theorem 2.7 holds unless there is a decomposition $V(G) = Z_0 \cup Z_1 \cup Z_2 \cup Z_3$ such that:*

- *All edges of $G[Z_0, Z_1]$ and $G[Z_2, Z_3]$ are in E_1.*
- *All edges of $G[Z_0, Z_2]$ and $G[Z_1, Z_3]$ are in E_2.*
- *All edges of $G[Z_0, Z_3]$ and $G[Z_1, Z_2]$ are in E_3.*

Proof. Define the parts as follows: $Z_0 = S_2 \cap S_3$, $Z_1 = \overline{S_2} \cap \overline{S_3}$, $Z_2 = S_2 \cap \overline{S_3}$, and $Z_3 = \overline{S_2} \cap S_3$.

Because S_2 and S_3 induce components in G_2 and G_3 respectively, the edges out of S_2 cannot be in E_2, and the edges out of S_3 cannot be in E_3. In particular, this implies that all edges in $G[Z_0, Z_1]$

and $G[Z_2, Z_3]$ are in E_1. The union of the complete bipartite graphs $G[Z_0, Z_1]$ and $G[Z_2, Z_3]$ is a subgraph of G_1. A vertex cover of this bipartite graph has to include either the entire Z_0 or the entire Z_1, and it has to include either the entire Z_2 or the entire Z_3. This means a vertex cover contains one of $Z_0 \cup Z_2 = S_2$, or $Z_0 \cup Z_3 = S_3$, or $Z_1 \cup Z_2 = \overline{S_3}$, or $Z_1 \cup Z_3 = \overline{S_2}$. Each of them has size at least $x_1 + 1$ by Claims 2.21 and 2.22.

So this bipartite graph has minimum vertex cover of order at least x_1+1; by Theorem 2.14 theorem, its maximum matching has size at least $x_1 + 1$. This maximum matching is connected if there is at least one edge from E_1 in any of $G[Z_0, Z_2]$, $G[Z_0, Z_3]$, $G[Z_1, Z_2]$, or $G[Z_1, Z_3]$. If this happens, then $\alpha'_*(G_1) \geq x_1 + 1$ and we obtain the conclusion of Theorem 2.7.

If not, then $G[Z_1, Z_2]$ and $G[Z_0, Z_3]$ cannot contain edges from E_1. We already know they cannot contain edges from E_2, so they must all be in E_3. Similarly, $G[Z_1, Z_3]$ and $G[Z_0, Z_2]$ cannot contain edges from E_1 or E_3, so they must all be in E_2, and the partition has the structure we wanted. □

Now we complete the proof of Theorem 2.7.

Proof of Theorem 2.7. Induct on $\min\{x_1, x_2, x_3\}$. The base case is when $\min\{x_1, x_2, x_3\} = 0$, which holds because we can always find a connected matching of size 0.

If the theorem holds for all smaller $\min\{x_1, x_2, x_3\}$, then it holds for the triple (x_1-1, x_2-1, x_3-1), so assume this case as the inductive hypothesis.

For the triple (x_1, x_2, x_3), let $G = K_{2x_1+x_2+x_3-2}$ with a 3-edge-coloring as in Theorem 2.7. If the hypotheses of any of the Claims 2.21–2.23 hold for G, then we are done. Otherwise, G has the decomposition (Z_0, Z_1, Z_2, Z_3) described in Claim 2.23.

Construct a 3-edge-colored subgraph G' of G by deleting a vertex v_0, v_1, v_2, v_3 from each of Z_0, Z_1, Z_2, Z_3. G' still has

$$N - 4 = 2(x_1 - 1) + (x_2 - 1) + (x_3 - 1) - 2$$

vertices, so the inductive hypothesis applies. We find a connected matching in G'_i of size $x_i - 1$ for some i. The vertices of this matching have to be contained in two of the parts Z_j, Z_k, with the edges between Z_j and Z_k all having color i. So we can add the edge $v_j v_k$ to this matching, getting a connected matching of size x_i in the original G_i. □

2.1.5 Stability for 2-edge-colorings (Theorem 2.11)

2.1.5.1 Proof setup

Among counter-examples for fixed n, γ and ϵ such that $0 < \epsilon < 10^{-3}\gamma < 10^{-6}$ and $n > 100/\gamma$, choose a 2-edge-colored (n, s, ϵ)-suitable graph G with the fewest vertices and modulo this, with the smallest s.

If both (2.5) and (2.6) are strict inequalities, we can delete a vertex from V_s and still have a 2-edge-colored (n, s, ϵ)-suitable graph contradicting the minimality of N.

If $N = 3n - 1$ and (2.6) is strict, then $s \geq 3$ and $n_{s-1} + n_s > n$, since otherwise we can consider the $(s-1)$-partite graph obtained from G by deleting all edges between V_{s-1} and V_s. This also yields that for $s \geq 6$, also $n_1 + n_2 \geq n_3 + n_4 \geq n_{s-1} + n_s > n$ implying $N > 3n$. This contradicts the condition $N = 3n - 1$. Thus, if $N - n_1 > 2n - 1$, then $N = 3n - 1$, $s \leq 5$ and $n_1 < n$.

On the other hand, if $N > 3n - 1$ and $N - n_1 = 2n - 1$, then $n_1 = n_2$, since otherwise by deleting a vertex from V_1 we get a smaller (n, s, ϵ)-suitable graph. Furthermore, in this case $n_1 = n_2 > (3n - 1) - (2n - 1) = n$ and hence $n_3 + \ldots + n_s < (2n - 1) - n = n - 1$. So, if $s \geq 4$, then we can replace the parts $V_3, \ldots, V_s$ with one part $V_3' = V_3 \cup \ldots \cup V_s$. If $s = 2$, then $n_1 = n_2 = 2n - 1$.

Summarizing, we will replace (2.5) and (2.6) with the following more restrictive conditions:

$$N \geq 3n - 1; \textit{ moreover, if } N > 3n - 1, \textit{ then } N - n_1 = 2n - 1, n_1 = n_2 > n \textit{ and } s \leq 3. \tag{2.9}$$

$$N - n_1 \geq 2n - 1; \textit{ and if } N - n_1 > 2n - 1, \textit{ then } N = 3n - 1, n_1 < n, s \leq 5, n_{s-1} + n_s > n. \tag{2.10}$$

Conditions (2.9) and (2.10) imply

$$N = \max\{n_1, n\} + 2n - 1 \leq 4n - 2, \text{ and } 2n - 1 \geq n_1 \geq \ldots \geq n_{s-1} > n/2. \tag{2.11}$$

We obtain G' by deleting from G the set $\widetilde{V}$ and in the case $|V_s - \widetilde{V}| < 4\epsilon n$ also deleting $V_s - \widetilde{V}$. Let $s' = s - 1$ if we have deleted $V_s - \widetilde{V}$ and $s' = s$ otherwise. Let $V' = V(G')$ and $N' = |V'|$. By (2.7) and the construction of V', $N' > N - 5\epsilon n$. For $j \in [s']$, let $V_j' = V_j - \widetilde{V}_j$ and $n_j' = |V_j'|$. We also reorder V_j' and n_j' so that

$$n_1' \geq n_2' \geq \ldots \geq n_{s'}'. \tag{2.12}$$

For $i \in [2]$, we let $G_i' = G_i - \widetilde{V} - V_s$ if $|V_s - \widetilde{V}| < 4\epsilon n$, and $G_i' = G_i - \widetilde{V}$ otherwise.

By construction, (2.12) and (2.11), $n_{s'}' \geq 4\epsilon n$. In particular,

$$\textit{for } j \in [s'], \textit{ every } v \in V_j' \textit{ is adjacent to more than half of } V_{j'}' \textit{ for each } j' \in [s'] - \{j\}. \tag{2.13}$$

The structure of the proof resembles that of the proof of Theorem 2.6, but everything becomes more complicated. For example, instead of a simple Claim 2.17, we need a 2-page Section 2.1.5.2 below considering the case of almost bipartite graphs. After this, in Section 2.1.5.3 we prove three important claims, and present the main proof in Section 2.1.5.4 We will many times use that $\gamma > 1000\epsilon$.

2.1.5.2 Almost bipartite graphs

Suppose G is an (n, s, ϵ)-suitable graph satisfying also (2.9), (2.10) and (2.11), and that $s' = 2$, i.e., G' is bipartite. This means $0 \le |V_3| \le 4\epsilon n$. By (2.6) and the definition of G',

$$|V_1'| \ge |V_2'| \ge 2n - 1 - 5\epsilon n. \tag{2.14}$$

Suppose neither of G_1' and G_2' has a connected matching of size at least $(1+\gamma)n$. Let F be a largest component over all components in G_1' and G_2'. By symmetry, we may think that F is a component of G_1'. Let R be the smallest of the sets $V_1' - V(F)$ and $V_2' - V(F)$, and let $r = |R|$. For $j = 1, 2$, let $F_j = V(F) \cap V_j'$.

Case 1: $r \le 2\epsilon n$. Since F is the only nontrivial component of $G_1' - R$,

$$\alpha'(G_1' - R) = \alpha_*'(G_1' - R) \le \alpha_*'(G_1') < (1+\gamma)n.$$

Hence by Theorem 2.14, F has a vertex cover Q with $|Q| \le (1+\gamma)n$. Choose $j \in \{1, 2\}$ so that $|Q \cap V_j'| \le |Q \cap V_{3-j}'|$. Then by (2.14),

$$|V_{3-j}' - Q| \ge 2n - 1 - 5\epsilon n - (1+\gamma)n = (1-\gamma-5\epsilon)n - 1 \text{ and } |V_j' - Q| \ge (1.5 - \frac{\gamma}{2} - 5\epsilon)n - 1. \tag{2.15}$$

Furthermore, since Q is a vertex cover in F,

each vertex in $G_2' - Q - R = G' - Q - R$ is not adjacent to at most ϵn vertices in the other part. (2.16)

In particular, (2.15) together with $r \le 2\epsilon n$ implies that $|V_i - R - Q| \ge n/2$ for $i = 1, 2$. Hence (2.16) yields that $G_2' - R - Q$ is connected, and therefore

every matching in G_2' such that each edge intersects $V' - Q - R$ is a connected matching. (2.17)

Suppose first that $|F_{3-j} - Q| \ge (1+\gamma)n$. By (2.15) and the assumption $r \le 2\epsilon n$, we have $|V_j' - Q - R| \ge (1.5 - \frac{\gamma}{2} - 7\epsilon)n - 1$. Hence by (2.16), we can greedily construct a matching of size at least $(1+\gamma)n$ in $G_2'[F_{3-j} - Q, V_j' - Q - R]$. This matching is connected by (2.17).

Thus we may assume that $|F_{3-j} - Q| < (1+\gamma)n$. Let $U_1 = Q \cap F_{3-j}$ and $U_2 = (V_{3-j}' - U_1) \cup V_3 \cup R \cup \widetilde{V}$ (possibly, $V_3 = \emptyset$). By the assumption,

$$|U_2| = |F_{3-j} - Q| + |V_3| + r + |\widetilde{V}| < (1+\gamma)n + 7\epsilon n.$$

Thus by (2.14), $|U_1| \ge (2 - 1 - \gamma - 12\epsilon)n - 1$. On the other hand, $|U_1| \le |Q| \le (1+\gamma)n$, and symmetrically, $|U_2| \ge (2 - 1 - \gamma)n - 1$. Thus Conditions (iv) and (v) in the definition of an $(8\gamma, 2, 2)$-bad partition (V_j, U_1, U_2) are satisfied.

Condition (iii) of the definition holds by (2.14). Since Q is a vertex cover in F, every edge in G_1'

connecting V_j with U_2 intersects $Q \cap V_j$ or $V_3 \cup \widetilde{V} \cup R$. Since $|V_3 \cup \widetilde{V} \cup R| \le 7\epsilon n$, $\gamma > 1000\epsilon$ and

$$|Q \cap V_j| = |Q| - |U_1| \le (1+\gamma)n - (1-\gamma-12\epsilon)n + 1 < (2\gamma+13\epsilon)n, \tag{2.18}$$

we get $|E(G_1[V_j, U_2])| \le 2n(7\epsilon n + (2\gamma+13\epsilon)n) \le 6\gamma n^2$. So Condition (ii) also holds for (V_j, U_1, U_2).

Suppose now that $|E(G_2[V_j, U_1])| > 8\gamma n^2$. By (2.7) and the fact that $|Q| \le (1+\gamma)n$, $|E(G_2[\widetilde{V}_j \cup R, U_1])| \le (3\epsilon n)|Q| \le 3\epsilon(1+\gamma)n^2$. Similarly, by (2.18),

$$|E(G_2[F_j \cap Q, U_1])| \le |F_j \cap Q| \cdot |Q| \le (2\gamma+13\epsilon)n(1+\gamma)n.$$

Hence

$$|E(G_2[F_j - Q, U_1])| > (8\gamma - (2\gamma+13\epsilon)(1+\gamma) - 3\epsilon(1+\gamma))n^2 > 5\gamma n^2.$$

Since the degree of each vertex in $G[(F_j - Q) \cup U_1]$ is at most $\max\{|F_j - Q|, |U_1|\} < 2n$, this implies that the size β of a minimum vertex cover in $G_2[V_j - Q, U_1]$ is at least $2.5\gamma n$. Then by Theorem 2.14, $G_2[F_j - Q, U_1]$ has a matching M_1 of size $\beta \ge 2.5\gamma n$. Let Z_1 be the set of the ends of the edges in M_1 that are in $F_j - Q$. By (2.16), each vertex in $F_{3-j} - Q$ has in G_2' at least $|F_j - Q - Z_1| - \epsilon n$ neighbors in $F_j - Q - Z_1$. By (2.14) and (2.18), this is at least

$$2n - 1 - 7\epsilon n - (2\gamma+13\epsilon)n - 2.5\gamma n - \epsilon n > (2-5\gamma)n.$$

Thus, $G_2'[F_{3-j} - Q, F_j - Q - Z_1]$ has a matching M_2 covering $F_{3-j} - Q$. By (2.17), $M_1 \cup M_2$ is a connected matching in G_2'. And by (2.14),

$$|M_1 \cup M_2| = 2.5\gamma n + |F_{3-j} - Q| \ge 2.5\gamma n + 2n - 1 - 7\epsilon n - (1+\gamma)n > (1+\gamma)n,$$

a contradiction. Thus $|E(G_2[V_j, U_1])| \le 8\gamma n^2$, which means Condition (i) for a $(8\gamma, 2, 2)$-bad partition also holds. So, partition (V_j, U_1, U_2) is $(8\gamma, 2, 2)$-bad.

Case 2: $r > 2\epsilon n$. For $j = 1, 2$, let $\overline{F}_j = V_j' - F_j$. By the case,

$$\min\{|\overline{F}_1|, |\overline{F}_2|\} \ge r \ge 2\epsilon n. \tag{2.19}$$

In this case, we choose $j \in \{1, 2\}$ so that $|F_j| \ge |F_{3-j}|$.

Case 2.1: $|F_j| \le n/2$. Then each vertex $w \in F_j$ is adjacent in G_2' to at least $|V_{3-j}'| - |F_{3-j}| - \epsilon n$ vertices in $\overline{F}_{3-j}$. Hence by (2.14), the component of G_2' containing w has at least

$$1 + (2n - 1 - 5\epsilon n) - \frac{n}{2} - \epsilon n \ge (1.5 - 6\epsilon)n > n \ge |F|$$

vertices, contradicting the choice of F.

Case 2.2: $|F_j| > n/2$ and $|F_{3-j}| \le (1-5\epsilon)n$. Now each vertex in $\overline{F}_{3-j}$ is adjacent in G_2' to at least

$|F_j| - \epsilon n$ vertices in F_j, and by (2.14), each vertex in F_j is adjacent in G'_2 to at least

$$|V'_{3-j} - F_{3-j}| - \epsilon n \geq (2-5\epsilon)n - 1 - (1-5\epsilon)n = n-1$$

vertices in $\overline{F}_{3-j}$. Hence G'_2 has a component containing $F_j \cup \overline{F}_{3-j}$, and the size of this component is larger than $|F|$, a contradiction to the choice of F.

Case 2.3: $|F_j| \geq |F_{3-j}| > (1-5\epsilon)n$, and G'_2 has an edge xy with $x \in F_j$ and $y \in F_{3-j}$. By (2.19), as in Case 2.2, G'_2 has a component H_1 containing $F_j \cup \overline{F}_{3-j}$, and symmetrically G'_2 has a component H_2 containing $F_{3-j} \cup \overline{F}_j$. Since $x \in F_j \subset V(H_1)$ and $y \in F_{3-j} \subset V(H_2)$, $H_1 = H_2$; thus $H_1 = G'_2$, contradicting the maximality of F.

Case 2.4: $|F_j| \geq |F_{3-j}| > (1+\gamma)n$, and Case 2.3 does not hold. Then $G'[V(F)] = F$. By (2.16), for every $A \subseteq F_{3-j}$ with $|A| > \epsilon n$, $N_{G'_1}(A) = F_j$. Thus $\alpha'(F) \geq (1+\gamma)n$, a contradiction.

Case 2.5: $|F_j| \geq |F_{3-j}| > (1-5\epsilon)n$, $|F_{3-j}| \leq (1+\gamma)n$, and Case 2.3 does not hold. Let $W_1 = V(F)$ and $W_2 = V(G) - W_1$. We will show that (W_1, W_2) is a $(2\gamma, 1, 1)$-bad partition of $V(G)$. Indeed, since $|F_{3-j}| \leq (1+\gamma)n$, by (2.14),

$$|W_2| \geq |V'_{3-j} - F_{3-j}| \geq 2n - 1 - 5\epsilon n - (1+\gamma)n > (1-2\gamma)n,$$

proving the left part of Condition (i) of a $(2\gamma, 1, 1)$-bad partition. On the other hand, since $|F_j| \geq |F_{3-j}| > (1-5\epsilon)n$, using (2.14),

$$|W_2| \leq N - 2(1-5\epsilon)n \leq (4 - 2 + 10\epsilon)n - 2$$

$$\leq (n_1 - (2-5\epsilon)n + 1) + (2-10\epsilon)n - 2 \leq n_1 + 15\epsilon n - 1 < (1+\gamma)n_1,$$

proving the right part of Condition (i).

Since Case 2.3 does not hold, $E(G_2[W_1]) = \emptyset$, implying Condition (iii) of a $(2\gamma, 1, 1)$-bad partition. For every edge e in $G_1[W_1, W_2]$, one of the ends must be in $V_3 \cup \widetilde{V}$. Since $|V_3 \cup \widetilde{V}| \leq 5\epsilon n$, $|E(G_1[W_1, W_2])| \leq 5\epsilon n|W_1| \leq 20\epsilon n^2 < 2\gamma n^2$. Thus Condition (ii) also holds. This proves Theorem 2.11 for $s' = 2$.

2.1.5.3 General claims

We start from finding large matchings in G'_{3-i} between different components of G'_i.

Claim 2.24. *Fix an $i \in [2]$. Let (W_1, W_2) be a partition of V' with $0 < |W_1| \leq |W_2|$. Write $|W_1|$ in the form $|W_1| = n - r$, where $-(n-1)/2 \leq r \leq n-1$. Then for every $R \subset W_2$ with $|R| \leq \min\{r, 2r\} + n - 1$ such that $G'_i[W_1, W_2 - R]$ has no edges, the graph $G'_{3-i}[W_1, W_2 - R]$ has a matching of size at least $|W_1| - 7\epsilon n$.*

Proof. By symmetry, let $i=1$. By Theorem 2.14, it is enough to show that for every $A \subseteq W_1$,

$$|N_{G'_2}(A) \cap (W_2 - R)| \geq |A| - 7\epsilon n. \tag{2.20}$$

Suppose first that A intersects at least two distinct V'_js, say contains vertices $v_1 \in V'_{j_1}$ and $v_2 \in V'_{j_2}$. Then $N_{G'_2}(v_1)$ contains all but ϵn vertices in $(W_2 - R) - V'_{j_1}$, and $N_{G'_2}(v_2)$ contains all but ϵn vertices in $(W_2 - R) \cap V'_{j_1}$. So $|(W_2 - R) - N_{G'_2}(A)| < 2\epsilon n$. But

$$|W_2 - R| = N' - |W_1| - |R| \geq 3n - 1 - 5\epsilon n - |W_1| - |R| \geq (3n-1) - 5\epsilon n - (n-r) - \min\{r, 2r\} - n + 1$$

$$= n - 5\epsilon n + r - \min\{r, 2r\} \geq n - r - 5\epsilon n = |W_1| - 5\epsilon n \geq |A| - 5\epsilon n,$$

i.e., (2.20) holds for A.

Suppose now that $A \subseteq V'_j$. Then $N' - |V'_j| \geq 2n - 1 - 5\epsilon n$, and at most $|W_1 - A|$ vertices of W_1 are in $V' - V_j$. So, $W_2 - R$ has at least $2n - 1 - 5\epsilon n - |W_1 - A| - |R|$ vertices in $V' - V_j$. Let $v \in A$. Since v has at most ϵn non-neighbors in $V' - V_j$,

$$|N_{G'_2}(v) \cap (W_2 - R)| \geq (2n-1) - 5\epsilon n - |W_1 - A| - \epsilon n - |R| \geq |A| - 6\epsilon n + r - \min\{r, 2r\} \geq |A| - 6\epsilon n;$$

and again (2.20) holds for A. $\square$

A similar proof gives the following.

Claim 2.25. *Suppose that for some $i \in [2]$, V' has a partition (W_1, W_2, W_3) such that $G'_i(W_1, W_3)$ has no edges, and $\min\{|W_1|, |W_3|\} > (1+\gamma+4\epsilon)n$. If $\alpha'_*(G'_{3-i}) < (1+\gamma)n$, then either*
(a) there is $j \in [s']$ such that $|(W_1 \cup W_3) - V'_j| < (1+\gamma+4\epsilon)n$, or
(b) there are $j, j' \in [s']$ such that $W_1 \cup W_3 \subseteq V'_j \cup V'_{j'}$ and $G'_{3-i}[W_1 \cup W_3]$ is disconnected.

Proof. Suppose V' has a partition (W_1, W_2, W_3) such that $G'_1[W_1, W_3]$ has no edges, $\min\{|W_1|, |W_3|\} > (1+\gamma+4\epsilon)n$, and neither of (a) and (b) holds.

Case 1: There is $j \in [s']$ such that $|W_1 - V'_j| < 4\epsilon n$ or $|W_3 - V'_j| < 4\epsilon n$. For definiteness, suppose $|W_1 - V'_j| < 4\epsilon n$. Then $|W_1 \cap V'_j| \geq (1+\gamma)n$. Since (a) does not hold, $|W_3 - V'_j| > (1+\gamma)n$. Let $U_1 = W_1 \cap V'_j$ and $U_3 = W_3 - V'_j$. By the construction of G',

$$\text{for } k \in \{1,3\}, \text{ each vertex of } U_k \text{ is adjacent in } G'_2 \text{ to all but at most } \epsilon n \text{ vertices in } U_{4-k}. \tag{2.21}$$

So the graph $F = G'_2[U_1 \cup U_3]$ is connected. Also by (2.21), for every $U \subseteq U_1$, $|N_{G'_2}(U) \cap U_3| \geq |U_3| - \epsilon n$, and moreover, for every $U \subseteq U_1$ with $|U| \geq \epsilon n$, $N_{G'_2}(U) \supseteq U_3$. Hence for every $U \subseteq U_1$, $|N_{G'_2}(U) \cap U_3| \geq |U| + \min\{0, |U_3| - |U_1|\}$. Then by Theorem 2.14, F has a matching of size $\min\{|U_1|, |U_3|\} \geq (1+\gamma)n$.

Case 2: Case 1 does not hold and there are distinct $j_1, j_2, j_3 \in [s']$ such that $W_1 \cap V'_{j_h} \neq \emptyset$ for all

$h \in [3]$. Suppose there are $j, j' \in [s']$ such that

$$|W_3 - (V'_j \cup V'_{j'})| < 2\epsilon n. \tag{2.22}$$

Since Case 1 does not hold, we have $|W_3 \cap V'_j| > 2\epsilon n$ and $|W_3 \cap V'_{j'}| > 2\epsilon n$. Thus (2.22) may hold for at most one pair of $j, j' \in [s']$. For every other pair (j_1, j_2), any vertices $v_1 \in W_1 \cap V'_{j_1}$ and $v_2 \in W_1 \cap V'_{j_2}$ have a common neighbor in $W_3 - (V'_{j_1} \cup V'_{j_2})$. This means $G'_2[W_1 \cup W_3]$ has a component D containing W_1. Furthermore, since Case 1 does not hold, each $w \in W_3$ has in G'_2 a neighbor in W_1. Thus $G'_2[W_1 \cup W_3]$ is connected, and it is enough to show that $\alpha'(G'_2) \geq (1+\gamma)n$. By Theorem 2.14, it is sufficient to prove that

$$\textit{for every } W \subseteq W_1, \quad |N_{G'_2}(W) \cap W_3| \geq |W| + (1+\gamma)n - |W_1|. \tag{2.23}$$

Let $\emptyset \neq W \subseteq W_1$. If $W \subseteq V'_j$ for some $j \in [s']$, then since (a) does not hold,

$$|N_{G'_2}(W) \cap W_3| \geq |(W_1 \cup W_3) - V'_j| - |W_1 - W| - \epsilon n \geq (1+\gamma+4\epsilon)n - |W_1| + |W| - \epsilon n,$$

and (2.23) holds. If W intersects two distinct V'_js, then

$$|N_{G'_2}(W) \cap W_3| \geq |W_3| - 2\epsilon n \geq (1+\gamma+4\epsilon)n - 2\epsilon n \geq (1+\gamma+2\epsilon)n + (|W| - |W_1|),$$

and again (2.23) holds.

Case 3: Case 1 does not hold, and for $k \in \{1, 3\}$ there are $j_{k,1}, j_{k,2} \in [s']$ such that $W_k \subseteq V_{j_{k,1}} \cup V_{j_{k,2}}$. If $\{j_{1,1}, j_{1,2}\} \neq \{j_{3,1}, j_{3,2}\}$, then repeating the argument of Case 2, we again find a connected matching of size at least $(1+\gamma)n$ in G'_2. So, suppose $W_1 \cup W_3 \subseteq V'_{j_1} \cup V'_{j_2}$. Since (b) does not hold, $G'_2[W_1 \cup W_3]$ is connected. For $k \in \{1, 3\}$ and $h \in [2]$, let $W_{k,h} = W_k \cap V'_{j_h}$. Since Case 1 does not hold, $|W_{k,h}| \geq 4\epsilon n$ for all $k \in \{1, 3\}$ and $h \in [2]$. Then $G'_2[W_{1,1} \cup W_{3,2}]$ has a matching of size $\min\{|W_{1,1}|, |W_{3,2}|\}$ for the same reason as the graph F in Case 1 has a matching of size $\min\{|U_1|, |U_3|\}$. Similarly, $G'_2[W_{1,2} \cup W_{3,1}]$ has a matching of size $\min\{|W_{1,2}|, |W_{3,1}|\}$. Thus,

$$\alpha'_*(G'_2[W_1 \cup W_3]) \geq \min\{|W_{1,1}|, |W_{3,2}|\} + \min\{|W_{1,2}|, |W_{3,1}|\}.$$

Note that the last sum of the minima is always at least $(1+\gamma)n$: if it has the form $|W_{k,1}| + |W_{k,2}|$, then it is equal to $|W_k| > (1+\gamma)n$; otherwise this holds because (a) is false. □

Now we discuss largest components in G'_1 and G'_2.

Claim 2.26. *Suppose $s' \geq 3$. For $i \in \{1, 2\}$, let C_i be the vertex set of a largest component in G'_i. If $|V' - C_i| \geq 4\epsilon n$, then G'_{3-i} has only one nontrivial component D, and there is some $j \in [s']$ such that $D \supseteq V' - V'_j$. In particular, if $|V' - C_i| \geq 4\epsilon n$, then $\alpha'(G'_{3-i}) = \alpha'_*(G'_{3-i})$.*

Proof. Suppose $|V' - C_1| \geq 4\epsilon n$. If $|C_1| \geq n$, then let $W_2 = V' - C_1$. Otherwise, let W_2 be obtained from $V' - C_1$ by deleting vertex sets of several components of G'_1 so that $n \leq |V' - W_2| < 2n$. Let

$W_1 = V' - W_2$. In any case,

$$|W_2| \geq 4\epsilon n \text{ and } |W_1| \geq n. \tag{2.24}$$

Case 1. There are $k \in [2]$ and $j, j' \in [s']$ such that $W_k \subseteq V'_j \cup V'_{j'}$. Suppose $|V'_j \cap W_k| \geq |V'_{j'} \cap W_k|$. Since $s' \geq 3$, there is $j'' \in [s'] - \{j, j'\}$. By the case, $V'_{j''} \subseteq W_{3-k}$. Then each $v \in W_k$ is non-adjacent in G'_2 to fewer than ϵn vertices in $V'_{j''}$. Since $|V'_{j''}| \geq 4\epsilon n$, every two vertices in W_k have a common neighbor in G'_2. So, G'_2 has a component D containing W_k. By (2.24) and the choice of j, each vertex in $V(G'_2) - V'_j$ has a neighbor in W_k and hence belongs to D. So, $V' - D \subset V'_j$ and thus $\alpha'(G'_2) = \alpha'_*(G'_2)$.

Case 2. Case 1 does not hold. Since $s' \geq 3$ and $|V'_j| \geq 4\epsilon n$ for each $j \in [s']$, there are $k \in [2]$ and $j, j' \in [s']$ such that $|W_k \cap V'_j| \geq 2\epsilon n$ and $|W_k \cap V'_{j'}| \geq 2\epsilon n$. Since $|W_k \cap V'_j| \geq 2\epsilon n$, every two vertices in $W_{3-k} - V'_j$ have a common neighbor in $W_k \cap V'_j$ in G'_2. So, G'_2 has a component D containing $W_{3-k} - V'_j$. Similarly, G'_2 has a component D' containing $W_{3-k} - V'_{j'}$. Since Case 1 does not hold, there is $v \in W_{3-k} - V'_j - V'_{j'}$. This means $D = D'$ and $D \supset W_{3-k}$. By (2.24), there is at most one $j'' \in [s']$ such that $|W_{3-k} - V'_{j''}| < \epsilon n$ (maybe $j'' \in \{j, j'\}$). Each vertex in $W_k - V'_{j''}$ has a neighbor in W_k and hence belongs to D. So, $V(G'_2) - D \subset V'_{j''}$ and thus $\alpha'(G'_2) = \alpha'_*(G'_2)$. □

2.1.5.4 Main part

We work with $s' \geq 3$. For $i \in [2]$, let C_i denote the vertex set of the largest component in G'_i and $c_i = |C_i|$. From now on, we assume $c_1 \geq c_2$. Let $B = V' - C_1$ and $b = |B| = N' - c_1$.

Claim 2.27. $b \leq n'_1/2$.

Proof. Suppose $b > n'_1/2$. Then $b > 4\epsilon n$, so by Claim 2.26 applied to G'_2, there is $j \in [s']$ such that $B \subset V'_j$. Since $V' - V'_j \subseteq C_1$ and $|V(G') - V'_j| \geq 2n - 1 - 5\epsilon n$, every two vertices in B have in G'_2 a common neighbor in $V' - V'_j$, and every two vertices in $V' - V'_j$ have a common neighbor in B. Thus G'_2 has a component D that includes B and $V' - V'_j$. So

$$N' - b = c_1 \geq c_2 \geq |D| \geq N' - |V'_j - B| \geq N' - n'_1 + b.$$

Comparing the first and the last expressions in the chain, we get $n'_1 \geq 2b$. □

Since by Claim 2.27,

$$c_1 \geq N' - \frac{n'_1}{2} = \frac{1}{2}(N' + (N' - n'_1)) \geq \frac{1}{2}(3n - 1 - 5\epsilon n + 2n - 1 - 5\epsilon n) > 2(1+\gamma)n,$$

and $\alpha'_*(G_1) < (1+\gamma)n$, we conclude that $G'_1[C_1]$ has no perfect matching. Then there is a partition $C_1 = A \cup C \cup \bigcup_{j=1}^{k} D_j$ satisfying Theorem 2.13. Let $a = |A|$.

If $N' - c_1 \geq 4\epsilon n$, then also $N' - c_2 \geq 4\epsilon n$, and by Claim 2.26 each vertex in B is isolated in G'_1. In this case, we view $V' - A$ as the union $\bigcup_{i=0}^{k'} D'_i$, where $k' = k + b$, $D_0 = C$, for $1 \le i \le k$ we define $D'_i = D_i$, and for $k+1 \le i \le k'$, each D_i is a vertex in B. By definition, D_0 could be empty.

If $N' - c_1 < 4\epsilon n$, then we view $V' - A$ as the union $\bigcup_{i=0}^{k'} D'_i$, where $k' = k$, $D_0 = C \cup B$, and $D'_i = D_i$ for $1 \le i \le k$. In both cases, we reorder D'_is so that $|D'_1| \ge \ldots \ge |D'_{k'}|$ and define $d_i = |D'_i|$ for $i \in [k']$.

Then by Theorem 2.13,

$$\alpha'_*(G'_1) = \alpha'(G'_1[C_1]) = \frac{N' - b - k + a}{2} \ge \frac{N' - k' + a}{2} - 2\epsilon n. \tag{2.25}$$

Since $N' \ge 3n - 1 - 5\epsilon n$ and $\alpha'(G'_1) < (1+\gamma)n$, (2.25) yields a lower bound on k':

$$k' \ge a + N' - 4\epsilon n - \alpha'_*(G'_1) > a + N' - 2(1 + \gamma + 2\epsilon)n > (1 - 3\gamma)n + a + 2. \tag{2.26}$$

Claim 2.28. *$G'_2 - A$ has only one nontrivial component. Moreover, if $G'_2 - A$ is disconnected, then $a \le 3\gamma n$, and all isolated vertices of $G'_2 - A$ are in the same V'_j.*

Proof. Suppose $G'_2 - A$ is disconnected. Recall that $D'_{k'}$ is a smallest of $D'_1, \ldots, D'_{k'}$. Since $N' \ge 3n - 1 - 5\epsilon n$, (2.26) yields

$$\frac{k'}{N'} \ge \frac{(1 - 3\gamma)n + a}{3n - 1 - 5\epsilon n} > \frac{1}{4}.$$

Thus $|D'_{k'}| < 4$. Since $G'_1[D'_{k'}]$ is factor-critical, if $|D'_{k'}| = 3$, then $G'_1[D'_{k'}] = K_3$. Pick $u \in D'_{k'}$. Suppose $u \in V'_j$. Let Q be the component of $G'_2 - A$ containing u. Let $R = V' - V'_j - Q - A$ and $R' = V'_j - Q - A$. Since $R \cap N_{G'_2}(u) = \emptyset$, $|R| < \epsilon n + 2$. Suppose $G'_2 - Q - A$ has vertices v_1 and v_2 in different parts of G', say in V_{j_1} and V_{j_2}. Then the set $\{v_1, v_2\}$ is adjacent in G' to all but $2\epsilon n$ vertices. For $h \in [2]$, let $v_h \in D'_{i_h}$ (possibly, $i_1 = i_2$). Then $\{v_1, v_2\}$ is adjacent in G'_2 to all but $2\epsilon n$ vertices of the set $\widetilde{D} = \left(\bigcup_{i=0}^{k'} D'_i\right) - D'_{i_1} - D'_{i_2}$. This means $|Q \cap \widetilde{D}| \le 2\epsilon n$ and hence

$$|\widetilde{D} - R'| < 3\epsilon n + 2. \tag{2.27}$$

It follows that

$$|D'_{i_1} \cup D'_{i_2} \cup A| \ge |V' - V'_j| - 3\epsilon n - 2 \ge 2n - 3 - 8\epsilon n. \tag{2.28}$$

By (2.28), $N' \ge |D'_{i_1} \cup D'_{i_2} \cup A| + (k' - 2) \ge 2n - 3 - 8\epsilon n + (k' - 2)$. Hence by (2.26),

$$k' \ge a + N' - 2(1 + \gamma + 2\epsilon)n \ge a + (2n - 3 - 8\epsilon n + (k' - 2)) - 2(1 + \gamma + 2\epsilon)n \ge a + k' - 3\gamma n.$$

Comparing the first and the last expressions, we get $a \le 3\gamma n$. The number of components in $\widetilde{D}$ is at least $k' - 2$, and by (2.27), fewer than $3\epsilon n + 2$ of these components contain vertices not in V'_j. Hence by (2.26), at least $(1 - 3\gamma - 3\epsilon)n - 2$ components of $G'_1 - A$ in $\widetilde{D}$ are singletons and belong to V'_j. But each of them is adjacent in G'_2 to all but ϵn vertices in the set $V' - V'_j - A$ of size at least

$2n-1-5\epsilon n-3\gamma n>n$. This means all of them are in Q, a contradiction. Thus all vertices outside of Q are in the same part of G'. In particular, Q is the only nontrivial component of $G_2'-A$. $\square$

We will finish with two lemmas that, together, complete the proof of Theorem 2.11.

Lemma 2.29. *If $a\le(1-3\gamma)n-1$, then G' has a $(16\gamma,1,1)$-bad partition.*

Lemma 2.30. *If $a\ge(1-3\gamma)n-1$, then G' has a $(68\gamma,2,1)$-bad or a $(35\lambda,2,2)$-bad partition.*

2.1.5.4.1 Small a: proof of Lemma 2.29

Case 1: $(1+\gamma+4\epsilon)n+1\le|D_1'|\le N'-a-(1+\gamma+4\epsilon)n-1$. Let $W_1=D_1'$, $W_2=A$, and $W_3=V'-W_2-W_1$. By the case, $|W_3|=N'-a-|D_1'|\ge(1+\gamma+4\epsilon)n+1$. Hence we obtain a partition (W_1,W_2,W_3) of V' satisfying conditions in Claim 2.25 with $i=1$. Thus either G_2' has a matching of size $(1+\gamma)n$ which by Claim 2.28 is connected, or
(a) there is $j_1\in[s']$ such that $|(V'-A)-V_{j_1}'|<(1+\gamma+4\epsilon)n$, or
(b) there are $j_1,j_2\in[s']$ such that $V'-A\subseteq V_{j_1}'\cup V_{j_2}'$ and $G_2'[V'-A]$ is disconnected.

If (a) holds, then by (2.10), $|V'-V_{j_1}'|\ge 2n-1-5\epsilon n$. So,

$$(2n-1-5\epsilon n)-a\le|(V'-A)-V_{j_1}'|<(1+\gamma+4\epsilon)n,$$

and $a>(1-\gamma-9\epsilon)n$, contradicting the condition $a\le(1-3\gamma)n-1$.

So, suppose (b) holds, in particular, $G'-A$ is bipartite. Since every factor-critical graph is either a singleton or contains an odd cycle, each of $D_1',\dots,D_{k'}'$ is a singleton, and only D_0 may have more than one vertex. Recall that either $D_0=C$ or $b\le 4\epsilon n$ and $D_0=C\cup B$. Since $G_1'[C]$ has a perfect matching, C is a bipartite graph with equal parts. So, $|C|\le 2(1+\gamma)n-a$ and $|V_{j_1}'\cap C|=|V_{j_2}'\cap C|\le(1+\gamma)n-a/2$. By (2.10), for $h\in[2]$,

$$|V_{j_h}'-C-A-B|\ge(N'-n_{j_{3-h}}')-|V_{j_h}'\cap C|-a-b$$

$$\ge 2n-1-((1+\gamma)n-\frac{a}{2})-a-4\epsilon n\ge(\frac{1}{2}-\frac{5}{2}\gamma-4\epsilon)n-1>(\frac{1}{2}-3\gamma)n.$$

Recall that all components of $G_1'-A-C$ are singletons. This means that for $h\in[2]$, each vertex in $V_{j_h}'-A$ is adjacent to all but ϵn vertices in the set $V_{j_{3-h}}'-C-A-B$ of size at least $(\frac{1}{2}-3\gamma)n$. But then $G_2'-A$ is connected, and so does not satisfy (b).

Case 2: $|D_1'|\ge N'-a-(1+\gamma+4\epsilon)n-1$. Since $k'\le N'-|D_1'|+1$, in our case $k'\le(1+\gamma+4\epsilon)n+1+1$. This together with (2.26) yields

$$a\le 2(1+\gamma+2\epsilon)n-N'+k'\le 2(1+\gamma+2\epsilon)n-3n+1+5\epsilon n+(1+\gamma+4\epsilon)n+2$$

$$\le(3\gamma+13\epsilon)n+5<4\gamma n. \tag{2.29}$$

Let $W_1=D_1'\cup A$ and $W_2=V'-W_1$. We show (W_1,W_2) is a $(16\gamma,1,1)$-bad partition for G'. We will check that all conditions (i)–(iii) of the definition of a $(16\gamma, 1, 1)$-bad partition hold.

Part 1: Checking (i). By (2.26), $|W_2| \geq k' - 1 > (1-3\gamma)n$. By the case, $|W_2| = N' - |D'_1| - a \leq (1+\gamma+4\epsilon)n + 1 < (1+2\gamma)n$.

Part 2: Checking (ii). Since D'_1 has no neighbors in W_2 in G'_1, (2.29) yields

$$|E_{G'_1}[W_1, W_2]| \leq a|W_2| \leq (4\gamma n)|W_2| \leq (4\gamma n)(1+2\gamma)n < 5\gamma n^2.$$

Part 3: Checking (iii). Suppose $\alpha'(G'_2[W_1]) \geq (4\gamma + 7\epsilon)n$. Let Q be a matching in $G'_2[W_1]$ of size $(4\gamma+7\epsilon)n$ and $V(Q)$ be the vertex set of Q. Let $R = A \cup V(Q)$. Since $a \leq 4\gamma n$, $|R| \leq (12\gamma + 14\epsilon)n$. We apply to G'_1 Claim 2.24 with the roles of W_1 and W_2 switched and $r = 3\gamma n$ (using (2.26)). Since $|R| \leq (12\gamma+14\epsilon)n \leq n-1+r$, graph $G'_2[W_1, W_2]-R$ has a matching P of size $|W_2|-7\epsilon n \geq k'-1-7\epsilon n$. By this and (2.26), $Q \cup P$ is a matching in G'_2 of size at least

$$|P| + |Q| \geq (k' - 1 - 7\epsilon n) + (4\gamma + 7\epsilon)n \geq (1-3\gamma)n + 4\gamma n = (1+\gamma)n,$$

and by Claim 2.28, it is connected, a contradiction. So, $\alpha'(G'_2[W_1]) < (4\gamma + 7\epsilon)n$. Hence, by the Erdős-Gallai Theorem and (2.11),

$$|E(G'_2[W_1])| \leq (4\gamma + 7\epsilon)n|W_1| < 16\gamma n^2.$$

Case 3: $|D'_1| \leq (1+\gamma+4\epsilon)n + 1$. We will construct a partition of V' satisfying the conditions in Claim 2.25. We start by letting $W_2 = A$, $W_1 = W_3 = \emptyset$, and then in steps add sets to W_1 and W_3. On Step 1 we add D'_1 to W_1 and on Step 2 add D'_2 to W_3. Now, for $i = 3, 4, \ldots$ we do as follows:

- **Step** i**:** If $|W_1| \leq |W_3|$, then we add D'_i to W_1. Otherwise we add D'_i to W_3. Stop if $\max\{|W_1|, |W_3|\} \geq (1+\gamma+4\epsilon)n$ and put the remaining sets in the smaller one of W_1 and W_3.

Since

$$N' - a \geq (3n - 1 - 5\epsilon n) - ((1-3\gamma)n - 1) > 2(1+\gamma+4\epsilon)n,$$

the algorithm stops sooner or later. Suppose it stopped after Step h. If both W_1 and W_3 are of size at least $(1+\gamma+4\epsilon)n$, then the partition satisfies the conditions of Claim 2.25. So, assume first that $D'_h \subset W_3$ (the argument in the case $D'_h \subset W_1$ is exactly the same with switching indices). Then $|W_1| < (1+\gamma+4\epsilon)n$ and $|W_3 - D'_h| < (1+\gamma+4\epsilon)n$, but $|W_3| \geq (1+\gamma+4\epsilon)n$.

Case 3.1: $|D'_h| \leq \gamma n/2$. Then

$$N' = |W_3 - D'_h| + |D'_h| + |W_2| + |W_1| < (1+\gamma+4\epsilon)n + \gamma n/2 + (1-3\gamma)n + (1+\gamma+4\epsilon)n$$

$$= (3 + 2.5\gamma + 8\epsilon)n < (3 - 6\epsilon)n < N',$$

a contradiction.

Case 3.2: $|D'_h| > \frac{\gamma n}{2}$. Let h' be the largest index such that $|D'_{h'}| > \frac{\gamma n}{2}$. By (2.11) and the definition of h', $4n > N' - a \geq h' \frac{\gamma n}{2}$, so

$$h \leq h' < 4n \cdot \frac{2}{\gamma n} = \frac{8}{\gamma} < \frac{n}{3}.$$

By (2.26), $k' \geq (1-3\gamma)n$, so $G'_1 - A$ has at least $k' - h' \geq (1-3\gamma)n - \frac{n}{3} > 0.6n$ components of size at most $\frac{\gamma n}{2}$. Since

$$N' - a - (1+\gamma+4\epsilon)n \geq (3n - 1 - 5\epsilon n) - (1-3\gamma)n + 1 - (1+\gamma+4\epsilon)n \geq (1+1.8\gamma)n,$$

if we add a component of size at most $\frac{\gamma n}{2}$ to a set of size at most $(1+\gamma+4\epsilon)n$, the remaining set in $V' - A$ has size at least $(1+1.3\gamma)n > (1+\gamma+4\epsilon)n$. Therefore, if we could not get a partition satisfying Claim 2.25 by adding to $W_3 - D'_h$ one by one components of G'_1 of size at most $\frac{\gamma n}{2}$, then

$$(1+\gamma+4\epsilon)n - |W_3 - D'_h| \geq |\bigcup_{i=h'+1}^{k'} D'_i| \geq k' - h' > \frac{2n}{3}.$$

This means $|D'_2| \leq |W_3 - D'_h| < (1/3+\gamma+4\epsilon)n$. On the other hand, $|D'_h| \geq \frac{2n}{3}$. This contradicts to the fact that $|D'_h| \leq |D'_2|$.

If follows that we did construct a partition satisfying conditions in Claim 2.25. Thus either G'_2 has a matching of size $(1+\gamma)n$ which by Claim 2.28 is connected, or
(a) there is $j_1 \in [s']$ such that $|(V'-A) - V'_{j_1}| < (1+\gamma+4\epsilon)n$, or
(b) there are $j_1, j_2 \in [s']$ such that $V' - A \subseteq V'_{j_1} \cup V'_{j_2}$ and $G'_2[V'-A]$ is disconnected.

Repeating the argument of the end of Case 1 word by word, we see that neither (a) nor (b) is possible.

2.1.5.4.2 Large a: proof of Lemma 2.30

By (2.26) and (2.11),

$$k' \geq N' + a - 2(1+\gamma+2\epsilon)n \geq \max\{n_1, n\} + 2n - 1 - 9\epsilon n + (1-3\gamma)n - 1 - 2(1+\gamma)n.$$

So,

$$k' \geq \max\{n_1, n\} + n - (5\gamma + 9\epsilon)n - 2. \tag{2.30}$$

Construct an independent set I in $G'_1 - A - D_0$ of size k' by choosing one vertex from each component of $G'_1 - A - D_0$. Let $Q = V' - A - I$. Then by (2.11),

$$|V' - A| \leq \max\{n_1, n\} + 2n - 1 - a \leq \max\{n_1, n\} + 2n - 1 - ((1-3\gamma)n - 1),$$

and thus by (2.30),

$$|Q| \leq N' - a - k' \leq \max\{n_1, n\} + 2n - 1 - ((1-3\gamma)n - 1) - (\max\{n_1, n\} + n - (5\gamma + 9\epsilon)n - 2).$$

Hence

$$|Q| \le 8\gamma n + 9\epsilon n + 2 < 9\gamma n. \tag{2.31}$$

Case 1: $\alpha'(G_2'[A, V'-A]) \le 8\gamma n$. Since $G_2'[A, V'-A]$ is bipartite, by Theorem 2.14, it has a vertex cover X with $|X| \le 8\gamma n$. Let $W_2 = A - X$, and $W_1 = V' - W_2$. We will show that (W_1, W_2) is a $(68\gamma, 2, 1)$-bad partition for G' by checking all conditions.

Part 1: Checking (i). Since $a \ge (1-3\gamma)n - 1$ and $|X| \le 8\gamma n$,

$$|W_2| = |A - X| \ge a - |X| \ge (1 - 3\gamma n) - 1 - 8\gamma n \ge (1 - 12\gamma)n.$$

On the other hand, $|W_2| = |A - X| \le a \le (1+\gamma)n$.

Part 2: Checking (ii). Since X is a vertex cover in $G_2'[A, V'-A]$, G_2' has no edge in G_2 between $W_2 - X = W_2$ and $W_1 - X$. Thus,

$$|E(G_2'[W_1, W_2])| \le |X \cap W_1| \cdot |W_2| \le 8\gamma n \cdot a < 16\gamma n^2.$$

Part 3: Checking (iii). Since I is an independent set in G_1', by (2.31),

$$|E(G_1'[W_1])| \le |Q \cup (A \cap X)| \cdot |W_1| \le 17\gamma n N' \le 68\gamma n^2.$$

Case 2: $\alpha'(G_2'[A, V'-A]) \ge 8\gamma n$. We will need the following claim.

Proposition 2.31. *Let $s \ge 2$ and $k_1, k_2, \ldots, k_s$ be positive integers. Let $S = k_1 + \ldots + k_s$ and $m = \max\{k_1, k_2, \ldots, k_s\}$. Let H be obtained from a complete s-partite graph $K_{k_1,k_2,\ldots,k_s}$ by deleting some edges in such a way that each vertex loses less than ϵn neighbors. Then*

$$\alpha'(H) \ge g(H) = \min\{\lfloor \frac{S}{2} \rfloor, S - m\} - \epsilon n. \tag{2.32}$$

Proof. Let H be a vertex-minimal counter-example to the claim. If $S \le 2\epsilon n$, then $\frac{S}{2} - \epsilon n \le 0$, and (2.32) holds trivially, so $S > 2\epsilon n$. Let the parts of H be $Z_1, \ldots, Z_s$ with $|Z_i| = k_i$ for $i \in [s]$. Suppose $m = k_1$. Since $S > 2\epsilon n$, either $k_1 > \epsilon n$ or $S - k_1 > \epsilon n$. In both cases, H has an edge xy connecting Z_1 with $V(H) - Z_1$. Let $H' = H - x - y$.

We claim that $g(H') \ge g(H) - 1$. Indeed, $\lfloor \frac{S}{2} \rfloor$ decreases by exactly 1, and if $S - m$ decreases by 2, then m does not change, which means there is $k_2 = k_1$ such that neither x nor y is in Z_2. But in this case, since $|\{x, y\} \cap Z_1| = 1$, $S \ge 2m + 1$, which yields $S - m \ge \lfloor \frac{S}{2} \rfloor + 1 = \min\{\lfloor \frac{S}{2} \rfloor, S - m\} + 1$, and hence $g(H') \ge g(H) - 1$.

So, by the minimality of H, $\alpha'(H') \ge g(H') \ge g(H) - 1$. Adding edge xy to a maximum matching in H', we complete the proof. $\square$

Take a matching X of size $8\gamma n$ in G_2' connecting A with $V' - A$. Denote the set of the endpoints of

X by $V(X)$. Since $|I| = k'$, by (2.30),

$$|I - V(X)| \geq \max\{n_1, n\} + n - (5\gamma + 9\epsilon)n - 2 - 8\gamma n = \max\{n_1, n\} + (1 - 13\gamma - 9\epsilon)n - 2. \quad (2.33)$$

Let R be a matching of size $\alpha'(G'_2[I - V(X)])$ in $I - V(X)$ in G'_2. Since $a > 3\gamma n$, by Claim 2.28, $G'_2 - A$ is connected, and hence $R \cup X$ is a connected matching in G'_2. Since $\alpha'_*(G'_2) < (1+\gamma)n$,

$$|R| + |X| = \alpha'(G'_2[I - V(X))]) + 8\gamma n < (1+\gamma)n;$$

therefore,

$$\alpha'(G'_2[I - V(X)]) < (1 - 7\gamma)n. \quad (2.34)$$

Let $X_j = V'_j \cap V(X) \cap I$, and $Y_j = V'_j \cap I - V(X)$ for $j \in [s']$. We assume that $|Y_{j_1}| = \max\{|Y_j| : j \in [s']\}$. By Proposition 2.31,

$$\alpha'(G_2[I - V(X]) \geq \min\left\{\left\lfloor \frac{|I - V(X)|}{2} \right\rfloor, |I - V(X) - Y_{j_1}|\right\} - \epsilon n. \quad (2.35)$$

Since by (2.33) and (2.34),

$$\lfloor \frac{|I - V(X)|}{2} \rfloor \geq \lfloor \frac{k' - 8\gamma n}{2} \rfloor \geq n - 1 - \frac{(13\gamma + 9\epsilon)n}{2} > (1 - 7\gamma + 2\epsilon)n \geq \alpha'(G_2[I - V(X)]) + 2\epsilon n,$$

(2.34) and (2.35) yield

$$|I - V(X) - Y_{j_1}| - 2\epsilon n \leq \alpha'(G_2[I - V(X)]) \leq (1 - 7\gamma)n. \quad (2.36)$$

Again by (2.33),

$$|Y_{j_1}| \geq \max\{n_1, n\} + (1 - 13\gamma - 9\epsilon)n - 2 - (1 - 7\gamma)n \geq \max\{n_1, n\} - 6.5\gamma n. \quad (2.37)$$

Let $U_1 = A - V'_{j_1}$ and $U_2 = V(G) - A - V'_{j_1}$. We now show that (V'_{j_1}, U_1, U_2) is a $(35\gamma, 1, 2)$-bad partition.

Part 1: Checking (i). By (2.37), we have

$$|A \cap V'_{j_1}| \leq |V'_{j_1}| - |Y_{j_1}| \leq n_1 - (n_1 - 6.5\gamma n) = 6.5\gamma n. \quad (2.38)$$

Since by (2.36) and (2.31),

$$|U_2| \leq |I - V(X) - Y_{j_1}| + |Q| + |X| \leq (1 - 7\gamma + 2\epsilon)n + 9\gamma n + 8\gamma n \leq (1 + 10\gamma + 2\epsilon)n, \quad (2.39)$$

we have

$$|E(G'_1[V'_{j_1}, U_2])| \leq |A \cap V_{j_1}| \cdot |U_2| + |Q| \cdot |U_2| + |Q| \cdot |Y_{j_1}|$$
$$\leq (6.5\gamma n)(1 + 10\gamma + 2\epsilon)n + 9\gamma n(1 + 10\gamma + 2\epsilon)n + 9\gamma n(2n - 1) \leq 35\gamma n^2.$$

Part 2: Checking (ii). We need a refined choice of X:

Claim 2.32. *G_2' has a matching X with $|X| = 8\gamma n$ from A to $V(G) - A$ such that $\alpha'(G_2'[U_1, (V_{j_1} - A)]) = |X_{j_1}|$ and $\alpha'(G_2'[U_1, (V_{j_1} - A)]) \leq 7\gamma n$.*

Proof. Let M_j be the subset of matching edges of X with an endpoint in X_j. By definition, $|M_{j_1}| = |X_{j_1}|$. Suppose $\alpha'(G_2'[U_1, (V_{j_1} - A)]) > |X_{j_1}|$ and S is a largest matching in $G_2'[U_1, (V_{j_1} - A)]$. Each component of $S \cup M_{j_1}$ is a path or a cycle. Since $|S| > |M_{j_1}|$, there is a component C (a path) of $S \cup M_{j_1}$ with one more edge in S than in M_{j_1}. Say the endpoints of C are w_1 and w_2. Then we can assume $w_1 \in Y_{j_1}$ and $w_2 \in A$. If w_2 is incident with an edge $e \in X - M_{j_1}$, then we switch the edges in C (if an edge was originally in S then now it is in M_{j_1} and vice versa) and delete e from X. If w_2 is not incident with any matching edge in $X - M_{j_1}$, then we switch the edges in C and delete any edge $e \in X - M_{j_1}$. In both cases, we obtain a new matching X' with size $8\gamma n$ and $|X'_{j_1}| = |X_{j_1}| + 1$. Note that (2.37) still works for X' and by (2.38),

$$|X'_{j_1}| \leq |V_{j_1}| - |Y'_{j_1}| < 7\gamma n. \tag{2.40}$$

Thus repeating the procedure, on every step we increase $|X'_{j_1}|$, but preserve (2.40). Eventually we construct a matching X'' with $|X''_{j_1}| = \alpha'(G_2[U_1, (V'_{j_1} - A)]) < 7\gamma n$. □

By Claim 2.32 and (2.38),

$$|E(G_2'[U_1, V_{j_1}])| \leq 7\gamma n \cdot n_1 + |A \cap V'_{j_1}| \cdot |U_1| \leq 7\gamma n(2n-1) + 6.5\gamma n(1+\gamma)n < 22\gamma n^2.$$

Part 3: Checking (iii). By (2.37), $|V'_{j_1}| \geq |Y_{j_1}| \geq (1 - 6.5\gamma)n$.

Part 4: Checking (iv). Since $a \geq (1-3\gamma)n - 1$, by (2.38),

$$(1-10\gamma)n - 1 \leq (1-3\gamma)n - 1 - 6.5\gamma n \leq a - |A \cap V_{j_1}| = |U_1| \leq a \leq (1+\gamma)n.$$

Part 5: Checking (v). By (2.37),

$$|U_2| = N' - |V_{j_1}| - |U_1| \geq (n_1 + 2n - 1 - 5\epsilon n) - n_1 - (1+\gamma)n = (1-2\gamma)n.$$

On the other hand, by (2.39), $|U_2| \leq (1 + 11\gamma)n$.

2.2 Long monochromatic paths and cycles in 2-edge-colored multipartite graphs

The goal of Section 2.2 is to prove for large n Conjecture 2.5 and similar exact bounds for paths P_{2n} (parity matters here) and cycles C_{2n}. We do it in a more general setting: for multipartite graphs with possibly different part sizes. In Section 2.2.1, we discuss extremal examples, define some notions and state our main results. In Section 2.2.2, we describe our tools. In Sections 2.2.3–2.2.7, we prove the main part, namely, the result for even cycles C_{2n}. In Sections 2.2.8, 2.2.9 and 2.2.10 we use the main result to derive similar results for cycles $C_{\geq 2n}$, and paths P_{2n} and P_{2n+1}.

2.2.1 Examples and results

For a graph G and disjoint sets $A, B \subset V(G)$, by $G[A]$ we denote the subgraph of G induced by A, and by $G[A, B]$ – the bipartite subgraph of G with parts A and B formed by all edges of G connecting A with B.

Our edge-colorings always will be with red (color 1) and blue (color 2).

We consider necessary restrictions on $n_1 \geq n_2 \geq \ldots \geq n_s$ providing that each 2-edge-coloring of $K_{n_1,n_2,\ldots,n_s}$ contains (a) a monochromatic path P_{2n}, (b) a monochromatic path P_{2n+1}, (c) a monochromatic cycle C_{2n} and (d) a monochromatic cycle $C_{\geq 2n}$. Each condition we add is motivated by an example showing that the condition is necessary.

First, recall that each of P_{2n}, P_{2n+1}, C_{2n} and $C_{\geq 2n}$ contains a connected matching M_n. Thus a graph with no M_n also contains neither P_{2n} nor P_{2n+1} nor $C_{\geq 2n}$.

2.2.1.1 Example with no monochromatic M_n: too few vertices

Let $G = K_{3n-2}$. Clearly, $G \supseteq K_{n_1,n_2,\ldots,n_s}$ for each $n_1, \ldots, n_s$ with $n_1 + \ldots + n_s = 3n - 2$. Partition $V(G)$ into sets U_1 and U_2 with $|U_1| = 2n-1$ and $|U_2| = n-1$. Color the edges of $G[U_1, U_2]$ with red and the rest of the edges with blue. Since neither K_{2n-1} nor $K_{n-1,2n-1}$ contains M_n, we conclude $G \not\mapsto (M_n, M_n)$; see Figure 2.3. Let $N = n_1 + \ldots + n_s$.

To rule out this example, we add the condition

$$N \geq 3n - 1. \tag{2.41}$$

2.2.1.2 Example with no monochromatic M_n: too few vertices outside V_1

Choose any n_1 and let $N = n_1 + 2n - 2$. Let G be obtained from K_N by deleting the edges inside a vertex subset U_1 with $|U_1| = n_1$. Graph G contains every $K_{n_1,n_2,\ldots,n_s}$ with $n_2 + \ldots + n_s = 2n - 2$. Partition $V(G) - U_1$ into sets U_2 and U_3 with $|U_2| = |U_3| = n - 1$. Color all edges incident with U_2 red, and the remaining edges of G blue. Since the red and blue subgraphs of G have vertex covers of size $n - 1$ (namely, U_2 and U_3), neither of them contains M_n. Thus $G \not\mapsto (M_n, M_n)$; see Figure 2.4.

To rule out this example, we add the condition

$$N - n_1 = n_2 + \ldots + n_s \geq 2n - 1. \tag{2.42}$$

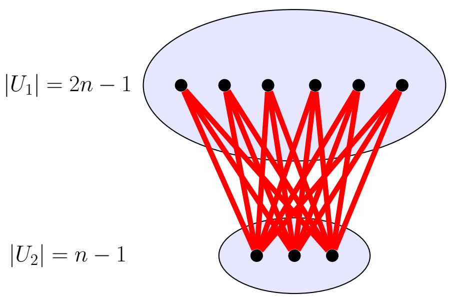


Figure 2.3: Example 2.2.1.1.

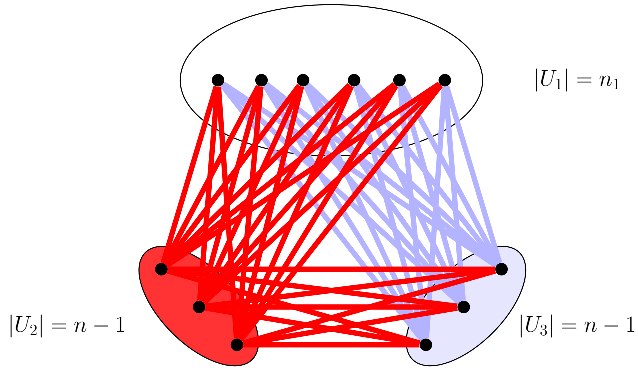


Figure 2.4: Example 2.2.1.2.

2.2.1.3 Example with no red M_n and no blue P_{2n+1}: too few vertices

Let $G = K_{3n-1}$. Partition $V(G)$ into sets U_1 and U_2 with $|U_1| = 2n$ and $|U_2| = n-1$. Color the edges of $G[U_1, U_2]$ with red and the rest of the edges with blue. Since the red subgraph of G has vertex cover U_2 with $|U_2| = n - 1$, it does not contain M_n. Since each component of the blue subgraph of G has fewer than $2n+1$ vertices, it does not contain P_{2n+1}.

Therefore
$$R(P_{2n}, P_{2n+1}) \geq R(M_n, P_{2n+1}) \geq 3n,$$
which yields for P_{2n+1} the following strengthening of (2.41):

$$\text{for } P_{2n+1}, \quad N \geq 3n. \tag{2.43}$$

2.2.1.4 Example with no monochromatic $C_{\geq 2n}$ when $N - n_1 - n_2 \leq 2$

This example, and all the ones that follow, show that additional restrictions are necessary when G is bipartite or close to bipartite.

Let $G = K_{n_1,\ldots,n_s}$ satisfy (2.41) and (2.42) with $N - n_1 - n_2 \leq 2$ such that $n_1 \leq 2n-2$. Then also $n_2 \leq 2n-2$, so $G \subseteq K_{2n-2,2n-2,1,1}$. Thus we assume $G = K_{2n-2,2n-2,1,1}$ with $V_1 = \{v_1, \ldots, v_{2n-2}\}$, $V_2 = \{u_1, \ldots, u_{2n-2}\}$, $V_3 = \{x\}$ and $V_4 = \{y\}$. Let $V_1' = \{v_1, \ldots, v_{n-1}\}$, $V_1'' = V_1 - V_1'$, $V_2' = \{u_1, \ldots, u_{n-1}\}$, $V_2'' = V_2 - V_2'$. Color the edges in $G[V_1', V_2']$, $G[V_1'', V_2'']$ and in $G[V_3, V_1 \cup V_2 \cup V_4]$ with red, and all other edges with blue. Then the red graph G_1 has cut vertex x, and the components of $G_1 - x$ have sizes $2n-2$, $2n-2$ and 1, so G_1 has no $C_{\geq 2n}$. Similarly, G_2 contains no $C_{\geq 2n}$; see Figure 2.5.

To rule out this example, we add the condition

$$\text{For } C_{\geq 2n},\ \text{if } N - n_1 - n_2 \leq 2,\ \text{then } \ n_1 \geq 2n-1. \tag{2.44}$$

2.2.1.5 Example with no monochromatic $C_{\geq 2n}$ when $N - n_1 - n_2 \leq 1$

Let $G = K_{n_1,\ldots,n_s}$ satisfying (2.41), (2.42) and (2.44) with $N - n_1 - n_2 \leq 1$ such that $N + n_1 \leq 6n-3$. Since by (2.44), $n_1 \geq 2n-1$, we get $N - n_1 \leq (6n-3) - 2(2n-1) = 2n-1$, but (2.42) implies $N - n_1 \geq 2n-1$; therefore both inequalities are tight and $N - n_1 = n_1 = 2n-1$. Hence $G \subseteq K_{2n-1,2n-2,1}$, which is a subgraph of the graph $K_{2n-2,2n-2,1,1}$ considered in Example 2.2.1.4.

This example is not ruled out by (2.44), so we add the condition

$$\text{For } C_{\geq 2n},\ \text{if } N - n_1 - n_2 \leq 1,\ \text{then } \ n_1 + N \geq 6n-2. \tag{2.45}$$

2.2.1.6 Example with no monochromatic P_{2n+1} when G is bipartite

Suppose $n_3 = 0$ and $n_1 \leq 2n$. Then $n_2 \leq 2n$ as well, so $G \subseteq K_{2n,2n}$. Thus we assume $G = K_{2n,2n}$ with $V_1 = \{v_1, \ldots, v_{2n}\}$ and $V_2 = \{u_1, \ldots, u_{2n}\}$. Let $V_1' = \{v_1, \ldots, v_n\}$, $V_1'' = V_1 - V_1'$, $V_2' = \{u_1, \ldots, u_n\}$, $V_2'' = V_2 - V_2'$. Color the edges in $G(V_1', V_2')$ and $G(V_1'', V_2'')$ with red, and all other edges with blue. Then each component in the red graph and each component in the blue graph has $2n$ vertices and thus does not contain P_{2n+1}; see Figure 2.6.

To rule out this example, we add the condition

$$\text{For } P_{2n+1},\ \text{if } n_3 = 0,\ \text{then } \ n_1 \geq 2n+1. \tag{2.46}$$

2.2.1.7 Example with no monochromatic C_{2n} when $N - n_1 - n_2 \leq 2$

Let $G = K_{n_1,\ldots,n_s}$ satisfying (2.41), (2.42) and (2.44) with $N - n_1 - n_2 = 2$ such that $N \leq 4n-2$. By (2.44), $N - n_1 \leq 2n-1$. Now (2.42) implies $N - n_1 = 2n-1 = n_1$. Hence $G \subseteq K_{2n-1,2n-3,1,1}$. Thus we assume $G = K_{2n-1,2n-3,1,1}$ with $V_1 = \{v_1, \ldots, v_{2n-1}\}$, $V_2 = \{u_1, \ldots, u_{2n-3}\}$, $V_3 = \{x\}$ and $V_4 = \{y\}$. Define $A = \{v_2, v_3, \ldots, v_n\}$, $B = \{v_{n+1}, v_{n+2}, \ldots, v_{2n-1}\}$, $C = \{u_1, u_2, \ldots, u_{n-1}\}$ and $D = \{u_n, u_{n+1}, \ldots, u_{2n-3}\}$. We assign the colors to the edges of G as follows.

1. $G[A, C]$ and $G[B, D]$ are complete bipartite red graphs.

2. $G[A, D]$ and $G[B, C]$ are complete bipartite blue graphs.

3. v_1 has all blue edges to V_2.

4. x has all red edges to $V_1 \cup V_2 \cup \{y\}$.

5. y has all red edges to $B \cup D \cup \{x\}$ and all blue edges to $A \cup C \cup \{v_1\}$.

We claim that G has no monochromatic cycle of length $2n$. Indeed, consider first the red graph G_1. The graph $G_1 - x$ has three components: a) $A \cup C$ of size $2n-2$, b) $\{v_1\}$ of size 1, and c) $B \cup D \cup \{y\}$ of size $2n-2$. Thus G has no red cycle of length $2n$ since the largest block of G_1 has order $2n-1$.

Consider now the blue graph G_2. We ignore x since it is isolated. Suppose G_2 contains a $2n$-cycle F. Since v_1 is a cut vertex of $G_2 - \{y\}$ with the components of $G_2 - \{y, v_1\}$ of order $2n-3$ and $2n-2$, F contains y.

If we delete from G_2 all edges in $G_2[\{y\}, C]$, then the blocks in the remaining blue graph will be of order $2n-1$ and $2n-1$; thus F contains an edge from y to C, say yz. Furthermore, if yz is the only edge in F connecting y to C, then all other edges in F belong to the bipartite graph $H = G_2[A \cup B \cup \{v_1\}, D \cup \{y\} \cup C]$. But this bipartite graph H cannot have a path of odd length $2n-1$ between the vertices y and z in the same part.

Thus, F has to use two edges from y to C, say yz_1 and yz_2. Then the problem is reduced to finding a blue path from z_1 to z_2 of length $2n-2$ in $G_2[C, B \cup \{v_1\}]$. However, it is impossible because $|C| = n-1$ and the longest path from z_1 to z_2 in $G_2[C, B \cup \{v_1\}]$ has $2n-3$ vertices.

Note that this example has cycles of length greater than $2n-1$, but all such cycles are odd.

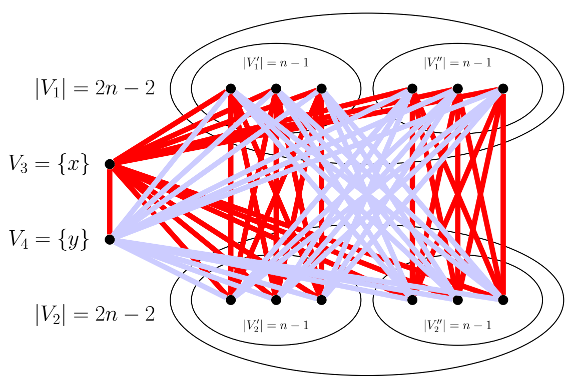


Figure 2.5: Example 2.2.1.4.

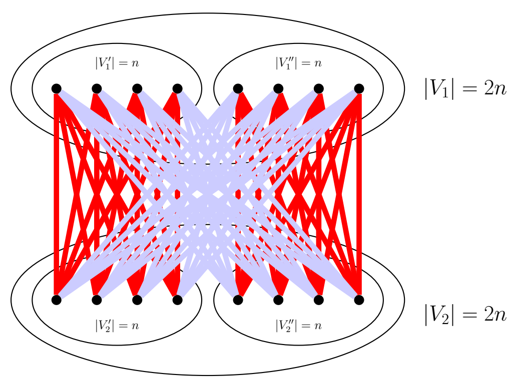


Figure 2.6: Example 2.2.1.5.

To rule out this example, we add the condition

$$\textit{For } C_{2n}, \textit{ if } N - n_1 - n_2 \leq 2, \textit{ then } N \geq 4n - 1. \tag{2.47}$$

2.2.1.8 Results

Our key result is that for large n, the necessary conditions (2.41), (2.42) and (2.47) for the presence in a 2-edge-colored $K_{n_1,\ldots,n_s}$ of a monochromatic C_{2n}, together are also sufficient for this.

Theorem 2.33. *Let $s \geq 2$ and n be sufficiently large. Let $n_1 \geq \ldots \geq n_s$ and $N = n_1 + \ldots + n_s$ satisfy (2.41), (2.42) and (2.47). Then for each 2-edge-coloring f of the complete s-partite graph $K_{n_1,\ldots,n_s}$, there exists a monochromatic cycle C_{2n}.*

Based on Theorem 2.33, we derive our other results. The first of them is on cycles of length at least $2n$ (it extends a result by DeBiasio and Krueger [34]). Recall that (2.47) is not necessary for the existence of a monochromatic $C_{\geq 2n}$, but (2.41), (2.42), (2.44) and (2.45) are.

Theorem 2.34. *Let $s \geq 2$ and n be sufficiently large. Let $n_1 \geq \ldots \geq n_s$ and $N = n_1 + \ldots + n_s$ satisfy (2.41), (2.42), (2.44) and (2.45). Then for each 2-edge-coloring f of the complete s-partite graph $K_{n_1,\ldots,n_s}$, there exists a monochromatic cycle $C_{\geq 2n}$.*

The results for paths of even and odd length are somewhat different. The first of them shows that for large n, the necessary conditions (2.41) and (2.42) for the presence in a 2-edge-colored $K_{n_1,\ldots,n_s}$ of a monochromatic connected matching M_n, together are sufficient for the presence of the monochromatic path P_{2n}.

Theorem 2.35. *Let $s \geq 2$ and n be sufficiently large. Let $n_1 \geq \ldots \geq n_s$ and $N = n_1 + \ldots + n_s$ satisfy (2.41) and (2.42). Then for each 2-edge-coloring f of the complete s-partite graph $K_{n_1,\ldots,n_s}$, there exists a monochromatic path P_{2n}.*

Our last result implies Conjecture 2.5:

Theorem 2.36. *Let $s \geq 2$ and n be sufficiently large. Let $n_1 \geq \ldots \geq n_s$ and $N = n_1 + \ldots + n_s$ satisfy (2.42), (2.43) and (2.46). Then for each 2-edge-coloring f of the complete s-partite graph $K_{n_1,\ldots,n_s}$, there exists a monochromatic path P_{2n+1}.*

In Section 2.2.2, we describe our main tools: the Szemerédi Regularity Lemma, connected matchings and theorems on the existence of Hamiltonian cycles in dense graphs. In Section 2.2.3 we set up and describe the structure of the proof of Theorem 2.33, and in Sections 2.2.4, 2.2.5, 2.2.6, and 2.2.7, we present this proof. In Sections 2.2.8, 2.2.9 and 2.2.10, we prove Theorems 2.34, 2.35 and 2.36.

2.2.2 Tools

As in many recent papers on Ramsey numbers of paths (see [11, 27, 34, 39, 55, 56, 64, 74, 82] and some references in them), our proof heavily uses the Szemerédi Regularity Lemma [85] and the idea of connected matchings in regular partitions of reduced graphs due to Łuczak [73].

2.2.2.1 Regularity

We say that a pair (V_1, V_2) of two disjoint vertex sets $V_1, V_2 \subseteq V(G)$ is (ϵ, G)-*regular* if

$$\left| \frac{|E(X,Y)|}{|X||Y|} - \frac{|E(V_1,V_2)|}{|V_1||V_2|} \right| < \epsilon$$

for all $X \subseteq V_1$ and $Y \subseteq V_2$ with $|X| > \epsilon |V_1|$ and $|Y| > \epsilon |V_2|$.

We use a 2-color version of the Regularity Lemma, following Gyárfás, Ruszinkó, Sárközy, and Szemerédi [55].

Lemma 2.37 (2-color version of the Szemerédi Regularity Lemma)**.** *For every $\epsilon > 0$ and integer $m > 0$, there are positive integers M and n_0 such that for $n \geq n_0$ the following holds. For all graphs G_1 and G_2 with $V(G_1) = V(G_2) = V$, $|V| = n$, there is a partition of V into $L+1$ disjoint classes (clusters) $(V_0, V_1, V_2, \ldots, V_L)$ such that*

- *$m \leq L \leq M$,*
- *$|V_1| = |V_2| = \ldots = |V_L|$,*
- *$|V_0| < \epsilon n$,*
- *Apart from at most $\epsilon\binom{L}{2}$ exceptional pairs, the pairs $\{V_i, V_j\}$ are (ϵ, G_q)-regular for $q = 1$ and 2.*

Additionally, if $G_1 \cup G_2$ is a multipartite graph with partition $V = V_1^* \cup V_2^* \cup \ldots \cup V_s^*$, with $s < 6$, we can guarantee that *each of the clusters $V_1, V_2, \ldots, V_L$ is contained entirely in a single part of this partition.*

To do so, for a given $\epsilon > 0$, we begin by arbitrarily partitioning each V_i^* into parts $V_{i,1}^*, V_{i,2}^*, \ldots$, each of size $\lfloor \frac{\epsilon}{10} n \rfloor$, with a part $V_{i,0}^*$ of size at most $\frac{\epsilon}{10} n$ left over. This is an equitable partition of $V - \bigcup_{i=1}^k V_{i,0}^*$, a set of at least $(1 - \frac{9\epsilon}{10})n$ vertices. The Regularity Lemma allows us to refine any equitable partition into one that satisfies the conclusions of Lemma 2.37. Working with the subgraphs of G_1 and G_2 excluding the vertices in $\bigcup_{i=1}^k V_{i,0}^*$, take such a refinement with parameters $\frac{\epsilon}{9}$ and m, then add $\bigcup_{i=1}^k V_{i,0}^*$ to its exceptional cluster V_0. The resulting exceptional cluster still has size at most ϵn, so we have obtained a partition satisfying the conditions of Lemma 2.37 in which each of $V_1, V_2, \ldots, V_L$ is entirely contained in one of $V_1^*, V_2^*, \ldots, V_k^*$.

2.2.2.2 Connected matchings

Let $\alpha'(G)$ denote the size of a largest matching and $\alpha'_*(G)$ denote the size of a largest connected matching in G. Let $\alpha(G)$ denote the independence number and $\beta(G)$ denote the size of a smallest vertex cover in G.

Łuczak [73] was the first to use the fact that the existence of large connected matchings in the reduced graph of a regular partition of a large graph G implies the existence of long paths and cycles in G. A flavor of it is illustrated by the following fact.

Lemma 2.38 (Lemma 8 in [74] and Lemma 1 in [64]). *Let a real number $c > 0$ and a positive integer k be given. If for every $\epsilon > 0$ there exists a $\delta > 0$ and an n_0 such that for every even $n > n_0$ and each graph G with $v(G) > (1+\epsilon)cn$ and $e(G) \geq (1-\delta)\binom{v(G)}{2}$ and each k-edge-coloring of G has a monochromatic connected matching $M_{n/2}$, then for large N, $R_k(C_N) \leq (c + o(1))N$ (and hence $R_k(P_N) \leq (c + o(1))N$).*

We use the following property of (ϵ, G)-regular pairs:

Lemma 2.39 (Lemma 3 in [55]). *For every $\delta > 0$ there exist $\epsilon > 0$ and t_0 such that the following holds. Let G be a bipartite graph with bipartition (V_1, V_2) such that $|V_1| = |V_2| = t \geq t_0$, and let the pair (V_1, V_2) be (ϵ, G)-regular. Moreover, assume that $\deg_G(v) > \delta t$ for all $v \in V(G)$.*

Then for every pair of vertices $v_1 \in V_1, v_2 \in V_2$, G contains a Hamiltonian path with endpoints v_1 and v_2.

Since we are aiming at an exact bound, we need a stability version of a result similar to Lemma 2.38. To state it, we need some definitions.

Definition 2.40. *For $\epsilon > 0$, an N-vertex s-partite graph G with parts $V_1, \ldots, V_s$ of sizes $n_1 \geq n_2 \geq \ldots \geq n_s$, and a 2-edge-coloring $E = E_1 \cup E_2$, is (n, s, ϵ)-*suitable *if the following conditions hold:*

$$N = n_1 + \ldots + n_s \geq 3n - 1, \tag{S1}$$

$$n_2 + n_3 + \ldots + n_s \geq 2n - 1, \tag{S2}$$

and if $\widetilde{V}_i$ is the set of vertices in V_i of degree at most $N - \epsilon n - n_i$ and $\widetilde{V} = \bigcup_{i=1}^{s} \widetilde{V}_i$, then

$$|\widetilde{V}| = |\widetilde{V}_1| + \ldots + |\widetilde{V}_s| < \epsilon n. \tag{S3}$$

We do not require $E_1 \cap E_2 = \emptyset$; an edge can have one or both colors. We write $G_i = G[E_i]$ for $i = 1, 2$.

Our stability theorem gives a partition of the vertices of near-extremal graphs called a (λ, i, j)-*bad partition*. There are two types of bad partitions.

Definition 2.41. *For $i \in \{1, 2\}$, $\lambda > 0$, and an (n, s, ϵ)-suitable graph G, a partition $V(G) = W_1 \cup W_2$ of $V(G)$ is $(\lambda, i, 1)$-*bad *if the following holds:*

(i) $(1 - \lambda)n \leq |W_2| \leq (1 + \lambda)n_1$;

(ii) $|E(G_i[W_1, W_2])| \leq \lambda n^2$;

(iii) $|E(G_{3-i}[W_1])| \leq \lambda n^2$.

Definition 2.42. *For $i \in \{1,2\}$, $\lambda > 0$, and an (n,s,ϵ)-suitable graph G, a partition $V(G) = V_j \cup U_1 \cup U_2$, $j \in [s]$, of $V(G)$ is $(\lambda, i, 2)$-bad if the following holds:*

(i) $|E(G_i[V_j, U_1])| \leq \lambda n^2$;

(ii) $|E(G_{3-i}[V_j, U_2])| \leq \lambda n^2$;

(iii) $n_j = |V_j| \geq (1-\lambda)n$;

(iv) $(1-\lambda)n \leq |U_1| \leq (1+\lambda)n$;

(v) $(1-\lambda)n \leq |U_2| \leq (1+\lambda)n$.

Our stability theorem [6] is:

Theorem 2.43 (Theorem 9 [6])**.** *Let $0 < \epsilon < 10^{-3}\gamma < 10^{-6}$, $n \geq s \geq 2$, $n > \frac{100}{\gamma}$. Let G be an (n,s,ϵ)-suitable graph. If $\max\{\alpha'_*(G_1), \alpha'_*(G_2)\} \leq n(1+\gamma)$, then for some $i \in [2]$ and $j \in [2]$, $V(G)$ has a $(68\gamma, i, j)$-bad partition.*

2.2.2.3 Theorems on Hamiltonian cycles in bipartite graphs

Theorem 2.44 (Chvátal [28], see also Corollary 5 in Chapter 10 in [12])**.** *Let H be a $2n$-vertex bipartite graph with vertices $u_1, u_2, \ldots, u_n$ on one side and $v_1, v_2, \ldots, v_n$ on the other, such that $d(u_1) \leq \ldots \leq d(u_n)$ and $d(v_1) \leq \ldots \leq d(v_n)$.*

If $d_H(u_i) \leq i < n \implies d_H(v_{n-i}) \geq n-i+1$, then H is Hamiltonian.

Theorem 2.45 (Berge [12])**.** *Let H be a $2m$-vertex bipartite graph with vertices $u_1, u_2, \ldots, u_m$ on one side and $v_1, v_2, \ldots, v_m$ on the other, such that $d(u_1) \leq \ldots \leq d(u_m)$ and $d(v_1) \leq \ldots \leq d(v_m)$. Suppose that for the smallest two indices i and j such that $d(u_i) \leq i+1$ and $d(v_j) \leq j+1$, we have $d(u_i) + d(v_j) \geq m+2$.*

Then H is Hamiltonian bi-connected: for every i and j, there is a Hamiltonian path with endpoints u_i and v_j.

Theorem 2.46 (Las Vergnas [68], see also Theorem 11 on page 214 in [12])**.** *Let H be a $2n$-vertex bipartite graph with vertices $u_1, u_2, \ldots, u_n$ on one side and $v_1, v_2, \ldots, v_n$ on the other, such that $d(u_1) \leq \ldots \leq d(u_n)$ and $d(v_1) \leq \ldots \leq d(v_n)$. Let q be an integer, $0 \leq q \leq n-1$.*

If, whenever $u_i v_j \notin E(H)$, $d(u_i) \leq i+q$, and $d(v_j) \leq j+q$, we have

$$d(u_i) + d(v_j) \geq n+q+1,$$

then each set of q edges that form vertex-disjoint paths is contained in a Hamiltonian cycle of G.

2.2.2.4 Using the tools

Our strategy to prove Theorem 2.33 is: We first apply a 2-colored version of the Regularity Lemma to G to obtain a reduced graph G^r. If G^r has a large monochromatic connected matching then we find a long monochromatic cycle using Lemma 2.38. If G^r does not have a large monochromatic connected matching, then we use Theorem 2.43 to obtain a bad partition of G^r. We then transfer the bad partition of G^r to a bad partition of G and work with this partition. In some important cases, theorems on Hamiltonian cycles help to find a monochromatic cycle C_{2n} in G.

2.2.3 Setup of the proof of Theorem 2.33

Formally, we need to prove the theorem for every N-vertex complete s-partite graph G with parts $(V_1^*, V_2^*, \ldots, V_s^*)$ such that the numbers $n_i = |V_i^*|$ satisfy $n_1 \geq n_2 \geq \ldots \geq n_s$ and the following three conditions:

$(S1')$ $N = n_1 + \ldots + n_s \geq 3n - 1$;

$(S2')$ $N - n_1 = n_2 + \ldots + n_s \geq 2n - 1$;

$(S3')$ If $N - n_1 - n_2 \leq 2$, then $N \geq 4n - 1$.

For a given large n, we consider a possible counterexample with the minimum $N + s$. In view of this, it is enough to consider the lists $(n_1, \ldots, n_s)$ satisfying $(S1')$, $(S2')$ and $(S3')$ such that

(a) for each $1 \leq i \leq s$, if $n_i > n_{i+1}$, then the list $(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_s)$ does not satisfy some of $(S1')$, $(S2')$ and $(S3')$;

(b) if $s \geq 4$, then the list $(n_1, \ldots, n_{s-2}, n_{s-1} + n_s)$ (possibly with the entries rearranged into a non-increasing order) does not satisfy some of $(S1')$, $(S2')$ and $(S3')$.

Case 1: $N - n_1 - n_2 \geq 3$ and $N > 3n - 1$. Then $(S3')$ holds by default. If $n_1 > n_2$, then the list $(n_1 - 1, n_2, n_3, \ldots, n_s)$ still satisfies the conditions $(S1')$, $(S2')$ and $(S3')$, a contradiction to (a). Hence $n_1 = n_2$. Choose the maximum i such that $n_1 = n_i$. If $N - n_1 > 2n - 1$, consider the list $(n_1, \ldots, n_{i-1}, n_i - 1, n_{i+1}, \ldots, n_s)$. In this case $(S1')$ and $(S2')$ still are satisfied for this list; so by (a), $(S3')$ fails for it. As we assumed $N - n_1 - n_2 \geq 3$, we must have $i \geq 3$ and $N - n_1 - n_2 = 3$ for $(S3')$ to fail for this list; this further implies $n_1 = n_i \leq 3$, so $N = n_1 + n_2 + 3 \leq 9$, a contradiction. Thus in this case $N - n_1 = 2n - 1$. Therefore, $n_1 = N - (N - n_1) \geq 3n - (2n - 1) = n + 1$ and hence $n_2 \geq n + 1$, so $N - n_1 - n_2 \leq (2n - 1) - (n + 1) = n - 2$. Then the list $(n_1, n_1, N - 2n_1)$ satisfies $(S1')$–$(S3')$. Summarizing, we get

$$\textit{if } N - n_1 - n_2 \geq 3 \textit{ and } N > 3n - 1, \textit{ then} \quad s = 3,\ n_2 + n_3 = 2n - 1 \quad \textit{and} \quad n_1 = n_2. \tag{2.48}$$

Case 2: $N - n_1 - n_2 \geq 3$ and $N = 3n - 1$. Again $(S3')$ holds by default. By $(S2')$, $n_1 \leq n$, hence $N - n_1 - n_2 \geq n - 1$. If $s \geq 4$ and $n_{s-1} + n_s \leq n$, then let L be the list obtained from $(n_1, \ldots, n_s)$ by replacing the two entries n_{s-1} and n_s with $n_{s-1} + n_s$ and then possibly rearrange the entries into non-increasing order. By construction, L satisfies $(S1')$–$(S3')$, a contradiction to (b). Hence $n_{s-1} + n_s \geq n + 1$. We also have $n_{s-1} + n_s \geq n + 1$ if $s = 3$, since in this case $n_{s-1} + n_s = N - n_1 \geq 2n - 1$. If $s \geq 6$, then $N \geq 3(n_{s-1} + n_s) \geq 3n + 3$, contradicting $N = 3n - 1$. Thus

$$\textit{if } N - n_1 - n_2 \geq 3 \textit{ and } N = 3n - 1, \textit{ then } n_1 \leq n, \quad s \leq 5, \quad n_{s-1} + n_s \geq n + 1. \tag{2.49}$$

Case 3: $N - n_1 - n_2 \leq 2$. Then $N \leq 2n_1 + 2$, so by $(S3')$, $2n_1 + 2 \geq N \geq 4n - 1$, implying $n_1 \geq 2n - 1$. If $n_1 \geq 2n$, then $(S2')$ implies that $G \supseteq K_{2n,2n-1}$. If $n_1 = 2n - 1$, then by $(S3')$, $N - n_1 \geq 2n$, so again $G \supseteq K_{2n,2n-1}$. Thus we can assume that

$$\textit{if } N - n_1 - n_2 \leq 2, \textit{ then } \quad G = K_{2n,2n-1}. \tag{2.50}$$

As we have seen,

$$\text{in each of Cases 1, 2 and 3 we have } s \leq 5. \tag{2.51}$$

Fix an arbitrary 2-edge-coloring $E(G) = E_1 \cup E_2$ of G. For $i \in [2]$ and $v \in V(G)$, let $G_i := (V(G), E_i)$ and $d_i(v)$ denote the degree of v in G_i.

2.2.4 Regularity

2.2.4.1 Applying the 2-colored version of the Regularity Lemma

We first choose parameter α so that $0 < \alpha < 10^{-10}$ and then choose ϵ such that $\epsilon < 10^{-20}$ and $0 < 10^6\epsilon < \alpha$ so that the pair $(\frac{\alpha}{2}, 3\epsilon)$ satisfies the relation of (δ, ϵ) in Lemma 2.39 with $\frac{\alpha}{2}$ playing the role of δ. Here, ϵ is the parameter for the Regularity Lemma, and α is our cutoff for the edge density at which we give an edge of the reduced graph a color.

We apply Lemma 2.37 to obtain a partition $(V_0, V_1, \ldots, V_L)$ of $V(G)$, with each of $V_1, V_2, \ldots, V_L$ contained entirely in one of $V_1^*, V_2^*, \ldots, V_k^*$. Define the k-partite *reduced graph* G^r as follows:

- The vertices of G^r are v_i for $i = 1, 2, \ldots, L$. A k-partition $(V_1', V_2', \ldots, V_k')$ of $V(G^r)$ is induced by the k-partition of G, and reordered if necessary so that $|V_1'| \geq |V_2'| \geq \ldots \geq |V_k'|$.
- There is an edge between v_i and v_j iff v_i and v_j are in different parts of the k-partition, and the pair $\{V_i, V_j\}$ is (ϵ, G_q)-regular for both $q = 1$ and $q = 2$.
- The reduced graph G^r is missing at most $\epsilon\binom{L}{2}$ edges between distinct pairs $\{V_i', V_j'\}$.
- We give G^r a 2-edge-multicoloring: two graphs (G_1^r, G_2^r) whose union include every edge of

G^r, but are not necessarily edge-disjoint. We add edge $v_iv_j \in E(G^r)$ to G_q^r if G_q contains at least $\alpha|V_i||V_j|$ of the edges between V_i and V_j. Since $G = G_1 \cup G_2$ contains all $|V_i||V_j|$ edges between V_i and V_j, each edge of G^r is added to either G_1^r or G_2^r, and possibly to both.

Let $t = |V_1| = |V_2| = \ldots = |V_L|$, $\ell_i = |V_i'|$ for $i = 1, \ldots, k$, and $\ell := \frac{n-\epsilon N}{t}$; since $N \leq 4n - 1$, we have $\ell t \geq (1 - 5\epsilon)n$.

Because $|V_0| \leq \epsilon N$, we have $(1-\epsilon)N \leq Lt \leq N$ and $n_i - \epsilon N \leq \ell_i t \leq n_i$. Therefore,

- $Lt \geq (1-\epsilon)N \geq 3n - 1 - \epsilon N = 3(\ell t + \epsilon N) - 1 - \epsilon N \geq 3\ell t - 1 + 2\epsilon n$, which means $L \geq 3\ell - 1$.
- $Lt \leq N \leq 4n - 1 = 4(\ell t + \epsilon N) - 1 \leq 5\ell t$, which means $L \leq 5\ell$.
- $Lt - \ell_1 t \geq N - n_1 - \epsilon N \geq 2n - 1 - \epsilon N \geq 2(\ell t + \epsilon N) - 1 - \epsilon N \geq 2\ell t - 1 + \epsilon N$, which means $L - \ell_1 \geq 2\ell - 1$.

Recall that G^r is missing at most $\epsilon\binom{L}{2} \leq \epsilon\frac{L^2}{2} < 16\epsilon L^2$ edges between distinct pairs $\{V_i', V_j'\}$. Since the number of V_i's missing at least $4\sqrt{\epsilon}\ell$ edges is less than $4\sqrt{\epsilon}\ell$, G^r is $(\ell, k, 4\sqrt{\epsilon})$-suitable. We apply Theorem 2.43 to the graph G^r with γ such that $10^{-6} > \gamma > 1000\alpha$ and $\gamma > 4000\sqrt{\epsilon}$. Then we conclude that either G^r has a monochromatic connected matching of size $(1+\gamma)\ell$, or else $V(G)$ has a $(68\gamma, i, j)$-bad partition for some $i \in [2]$ and $j \in [2]$.

2.2.4.2 Handling a large connected matching in the reduced graph

For every edge $v_iv_j \in G_1^r$, the corresponding pair (V_i, V_j) is (ϵ, G_1)-regular and contains at least αt^2 edges of G_1. Let $X_{ij} \subseteq V_i$ be the set of all vertices of V_i with fewer than $\frac{\alpha}{2}t$ edges of G_1 to V_j, and let $Y_{ij} \subseteq V_j$ the set of all vertices of V_j with fewer than $\frac{\alpha}{2}t$ edges of G_1 to V_i. Note we have $Y_{ij} = X_{ji}$ but we keep using the notation Y_{ij} for emphasising they are in different parts. Then $\frac{|E(X_{ij}, V_j)|}{|X_{ij}||V_j|} \leq \frac{\alpha}{2}$, so $|X_{ij}| \leq \epsilon t$ to avoid violating (ϵ, G_1)-regularity; similarly, $|Y_{ij}| \leq \epsilon t$. Call vertices of $V_i \cup V_j$ which are not in $X_{ij} \cup Y_{ij}$ *typical* for the pair (V_i, V_j) (or for the edge v_iv_j of G_1).

Let $\mathcal{M}$ be a connected matching in G_1^r of size $(1+\gamma)\ell$. Give the edges in $\mathcal{M}$ an arbitrary cyclic ordering.

If $v_{i_1}v_{j_1}$ and $v_{i_2}v_{j_2}$ are edges of $\mathcal{M}$ which are consecutive in the ordering, we shall find a path $P(j_1, i_2)$ in G_1 joining a vertex of $V_{j_1} \setminus Y_{i_1j_1}$ to a vertex of $V_{i_2} \setminus X_{i_2j_2}$. To do so, we begin by finding a path P^r from v_{j_1} to v_{i_2} in G_1^r, then find a realization of that path in G_1. Pick a starting point of $P(j_1, i_2)$ typical both for the edge $v_{i_1}v_{j_1}$ and for the first edge of P^r. Next, choose the path greedily, making sure to satisfy the following conditions:

- Choose a neighbor of the previous vertex chosen which is typical for the next edge of P^r (or for $v_{i_2j_2}$ when we reach the end of P^r).
- Choose a vertex which has not been chosen for any previous paths.

As we construct $P(j_1, i_2)$, the last vertex we have chosen is always typical for the edge of P^r we are about to realize; therefore we have at least $\frac{\alpha}{2}t$ options for its neighbors. At most ϵt of them are eliminated because they are not typical for the next edge, and at most L^2 are eliminated because they have been chosen for previous paths. Since L is upper bounded by M which is independent of n, and $\epsilon < 10^{-6}\alpha$, we can always choose such a vertex.

Moreover, we may choose the paths such that their total length has the same parity as $|\mathcal{M}|$. If the component of G_1^r containing $\mathcal{M}$ is not bipartite, then each path can be chosen to have any parity we like. If the component of G_1^r containing $\mathcal{M}$ is bipartite, then this condition is satisfied automatically: if we join the paths of P^r we chose by the edges of $\mathcal{M}$, we get a closed walk, which must have even length.

Once all these paths are chosen, we combine them into a long even cycle in G_1. For each edge $v_i v_j$ in the matching $\mathcal{M}$, we have vertices $x \in V_i$ and $y \in V_j$, both typical for (V_i, V_j), which are the endpoints of two paths we have constructed. We show that we can find a path from x to y using only edges of G_1 between V_i and V_j of any odd length between $t-1$ and $(1-3\epsilon)2t-1$.

To do so, we choose any $X \subseteq V_i$ with $|X| \geq \frac{t}{2}$ that contains x and at least $\frac{\alpha}{2}t$ neighbors of y; similarly, we choose $Y \subseteq V_j$ with $|Y| = |X|$ that contains y and at least $\frac{\alpha}{2}t$ neighbors of x. If we want the path to have length $2Ct - 1$ where $C \in [\frac{1}{2}, 1-3\epsilon]$, we begin by choosing X and Y of size $(C+3\epsilon)t$. The pair (X, Y) is $(2\epsilon, G_1)$-regular with density at least $\alpha - \epsilon$, so there are at most 2ϵ vertices in each of X and Y which have fewer than $\frac{\alpha}{2}t$ neighbors on the other side; by our construction of X and Y, x and y are not among them.

Let $X' \subseteq X$ and $Y' \subseteq Y$ be the subsets obtained by deleting these low-degree vertices, leaving at least $(C+\epsilon)t$ vertices on each side, and then deleting enough vertices from each part to make $|X'| = |Y'| = Ct$. The pair (X', Y') is $(3\epsilon, G_1)$-regular, and all vertices have minimum degree at least $(\alpha - 3\epsilon)t$, so by Lemma 2.39, there is a path from x to y using all vertices of X' and Y', which has the desired length $2Ct - 1$.

If we use $C = 1 - 3\epsilon$ for each edge $v_i v_j$ in the matching $\mathcal{M}$, then the cycle contains at least $2(1-3\epsilon)t$ vertices for each edge of $\mathcal{M}$, even ignoring the paths we constructed between them, while $|\mathcal{M}| \geq (1+10\epsilon)\ell$; therefore the total length is at least

$$2(1-3\epsilon)(1+10\epsilon)\ell t \geq 2(1-3\epsilon)(1+10\epsilon)(1-5\epsilon)n \geq (1+\epsilon)2n.$$

If we use $C = \frac{1}{2}$ each edge $v_i v_j$, then the cycle contains only t vertices for each edge of $\mathcal{M}$, giving approximately half as many edges. Up to parity, we are free to choose any length in this range, and therefore it is possible to construct a path in G_1 of length exactly $2n$.

2.2.4.3 Handling a bad partition of the reduced graph

We will show in Sections 2.2.5 and 2.2.6 how to find a long monochromatic cycle in a bad partition of G. In Section 2.2.4.3, we show that a bad partition of G^r corresponds to a bad partition of G.

1. If $X \subseteq V(G^r)$ has size $C\ell$, then the corresponding set of vertices in G is $\bigcup_{v_i \in X} V_i$. It has size $C\ell t$, which is in the range $[(1-5\epsilon)Cn, Cn]$.
2. If $|E_{G_i^r}(X)| \leq \lambda \ell^2$, then each of those $\lambda \ell^2$ edges of G_i^r corresponds to at most t^2 edges of G_i, for $\lambda \ell^2 t^2 \leq \lambda n^2$ edges.

   Additionally, edges not in G_i^r may appear in G_i; across all of G_i there are at most $\alpha t^2 \binom{L}{2} \leq \frac{1}{2}\alpha N^2 \leq 10\alpha n^2$ edges that occur in this way.

   Moreover, edges from at most $\epsilon \binom{L}{2}$ exceptional pairs may appear in G_i, contributing at most $10\epsilon n^2$ edges in total by the same calculation.

   To summarize, there are at most $(\lambda + 10\alpha + 10\epsilon)n^2$ edges in G_i corresponding to $E_{G_i^r}(X)$. A similar argument applies to a bound on $|E_{G_i^r}(X, Y)|$ for $X, Y \subseteq V(G^r)$.
3. There are fewer than $\epsilon N \leq 5\epsilon n$ vertices from the exceptional part V_0, which can generally be assigned to any part of any bad partition without changing the approximate structure.

Thus, for $10^{-3} > \lambda > 1000\alpha > 10^9\epsilon > 0$, if G^r has a $(\lambda, i, 1)$-bad partition ($i \in [2]$) $V(G^r) = W_1^r \cup W_2^r$, then G has a corresponding $(2\lambda, i, 1)$-bad partition with

(0):

$$W_1 := \left(\bigcup_{v_i \in W_1^r} V_i \right) \cup V_0 \text{ and } W_2 := \bigcup_{v_i \in W_2^r} V_i.$$

(i):

$$(1-2\lambda)n \leq (1-\lambda)(1-5\epsilon)n \leq (1-\lambda)\ell t \leq |W_2| \leq (1+\lambda)\ell_1 t \leq (1+\lambda)n_1.$$

(ii):

$$|E(G_i[W_1, W_2])| \leq (\lambda + 10\alpha + 10\epsilon + 5\epsilon)n^2 \leq 2\lambda n^2.$$

(iii):

$$|E(G_{3-i}[W_1])| \leq (\lambda + 10\alpha + 10\epsilon + 5\epsilon + \frac{25\epsilon^2}{2})n^2 \leq 2\lambda n^2.$$

If G^r has a $(\lambda, i, 2)$-bad partition ($i \in [2]$) $V(G^r) = V_j' \cup U_1^r \cup U_2^r$ then G has a corresponding $(2\lambda, i, 2)$-bad partition with

(0):

$$U_1 := \bigcup_{v_i \in U_1^r} V_i \cup (V_0 - V_j^*) \text{ and } U_2 := \bigcup_{v_i \in U_2^r} V_i.$$

(i):

$$|E(G_i[V_j^*, U_1])| \leq (\lambda + 10\alpha + 10\epsilon + 5\epsilon)n^2 \leq 2\lambda n^2.$$

(ii):

$$|E(G_{3-i}[V_j, U_2])| \leq (\lambda + 10\alpha + 10\epsilon + 5\epsilon)n^2 \leq 2\lambda n^2.$$

(iii):

$$n_j = |V_j^*| \geq \ell_j t \geq (1-\lambda)\ell t \geq (1-\lambda)(1-5\epsilon)n \geq (1-2\lambda)n.$$

(iv):

$$(1+2\lambda)n \geq (1+\lambda)n + 5\epsilon n \geq (1+\lambda)\ell t + 5\epsilon n \geq |U_1| \geq (1-\lambda)\ell t \geq (1-\lambda)(1-5\epsilon)n \geq (1-2\lambda)n.$$

(v):

$$(1+\lambda)n \geq (1+\lambda)\ell t \geq |U_2| \geq (1-\lambda)\ell t \geq (1-\lambda)(1-5\epsilon)n \geq (1-2\lambda)n.$$

Therefore, a $(68\gamma, i, j)$-bad partition of G^r corresponds to a $(136\gamma, i, j)$-bad partition of G for some $i \in [2]$ and $j \in [2]$. In Sections 2.2.5, 2.2.6, and 2.2.7, we show how to find a monochromatic cycle of length exactly $2n$ when G has a (λ, i, j)-bad partition for some $i \in [2]$ and $j \in [2]$, where $\lambda = 136\gamma$.

2.2.5 Dealing with $(\lambda, i, 1)$-bad partitions when $N - n_1 - n_2 \geq 3$

2.2.5.1 Setup

Without loss of generality, let $i = 1$. Recall that $d_k(v)$ is the degree of v in G_k, where $k \in [2]$. We assume that for some $\lambda < 0.01$, there is a partition $V(G) = W_1 \cup W_2$ such that:

$$(1-\lambda)n \leq |W_2| \leq (1+\lambda)n_1; \tag{2.52}$$

$$|E(G_1[W_1, W_2])| \leq \lambda n^2; \tag{2.53}$$

$$|E(G_2[W_1])| \leq \lambda n^2. \tag{2.54}$$

If G has at least 4 parts then $n_1 \leq n$ by (2.48) and (2.49). If G is tripartite, then we could have n_1 much larger than n, but in Section 2.2.5.1, we will assume $n_1 < \frac{5}{3}n$. The alternative, that G is tripartite and $n_1 \geq \frac{5}{3}n$, is handled in Section 2.2.5.2.

We know that $|W_1| \geq N - (1+\lambda)n_1 \geq 2n - 1 - \lambda n_1 \geq (2-5\lambda)n$ since $n_1 \leq 2n$. For any vertex x, fewer than $\frac{5}{3}n$ vertices of W_1 can be in the same part V_i of G as x, so at least $(\frac{1}{3} - 5\lambda)n > \frac{1}{4}n$ are in other parts of G. In other words, we have $d(x, W_1) \geq \frac{1}{4}n$ for all $x \in V(G)$.

We call a vertex $x \in V(G)$ *W_1-typical* if $d_1(x, W_1) \geq \frac{3}{4}d(x, W_1)$, and *$W_2$-typical* if $d_1(x, W_1) <$

$\frac{3}{4}d(x, W_1)$.

If x is W_1-typical, then $d_1(x, W_1) \geq \frac{3}{4} \cdot \frac{1}{4}n = \frac{3}{16}n$. Since

$$\sum_{x \in W_2} d_1(x, W_1) = |E(G_1[W_1, W_2])| \leq \lambda n^2,$$

the number of W_1-typical vertices in W_2 is at most $\frac{\lambda n^2}{3n/16} < 6\lambda n$.

Similarly, if x is W_2-typical, then $d_2(x, W_1) \geq \frac{1}{4} \cdot \frac{1}{4}n = \frac{1}{16}n$. Since

$$\sum_{x \in W_1} d_2(x, W_1) = 2|E(G_2[W_1])| \leq 2\lambda n^2,$$

the number of W_2-typical vertices in W_1 is at most $\frac{2\lambda n^2}{n/16} = 32\lambda n$.

Let W_1' be the set of all W_1-typical vertices, and W_2' be the set of all W_2-typical vertices. The partition (W_1', W_2') is almost exactly the same as the partition (W_1, W_2): at most $40\lambda n$ vertices have been moved from one part to the other part to obtain (W_1', W_2') from (W_1, W_2). Therefore, if $x \in W_1'$, we still have $d_1(x, W_1') \geq \frac{3}{4}d(x, W_1) - 40\lambda n$, and if $x \in W_2'$, we still have $d_1(x, W_1') < \frac{3}{4}d(x, W_1) + 40\lambda n$. In either case, we still have $d(x, W_1') \geq \frac{1}{4}n - 40\lambda n$ for all x.

Moreover, W_1' and W_2' still satisfy similar conditions to W_1 and W_2:

1. $(1 - 41\lambda)n \leq |W_2'| \leq (1 + \lambda)n_1 + 40\lambda n \leq (1 + 81\lambda)n_1$ (since $n_1 \geq \frac{n}{2}$ in all cases).
2. $|E(G_1[W_1', W_2'])| \leq \lambda n^2 + N \cdot (40\lambda n) \leq 161\lambda n^2$, since we move at most $40\lambda n$ vertices with degree less than N.
3. $|E(G_2[W_1'])| \leq \lambda n^2 + N \cdot (6\lambda n) \leq 25\lambda n^2$, since we move at most $6\lambda n$ vertices with degree less than N into W_1'.

For convenience, let $\delta = 200\lambda$, which is at least as large as all multiples of λ used above.

Our goal is to find a cycle of length $2n$ in either G_1 or G_2. We decide which type of cycle we will attempt to find based on the relative sizes of W_1' and W_2'.

Suppose that $|W_1'| \geq 2n$ and, moreover, $|W_1' \setminus V_i| \geq n$ for all i. In this case, we find a cycle of length $2n$ in G_1; this is done in Section 2.2.5.3.

Otherwise, we must have $|W_2'| \geq n$: either $|W_1'| \leq 2n - 1$, and $|W_2'| = N - |W_1'| \geq n$, or else $|W_1' \setminus V_i| \leq n - 1$ for some i, and

$$|W_2'| \geq |W_2' \setminus V_i| = |V \setminus V_i| - |W_1' \setminus V_i| \geq (N - n_i) - (n - 1) \geq (2n - 1) - (n - 1) = n.$$

In this case, we find a cycle of length $2n$ in G_2; this is done in Section 2.2.5.4.

We use the following lemma to pick out "well-behaved" vertices in W_1' and W_2'. For example, we commonly apply it to $G_2[W_1']$ or to $G_1[W_1', W_2']$.

Lemma 2.47. *Let H be an n-vertex graph with at most ϵn^2 edges, for some $\epsilon > 0$, and let $S \subseteq V(H)$. If $S' \subseteq S$ is any subset that excludes the k vertices of S with the highest degree, then every $v \in S'$ satisfies $d_H(v) < \frac{2\epsilon n^2}{k}$.*

Additionally, when H is bipartite, and S is entirely contained in one part of H, every $v \in S'$ satisfies $d_H(v) < \frac{\epsilon n^2}{k}$.

Proof. In the first case, if we have $d_H(v) \geq \frac{2\epsilon n^2}{k}$ for any $v \in S'$, then we also have $d_H(v) \geq d$ for the k vertices of S with the highest degree, which we excluded from S'. The sum of degrees of these $k+1$ vertices exceeds $2\epsilon n^2$, so it is greater than twice the number of edges in H, a contradiction.

In the second case, if we have $d_H(v) \geq \frac{\epsilon n^2}{k}$ for any $v \in S'$, the same sum of degrees exceeds ϵn^2. But since the vertices of S are all on one side of the bipartition of H, this sum of degrees cannot be greater than the number of edges in H, which is again a contradiction. □

2.2.5.2 The nearly-bipartite subcase

In Section 2.2.5.2, we assume that G is tripartite with $n_1 \geq \frac{5}{3}n$. Recall that when G is tripartite we have $n_1 = n_2$ and $n_1 + n_3 = n_2 + n_3 = 2n - 1$, and that throughout Section 2.2.5 we assume $N - n_1 - n_2 \geq 3$, or in this case that $n_3 \geq 3$.

Case 1: $|W_1 \cap V_i| \geq (1 + 10\lambda)n$ for $i = 1$ or $i = 2$. We assume $i = 1$; the proof for the case $i = 2$ is the same. In this case, let X be an n-vertex subset of $V_1 \cap W_1$ avoiding the $5\lambda n$ vertices of $V_1 \cap W_1$ with the most edges of G_2 to $W_1 \setminus V_1$ and the $5\lambda n$ vertices of $V_1 \cap W_1$ with the most edges of G_1 to $W_2 \setminus V_1$.

For any vertex $v \in X$, we have $d_2(v, W_1 \setminus V_1) \leq \frac{\lambda n^2}{5\lambda n} = \frac{1}{5}n$ and $d_1(v, W_2 \setminus V_1) \leq \frac{1}{5}n$ by Lemma 2.47.

We partition $V_2 \cup V_3$ into sets Y_1 and Y_2 by the following procedure.

1. The $2\lambda n$ vertices of $W_1 \setminus V_1$ with the most edges of G_2 to X are set aside, and the remaining vertices of $W_1 \setminus V_1$ are assigned to Y_1.

   By Lemma 2.47, any vertex v assigned to Y_1 in this step has $d_2(v, X) \leq \frac{1}{2}n$.

2. The $2\lambda n$ vertices of $W_2 \setminus V_1$ with the most edges of G_1 to X are set aside, and the remaining vertices of $W_2 \setminus V_1$ are assigned to Y_2.

   By Lemma 2.47, any vertex v assigned to Y_2 in this step has $d_1(v, X) \leq \frac{1}{2}n$.

3. Each remaining vertex v is assigned to Y_1 if $d_1(v, X) \geq \frac{n}{2}$ and to Y_2 otherwise (in which case $d_2(v, X) \geq \frac{n}{2}$).

Since $|V_2 \cup V_3| = 2n - 1$, we must have $|Y_1| \geq n$ or $|Y_2| \geq n$. Let Y_j' be an n-vertex subset of Y_j, where $j \in [2]$ and $|Y_j| \geq n$. We apply Theorem 2.44 to find a Hamiltonian cycle in the bipartite graph $H = G_j[X, Y_j']$.

The minimum H-degree in X is $\frac{4}{5}n - 2\lambda n$, since each $v \in X$ had at most $\frac{1}{5}n$ edges to $W_j \setminus V_1$ which were not in G_j, and at most $2\lambda n$ vertices of Y_j' did not come from $W_j \setminus V_1$ originally. The minimum H-degree in Y_j' is $\frac{n}{2}$, so the condition of Theorem 2.44 is satisfied: whenever $d_H(u_i) \le i$, we have $i \ge (\frac{4}{5} - 2\lambda)n$, so $d_H(v_{n-i}) \ge \frac{n}{2} \ge (\frac{1}{5} + 2\lambda)n + 1$.

Case 2: $|V_i \cap W_1| < (1+10\lambda)n$ for $i = 1$ and $i = 2$. By (2.52), we must have $|W_1| \ge N - (1+\lambda)n_1 = 2n - 1 - \lambda n_1 > 2n - 3\lambda n$. Since $n_1 = n_2 \ge \frac{5n}{3}$ and $n_2 + n_3 = 2n - 1$, fewer than $\frac{1}{3}n$ vertices of W_1 are in V_3, so at least $(\frac{5}{3} - 3\lambda)n$ of them are in $V_1 \cup V_2$; therefore $|W_1 \cap V_1| > (\frac{2}{3} - 13\lambda)n$ and $|W_1 \cap V_2| > (\frac{2}{3} - 13\lambda)n$.

Because $2n > n_1 = n_2 \ge \frac{5}{3}n$, we have $(\frac{2}{3} - 10\lambda)n < |V_i \cap W_2| < (\frac{4}{3} + 13\lambda)n$ for $i = 1, 2$, as well.

Next, we choose subsets $X_{ij} \subseteq V_i \cap W_j$ with $|X_{11}| = |X_{21}| = |X_{12}| = |X_{22}| = \frac{n}{2} + 10$. To choose X_{11} and X_{21}, avoid the $\frac{1}{20}n$ vertices with the most edges in G_1 to W_2 and the $\frac{1}{20}n$ vertices with the most edges in G_2 to W_1, so that each chosen vertex has at most $20\lambda n$ edges of each kind by Lemma 2.47. To choose X_{12} and X_{22}, avoid the $\frac{1}{10}n$ vertices with the most edges in G_1 to W_1, so that each chosen vertex has at most $10\lambda n$ such edges by Lemma 2.47.

First, we observe that if H is any of the graphs $G_1[X_{11}, X_{21}]$, $G_2[X_{12}, X_{21}]$, or $G_2[X_{11}, X_{22}]$, then given any vertices v, w in H, we can find a (v, w)-path in H on m vertices, provided that $n - 10 \le m \le n + 10$ (this is not optimal, but it is more than we need) and that the parity of m is correct.

To do so, we apply Theorem 2.46. If v and w are on the same side of H, add a vertex x to the other side adjacent to all vertices in the side containing v and w; if not, add an edge vw. Then take a subgraph containing $\lceil \frac{m}{2} \rceil$ vertices from each side, making sure to include v, w and if applicable x. In this subgraph, the minimum degree is at least $\lceil \frac{m}{2} \rceil - 20\lambda n$, so we can use Theorem 2.46 to find a Hamiltonian cycle in this graph containing either the edge vw or the edges vx and xw. Deleting the vertex x or the edge vw, whichever applies, creates a (v, w)-path in H of the correct length.

Suppose that $G_2[X_{12}, X_{22}]$ contains a matching $M = \{u_1u_2, v_1v_2\}$ of size 2, where $u_1, v_1 \in X_{12}$ and $u_2, v_2 \in X_{22}$. In that case, we can find a (u_1, v_1)-path P in $G_2[X_{12}, X_{21}]$ on $2\lceil \frac{n}{2} \rceil + 1$ vertices and a (u_2, v_2)-path Q in $G_2[X_{11}, X_{22}]$ on $2\lfloor \frac{n}{2} \rfloor - 1$ vertices by the previous observation. Joining the paths P and Q using the edges of the matching M, we find a cycle of length $2n$ in G_2.

Now we assume $G_2[X_{12}, X_{22}]$ does not contain a matching of size 2. If the size of a maximum matching in this graph is one, then there is a vertex cover of size one since $G_2[X_{12}, X_{22}]$ is bipartite. We delete this vertex cover from X_{12} or X_{22} (it depends on where this vertex cover is). Having changed X_{12} and X_{22} in this way, $G_1[X_{12}, X_{22}]$ is a complete bipartite graph, so it also has the property that any two vertices in it can be joined by a path on m vertices, provided that $n - 10 \le m \le n + 10$ and that the parity of m is correct.

Note that there are at least three vertices in V_3.

We say that a vertex $v \in V_3$:

- is j-*adjacent* to a set S if it has at least two edges in G_j to S.

- *S-connects* G_j if it is j-adjacent to both X_{11} and X_{12}, or if it is j-adjacent to both X_{21} and X_{22}. ("S-connects" because it is j-adjacent to two sets in the *same* part of V_1 or V_2.)
- *C-connects* G_1 if it is 1-adjacent to both X_{11} and X_{22}, or if it is 1-adjacent to both X_{12} and X_{21}. ("C-connects" because the j-adjacency *crosses* from V_1 to V_2.)
- *C-connects* G_2 if it is 2-adjacent to both X_{11} and X_{21}, or if it is 2-adjacent to both X_{12} and X_{22}.
- *Folds into* G_1 if it is 1-adjacent to both X_{11} and X_{21}, or if it is 1-adjacent to both X_{12} and X_{22}.
- *Folds into* G_2 if it is 2-adjacent to both X_{11} and X_{22}, or if it is 2-adjacent to both X_{12} and X_{21}.

Some comments on these definitions: first, a vertex that is j-adjacent to at least three of $X_{11}, X_{12}, X_{21}, X_{22}$ is guaranteed to both S-connect and C-connect G_j. Second, a vertex that is j-adjacent to only two of $X_{11}, X_{12}, X_{21}, X_{22}$ for each value of j may S-connect both G_1 and G_2, or C-connect G_1 and fold into G_2, or C-connect G_2 and fold into G_1. In particular, each vertex either S-connects or C-connects some G_j.

If there are two vertices in V_3 that both S-connect G_j, or both C-connect G_j, then we can find a cycle of length $2n$ in G_j. The cases are all symmetric; without loss of generality, suppose $v, w \in V_3$ both S-connect G_1. We can find a path P in $G_1[X_{11}, X_{21}]$ on $2\lceil \frac{n}{2} \rceil - 1$ vertices that starts at a G_1-neighbor of v and ends at a G_1-neighbor of w, and a path Q in $G_1[X_{12}, X_{22}]$ on $2\lfloor \frac{n}{2} \rfloor - 1$ vertices that starts at a G_1-neighbor of v and ends at a G_1-neighbor of w. Joining P and Q via v at one endpoint and via w on the other creates a cycle of length $2n$ in G_1.

If we cannot find two vertices as in the previous paragraph, then the best we can do is to find, for some j, a vertex $v \in V_3$ that S-connects G_j and another vertex $w \in V_3$ that C-connects G_j. Since v does not C-connect G_j, it must also S-connect G_{3-j}.

There is at least one more vertex $x \in V_3$. By assumption, it does not S-connect G_{3-j} and neither S-connects nor C-connects G_j, so it must fold into G_j (and C-connect G_{3-j}).

Without loss of generality, suppose that $j = 1$ and x has a G_1-neighbor in both X_{11} and X_{21}. We add an artificial edge e_x between a pair of such neighbors of x.

As before, we can find a path P in $G_1[X_{11}, X_{21}]$ joining a neighbor of v to a different neighbor of w; we add the requirement that it uses the edge e_x, which is still possible by Theorem 2.46. We can also find a path Q in $G_1[X_{12}, X_{22}]$ joining a neighbor of v to a different neighbor of w. Since v S-connects G_1 and w C-connects G_1, one of these paths will have even length and the other will have odd length, but we can choose them to have $2n - 3$ vertices total.

Now join the paths P and Q using the vertices v and w, then replace the artificial edge e_x by two edges to x from its endpoints. The result is a cycle of length $2n$ in G_1.

2.2.5.3 Finding a cycle in G_1

In Section 2.2.5.3, we are considering a 2-edge-colored graph G and a partition $W_1' \cup W_2'$ of $V(G)$ satisfying the following properties:

1. G is a complete s-partite graph with parts $V_1, V_2, \ldots, V_s$ of size $n_1, n_2, \ldots, n_s$, with $s \geq 3$ and $n_1 + \ldots + n_s \leq 4n$.

2. $(1-\delta)n \leq |W_2'| \leq (1+\delta)n_1$.

3. $|E(G_1[W_1', W_2'])| \leq \delta n^2$ and $|E(G_2[W_1'])| \leq \delta n^2$.

4. If $x \in W_1'$, then $d_1(x, W_1') \geq \frac{3}{4} d(x, W_1) - \delta n$.

5. $|W_1'| \geq 2n$ and $|W_1' \setminus V_i| \geq n$ for all i. (This is the assumption that leads to this Section (Section 2.2.5.3) as opposed to Section 2.2.5.4.)

We can deduce a further degree condition that holds for all vertices $x \in W_1'$:

6. By Properties 1 and 2, $|W_1'| = |V(G)| - |W_2'| \leq 4n - (1-\delta)n = (3+\delta)n$, so $d(x, W_1') \leq (3+\delta)n$. By Property 4, we have $d_2(x, W_1) \leq \frac{1}{4}(3+\delta)n + \delta n \leq (\frac{3}{4} + 2\delta)n$.

To find a cycle of length $2n$ in G_1, we will choose two disjoint sets $X, Y \subseteq W_1'$ of size n, then apply Theorem 2.44 to find a Hamiltonian cycle in $H = G_1[X, Y]$.

Let $a, b \in \{1, 2, \ldots, s\}$ be such that $V_a \cap W_1'$ is the largest part of $G_1[W_1']$ and $V_b \cap W_1'$ is the second largest part of $G_1[W_1']$. To define X and Y, we begin by assigning $V_a \cap W_1'$ to X and $V_b \cap W_1'$ to Y. If either of these exceeds n vertices, we choose n of the vertices arbitrarily.

Continue by assigning the parts $V_i \cap W_1'$ to either X or Y arbitrarily, for as long as this does not make $|X|$ or $|Y|$ exceed n. Once this is no longer possible, then:

- If there are still at least two parts $V_i \cap W_1'$ left unassigned, then each of them must have more than $\max\{n - |X|, n - |Y|\}$ vertices. Therefore we can add vertices from one of them to X to make $|X| = n$ (if necessary), and add vertices from the other to Y to make $|Y| = n$ (if necessary).

- If there is only one part of $G_1[W_1']$ left unassigned, call it $V_{\text{split}} \cap W_1'$. We assign $n - |X|$ vertices of $V_{\text{split}} \cap W_1'$ to X and $n - |Y|$ other vertices of $V_{\text{split}} \cap W_1'$ to Y.

- If there are no parts left unassigned, then we must have $|X| = |Y| = n$.

We must show that we do not run out of vertices in either of the last two cases. If $|V_a \cap W_1'| \leq n$, then we do not run out because $|W_1'| \geq 2n$ (by Property 5) and all vertices in $W_1' \setminus V_{\text{split}}$ are assigned to either X or Y, so either $V_{\text{split}} \cap W_1'$ must contain enough vertices to fill X and Y or X and Y are already full. If $|V_a \cap W_1'| > n$, then we do not run out because $|W_1' \setminus V_a| \geq n$ (again, by Property 5), and after $V_a \cap W_1'$ is assigned, all vertices of W_1' are added to Y until it is full.

The most difficult case for us is the one in which some part $V_{\text{split}} \cap W_1'$ is divided between X and Y. To handle all cases at once, we assume this happens; if necessary, we choose some part $V_i \cap W_1'$ ($i \neq a, b$) to be a degenerate instance of V_{split} which is entirely in X or Y.

Let $n_x = |V_{\text{split}} \cap X|$ and $n_y = |V_{\text{split}} \cap Y|$. We assigned the largest part of $G[W_1']$ to X and the second-largest to Y; therefore X and Y both contain at least $n_x + n_y$ vertices not in V_{split}. Since $|X| = |Y| = n$, we must have $n_x + (n_x + n_y) \leq n$ and $n_y + (n_x + n_y) \leq n$; therefore $n_x + n_y \leq \frac{2}{3}n$, while individually $n_x \leq \frac{n}{2}$ and $n_y \leq \frac{n}{2}$.

We first prove some bounds on $d_1(x, Y)$ for $x \in X$ (and, by symmetry, $d_1(y, X)$ for $y \in Y$). If $x \notin V_{\text{split}}$, then $d(x, Y) = n$ (since there are no vertices of Y in the same part of G as x) while $d_2(x, W_1') \leq (\frac{3}{4} + 2\delta)n$ by Property 6, so $d_1(x, Y) \geq (\frac{1}{4} - 2\delta)n$. If $x \in V_{\text{split}}$, then $d(x, W_1') = (n - n_x) + (n - n_y)$, since all vertices of W_1' outside V_{split} have been assigned to either X or Y, so $d_2(x, W_1') \leq \frac{1}{4}(2n - n_x - n_y) + \delta n$ by Property 4. This leaves $d_1(x, Y) \geq \frac{1}{2}n - \frac{3}{4}n_y - \delta n \geq (\frac{1}{8} - \delta)n$.

If we exclude the $\frac{1}{10}n$ vertices of X with the most edges to W_1' in G_2, then by Lemma 2.47, the remaining vertices $x \in X$ have $d_2(x, W_1') \leq 20\delta n$. If $x \notin V_{\text{split}}$, this means $d_1(x, Y) \geq (1 - 20\delta)n$, and if $x \in V_{\text{split}}$, this means that $d_1(x, Y) \geq n - n_y - 20\delta n$.

Let $H = G_1[X, Y]$, let $u_1, u_2, \ldots, u_n$ be the vertices of X ordered so that $d_H(u_1) \leq \ldots \leq d_H(u_n)$, and let $v_1, v_2, \ldots, v_n$ be the vertices of Y ordered so that $d_H(v_1) \leq \ldots \leq d_H(v_n)$.

Suppose $u_i \in X$ satisfies $d_H(u_i) \leq i < n$. We have shown $d_1(x, Y) \geq (\frac{1}{8} - \delta)n$, so among $u_1, u_2, \ldots, u_i$, there must be a vertex not among the $\frac{1}{10}n$ vertices of X with the most edges to W_1' in G_2. For such a vertex, $d_1(x, Y) \geq n - n_y - 20\delta n$, so in particular $d_H(u_i) \geq n - n_y - 20\delta n$, which means $i \geq n - n_y - 20\delta n$.

If we had $d_H(v_{n-i}) \leq n - i$, then by repeating this argument for vertices in Y, we would have $d_H(v_{n-i}) \geq n - n_x - 20\delta n$, which would mean $n - i \geq n - n_x - 20\delta n$. Adding this to the inequality on i, we would get $n \geq 2n - n_x - n_y - 40\delta n$, which is impossible since $n_x + n_y \leq \frac{2}{3}n$. So we must have $d_H(v_{n-i}) \geq n - i + 1$, and by Theorem 2.44, H contains a Hamiltonian cycle. This gives a cycle of length $2n$ in G_1.

2.2.5.4 Finding a cycle in G_2

In this Section (Section 2.2.5.4), we are considering a 2-edge-colored graph G and a partition $W_1' \cup W_2'$ of $V(G)$ satisfying the following properties:

1. G is a complete s-partite graph with parts $V_1, V_2, \ldots, V_s$ of size $n_1, n_2, \ldots, n_s$, with $s \geq 3$ and $n_1 + \ldots + n_s \leq 4n$. Morever, $\frac{5}{3}n > n_1 \geq \cdots \geq n_s$; we considered the case $n_1 \geq \frac{5}{3}n$ in Section 2.2.5.2.

2. Either $N - n_1 > 2n - 1$ and $|V_i| \leq n$ for all i, or $n_1 = n_2 \geq n$, $s = 3$, and $N - n_1 = N - n_2 = 2n - 1$.

3. $|E(G_1[W_1', W_2'])| \leq \delta n^2$ and $|E(G_2[W_1'])| \leq \delta n^2$.

4. If $x \in W_2'$, then $d(x, W_1') \geq \frac{1}{4}n - \delta n$, and $d_2(x, W_1') \geq \frac{1}{4}d(x, W_1) - \delta n$.

5. $n \leq |W_2'| \leq (1 + \delta)n_1$. (The lower bound is the assumption that leads to this Section (Section 2.2.5.4) as opposed to Section 2.2.5.3.)

Let Bad consist of the $\sqrt{\delta}n$ vertices of W_2' that maximize $d_1(x, W_1')$; let $\mathsf{Good} = W_2' \setminus \mathsf{Bad}$. By Lemma 2.47, $d_1(x, W_1') \leq \sqrt{\delta}n$ for all $x \in \mathsf{Good}$.

Our strategy is to handle the vertices in Bad: first by finding short vertex-disjoint paths contaning the vertices in Bad, then by combining them into a single path. Finally, we extend this path to a cycle of length $2n$ in $G_2[W_1', W_2']$.

2.2.5.4.1 Constructing paths containing each vertex of Bad

For every vertex $x \in \mathsf{Bad}$, we find a four-edge path $P(x)$ in G_2, which contains x, but begins and ends at a vertex of Good. We construct these paths one at a time; for each vertex x, we must keep in mind that in each of W_1' and W_2', up to $2\sqrt{\delta}n$ vertices may have been used for previously chosen paths.

This is not always possible; when it is not, we find a cycle of length $2n$ in another way.

Lemma 2.48. *One of the following holds:*

1. *G_2 contains a collection $\{P(x) : x \in \mathsf{Bad}\}$ of vertex-disjoint paths of length 4, such that for all $x \in \mathsf{Bad}$, $P(x)$ begins and ends at a vertex of Good, and also contains x and two vertices in W_1'.*

2. *G_2 contains a cycle of length $2n$.*

Proof. We attempt to find the collection of vertex-disjoint paths, one vertex of Bad at a time.

By Property 4 at the beginning of this section (Section 2.2.5.4), even if $x \in \mathsf{Bad}$, we have $d(x, W_1') \geq (\frac{1}{4} - \delta)n$ and $d_2(x, W_1') \geq \frac{1}{4}d(x, W_1') - \delta n$, so $d_2(x, W_1') \geq (\frac{1}{16} - \frac{5}{4}\delta)n$. There is a part V_i with $d_2(x, W_1' \cap V_i) \geq (\frac{1}{64} - \frac{5}{16}\delta)n$.

First we consider the first case of Property 2. That is, suppose $N - n_1 > 2n - 1$; then we have $|V_i| = n_i \leq n_1 \leq n$, so $|W_2' \cap V_i| \leq (\frac{63}{64} + \frac{5}{16}\delta)n$. But $|W_2'| \geq n$ in total, so there must be another part V_j with $|W_2' \cap V_j| \geq \frac{1}{4}(\frac{1}{64} - \frac{5}{16}\delta)n$. We can choose two vertices $v, w \in V_j$ to use as the endpoints of $P(x)$: ruling out the vertices of $V_j \cap \mathsf{Bad}$ (at most $\sqrt{\delta}n$) and previously used vertices of W_2' in V_j (at most $2\sqrt{\delta}n$) we still have a number of choices linear in n.

Now we know not just the center vertex x of the path $P(x)$ but also its two endpoints v and w. To complete $P(x)$, we must find a common neighbor of v and x, and another common neighbor of w and x. This is possible, since there are at least $(\frac{1}{64} - \frac{5}{16}\delta)n$ neighbors of x in $W_1' \cap V_i$; v and w have edges in G_2 to all but at most $\sqrt{\delta}n$ of them, and we exclude at most $2\sqrt{\delta}n$ more that have been already used.

We call the method above of choosing the collection $\{P(x) : x \in \mathsf{Bad}\}$ the *greedy strategy*. As we have seen, it always works in the first case of Property 2; it remains to see when it works in the second case. Now, we assume that G is tripartite, $n_1 = n_2 \geq n$, and $N - n_1 = N - n_2 = 2n - 1$.

The greedy strategy continues to work if we can always choose the part V_j from which to pick the endpoints of $P(x)$. For this choice to always be possible, it is enough that at least two parts of G contain $3\sqrt{\delta}n$ vertices of W_2': both of them will have vertices outside Bad not previously chosen for any path, and one of them will not be the same as V_i.

If this does not occur, then one part V_a of G contains all but $6\sqrt{\delta}n$ vertices of W_2', and each of the other two parts contains fewer than $3\sqrt{\delta}n$ vertices of W_2'. If V_a contains fewer than $\frac{1}{20}n$ vertices of W_1', then the greedy strategy still works: for any $x \in \mathsf{Bad}$, we have $d_2(x, W_1') \geq (\frac{1}{16} - \frac{5}{4}\delta)n > |V_a \cap W_1'| + 2\sqrt{\delta}n$, so we can always choose a part of G other than V_a to play the part of V_i. In this case, it does not matter that only V_a contains many vertices of W_2', because we only need to choose the endpoints of $P(x)$ from vertices in V_a.

The greedy strategy fails in the remaining case: when V_a contains all but $6\sqrt{\delta}n$ vertices of W_2' and at least $\frac{1}{20}n$ vertices of W_1'. Then $|V_a| > n$, so without loss of generality, $V_a = V_2$. In this case, we do not try to find the paths $P(x)$ and instead find a cycle of length $2n$ in G_1 or G_2 directly.

We have a lower bound on $n_1 = n_2 = |V_2|$: it is $|V_2 \cap W_1'| + |V_2 \cap W_2'| \geq (1 + \frac{1}{20} - 6\sqrt{\delta})n$. Since $|V_1 \cap W_2'| \leq 3\sqrt{\delta}n$, we have $|V_1 \cap W_1'| \geq (\frac{21}{20} - 9\sqrt{\delta})n > n$.

Let Y_1 be a subset of exactly n vertices of $V_1 \cap W_1'$, chosen to avoid the $\sqrt{\delta}n$ vertices of $V_1 \cap W_1'$ with largest degree in $G_1[W_1', W_2']$ and the $\sqrt{\delta}n$ vertices of $V_1 \cap W_1'$ with largest degree in $G_2[V_1 \cap W_1', W_1' \setminus V_1]$. (This is possible since $(\frac{21}{20} - 11\sqrt{\delta})n > n$ as well.) In both cases, if a vertex $x \in Y_1$ has degree d in the corresponding graph, we get at least $\sqrt{\delta}nd$ edges in either $G_1[W_1', W_2']$ or $G_2[W_1']$ by looking at the vertices we deleted; therefore $\sqrt{\delta}nd \leq \delta n^2$ and $d \leq \sqrt{\delta}n$.

Redistribute vertices of $V_2 \cup V_3$ into two parts (X_1, X_2) as follows:

- All vertices of $W_1' \setminus V_1$, except the $\sqrt{\delta}n$ vertices v maximizing $d_2(v, Y_1)$, are put in X_1. A vertex v of this type is guaranteed to have $d_2(v, Y_1) \leq \sqrt{\delta}n$.

- All vertices of $W_2' \setminus V_1$, except the vertices in Bad, are put in X_2. A vertex v of this type is guaranteed to have $d_1(v, Y_1) \leq \sqrt{\delta}n$.

- The remaining vertices, of which there are at most $2\sqrt{\delta}n$, are assigned to X_1 or X_2 based on their edges to Y_1. If $d_1(v, Y_1) \geq \frac{n}{2}$, then v is put into X_1; otherwise, $d_2(v, Y_1) \geq \frac{n}{2}$, and v is put into X_2.

The sets X_1, X_2, Y_1 satisfy the following properties. For any $v \in X_1$, $d_1(v, Y_1) \geq \frac{n}{2}$. For any $v \in X_2$, $d_2(v, Y_1) \geq \frac{n}{2}$. For any $v \in Y_1$, $d_2(v, X_1) \leq 4\sqrt{\delta}n$, since $d_2(v, W_1') \leq \sqrt{\delta}n$ and X_1 contains at most $3\sqrt{\delta}n$ vertices of W_2'; similarly, for any $v \in Y_1$, $d_1(v, X_2) \leq 4\sqrt{\delta}n$.

Since $|X_1| + |X_2| = |V_2 \cup V_3| = 2n - 1$, either $|X_1| \geq n$ or $|X_2| \geq n$.

If $|X_1| \geq n$, then we let X_1' be a subset of exactly n vertices of X_1, and find a cycle of length $2n$ in $H = G_1[X_1', Y_1]$ by applying Theorem 2.44. The hypotheses of the theorem are satisfied by the minimum degree conditions above: for $u \in X_1'$, $d_H(u) \geq \frac{1}{2}n$, and for $v \in Y_1$, $d_H(v) \geq (1 - 4\sqrt{\delta})n$.

Similarly, if $|X_2| \geq n$, then we let X_2' be a subset of exactly n vertices of X_2, and find a cycle of length $2n$ in $H = G_2[X_2', Y_1]$ by applying Theorem 2.44. The argument is the same as in the previous paragraph. □

2.2.5.4.2 Finding a cycle using Theorem 2.46

Applying Lemma 2.48, each of the $\sqrt{\delta}n$ vertices $x \in \mathsf{Bad}$ is the center of a length-4 path $P(x)$. Let A be the $2\sqrt{\delta}n$ vertices of W_1' in these paths and B be the $3\sqrt{\delta}n$ vertices of W_2' in these paths (including the vertices in Bad). Additionally, let C be the set of $\sqrt{\delta}n$ vertices of $W_1' \setminus A$ with the most edges to W_2' in G_1; by Lemma 2.47, every $x \in W_1' \setminus (A \cup C)$ satisfies $d_1(x, W_2') \leq \sqrt{\delta}n$.

Next, we will construct a bipartite graph H by choosing subsets $W_1'' \subseteq W_1' \setminus (A \cup C)$ of size $n - 2\sqrt{\delta}n$, and $W_2'' \subseteq W_2' \setminus B$ of size $n - 3\sqrt{\delta}n$; the edges of H are the edges of $G_2[W_1'' \cup A, W_2'' \cup B]$, except that we artificially join every internal vertex of every path $P(x)$ to every vertex on the other side of H. We will apply Theorem 2.46 to find a Hamiltonian cycle in H containing all $q = 4\sqrt{\delta}n$ edges belonging to the paths $P(x)$, after choosing W_1'' and W_2'' to make sure that the hypotheses of this theorem hold.

In terms of our future choice of (W_1'', W_2''), let $n_{i,j} = |V_i \cap W_j''|$. If $u \in V_i \cap W_1''$, then the degree of u in H is at least $n - n_{i,2} - \sqrt{\delta}n$: u has at most $\sqrt{\delta}n$ edges to W_2'' that are in G_1, not G_2, and its degree is further reduced by the $n_{i,2}$ vertices of W_2'' that are also in V_i. Similarly, if $v \in V_i \cap W_2''$, then the degree of v in H is at least $n - n_{i,1} - \sqrt{\delta}n$.

Let $n_{*,1} \geq n_{**,1}$ be the two largest values of $n_{i,1}$ and let $n_{*,2} \geq n_{**,2}$ be the two largest values of $n_{i,2}$. As in the statement of Theorem 2.46 let $u_1, u_2, \ldots, u_n$ be the vertices of $W_1'' \cup A$ and let $v_1, v_2, \ldots, v_n$ be the vertices of $W_2'' \cup B$, ordered by degree in H.

We begin with a lemma showing that some choices of (W_1'', W_2'') are guaranteed to satisfy the conditions of Theorem 2.46:

Lemma 2.49. *Theorem 2.46 can be applied, letting us find a cycle of length $2n$ in H, if we can choose W_1'' and W_2'' to satisfy the following two conditions:*

1. *For each i, either $n_{i,1} + n_{i,2} \leq n - 10\sqrt{\delta}n$, or $n_{i,1} = 0$.*
2. *For either $j = 1$ or $j = 2$, at most one value of $n_{i,j}$ exceeds $(\frac{1}{2} - 10\sqrt{\delta})n$.*

Proof. Suppose that $u_i \in W_1'' \cup A$ and $d(u_i) \le i + q = i + 4\sqrt{\delta}n$. The minimum H-degree of vertices in $W_1'' \cup A$ is $n - n_{*,2} - \sqrt{\delta}n$, so we must have $i \ge n - n_{*,2} - 5\sqrt{\delta}n$. By Condition 1, at most $n - n_{*,2} - 10\sqrt{\delta}n$ vertices in W_1'' are in the same part as the largest part of W_2''; at most $2\sqrt{\delta}n$ vertices are endpoints of paths $P(x)$, so together these make up at most $n - n_{*,2} - 8\sqrt{\delta}n < i$ vertices. Therefore some of the vertices $u_1, \dots, u_i$ are vertices of W_1'' in a different part, and therefore $d(u_i) \ge n - n_{**,2} - \sqrt{\delta}n$.

Similarly, suppose that $v_j \in W_2'' \cup B$ and $d(v_j) \le j + q \le j + 4\sqrt{\delta}n$. The minimum H-degree of vertices in $W_2'' \cup B$ is $n - n_{*,1} - \sqrt{\delta}n$, so we must have $j \ge n - n_{*,1} - 5\sqrt{\delta}n$. By Condition 1, at most $n - n_{*,1} - 10\sqrt{\delta}n + |B|$ vertices in W_2'' are in the same part as the largest part of W_1'', which is fewer than j. Therefore some of the vertices $v_1, \dots, v_j$ are vertices of W_2'' in a different part, and therefore $d(v_j) \ge n - n_{**,1} - \sqrt{\delta}n$.

In such a case, we have $d(u_i) + d(v_j) \ge 2n - n_{**,1} - n_{**,2} - 2\sqrt{\delta}n$. We have $n_{**,1}, n_{**,2} \le \frac{1}{2}n$, and additionally by Condition 2, $n_{**,j} \le \frac{1}{2}n - 10\sqrt{\delta}n$ for some j. Therefore $d(u_i) + d(v_j) \ge n + 8\sqrt{\delta}n \ge n + 4\sqrt{\delta}n + 1$, and the hypothesis of Theorem 2.46 holds. $\square$

It remains to choose W_1'' and W_2'' so that they satisfy the conditions of Lemma 2.49, or to deal separately with the cases where this is impossible.

First, we consider the case in which all parts of G have size at most $\frac{5}{4}n$. (By Property 2, this automatically holds when G has more than 3 parts: if so, all parts of G have size at most n.) Choose W_2'' arbitrarily. W_1' must contain at least $N - (1+\delta)n_1 \ge N - n_1 - \delta n_1 \ge 2n - 1 - 2\delta n$ vertices, of which only $2\sqrt{\delta}n$ vertices have been used by paths and $\sqrt{\delta}n$ more have been thrown away as C; therefore we have at least $2n - 1 - 3\sqrt{\delta}n - 2\delta n$ choices for vertices in W_1''.

We set aside vertices of W_1' which we forbid from being in W_1''. From each part, V_i, forbid either at least $|V_i| - (1 - 10\sqrt{\delta})n$ vertices, or else all vertices of $V_i \cap W_1'$, whichever is smaller. This forbids at most $(\frac{1}{4} + 10\sqrt{\delta})n$ vertices from each part, and at most $10\sqrt{\delta}n$ vertices in the case $n_i \le n$. There are at most two parts with $n_i > n$, so we forbid at most $(\frac{1}{2} + 50\sqrt{\delta})n$ vertices. Now Condition 1 of Lemma 2.49 will be satisfied no matter what: for each part i, we will either have $n_{i,1} + n_{i,2} \le (1 - 10\sqrt{\delta})n$, or else $n_{i,1} = 0$.

Next, we attempt to ensure that Condition 2 of Lemma 2.49 holds. Call a part V_i of G *W_1''-rich* if, after excluding the forbidden vertices, and vertices of $A \cup C$, there are still at least $20\sqrt{\delta}n$ vertices of W_1' left in V_i; call it *W_1''-poor* otherwise.

If there are at least three W_1''-rich parts, then we can choose $20\sqrt{\delta}n$ vertices from each of them for W_1'', and complete the choice of W_1'' arbitrarily. Condition 2 of Lemma 2.49 must now hold for $j = 1$: if we had $n_{*,1} \ge (\frac{1}{2} - 10\sqrt{\delta})n$ and $n_{**,1} \ge (\frac{1}{2} - 10\sqrt{\delta})n$, then together these two parts would contain all but $20\sqrt{\delta}n$ vertices of W_1''. This is impossible, since there is a third W_1''-rich part containing at least that many vertices of W_1''.

If there are not at least three W_1''-rich parts, we give up on Lemma 2.49, and satisfy the conditions of Theorem 2.46 by a different strategy.

If V_i is W_1''-poor, it must have many vertices of W_2''. More precisely, V_i has at least $\min\{n, n_i\} - 10\sqrt{\delta}n$ vertices that we have not forbidden. Among these, there are up to $3\sqrt{\delta}n$ vertices which are in $A \cup C$, up to $3\sqrt{\delta}n$ vertices which are in B, and fewer than $20\sqrt{\delta}n$ vertices that can be added to W_1'', so the remaining $\min\{n, n_i\} - 36\sqrt{\delta}n$ vertices must be in $W_2' \setminus B$.

Moreover, when G is tripartite, $n_i \geq \frac{3}{4}n - 1$ for any part, so if a part is W_1''-poor, it contains at least $\frac{3}{4}n - 36\sqrt{\delta}n - 1$ vertices of $W_2' \setminus B$. When G has more than three parts, at least two parts must be W_1''-poor; any two parts V_i, V_j have $n_i + n_j > n$, so together, two W_1''-poor parts have at least $n - 72\sqrt{\delta}n$ vertices of $W_2' \setminus B$. In either case, there are one or two W_1''-poor parts which together contain at least $\frac{2}{3}n$ vertices of $W_2' \setminus B$.

We change our choice of W_2'', if necessary, to include at least $\frac{2}{3}n$ vertices from this W_1''-poor part or parts; otherwise, the choice is still arbitrary. Meanwhile, we choose no vertices from these parts from W_1''; this rules out at most $40\sqrt{\delta}n$ vertices in addition to our previous restrictions. Completing the choice of W_1'' arbitrarily, we are left with a pair (W_1'', W_2'') that satisfies Condition 1 of Lemma 2.49, but possibly not Condition 2.

From Condition 1, we know that if $v_j \in W_2''$ satisfies $d(v_j) \leq j+q$, we have $d(v_j) \geq n - n_{**,2} - \sqrt{\delta}n \geq \frac{1}{2}n - \sqrt{\delta}n$. Additionally, we know that for any $u_i \in W_1''$, $d(u_i) \geq \frac{2}{3}n - \sqrt{\delta}n$, since there are at least $\frac{2}{3}n$ vertices of W_2'' in a different part of G. Then $d(u_i) + d(v_j) \geq \frac{7}{6}n - 2\sqrt{\delta}n \geq n + q + 1$, satisfying the hypothesis of Theorem 2.46.

Next, we consider the case where G has at most 3 parts and $n_1 > \frac{5}{4}n$. By (2.49), $N > 3n-1$. Hence by (2.48) we know that $n_1 = n_2$ and $N - n_1 = 2n - 1$. The case of $n_1 \geq \frac{5}{3}n$ was handled in Section 2.2.5.2. Thus, we may assume $n_1 < \frac{5}{3}n$, so $n_3 = (2n-1) - n_2 > \frac{1}{3}n - 1$.

Assume first that one of $W_1' \setminus (A \cup C)$ or $W_2' \setminus B$ intersects each part of G in at least $20\sqrt{\delta}n$ vertices, and the other has at least $30\sqrt{\delta}n$ vertices outside each part of G; we will consider departures from this assumption later. This implies that for $j = 1$ or $j = 2$, we can choose $20\sqrt{\delta}n$ vertices from each part to add to W_j'', and match these by choosing $60\sqrt{\delta}n$ vertices to add to W_{3-j}'' with no more than $30\sqrt{\delta}n$ of these from one part. (No V_i has more than $50\sqrt{\delta}n$ vertices chosen from it at this point.)

Then proceed by an iterative strategy. At each step, choose one vertex from $W_1' \setminus (A \cup C)$ not previously added to W_1'', and a vertex from $W_2' \setminus B$ not previously added to W_2'', so that these vertices are in different parts of G. Then add them to W_1'' and W_2'' respectively. This step is always possible when $|W_1'' \cup A|, |W_2'' \cup B| < n$: in this case, at least two parts still have unchosen vertices, since $|V_1|, |V_2| \geq \frac{5}{4}n$ but fewer than n vertices have been chosen. Additionally, choosing a pair of vertices, one from W_1' and one from W_2', is only impossible if $W_2' \setminus B$ has no more vertices, in which case W_2'' has reached its desired size.

Stop when $|W_2'' \cup B| = n$. When this happens, W_1'' still needs $\sqrt{\delta}n$ more vertices, and these can be chosen arbitrarily.

This process guarantees that Conditions 1 and 2 of Lemma 2.49 hold. Before we begin iterating, we have chosen $60\sqrt{\delta}n$ vertices, but at most $50\sqrt{\delta}n$ from each part. After we begin iterating, we add at most one vertex from each part at each step. Therefore in the end, $n_{i,1} + n_{i,2} \leq n - 10\sqrt{\delta}n$ for each

i, satisfying Condition 1. Moreover, for some j, we added at least $20\sqrt{\delta}n$ vertices from each part to W_j'', ensuring that at most one value of $n_{i,j}$ can exceed $(\frac{1}{2} - 10\sqrt{\delta})n$ and satisfying Condition 2.

Now we consider alternatives to our initial assumptions in this case. We cannot have $W_1' \setminus (A \cup C)$ have fewer than $30\sqrt{\delta}n$ vertices outside V_i for any i, since it contains at least $2n - 1 - 4\sqrt{\delta}n - 2\delta n$ vertices, and no V_i is larger than $\frac{5}{3}n$. But it is possible that one of V_1 or V_2 contains all but $30\sqrt{\delta}n$ vertices of $W_2' \setminus B$; without loss of generality, it is V_1.

In this case, if $|V_1 \cap W_2' \setminus B| > n$, then let W_2'' be any n-element subset of $V_1 \cap W_2' \setminus B$; otherwise, let W_2'' be any n-element subset of $W_2' \setminus B$ containing $V_1 \cap W_2' \setminus B$. The set $V_2 \cup V_3$ has $2n - 1$ vertices, at most $30\sqrt{\delta}n + |B| = 33\sqrt{\delta}n$ of which are in W_2', so we can pick all n vertices of W_1'' from $V_2 \cup V_3$. Choose at least $10\sqrt{\delta}n$ of them from V_3 to satisfy Condition 1 of Lemma 2.49 for $i = 2$. Condition 1 also holds for $i = 1$ (since $n_{i,1} = 0$) and $i = 3$ (since $n_3 < \frac{3}{4}n$); Condition 2 holds for $j = 2$.

Finally, we also violate the assumptions at the beginning of this case when neither $W_1' \setminus (A \cup C)$ nor $W_2' \setminus B$ have at least $20\sqrt{\delta}n$ vertices from each part of G. It is impossible that both of them have at most $20\sqrt{\delta}n$ vertices from V_3, so one of them has at most $20\sqrt{\delta}n$ vertices from one of V_1 or V_2.

If one of them (without loss of generality, V_1) contains at most $20\sqrt{\delta}n$ vertices of $W_1' \setminus (A \cup C)$, it must have at least n vertices of $W_2' \setminus B$, since $|V_1| \geq \frac{5}{4}n$, so choose all remaining vertices out of W_2'' from there. Outside V_1, we have at least $(2n - 1 - 4\sqrt{\delta}n - 2\delta n) - 20\sqrt{\delta}n$ vertices of $W_1' \setminus (A \cup C)$, which leaves at most $24\sqrt{\delta}n + 2\delta n$ vertices we *cannot* choose for W_1''. Choose n vertices outside V_1 for W_1'', including at least $10\sqrt{\delta}n$ vertices of V_3. This satisfies Condition 1 for $i = 1$ (since $n_{i,1} = 0$), $i = 2$ (since $n_{i,1} = 0$ and $n_{i,2} < n - 10\sqrt{\delta}n$), and $i = 3$ (since $n_3 < \frac{3}{4}n$); Condition 2 holds for $j = 2$.

If one of V_1 or V_2 (without loss of generality, V_1) contains at most $20\sqrt{\delta}n$ vertices of $W_2' \setminus B$, choose $n - 30\sqrt{\delta}n$ vertices of W_1'' from V_1 (satisfying Condition 1 for $i = 1$ and Condition 2 by taking $j = 1$). If V_3 contains at least $30\sqrt{\delta}n$ vertices of $W_1' \setminus (A \cup C)$, take the remaining vertices of W_1'' from W_3. Otherwise, V_3 contains at least $60\sqrt{\delta}n$ vertices of $W_2' \setminus B$; choosing as many vertices as possible from $V_1 \cup V_3$ to add to W_2'', and the remaining vertices of W_1'' arbitrarily, we end up choosing no more than $n - 10\sqrt{\delta}n$ vertices from V_2. So Condition 1 holds for $i = 2$ either because $n_{i,1} = 0$ or because $n_{i,1} + n_{i,2} \leq n - 10\sqrt{\delta}n$; Condition 1 holds for $i = 3$ because $n_3 < \frac{3}{4}n$.

2.2.6 Dealing with $(\lambda, i, 2)$-bad partitions when $N - n_1 - n_2 \geq 3$

A *cherry* is a path on three vertices. The *center* of a cherry is the vertex with degree 2.

Suppose $N - n_1 - n_2 \geq 3$. By (2.48)–(2.50), we have two cases:

1) $N > 3n - 1$, $s = 3$, $n_2 + n_3 = 2n - 1$ and $n_1 = n_2$ (i.e., (2.48) holds), or

2) $N = 3n - 1$, $n_1 \leq n$, $s \leq 5$, and if $s \geq 4$, then $n_{s-1} + n_s \geq n + 1$ (i.e., (2.49) holds).

2.2.6.1 The case when (2.48) holds

By (2.48), $n_1 = n_2 > n$, $s = 3$, and $0 < n_3 = 2n - 1 - n_2 < n$.

Lemma 2.50. *Let $G = K_{n_1,n_2,n_3}$ with $n_1 = n_2$ and $n_2 + n_3 = 2n - 1$ be 2-edge-colored with a $(\lambda, i, 2)$-bad partition. Then G has a monochromatic cycle of length $2n$.*

In Section 2.2.6.1, we prove Lemma 2.50, but postpone technical details of how the monochromatic cycles are constructed in each of four cases; these details are given in Claims 2.51–2.54.

Proof of Lemma 2.50. Without loss of generality, let $i = 2$; we call color 1 red, color 2 blue, and use d_1 (d_2) to denote the red (blue) degree.

We begin by assuming that in the $(\lambda, 2, 2)$-bad partition (V_j, U_1, U_2), $j = 3$. Later, in Section 2.2.6.1.5, we discuss the modifications to the proof when $j \neq 3$.

Since (V_j, U_1, U_2) is a 2-bad partition, we know the following conditions hold:

(i) $|V_3| \geq (1 - \lambda)n$.

(ii) $(1 - \lambda)n \leq |U_1| \leq (1 + \lambda)n$.

(iii) $(1 - \lambda)n \leq |U_2| \leq (1 + \lambda)n$.

(iv) $E(G_2[V_3, U_1]) \leq \lambda n^2$.

(v) $E(G_1[V_3, U_2]) \leq \lambda n^2$.

If a vertex u_1 in U_1 has blue degree at least $\frac{n_3}{2}$ to V_3 then we move u_1 to U_2. If a vertex u_2 in U_2 has red degree at least $\frac{n_3}{2}$ to V_3 then we move u_2 to U_1. Since there are at most $3\lambda n$ vertices in U_1 with blue degree at least $\frac{n_3}{2}$ to V_3 and there are at most $3\lambda n$ vertices in U_2 with red degree at least $\frac{n_3}{2}$ to V_3, we moved at most $3\lambda n$ vertices out of U_1 and U_2 respectively and moved at most $3\lambda n$ vertices into U_1 and U_2 respectively. Thus, we may assume $|U_1| \geq |U_2|$, $|U_1| = n + a_1$, $|U_2| = n + a_2$, and $a_1 \geq 0$.

Note that (iv) and (v) change to:

(iv) $|E(G_2[V_3, U_1])| \leq 4\lambda n^2$,

(v) $|E(G_1[V_3, U_2])| \leq 4\lambda n^2$.

Let $|V_3| = n - a_3$, where $a_3 \leq 10\lambda n$. Let B be the set of vertices in V_3 with blue degree at least $0.9n$ to U_1 and $|B| = b$. Let R be the set of vertices in V_3 with blue degree at most $0.05n$ to U_1. By Condition (iv), we know
$$|B| \leq 5\lambda n \text{ and } |R| \geq n - a_3 - 80\lambda n.$$

Let C be a maximum collection of vertex-disjoint red cherries with center in U_2 and leaves in U_1. If there at least $m := a_3 + b$ cherries in C, then we use them, together with the edges between U_1 and V_3, to find a red cycle of length $2n$; this is done in Claim 2.51.

Otherwise, we assume that $|C| \le m - 1$: there are at most $m - 1$ red cherries from U_2 to U_1. Every vertex in $U_2 - V(C)$ has red degree at most $2m - 1$ to U_1, since otherwise we have a larger collection of red cherries.

When $|U_2| = n + a_2 \ge n - b$, we can find a blue cycle using edges between U_2 and V_3, as well as enough edges between U_1 and B to make up for the size of U_2 when $|U_2| < n$. This is done in Claim 2.52.

Otherwise, we assume that $|U_2| \le n - b - 1$; in other words,

$$a_2 \le -(b+1). \tag{2.55}$$

Our goal is now to use edges within U_1 to find a monochromatic cycle. Without loss of generality, we may assume that $|U_1 \cap V_1| \ge |U_1 \cap V_2|$. We first argue that $U_1 \cap V_2$ cannot be too small.

Earlier, we defined $|U_1| = n + a_1$, $|U_2| = n + a_2$, $|V_3| = n - a_3$. Since $|V_1| + |V_3| = |V_2| + |V_3| = 2n - 1$ and $U_1 \cup U_2 = V_1 \cup V_2$, we have

$$2n + a_1 + a_2 = |V_1| + |V_2| = 4n - 2 - 2|V_3| = 2n + 2a_3 - 2$$

or

$$a_1 + a_2 = 2a_3 - 2. \tag{2.56}$$

Therefore

$$\begin{aligned} |U_1 \cap V_2| \ge |U_1| - |V_1| = |U_1| - \frac{|U_1| + |U_2|}{2} &= n + a_1 - n - \frac{a_1 + a_2}{2} \\ = \frac{a_1 - a_2}{2} &= a_3 - a_2 - 1 = (b + a_3) + (-b - a_2) - 1. \end{aligned}$$

There are two possibilities for the vertices of $U_1 \cap V_2$:

- There are at least $m = b + a_3$ vertices in $U_1 \cap V_2$ which have red degree at least $0.1n$ to $U_1 \cap V_1$. In this case, we use Claim 2.53 to find a red cycle of length exactly $2n$.
- There are at least $m' := -b - a_2$ vertices in $U_1 \cap V_2$ which have blue degree at least $|U_1 \cap V_1| - 0.1n \ge 0.4n$ to $U_1 \cap V_1$. In this case, we use Claim 2.54 to find a blue cycle of length exactly $2n$.

One of these must hold, since $|U_1 \cap V_2| \ge m + m' - 1$, while by (2.55), $m' = -b - a_2 \ge 1$: therefore there are either m vertices for Claim 2.53 or m' vertices for Claim 2.54. In either case, we obtain a monochromatic cycle of length exactly $2n$, completing the proof. $\square$

2.2.6.1.1 The case of many cherries: $|C| \geq m$

Recall that C is a maximum collection of vertex-disjoint red cherries with centers in U_2 and leaves in U_1; $m = b + a_3$, where $b = |B|$ and $a_3 = n - |V_3|$.

Claim 2.51. *If $|C| \geq m$, then we have a red cycle of length exactly $2n$.*

Proof. We do the following steps. Let $C' \subseteq C$ be a collection of m red cherries with centers in U_2 and leaves in U_1. Let $\{u_1, \ldots, u_m\} = V(C') \cap U_2$ and $\{v_1, \ldots, v_{2m}\} = V(C') \cap U_1$ such that each $v_{2i-1}u_iv_{2i}$ is a cherry with center u_i, where $1 \leq i \leq m$.

To find a cycle of length $2n$ in G_1 that contains the edges of C', we will apply Theorem 2.46 to an appropriately chosen bipartite graph.

First, create an auxiliary graph G_1' by starting with G_1 and adding every edge between $\{u_1, \ldots, u_m\}$ and U_1. This will help us to satisfy the degree conditions of Theorem 2.46; however, these artificial edges will never be used by a cycle containing all the edges of C', since each of $\{u_1, \ldots, u_m\}$ already has degree 2 in C'.

Second, let $X = (V_3 - B) \cup \{u_1, u_2, \ldots, u_m\}$ (a set of n vertices total) and let $Y \subseteq U_1$ be any set of size n such that $\{v_1, \ldots, v_{2m}\} \subseteq Y$. We check that the hypotheses of Theorem 2.46 apply to $G_1'[X, Y]$.

Order vertices in X and Y separately by their degree from smallest to largest. Since vertices in Y have red degree at least $\frac{n_3}{2} - b \geq 0.4n$ to X and at most $100\lambda n \ll 0.001n$ vertices in Y have blue degree at least $0.04n$ to X, the smallest index k such that $d_1(y_k) \leq k + q$ satisfies $d_1(y_k) \geq 0.95n$. Since vertices in X have blue degree at most $0.9n$ to U_1, they have red degree at least $n - 0.9n = 0.1n \gg 0.09n$ to Y. The smallest index j such that $d_1(x_j) \leq j + q$ satisfies $d_1(x_j) \geq 0.09n$. By Theorem 2.46 and $0.09n + 0.95n \gg n + q + 1$, we can find a Hamiltonian cycle in $G_1'[X, Y]$ of length $2n$ containing the edges of C', which is a cycle of length $2n$ in G_1. $\square$

2.2.6.1.2 The case of large U_2: $|U_2| \geq n - b$

Recall that $|U_2| = n + a_2$, B is the set of vertices in V_3 with blue degree at least $0.9n$ to U_1, and $b = |B|$.

Claim 2.52. *If $b \geq -a_2$ (in other words, if $|U_2| = n + a_2 \geq n - b$), then we have a blue cycle of size exactly $2n$.*

Proof. Let $c := |C|$; let $V(C) \cap U_2 = \{u_1, \ldots, u_c\}$ and $V(C) \cap U_1 = \{v_1, v_2, \ldots, v_{2c}\}$. Let B_2 be the collection of vertices in $V_3 - B$ with red degree at most $0.1n$ to U_2. By Condition (v),

$$q := |B_2| \geq n - a_3 - 40\lambda n - b.$$

Since $2n_1 = |U_1| + |U_2| = 2n + a_1 + a_2$, we know

$$|U_2 \cap V_2| = n_1 - |U_1 \cap V_2| \geq n_1 - \frac{n + a_1}{2} = n + \frac{a_1 + a_2}{2} - \frac{n}{2} - \frac{a_1}{2} = \frac{n + a_2}{2}$$

and thus

$$|U_2 \cap V_1| \leq n + a_2 - \frac{n + a_2}{2} = \frac{n + a_2}{2}. \tag{2.57}$$

Step 1: We first find a path to include $0.8n$ vertices in V_3 and $0.8n$ vertices in U_2 (all of $U_2 \cap V_1$ and $V(C)$) by Theorem 2.45.

Details: Since $|B_2| \geq n - a_3 - 40\lambda n - b$, we take a set $X \subseteq B_2$ such that $|X| = 0.8n$. By (2.57), we can take a set $Y \subseteq U_2$ such that $U_2 \cap V_1 \subseteq Y$, $V(C) \cap U_2 \subset Y$, and $Y = 0.8n$.

Now we consider $G_2[X, Y]$ and we order vertices in X and Y separately by their degree from smallest to largest. Since vertices in Y have blue degree at least $0.8n - \frac{n_3}{2} > 0.2n$ to X, the smallest index k such that $d_2(y_k) \leq k + 1$ satisfies $d_2(y_k) \geq 0.2n$. Since vertices in X have red degree at most $0.1n$ to U_2, they have blue degree at least $0.8n - 0.1n = 0.7n$ to Y. The smallest index j such that $d_2(x_j) \leq j + 1$ satisfies $d_2(x_j) \geq 0.7n$. By Theorem 2.45 and $0.7n + 0.2n > 0.8n + 2$, we can find a Hamiltonian red path P_1' from $x \in X$ to some vertex $y \in Y - V_1 - V(C)$ in $G_2[X, Y]$ of length $1.6n - 1$.

Since $x \in X \subseteq B_2$,

$$d_2(x, U_2 - Y) \geq n + a_2 - 0.8n - 0.1n > 0.05n.$$

We extend the path P_1' to P_1 of length $1.6n$ by adding a blue edge xy' such that $y' \in U_2 - Y$.

Step 2: Use $\min\{0, -a_2\}$ vertices in B to obtain a blue path. (We can skip this step if $a_2 \geq 0$.)

Details: Assume $a_2 < 0$; since $b \geq -a_2$, let $Z := \{z_1, \ldots, z_{|a_2|}\} \subseteq B$.

Since

$$|U_1 \cap V_1| \geq \frac{n + a_1}{2} \geq |U_1 \cap V_2|,$$

each vertex in B has blue degree at least $0.9n - |U_1 \cap V_2|$ to $U_1 \cap V_1$. Therefore,

$$0.9n - |U_1 \cap V_2| \geq 0.9n - (n + a_1 - |U_1 \cap V_1|) = |U_1 \cap V_1| - a_1 - 0.1n \geq \frac{3}{4}|U_1 \cap V_1|.$$

We can find for each pair (z_i, z_{i+1}) a common neighbor $r_i \in U_1 \cap V_1 - V(C)$ where $1 \leq i \leq |a_2| - 1$, a blue neighbor r_0 of z_1, a blue neighbor $r_{|a_2|}$ of $z_{|a_2|}$ such that $r_0, \ldots, r_{|a_2|}$ are all distinct.

We obtain a blue path

$$P_2 = r_0 z_1 r_1 \ldots z_i r_i \ldots z_{|a_2|} r_{|a_2|}$$

of length $2|a_2|$.

Since y' has at most one red neighbor to $U_1 - V(C)$, at least one of $\{r_0, r_{|a_2|}\}$ is a blue neighbor of y'. We may assume $r_{|a_2|} y'$ is blue.

Step 3: Include the rest of vertices in U_2 to U_1.

Details: We proceed differently depending on whether $a_2 \geq 0$.

- If $a_2 < 0$ then we do the following. Let $K := (U_2 - Y - \{y'\}) \cup \{y\} = \{y, f_1, \ldots, f_{k-1}\}$. Note that $k = |K| = n + a_2 - 0.8n = 0.2n + a_2$ and $K \subseteq U_2 \cap V_2 - V(C)$. Since each vertex in K has at most one red neighbor to $U_1 - V_2 - V(C) - \{r_0, r_1, \ldots, r_{|a_2|}\}$, we find for (y, f_1) a blue common neighbor $h_0 \in U_1 - V_2 - V(C) - \{r_0, r_1, \ldots, r_{|a_2|}\}$ and each pair (f_i, f_{i+1}) a distinct blue common neighbor, h_i, in $U_1 - V_2 - V(C) - \{r_0, r_1, \ldots, r_{|a_2|}\}$ where $1 \leq i \leq k-2$. We obtain a blue path
$$P_3 = yh_0f_1 \ldots f_ih_if_{i+1} \ldots f_{k-1}$$
of size $2k - 2 = 0.4n + 2a_2 - 2$.

  We may assume $f_{k-1}r_0$ is blue since f_{k-1} has only one red neighbor to $U_1 \cap V_1 - V(C)$ and there are many choices when we choose r_0 to connect with z_1.

  Finally, we connect P_2 and P_1 by adding the edge $r_{|a_2|}y'$, glue the paths P_1 and P_3 at y, then add the edge $f_{k-1}r_0$ to complete a blue cycle of length exactly
$$2|a_2| + 1 + 1.6n + 0.4n + 2a_2 - 2 + 1 = 2n.$$

- If $a_2 \geq 0$ then in the previous argument we take $K = \{y, y', f_1, \ldots, f_{k-2}\}$ of size $0.2n + 1$ and find common neighbors h_0 for (y, f_1), h_i for (f_i, f_{i+1}) where $1 \leq i \leq k-3$, and h_{k-2} for (f_{k-2}, y').

In either case, we obtain a path
$$P_3 = yh_0f_1 \ldots f_ih_if_{i+1} \ldots f_{k-2}h_{k-2}y'$$
of size $2k - 2 = 0.4n$. We glue P_1 and P_3 at y and y' to obtain a blue cycle of length exactly $1.6n + 0.4n = 2n$. □

2.2.6.1.3 Handling many vertices in $U_1 \cap V_2$ incident to red edges

We will find a red cycle. Note that the size of $U_1 \cap V_2$ is at least $n + a_1 - n_1$.

Claim 2.53. *If there are at least $m = b + a_3$ vertices in $U_1 \cap V_2$ of red degree at least $0.1n$ to $U_1 \cap V_1$, then we have a red cycle of length exactly $2n$.*

Proof. Let B' be the collection of vertices in U_1 with blue degree at least $0.05n$ to V_3. By (iv), we have
$$|B'| \leq 80\lambda n.$$

Step 1: We first find a collection of red cherries C_3 with center in $U_1 \cap V_2$ and leaves in $U_1 \cap V_1 - B'$ of size $b + a_3 =: m$.

Details: Since there are at least m vertices in $U_1 \cap V_2$ of red degree at least $0.1n$ to $U_1 \cap V_1$ and $0.1n - 80\lambda n \gg 2m$, we can find a collection of red cherries C_3 with centers in $U_1 \cap V_2$ and leaves in $U_1 \cap V_1 - B'$ of size m. Let $V(C_3) \cap V_2 = \{u_1, \ldots, u_m\}$ and $V(C_3) \cap V_1 = \{v_1, \ldots, v_{2m}\}$.

Recall that $R \subseteq V_3$ is the collection of vertices in V_3 with blue degree at most $0.05n$ to U_1.

Step 2: Then by Hall's Theorem we find matching M for $V(C_3) \cap V_1$ to R and then find common neighbor back to connect those vertices.

Details: Since $\{v_2, \ldots, v_{2m}\} \cap B' = \emptyset$, each of them has red degree at least $n - a_3 - 0.05n - 80\lambda n > 0.9n$ to R. Thus, we can find a matching M for $\{v_2, \ldots, v_{2m}\}$ such that $V(M) \cap V_3 = \{w_2, \ldots, w_{2m}\}$ and each $v_i w_i$ is a matching edge, where $2 \leq i \leq 2m$.

Since $V(M) \cap V_3 \subseteq R$, we can find for each pair (w_{2i}, w_{2i+1}) a common red neighbor $g_i \in U_1$, where $1 \leq i \leq m - 1$.

Therefore, we obtained a path

$$P_1 = v_1 u_1 v_2 w_2 g_1 w_3 v_3 u_2 v_4 w_4 \ldots v_{2m-1} u_m v_{2m} w_{2m}$$

of length $6m - 3$.

Step 3: We use Theorem 2.45 to get a path saturating all vertices left in $V_3 - B - V(M)$.

Details: Let $X = V_3 - B - \{w_2, \ldots, w_{2m-1}\}$ and we know

$$|X| = n - a_3 - b - (2m - 2) = n - 3m + 2.$$

Choose $Y \subseteq U_1 - \{u_1, \ldots, u_m\} - \{v_2, \ldots, v_{2m}\} - \{g_1, \ldots, g_{m-1}\}$ such that $v_1 \in Y$. By (2.56),

$$a_1 = -a_2 + 2a_3 - 2 \geq b + 1 + a_3 + a_3 - 2 = m + a_3 - 1 \geq m \tag{2.58}$$

and thus

$$n + a_1 - m - (2m - 1) - (m - 1) \geq n - 3m + 2.$$

Hence we can require $|Y| = n - 3m + 2$.

Now we consider $G_1[X, Y]$ and we order vertices in X and Y separately by their degree from smallest to largest. Since vertices in U_1 have red degree at least $\frac{n_3}{2}$ to V_3, they have red degree at least $\frac{n_3}{2} - b - (2m - 2) > 0.4n$ to X.

By Condition (iv), there are at most $80\lambda n$ vertices in U_1 with blue degree at least $0.05n$ to V_3. Thus, at least $|Y| - 80\lambda n$ vertices in Y have red degree at least $|X| - 0.05n > 0.94n$ to X, the smallest index k such that $d_1(y_k, X) \leq k + 1$ satisfies $d_1(y_k, X) \geq 0.94n - 1$. Since vertices in X have blue degree at most $0.9n$ to U_1, they have red degree at least $n + a_1 - m - (2m - 1) - (m - 1) - 0.9n > 0.09n$ to

Y. The smallest index j such that $d_1(x_j, Y) \le j+1$ satisfies $d_1(x_j, Y) \ge 0.09n$. By Theorem 2.45 and $0.09n + 0.94n \gg n+2$, we can find a Hamiltonian red path P_2 from v_1 to w_{2m} in $G_1[X, Y]$ of length

$$2(n-3m+2)-1 = 2n-6m+3.$$

We glue P_1 and P_2 at v_1 and w_{2m} to obtain a red cycle of size exactly

$$6m-3+2n-6m+3 = 2n. \qquad \square$$

2.2.6.1.4 Handling many vertices in $U_1 \cap V_2$ incident to blue edges

In this case, there are many disjoint blue cherries inside U_1, and we will find a blue cycle. Recall that C is a collection of at most $m-1$ cherries with centers in U_2 and leaves in U_1, which is defined three paragraphs ahead of equation (2.55).

Claim 2.54. *If there are at least $-a_2-b$ vertices in $U_1 \cap V_2$ of blue degree at least $|U_1 \cap V_1| - 0.1n \ge 0.4n$ to $U_1 \cap V_1$, then we find a blue cycle of length exactly $2n$.*

Proof. **Step 1:** We find $m' = -a_2 - b$ blue cherries with centers in $U_1 \cap V_2$ and leaves in $U_1 \cap V_1$. Possibly avoiding bad vertices. Then find common neighbors in $U_2 \cap V_2$ to connect those cherries.

Details: Since vertices in $U_2 \cap V_2 - V(C)$ have red degree at most one to $U_1 \cap V_1 - V(C)$, there are at most $|U_2 \cap V_2| \le \lambda n^2$ red edges between $U_2 \cap V_2 - V(C)$ and $U_1 \cap V_1 - V(C)$. Therefore, there are at most $20\lambda n$ vertices in $U_1 \cap V_1 - V(C)$ with red degree at least $0.05n$ to $U_2 \cap V_2 - V(C)$ and at least $|U_1 \cap V_1| - |V(C) \cap U_1| - 20\lambda n$ vertices in $U_1 \cap V_1 - V(C)$ with blue degree at least $|U_2 \cap V_2| - |V(C)| - 0.05n > \frac{3}{4}|U_2 \cap V_2|$ to $U_2 \cap V_2 - V(C)$, we call those vertices B_3.

Since there are m' vertices in $U_1 \cap V_2$ of blue degree at least $|U_1 \cap V_1| - 0.1n - |V(C)| - 20\lambda n > 0.3n$ to B_3, we find m' blue cherries, C_4, with center in $U_1 \cap V_2$ and leaves in B_3. Let $V(C_4) \cap V_2 = \{u_1, \ldots, u_{m'}\}$ and $V(C_4) \cap V_1 = \{v_1, \ldots, v_{2m'}\}$.

We can find for each pair (v_{2i}, v_{2i+1}) a common blue neighbor, w_i, in $U_2 \cap V_2 - V(C)$, where $1 \le i \le m'-1$. We also find for v_1 a blue neighbor w_0 and $v_{2m'}$ a blue neighbor $w_{m'}$ distinct from $\{w_1, \ldots, w_{m'-1}\}$ and $V(C)$.

We obtain a blue path

$$P_1 = w_0 v_1 u_1 v_2 w_1 \ldots v_{2m'-1} u_{m'} v_{2m'} w_{m'}$$

of length $4m'$.

Step 2: We find for vertices in B common neighbors in $U_1 \cap V_1$, avoiding vertices already used.

Details: Since

$$|U_1 \cap V_1| \ge \frac{n+a_1}{2} \ge |U_1 \cap V_2|, \tag{2.59}$$

each vertex in B has blue degree at least $0.9n - 2m' - |U_1 \cap V_2| - |V(C) \cap U_1|$ to $U_1 \cap V_1 - V(C)$.

Therefore,

$$0.9n - 2m' - |U_1 \cap V_2| - |V(C) \cap U_1| \geq 0.9n - 2m' - (n + a_1 - |U_1 \cap V_1|) - 2(m-1)$$

$$= |U_1 \cap V_1| - a_1 - 2m' - 0.1n - 2m + 2 \geq \frac{3}{4}|U_1 \cap V_1|.$$

Let $B = \{z_1, \ldots, z_b\}$. We can find for each pair (z_i, z_{i+1}) a common neighbor r_i where $1 \leq i \leq b-1$, a blue neighbor r_0 of z_1, a blue neighbor r_b of z_b such that $r_0, \ldots, r_b$ are all distinct and in $U_1 \cap V_1 - V(C)$.

We obtain a blue path

$$P_2 = r_0 z_1 r_1 \ldots z_i r_i \ldots z_b r_b$$

of length $2b$.

Step 3: Take $0.9n$ vertices in V_3 and $0.9n$ vertices in U_2 including $U_2 \cap V_1$ and $V(C)$. Use Theorem 2.45 to find a path.

Details: Recall that B_2 is the collection of vertices in V_3 with red degree at most $0.1n$ to U_2 and $|B_2| \geq n - a_3 - 40\lambda n - b$. Since $|B_2| \geq n - a_3 - 40\lambda n - b$, we take a set $X \subseteq B_2$ such that $|X| = 0.9n$. By (2.59), $|U_2 \cap V_1| \leq 0.6n$ and we can take a set $Y \subseteq U_2 - \{w_0, w_1, \ldots, w_{m'-1}\}$ such that $U_2 \cap V_1 \subseteq Y$, $V(C) \subseteq Y$, $w_{m'} \in Y$, and $Y = 0.9n$.

First we find a blue edge $v'u'$ with $v' \in X$ and $u' \in U_2 - Y$. Now we consider $G_2[X, Y]$ and we order vertices in X and Y separately by their degree from smallest to largest. Since vertices in Y have blue degree at least $0.9n - \frac{n_3}{2} > 0.3n$ to X, the smallest index k such that $d_2(y_k, X) \leq k+1$ satisfies $d_2(y_k, X) \geq 0.3n$. Since vertices in X have red degree at most $0.1n$ to U_2, they have blue degree at least $0.9n - 0.1n = 0.8n$ to Y. The smallest index j such that $d_2(x_j, Y) \leq j+1$ satisfies $d_2(x_j, Y) \geq 0.8n$. By Theorem 2.45 and $0.8n + 0.3n > 0.9n + 2$, we can find a Hamiltonian blue path P_3' from $w_{m'}$ to v' in $G_2[X, Y]$ of length $1.8n - 1$. We then extend the path P_3' to P_3 by adding the edge $v'u'$. Thus, the path P_3 has length $1.8n$.

Step 4: Finally, the rest of vertices in $U_2 \cap V_2$ have large blue degree to $U_1 \cap V_1$, and we find common neighbors to include them.

Details: Let $K := (U_2 - Y - \{w_0, w_1, \ldots, w_{m'-1}\}) = \{u', f_1, \ldots, f_{k-1}\}$. Note that $k = |K| = n + a_2 - 0.9n - m' = 0.1n + a_2 - m'$ and $K \subseteq U_2 \cap V_2 - V(C)$. Since each vertex in K has at most one red neighbor to $U_1 \cap V_1 - V(C) - \{v_1, \ldots, v_{2m'}\} - \{r_0, \ldots, r_b\}$, we find for (u', f_1) a distinct blue common neighbor h_0, each pair (f_i, f_{i+1}) a distinct blue common neighbor, h_i, in $U_1 \cap V_1 - V(C) - \{v_1, \ldots, v_{2m'}\} - \{r_0, \ldots, r_b\}$ where $1 \leq i \leq k-2$. We may assume that $r_0 f_{k-1}$ is blue (since f_{k-1} has at most one red neighbor to $U_1 \cap V_1$ and z_1 has very large blue degree to $U_1 \cap V_1$, if $r_0 f_{k-1}$ was not blue then we choose r_0 such that $r_0 f_{k-1}$ is blue).

We obtain a blue path

$$P_4 = u' h_0 f_1 \ldots f_i h_i f_{i+1} \ldots h_{k-2} f_{k-1}$$

of size $2k - 2 = 0.2n + 2a_2 - 2m' - 2$.

Finally, we add the edge $r_b w_0$ to connect P_2 and P_1, glue P_1 and P_3 at $w_{m'}$, glue P_3 and P_4 at u', and add the edge $r_0 f_{k-1}$ to complete the cycle of length

$$1 + 4m' + 2b + 1.8n + 0.2n + 2a_2 - 2m' + 1 = 2n.$$

□

2.2.6.1.5 Changes of the proof when $j \neq 3$

When $j \neq 3$, essentially the same proof works, with minor modifications.

Without loss of generality, we assume $j = 1$. We use the same setup as in the case when $j = 3$ but replace every place of V_3 by V_1 and n_3 by n_1.

Case 1: $n_1 \geq n + b$.

Since $n_1 \geq n + b$ and $|U_1| \geq n$, we take a set of vertices $X \subseteq V_1 - B$ of size n and a set of vertices $Y \subseteq U_1$ of size n.

Now we consider $G_1[X, Y]$ and we order vertices in X and Y separately by their degree from smallest to largest. Since vertices in Y have red degree at least $0.5n_1$ to X and there are at most $80\lambda n$ vertices with blue degree at least $0.05n$ to V_1, the smallest index k such that $d_1(y_k, X) \leq k + 1$ satisfies $d_1(y_k, X) \geq 0.95n$. Since vertices in X have blue degree at most $0.9n$ to U_1, they have red degree at least $0.1n$ to Y. The smallest index j such that $d_1(x_j, Y) \leq j+1$ satisfies $d_1(x_j, Y) \geq 0.1n$. By Theorem 2.46 and $0.1n + 0.95n \gg n + 1$, there is a Hamiltonian cycle in $G_1[X, Y]$ of length $2n$.

Case 2: $n + 1 \leq n_1 \leq n + b - 1$.

We still assume $n_1 = n - a_3$ with $a_3 < 0$. It is included in Case 1 by replacing n_3 with n_1, V_3 with V_1, V_1 with V_2, and V_2 with V_3. Note that in this case we have

$$n + a_1 + n + a_2 = 2n - 1$$

and thus

$$a_1 + a_2 = -1. \tag{2.60}$$

Equation (2.57) changes to

$$|U_2 \cap V_3| = n_3 - |U_1 \cap V_3| \geq 2n - 1 - n + a_3 - \frac{n + a_1}{2} = \frac{n}{2} - 1 + a_3 - \frac{a_1}{2}$$

and thus

$$|U_2 \cap V_2| \leq n + a_2 - (\frac{n}{2} - 1 + a_3 - \frac{a_1}{2}) = \frac{n}{2} + 1 + a_2 - a_3 + \frac{a_1}{2} = \frac{n}{2} - a_3 - \frac{a_1}{2}.$$

Moreover, by $a_3 < 0$, the inequality $a_1 \geq m$ in (2.58) still holds under the assumption $a_2 \leq -b - 1$

since

$$a_1 = -1 - a_2 \geq b \geq b + a_3 = m.$$

When choosing between Claim 2.53 and Claim 2.54, we still have by (2.60)

$$|U_1| - |V_2| \geq n + a_1 - n + a_3 = a_1 + a_3 = -1 - a_2 + a_3 = (b + a_3) + (-b - a_2) - 1$$

and therefore one of the two claims can still be applied.

2.2.6.2 The case when (2.49) holds

2.2.6.2.1 Statement and setup of the main lemma

In this case, we have

$$n_1 + n_2 + \ldots + n_s = 3n - 1 \tag{2.61}$$

and

$$n_2 + \ldots + n_s \geq 2n - 1. \tag{2.62}$$

By (2.51), $s \leq 5$. Our main lemma in Section 2.2.6.2 is:

Lemma 2.55. *Let $G = K_{n_1,n_2,\ldots,n_s}$ satisfying (2.61) and (2.62) be 2-edge-colored with a $(\lambda, i, 2)$-bad partition. Then G has a monochromatic cycle of length $2n$.*

Proof. Without loss of generality, let $i = 2$. By the definition of a $(\lambda, i, 2)$-bad partition, there is a $j \in [s]$ such that

(i) $n \geq |V_j| \geq (1 - \lambda)n$.

(ii) $(1 - \lambda)n \leq |U_1| \leq (1 + \lambda)n$.

(iii) $(1 - \lambda)n \leq |U_2| \leq (1 + \lambda)n$.

(iv) $E(G_2[V_j, U_1]) \leq \lambda n^2$.

(v) $E(G_1[V_j, U_2]) \leq \lambda n^2$.

Our plan is as follows. In Sections 2.2.6.2.1, 2.2.6.2.2, 2.2.6.2.3, and 2.2.6.2.4, we handle the case $s = 4$ and renumber the parts so that $j = 1$ and $n_2 \geq n_3 \geq n_4$. Later, in Section 2.2.6.2.5, we return to the original numbering of the parts ($n_1 \geq \ldots \geq n_s$) and describe modifications to the proof for $s \neq 4$.

Since (2.49) holds, we have $n_i \leq n$ for all i; we also know that $n_2 \geq n_3 \geq n_4$, $n_1 = |V_j| \geq (1 - \lambda)n$, and

$$|U_1| + |U_2| = n_2 + n_3 + n_4 = 3n - 1 - n_1 \leq 2n + \lambda n - 1,$$

so $n_2 \geq \frac{n_2+n_3+n_4}{3} \geq \frac{2n}{3}$.

We move vertices as we did in Section 2.2.6.1 so that for each $u \in U_1$, $d_1(u, V_1) \geq \frac{n_1}{2}$ and for each $v \in U_2$, $d_2(v, V_1) \geq \frac{n_1}{2}$. Note that (iv) and (v) change to (iv) $|E(G_2[V_1, U_1])| \leq 4\lambda n^2$ and (v) $|E(G_1[V_1, U_2])| \leq 4\lambda n^2$.

Let $|U_1| = n + a_1$, $|U_2| = n + a_2$, and $|V_1| = n - a_3$. Let B be the set of vertices in V_1 with blue degree at least $0.9n$ to U_1, and let $b := |B|$. By Condition (iv), we know $b \leq 5\lambda n$.

Let C be a maximum collection of vertex-disjoint red cherries with center in U_2 and leaves in U_1. If there are at least $m := a_3 + b$ cherries in C, then we use them, together with the edges between U_1 and V_1, to find a red cycle of length $2n$. This is done in exactly the same way as in Claim 2.51, except with V_1 playing the role of V_3.

Otherwise, we assume that $c := |C| \leq m - 1$, which means every vertex in $U_2 - V(C)$ has red degree at most $2m - 1$ to U_1.

When $|U_2| = n + a_2 \geq n - b$, we can find a blue cycle in almost the same way as in Claim 2.52; the updated proof is given in Claim 2.56.

Otherwise, we may assume that $|U_2| \leq n - b - 1$, in which case (2.55) holds.

As before, to proceed, we want to use edges within U_1. Let k be such that $|U_1 \cap V_k|$ is maximized. This intersection is still at most $|V_k| \leq n$, while $|U_1| = n + a_1$, so $|U_1 - V_k| \geq a_1$.

Since $(n + a_1) + (n + a_2) = |U_1| + |U_2| = 3n - 1 - |V_1| = 2n + a_3 - 1$, we have $a_1 + a_2 = a_3 - 1$, and therefore

$$|U_1 - V_k| \geq a_3 - a_2 - 1 = (b + a_3) + (-a_2 - b) - 1.$$

There are two possibilities.

- There are at least $m = b + a_3$ vertices in $U_1 - V_k$ of red degree at least $0.1n$ to $U_1 \cap V_k$. In this case, we will find a red cycle of length exactly $2n$ by Claim 2.57.
- There are at least $m' = -a_2 - b$ vertices in $U_1 - V_k$ of blue degree at least $|U_1 \cap V_k| - 0.1n \geq 0.2n$ to $U_1 \cap V_k$. In this case, we find a blue cycle of length exactly $2n$ by Claim 2.58.

One of these must hold, since $|U_1 - V_k| \geq m + m' - 1$, while by (2.55), $m' \geq 1$; therefore there are either m vertices for Claim 2.57 or m' vertices form Claim 2.58. In either case, we obtain a monochromatic cycle of length exactly $2n$, completing the proof. □

2.2.6.2.2 The case of large U_2: $|U_2| \geq n - b$

Claim 2.56. *If $|U_2| = n + a_2 \geq n - b$, then we have a blue cycle of size exactly $2n$.*

Proof. Since $|U_2| = n + a_2 \geq n - 4\lambda n$, we know that the largest among $U_2 \cap V_2$, $U_2 \cap V_3$, $U_2 \cap V_4$

has size at least $0.33n$. We assume $|U_2 \cap V_p|$ is the largest and

$$|U_2 \cap V_p| \geq 0.33n. \tag{2.63}$$

By (2.63) and $|V_p| \leq n$, we have

$$|U_1 \cap V_p| \leq 0.67n$$

and there is a $q \in \{2,3,4\} - \{p\}$ such that

$$|U_1 \cap V_q| \geq 0.16n. \tag{2.64}$$

Step 1: We first find a path to include say $0.8n$ vertices in V_1 and $0.8n$ vertices in U_2 (all of $(V - V_p) \cap U_2$ and $V(C)$) by Theorem 2.45.

Details: The details are almost the same as in Step 2 of Claim 2.52 except every place of n_3 is replaced by n_1, every place of V_3 is replaced by V_1, V_1 is replaced by $(V - V_p)$.

• If $a_2 \geq 0$, then we do not need step 2 and go to step 3 directly.

Step 2: Use $|a_2|$ vertices in B to obtain a blue path.

Details: Since $b \geq |a_2|$, let $Z := \{z_1, \ldots, z_{|a_2|}\} \subseteq B$.

By (2.64) and each vertex v in B having blue degree at least $0.9n \gg \frac{1}{2}|U_1|$ to U_1, we can find for each pair (z_i, z_{i+1}) a blue common neighbor $r_i \in U_1 - V(C)$ where $1 \leq i \leq |a_2| - 1$, a blue neighbor r_0 of z_1 such that $r_0 \in V_q \cap U_1 - V(C)$, a blue neighbor $r_{|a_2|}$ of $z_{|a_2|}$ such that $r_{|a_2|} \in V_q \cap U_1 - V(C)$ and $r_0, \ldots, r_{|a_2|}$ are all distinct.

Since y' has at most one red neighbor to $U_1 - V(C)$, we choose $r_{|a_2|}$ to be in $U_1 \cap V_q - V(C)$ and such that $r_{|a_2|}y'$ is blue.

We obtain a blue path

$$P_2 = r_0 z_1 r_1 \ldots z_i r_i \ldots z_{|a_2|} r_{|a_2|}$$

of length $2|a_2|$.

Step 3: Include the rest of vertices in U_2 to U_1 by Theorem 2.45.

Details: The details are almost the same as in Step 3 of Claim 2.52 except every place of V_2 is replaced by V_p. □

2.2.6.2.3 Handling many vertices in $U_1 - V_k$ incident to red edges

Claim 2.57. *If there are at least $m = b + a_3$ vertices in $(V - V_k) \cap U_1$ of red degree at least $0.1n$ to $U_1 \cap V_k$, then we have a red cycle of length exactly $2n$.*

Proof. Let B' be the collection of vertices in U_1 with blue degree at least $0.05n$ to V_1. Since there are at most $4\lambda n^2$ blue edges between U_1 and V_1, we have

$$|B'| \leq 80\lambda n.$$

Step 1: We first find a collection of red cherries C_3 with center in $U_1 \cap (V - V_k)$ and leaves in $U_1 \cap V_k - B'$ of size m.

Details: The details are almost the same as in Step 1 of Claim 2.53 except we replace everywhere V_2 by $V - V_k$, V_1 by V_k and V_3 by V_1.

Step 2: By Hall's Theorem we find matching M for $V(C_3) \cap V_k$ to R and then find common neighbor back to connect those vertices.

Details: The details are almost the same as in Step 2 of Claim 2.53 except we replace everywhere V_3 by V_1 and n_3 by n_1.

Step 3: Use Theorem 2.45 to get a path saturating all vertices left in $V_1 - B - V(M)$.

Details: Let $X = V_1 - B - \{w_2, \ldots, w_{2m-1}\}$ and we know $|X| = n - a_3 - b - (2m-2) = n - 3m + 2$. We have $a_1 = a_3 - a_2 - 1 = m - a_2 - b - 1 \geq m$, and therefore

$$n + a_1 - m - (2m-1) - (m-1) = n + a_1 - 4m + 2 \geq n - 3m + 2.$$

We can take $Y \subseteq U_1 - \{u_1, \ldots, u_m\} - \{v_2, \ldots, v_{2m}\} - \{g_1, \ldots, g_{m-1}\}$ such that $v_1 \in Y$ and $|Y| = n - 3m + 2$.

The rest of details are almost the same as in Step 3 of Claim 2.53 except we replace everywhere V_3 by V_1 and n_3 by n_1.

$\square$

2.2.6.2.4 Handling many vertices in $U_1 - V_k$ incident to blue edges

In the case when many vertices in $U_1 - V_k$ are incident to blue edges, there are many disjoint blue cherries inside U_1, and we find a blue cycle.

Claim 2.58. *If there are at least $m' = -a_2 - b$ vertices in $U_1 - V_k$ of blue degree at least $|U_1 \cap V_k| - 0.1n$ to $U_1 \cap V_k$, then we have a blue cycle of length exactly $2n$.*

Proof. Since $U_1 \cap V_k$ is the largest among $U_1 \cap V_2$, $V_3 \cap U_1$, and $V_4 \cap U_1$, we know

$$|U_1 \cap V_k| \geq 0.33n,\ |U_2 \cap V_k| \leq 0.67n \text{ and } |U_2 - V_k| \geq 0.32n. \tag{2.65}$$

Step 1: We find m' blue cherries from $U_1 \cap (V - V_k)$ to $U_1 \cap V_k$, possibly avoiding bad vertices. Then we find common neighbors in U_2 to connect those cherries.

Details: The details are almost the same as in Step 1 of Claim 2.54 until the following sentence except that we replace everywhere V_2 by $V - V_k$ and V_1 by V_k.

For all pairs (v_{2i}, v_{2i+1}) we can find distinct common blue neighbors, w_i, in $(V - V_k) \cap U_2 - V(C)$, where $1 \leq i \leq m' - 1$.

By (2.65), there is an $\ell \in \{2, 3, 4\} - \{k\}$ such that

$$|V_\ell \cap U_2| \geq 0.16n. \tag{2.66}$$

We also find for v_1 a blue neighbor $w_0 \in V_\ell \cap U_2$ and $v_{2m'}$ a blue neighbor $w_{m'} \in V_\ell \cap U_2$ distinct from $\{w_1, \ldots, w_{m'-1}\}$ and $V(C)$.

We obtain a blue path

$$P_1 = w_0 v_1 u_1 v_2 w_1 \ldots v_{2m'-1} u_{m'} v_{2m'} w_{m'}$$

of length $4m'$.

Step 2: We find for vertices in B common neighbors in $U_1 \cap V_k$, avoiding vertices already used.

Details: By (2.65) and each vertex v in B having red degree at most $0.1n + a_1$ to U_1, v has at least

$$|U_1 \cap V_k| - 2m' - 0.1n - a_1 > 0.6|U_1 \cap V_k - V(C)| \tag{2.67}$$

edges to $U_1 \cap V_k - V(C)$. We can find for each pair (z_i, z_{i+1}) a common neighbor r_i where $1 \leq i \leq b-1$, a blue neighbor r_0 of z_1, a blue neighbor r_b of z_b such that $\{r_0, \ldots, r_b\} \subseteq U_1 \cap V_k - V(C)$ are all distinct and $w_0 r_b$ is blue.

We obtain a blue path

$$P_2 = r_0 z_1 r_1 \ldots z_i r_i \ldots z_b r_b$$

of length $2b$.

Step 3: Take $0.9n$ vertices in V_1 and $0.9n$ vertices in U_2 including $(V - V_\ell) \cap U_2$ and $V(C)$. Use Theorem 2.45 to find a path.

Details: The details are almost the same as in Step 3 of Claim 2.54 except we replace everywhere V_1 by $V - V_\ell$, V_3 by V_1 and n_3 by n_1.

Step 4: Finally, the rest of vertices in $U_2 \cap V_\ell$ have large blue degree to $(V - V_\ell) \cap U_1$, and we find common neighbors to include them.

Details: The details are almost the same as in Step 4 of Claim 2.54 except we replace everywhere V_1 by $V - V_\ell$, V_2 by V_ℓ, V_3 by V_1 and n_3 by n_1. □

2.2.6.2.5 Changes in the proof when $s \neq 4$

When $s \neq 4$, essentially the proof for $s = 4$ works, with minor modifications.

Case 1: $s = 3$. Then $n_2 + n_3 \geq 2n - 1$ implies $n_1 \geq n_2 \geq n$ and therefore

$$n_1 = n_2 = n \text{ and } n_3 = n - 1.$$

This case is addressed in Lemma 2.50.

Case 2: $s = 5$. If $j = 2$, then since $n_4 + n_5 > n$, $n_1 \geq n_2 \geq (1 - \lambda)n$ and $n_3 > \frac{n}{2}$, we have

$$N = n_1 + n_2 + n_3 + n_4 + n_5 \geq 2(1 - \lambda)n + \frac{3n}{2} > 3n,$$

which is not the case. By a similar argument, $j \notin \{3, 4, 5\}$. Thus, we may assume $j = 1$.

The argument is almost the same as for $s = 4$. We only mention differences.

In our case, $n_4 + n_5 > n$ implies that

$$n_1 \geq n_2 \geq n_3 \geq n_4 > \frac{n}{2}, \tag{2.68}$$

thus

$$n_2 + n_3 = 3n - 1 - n_1 - n_4 - n_5 < n + \lambda n - 1. \tag{2.69}$$

By (2.68) and (2.69), we have

$$\frac{n}{2} - \lambda n \leq n_5 \leq n_4 \leq n_3 \leq n_2 \leq \frac{n}{2} + \lambda n. \tag{2.70}$$

In Section 2.2.6.2.2, in (2.63) we now can only guarantee $|U_2 \cap V_p| \geq 0.24n$ instead of $0.33n$. By (2.70), we can find a $q \in \{2, 3, 4, 5\} - \{p\}$ such that $|U_1 \cap V_q| \geq 0.16n$.

In Section 2.2.6.2.4, in (2.65) we can now only guarantee the largest $|U_1 \cap V_k| \geq 0.24n$. Equation (2.66) still holds with $\ell \in \{2, 3, 4, 5\} - \{k\}$. Everything else is the same.

2.2.7 Completion of the proof of Theorem 2.33

In Sections 2.2.4, 2.2.5, and 2.2.6, we proved Theorem 2.33 in the cases when $N - n_1 - n_2 \geq 3$. By (2.50), in the case $N - n_1 - n_2 \leq 2$, it is sufficient to show that for every 2-edge-coloring of $K_{2n,2n-1}$, there is a monochromatic cycle of length exactly $2n$. Thus, the next lemma completes the proof of Theorem 2.33.

Lemma 2.59. *If n is sufficiently large, then for every 2-edge-coloring of $K_{2n,2n-1}$, there is a monochromatic cycle of length exactly $2n$.*

Proof. Let $G = K_{2n,2n-1}$. From Section 2.2.4, we know that if the reduced graph G^r has a connected matching of size at least $(1 + \gamma)n$, then we can find a monochromatic cycle of length exactly $2n$. Suppose G^r has no connected matching of size $(1 + \gamma)n$ and thus, by Section 2.2.4 again, G has a

(λ, i, j)-bad partition for some $i \in [2]$ and $j \in [2]$.

Without loss of generality, we assume $i = 1$ and discuss separately cases $j = 1$ and $j = 2$.

Case 1: G has a $(\lambda, 1, 1)$-bad partition. By the setup in Section 2.2.5, we have a partition $W_1 \cup W_2$ of $V(G)$ such that

(i): $(1-\lambda)n \le |W_2| \le (1+\lambda)n_1 = (1+\lambda)\cdot 2n$.

(ii): $|E(G_1[W_1, W_2])| \le \lambda n^2$.

(iii): $|E(G_2[W_1])| \le \lambda n^2$.

We know $|W_1| = N - |W_2| = 4n - 1 - |W_2|$, so by Condition **(i)**,

$$(2-3\lambda)n \le |W_1| \le (3+\lambda)n. \tag{2.71}$$

For simplicity, let $A := W_1 \cap V_1$, $B := W_2 \cap V_1$, $C := W_1 \cap V_2$ and $D := W_2 \cap V_2$. Let A^* be the collection of vertices in A with less than $0.6|C|$ red edges to C, B^* be the collection of vertices in B with at least $0.6|C|$ red edges to C, C^* be the collection of vertices in C with less than $0.6|A|$ red edges to A, and D^* be the collection of vertices in D with at least $0.6|A|$ red edges to A. Let $A = (A - A^*) \cup B^*$, $B = (B - B^*) \cup A^*$, $C = (C - C^*) \cup D^*$, and $D = (D - D^*) \cup C^*$. By Condition **(ii)** and **(iii)**, $|A^*| \le \frac{5}{2|C|}\lambda n^2$, $|B^*| \le \frac{5}{3|C|}\lambda n^2$, $|C^*| \le \frac{5}{2|A|}\lambda n^2$, and $|D^*| \le \frac{5}{3|A|}\lambda n^2$. Let $\lambda' = 10\lambda$, $W_1 = A \cup C$, and $W_2 = B \cup D$.

Remark 2.60. *Conditions **(i)-(iii)** still hold with λ' replacing λ and every vertex in A has red degree at least $0.59|C|$ to C, every vertex in B has blue degree at least $0.39|C|$ to C, every vertex in C has red degree at least $0.59|A|$ to A, and every vertex in D has red degree at least $0.39|A|$ to A.*

Case 1.1: $|A| \ge n$ and $|C| \ge n$. Let $X \subseteq A$ and $Y \subseteq C$ such that $|X| = |Y| = n$. For each $x \in X$ and $y \in Y$, by $|A|, |C| \le 2n$ and Remark 2.60,

$$d_1(x, Y) \ge |Y| - 0.41|C| \ge n - 0.82n = 0.18n \text{ and similarly } d_1(y, X) \ge |X| - 0.41|A| \ge 0.18n.$$

By Condition **(iii)**, we know that the number of vertices in X with at least $0.95n$ edges to Y in G_1 is at least $n - 20\lambda' n$ and the number of vertices in Y with at least $0.95n$ edges to X in G_1 is at least $n - 20\lambda' n$. Therefore, if we order vertices in X by their degrees in non-decreasing order, say the ordering follows from $d(x_1) \le \ldots \le d(x_n)$, then the smallest index i such that $d(x_i) \le i + 1$ has the property that $d(x_i) \ge 0.95n$. Similarly, if we order vertices in Y by their degree in non-decreasing order, say the ordering follows from $d(y_1) \le \ldots \le d(y_n)$, then the smallest index j such that $d(y_j) \le j+1$ has the property that $d(y_j) \ge 0.95n$. Since $d(x_i)+d(y_j) \gg n+2$, by Theorem 2.45, we know $G_1[X, Y]$ is Hamiltonian bi-connected and we can find a cycle in G_1 of length exactly $2n$.

Remark 2.61. *The same proof shows that there is a red cycle of length exactly* $\min\{|A|, |C|\}$.

Case 1.2: $|A| \leq (1 - 30\lambda')n$. By equation (2.71) and $|V_1| = 2n$,

$$|C| \geq (1 + 27\lambda')n \text{ and } |B| \geq (1 + 30\lambda')n. \tag{2.72}$$

By Condition **(ii)**, there are at most $20\lambda' n$ vertices in C with red degree at least $0.05n$ to B. Let C' be the $20\lambda' n$ vertices in C of largest red degree to B. Let Y be a subset of $C - C'$ with size n. Similarly, let B' be the $20\lambda' n$ vertices in B of largest red degree to C and we define $X \subseteq B - B'$ of size n. We show there is a blue cycle of length exactly $2n$ in $G_2[X, Y]$.

By the definitions of X and Y, we know that $d_2(x, Y) \geq 0.95n$ for $x \in X$ and $d_2(y, X) \geq 0.95n$ for $y \in Y$. By a similar argument with the last paragraph of **Case 1.1**, we can find a blue cycle of length exactly $2n$ in $G_2[X, Y]$.

Case 1.3: $|C| \leq (1 - 30\lambda')n$. We find a blue cycle by an argument similar to **Case 1.2**.

Case 1.4: $|A| \geq (1 + 30\lambda')n$ and $|D| \geq n$. By Condition **(iii)**, there are at most $20\lambda' n$ vertices in A of red degree at least $0.05n$ to D. Let X' be the $20\lambda' n$ vertices in A of largest red degree to D.

By Condition **(ii)**, there are at most $20\lambda' n$ vertices in D of red degree at least $0.05n$ to A. Let R be the $20\lambda' n$ vertices in D of largest red degree to A. Since $d_2(v, A) \geq 0.39|A| > 0.39n$ for each $v \in R$ and $|R| = 20\lambda' n =: m$, we can order vertices in R so that $R = \{r_1, \ldots, r_m\}$ and find for R a distinct collection of blue cherries to $A - X'$. We may assume the other ends of the cherries are $S = \{s_1, \ldots, s_{2m}\}$ so that each $s_{2i-1} r_i s_{2i}$ is a cherry. Since $S \subseteq A - X'$, each s_i has blue degree at least $|D| - 0.05n$ to D and we can find for each (s_{2i}, s_{2i+1}) a distinct common blue neighbor f_i in $D - R$, where $1 \leq i \leq m - 1$. and thus form a blue path

$$P_1 = s_1 r_1 s_2 f_1 s_3 \ldots s_{2m}$$

from s_1 to s_{2m}. We then extend the path P_1 by finding a blue neighbor r_0 of s_1 in $D - R$ distinct from each vertex chosen in P_1. Note now P_1 has length $4m - 1$ from r_0 to s_{2m}.

Let $X \subseteq (A - X' - V(P_1)) \cup \{s_{2m}\}$ such that $s_{2m} \in X$ and $|X| = n - 2m + 1$. Let $Y \subseteq (D - R - V(P_1)) \cup \{r_0\}$ such that $|Y| = n - 2m + 1$. Since $d_2(y, X) \geq 0.9n$ for $y \in Y$ and $d_2(x, Y) \geq 0.9n$ for $x \in X$, we claim that $G_2[X, Y]$ is Hamiltonian bi-connected by an argument similar to the last paragraph of **Case 1.2**. Therefore, we can find a blue path P_2 of length $2n - 4m + 1$ from r_0 to s_{2m}.

Finally, we glue P_1 and P_2 at r_0 and s_{2m} to complete a blue cycle of length exactly $2n$.

Case 1.5: $|C| \geq (1 + 30\lambda')n$ and $|B| \geq n$. It is similar to **Case 1.4**.

Case 1.6: $|B| \geq n$ and $|D| \geq n$.

• If there is no blue edge in $G[B, D]$, then $G_1[B, D]$ is a complete bipartite graph and thus we can find a red cycle of length exactly $2n$.

• If there is a blue matching of size 2 in $G_2[B, D]$, say the two matching edges are $v_1 v_2$ and $u_1 u_2$, where $v_1, u_1 \in V_1$ and $v_2, u_2 \in V_2$, then by **Case 1.2** and **Case 1.3**, we know $|A| \geq (1 - 30\lambda')n$ and

$|C| \geq (1-30\lambda')n$. By Condition **(ii)**, there are at most $20\lambda' n$ vertices in A such that the red degree to D is at least $0.05n$ and there are at most $20\lambda' n$ vertices in D such that the red degree to A is at least $0.05n$. Similarly, there are at most $20\lambda' n$ vertices in C such that the red degree to B is at least $0.05n$ and there are at most $20\lambda' n$ vertices in B such that the red degree to C is at least $0.05n$.

Let $A' \subseteq A$ be the $|A| - 20\lambda' n$ vertices with the largest blue degree to D, $D' \subseteq D$ be the $|D| - 20\lambda' n$ vertices with the largest blue degree to A, $C' \subseteq C$ be the $|C| - 20\lambda' n$ vertices with the largest blue degree to B and $B' \subseteq B$ be the $|B| - 20\lambda' n$ vertices with largest blue degree to C.

By Condition **(i)** and $|W_2| = |B| + |D| \geq 2n$, $|A| \geq n - 2\lambda' n$. Thus, by Remark 2.60,

$$d_2(u_2, A) \geq 0.39|A| \geq 0.38n.$$

We find a blue neighbor $w_1 \in A'$ of u_2. Let $A'' \subseteq A$ such that $w_1 \in A''$ and $|A''| = \lfloor n/2 \rfloor$. Let $D'' \subseteq D'$ such that $v_2 \in D''$ and $|D''| = \lfloor n/2 \rfloor$. By $A'' \subset A'$ and $D'' \subseteq D'$, $d_2(v, A'') \geq 0.4n$ for every $v \in D''$ and $d_2(v, D'') \geq 0.4n$ for every $v \in A''$. Since $0.4n + 0.4n > 0.5n + 1$, we can use Theorem 2.45 to find a blue path P_1 of length $2 \cdot (\lfloor n/2 \rfloor - 1)$ from v_2 to w_1 and then extend P_1 by adding $w_1 u_2$. Similarly, we can find a blue path P_2 with vertices in $B \cup C$ from v_1 to u_1 of length exactly $2 \cdot (\lceil n/2 \rceil - 1)$.

Finally, we connect P_1 and P_2 by adding the edge $v_1 v_2$ and $u_1 u_2$ to form a blue cycle of length exactly $2n$.

Remark 2.62. *The argument also works whenever all of A, B, C, D are of size in $[n - 100\lambda', n + 100\lambda' n]$.*

• If the size of a maximum matching in $G_2[B, D]$ is exactly one, then let $v_1 v_2$ be a blue edge, and say $\{v_2\} \subseteq D$ be a smallest vertex cover in $G_2[B, D]$ (the case $\{v_1\}$ is a smallest vertex cover has a similar proof and is simpler). If we delete v_2, then the remaining graph is a complete bipartite graph in G_1. If $|D| \geq n+1$ then we can find a red cycle of length $2n$ in $G_1[B, D - \{v_2\}]$. Thus, we may assume $|D| = n$ and $|C| = n - 1$.

Let $B'' \subseteq B$ such that $|B''| = n$. We find a blue cycle in $G_2[B'', C \cup \{v_2\}]$. By Condition **(i)** and $|W_2| = |B| + |D| \geq 2n$, $|C| \geq n - 2\lambda' n$. Thus, by Remark 2.60, for each $v \in B''$ we have

$$d_2(v, C) \geq 0.39|C| \geq 0.38n.$$

We also know that each vertex v_c in $C \cup \{v_2\}$ can have red degree at most one to B (so it has blue degree at least $n-1$ to B'') since otherwise with vertices in $D - \{v_2\}$ we can find a red cycle of length $2n$. Since $n - 1 + 0.19n > n + 1$, we can use Theorem 2.45 to find a blue cycle of length exactly $2n$.

Case 1.7: $n + 1 \leq |A| \leq (n + 30\lambda' n)$ and $n \leq |D| \leq n + 30\lambda' n$. By Remark 2.62, the size of a maximum matching in $G_2[B, D]$ is at most one. Let $v_1 v_2 \in G_2$ such that $v_1 \in B$ and $v_2 \in D$. We may also assume that $\{v_2\}$ is a minimum vertex cover of $G_2[B, D]$ (the case $\{v_1\}$ is a smallest vertex cover has a similar proof and is simpler). Let $R \subseteq A$ be the set of vertices with red degree at least $0.8n$ to D. By Condition **(ii)**, we know $|R| \leq 2\lambda' n$.

We first show that $|D| = n$. Assume not, i.e., $|D| \geq n+1$. Then $|D - \{v_2\}| \geq n$.

If $|A - R| \geq n$, then we find a blue cycle of length $2n$ in $G_2[A - R, D]$. To do so, take a subset $A' \subseteq A - R$ of size n and $D' \subseteq D - \{v_2\}$ of size n. By Remark 2.60, for every $v \in D$ we have

$$d_2(v, C) \geq 0.39|C| = 0.39(2n - |D|) \geq 0.38n.$$

Thus, $d_2(v, A') \geq$ for $v \in D'$. By the definition of A', we know $d_2(v, D') \geq 0.2n$ for $v \in A'$. By Condition **(ii)**, we also know there are at most $20\lambda' n$ vertices in A' of red degree at least $0.05n$ to D and thus if we order vertices in A' and D' in non-decreasing order respectively, say $A' = \{u_1, \ldots, u_n\}$ and $D' = \{w_1, \ldots, w_n\}$, then the smallest index such that $d_2(u_i) \leq i+1$ has $d_2(u_i) \geq 0.95n$ and the smallest index such that $d_2(w_j) \leq j+1$ has $d_2(u_j) \geq 0.19n$. Since $0.95n + 0.19n > n+1$, we can use Theorem 2.45 to find a blue cycle of length exactly $2n$ in $G_2[A', D']$.

If $|A - R| \leq n-1$, then we find a red cycle of length exactly $2n$ in $G_1[B \cup R, D - \{v_2\}]$. To do so, note that 1) $|B \cup R| = 2n - |A - R| \geq n+1$, 2) $G_1[B, D - \{v_2\}]$ is a red complete bipartite graph and 3) each vertex in R has degree at least $0.8n$ to $D - \{v_2\}$. We can use Theorem 2.45 to find a red cycle of length exactly $2n$, since this red graph is very dense and has both parts large enough.

Remark 2.63. *The proof also shows that we can find a monochromatic cycle whenever* $|A| \in [n - 100\lambda' n, n + 100\lambda' n]$ *and* $n + 1 \leq |D| \leq (1 + 100\lambda')n$*.*

We assume $|D| = n$ from now on. Since each vertex in R has red degree at least $0.8n$ to D, if there are at least two vertices in R, say r_1 and r_2, then we find a red common neighbor $w \in D$ for r_1 and r_2. Note that by Remark 2.61, $G_1[A, C]$ is Hamiltonian-bi connected. Therefore, we can find a red cycle of length exactly $2n$ from a path P_1 from r_1 to r_2 of length $2n-2$ glued with the path $P_2 = r_1 w r_2$. The only case remained is $|R| \leq 1$. Then we have $|A - R| \geq n$ and we find a blue cycle of length $2n$ by the same argument as in two paragraphs ahead of this paragraph.

Remark 2.64. *Note that the last sentence of the previous paragraph shows why we need* $|A| \geq n+1$*.*

The only uncovered case is :
Case 1.8: $n \leq |C| \leq (1 + 30\lambda')n$ and $(1 - 30\lambda')n \leq |A| \leq n-1$. We define R to be vertices in C with red degree at least $0.8n$ to B. By Remark 2.62, we may assume that the size of a maximum matching in $G_2[B, D]$ is at most one.

If $|C - R| \geq n$, then we find a blue cycle of length exactly $2n$ in $G_2[B, C - R]$. Thus, we may assume that

$$|C - R| \leq n - 1. \tag{2.73}$$

• If there is no edge in $G_2[B, D]$, then $G_1[B, D]$ is a complete bipartite graph and we are done if $|D \cup R| \geq n$. Thus, we may assume that $|D \cup R| \leq n-1$. Since $|C - R| + |R| + |D| = 2n - 1$,

$|C-R| \geq n$ and we have a contradiction.

• If the size of a maximum matching in $G_2[B,D]$ is exactly one, say v_1v_2 is such a matching with $v_1 \in B$ and $v_2 \in D$, then one of $\{v_1\}$ or $\{v_2\}$ is a minimum vertex cover of $G_2[B,D]$. We may assume that $\{v_2\}$ is a minimum vertex cover of $G_2[B,D]$, and the case when $\{v_1\}$ is a minimum vertex cover has a similar proof and is simpler.

Since $G_1[B, D-\{v_2\}]$ is a complete bipartite graph, we are done if $|D| \geq n+1$. Thus, we may assume $|D| \leq n$. Moreover, if $|D \cup R - \{v_2\}| \geq n$ then we can find a red cycle of length $2n$ in $G_1[D \cup R - \{v_2\}, B]$, hence we may assume

$$|D| + |R| - 1 \leq n-1.$$

But we also know that $|D| + |R| + |C-R| = 2n-1$. Thus,

$$|C-R| \geq n-1,$$

and by (2.73) we know

$$|C-R| = n-1 \quad \text{and} \quad |D \cup R| = n.$$

If v_2 has at least two red edges to B then we can find a red cycle in $G_1[B, D \cup R]$ by first considering the two edges incident with v_2. Thus, v_2 has at most one red edge to B and thus has at least $|B|-1$ blue edges to B. We can find a blue cycle in $G_2[(C-R) \cup \{v_2\}, B]$.

Case 2: G has a $(\lambda, 1, 2)$-bad partition. This case is covered in **Case 1** in Section 2.2.6.1.5 (with the same proof). $\square$

2.2.8 Proof of Theorem 2.34 on monochromatic $C_{\geq 2n}$

For large n, we need to prove the theorem for every N-vertex complete s-partite graph G with parts $(V_1^*, V_2^*, \ldots, V_s^*)$ such that the numbers $n_i = |V_i^*|$ satisfy $n_1 \geq n_2 \geq \ldots \geq n_s$ and Conditions (2.41), (2.42), (2.44) and (2.45).

Consider a possible counterexample G with a 2-edge-coloring f and the minimum $N+s$. If $N - n_1 - n_2 \geq 3$, then restriction (2.47) does not apply, so by Theorem 2.33, G has a monochromatic C_{2n}, a contradiction. If $N - n_1 - n_2 \leq 2$ and (2.47) holds, then again by Theorem 2.33, G has a monochromatic C_{2n}. Hence we need to consider only the case that $N - n_1 - n_2 \leq 2$, all (2.41), (2.42), (2.44) and (2.45) hold, but (2.47) does not hold. In particular, $n_1 \geq 2n-1$, but $N \leq 4n-2$. This means $N - n_1 \leq (4n-2) - (2n-1) = 2n-1$, so by (2.42), $N = 4n-2$ and $n_1 = 2n-1$. If $N - n_1 - n_2 \leq 1$, this does not satisfy (2.45). Thus $N - n_1 - n_2 = 2$, and hence $G \supseteq K_{2n-1,2n-3,2}$. Therefore, the following lemma implies Theorem 2.34.

Lemma 2.65. *If n is sufficiently large, then for every 2-edge-coloring of $K_{2n-1,2n-3,2}$, there is a monochromatic cycle of length at least $2n$.*

Proof. The set-up of the proof is similar to the proof of Lemma 2.59. We only show the differences.

Let $V_3 = \{u_1, u_2\}$. Define $V_1' = V_1$ and $V_2' = V_2 \cup V_3$. We first consider $G[V_1', V_2']$ and then use the fact that $V_2' = V_2 \cup V_3$. Note that we have $|V_1'| = |V_2'| = 2n - 1$.

By the proof in Lemma 2.59, we narrow the uncovered cases to 1) $|A| = n - 1$ and $n \leq |C| \leq (1 + 30\lambda')n$ and 2) $n \leq |A| \leq (1 + 30\lambda')n$ and $|C| = n - 1$.

Case 1: $|A| = n - 1$ and $n \leq |C| \leq (1 + 30\lambda')n$.

Then we know $|B| = n$ and $(1 - 30\lambda')n - 1 \leq |D| \leq n - 1$. By Remark 2.62, we know the size of a maximum matching, α', in $G_2[B, D]$ is at most one. Let R be the set of vertices in C with at least $0.8n$ red neighbours in B. By Condition **(ii)**, $|R| \leq 2\lambda' n$.

Claim 2.66. *If $|C - R| \geq n$ then we find a blue cycle of length $2n$ in $G_2[B, C - R]$.*

Proof. We pick $C' \subseteq C - R$ of size n. We know

1) By Remark 2.60 and the definition of R, each vertex in B has blue degree at least $0.38n$ to C' and each vertex in C' has blue degree at least $0.2n$ to B,

2) By Condition **(ii)**, all but at most $20\lambda' n$ vertices in B has red degree at most $0.05n$ to C' and all but at most $20\lambda' n$ vertices in C has red degree at most $0.05n$ to B, and

3) If we order vertices in C' and B in non-decreasing order by their degree in $G_2[C', B]$ respectively, then the smallest index with $d(x_i) \leq i + 1$ and the smallest index with $d(y_j) \leq j + 1$ satisfies $d(x_i) \geq 0.95n$ and $d(y_j) \geq 0.95n$.

Since $0.95n + 0.95n > n + 1$, we can use Theorem 2.45 to show $G_2[C', B]$ is Hamiltonian bi-connected and thus we can find a cycle by fixing an edge e first and then find a Hamiltonian path in $G_2[C', B]$ without e, which is still Hamiltonian bi-connected. □

Remark 2.67. *Similarly to Claim 2.66, we can show:*

1) For any two vertices $c_1 \in C, a_1 \in A$, graph $G_1[A, C]$ has a red path of length $2n - 3$ from c_1 to a_1.

2) For any two vertices $c_1, c_2 \in C$, graph $G_1[A, C]$ has a red path of length $2n - 2$ from c_1 to c_2.

3) For any two vertices $b_1, b_2 \in B$, graph $G_2[B, C - R]$ has a blue path of length $2n - 2$ from b_1 to b_2.

4) For any two vertices $c_1 \in C - R, b_1 \in B$, graph $G_2[B, C - R]$ has a blue path of length $2n - 3$ from c_1 to b_1.

Therefore, we may assume

$$|C - R| \leq n - 1 \text{ and thus } |D \cup R| \geq n. \tag{2.74}$$

If $|R| \geq 2$, say $r_1, r_2 \in R$, then we find a common neighbour $r_b \in B$ for them. By Remark 2.67, we can find a red path P_1 of length $2n-2$ in $G_1[C, A]$ and then extend P_1 to a red cycle of length $2n$ by adding $r_1 r_b r_2$. Thus, we may assume

$$|C-R| = n-1, \ |R| = 1 \text{ and } |D| = n-1. \tag{2.75}$$

Let $R = \{r\}$. If $\alpha' = 0$, then $G_1[B, D]$ is a complete bipartite graph. We can find a red cycle of length $2n$ in $G_1[B, D \cup R]$ by first fixing two neighbours in B for r.

If $\alpha' = 1$, say $v_1 v_2$ is a maximum matching in $G_2[B, D]$ where $v_1 \in B$ and $v_2 \in D$. If $\{v_2\}$ is a minimum vertex cover, then v_2 has at most one red edge to B since otherwise we find a red cycle by (2.75) in $G_1[D \cup R, B]$ by first fixing two neighbours in B for v_2. Thus, we may assume v_2 has at least $|B|-1$ blue edges to B and thus we can find a blue cycle in $G_2[(C-R) \cup \{v_2\}, B]$ by Remark 2.67.

We may assume $\{v_1\}$ is a minimum vertex cover. Note that v_1 has at most one red edge to D since otherwise we find a red cycle in $G_1[B, D \cup R]$ by first fixing two red neighbours for v_1. For the same reason, each vertex in A has at most one red edge to D. We use vertices in V_3 to find a monochromatic cycle.

If there is a red edge from D to $C-R$, say $u_1 y_1$ with $u_1 \in D$ and $y_1 \in C$, then we find a red cycle of length at least $2n$. To do so, by Remark 2.67, we first find a red path P_1 from y_1 to r of length $2n-2$ in $G_1[A, C]$. Since r has at least $0.8n$ red neighbours in B and $G_1[B - \{v_1\}, D]$ is complete bipartite, we find for r and u_1 a red common neighbour in $B - \{v_1\}$, say r_b. Finally, we extend P_1 to a red cycle of length $2n+1$ by adding the red path $r r_b u_1 y_1$. Since at least one of u_1 and u_2 are not in R, say $u_1 \notin R$, we may assume there is a blue edge $u_1 y_1$ from $C-R$ to D with $u_1 \in C-R$ and $y_1 \in D$.

We find a blue cycle of length at least $2n$ by using u_1. To do so, by Remark 2.60, each vertex in D has blue degree at least $0.38n$ to $A \cup \{v_1\}$ and each vertex in $C-R$ has blue degree at least $0.2n-1$ to B. We first fix a blue neighbour z_1 of y_1 with $z_1 \in A$ and then find a common blue neighbour, say $y_2 \in D - \{y_1\}$, for v_1 and z_1. We can find a blue path P_1 of length $2n-3$ from u_1 to v_1 in $G_2[C-R, B]$ by Remark 2.67 and then extend P_1 by adding the path $v_1 y_2 z_1 y_1 u_1$ to obtain a blue cycle of length $2n+1$.

Case 2: $n \leq |A| \leq (1+30\lambda')n$ and $|C| = n-1$. It is symmetric to **Case 1** until we use vertices in V_3. Thus, we may assume the maximum size of a matching in $G_2[B, D]$ is one, $v_1 v_2$ is one maximum matching and $\{v_2\}$ is a minimum vertex cover and every vertex in $C \cup \{v_2\}$ has blue degree at least $|B|-1$ to B. Moreover, we may define $R \subseteq A$ similarly to **Case 1**, i.e. R is the collection of vertices in A with at least $0.8n$ red degrees to D, and assume

$$|A-R| = n-1, \ |R| = 1 \text{ and } |B| = n-1. \tag{2.76}$$

Let $R = \{r\}$. If there is a red edge from C to $D - \{v_2\}$, say $u_1 y_1$ with $u_1 \in C$ and $y_1 \in D$, then we can find a red cycle of length at least $2n$. To do so, we first find a red path P_1 of length $2n-3$ from

u_1 to r by Remark 2.67. Then we find a red neighbour r_d of r in $D - \{v_2, y_1\}$ and a common red neighbour r_b of r_d and y_1 in B. We extend the path P_1 to a red cycle of length $2n + 1$ by adding the red path $rr_dr_by_1u_1$ to P_1.

Then we may assume there is a blue edge from C to $D - \{v_2\}$, say u_1y_1 with $u_1 \in C$ and $y_1 \in D - \{v_2\}$. We first find a blue path of length $2n - 2$ from y_1 to v_2 in $G_2[A - R, D]$ by Remark 2.67 and then find a common blue neighbour $y \in B$ for v_2 and u_1. Finally, we add the path $y_1u_1yv_2$ to P_1 to obtain a blue cycle of length $2n + 1$. □

2.2.9 Proof of Theorem 2.35 on monochromatic P_{2n}

2.2.9.1 A useful lemma

If G contains a monochromatic C_{2n}, then it certainly contains a monochromatic P_{2n}. So suppose $G = K_{n_1,\ldots,n_s}$ does not have a monochromatic C_{2n}. The lemma below is very helpful in Sections 2.2.9 and 2.2.10.

Lemma 2.68. *Let $s \geq 3$ and n be sufficiently large. Let $n_1 \geq \ldots \geq n_s$ and $N = n_1 + \ldots + n_s$ satisfy (2.41) and (2.42). Suppose that for some 2-edge-coloring f of the complete s-partite graph $G = K_{n_1,\ldots,n_s}$, there are no monochromatic cycles C_{2n}. Then G contains a monochromatic P_{2n+1}.*

Proof. By Theorem 2.33, if (2.41) and (2.42) hold but G does not have a monochromatic C_{2n}, then (2.47) fails. In particular, $N - n_1 - n_2 \leq 2$. Since $s \geq 3$, $N - n_1 - n_2 \geq 1$. We may assume $s = 3$: if $s > 3$, then $N - n_1 - n_2 \leq 2$ yields $s = 4$ and $n_3 = n_4 = 1$. In this case, deleting the edges between V_3 and V_4 and combining them into one part (of size 2) only makes the case harder.

We use Condition (2.47) to find a monochromatic C_{2n} only in the nearly-bipartite subcase of Section 2.2.5: in Section 2.2.5.2. Therefore, if there is no monochromatic C_{2n}, but (2.41) and (2.42) hold, we have a graph G that falls under this subcase.

In this case, we have found disjoint subsets $X_{11}, X_{12} \subseteq V_1$ and $X_{21}, X_{22} \subseteq V_2$ with $|X_{11}| = |X_{21}| = |X_{12}| = |X_{22}| = \frac{n}{2} + 10$ satisfying the following property: if H is any of the graphs $G_1[X_{11}, X_{21}]$, $G_1[X_{12}, X_{22}]$, $G_2[X_{12}, X_{21}]$, or $G_2[X_{11}, X_{22}]$, then given any vertices v, w in H, we can find a (v, w)-path in H on m vertices, provided that $n - 10 \leq m \leq n + 10$ and that the parity of m is correct.

Now let $x \in V_3$ be an arbitrary vertex (since we know that $1 \leq n_3 \leq 2$). Without loss of generality, we may assume that x has an edge in G_1 to X_{11}. If x also has an edge in G_1 to $X_{12} \cup X_{22}$, then we obtain a long path in G_1 as follows:

- Let P_1 be a path in $G_1[X_{11}, X_{21}]$ of length at least n starting from a neighbor of x in X_{11}.
- Let P_2 be a path in $G_1[X_{12}, X_{22}]$ of length at least n starting from a neighbor of x.
- Use x to join P_1 and P_2 into a path.

Otherwise, all edges of x to $X_{12} \cup X_{22}$ are in G_2; in particular, x has a neighbor in G_2 in both X_{12} and X_{22}. We obtain a long path in G_2 in a similar way:

- Let P_1 be a path in $G_2[X_{12}, X_{21}]$ of length at least n starting from a neighbor of x in X_{12}.
- Let P_2 be a path in $G_2[X_{11}, X_{22}]$ of length at least n starting from a neighbor of x in X_{22}.
- Use x to join P_1 and P_2 into a path.

In either case, G contains a monochromatic P_{2n+1}. $\square$

2.2.9.2 Completion of the proof of Theorem 2.35

As observed above, if G has a monochromatic C_{2n}, then we are done. Otherwise, by Theorem 2.33 and Lemma 2.68, G is bipartite. In this case, (2.42) yields $n_2 \geq 2n-1$. Hence $n_1 \geq 2n-1$, and $G \supseteq K_{2n-1,2n-1}$. In this case, Theorem 2.3 yields the result. $\square$

2.2.10 Proof of Theorem 2.36 on monochromatic P_{2n+1}

2.2.10.1 Setup of the proof

For large n, we need to prove the theorem for each complete s-partite graph $G = K_{n_1,\dots,n_s}$ such that the numbers n_i satisfy $n_1 \geq n_2 \geq \dots \geq n_s$ and the following three conditions:

($T1'$) $N = n_1 + \dots + n_s \geq 3n$;

($T2'$) $N - n_1 = n_2 + \dots + n_s \geq 2n - 1$;

($T3'$) If $s = 2$, then $n_1 \geq 2n + 1$.

For a given large n, we consider a possible counterexample with the minimum $N + s$. In view of this, it is enough to consider the lists $(n_1, \dots, n_s)$ satisfying ($T1'$), ($T2'$) and ($T3'$) such that

(a) for each $1 \leq j \leq s$, if $n_i > n_{i+1}$, then the list $(n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_s)$ does not satisfy some of ($T1'$), ($T2'$) and ($T3'$);

(b) if $s \geq 4$, then the list $(n_1, \dots, n_{s-2}, n_{s-1} + n_s)$ (possibly with the entries rearranged into a non-increasing order) does not satisfy some of ($T1'$), ($T2'$) and ($T3'$).

Case 1: $s \geq 3$ and $N > 3n$. Then ($T3'$) holds by default. If $n_1 > n_2$, then the list $(n_1 - 1, n_2, n_3, \dots, n_s)$ still satisfies the conditions ($T1'$), ($T2'$) and ($T3'$), a contradiction to (a). Hence $n_1 = n_2$. Choose the maximum i such that $n_1 = n_i$. If $N - n_1 > 2n - 1$, consider the list $(n_1, \dots, n_{i-1}, n_i - 1, n_{i+1}, \dots, n_s)$. In this case ($T1'$) and ($T2'$) still are satisfied; so by (a), ($T3'$)

fails. But this means $s = 3$ and $n_1 = n_i = 1$, so $N \leq 3$, a contradiction. Thus in this case $N - n_1 = 2n - 1$. Therefore, $n_1 = N - (N - n_1) \geq 3n + 1 - (2n - 1) = n + 2$ and hence $n_2 \geq n + 2$, so $N - n_1 - n_2 \leq (2n - 1) - (n + 2) = n - 3$. Then the list $(n_1, n_1, N - 2n_1)$ satisfies $(T1')$–$(T3')$. Summarizing, we get

$$\textit{if } s \geq 3 \textit{ and } N > 3n, \textit{ then} \quad s = 3,\ n_2 + n_3 = 2n - 1 \quad \textit{and} \quad n_1 = n_2 \geq n + 2. \tag{2.77}$$

Case 2: $s \geq 3$ and $N = 3n$. Again $(T3')$ holds by default. By $(T2')$, $n_1 \leq n+1$, hence $N - n_1 - n_2 \geq n-2$. If $s \geq 4$ and $n_{s-1} + n_s \leq n+1$, then let L be the list obtained from $(n_1, \ldots, n_s)$ by replacing the two entries n_{s-1} and n_s with $n_{s-1} + n_s$ and then possibly rearrange the entries into non-increasing order. By construction, L satisfies $(T1')$–$(T3')$, a contradiction to (b). Hence $n_{s-1} + n_s \geq n + 2$. If $s \geq 6$, then $N \geq 3(n_{s-1} + n_s) \geq 3n + 6$, contradicting $N = 3n$. Thus

$$\textit{if } s \geq 3 \textit{ and } N = 3n, \textit{ then} \quad s \leq 5 \quad \textit{and if } s \geq 4, \textit{ then} \quad n_{s-1} + n_s \geq n + 2. \tag{2.78}$$

Case 3: $s = 2$. Then by $(T3')$, $n_1 \geq 2n + 1$ and by $(T2')$, $n_2 \geq 2n - 1$. Thus $G \supseteq K_{2n+1,2n-1}$, and we can assume that

$$\textit{if } s = 2, \textit{ then} \quad G = K_{2n+1,2n-1}. \tag{2.79}$$

As we have seen, always $s \leq 5$.

2.2.10.2 Completion of the proof

Suppose G satisfies (2.77)–(2.79), and f is a 2-edge-coloring G such that there is no monochromatic P_{2n+1}.

If G has no monochromatic C_{2n}, then by Lemma 2.68, G is bipartite. So by (2.79), $G = K_{2n+1,2n-1}$. But by Lemma 2.59, $K_{2n,2n-1} \mapsto (C_{2n}, C_{2n})$. Therefore, below we assume that the 2-edge-coloring f of G is such that G contains a red cycle C with $2n$ vertices (i.e. G_1 contains C).

Let $V' = V(C)$ and $V'' = V(G) - V'$. Similarly, for $j = 1, \ldots, s$, let $V'_j = V_j \cap C$ and $V''_j = V_j - V'_j$. If some red edge e connects V' with V'', then $C + e$ contains a red P_{2n+1}, so below we assume that

$$\textit{all the edges in } G[V', V''] \textit{ are blue, i.e., } G_2[V', V''] = G[V', V'']. \tag{2.80}$$

Case 1: $s = 2$. Then $|V'_1| = |V'_2| = n$. By (2.79), $|V''_1| = n + 1$. By (2.80), $G_2[V''_1, V'_2] = K_{n+1,n}$, but $K_{n+1,n}$ contains P_{2n+1}.

Case 2: $s \geq 3$ and $n_1 \geq n$. If $V_1 \supseteq V''$, then (since $|V''| \geq n$ by (2.78))

$$G_2[V'', V(G) - V_1] = G[V'', V(G) - V_1] = K_{n,N-n_1} \supseteq K_{n,2n-1} \supseteq P_{2n+1}.$$

Because C is a cycle of length $2n$ and V_1' is an independent set, $|V_1'| \le n$. In particular, since $s \ge 3$,

there are distinct $2 \le j_1, j_2 \le s$ *such that there are vertices* $v_1 \in V'_{j_1}$ *and* $v_2 \in V''_{j_2}$.

If $|V_1''| \ge n$, then $G_2[V_1'', V' - V_1']$ is a complete bipartite graph with parts of size at least n, so it contains a path P with $2n$ vertices, starting from v_1. Adding to it edge v_1v_2, we get a blue P_{2n+1}.

Suppose now $|V_1''| \le n-1$. Then the complete bipartite graph $G_2[V_1'', V' - V_1']$ has a path Q_1 with $2|V_1''| + 1$ vertices starting from v_1 and ending in $V' - V_1$. Also since $n_1 \ge n$ and $|V''| \ge n$, the complete bipartite graph $G[V_1', V'' - V_1]$ contains $K_{n-|V_1''|, n-|V_1''|}$ and hence contains a path Q_2 with $2(n - |V_1''|)$ vertices starting from v_2. Then connecting Q_1 with Q_2 by the edge v_1v_2 we create a P_{2n+1}.

Case 3: $s \ge 3$ and $n_1 \le n-1$. In this case, $N/n_1 > 3$, so $s \ge 4$. Then (2.77)–(2.79) imply that $N = 3n$ and $4 \le s \le 5$. In particular,

$$N - n_i \ge 3n - (n-1) = 2n + 1 \quad \text{for every } 1 \le i \le s. \tag{2.81}$$

Relabel V_is so that $|V_1''| \ge \ldots \ge |V_s''|$. Let s' be the largest i such that $V_i'' \ne \emptyset$. We construct a path Q with $2n+1$ vertices greedily in two stages.

Stage 1: For $i = 1, \ldots, s'-1$, find a vertex $w_i \in V' - V_i - V_{i+1}$ so that all $s'-1$ of them are distinct. We can do it, because V_i'' and V_{i+1}'' are non-empty, so

$$|V_i' \cup V_{i+1}'| \le (n_i - 1) + (n_{i+1} - 1) \le 2n - 4 = |V'| - 4.$$

At least four choices for each of the $s' - 1 \le 4$ vertices w_i allow us to choose them all distinct. Then we choose $w_0 \in V' - V_1$ and $w_{s'} \in V' - V_{s'}$ so that all $w_0, \ldots, w_{s'}$ are distinct.

Stage 2: For $i = 0, \ldots, s'-1$ we find a (w_i, w_{i+1})-path Q_i such that
(i) $V(Q_i) \cap V'' = V_{i+1}''$; and (ii) all paths $Q_0, \ldots, Q_{s'-1}$ are internally disjoint.

If we succeed, then $\bigcup_{i=0}^{s'-1} Q_i$ is a path that we are seeking.

Suppose we are constructing Q_i and $V_{i+1}'' = \{u_1, \ldots, u_q\}$. We start Q_i by the edge w_iu_1. Then on Step j for $j = 1, \ldots, q$, do as follows.

If $j = q$, then add edge u_qw_{i+1} and finish Q_i. Otherwise, find a vertex $z_j \in V' - V_{i+1}$ not yet used in any $Q_{i'}$, then add to Q_i edges u_jz_j and z_ju_{j+1}, and then go to Step $j+1$. We can find this z_j because by (2.81), $|V - V_i| \ge 2n+1$, at most $n-2$ of these vertices are in V'', and at most n vertices of all paths $Q_{i'}$ are already chosen in V'. Since we always can choose z_j, our greedy procedure constructs Q_i, and all Q_i together form the promised path Q. □

2.3 Monochromatic paths and cycles in 2-edge-colored graphs with large minimum degree

2.3.1 Introduction

The *circumference*, $c(G)$, of a graph G is the length of a longest cycle in G. A graph G on n vertices is *pancyclic* if it contains cycles of every integer length in $[3, n]$. A balanced bipartite graph G on $2n$ vertices is *bipancyclic* if it contains cycles of every even length in $[4, 2n]$.

The study of Ramsey-type problems of paths started from the seminal paper by Gerencsér and Gyárfás [45]. They proved that *for any choice of positive integers $k \geq \ell$, $R(P_k, P_\ell) = k - 1 + \lfloor \frac{\ell}{2} \rfloor$.*

Later, there was a series of papers proving that not only K_{3n-1} arrows P_{2n}, but also some dense subgraphs of K_{3n-1}. In particular, Gyárfás, Ruszinkó, Sárközy and Szemerédi [54] showed that $K_{n,n,n} \mapsto (P_{2n-o(n)}, P_{2n-o(n)})$. Their conjecture that $K_{n,n,n} \mapsto (P_{2n+1}, P_{2n+1})$ was recently proved for large n in [7].

More generally, Schelp had the idea that for many graphs H, if K_n arrows H then each "sufficiently dense" subgraph of K_n also arrows H. In [83] he discussed some specific graphs H, and different notions of density. One natural measure of density is the minimum degree. Schelp asked some questions and outlined possible directions of study of this phenomenon. In particular, having in mind that $R(P_{2n}, P_{2n}) = 3n - 1$, Schelp [83] posed the following conjecture.

Conjecture 2.69 ([83]). *Suppose that n is large enough and G is a graph on $3n - 1$ vertices with minimum degree larger than $3|V(G)|/4$. Then G arrows P_{2n}.*

Gyárfás and Sárközy [57] and independently Benevides, Łuczak, Scott, Skokan and White [11] proved an asymptotic version of this conjecture. In fact, Benevides et al. [11] proved more:

Theorem 2.70 (Theorem 1.8 in [11]). *For every $0 < \delta \leq 1/180$, there exists an integer $n_0 = n_0(\delta)$ such that the following holds. Let G be a graph of order $n > n_0$ with $\delta(G) \geq 3n/4$. Suppose that $E(G) = E(R_G) \cup E(B_G)$ is a 2-edge-coloring of G. Then either G has monochromatic circumference at least $(2/3 + \delta/2)n$ or one of R_G and B_G contains cycles of all lengths $\ell \in [3, (2/3 - \delta)n]$.*

Theorem 2.70 implies an asymptotic version of Schelp's conjecture and provides not only monochromatic paths but also equally long monochromatic cycles. Thus Theorem 2.70 yields a partial result towards the following question of Li, Nikiforov and Schelp [69]:

Question 2.71 ([69]). *Let $0 < c < 1$ and n be sufficiently large integer and G be a 2-edge-colored graph of order n with $\delta(G) > cn$. What is the minimum possible monochromatic circumference of G?*

Benevides et al [11] also conjectured the following.

Conjecture 2.72 ([11]). *Let G be a graph of order n with $\delta(G) \geq 3n/4$. Let $n = 3t + r$, where $r \in \{0, 1, 2\}$. Every 2-edge-coloring $E(G) = E(R_G) \cup E(B_G)$ of G has monochromatic circumference at least $2t + r$.*

2.3.2 Results

Our main result is the following theorem in the spirit of works of Benevides et al. [11] and Li et al. [69].

Theorem 2.73. *There exists a positive integer n_0 with the following property. Let $n = 3t + r > n_0$, where $r \in \{0, 1, 2\}$. Let G be a graph of order n with $\delta(G) \geq (3n - 1)/4$. Then for every 2-edge-coloring of G, either there are cycles of every length in $\{3, 4, 5, \ldots, 2t + r\}$ of the same color, or there are cycles of every even length in $\{4, 6, 8, \ldots, 2t + 2\}$ of the same color.*

The following examples show that our result is best possible.

Example 1: Let G be the complete graph on $3t + r$ vertices, where t is a positive integer and $r \in \{0, 1, 2\}$. We partition the vertex set of G into U_1 and U_2 such that $|U_1| = 2t + r$ and $|U_2| = t$. Color all edges inside U_1 and U_2 blue and all edges in $G[U_1, U_2]$ red. There is a blue cycle of length $2t + r$ but no monochromatic cycle of length larger than $2t + r$.

Example 2: Let G be a graph on $n = 3t + r$ vertices, where t is a positive integer and $r \in \{0, 1, 2\}$. We partition $V(G)$ into $U_1 \cup U_2 \cup U_3 \cup U_4 \cup \{x, y\}$, where $\lfloor \frac{n-2}{4} \rfloor \leq |U_1| \leq |U_2| \leq |U_3| \leq |U_4| \leq \lceil \frac{n-2}{4} \rceil$. We obtain G from K_{3t+r} by deleting all edges between U_1 and U_4, and all edges between U_2 and U_3. Color all edges in $G[U_1, U_2]$ and $G[U_3, U_4]$ blue, all edges in $G[U_1, U_3]$ and $G[U_2, U_4]$ red, all edges incident with x red and all edges incident with y apart from the edge xy blue. The minimum degree in G is

$$n - 1 - \left\lceil \frac{n-2}{4} \right\rceil = \left\lfloor \frac{3n-2}{4} \right\rfloor,$$

and a longest monochromatic cycle has length at most $2\lceil \frac{n-2}{4} \rceil + 1$, which is strictly less than $2t + r$ for $n \geq 8$.

Example 3: Let G be a complete graph with $n = 3t + 1$ vertices, where t is a positive integer. We partition the vertex set of G into U_1 and U_2 such that $|U_1| = \lfloor \frac{n}{2} \rfloor$ and $|U_2| = \lceil \frac{n}{2} \rceil$. Color all edges inside U_1 and U_2 blue and all edges in $G[U_1, U_2]$ red. Although G has a red cycle of length $2\lfloor \frac{n}{2} \rfloor$, there is no monochromatic cycle of length exactly $2t + 1$, since the red graph is bipartite and the largest component in the blue graph only contains $\lceil \frac{n}{2} \rceil < 2t + 1$ vertices. In particular, there is no monochromatic cycle of length exactly $2t + 1$ in G.

Thus, the conditions of Theorem 2.73 for $r = 1$ can neither guarantee a monochromatic cycle of length $2t + 1$ in G nor a monochromatic cycle of length $2t + 2$. However, they imply that G has a monochromatic cycle of at least one of these lengths, and of many other lengths.

Since $2t + 2 \geq 2t + r$, Theorem 2.73 immediately yields a slightly stronger version of Conjecture 2.72 (with restriction $\delta(G) \geq (3n - 1)/4$ in place of $\delta(G) \geq 3n/4$) for every sufficiently large n:

Theorem 2.74. *There exists a positive integer n_0 with the following property. Let G be a graph of order $n > n_0$ with $\delta(G) \geq (3n-1)/4$. Let $n = 3t+r$, where $r \in \{0, 1, 2\}$. Then every 2-edge-coloring of G contains a monochromatic cycle of length at least $2t + r$.*

Observe that although Conjecture 2.72 is stated for all n, it is not true for $n \in \{4, 5\}$. Indeed, $E(K_4)$ decomposes into two Hamiltonian paths, and so the corresponding edge-coloring of K_4 does not have monochromatic cycle at all. Also, $E(K_5)$ decomposes into two *bull graphs* (paths of length 4 with the chord connecting the second and the fourth vertices); hence the corresponding edge-coloring of K_5 does not have a monochromatic cycle of length at least 4.

Theorem 2.74 in turn implies Conjecture 2.69:

Theorem 2.75. *Suppose that n is large enough and G is a graph on $3n - 1$ vertices with minimum degree at least $(3|V(G)| - 1)/4$. Then G arrows P_{2n}.*

Proof. We have $3n - 1 = 3t + r$ for $t = n - 1$ and $r = 2$. Theorem 2.74 yields that G has a monochromatic cycle of length at least $2t + r = 2n$, so in particular G contains a monochromatic P_{2n}. □

Gyárfás and Sárközy [57] suggested that maybe the claim in Conjecture 2.69 holds for all n. Theorem 2.73 is also a (small) step toward a resolution of Question 2.71.

Our proof of Theorem 2.73 uses the Szemerédi Regularity Lemma [85], the idea of connected matchings in regular partitions due to Łuczak [73], a stability theorem of Benevides et al. (Lemma 4.1 in [11], see Lemma 2.86 in Section 2.3.4 below), and several classical theorems on existence of cycles in graphs, including theorems of Berge [12] and Jackson [61].

We first apply the 2-color version of the Regularity Lemma to G to obtain a reduced graph H. Then we apply Lemma 4.1 to obtain three cases. In Case (i) of Lemma 4.1, it is already shown in [11] that there is a long monochromatic cycle, and some additional work yields the conclusions of Theorem 2.73 as well. The remaining two cases describe near-extremal graphs, which we handle separately: we deal with Case (ii) in Section 2.3.5 and with Case (iii) in Section 2.3.6.

2.3.3 Tools

2.3.3.1 The Regularity Lemma

For the sake of consistency, we use the same form of the Szemerédi Regularity Lemma [85] as in [11]; the definitions and theorems given there are reproduced below.

Definition 2.76. *Let G be a graph and X and Y be disjoint subsets of $V(G)$. The* density *of the pair (X, Y) is the value*

$$d(X, Y) := \frac{e(X, Y)}{|X||Y|}.$$

Let $\epsilon > 0$ and G be a graph and X and Y be disjoint subsets of $V(G)$. We call (X, Y) an ϵ-regular pair for G *if, for all $X' \subseteq X$ and $Y' \subseteq Y$ satisfying $|X'| \geq \epsilon|X|$ and $|Y'| \geq \epsilon|Y|$, we have*

$$|d(X, Y) - d(X', Y')| < \epsilon.$$

Theorem 2.77 (Theorem 2.4 in [11])**.** *For every $\epsilon > 0$ and positive integer k_0, there is an $M = M(\epsilon, k_0)$ such that if $G = (V, E)$ is an arbitrary 2-edge-colored graph and $d \in [0, 1]$, then there is $k_0 \leq k \leq M$, a partition $(V_i)_{i=0}^k$ of the vertex set V and a subgraph $G' \subseteq G$ with the following properties:*

(R1) $|V_0| \leq \epsilon|V|$,

(R2) all clusters V_i, $i \in [k] := \{1, 2, \ldots, k\}$, are of the same size $m \leq \lceil \epsilon|V| \rceil$,

(R3) $\deg_{G'}(v) > \deg_G(v) - (2d + \epsilon)|V|$ *for all $v \in V$,*

(R4) $e(G'[V_i]) = 0$ *for all $i \in [k]$,*

(R5) for all $1 \leq i < j \leq k$, the pair (V_i, V_j) is ϵ-regular for $R_{G'}$ with a density either 0 or greater than d and ϵ-regular for $B_{G'}$ with a density either 0 or greater than d, where $E(G') = E(R_{G'}) \cup E(B_{G'})$ is the inherited 2-edge-coloring of G'.

Definition 2.78. *Given a graph $G = (V, E)$ and a partition $(V_i)_{i=0}^k$ of V satisfying conditions (R1)–(R5) above, we define the* (ϵ, d)-reduced 2-edge-colored graph H *on vertex set $\{v_i : 1 \leq i \leq k\}$ as follows. For $1 \leq i < j \leq k$,*

- *let $v_i v_j$ be a blue edge of H when $B_{G'}[V_i, V_j]$ has density at least d;*
- *let $v_i v_j$ be a red edge of H when $R_{G'}[V_i, V_j]$ has density at least d.*

Our definition of the reduced graph departs slightly from the definition of [11]: we let an edge $v_i v_j$ of H have both red and blue colors when $B_{G'}[V_i, V_j]$ and $R_{G'}[V_i, V_j]$ both are ϵ-regular and have density at least d, while in such cases, it is only a blue edge in [11].

2.3.3.2 Extremal results for matchings, paths, and cycles.

Theorem 2.79 (Bagga and Varma [2])**.** *Let G be a bipartite balanced graph of order $2n$ such that the sum of the degrees of any two non-adjacent vertices from different parts is at least $n + 1$. Then G is bipancyclic.*

Theorem 2.80 (Berge [12])**.** *Let H be a $2m$-vertex bipartite graph with vertices $u_1, u_2, \ldots, u_m$ on one side and $v_1, v_2, \ldots, v_m$ on the other, such that $\deg(u_1) \leq \ldots \leq \deg(u_m)$ and $\deg(v_1) \leq \ldots \leq \deg(v_m)$. Suppose that for the smallest two indices i and j such that $\deg(u_i) \leq i + 1$ and $\deg(v_j) \leq j + 1$, we have $\deg(u_i) + \deg(v_j) \geq m + 2$. Then H is Hamiltonian bi-connected: for every i and j, there is a Hamiltonian path with endpoints u_i and v_j.*

Theorem 2.81 (Bondy)**.** *Let G be a graph of order n such that for every pair of non-adjacent vertices has their degree sum at least n. Then G is either pancyclic or G is the bipartite complete graph $K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}$.*

Theorem 2.82 (Bondy and Simonovits [26])**.** *Let G be a graph on n vertices with $|E(G)| > 100qn^{1+\frac{1}{q}}$. Then G contains a cycle of every even length from $[2q, 2n^{\frac{1}{q}}]$.*

Theorem 2.83 (Chvátal [28]; see also Corollary 5 in Chapter 10 in [12])**.** *Let G be a graph of order $n \geq 3$ with degree sequence $d_1 \leq d_2 \leq \ldots \leq d_n$ such that*

$$d_k \leq k < \frac{n}{2} \implies d_{n-k} \geq n-k.$$

Then G contains a Hamiltonian cycle.

Theorem 2.84 (Jackson [61])**.** *Let G be a bipartite graph with bipartition (X, Y) in which every vertex of X has degree at least k. If $2 \leq |X| \leq k$ and $|Y| \leq 2k-2$, then G contains a cycle of length $2|X|$.*

Theorem 2.85 (Hall)**.** *Let H be a bipartite graph with bipartition (X, Y) with $|X| \leq |Y|$. If $|N(S)| \geq |S|$ for every $S \subseteq X$, then H has a matching saturating X.*

2.3.4 Main part of the proof of Theorem 2.73

We begin with a stability result from [11]:

Lemma 2.86 (Lemma 4.1 in [11])**.** *Let $0 < \delta < 1/36$ and let G be a graph of sufficiently large order k with $\delta(G) \geq (3/4 - \delta)k$. Suppose that we are given a 2-edge-coloring $E(G) = E(R) \cup E(B)$. Then one of the following holds.*

- *(i) There is a component of R or B that contains a matching on at least $(2/3 + \delta)k$ vertices.*
- *(ii) There is a set S of order at least $(2/3 - \delta/2)k$ such that either $\Delta(R[S]) \leq 10\delta k$ or $\Delta(B[S]) \leq 10\delta k$.*
- *(iii) There is a partition $V(G) = U_1 \cup U_2 \cup U_3 \cup U_4$ with $\min_i\{|U_i|\} \geq (1/4 - 3\delta)k$ such that there are no red edges from $U_1 \cup U_2$ to $U_3 \cup U_4$ and no blue edges from $U_1 \cup U_3$ to $U_2 \cup U_4$.*

Because our definition of the reduced graph is slightly different from the one in [11], we will need to apply Lemma 2.86 to a slightly more general class of graphs: 2-edge-colored graphs in which an edge can potentially be colored both red and blue. It could be checked that the proof of Lemma 2.86 in [11] continues to work: it never uses the existence of an edge $vw \in E(R)$ to conclude that $vw \notin E(B)$. But then the reader would need to read a 4-page proof in [11]. To avoid this, in Section 2.3.7, we present a different argument to extend Lemma 2.86. It results in a smaller maximum value of δ, but this will not matter, since below, we will take $\delta < \frac{1}{1000}$.

Choose $0 < \epsilon \ll d \ll \delta < \frac{1}{1000}$ and a sufficiently large n_0 as in [11]. We let G be a graph satisfying the hypotheses of Theorem 2.74, and apply Theorem 2.77 to get an ϵ-regular partition of $V(G)$ and a reduced graph H. For appropriately chosen d and ϵ, the minimum degree in H is at least $(\frac{3}{4} - \delta)k$, and if $2d + \epsilon < \delta$, then each $v \in V(G)$ is incident to at most δn edges not present in the subgraph G' provided by Theorem 2.77.

When we apply Lemma 2.86 to H, there are three possibilities.

If Case (i) of Lemma 2.86 holds, then it is already shown in [11] that G contains monochromatic cycle, say red, of length ℓ for all even ℓ such that $4k \leq \ell \leq (2/3 + \delta/2)n$; in particular, of every even length from $[4k, 2t + 2]$.

Since a red matching edge in H corresponds to an ϵ-regular d-dense pair (V_i, V_j) in G, where $(1-\epsilon)\frac{n}{k} \leq |V_i| = |V_j| \leq \frac{n}{k}$, there are at least $d|V_i||V_j| \geq d(1-\epsilon)^2\frac{n^2}{k^2} > 200(\frac{2n}{k})^{\frac{3}{2}}$ edges in $R[V_i, V_j]$. By Theorem 2.82, we have a red cycle of every even length in $[4, 2\sqrt{\frac{2n(1-\epsilon)}{k}}]$. Since $4k \ll 2\sqrt{\frac{2n(1-\epsilon)}{k}}$, there is a red cycle of every even length in $[4, 2t + 2]$.

Suppose that Case (ii) of Lemma 2.86 holds. Let $L \subseteq V(G)$ be the union of all clusters V_i such that the vertex v_i of the reduced graph was an element of the set S found in Case (ii). We have $|L| \geq (2/3 - \delta/2)k|V_i|$ (where $i \in [k]$ is arbitrary), and $|V_i| = (n - |V_0|)/k \geq (1-\epsilon)n/k$, hence $|L| \geq (2/3 - \delta/2 - \epsilon)n \geq (2/3 - \delta)n$.

Without loss of generality, it is the red edges that are sparse inside S, in which case $\Delta(R_H[S]) \leq 10\delta k$. For a cluster $V_i \subseteq L$, there are at most $10\delta k$ parts V_j, $1 \leq j \leq k$, such that $V_j \subseteq L$ and the density of the ϵ-regular pair (V_i, V_j) is greater than d. They contribute at most $10\delta k \cdot \frac{n}{k} = 10\delta n$ to the red degree of a vertex in V_i. For all other parts $V_j \subseteq L$, the pair (V_i, V_j) is ϵ-regular with density 0 in $R_{G'}$, which means that there are no red edges between V_i and V_j in G'; neither are there edges within V_i. Finally, each $v \in L$ has at most δn edges in G which are not in G'. Therefore $L \subseteq V(G)$ satisfies $\Delta(R_G[L]) \leq 11\delta n$.

We complete this case of the proof of Theorem 2.73 with the following lemma, whose proof is given in Section 2.3.5.

Lemma 2.87. *Let $0 < \delta < \frac{1}{1000}$, and let G be a graph of order n with $\delta(G) \geq (3n-1)/4$ with a 2-edge-coloring $E(G) = E(R) \cup E(B)$. Let $n = 3t + r$, where $r \in \{0, 1, 2\}$. Suppose that there is a set $L \subseteq V(G)$ of order at least $(2/3 - \delta)n$ such that $\Delta(R[L]) \leq 11\delta n$. Then either one of R and B contains cycles of every integer length in $[3, 2t + r]$ or one of R and B contains cycles of every even length in $[4, 2t + 2]$.*

Finally, suppose that Case (iii) of Lemma 2.86 holds. In this case, for $j = 1, 2, 3, 4$, let $\mathcal{U}_j$ be the union of all clusters V_i such that the vertex v_i of the reduced graph was an element of the set U_j found in Case (iii).

For each j, we have

$$|\mathcal{U}_j| \geq (1/4 - 3\delta)k \cdot \frac{n - |V_0|}{k} \geq (1/4 - 3\delta)(1-\epsilon)n \geq (1/4 - 4\delta)n.$$

The graph $R_{G'}[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ is empty: if $v \in V_i \subseteq \mathcal{U}_1 \cup \mathcal{U}_2$ and $w \in V_j \subseteq \mathcal{U}_3 \cup \mathcal{U}_4$, then there cannot be a red edge between v_i and v_j in H, which means that the pair (V_i, V_j) is ϵ-regular with density 0 in $R_{G'}$: there are no red edges in G' between V_i and V_j. In particular, vw cannot be a red edge in G'. Every vertex in G is incident to at most δn edges not in G'. Therefore $R_G[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ has maximum degree at most δn. Similarly, $B_G[\mathcal{U}_1 \cup \mathcal{U}_3, \mathcal{U}_2 \cup \mathcal{U}_4]$ has maximum degree at most δn.

The set V_0 in G is not a part of any $\mathcal{U}_j$, but $|V_0| \leq \epsilon |V| \leq \delta |V|$ by (R1).

We complete this case of the proof of Theorem 2.73 by the following lemma, whose proof is given in Section 2.3.6.

Lemma 2.88. *Let $0 < \delta < \frac{1}{1000}$, and let G be a graph of order $n = 3t + r$, where $r \in \{0, 2\}$, with $\delta(G) \geq (3n-1)/4$ and a 2-edge-coloring $E(G) = E(R) \cup E(B)$. Suppose that there is a partition $V(G) = \mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4 \cup V_0$ such that*

- *$(1/4 - 4\delta)n \leq |\mathcal{U}_j|$ for each j, $|V_0| \leq \delta n$, and*
- *$R[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ and $B[\mathcal{U}_1 \cup \mathcal{U}_3, \mathcal{U}_2 \cup \mathcal{U}_4]$ have maximum degree at most δn.*

Then one of R and B contains cycles of every even length in $[4, 2t + 2]$.

2.3.5 Proof of Lemma 2.87

In Section 2.3.5, we assume that there is a set $L \subseteq V(G)$ of order at least $\ell = (2/3 - \delta)n$ such that $\Delta(R[L]) \leq 11\delta n$. We write $n = 3t + r$, where $r \in \{0, 1, 2\}$.

We consider two cases; in the first case, we find blue cycles and in the other case, red cycles.

Case 1: Either $|L| \geq 2t + r$ or there are at least $2t + r - |L|$ vertices in $V(G) - L$ with at least $\delta n + 2$ blue edges into L.

We begin by finding blue cycles of every length from $\{3, 4, \ldots, |L|\}$. Since $\Delta(R[L]) \leq 11\delta n$, minimum degree in $B[L]$ is at least $|L| - 1 - \frac{n-3}{4} - 11\delta n \geq 0.6|L|$. For any two vertices $u, v \in L$, their degrees in $B[L]$ sum to more than $|L|$. Hence $B[L]$ is pancyclic by Theorem 2.6.

If $|L| < 2t + r$, then for every length k from $\{|L| + 1, |L| + 2, \ldots, 2t + r\}$, we still will find a blue cycle of length k. Let Y be obtained form L by the addition of $k - |L|$ vertices of $V(G) - L$ with at least $\delta n + 2$ blue edges into L. Let $d_1 \leq d_2 \leq \ldots \leq d_k$ be the degree sequence of $B[Y]$.

We verify that $d_i \geq i + 1$ for all $i \leq k/2$. If the vertex of degree d_i was originally in L, then

$$d_i \geq |L| - 11\delta n - (n-3)/4 \geq (5/12 - 11\delta)n \geq 0.405n,$$

while $k/2 + 1 \leq 0.334n$, so $d_i \geq k/2 + 1 \geq i + 1$. Therefore, if $d_i \leq k/2$, then we are looking at a vertex of $v_i \in Y - L$ and $i \leq k - |L| \leq k - \ell \leq \delta n + r/3 \leq \delta n + 1$. But then, $d_i \geq \delta n + 2 \geq i + 1$ by our choice of vertices to add to Y. By Theorem 2.83, $B[Y]$ contains a Hamiltonian cycle, which is a blue cycle of length exactly k, as desired. Thus, we find blue cycles of every length from $\{3, 4, \dots, 2t+r\}$.

Case 2: Both, $|L| \leq 2t + r - 1$ and there are fewer than $2t + r - |L|$ vertices in $V(G) - L$ with at least $\delta n + 2$ blue edges into L. This leaves at least $t + 1$ vertices in $V(G) - L$ that do not have this property.

Let $2m \in \{4, 6, \dots, 2t+2\}$. Let $X \subseteq V(G) - L$ consist of m vertices, each with fewer than $\delta n+2$ blue edges into L. In the bipartite graph $R[X, L]$, every vertex $x \in X$ has degree at least $\ell - n/4 - \delta n - 2$: there are at least ℓ vertices of L, x is not adjacent in G to at most $n-1-(3n-1)/4 < n/4$ of them, and x has blue edges to fewer than $\delta n + 2$ vertices. We have $\ell - n/4 - \delta n - 2 \geq (5/12 - 3\delta)n$.

Our goal is to apply Theorem 2.84 with $k = (5/12 - 3\delta)n$ to the graph $R[X, L]$. We have already checked that every vertex of X has degree at least k. We verify the other two conditions:

$$|X| \leq m \leq t + 1 \leq n/3 + 1 \leq (5/12 - 3\delta)n = k,$$

and

$$|L| \leq n - (t + 1) \leq 2n/3 \leq (5/6 - 6\delta)n - 2 = 2k - 2.$$

Therefore $R[X, L]$ contains a cycle of length $2m$, and as m varies, we obtain a red cycle of every even length from $\{4, 6, \dots, 2t + 2\}$.

2.3.6 Proof of Lemma 2.88

We have a partition of $V(G)$ into $\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4 \cup V_0$ such that

$$(1/4 - 4\delta)n \leq |\mathcal{U}_j| \text{ for each } j,\ |V_0| \leq \delta n, \text{ and} \tag{2.82}$$

$$\text{each of } R[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4] \text{ and } B[\mathcal{U}_1 \cup \mathcal{U}_3, \mathcal{U}_2 \cup \mathcal{U}_4] \text{ has maximum degree at most } \delta n. \tag{2.83}$$

Definition 2.89. *Let G be a bipartite graph with parts X and Y. The* deficiency, *$\overline{d}(v)$ of a vertex v is $|Y| - \deg(v)$ when $v \in X$ and $|X| - \deg(v)$ when $v \in Y$.*

Lemma 2.90. *In each of the graphs $R[\mathcal{U}_1, \mathcal{U}_2]$, $R[\mathcal{U}_3, \mathcal{U}_4]$, $B[\mathcal{U}_1, \mathcal{U}_3]$, and $B[\mathcal{U}_2, \mathcal{U}_4]$, every vertex has deficiency at most $7\delta n$.*

Proof. Without loss of generality, consider the graph $R[\mathcal{U}_1, \mathcal{U}_2]$ and let $v \in \mathcal{U}_1$. An edge from v to $\mathcal{U}_4$ would be in either $R[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ or $B[\mathcal{U}_1 \cup \mathcal{U}_3, \mathcal{U}_2 \cup \mathcal{U}_4]$, each of which by (2.83) has maximum degree at most δn; so there can be at most $2\delta n$ such edges. Since $|\mathcal{U}_4| \geq (1/4 - 4\delta)n$, there are at least $(1/4 - 6\delta)n$ vertices in $\mathcal{U}_4$ not adjacent to v. Since $\delta(G) \geq (3n - 1)/4$, there are at most $(n - 3)/4 < n/4$ vertices not adjacent to v; therefore v has deficiency at most $6\delta n$ in $G[\mathcal{U}_1, \mathcal{U}_2]$. Finally, each blue edge of v in $G[\mathcal{U}_1, \mathcal{U}_2]$ is also in $B[\mathcal{U}_1 \cup \mathcal{U}_3, \mathcal{U}_2 \cup \mathcal{U}_4]$, so by (2.83) there are at most δn such edges, and the deficiency of v in $R[\mathcal{U}_1, \mathcal{U}_2]$ is at most $7\delta n$. □

We first find monochromatic cycles of every even length from $[4,(\frac{1}{2}-8\delta)n]$, in both, R and B. For red cycles, consider $R[\mathcal{U}_1,\mathcal{U}_2]$. We pick a set $X \subseteq \mathcal{U}_1$ and a set $Y \subseteq \mathcal{U}_2$ such that $|X|=|Y|=(\frac{1}{4}-4\delta)n$. By Lemma 2.90, each vertex in $\mathcal{U}_1$ has red degree at least $(\frac{1}{4}-11\delta)n$ to $\mathcal{U}_2$ and each vertex in $\mathcal{U}_2$ has red degree at least $(\frac{1}{4}-11\delta)n$ to $\mathcal{U}_1$. Since the degrees of any pair of non-adajcent vertices in $R[\mathcal{U}_1,\mathcal{U}_2]$ sum to at least $(\frac{1}{4}-4\delta)n+1$, $R[\mathcal{U}_1,\mathcal{U}_2]$ is bipancyclic by Theorem 2.79. For blue cycles, $B[\mathcal{U}_1,\mathcal{U}_3]$ is bipancyclic by the same argument.

In the remainder of Section 2.3.6, we show that either R or B contains cycles of every even length from $[(\frac{1}{8}-\delta)n, 2t+2]$. First, we need to prove some preliminary lemmas.

Lemma 2.91. *Let H be a bipartite graph with parts A_1 and A_2, where $|A_1|,|A_2| \geq (\frac{1}{4}-5\delta)n$, and assume every vertex of H has deficiency at most $10\delta n$. Then*

1. *For each odd $\ell \in [(\frac{1}{4}-4\delta)n-5, t+5]$ and any vertices $x_1 \in A_1$, $x_2 \in A_2$, there is an (x_1,x_2)-path in H of length exactly ℓ.*

2. *For each even $\ell \in [(\frac{1}{4}-4\delta)n-5, t+5]$ and any vertices $x_1, x_1' \in A_1$, there is an (x_1,x_1')-path in H of length exactly ℓ.*

Proof. The exact condition required on ℓ is that $80\delta n+3 \leq \ell \leq (\frac{1}{2}-10\delta)n-1$, which is implied by the requirements above.

To prove 1, we pick a set of vertices $X_1 \subseteq A_1$ such that $|X_1| = \frac{1}{2}(\ell+1)$ and $x_1 \in X_1$, and a set of vertices $X_2 \subseteq A_2$ such that $|X_2| = \frac{1}{2}(\ell+1)$ and $x_2 \in A_2$, noting that $|A_i| \geq (\frac{1}{4}-5\delta)n \geq \frac{1}{2}(\ell+1)$ for $i=1,2$. Since every vertex in H has deficiency at most $10\delta n$, the same is true for $H' := H[X_1,X_2]$, and therefore every vertex of H' has degree at least $\frac{1}{2}(\ell+1)-10\delta n$.

In particular, for any two vertices $u \in X_1$, $v \in X_2$,

$$\deg_{H'}(u)+\deg_{H'}(v) \geq (\ell+1)-20\delta n \geq \frac{1}{2}(\ell+1)+2,$$

and therefore H' is Hamiltonian bi-connected by Theorem 2.80. In particular, H' contains a Hamiltonian (x_1,x_2)-path, which has length ℓ.

To prove 2, we first pick any $x_2 \in A_2$ adjacent to x_1', then proceed as above with subsets $X_i \subseteq A_i$ of size $\frac{1}{2}\ell$, making sure that $x_1' \notin A_1$. The same argument finds an (x_1,x_2)-path of length $\ell-1$, which extends to an (x_1,x_1')-path of length ℓ with the addition of the edge x_2x_1'. □

Lemma 2.92. *For every even length $2\ell \in [(\frac{1}{2}-8\delta)n, 2t+2]$, we can find a red cycle (blue in Case 4) of length exactly 2ℓ in G in the following cases.*

1. *Both $R[\mathcal{U}_1,\mathcal{U}_3]$ and $R[\mathcal{U}_2,\mathcal{U}_4]$ contain at least one edge: two red edges x_1y_1 and x_2y_2 with $x_1 \in \mathcal{U}_1$, $y_1 \in \mathcal{U}_3$, $x_2 \in \mathcal{U}_2$ and $y_2 \in \mathcal{U}_4$.*

2. *We find an edge in each of $R[\mathcal{U}_1,\mathcal{U}_4]$ and $R[\mathcal{U}_2,\mathcal{U}_3]$, or we find a matching of size 2 in any of $R[\mathcal{U}_i,\mathcal{U}_j]$ where $i \in \{1,2\}$ and $j \in \{3,4\}$.*

3. *We replace the two edges (or a matching of size* 2*) by two vertex-disjoint paths of length* 2 *with no interior vertices in* $\mathcal{U}_1 \cup \mathcal{U}_2 \cup \mathcal{U}_3 \cup \mathcal{U}_4$*.*

4. *We find the corresponding blue edges between* $\mathcal{U}_1 \cup \mathcal{U}_3$ *and* $\mathcal{U}_2 \cup \mathcal{U}_4$*.*

Proof. We prove only Case 1, since the proofs in Cases 2, 3, and 4 are similar. If ℓ is even, then by Lemma 2.91, we can find a red (x_1, x_2)-path P_1 of length $\ell - 1$ in $R[\mathcal{U}_1, \mathcal{U}_2]$ and a red (y_1, y_2)-path P_2 of length $\ell - 1$ in $R[\mathcal{U}_3, \mathcal{U}_4]$. If ℓ is odd, we find paths of length ℓ and $\ell - 2$ instead. We then connect P_1 and P_2 by adding the edges x_1y_1 and x_2y_2 to obtain a red cycle of length exactly 2ℓ. □

Suppose $R[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ contains a matching M of size 3. We claim that in this case one of the cases in Lemma 2.92 occurs. Suppose otherwise. Since Case 2 of the lemma does not hold, all edges of M are in distinct $R[\mathcal{U}_i, \mathcal{U}_j]$ where $i \in \{1, 2\}$ and $j \in \{3, 4\}$. By symmetry, we may assume an edge in M is in $R[\mathcal{U}_1, \mathcal{U}_3]$. Then by Case 1, the other two are not in $R[\mathcal{U}_2, \mathcal{U}_4]$, and we have Case 2 of the lemma. Thus, if $R[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ has a matching of size 3, then we have a red cycle of every even length from $[(\frac{1}{2} - 8\delta)n, 2t + 2]$.

Thus, it is enough to consider the situation when neither $R[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ nor (by symmetry) $B[\mathcal{U}_1 \cup \mathcal{U}_3, \mathcal{U}_2 \cup \mathcal{U}_4]$ has a matching of size 3. In this case, each of them has a vertex cover of size at most 2. Move the vertices in these vertex covers to V_0. Increasing $|V_0|$ by at most 4, we ensure that both $R[\mathcal{U}_1 \cup \mathcal{U}_3, \mathcal{U}_2 \cup \mathcal{U}_4]$ and $B[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ are empty.

Next, let $X_R = X_B = \emptyset$. We will process the vertices of V_0 one at a time, adding each of them to one of $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4, X_R, X_B$.

Pick a vertex $v \in V_0$.

1. If v has at least three red edges to each of $\mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_3 \cup \mathcal{U}_4$, we move v from V_0 to X_R.

2. If v has at least three blue edges to each of $\mathcal{U}_1 \cup \mathcal{U}_3$ and $\mathcal{U}_2 \cup \mathcal{U}_4$, we move v from V_0 to X_B.

3. When v has at most two red edges to $\mathcal{U}_1 \cup \mathcal{U}_2$ and at most two blue edges to $\mathcal{U}_1 \cup \mathcal{U}_3$, we move v from V_0 to $\mathcal{U}_4$.

4. When v has at most two red edges to $\mathcal{U}_1 \cup \mathcal{U}_2$ and at most two blue edges to $\mathcal{U}_2 \cup \mathcal{U}_4$, we move v from V_0 to $\mathcal{U}_3$.

5. When v has at most two red edges to $\mathcal{U}_3 \cup \mathcal{U}_4$ and at most two blue edges to $\mathcal{U}_1 \cup \mathcal{U}_3$, we move v from V_0 to $\mathcal{U}_2$.

6. When v has at most two red edges to $\mathcal{U}_3 \cup \mathcal{U}_4$ and at most two blue edges to $\mathcal{U}_2 \cup \mathcal{U}_4$, we move v from V_0 to $\mathcal{U}_1$.

At each step, $R[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ and $B[\mathcal{U}_1 \cup \mathcal{U}_3, \mathcal{U}_2 \cup \mathcal{U}_4]$, which initially start out empty, gain at most two edges. Therefore once V_0 is processed, each of these graphs has at most $2(\delta n + 4)$ edges.

Now we show that after V_0 is processed,

$$\text{in each of } R[\mathcal{U}_1,\mathcal{U}_2], R[\mathcal{U}_3,\mathcal{U}_4],\ B[\mathcal{U}_1,\mathcal{U}_3],\ B[\mathcal{U}_2,\mathcal{U}_4], \text{ each vertex has deficiency at most } 8\delta n + 4. \tag{2.84}$$

To see this, say a vertex v from V_0 is moved to $\mathcal{U}_4$ and we consider $R[\mathcal{U}_3,\mathcal{U}_4]$, then v has at most four edges to $\mathcal{U}_1$ thus v is not adjacent to at least $|\mathcal{U}_1| - 4 \geq (\frac{1}{4} - 4\delta)n - 4$ vertices in $\mathcal{U}_1$. Since there are at most $\frac{n-3}{4} < \frac{n}{4}$ vertices not adjacent to v and v has at most two blue edges to $\mathcal{U}_3$, v has deficiency at most $4\delta n + 4 + 2 = 4\delta n + 6$ in $R[\mathcal{U}_3,\mathcal{U}_4]$. For vertices which are originally in $\mathcal{U}_4$, there are at most $\delta n + 4$ vertices in V_0 processed by previous process and thus by Lemma 2.90 the deficiency is at most $7\delta n + \delta n + 4 = 8\delta n + 4$. This proves (2.84).

By (2.84), Lemma 2.91 can be applied to the new $\mathcal{U}_1,\mathcal{U}_2,\mathcal{U}_3,\mathcal{U}_4$.

Except for at most $4(\delta n + 4)$ vertices incident to an edge in either $R[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ or $B[\mathcal{U}_1 \cup \mathcal{U}_3, \mathcal{U}_2 \cup \mathcal{U}_4]$, every vertex $v \in \mathcal{U}_{5-j}$, where $j \in [4]$, has no neighbors in $\mathcal{U}_j$, so $\deg(v) \leq n - 1 - |\mathcal{U}_j|$. Recall that we have $\deg(v) \geq (3n-1)/4$ for every $v \in V(G)$. Therefore, $|\mathcal{U}_j| \leq (n-3)/4$ for every j. We have

$$|\mathcal{U}_1| + |\mathcal{U}_2| + |\mathcal{U}_3| + |\mathcal{U}_4| \leq n - 3,$$

leaving $|X_R| + |X_B| \geq 3$. Without loss of generality, we assume $|X_R| \geq |X_B|$; in particular, $|X_R| \geq 2$.

Give a vertex *type* (i, j) with $i \in \{1, 2\}$ and $j \in \{3, 4\}$ if x has two or more red edges to each of $\mathcal{U}_i$ and $\mathcal{U}_j$. A vertex can be given more than one type, but each vertex in X_R has three red edges to each of $\mathcal{U}_1 \cup \mathcal{U}_2$ and $\mathcal{U}_3 \cup \mathcal{U}_4$, and therefore each vertex in X_R is given at least one type.

If there are two vertices in X_R with the same type (i, j) then we can use them to form two red vertex-disjoint paths of length 2 from $\mathcal{U}_i$ to $\mathcal{U}_j$. By Lemma 2.92, we can find a red cycle of every even length from $[(\frac{1}{2} - 8\delta)n, 2t + 2]$, in which case we are done. The same happens if there are two vertices $x, x' \in X_R$ with types (i, j) and (i', j') respectively, where $i \neq i'$ and $j \neq j'$.

The outcome in the previous paragraph can only be avoided if $|X_R| = 2$. In this case, the two vertices in X_R must each have only one type, and the two types agree in only one index. Without loss of generality, the two vertices are x and x' with types $(1, 3)$ and $(1, 4)$ respectively.

Claim 2.93. *In this case, either every edge with both endpoints in $\mathcal{U}_j$, where $j \in \{2, 3, 4\}$, is blue, or we find a red cycle of every even length from $[(\frac{1}{2} - 8\delta)n, 2t + 2]$.*

Proof. Suppose that there is a red edge uv in some part $\mathcal{U}_j$, where $j \in \{2, 3, 4\}$. Consider first the case where the edge is in $\mathcal{U}_2$. Let xa and xb be red edges from x to $\mathcal{U}_1$ and $\mathcal{U}_3$; let $x'a'$ and $x'b'$ be red edges from x' to $\mathcal{U}_1$ and $\mathcal{U}_4$, with $a' \neq a$.

We could use Lemma 2.92 to find a red (a, a')-path in $R[\mathcal{U}_1,\mathcal{U}_2]$ and a red (b, b')-path in $R[\mathcal{U}_3,\mathcal{U}_4]$; however, they would join together to a cycle of odd length. To obtain a cycle of even length, we need to use the red edge uv.

More precisely, let 2ℓ be an even length in $[(\frac{1}{2} - 8\delta)n, 2t + 2]$. By Lemma 2.92, there is a red (b, b')-path P_1 of length $2\lceil \ell/2 \rceil - 1$. To extend P_1 to a red cycle of length 2ℓ, we will find a red (a, a')-path of length $2\lfloor \ell/2 \rfloor - 3$ in $R[\mathcal{U}_1, \mathcal{U}_2] \cup \{uv\}$.

Let c be a red neighbor of v in $\mathcal{U}_1$. By (2.84), a and c have a common neighbor d in $\mathcal{U}_2$. Excluding vertices $\{a, c, d, v\}$ from $R[\mathcal{U}_1, \mathcal{U}_2]$, we still have a graph to which Lemma 2.92 applies, and we can find an (a', u)-path P_2 in that graph of length $2\lfloor \ell/2 \rfloor - 7$. Now we obtain a cycle of length 2ℓ as the concatenation $P_1, b'x', x'a', P_2, uv, vc, cd, da, ax, xb$.

A similar argument can be applied if the red edge uv is in $\mathcal{U}_3$ or $\mathcal{U}_4$, except that we find a red (b, b')-path in $R[\mathcal{U}_3, \mathcal{U}_4] \cup \{uv\}]$ using edge uv instead. It is possible that u or v may coincide with b or b', in which case finding the path is even easier. □

From now on, we assume that the first condition of Claim 2.93 holds: $R[\mathcal{U}_2], R[\mathcal{U}_3], R[\mathcal{U}_4]$ are empty.

Suppose that one of the vertices in X_R, either x or x', could also have been placed in X_B instead, and if we had done so, we would have $|X_B| \geq 2$. If this argument were repeated for the blue graph B, it would be impossible that three out of $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \mathcal{U}_4$ also contain no blue edges, because we know three of them contain no red edges, and all four of them are very dense to satisfy degree conditions (2.83).

Therefore our case is that $|X_B| = 1$ and none of the vertices in X_R could belong in X_B. In particular, x', which has type $(1, 4)$, could not belong to X_B: it either has at most two blue edges to $\mathcal{U}_1 \cup \mathcal{U}_3$, or at most two blue edges to $\mathcal{U}_2 \cup \mathcal{U}_4$. Either way, for some $j \in \{2, 3\}$, x' has at most three edges to $\mathcal{U}_j$: at most one red edge and at most two blue edges.

We now show that this is impossible, ruling out this final case and finishing the proof.

We have $|X_R| = 2$ and $|X_B| = 1$, so $|\mathcal{U}_1| + |\mathcal{U}_2| + |\mathcal{U}_3| + |\mathcal{U}_4| = n - 3$, which can only happen if $|\mathcal{U}_j| = (n-3)/4$ for all j. Except for at most $4(\delta n + 4)$ vertices incident to an edge in either $R[\mathcal{U}_1 \cup \mathcal{U}_2, \mathcal{U}_3 \cup \mathcal{U}_4]$ or $B[\mathcal{U}_1 \cup \mathcal{U}_3, \mathcal{U}_2 \cup \mathcal{U}_4]$, every vertex in $v \in \mathcal{U}_j$ has no neighbors in $\mathcal{U}_{5-j}$, so it is already missing $(n-3)/4$ edges, and can reach degree $(3n-1)/4$ only if it is adjacent to every vertex in $X_R \cup X_B$. In particular, almost all vertices in both $\mathcal{U}_2$ and $\mathcal{U}_3$ must be adjacent to x', contradicting the assumption that x' has at most three edges to one of these parts.

2.3.7 Extension of Lemma 2.86

In Section 2.3.7, we show that Lemma 2.86 still holds for 2-edge-colored graphs G if we allow an edge to be both red and blue simultaneously.

Let $0 < \delta < \frac{1}{1000}$ and let G be a graph of sufficiently large order k with $\delta(G) \geq (3/4 - \delta)k$. Suppose that we are given a 2-edge-coloring $E(G) = E(R) \cup E(B)$ where $E(R)$ and $E(B)$ are not necessarily disjoint.

For any 2-edge-coloring $E(G) = E(R') \cup E(B')$ with $E(R') \cap E(B') = \emptyset$, obtained by assigning edges of $E(R) \cap E(B)$ to just one or the other color, we know that Lemma 2.86 holds.

If Case (i) of Lemma 2.86 holds for any coloring (R', B'), then it also holds for the coloring (R, B), since R' and B' are subgraphs of R and B, and we are done.

If Case (iii) of Lemma 2.86 holds for a coloring (R', B') but does not hold for the coloring (R, B), let $V(G) = U_1 \cup U_2 \cup U_3 \cup U_4$ be the partition we obtain for the coloring (R', B'). There are no edge in G between U_1 and U_4, or between U_2 and U_3, because there are neither edges in R' nor in B' between those pairs. Therefore, each vertex of G has at least $(1/4 - 3\delta)k$ missing edges coming from $G[U_1, U_4]$ or $G[U_2, U_3]$; however, $\delta(G) \geq (3/4 - \delta)k$, so each vertex of G can have at most $4\delta k$ other missing edges. In particular, in the subgraphs $R'[U_1, U_2]$, $R'[U_3, U_4]$, $B'[U_1, U_3]$, and $B'[U_2, U_4]$, the minimum degree is $\min_j\{|U_j|\} - 4\delta k \geq (1/4 - 7\delta)k$.

By Theorem 2.85, each of these bipartite subgraphs has a matching saturating the smallest part. To see this, consider without loss of generality $R'[U_1, U_2]$ and assume $|U_1| \leq |U_2|$. For $S \subseteq U_1$ with $1 \leq |S| \leq (1/4 - 7\delta)k$, $|N(S)| \geq (1/4 - 7\delta)k \geq |S|$ because any vertex in S has at least $(1/4 - 7\delta)k$ neighbors in U_2. For $S \subseteq U_1$ with $|S| > 7\delta k$, $|N(S)| = |U_2| \geq |S|$ because any vertex in U_2 has fewer than $|S|$ non-neighbors in U_1. This covers all possibilities, so Hall's condition holds. Moreover, each of these bipartite subgraphs is connected; two vertices in one part share all but at most $14\delta k \leq 0.014k$ neighbors in the other part, which has at least $(1/4 - 3\delta)k \geq 0.247k \geq 2 \cdot 0.014k + 1$ vertices. So each of R' and B' has two connected components, each with a large matching.

By assumption, there is an edge of the coloring (R, B) that violates the condition in Case (iii): a blue edge from $U_1 \cup U_2$ to $U_3 \cup U_4$ that is also red, or a red edge from $U_1 \cup U_3$ to $U_2 \cup U_4$ that is also blue. In the first case, this edge connects the two components of R'; in the second case, this edge connects the two components of B'. In either case, R' or B' becomes connected, and has a matching saturating at least two of U_1, U_2, U_3, U_4. We must have $|U_j| \leq (1/4 + \delta)k$ for all j, otherwise the vertices of U_{5-j} would have degree less than $(3/4 - \delta)k$. So the matching contains at least $k - 2(1/4 + \delta)k = (1/2 - 2\delta)k$ edges, and $(1 - 4\delta)k \geq 0.996k \geq 0.668k \geq (2/3 + \delta)k$ vertices, and Case (i) of Lemma 2.86 holds for the coloring (R, B).

Finally, suppose that for every choice of (R', B'), Case (ii) of Lemma 2.86 holds. We first consider the possibility that for different choices of (R', B') the color in which the sets S have small maximum degree varies. Then there are two choices of (R', B'), say (R_1, B_1) and (R_2, B_2), that differ only in the color of one edge, for which sets S_1, S_2 exist of order at least $(2/3 - \delta/2)k$ with $\Delta(R_1[S_1]) \leq 10\delta k$ and $\Delta(B_2[S_2]) \leq 10\delta k$. We have $|S_1 \cap S_2| \geq (1/3 - \delta)k$; let v be a vertex of $S_1 \cap S_2$ such that the two colorings (R_1, B_1) and (R_2, B_2) agree on the edges incident to v. (All but at most two vertices of $S_1 \cap S_2$ have this property, since the two colorings only disagree on one edge.) Then v has at most $10\delta k$ edges of R_1 to $S_1 \cap S_2$, and at most $10\delta k$ edges of B_1 to $S_1 \cap S_2$: altogether v has at most $20\delta k$ neighbors in $S_1 \cap S_2$. Therefore

$$\deg(v) \leq k - (1/3 - 21\delta)k = (2/3 + 21\delta)k \leq 0.687k < 0.749k \leq (3/4 - \delta)k,$$

contradicting our assumption about the minimum degree of G.

Therefore Case (ii) always holds with the sets S inducing small maximum degree in the same color:

without loss of generality, red. Choose the coloring (R', B') in which every edge of $E(R) \cap E(B)$ is red. There is a set S of order at least $(2/3 - \delta/2)k$ such that $\Delta(R'[S]) \leq 10\delta k$; then $\Delta(R[S]) \leq 10\delta k$ as well, and Case (ii) of Lemma 2.86 holds for the coloring (R, B).

Chapter 3

Packing colorings of subcubic graphs

Results in Chapter 3.1, 3.2, and 3.3 are joint work with Balogh and Kostochka; results in Chapter 3.4 are joint work with Liu, Rolek, and Yu.

3.1 Packing chromatic number of cubic graphs

The main result of Section 3.1 answers the question whether the packing chromatic number of all subcubic graphs is bounded by a constant in full: Indeed, there are cubic graphs with arbitrarily large packing chromatic number. Moreover, we prove that 'many' cubic graphs have 'high' packing chromatic number:

Theorem 3.1. *For each fixed integer $k \geq 12$ and $g \geq 2k+2$, almost every n-vertex cubic graph G of girth at least g satisfies $\chi_p(G) > k$.*

The theorem will be proved in the language of the so-called *Configuration model*, $\mathcal{F}_3(n)$.

3.1.1 Preliminaries

3.1.1.1 Notation

We mostly use standard notation. If G is a (multi)graph and $v, u \in V(G)$, then $E_G(v,u)$ denotes the set of all edges in G connecting v and u, $e_G(v,u) := |E_G(v,u)|$, and $\deg_G(v) := \sum_{u \in V(G)\setminus\{v\}} e_G(v,u)$. For $A \subseteq V(G)$, $G[A]$ denotes the sub(multi)graph of G induced by A. The independence number of G is denoted by $\alpha(G)$. For $k \in \mathbb{Z}_{>0}$, $[k]$ denotes the set $\{1, \ldots, k\}$.

3.1.1.2 The Configuration Model

The configuration model is due in different versions to Bender and Canfield [10] and Bollobás [14]. Our work is based on the version of Bollobás. Let V be the vertex set of the graph, we are going to associate a 3-element set to each vertex in V. Let n be an even positive integer. Let $V_n = [n]$ and consider the Cartesian product $W_n = V_n \times [3]$. A *configuration/pairing* (of order n and degree 3) is a partition of W_n into $3n/2$ pairs, i.e., a perfect matching of elements in W_n. There are

$$\frac{\binom{3n}{2} \cdot \binom{3n-2}{2} \cdot \dots \cdot \binom{2}{2}}{(3n/2)!} = (3n-1)!!$$

such matchings. Let $\mathcal{F}_3(n)$ denote the collection of all $(3n-1)!!$ possible pairings on W_n. We project each pairing $F \in \mathcal{F}_3(n)$ to a multigraph $\pi(F)$ on the vertex set V_n by ignoring the second coordinate. Then $\pi(F)$ is a 3-regular multigraph (which may or may not contain loops and multi-edges). Let $\pi(\mathcal{F}_3(n)) = \{\pi(F) : F \in \mathcal{F}_3(n)\}$ be the set of 3-regular multigraphs on V_n. By definition,

$$\textit{each simple graph } G \in \pi(\mathcal{F}_3(n)) \textit{ corresponds to } (3!)^n \textit{ distinct pairings in } \mathcal{F}_3(n). \tag{3.1}$$

We will call the elements of V_n - *vertices*, and of W_n - *points*.

Definition 3.2. *Let $\mathcal{G}_g(n)$ be the set of all cubic graphs with vertex set $V_n = [n]$ and girth at least g and $\mathcal{G}'_g(n) = \{F \in \mathcal{F}_3(n) : \pi(F) \in \mathcal{G}_g(n)\}$.*

We will use the following result:

Theorem 3.3 (Wormald [89], Bollobás [14])**.** *For each fixed $g \geq 3$,*

$$\lim_{n\to\infty} \frac{|\mathcal{G}'_g(n)|}{|\mathcal{F}_3(n)|} = \exp\left\{-\sum_{k=1}^{g-1} \frac{2^{k-1}}{k}\right\}. \tag{3.2}$$

Remark. When we say that *a pairing F has a multigraph property $\mathcal{A}$*, we mean that $\pi(F)$ has property $\mathcal{A}$.

Since dealing with pairings is simpler than working with labeled simple regular graphs, we need the following well-known consequence of Theorem 3.3.

Corollary 3.4 ([76](Corollary 1.1), [62](Theorem 9.5))**.** *For fixed $g \geq 3$, any property that holds for $\pi(F)$ for almost all pairings $F \in \mathcal{F}_3(n)$ also holds for almost all graphs in $\mathcal{G}_g(n)$.*

Proof. Suppose property $\mathcal{A}$ holds for $\pi(F)$ for almost all $F \in \mathcal{F}_3(n)$. Let $\mathcal{H}(n)$ denote the set of graphs in $\mathcal{G}_g(n)$ that do *not* have property $\mathcal{A}$ and $\mathcal{H}'(n) = \{F \in \mathcal{F}_3(n) : \pi(F) \in \mathcal{H}(n)\}$. Let $\mathcal{B}(n)$ denote the set of pairings $F \in \mathcal{F}_3(n)$ such that $\pi(F)$ does not have property $\mathcal{A}$. Then $\mathcal{H}'(n) \subseteq \mathcal{B}(n)$. Hence by the choice of $\mathcal{A}$,

$$\frac{|\mathcal{H}'(n)|}{|\mathcal{F}_3(n)|} \leq \frac{|\mathcal{B}(n)|}{|\mathcal{F}_3(n)|} \to 0 \qquad \text{as } n \to \infty. \tag{3.3}$$

By (3.1), we have

$$\frac{|\mathcal{H}(n)|}{|\mathcal{G}_g(n)|} = \frac{|\mathcal{H}(n)|}{|\mathcal{H}'(n)|} \cdot \frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|} \cdot \frac{|\mathcal{G}'_g(n)|}{|\mathcal{G}_g(n)|} = \frac{1}{(3!)^n} \cdot \frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|} \cdot (3!)^n = \frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|}.$$

Furthermore,

$$\frac{|\mathcal{H}'(n)|}{|\mathcal{G}'_g(n)|} = \frac{|\mathcal{H}'(n)|}{|\mathcal{F}_3(n)|} \cdot \frac{|\mathcal{F}_3(n)|}{|\mathcal{G}'_g(n)|}. \tag{3.4}$$

By (3.3) and Theorem 3.3, the right-hand side of (3.4) tends to 0 as n tends to infinity. □

3.1.2 Bounds for $c_1, c_2, \dots$

We will use the following theorem of McKay [76].

Theorem 3.5 (McKay [76]). *For every $\epsilon > 0$, there exists an $N > 0$ such that for each $n > N$,*

$$|\{F \in \mathcal{F}_3(n) \,:\, c_1(\pi(F)) > 0.45537n\}| \ < \epsilon \cdot (3n-1)!!.$$

Definition 3.6. *A* 3-regular tree *is a tree such that each vertex has degree* 3 *or* 1*. A* $(3,k,a)$-tree *is a rooted* 3*-regular tree T with root a of degree* 3 *such that the distance in T from each of the leaves to a is k.*

Definition 3.7. *For a positive integer s and a vertex a in a graph G, the* ball $B_G(a,s)$ in G of radius s with center a *is $\{v \in V(G) \,:\, d_G(v,a) \le s\}$, where $d_G(v,a)$ denotes the distance in G from v to a.*

We first prove simple bounds on $c_{2k}(G)$ and $c_{2k+1}(G)$ when $G \in \mathcal{G}_{2k+2}(n)$.

Lemma 3.8. *Let j be a fixed positive integer and $n > g \ge 2j+2$. Then for every $G \in \mathcal{G}_g(n)$,*

$$(i) \quad c_{2j}(G) \le \frac{n}{3 \cdot 2^j - 2},$$

and

$$(ii) \quad c_{2j+1}(G) \le \frac{c_1(G)}{2^{j+1}-1}.$$

Proof. (i) Let C_{2j} be a $2j$-independent set in G with $|C_{2j}| = c_{2j}(G)$. Since the distance between any distinct $a, b \in C_{2j}$ is at least $2j+1$, the balls $B_G(a,j)$ for all distinct $a \in C_{2j}$ are disjoint. Moreover, since $g \ge 2j+2$, each ball $B_G(a,j)$ induces a $(3,j,a)$-tree T_a, and hence has

$$1 + 3 + 3 \cdot 2 + 3 \cdot 2^2 + \ldots + 3 \cdot 2^{j-1} = 3 \cdot 2^j - 2$$

vertices. This proves (i).

(ii) Let C_{2j+1} be a $(2j+1)$-independent set in G with $|C_{2j+1}| = c_{2j+1}(G)$. As in the proof of (i), the balls $B_G(a,j)$ for distinct $a \in C_{2j}$ are disjoint, and each $B_G(a,j)$ induces a $(3,j,a)$-tree T_a. But in this case, in addition, the balls with centers in distinct vertices of C_{2j+1} are at distance at least 2 from each other. Let S_i be the set of vertices in T_a at distance i from a. Then $|S_0| = 1$, and for each $1 \le i \le j$, $|S_i| = 3 \cdot 2^{i-1}$. If follows that the set $I_a = \bigcup_{i=0}^{\lfloor j/2 \rfloor} S_{j-2i}$ is independent, and

$$|I_a| = \sum_{i=0}^{\lfloor j/2 \rfloor} |S_{j-2i}| = 2^{j+1} - 1.$$

Therefore $I := \bigcup_{a \in C_{2j+1}} I_a$ is an independent set in G and $|I| = (2^{j+1} - 1)c_{2j+1}(G)$. This implies (ii). $\square$

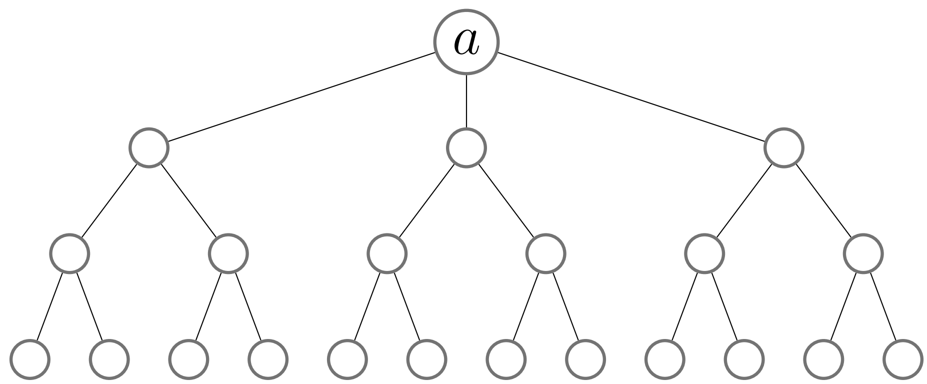


Figure 3.1: A $(3, 3, a)$-tree T_a.

Lemma 3.9. *Let k be a fixed positive integer and x be a real number with $0 < x < \frac{1}{3 \cdot 2^k - 2}$. The number of pairings $F \in \mathcal{G}'_{2k+2}(n)$ such that $\pi(F)$ has a $2k$-independent vertex set of size xn is at most*

$$q(n, k, x) := \binom{n}{xn} \cdot (3n - (6 \cdot 2^k - 6)xn - 1)!! \cdot \prod_{i=0}^{k-1} \binom{(1 - (3 \cdot 2^i - 2)x)n}{3 \cdot 2^i xn} \cdot (3 \cdot 2^i xn)! \cdot 3^{3 \cdot 2^i xn}.$$

Proof. To prove the lemma, we will show that the total number of $2k$-independent sets of size xn in $\pi(F)$ over all $F \in \mathcal{G}'_{2k+2}(n)$ does not exceed $q(n, k, x)$. Below we describe a procedure of constructing for every $C \subset [n]$ with $|C| = xn$ all pairings $F \in \mathcal{G}'_{2k+2}(n)$ for which C is $2k$-independent in $\pi(F)$. Not every obtained pairing will be in $\mathcal{G}'_{2k+2}(n)$, but every $F \in \mathcal{G}'_{2k+2}(n)$ such that C is a $2k$-independent set in $\pi(F)$ will be a result of this procedure:

1. We choose a vertex set C of size xn from $[n]$. There are $\binom{n}{xn}$ ways to do it.
2. In order C to be $2k$-independent and $\pi(F)$ to have girth at least $2k + 2$, all the balls of radius k with the centers in C must be disjoint, and for each $a \in C$, the ball $B_{\pi(F)}(a, k)$ must induce a $(3, k, a)$-tree. Thus, we have $\binom{(1-x)n}{3xn}$ ways to choose the neighbors of C, call it $N(C)$, $(3xn)!$ ways to determine which vertex in $N(C)$ will be the neighbor for each point in $\pi^{-1}(C)$, and 3^{3xn} ways to decide which point of each vertex in $N(C)$ is adjacent to the corresponding point in $\pi^{-1}(C)$. Each vertex of $N(C)$ will have 2 free points left at this moment, and in total the set $\pi^{-1}(N(C))$ has now $2 \cdot 3xn = 6xn$ free points.

3. Similarly to the previous step, consecutively for $i = 1, 2, \dots, k-1$, we will decide which vertices and points are in the set $\pi^{-1}(N^{i+1}(C))$ of the vertices at distance i from C, as follows. Before the ith iteration, we have $3x \cdot 2^i n$ free points in the $3x \cdot 2^{i-1} n$ vertices of $\pi^{-1}(N^i(C))$, and

$$|C \cup N^1(C) \cup \ldots \cup N^i(C)| = xn\left(1 + 3(1 + 2 + \ldots + 2^{i-1})\right) = (3 \cdot 2^i - 2)xn.$$

   We choose $3x \cdot 2^i n$ vertices out of the remaining $\left(1 - (3 \cdot 2^i - 2)x\right) n$ vertices to include into $N^{i+1}(C)$, then we have $(3x \cdot 2^i n)!$ ways to determine which vertex in $N^{i+1}(C)$ will be the neighbor for each free point in $\pi^{-1}(N^i(C))$, and $3^{3x \cdot 2^i n}$ ways to decide which point of each vertex in $N^{i+1}(C)$ is adjacent to the corresponding point in $\pi^{-1}(N^i(C))$.

4. Finally, there are $3n - (6 \cdot 2^k - 6)xn$ free points left and we have $(3n - (6 \cdot 2^k - 6)xn - 1)!!$ ways to pair them.

Multiplying the quantities in 1–4 above, we obtain $q(n, k, x)$. This proves the bound. $\square$

In the proofs below we will use Stirling's formula: For every $n \geq 1$,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}. \tag{3.5}$$

Corollary 3.10. *Let $g \geq 22$ be fixed. For every $\epsilon > 0$, there exists an $N > 0$ such that for each $n > N$,*

$$|\{G \in \mathcal{G}_g(n) : c_2(G) > 0.236n =: b_2 n\}| < \epsilon \cdot |\mathcal{G}_g(n)|, \tag{3.6}$$

$$|\{G \in \mathcal{G}_g(n) : c_4(G) > 0.082n =: b_4 n\}| < \epsilon \cdot |\mathcal{G}_g(n)|, \tag{3.7}$$

$$|\{G \in \mathcal{G}_g(n) : c_6(G) > 0.03n =: b_6 n\}| < \epsilon \cdot |\mathcal{G}_g(n)|, \tag{3.8}$$

$$|\{G \in \mathcal{G}_g(n) : c_8(G) > 0.011n =: b_8 n\}| < \epsilon \cdot |\mathcal{G}_g(n)|, \tag{3.9}$$

and

$$|\{G \in \mathcal{G}_g(n) : c_{10}(G) > 0.004n =: b_{10} n\}| < \epsilon \cdot |\mathcal{G}_g(n)|. \tag{3.10}$$

Proof. By Lemma 3.9,

$$q(n,k,x) = \binom{n}{xn} \cdot ((3n - (6 \cdot 2^k - 6)xn - 1)!!) \prod_{i=0}^{k-1} \binom{(1-(3 \cdot 2^i - 2)x)n}{3 \cdot 2^i xn} \cdot ((3 \cdot 2^i xn)!)(3^{3 \cdot 2^i xn})$$

$$= \frac{(3n-(6\cdot 2^k-6)xn-1)!!\cdot n!}{(xn)!\cdot((1-x)n)!}\cdot 3^{3xn+6xn+\ldots+3\cdot 2^{k-1}xn}$$

$$\cdot\frac{((1-x)n)!\cdot(3xn)!}{(3xn)!\cdot((1-4x)n)!}\cdot\frac{((1-4x)n)!\cdot(6xn)!}{(6xn)!\cdot((1-10x)n)!}\cdot\ldots\cdot\frac{((1-(3\cdot 2^{k-1}-2)x)n)!\cdot(3\cdot 2^{k-1}xn)!}{(3\cdot 2^{k-1}xn)!\cdot((1-(3\cdot 2^k-2)x)n)!}$$

$$= \frac{(3n-(6\cdot 2^k-6)xn-1)!!\cdot n!}{(xn)!\cdot((1-(3\cdot 2^k-2)x)n)!}\cdot 3^{(3\cdot 2^k-3)xn}.$$

We know that

$$(3n-1)!! = \frac{(3n)!!}{3n} \geq \frac{\sqrt{(3n)!}}{3n}$$

and

$$(3n-(6\cdot 2^k-6)xn-1)!! \leq \sqrt{(3n-(6\cdot 2^k-6)xn)!}.$$

Therefore,

$$\frac{q(n,k,x)}{(3n-1)!!} \leq (3n)\cdot\left(\frac{(3n-(6\cdot 2^k-6)xn)!}{(3n)!}\right)^{\frac{1}{2}}\cdot\frac{n!}{(xn)!\cdot((1-(3\cdot 2^k-2)x)n)!}\cdot 3^{(3\cdot 2^k-3)xn}.$$

Using Stirling's formula (3.5), we have

$$\frac{q(n,k,x)}{(3n-1)!!} = O(n^2)\cdot\frac{\left(\frac{n}{e}\right)^{\frac{1}{2}\cdot(3n-(6\cdot 2^k-6)xn)}\cdot\left(\frac{n}{e}\right)^n}{\left(\frac{n}{e}\right)^{\frac{3n}{2}}\cdot\left(\frac{n}{e}\right)^{xn}\cdot\left(\frac{n}{e}\right)^{(1-(3\cdot 2^k-2)x)n}}\cdot\left(\frac{(1-(2^{k+1}-2)x)^{1.5-(3\cdot 2^k-3)x}}{x^x(1-(3\cdot 2^k-2)x)^{1-(3\cdot 2^k-2)x}}\right)^n$$

$$= O(n^2)\cdot\left(\frac{(1-(2^{k+1}-2)x)^{1.5-(3\cdot 2^k-3)x}}{x^x(1-(3\cdot 2^k-2)x)^{1-(3\cdot 2^k-2)x}}\right)^n.$$

Let

$$f(x,k) = \frac{(1-(2^{k+1}-2)x)^{1.5-(3\cdot 2^k-3)x}}{x^x(1-(3\cdot 2^k-2)x)^{1-(3\cdot 2^k-2)x}}, \tag{3.11}$$

so that

$$\frac{q(n,k,x)}{|\mathcal{F}_3(n)|} = \frac{q(n,k,x)}{(3n-1)!!} = O(n^2)\,(f(x,k))^n. \tag{3.12}$$

By plugging $x = 0.236$ and $k = 1$ into (3.11) (using a computer or a good calculator), we see that $0 < f(0.236,1) < 0.9964$. Since $f(x,1)$ is a smooth function for $0 < x < 1$, there exists δ_1 such that $f(x,1) < 0.9964$ for all $x \in [0.236-\delta_1, 0.236]$. If $n > 1/\delta_1$, then there exists an $x_1 = x_1(n) \in [0.236-\delta_1, 0.236]$ such that $x_1 n$ is an integer. By (3.12),

$$\frac{q(n,1,x_1 n)}{|\mathcal{F}_3(n)|} = O(n^2)\,(0.9964)^n \to 0 \qquad \text{as } n \to \infty.$$

By the definition of $q(n,k,x)$, (3.2) and Corollary 3.4, this implies (3.6).

Similarly, by plugging the corresponding values of x and k into (3.11), one can check that $0 < f(0.082,2) < 0.9977$, $0 < f(0.03,3) < 0.9981$, $0 < f(0.011,4) < 0.996$, and $0 < f(0.004,5) < 0.995$. Thus repeating the argument of the previous paragraph, we obtain that (3.7), (3.8), (3.9), (3.10) also hold. $\square$

Lemma 3.11. *Let k be a fixed positive integer and $0 < x < \frac{0.45537}{2^{k+1}-1}$. The number of pairings $F \in \mathcal{G}'_{2k+2}(n)$ such that $\pi(F)$ has a $(2k+1)$-independent vertex set of size xn is at most*

$$r(n,k,x) := \frac{\binom{n}{xn} \cdot (3(n-(3\cdot 2^k-2)xn))! \cdot (3(n-(4\cdot 2^k-2)xn)-1)!!}{(3(n-(4\cdot 2^k-2)xn))!}$$

$$\times \prod_{i=0}^{k-1} \binom{(1-(3\cdot 2^i-2)x)n}{3\cdot 2^i xn} \cdot (3\cdot 2^i xn)! \cdot 3^{3\cdot 2^i xn}. \tag{3.13}$$

Proof. We will show that the total number of $(2k+1)$-independent sets of size xn in $\pi(F)$ over all $F \in \mathcal{G}'_{2k+2}(n)$ does not exceed $r(n,k,x)$. Below we describe a procedure of constructing for every set C of size xn in $[n]$ all pairings in $\mathcal{G}'_{2k+2}(n)$ for which C is $(2k+1)$-independent. Not every obtained pairing will be in $\mathcal{G}'_{2k+2}(n)$, but every $F \in \mathcal{G}'_{2k+2}(n)$ such that C is a $(2k+1)$-independent set in $\pi(F)$ will be a result of this procedure:

1. We choose a vertex set C of size xn from $[n]$. There are $\binom{n}{xn}$ ways to do it.

2. In order C to be $(2k+1)$-independent and $\pi(F)$ to have girth at least $2k+2$, all the balls of radius k with the centers in C must be disjoint, and for each $a \in C$, the ball $B_{\pi(F)}(a,k)$ must induce a $(3,k,a)$-tree. Thus, we have $\binom{(1-x)n}{3xn}$ ways to choose the neighbors of C, call it $N(C)$, $(3xn)!$ ways to determine which vertex in $N(C)$ will be the neighbor for each point in $\pi^{-1}(C)$, and 3^{3xn} ways to decide which point of each vertex in $N(C)$ is adjacent to the corresponding point in $\pi^{-1}(C)$. Each vertex of $N(C)$ will have 2 free points left at this moment, and in total the set $\pi^{-1}(N(C))$ has now $2\cdot 3xn = 6xn$ free points.

3. Similarly to the previous step, consecutively for $i = 1, 2, \dots, k-1$, we will decide which vertices and points are in the set $\pi^{-1}(N^{i+1}(C))$ of the vertices at distance i from C, as follows. Before the ith iteration, we have $3x\cdot 2^i n$ free points in the $3x\cdot 2^{i-1}n$ vertices of $\pi^{-1}(N^i(C))$, and

   $$|C \cup N^1(C) \cup \ldots \cup N^i(C)| = xn\left(1 + 3(1+2+\ldots+2^{i-1})\right) = (3\cdot 2^i - 2)xn.$$

   We choose $3x\cdot 2^i n$ vertices out of the remaining $\left(1-(3\cdot 2^i-2)x\right)n$ vertices to include into $N^{i+1}(C)$, then we have $(3x\cdot 2^i n)!$ ways to determine which vertex in $N^{i+1}(C)$ will be the neighbor for each free point in $\pi^{-1}(N^i(C))$, and $3^{3x\cdot 2^i n}$ ways to decide which point of each vertex in $N^{i+1}(C)$ is adjacent to the corresponding point in $\pi^{-1}(N^i(C))$.

4. Let $N^0(C) := C$ and $S := \cup_{i=0}^{k} N^i(C)$. In order the distance between each pair of vertices in C to be at least $2k+2$, $N^k(C)$ has to be an independent set. Therefore, each of the $3x\cdot 2^k n$

free points in the $3x \cdot 2^{k-1}n$ vertices of $\pi^{-1}(N^k(C))$ has to be paired with one of the remaining $3(n-(3\cdot 2^k-2)xn)$ free points of $\pi^{-1}([n]-S)$ and we have

$$\frac{(3(n-(3\cdot 2^k-2)xn))!}{(3(n-(4\cdot 2^k-2)xn))!}$$

ways to do that.

5. Finally, there are $3(n-(4\cdot 2^k-2)xn)$ free points left and we have $(3(n-(4\cdot 2^k-2)xn)-1)!!$ ways to pair them.

The product of the numbers of choices in the above Steps 1–5 equals $r(n,k,x)$, which proves the lemma. $\square$

Corollary 3.12. *Let $g \geq 24$ be fixed. For every $\epsilon > 0$, there exists an $N > 0$ such that for each $n > N$,*

$$|\{G \in \mathcal{G}_g(n) \ : \ c_3(G) > 0.1394n =: b_3 n\}| \ < \epsilon \cdot |\mathcal{G}_g(n)|, \tag{3.14}$$

$$|\{G \in \mathcal{G}_g(n) \ : \ c_5(G) > 0.05n =: b_5 n\}| \ < \epsilon \cdot |\mathcal{G}_g(n)|, \tag{3.15}$$

$$|\{G \in \mathcal{G}_g(n) \ : \ c_7(G) > 0.0182n =: b_7 n\}| \ < \epsilon \cdot |\mathcal{G}_g(n)|, \tag{3.16}$$

$$|\{G \in \mathcal{G}_g(n) \ : \ c_9(G) > 0.0063n =: b_9 n\}| \ < \epsilon \cdot |\mathcal{G}_g(n)|, \tag{3.17}$$

and

$$|\{G \in \mathcal{G}_g(n) \ : \ c_{11}(G) > 0.0022n =: b_{11} n\}| \ < \epsilon \cdot |\mathcal{G}_g(n)|. \tag{3.18}$$

Proof. By Lemma 3.11,

$$\begin{aligned} r(n,k,x) = {} & \frac{\binom{n}{xn} \cdot (3(n-(3\cdot 2^k-2)xn))! \cdot (3(n-(4\cdot 2^k-2)xn)-1)!!}{(3(n-(4\cdot 2^k-2)xn))!} \\ & \times \prod_{i=0}^{k-1} \binom{(1-(3\cdot 2^i-2)x)n}{3\cdot 2^i xn} \cdot (3\cdot 2^i xn)! \cdot 3^{3\cdot 2^i xn} \end{aligned} \tag{3.19}$$

$$= \frac{(3(n-(3\cdot 2^k-2)xn))! \cdot (3(n-(4\cdot 2^k-2)xn)-1)!!}{(3(n-(4\cdot 2^k-2)xn))!} \cdot \frac{n!}{(xn)! \cdot ((1-x)n)!} \cdot 3^{3xn+6xn+\ldots+3\cdot 2^{k-1}xn}$$

$$\cdot \frac{((1-x)n)! \cdot (3xn)!}{(3xn)! \cdot ((1-4x)n)!} \cdot \frac{((1-4x)n)! \cdot (6xn)!}{(6xn)! \cdot ((1-10x)n)!} \cdot \dots \cdot \frac{((1-(3\cdot 2^{k-1}-2)x)n)! \cdot (3\cdot 2^{k-1}xn)!}{(3\cdot 2^{k-1}xn)! \cdot ((1-(3\cdot 2^{k}-2)x)n)!}$$

$$= \frac{(3(n-(3\cdot 2^k-2)xn))! \cdot (3(n-(4\cdot 2^k-2)xn)-1)!!}{(3(n-(4\cdot 2^k-2)xn))!} \cdot \frac{n!}{(xn)! \cdot ((1-(3\cdot 2^k-2)x)n)!} \cdot 3^{(3\cdot 2^k-3)xn}$$

By the definition of the double factorial,

$$(3n-1)!! \geq \frac{(3n)!!}{3n} \geq \frac{\sqrt{(3n)!}}{3n}$$

and

$$(3(n-(4\cdot 2^k-2)xn)-1)!! \leq \sqrt{(3(n-(4\cdot 2^k-2)xn))!}.$$

Therefore,

$$\frac{r(n,k,x)}{(3n-1)!!} \leq (3n) \cdot \left(\frac{(3(n-(4\cdot 2^k-2)xn))!}{(3n)!}\right)^{\frac{1}{2}} \cdot \frac{(3(n-(3\cdot 2^k-2)xn))!}{(3(n-(4\cdot 2^k-2)xn))!}$$

$$\cdot \frac{n!}{(xn)! \cdot ((1-(3\cdot 2^k-2)x)n)!} \cdot 3^{(3\cdot 2^k-3)xn}.$$

By Stirling's formula (3.5),

$$\frac{r(n,k,x)}{(3n-1)!!} = O(n^3) \cdot \frac{\left(\frac{n}{e}\right)^{\frac{3}{2}\cdot(n-(4\cdot 2^k-2)xn)} \cdot \left(\frac{n}{e}\right)^{3(n-(3\cdot 2^k-2)xn)} \cdot \left(\frac{n}{e}\right)^{n}}{\left(\frac{n}{e}\right)^{\frac{3n}{2}} \cdot \left(\frac{n}{e}\right)^{3(n-(4\cdot 2^k-2)xn)} \cdot \left(\frac{n}{e}\right)^{xn} \cdot \left(\frac{n}{e}\right)^{(1-(3\cdot 2^k-2)x)n}}$$

$$\cdot \left(\frac{(1-(3\cdot 2^k-2)x)^{2-(6\cdot 2^k-4)x}}{x^x(1-(4\cdot 2^k-2)x)^{1.5-(6\cdot 2^k-3)x}}\right)^n$$

$$= O(n^3) \cdot \left(\frac{(1-(3\cdot 2^k-2)x)^{2-(6\cdot 2^k-4)x}}{x^x(1-(4\cdot 2^k-2)x)^{1.5-(6\cdot 2^k-3)x}}\right)^n.$$

Let

$$h(x,k) = \frac{(1-(3\cdot 2^k-2)x)^{2-(6\cdot 2^k-4)x}}{x^x(1-(4\cdot 2^k-2)x)^{1.5-(6\cdot 2^k-3)x}}, \tag{3.20}$$

so that

$$\frac{r(n,k,x)}{|\mathcal{F}_3(n)|} = \frac{r(n,k,x)}{(3n-1)!!} = O(n^3)(h(x,k))^n. \tag{3.21}$$

By plugging $x = 0.1394$ and $k = 1$ into (3.20) (using a computer or a calculator), we see that $0 < h(0.1394, 1) < 0.9974$. Since $h(x, 1)$ is a smooth function for $0 < x < 1$, there exists ν_1 such that $h(x, 1) < 0.9974$ for all $x \in [0.1394 - \nu_1, 0.1394]$. If $n > 1/\nu_1$, then there exists an $x_1 = x_1(n) \in [0.1394 - \nu_1, 0.1394]$ such that $x_1 n$ is an integer. By (3.21),

$$\frac{r(n, 1, x_1 n)}{|\mathcal{F}_3(n)|} = O(n^3)(0.9974)^n \to 0 \qquad \text{as } n \to \infty.$$

By the definition of $r(n, k, x)$, (3.2) and Corollary 3.4, this implies (3.14).

Similarly, by plugging the corresponding values of x and k into (3.20), one can check that $0 < h(0.05, 2) < 0.9985$, $0 < h(0.0182, 3) < 0.9973$, $0 < h(0.0063, 4) < 0.9986$, and $0 < h(0.0022, 5) < 0.9979$. Thus repeating the argument of the previous paragraph, we obtain that (3.15), (3.16), (3.17), (3.18) also hold.

□

3.1.3 Bound on $|C_1 \cup C_2 \cup C_4|$

Definition 3.13. *For a graph G, let $c_{1,2,4}(G)$ be the maximum size of $|C_1 \cup C_2 \cup C_4|$, where C_1, C_2 and C_4 are disjoint subsets of $V(G)$ such that C_i is i-independent for all $i \in \{1, 2, 4\}$.*

In this section we prove an upper bound on $c_{1,2,4}(G)$ in terms of $c_1(G)$ for cubic graphs G of girth at least 9. For every vertex a in such a graph G, the ball $B_G(a, 2)$ induces a $(3, 2, a)$-tree T_a. When handling such a tree T_a, we will use the following notation (see Fig 3.2):

$$V(T_a) = \{a\} \cup N_1(a) \cup N_2(a), \text{ where } N_1(a) = \{a_1, a_2, a_3\},\ N_2(a) = \{a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}, a_{3,1}, a_{3,2}\},$$

and

$$E(T) = \{aa_1, aa_2, aa_3, a_1a_{1,1}, a_1a_{1,2}, a_2a_{2,1}, a_2a_{2,2}, a_3a_{3,1}, a_3a_{3,2}\}.$$

Lemma 3.14. *Let G be an n-vertex cubic graph with girth at least 9 and*

$$c_1(G) < 0.456n. \tag{3.22}$$

Then $c_{1,2,4}(G) \leq 0.7174n =: b_{1,2,4}n$.

Proof. Let G satisfy the conditions of the lemma, and let C_1, C_2 and C_4 be disjoint subsets of $V(G)$ such that C_i is i-independent for $i \in \{1, 2, 4\}$ and $|C_1 \cup C_2 \cup C_4| = c_{1,2,4}(G)$.

The idea of the proof uses the fact that for a typical vertex $a \in C_4$, the tree T_a contains several vertices not in $C_1 \cup C_2$. For example, each vertex in G has at most one neighbor in C_2. Also for distinct $a_1, a_2 \in C_4$, the trees T_{a_1} and T_{a_2} are vertex-disjoint. For more accurate counting, we need a couple of new notions. Let Q be the set of vertices in C_1 that do not have neighbors in C_2, and

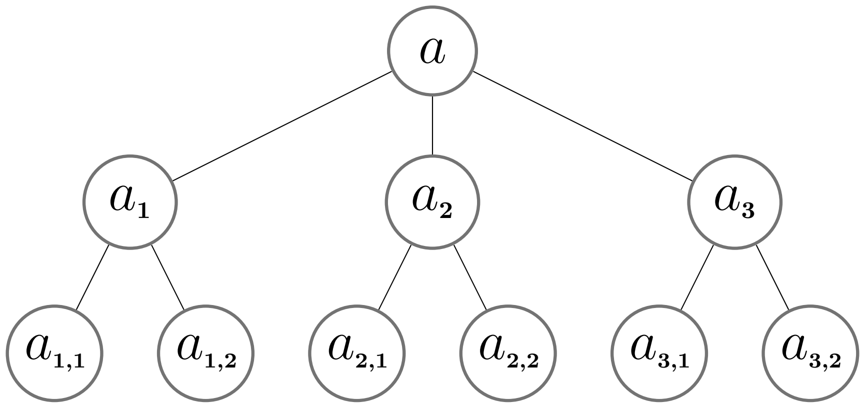


Figure 3.2: A $(3,2,a)$-tree T_a.

$q = |Q|$. Let L be the set of edges in $G - C_1 - C_2$, and $\ell = |L|$. For brevity, the vertices in Q will be called *Q-vertices*, and the edges in L will be called *L-edges*. Let $s = |C_1| + |C_2|$. It will turn out that $q + \ell$ is a convenient parameter helping to bound $|C_4|$ in terms of s and $|C_2|$. We will prove the lemma in a series of claims. Our first claim is:

$$s < 0.652n. \tag{3.23}$$

To show (3.23), we count the edges connecting $C_1 \cup C_2$ with $\overline{C_1 \cup C_2}$ in two ways:

$$3(n - s) - 2\ell = e[C_1 \cup C_2, \overline{C_1 \cup C_2}] = 3s - 2(|C_1| - q). \tag{3.24}$$

Solving for s, we get $s = \frac{n}{2} - \frac{1}{3}(\ell - |C_1| + q)$. Since $q, \ell \geq 0$ and $|C_1| \leq c_1$, this together with (3.22) yields

$$s \leq \frac{n}{2} - \frac{1}{3}(0 - |C_1| + 0) \leq \frac{n}{2} + \frac{c_1}{3} < 0.652n,$$

as claimed. $\square$

For $j \in \{0, 1, 2\}$, let

$$S_j = \{a \in C_4 : \text{ the total number of } L\text{-edges and } Q\text{-vertices in } T_a \text{ is } j\},$$

and let $U = C_4 - \bigcup_{j=0}^{2} S_j$.

Our next claim is:

For each $0 \leq j \leq 2$ and every $a \in S_j$, $|V(T_a) \cap C_2| \geq 3 - j$. (3.25)

Indeed, let $0 \leq j \leq 2$ and $a \in S_j$. If a vertex $a_i \in N_1(a)$ is not in $(C_1 \cup C_2) - Q$, then either $a_i \in Q$ or $aa_i \in L$. Thus, by the definition of S_j, $|N_1(a) \cap ((C_1 \cup C_2) - Q)| \geq 3 - j$. Since each

$a_i \in (C_1 \cup C_2) - Q$ either is in C_2 or has a neighbor in $C_2 \cap \{a_{i,1}, a_{i,2}\}$, we get at least $3 - j$ vertices in $C_2 \cap V(T_a)$. This proves (3.25).

For $0 \le j \le 2$, let $|S_j| = \alpha_j n$, and let $|U| = \beta n$. Then

$$(\alpha_1 + \alpha_2 + \alpha_3 + \beta)n = |C_4|. \tag{3.26}$$

By the definition of 4-independent sets, for all $a \in C_4$ the balls $B_G(a, 2)$ are disjoint and not adjacent to each other. For $0 \le j \le 2$ and every $a \in S_j$, the tree T_a contributes j to $\ell + q$, and for every $a \in U$, T_a contributes at least 3 to $\ell + q$. Therefore

$$\alpha_1 n + 2\alpha_2 n + 3\beta n \le \ell + q. \tag{3.27}$$

Also, (3.25) yields a lower bound on $|C_2|$:

$$3\alpha_0 n + 2\alpha_1 n + \alpha_2 n \le |C_2|. \tag{3.28}$$

Now (3.26), (3.27), and (3.28) yield

$$3|C_4| = (\alpha_1 n + 2\alpha_2 n + 3\beta n) + (3\alpha_0 n + 2\alpha_1 n + \alpha_2 n) \le \ell + q + |C_2|. \tag{3.29}$$

On the other hand, by (3.24)

$$2(\ell + q) = 3n - 6s + 2|C_1| = 3n - 4s - 2|C_2|,$$

so $2(\ell + q + |C_2|) = 3n - 4s$. Comparing with (3.29), we get

$$|C_4| \le \frac{3n - 4s}{6} = \frac{3n + 2s}{6} - s.$$

Hence by the definition of s and (3.23),

$$|C_1 \cup C_2 \cup C_4| = |C_4| + s \le \frac{3n + 2s}{6} \le \frac{n}{2} + \frac{0.652n}{3} \le 0.7174n. \quad \square$$

3.1.4 Proof of Theorem 1

For each fixed integer $k \ge 12$ and $g \ge 2k + 2$, let $J := \{3, 5, 6, 7, \ldots, 11\}$ and

$$\mathcal{B}_g(n) = \left\{ G \in \mathcal{G}_g(n) \;:\; c_{1,2,4}(G) + \sum_{j \in J} c_j(G) > 0.9785n \;\text{ or }\; \sum_{j=6}^{\lceil k/2 \rceil - 1} c_{2j+1}(G) > \frac{2 \cdot 0.45537n}{127} \right\}. \tag{3.30}$$

Lemma 3.15. *Let $k \ge 12$ be a fixed integer and $g \ge 2k + 2$. For every $\epsilon > 0$, there exists an*

$N = N(\epsilon) > 0$ *such that for each* $n > N$,

$$|\mathcal{B}_g(n)| < \epsilon \cdot |\mathcal{G}_g(n)|. \tag{3.31}$$

Proof. Let $\epsilon > 0$ be given. By Lemma 3.14, Theorem 3.5, and Corollary 3.4, there exists an $N_{1,2,4} > 0$ such that for each $n > N_{1,2,4}$,

$$|\{G \in \mathcal{G}_g(n) \ : \ c_{1,2,4}(G) > b_{1,2,4}n\}| \ < \frac{\epsilon}{10} \cdot |\mathcal{G}_g(n)|.$$

Let

$$M_{1,2,4}(n) := \{G \in \mathcal{G}_g(n) \ : \ c_{1,2,4}(G) > b_{1,2,4}n\}.$$

For each $j \in J$ and the constants b_j defined in Corollaries 3.10 and 3.12, let

$$M_j(n) := \{G \in \mathcal{G}_g(n) \ : \ c_j(G) > b_j n\}.$$

Let

$$\mathcal{B}'_g(n) = \left\{ G \in \mathcal{G}_g(n) : \ c_{1,2,4}(G) + \sum_{j \in J} c_j(G) > 0.9785n \right\}$$

$$\text{and} \quad \mathcal{B}''_g(n) = \{G \in \mathcal{G}_g(n) : c_1(G) > 0.45537n\}.$$

If $G \in \mathcal{B}'_g(n)$, then

$$G \in M_{1,2,4}(n) \cup \bigcup_{j \in J} M_j(n),$$

because $b_{1,2,4}n + \sum_{j \in J} b_j n = 0.9785n$ and $c_{1,2,4} + \sum_{j \in J} c_j > 0.9785n$.

Corollaries 3.10 and 3.12 imply that for each $j \in J$, there exists an $N_j > 0$ such that for each $n > N_j$,

$$|\{G \in \mathcal{G}_g(n) \ : \ c_j(G) > b_j n\}| \ < \frac{\epsilon}{10} \cdot |\mathcal{G}_g(n)|.$$

By Theorem 3.5, there exists an $N_1 > 0$ such that for each $n > N_1$, $|\mathcal{B}''_g(n)| \ < \frac{\epsilon}{10} \cdot |\mathcal{G}_g(n)|$.

Let $N = \max\{N_{1,2,4}, N_1, N_3, N_5, N_6, \ldots, N_{11}\}$. By the definition of N, for each $n > N$,

$$|\mathcal{B}'_g(n)| + |\mathcal{B}''_g(n)| < (1 + |J| + 1)\frac{\epsilon}{10} \cdot |\mathcal{G}_g(n)| = \epsilon \cdot |\mathcal{G}_g(n)|. \tag{3.32}$$

Every graph $G \in \mathcal{G}_g(n) \setminus \mathcal{B}''_g(n)$ satisfies $c_1(G) \leq 0.45537n$. Using this, Lemma 3.8(ii) implies that such a graph G satisfies

$$\sum_{j=6}^{\lceil k/2 \rceil - 1} c_{2j+1}(G) < \sum_{j=6}^{\lceil k/2 \rceil - 1} \frac{c_1(G)}{2^{j+1} - 1} < \sum_{j=6}^{\infty} \frac{0.45537n}{2^{j+1} - 1} \leq \frac{0.45537n}{127} \cdot \sum_{s=0}^{\infty} \frac{1}{2^s} = \frac{2 \cdot 0.45537n}{127}.$$

It follows that $\mathcal{B}_g(n) \subseteq \mathcal{B}'_g(n) \cup \mathcal{B}''_g(n)$. Thus (3.32) implies (3.31). $\square$

Now we are prepared to prove our main result.

Proof of Theorem 3.1. Let $k \geq 12$ be a fixed integer and $g \geq 2k+2$. We need to show that for every $\epsilon > 0$, there exists an $N > 0$ such that for each $n > N$,

$$|\{G \in \mathcal{G}_g(n) : \chi_p(G) \leq k\}| < \epsilon \cdot |\mathcal{G}_g(n)|. \tag{3.33}$$

Let $\epsilon > 0$ be given and $G \in \mathcal{G}_g(n)$ satisfy $\chi_p(G) \leq k$. Then there is a partition of $V(G)$ into $C_1, C_2, \ldots, C_k$ such that for each $i = 1, 2, \ldots, k$, C_i is i-independent. In particular, $|C_1| + |C_2| + \ldots + |C_k| = n$. By Lemma 3.8(i),

$$\sum_{j=6}^{\lfloor k/2 \rfloor} |C_{2j}| < \sum_{k=6}^{\infty} \frac{n}{3 \cdot 2^k - 2} < \frac{n}{190} \cdot \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{n}{95}. \tag{3.34}$$

Since $n - \frac{n}{95} > 0.9785n + \frac{2 \cdot 0.45537n}{127}$, this implies that $G \in \mathcal{B}_g(n)$, where $\mathcal{B}_g(n)$ is defined by (3.30). Thus, Lemma 3.15 implies (3.33). $\square$

3.2 Cubic graphs with small independence ratio

3.2.1 Introduction

A set S of vertices in a graph G is *independent* if no two vertices of S are joined by an edge. The *independence number*, $\alpha(G)$, is the maximum size of an independent set in G. The *independence ratio*, $i(G)$, of a graph G is the ratio $\frac{\alpha(G)}{|V(G)|}$. For positive integers r and g, $i(r,g)$ denotes the infimum of $i(G)$ over the r-regular graphs of girth at least g, and $i(r,\infty)$ denotes $\lim\limits_{g\to\infty} i(r,g)$. The first interesting upper bounds on $i(r,\infty)$ were obtained by Bollobás [13] in 1981. In particular, he proved $i(3,\infty) < \frac{6}{13}$. Refining the method, McKay [76] in 1987 showed

Theorem 3.16 (McKay [76])**.**

$$i(3,\infty) < 0.45537. \tag{3.35}$$

In the next 30 years, there were no improvements of Theorem 3.35, but recently some interesting lower bounds on $i(r,\infty)$ and in particular on $i(3,\infty)$ were proved. Hoppen [58] showed $i(3,\infty) \geq 0.4328$. Then Kardoś, Král and Volec [63] improved the bound to 0.4352. Csóka, Gerencsér, Harangi, and Virág [30] pushed the bound to 0.4361 and Hoppen and Wormald [59] — to 0.4375. Moreover, Csóka et al [30] claimed a computer assisted lower bound $i(3,\infty) \geq 0.438$, and Csóka [29] later improved the bound to 0.44533. Our result is an improvement of (3.35) to $i(3,\infty) \leq 0.454$. The improvement is small, but it decreases the gap between the upper and lower bounds on $i(3,\infty)$ by approximately 14%.

In Section 3.2, we prove the following theorem.

Theorem 3.17. *$i(3,\infty) \leq 0.454$.*

The proof uses the language of configurations introduced by Bollobás [14], and shows that "many" 3-regular configurations have "small" independence ratio. The proof of our improvement is based on analyzing the presence not of largest independent sets, but of larger structures, so called MAI-sets (defined in Section 3.2.3) that contain largest independent sets.

3.2.2 Preliminaries

We use notation similarly to Section 3.1.1.1 and see the introduction of the configuration model in Section 3.1.1.2.

Definition 3.18. *For a graph G, let $I(G)$ denote the total number of all independent sets in G, including the empty set. For all integer $r \geq 0$, $g \geq 3$, we define $I(r,g) = \inf I(G)^{1/|V(G)|}$, where the infimum is over all graphs G of maximum degree at most r and girth at least g.*

Recall that the *Fibonacci numbers* F_n are defined by $F_1 = F_2 = 1$, and $F_i = F_{i-1} + F_{i-2}$, for $i \geq 3$. The exact formula for F_i is

$$F_i = \frac{\varphi^i - \psi^i}{\sqrt{5}},$$

where $i \geq 0$, $\varphi = \frac{1+\sqrt{5}}{2}$, and $\psi = \frac{1-\sqrt{5}}{2}$.

Lemma 3.19 (McKay [76])**.** *For any $g \geq 4$, $I(2,g) = (F_{s-1} + F_{s+1})^{\frac{1}{s}}$, where $s = 2\lfloor g/2 \rfloor + 1$.*

Remark 3.20. *The numbers $s-1$ and $s+1$ in Lemma 3.19 are even. Therefore,*

$$I(2,g) = (F_{s-1} + F_{s+1})^{\frac{1}{s}} = \left(\frac{\varphi^{s-1} + \varphi^{s+1} - \varphi^{1-s} - \varphi^{-s-1}}{\sqrt{5}}\right)^{\frac{1}{s}}$$

$$= \varphi \cdot \left((1 - \varphi^{-2s})\frac{\varphi^{-1} + \varphi}{\sqrt{5}}\right)^{\frac{1}{s}} = \varphi(1 - \varphi^{-2s})^{1/s}.$$

Since the function $(1-\varphi^{-2s})^{1/s}$ monotonically increases for $s \geq 1$, and $\varphi(1-\varphi^{-18})^{1/9} \geq 1.618002$, we conclude that for each graph H with maximum degree at most 2 and girth at least 8,

$$1.618 \leq I(2,8) \leq I(H)^{1/|V(H)|}. \tag{3.36}$$

3.2.3 MAI sets in cubic graphs

Definition 3.21. *A vertex set A in a graph G is* an AI set *(an almost independent set), if every component of $G[A]$ is an edge or an isolated vertex. In other words, A is* an AI set *if $\Delta(G[A]) \leq 1$.*

Definition 3.22. *A vertex set A is a* maximum almost independent set (MAI set) *in a graph G if all of the following hold:*

M1. A is an AI set;

M2. A contains an independent set A' of size $\alpha(G)$;

M3. A is largest among all sets satisfying M1 and M2.

Let $G \in \mathcal{G}_{16}(n)$ and A be a MAI set. Denote $B = V(G) - A$.

Lemma 3.23. *B is an AI set.*

Proof. Let $b \in B$. We prove that $d_{G[B]}(b) \leq 1$. Let A' be a maximum independent set in A.

If $d_{G[B]}(b) = 3$, then there is no edge from b to A, and $A' \cup \{b\}$ is an independent set in G with size $|A'| + 1 = \alpha(G) + 1$, contradicting the definition of $\alpha(G)$.

If $d_{G[B]}(b) = 2$, then there is only one edge e from b to A, say ba. If $d_{G[A]}(a) = 0$, then $G[A \cup \{b\}]$ is an AI set in G larger than A containing A'. This contradicts the fact that A is a MAI set. If $d_{G[A]}(a) = 1$, then without loss of generality, we may assume $a \in A - A'$. Then b has no neighbors in A', and $A' \cup \{b\}$ is an independent set in G with size $|A'| + 1$, again contradicting the definition of $\alpha(G)$. $\square$

Let A be a MAI set in $G \in G_{16}(n)$. Denote the set of vertices with degree 1 in $G[A]$ by Y, the set of vertices with degree 1 in $G[B]$ by Z. We introduce notation for the sizes of the sets: Let $x := |A'|$, $s := |Y|/2$, $t := |Z|/2$, and $i := \frac{n}{2} - |A|$. Then $|A| = \frac{n}{2} - i$ and $|B| = \frac{n}{2} + i$.

Lemma 3.24. *$i \geq 0$ and $t \geq s$.*

Proof. We count the number of edges with one end in A and one end in B in two ways. We have

$$2s \cdot 2 + \left(\frac{n}{2} - i - 2s\right) \cdot 3 = e[A, B] = 2t \cdot 2 + \left(\frac{n}{2} + i - 2t\right) \cdot 3, \tag{3.37}$$

i.e.,

$$t - s = 3i. \tag{3.38}$$

We also know that $x = \alpha(G)$, so

$$x = \frac{n}{2} - i - s \geq \frac{n}{2} + i - t,$$

i.e.,

$$2i \leq t - s = 3i,$$

which implies that

$$i \geq 0 \text{ and } t \geq s. \quad \square$$

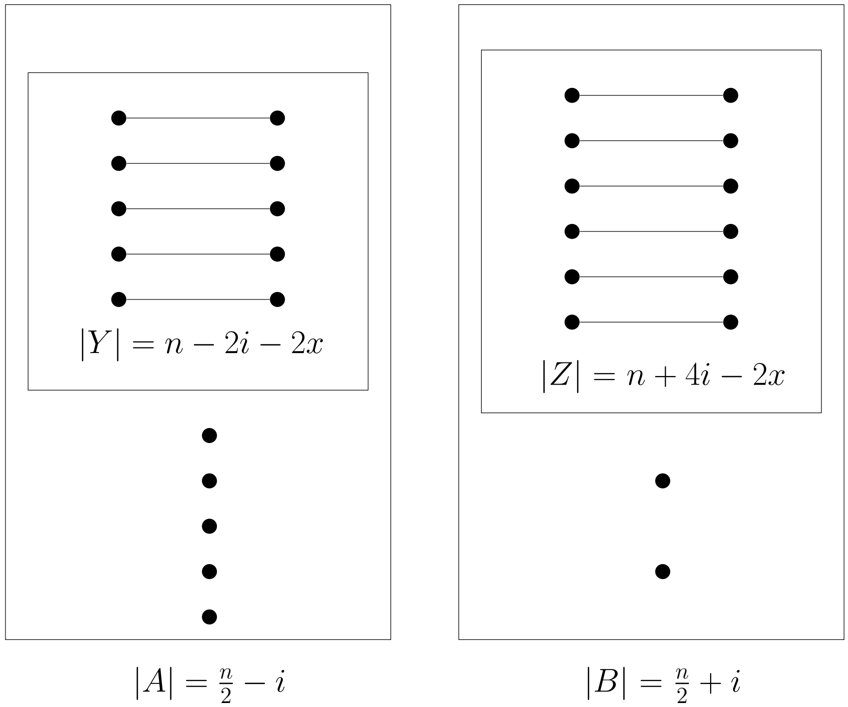


Figure 3.3: A MAI set A.

Lemma 3.25. *If $G \in \mathcal{G}_5(n)$, then*
(i) each vertex in Z has degree at most one to Y;
(ii) each vertex in Y has degree at most one to Z.

Proof. (i) Suppose $z \in Z$ and $N_G(z) = \{z', y_1, y_2\}$, where $z' \in Z$ and $y_1, y_2 \in Y$. Since $g(G) \geq 4$, $y_1 \neq y_2$, $y_1y_2 \notin E(G)$, and so $A - y_1 - y_2$ contains an independent set A' with $|A'| = \alpha(G)$. Thus the set $A' + z$ is an independent set of size $\alpha(G) + 1$ contradicting the definition of $\alpha(G)$.

(ii) Similarly, suppose $y \in Y$ and $N_G(y) = \{y', z_1, z_2\}$, where $y' \in Y$ and $z_1, z_2 \in Z$. Then $A - y$ contains an independent set A' with $|A'| = \alpha(G)$. For $i = 1, 2$, let $N_G(z_i) = \{z_i', y, a_i\}$, where $z_i' \in Z$. By Part (i), $a_1, a_2 \notin Y$. Since $g(G) \geq 5$, $a_2 \neq a_1$. Then $(A - y) \cup \{z_1, z_2\}$ is an AI set containing A' and is larger than A, a contradiction. $\square$

Let $J = \{y_1z_1, \ldots, y_jz_j\}$ be the set of all edges connecting Y with Z in G. By Lemma 3.25, J is a matching in G. Define an auxiliary graph $H = H(A)$ as follows: $V(H) = J$, and $y_\ell z_\ell$ is adjacent to $y_{\ell'}z_{\ell'}$ if $y_\ell y_{\ell'} \in E(G)$ or $z_\ell z_{\ell'} \in E(G)$. By construction, the maximum degree of H is at most 2 and a cycle of length c in H corresponds to a cycle of length $2c$ in G.

Lemma 3.26. *The graph G contains at least $I(H)$ distinct MAIs.*

Proof. Let $J' = \{y_1z_1, \ldots, y_{j'}z_{j'}\}$ be an arbitrary independent set in H. Then the sets $Y_1 =$

$\{y_1, \ldots, y_{j'}\}$ and $Z_1 = \{z_1, \ldots, z_{j'}\}$ are independent in G. By the definition of Y, $A - Y_1$ contains an independent set A' with $|A'| = \alpha(G)$. Let $A_1 = (A - Y_1) \cup Z_1$. By Lemma 3.25, the degree in $G[A_1]$ of every vertex in $(Y - Y_1) \cup Z_1$ is at most 1. If a vertex $a \in A - Y$ is adjacent to two vertices, say z_1, z_2 in Z_1, then the set $(A' - a) \cup \{z_1, z_2\}$ is independent and is larger than A', a contradiction. Thus, A_1 is an AI set. Since $|A_1| = |A|$, this proves the lemma. □

Remark 3.27. *Recall that* $|A| = \frac{n}{2} - i$, $|B| = \frac{n}{2} + i$, $|Y| = 2s = 2(\frac{n}{2} - i - x)$, *and* $|A - Y| = 2x - \frac{n}{2} + i$. *By* (3.38), *we know that* $t = 3i + s = \frac{n}{2} + 2i - xn$. *Therefore,* $|Z| = 2t = 2(\frac{n}{2} + 2i - x)$ *and* $|B - Z| = 2x - \frac{n}{2} - 3i$. *By* (3.37), $e[A, B] = 2x + \frac{n}{2} - i$.

3.2.4 The set up of the proof

3.2.4.1 Restating the theorem

We will use Theorem 3.16 of McKay in the following stronger form.

Theorem 3.28 (McKay [76]). *For every* $\epsilon > 0$, *there exists an* $N > 0$ *such that for each* $n > N$,

$$|\{F | F \in \mathcal{F}_3(n) \; : \; \alpha(\pi(F)) > 0.45537n\}| \; < \epsilon \cdot (3n-1)!!.$$

We will show that "almost all" cubic labeled graphs of girth at least 16 have independence ratio at most 0.454. In view of Theorem 3.3, the following more technical statement implies Theorem 3.17.

Theorem 3.29. *For every* $\epsilon > 0$, *there is an* $N > 0$ *such that for each* $n > N$,

$$|\{F \in \mathcal{G}'_{16}(n) \; : \; \alpha(\pi(F)) > 0.454n\}| \; < \epsilon \, (3n-1)!!. \tag{3.39}$$

The rest of Section 3.2 is a proof of Theorem 3.29. By definition, every graph has a MAI set. So, for large n, nonnegative integers $x \geq 0.454n$ and $i \leq \frac{n}{2} - x$, and each set A of size $\frac{n}{2} - i$ with a fixed matching of size $\frac{n}{2} - i - x$ we will estimate the total *x-weight* of configurations $F \in \mathcal{G}'_{16}(n)$ in which A forms a MAI set. The idea of the weight (used by McKay in [76]) is to decrease overcount of the configurations containing a given MAI set, but guarantee that the total weight of each configuration containing at least one MAI set with independence number x would be at least 1.

3.2.4.2 Setup of the proof of Theorem 3.29

An *AI-pair* on $[n]$ is a pair (A, R) consisting of a set $A \subset [n]$ and a matching R on a subset of A such that $E(G[A]) = R$. The *independence number*, $\alpha(A, R)$, of an AI-pair (A, R) is $|A| - |R|$. Let $\mathcal{P}(n, x)$ denote the family of all AI-pairs (A, R) on $[n]$ with $\alpha(A, R) = x$.

A *preimage* of an AI-pair (A, R) on $[n]$ is a pair $(\hat{A}, \hat{R})$ where $\hat{A} = A \times [3]$ and $\hat{R}$ is a matching on a subset of $\hat{A}$ with $|\hat{R}| = |R|$ such that for each edge $(i,j)(i',j') \in \hat{R}$, $ii' \in R$. In other words, each edge $e \in R$ is obtained from an edge in $\hat{e} \in \hat{R}$ by ignoring the second coordinates of the ends of $\hat{e}$, and this mapping is one-to-one.

By the x-weight of a configuration F we mean

$$\begin{aligned}&\omega_x(F) := \text{the reciprocal of the number of preimages } (\hat{A}, \hat{R}) \subseteq F \text{ of AI-pairs } (A,R) \text{ on } [n] \\ &\text{such that } A \text{ is an AI set in } \pi(F) \text{ with } E(\pi(F)[A]) = R \text{ and } \alpha(A,R) = x.\end{aligned} \tag{3.40}$$

By the definition of x-weight, each pairing $F \in \mathcal{G}'_{16}(n)$ with $\alpha(\pi(F)) = x$ contributes exactly 1 to

$$\sigma(n, x, 16) := \sum_{(A,R) \in \mathcal{P}(n,x)} \{\omega_x(F') : F' \in \mathcal{G}'_{16}(n) \text{ and } (\hat{A}, \hat{R}) \textit{ is an induced subpairing of } F'\}. \tag{3.41}$$

It follows that

$$\sigma(n, x, 16) \geq |\{F' \in \mathcal{G}'_{16}(n) \text{ with } \alpha(\pi(F')) = x\}|\,. \tag{3.42}$$

Lemma 3.30. *Let n be a positive even integer and x be an integer with $0.454n < x \leq 0.45537n$. The number of pairings $F \in \mathcal{G}'_{16}(n)$ such that $\pi(F)$ has a MAI set A with $|A'| = x$ is at most*

$$\begin{aligned} q(x,n) := \sum_{i=0}^{\frac{n}{2}-x} \binom{n}{\frac{n}{2}-i} \cdot \frac{(\frac{n}{2}-i)! \cdot 3^{(n-2x-2i)}}{(2x+i-\frac{n}{2})! \cdot 2^{\frac{n}{2}-x-i} \cdot (\frac{n}{2}-x-i)!} \\ \cdot \frac{(\frac{n}{2}+i)! \cdot 3^{n-2x+4i}}{(2x-3i-\frac{n}{2})! \cdot 2^{\frac{n}{2}-x+2i} \cdot (\frac{n}{2}-x+2i)!} \\ \cdot \sum_{j=0}^{n-2i-2x} \binom{n-2i-2x}{j} \cdot \binom{n-2x+4i}{j} \cdot 2^{2j} \cdot j! \cdot \left(\frac{1}{1.618}\right)^j \\ \cdot \frac{(3(2x-\frac{n}{2}-3i))! \cdot (3(2x-\frac{n}{2}+i))!}{(3(2x-\frac{n}{2}-3i)-2(n-2i-2x)+j)!}. \end{aligned}$$

Proof. By (3.42), it is enough to show that $\sigma(n, x, 16) \leq q(x, n)$. Below we describe a procedure of constructing for every AI-pair (A, R) on $[n]$ with $\alpha(A, R) = x$ all pairings in $F \in \mathcal{G}'_{16}(n)$ for which A is a MAI set. Not every obtained pairing will be in $\mathcal{G}'_{16}(n)$ and some pairings will have independence number larger than x, but every $F \in \mathcal{G}'_{16}(n)$ such that A is a MAI set in $\pi(F)$ will be a result of this procedure.

0. Choose nonnegative integers n, x, i, j such that n is even, $0.454n < x \leq 0.45537n$, $i \leq \frac{n}{2} - x$, and $j \leq \frac{n}{2} - x - i$.
1. Choose a set $A \subset [n]$ with $|A| = \frac{n}{2} - i$. There are $\binom{n}{\frac{n}{2}-i}$ ways to do it.
2. Choose a matching R on A with $|R| = \frac{n}{2} - x - i$. There are

$$\frac{(\frac{n}{2}-i)!}{(2x+i-\frac{n}{2})! \cdot 2^{\frac{n}{2}-x-i} \cdot (\frac{n}{2}-x-i)!}$$

ways to do it. Then there are $3^{n-2x-2i}$ ways to decide which point of each chosen end of an edge in R will be the end of the corresponding edge in F.

3. Similarly to Step 2, we have

$$\frac{(\frac{n}{2}+i)!}{(2x-3i-\frac{n}{2})!\cdot 2^{\frac{n}{2}-x+2i}\cdot(\frac{n}{2}-x+2i)!}$$

ways to construct a matching R' of $\frac{n}{2}-x+2i$ edges on $B := [n]-A$, since $|B| = \frac{n}{2}+i$. After that there are $3^{n-2x+4i}$ ways to decide which point of each chosen end of an edge in R' will be the end of the corresponding edge in F.

4. Let Y (respectively, Z) be the set of vertices covered by the matching R (respectively, R'). By Lemma 3.25, if A is a MAI-set in $\pi(F)$, then the set of edges connecting Y with Z is a matching. If this matching, say M has j edges, then there are $\binom{n-2i-2x}{j}$ ways to choose the set of the ends of M in Y and $\binom{n-2x+4i}{j}j!$ ways to choose the ends of M in Z. Since there are 2 free points left for each vertex in Y and Z, we have 2^{2j} ways to choose which point of each vertex in Y and Z to be used to form an edge in M.

5. By Lemma 3.26 each pairing $F \in \mathcal{G}'_{16}(n)$ containing a MAI set A with j edges between Y and Z contains at least $I(2,8)^j$ distinct MAI sets of the same cardinality. By Lemma 3.19, $I(2,8)^j \geq 1.618^j$. Hence by (3.40), $\omega_x(F) \leq 1.618^{-j}$.

6. Now we choose for each remaining free point p from vertices in Y a free point q in a vertex in $B-Z$ and add edge pq. There are

$$\frac{(3(2x-\frac{n}{2}-3i))!}{(3(2x-\frac{n}{2}-3i)-2(n-2i-2x)+j)!}$$

ways to do it.

7. Similarly to Step 6, we choose for each remaining free point q from vertices in Z a free point p in a vertex in $A-Y$ and add edge pq. There are

$$\frac{3(2x-\frac{n}{2}+i))!}{(3(2x-\frac{n}{2}+i)-2(n-2x+4i)+j)!}$$

ways to do it.

8. Finally, there are $3(2x-\frac{n}{2}+i)-2(n-2x+4i)+j = 10x-\frac{7n}{2}-5i+j$ free points left in A and $10x-\frac{7n}{2}-5i+j$ free points left in B. We have $(10x-\frac{7n}{2}-5i+j)!$ ways to complete a pairing on W_n. $\square$

In the proofs below we will use Stirling's formula: For every $n \geq 1$,

$$\sqrt{2\pi n}\left(\frac{n}{e}\right)^n \leq n! \leq \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^{1/12n}. \tag{3.43}$$

We will also use the notation $\frac{\partial}{\partial j}$ to denote the partial derivative with respect to j. Moreover, we use the domain $x \geq 0$ and define $\ln(0) = -\infty$ when we consider $\ln x$.

Lemma 3.31. *Let n be a positive even integer and x be an integer satisfying $0.454n < x \leq 0.45537n$. Let*

$$\Omega = \{(\chi, \zeta, \xi) : 0.454 < \chi \leq 0.45537,\ 0 \leq \zeta \leq \frac{1}{2} - \chi,\ 0 \leq \xi \leq 1 - 2\chi - 2\zeta\}. \tag{3.44}$$

Let

$$f(\chi, \zeta) :=$$

$$\frac{3^{\frac{1}{2}-4\chi+2\zeta} \cdot (1-2\chi-2\zeta)^{1-2\chi-2\zeta} \cdot (1-2\chi+4\zeta)^{1-2\chi+4\zeta} \cdot (6\chi-\frac{3}{2}+3\zeta)^{6\chi-\frac{3}{2}+3\zeta} \cdot (6\chi-\frac{3}{2}-9\zeta)^{6\chi-\frac{3}{2}-9\zeta}}{(2\chi+\zeta-\frac{1}{2})^{2\chi+\zeta-\frac{1}{2}} \cdot 2^{1-2\chi+\zeta} \cdot (\frac{1}{2}-\chi-\zeta)^{\frac{1}{2}-\chi-\zeta} \cdot (\frac{1}{2}-\chi+2\zeta)^{\frac{1}{2}-\chi+2\zeta} \cdot (2\chi-3\zeta-\frac{1}{2})^{2\chi-3\zeta-\frac{1}{2}}},$$

$$g(\chi, \zeta, \xi) :=$$

$$\frac{2^{2\xi} \cdot (\frac{1}{1.618})^{\xi}}{\xi^{\xi} \cdot (1-2\chi-2\zeta-\xi)^{1-2\chi-2\zeta-\xi} \cdot (1-2\chi+4\zeta-\xi)^{1-2\chi+4\zeta-\xi} \cdot (-\frac{7}{2}+10\chi-5\zeta+\xi)^{-\frac{7}{2}+10\chi-5\zeta+\xi}},$$

and

$$h(\chi, \zeta, \xi) := f(\chi, \zeta) \cdot g(x, \zeta, \xi).$$

Then

$$\frac{q(x,n)}{(3n-1)!!} = O(n^6) \cdot \max\{(h(\chi,\zeta,\xi))^n \ :\ (\chi,\zeta,\xi) \in \Omega\}. \tag{3.45}$$

Proof. We write $q(x, n)$ as a double sum of i and j and let $r(x, n, i, j)$ be the function inside the double sum of $q(x, n)$, i.e.,

$$q(x,n) = \sum_{i=0}^{\frac{n}{2}-x} \sum_{j=0}^{n-2x-2i} r(x,n,i,j).$$

Then certainly,

$$q(x,n) \leq n^2 \cdot \max\{r(x,n,i,j) : 0 \leq i \leq \frac{n}{2} - x,\ 0 \leq j \leq n - 2x - 2i\}.$$

So, it is enough to estimate $r(x, n, i, j)$. We know that

$$r(x,n,i,j) = \frac{n!}{(\frac{n}{2}-i)! \cdot (\frac{n}{2}+i)!} \cdot \frac{(\frac{n}{2}-i)! \cdot 3^{n-2x-2i}}{(2x+i-\frac{n}{2})! \cdot 2^{\frac{n}{2}-x-i} \cdot (\frac{n}{2}-x-i)!}$$

$$\cdot \frac{(\frac{n}{2}+i)! \cdot 3^{n-2x+4i}}{(2x-3i-\frac{n}{2})! \cdot 2^{\frac{n}{2}-x+2i} \cdot (\frac{n}{2}-x+2i)!} \cdot \frac{(n-2i-2x)!}{j! \cdot (n-2i-2x-j)!}$$

$$\cdot \frac{(n-2x+4i)!}{j! \cdot (n-2x+4i-j)!} \cdot 2^{2j} \cdot j! \cdot (\frac{1}{1.618})^j \cdot \frac{(6x-\frac{3n}{2}-9i)! \cdot (6x-\frac{3n}{2}+3i)!}{(10x-\frac{7n}{2}-5i+j)!}.$$

Recall that

$$(3n-1)!! \geq \frac{(3n)!!}{3n} \geq \frac{\sqrt{(3n)!}}{3n}.$$

Therefore,

$$\frac{r(x,n,i,j)}{(3n-1)!!} \le \frac{n!\cdot(3n)}{((3n)!)^{\frac{1}{2}}} \cdot \frac{3^{n-2x-2i}}{(2x+i-\frac{n}{2})!\cdot 2^{\frac{n}{2}-x-i}\cdot(\frac{n}{2}-x-i)!}$$

$$\cdot\frac{3^{n-2x+4i}}{(2x-3i-\frac{n}{2})!\cdot 2^{\frac{n}{2}-x+2i}\cdot(\frac{n}{2}-x+2i)!}\cdot\frac{(n-2i-2x)!}{j!\cdot(n-2i-2x-j)!}$$

$$\cdot\frac{(n-2x+4i)!}{(n-2x+4i-j)!}\cdot 2^{2j}\cdot(\frac{1}{1.618})^{j}\cdot\frac{(6x-\frac{3n}{2}-9i)!\cdot(6x-\frac{3n}{2}+3i)!}{(10x-\frac{7n}{2}-5i+j)!}.$$

Introducing new variables $\chi := \frac{x}{n}$, $\zeta := \frac{i}{n}$, and $\xi := \frac{j}{n}$ and using Stirling's formula (3.43), we get

$$\frac{r(x,n,i,j)}{(3n-1)!!} = O(n^4)\cdot\frac{\left(\frac{n}{e}\right)^{n}\cdot\left(\frac{n}{e}\right)^{(1-2\zeta-2\chi)n}\cdot\left(\frac{n}{e}\right)^{(1-2\chi+4\zeta)n}\cdot\left(\frac{n}{e}\right)^{(6\chi-\frac{3}{2}-9\zeta)n}\cdot\left(\frac{n}{e}\right)^{6\chi-\frac{3}{2}+3\zeta}}{\left(\frac{n}{e}\right)^{\frac{3}{2}n}\cdot\left(\frac{n}{e}\right)^{(2\chi+\zeta-\frac{1}{2})n}\cdot\left(\frac{n}{e}\right)^{(\frac{1}{2}-\chi-\zeta)n}\cdot\left(\frac{n}{e}\right)^{(2\chi-3\zeta-\frac{1}{2})n}\cdot\left(\frac{n}{e}\right)^{(\frac{1}{2}-\chi+2\zeta)n}\cdot\left(\frac{n}{e}\right)^{\xi n}}$$

$$\cdot\frac{1}{\left(\frac{n}{e}\right)^{(1-2\zeta-2\chi-\xi)n}\cdot\left(\frac{n}{e}\right)^{(1-2\chi+4\zeta-\xi)n}\cdot\left(\frac{n}{e}\right)^{(10\chi-\frac{7}{2}-5\zeta+\xi)n}}\cdot\left(f(\chi,\zeta)\cdot g(\chi,\zeta,\xi)\right)^{n}.$$

Therefore,

$$\frac{r(x,n,i,j)}{(3n-1)!!} = O(n^4)\cdot(h(\chi,\zeta,\xi))^{n}.$$

This proves the lemma. □

Recall that the domain of $h(\chi,\zeta,\xi)$ is Ω defined in (3.44). Our main goal now is to show that

$$\max_{(\chi,\zeta,\xi)\in\Omega} h(\chi,\zeta,\xi) \le 0.999983 < 1. \tag{3.46}$$

We do this in the next section, and then Theorem 3.29 easily follows.

3.2.5 Proof of (3.46)

In order to find the maximum value of $h(\chi,\zeta,\xi)$ for a fixed χ, we will maximize $\ln(h(\chi,\zeta,\xi))$. We first find the value of ξ in terms of χ and ζ that maximizes $\ln(g(\chi,\zeta,\xi))$. By definition,

$$\ln(g(\chi,\zeta,\xi)) = \xi\ln(\frac{4}{1.618}) - (\xi\ln(\xi) + (1-2\zeta-2\chi-\xi)\ln(1-2\zeta-2\chi-\xi)$$

$$+(1-2\chi+4\zeta-\xi)\ln(1-2\chi+4\zeta-\xi) + (10\chi-\frac{7}{2}-5\zeta+\xi)\ln(10\chi-\frac{7}{2}-5\zeta+\xi)).$$

Hence

$$\frac{\partial\ln(g(\chi,\zeta,\xi))}{\partial\xi} = \ln(1-2\chi-2\zeta-\xi) + \ln(1-2\chi+4\zeta-\xi) - \ln(10\chi-5\zeta+\xi-\frac{7}{2}) - \ln(\xi) + \ln(\frac{4}{1.618})$$

$$= \ln\left(\frac{(1-2\chi-2\zeta-\xi)\cdot(1-2\chi+4\zeta-\xi)\cdot\frac{4}{1.618}}{\xi\cdot(10\chi-5\zeta+\xi-\frac{7}{2})}\right).$$

In order to solve

$$\frac{\partial \ln(g(\chi,\zeta,\xi))}{\partial \xi} = 0,$$

we solve the equivalent equation

$$p(\xi) := 4\cdot(1-2\chi-2\zeta-\xi)\cdot(1-2\chi+4\zeta-\xi) - 1.618\cdot\xi\cdot(10\chi-5\zeta+\xi-\frac{7}{2}) = 0,$$

where $p(\xi)$ has domain $0 \le \xi \le 1-2\chi-2\zeta$. By the quadratic formula, the roots are

$$\xi_1 = \frac{-b-\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad \xi_2 = \frac{-b+\sqrt{b^2-4ac}}{2a},$$

where

$$a = 2.382,$$

$$b = -0.18\chi + 0.09\zeta - 2.337, \tag{3.47}$$

$$c = 16\chi^2 - 32\zeta^2 - 16\chi + 8\zeta - 16\chi\zeta + 4. \tag{3.48}$$

Moreover, for fixed χ and ζ satisfying $0.454 \le \chi \le 0.45537$ and $\chi+\zeta \le \frac{1}{2}$, $p(\xi)$ is a parabola opening upward with $\xi_1 \le 1-2\chi-2\zeta \le \xi_2$ because $p(1-2\chi-2\zeta) \le 0$, and $g(\chi,\zeta,\xi)$ is a continuous function on ξ. Therefore, the maximum of $g(\chi,\zeta,\xi)$ can only be attained at $\xi = \xi_1$.

Let $g_1(\chi,\zeta) = g(\chi,\zeta,\xi_1(\chi,\zeta))$. For each fixed χ, consider the maximum of

$$h_1(\chi,\zeta) := f(\chi,\zeta)\cdot g_1(\chi,\zeta).$$

By definition,

$$\ln(h_1) = (\frac{1}{2}-4\chi+2\zeta)\ln(3) + (1-2\chi-2\zeta)\ln(1-2\chi-2\zeta) + (1-2\chi+4\zeta)\ln(1-2\chi+4\zeta)$$

$$+(6\chi-\frac{3}{2}+3\zeta)\ln(6\chi-\frac{3}{2}+3\zeta) + (6\chi-\frac{3}{2}-9\zeta)\ln(6\chi-\frac{3}{2}-9\zeta) + \xi_1(\chi,\zeta)\cdot\ln(\frac{4}{1.618})$$

$$-(2\chi+\zeta-\frac{1}{2})\ln(2\chi+\zeta-\frac{1}{2}) - (1-2\chi+\zeta)\ln(2) - (\frac{1}{2}-\chi-\zeta)\ln(\frac{1}{2}-\chi-\zeta) - (\frac{1}{2}-\chi+2\zeta)\ln(\frac{1}{2}-\chi+2\zeta)$$

$$-(2x-3\zeta-\frac{1}{2})\ln(2\chi-3\zeta-\frac{1}{2}) - \xi_1(\chi,\zeta)\cdot\ln(\xi_1(\chi,\zeta)) - (1-2\zeta-2\chi-\xi_1(\chi,\zeta))\ln(1-2\zeta-2\chi-\xi_1(\chi,\zeta))$$

$$-(1-2\chi+4\zeta-\xi_1(\chi,\zeta))\ln(1-2\chi+4\zeta-\xi_1(\chi,\zeta)) - (10\chi-\frac{7}{2}-5\zeta+\xi_1(\chi,\zeta))\ln(10\chi-\frac{7}{2}-5\zeta+\xi_1(\chi,\zeta)),$$

and

$$\frac{\partial \ln(h_1)}{\partial \zeta} = -4\ln(3) + \ln(2) - 3 + 2\ln(2\chi+\zeta-\frac{1}{2}) - \ln(\frac{1}{2}-\chi-\zeta) + 2\ln(\frac{1}{2}-\chi+2\zeta) - 6\ln(2\chi-3\zeta-\frac{1}{2})$$

$$+\ln(\frac{4}{1.618})\cdot\frac{\partial\xi_1(\chi,\zeta)}{\partial\zeta}-\frac{\partial\xi_1(\chi,\zeta)}{\partial\zeta}\cdot(\ln(\xi_1(\chi,\zeta))+1)+(2+\frac{\partial\xi_1(\chi,\zeta)}{\partial\zeta})\cdot(\ln(1-2\zeta-2\chi-\xi_1(\chi,\zeta))+1)$$

$$+(\frac{\partial\xi_1(\chi,\zeta)}{\partial\zeta}-4)\cdot(\ln(1-2\chi+4\zeta-\xi_1(\chi,\zeta))+1)+(5-\frac{\partial\xi_1(\chi,\zeta)}{\partial\zeta})\cdot(\ln(10\chi-\frac{7}{2}-5\zeta+\xi_1(\chi,\zeta))+1),$$

where

$$\frac{\partial\xi_1(\chi,\zeta)}{\partial\zeta}=\frac{1}{2a}\cdot(-\frac{\partial b}{\partial\zeta}-\frac{1}{2}\cdot(b^2-4ac)^{-\frac{1}{2}}\cdot(2b\frac{\partial b}{\partial\zeta}-4a\frac{\partial c}{\partial\zeta})),$$

$$\frac{\partial b}{\partial\zeta}=0.09,$$

$$\frac{\partial c}{\partial\zeta}=-16\chi-64\zeta+8.$$

Lemma 3.32. *When $\chi=0.454$, the maximum of $h(\chi,\zeta,\xi)$ over $0\le\zeta\le0.046$ and $0\le\xi\le 0.092-2\zeta$ is at most 0.999983.*

Proof Fix $\chi=0.454$. For $0\le\zeta\le0.046$, denote

$$\xi_1'(\zeta):=\frac{\partial\xi_1(0.454,\zeta)}{\partial\zeta},\ \xi_1''(\zeta):=\frac{\partial^2\xi_1(0.454,\zeta)}{\partial\zeta^2},\text{ and }\xi_1(\zeta):=\xi_1(0.454,\zeta).$$

We have

$$\frac{\partial\ln(h_1(0.454,\zeta))}{\partial\zeta}=-4\ln(3)+\ln(2)-3+2\ln(0.408+\zeta)-\ln(0.046-\zeta)+2\ln(0.046+2\zeta)$$

$$-6\ln(0.408-3\zeta)+\ln(\frac{4}{1.618})\cdot\xi_1'(\zeta)-\xi_1'(\zeta)\cdot(\ln(\xi_1(\zeta))+1)+(2+\xi_1'(\zeta))\cdot(\ln(0.092-2\zeta-\xi_1(\zeta))+1)$$

$$+(\xi_1'(\zeta)-4)\cdot(\ln(0.092+4\zeta-\xi_1(\zeta))+1)+(5-\xi_1'(\zeta))\cdot(\ln(1.04-5\zeta+\xi_1(\zeta))+1),$$

$$\frac{\partial^2\ln(h_1(0.454,\zeta))}{\partial\zeta^2}=\frac{1}{0.046-\zeta}+\frac{4}{0.046+2\zeta}+\frac{2}{0.408+\zeta}+\frac{18}{0.408-3\zeta}$$

$$+\ln(\frac{4}{1.618})\cdot\xi_1''(\zeta)-\xi_1''(\zeta)\cdot(\ln(\xi_1(\zeta))+1)-(\xi_1'(\zeta))^2\cdot\frac{1}{\xi_1(\zeta)}$$

$$+\xi_1''(\zeta)\cdot(\ln(0.092-2\zeta-\xi_1(\zeta))+1)-(2+\xi_1'(\zeta))^2\cdot\frac{1}{0.092-2\zeta-\xi_1(\zeta)}$$

$$+\xi_1''(\zeta)\cdot(\ln(0.092+4\zeta-\xi_1(\zeta))+1)-(\xi_1'(\zeta)-4)^2\cdot\frac{1}{0.092+4\zeta-\xi_1(\zeta)}$$

$$-\xi_1''(\zeta)\cdot(\ln(1.04-5\zeta+\xi_1(\zeta))+1)-(\xi_1'(\zeta)-5)^2\cdot\frac{1}{1.04-5\zeta+\xi_1(\zeta)},$$

where

$$\xi_1''(\zeta)=\frac{1}{2a}\cdot(b^2-4ac)^{-\frac{1}{2}}\cdot\left(\frac{1}{4}\cdot(b^2-4ac)^{-1}\cdot(2b\frac{\partial b}{\partial\zeta}-4a\frac{\partial c}{\partial\zeta})^2-\frac{1}{2}\cdot(2(\frac{\partial b}{\partial\zeta})^2+1024a)\right).\quad(3.49)$$

We will show that

$$\frac{\partial^2\ln(h_1(0.454,\zeta))}{\partial\zeta^2}<0,\ \text{for all }0\le\zeta<0.046.\quad(3.50)$$

This will guarantee that if we find a solution $\zeta_0 \in [0, 0.046)$ of the equation $\frac{\partial \ln(h_1(0.454,\zeta))}{\partial \zeta} = 0$, then the maximum of $h_1(0.454, \zeta)$ over $\zeta \in [0, 0.046)$ is attained at ζ_0.

Claim 3.33. *For each* $\zeta \in [0, 0.046)$*,* $-27.336 \le \xi_1''(\zeta) < -24.822$*.*

Proof. By (3.47) and (3.48), for $\chi = 0.454$, the function $\Delta(\zeta) := b^2 - 4ac$ is quadratic in ζ with derivative

$$\Delta'(\zeta) = 2b\frac{\partial b}{\partial \zeta} - 4a\frac{\partial c}{\partial \zeta},$$

which is linear in ζ and has minimum at $\zeta = 0$ and maximum at $\zeta = 0.046$. Therefore,

$$-7.45 \le \Delta'(0) \le \Delta'(\zeta) \le \Delta'(0.046) \le 20.61$$

for each $\zeta \in [0, 0.046)$. Also for such ζ,

$$\Delta''(\zeta) = 2(\frac{\partial b}{\partial \zeta})^2 - 4a\frac{\partial^2 c}{\partial \zeta^2} = 2 \cdot 0.09^2 - 4 \cdot 2.382 \cdot (-64) \in (609.8, 609.81),$$

so $\Delta(\zeta)$ is a parabola opening upward with minimum attained at the unique root ζ_ξ of the equation $\Delta'(\zeta) = 0$. By $\Delta'(0.012213) < -0.00039$, $\Delta'(0.012214) > 0.000219$, and the above statements,

$$0.012213 \le \zeta_\xi \le 0.012214.$$

Hence $\Delta(\zeta_\xi)$ satisfies

$$5.4821 \le 5.48214 - 0.0004 \cdot 0.000001 \le \Delta(0.012213) + \Delta'(0.012213) \cdot 0.000001$$

$$\le \Delta(\zeta_\xi) \le \Delta(0.012213) \le 5.4822,$$

and the maximum of $\Delta(\zeta)$ over $\zeta \in [0, 0.046)$ is attained at $\zeta = 0.046$ and satisfies

$$5.83019 \le \Delta(0.046) \le 5.8302.$$

Therefore, for each $\zeta \in [0, 0.046)$,

$$0.41415 \le \frac{1}{\sqrt{5.8302}} \le (\Delta(\zeta))^{-\frac{1}{2}} \le \frac{1}{\sqrt{5.4821}} \le 0.427098,$$

$$0.17152 \le (\Delta(\zeta))^{-1} \le 0.182412.$$

Thus by (3.49),

$$-27.336 \le \frac{1}{2 \cdot 2.382} \cdot 0.427098 \cdot (0 - 0.5 \cdot 609.81) \le \xi_1''(\zeta) \tag{3.51}$$

$$= \frac{1}{2a} \cdot (\Delta(\zeta))^{-\frac{1}{2}} \cdot \left(\frac{1}{4} \cdot (\Delta(\zeta))^{-1} \cdot (\Delta'(\zeta))^2 - \frac{1}{2} \cdot \Delta''(\zeta)\right)$$

$$\le \frac{1}{2 \cdot 2.382} \cdot 0.41415 \cdot (0.25 \cdot 0.182412 \cdot (20.61)^2 - 0.5 \cdot 609.8) \le -24.822. \tag{3.52}$$

This proves Claim 3.33. $\square$

Claim 3.34. *For each* $\zeta \in [0, 0.046)$, $-0.91445 \leq \xi_1'(0.046) \leq \xi_1'(\zeta) \leq \xi_1'(0) \leq 0.31359$.

Proof. By Claim 3.33, $\xi_1'(\zeta)$ is a decreasing function on $0 \leq \zeta \leq 0.046$. $\square$

To prove (3.50), we write $\frac{\partial^2 \ln(h_1(0.454,\zeta))}{\partial \zeta^2}$ in a form

$$\frac{\partial^2 \ln(h_1(0.454,\zeta))}{\partial \zeta^2} = A_1(\zeta) + A_2(\zeta) + A_3(\zeta) + A_4(\zeta) + A_5(\zeta), \tag{3.53}$$

and then bound these expressions separately so that the sum of the upper bounds will be negative for each $\zeta \in [0, 0.046)$. By definition

$$\frac{\partial^2 \ln(h_1)}{\partial \zeta^2} = \frac{1}{0.046-\zeta} - \xi_1''(\zeta)\cdot\ln(\xi_1(\zeta)) - \frac{(\xi_1'(\zeta))^2}{\xi_1(\zeta)} + \frac{4}{0.046+2\zeta} + \frac{2}{0.408+\zeta}$$

$$+\frac{18}{0.408-3\zeta} + \ln(\frac{4}{1.618})\cdot\xi_1''(\zeta) + \xi_1''(\zeta)\cdot\ln(0.092-2\zeta-\xi_1(\zeta)) - \frac{(2+\xi_1'(\zeta))^2}{0.092-2\zeta-\xi_1(\zeta)}$$

$$+\xi_1''(\zeta)\cdot\ln(0.092+4\zeta-\xi_1(\zeta)) - \frac{(\xi_1'(\zeta)-4)^2}{0.092+4\zeta-\xi_1(\zeta)}$$

$$-\xi_1''(\zeta)\cdot\ln(1.04-5\zeta+\xi_1(\zeta)) - \frac{(\xi_1'(\zeta)-5)^2}{1.04-5\zeta+\xi_1(\zeta)}.$$

Let

$$A_1(\zeta) := \frac{1}{0.046-\zeta} - (\xi_1'(\zeta))^2\cdot\frac{1}{\xi_1(\zeta)} - \frac{(2+\xi_1'(\zeta))^2}{0.092-2\zeta-\xi_1(\zeta)}, \tag{3.54}$$

$$A_2(\zeta) := \xi_1''(\zeta)\cdot\ln(0.092-2\zeta-\xi_1(\zeta)) - \xi_1''(\zeta)\cdot\ln(\xi_1(\zeta)), \tag{3.55}$$

$$A_3(\zeta) := \ln(\frac{4}{1.618})\cdot\xi_1''(\zeta), \tag{3.56}$$

$$A_4(\zeta) := \frac{4}{0.046+2\zeta} + \frac{2}{0.408+\zeta} + \frac{18}{0.408-3\zeta}, \text{ and} \tag{3.57}$$

$$A_5(\zeta) := \xi_1''(\zeta)\cdot\ln(0.092+4\zeta-\xi_1(\zeta)) - \frac{(\xi_1'(\zeta)-4)^2}{0.092+4\zeta-\xi_1(\zeta)} \tag{3.58}$$

$$-\xi_1''(\zeta)\cdot\ln(1.04-5\zeta+\xi_1(\zeta)) - \frac{(\xi_1'(\zeta)-5)^2}{1.04-5\zeta+\xi_1(\zeta)},$$

so that (3.53) holds.

Claim 3.35. *For each* $\zeta \in [0, 0.046)$, $A_1(\zeta) < 0$.

Proof. Since $0.092 - 2\zeta - \xi_1(\zeta) \geq 0$ and $\xi_1(\zeta) \geq 0$, by Claim 3.34,

$$A_1(\zeta) = \frac{1}{0.046-\zeta} - (\xi_1'(\zeta))^2\cdot\frac{1}{\xi_1(\zeta)} - (2+\xi_1'(\zeta))^2\cdot\frac{1}{0.092-2\zeta-\xi_1(\zeta)}$$

$$\leq \frac{1}{0.046-\zeta} - (\xi_1'(\zeta))^2 \cdot \frac{1}{0.092-2\zeta} - (2+\xi_1'(\zeta))^2 \cdot \frac{1}{0.092-2\zeta}$$

$$= \frac{1}{0.046-\zeta} - \frac{(\xi_1'(\zeta)+1)^2+1}{0.046-\zeta} = -\frac{(\xi_1'(\zeta)+1)^2}{0.046-\zeta} < 0. \quad \square$$

Claim 3.36. *For each* $\zeta \in [0, 0.046)$, $A_2(\zeta) < 0$.

Proof. Let $\zeta \in [0, 0.046)$. By Claim 3.33, inequality $A_2(\zeta) < 0$ is equivalent to

$$0.092 - 2\zeta - \xi_1(\zeta) > \xi_1(\zeta).$$

Let $y(\zeta) = 0.092 - 2\zeta - 2\xi_1(\zeta)$. By Claim 3.34,

$$y'(\zeta) = -2 - 2\xi_1'(\zeta) < 0.$$

Therefore, $y(\zeta) > y(0.046) = 0$ for each $\zeta \in [0, 0.046)$. This proves the claim. $\square$

Claim 3.37. *For each* $\zeta \in [0, 0.046)$, $A_3(\zeta) \leq -22.46$.

Proof. This follows from the definition (3.56), since $\xi_1''(\zeta) \leq -24.822$ by Claim 3.33. $\square$

Claim 3.38. *The function* $A_4'(\zeta)$ *has exactly one root* d_ζ *in the interval* $[0, 0.046]$. *Furthermore,* $d_\zeta \in (0.0355167, 0.0355168)$, *and* $A_4(\zeta)$ *is decreasing on* $[0, d_\zeta]$ *and increasing on* $[d_\zeta, 0.046]$.

Proof. By Definition (3.57),

$$A_4'(\zeta) = -\frac{2}{(\zeta+0.023)^2} - \frac{2}{(\zeta+0.408)^2} + \frac{6}{(\zeta-0.136)^2}$$

and

$$A_4''(\zeta) = \frac{4}{(\zeta+0.023)^3} + \frac{4}{(\zeta+0.408)^3} - \frac{12}{(\zeta-0.136)^3}.$$

The last expression is positive for all $\zeta \in [0, 0.046]$, so function $A_4'(\zeta)$ may have at most one root on $[0, 0.046]$. On the other hand, $A_4'(0.0355167) < -0.002$ and $A_4'(0.0355168) > 0.0006$. This proves the claim. $\square$

Claim 3.39. *For each* $\zeta \in [0, 0.046)$, $A_4(\zeta) + A_5(\zeta) \leq 20$.

Proof. Let

$$z_1(\zeta) = 0.092 + 4\zeta - \xi_1(\zeta) \quad \text{and} \quad z_2(\zeta) = 1.04 - 5\zeta + \xi_1(\zeta).$$

By Claim 3.34, $z_1'(\zeta) = 4 - \xi_1'(\zeta) > 0$ and $z_2'(\zeta) = -5 + \xi_1'(\zeta) < 0$ for each $\zeta \in [0, 0.046)$. So,

$$z_1(\zeta) \text{ is increasing and } z_2(\zeta) \text{ is decreasing on } [0, 0.046). \tag{3.59}$$

Since

$$z_1(\zeta) < z_1(0.046) < z_2(0.046) < z_2(\zeta)$$

for each $\zeta \in [0, 0.046)$, Definitions (3.57) and (3.58) together with Claim 3.33 yield

$$A_4(\zeta) + A_5(\zeta) = A_4(\zeta) + \xi_1''(\zeta) \cdot \ln(z_1(\zeta)) - \frac{(\xi_1'(\zeta) - 4)^2}{z_1(\zeta)} - \xi_1''(\zeta) \cdot \ln(z_2(\zeta)) - \frac{(\xi_1'(\zeta) - 5)^2}{z_2(\zeta)}$$

$$\le A_4(\zeta) - 27.336 \cdot (\ln(z_1(\zeta)) - \ln(z_2(\zeta))) - \frac{(\xi_1'(\zeta) - 4)^2}{z_1(\zeta)} - \frac{(\xi_1'(\zeta) - 5)^2}{z_2(\zeta)} =: Q(\zeta).$$

Since $\zeta \in [0, 0.046)$, it belongs to the interval $[0.001k, 0.001(k+1))$ for some integer $0 \le k \le 45$. We consider 3 cases.

Case 1: $0 \le k \le 34$. Then by Claim 3.38 and (3.59), for each $\zeta \in [0.001k, 0.001(k+1))$,

$$A_4(0.001k) \ge A_4(\zeta),$$

$$z_1(\zeta) \ge z_1(0.001k), \text{ and } z_2(0.001k) \ge z_2(\zeta).$$

Therefore,

$$Q(\zeta) \le M_1(k) := A_4(0.001k) - 27.336 \cdot (\ln(z_1(0.001k)) - \ln(z_2(0.001k)))$$

$$-\frac{(\xi_1'(0.001k) - 4)^2}{z_1(0.001(k+1))} - \frac{(\xi_1'(0.001k) - 5)^2}{z_2(0.001k)}.$$

The bounds for $M_1(k)$ certifying that $M_1(k) < 20$ for each $0 \le k \le 34$ are given in Table 2 in Section 3.2.7.

Case 2: $k = 35$. Similarly to Case 1,

$$Q(\zeta) \le \max(A_4(0.035), A_4(0.036)) - 27.336 \cdot (\ln(z_1(0.035)) - \ln(z_2(0.035)))$$

$$-\frac{(\xi_1'(0.035) - 4)^2}{z_1(0.036)} - \frac{(\xi_1'(0.035) - 5)^2}{z_2(0.035)}$$

$$< 98.404 - 27.336 \cdot (-1.5 - (-0.135)) - 94 - 36.3 < 5.5 < 20.$$

Case 3: $36 \le k \le 45$. Again, similarly to Case 1,

$$Q(\zeta) \le M_3(k) := A_4(0.001(k+1)) - 27.336 \cdot (\ln(z_1(0.001k)) - \ln(z_2(0.001k)))$$

$$-\frac{(\xi_1'(0.001k) - 4)^2}{z_1(0.001(k+1))} - \frac{(\xi_1'(0.001k) - 5)^2}{z_2(0.001k)}.$$

The bounds for $M_1(k)$ certifying that $M_1(k) < 20$ for each $36 \le k \le 45$ are given in Table 1 in Section 3.2.7. □

Thus by (3.53) and Claims 3.35–3.39, for each $\zeta \in [0, 0.046)$,

$$\frac{\partial^2 \ln(h_1(0.454,\zeta))}{\partial \zeta^2} = \sum_{i=1}^{5} A_i(\zeta) < -22.46 + 20 = -2.46 < 0.$$

We also can check by plugging in the values that

$$\frac{\partial \ln(h_1(0.454, 0.0228718))}{\partial \zeta} < -9 \cdot 10^{-6}, \text{ and } \frac{\partial \ln(h_1(0.454, 0.0228719))}{\partial \zeta} > 7.54 \cdot 10^{-8}.$$

Thus, the derivative of $h_1(0.454,\zeta)$ equals 0 at a unique $\zeta_1 \in (0.0228718, 0.0228719)$.

Recall that $h_1(0.454,\zeta) > 0$ for $\zeta \in [0, 0.046)$. So, after comparing the value $h_1(0.454, 0.0228719)$ with the boundary values $h_1(0.454, 0)$ and $h_1(0.454, 0.46)$, we conclude that the maximum of $h_1(0.454,\zeta)$ is attained at ζ_1. We can plug in numbers into a computer and obtain that

$$h_1(0.454, 0.0228718) \leq 0.999982,$$

$$\frac{\partial \ln(h_1(0.454, 0.0228718))}{\partial \zeta} \leq 1 \cdot 10^{-7},$$

and

$$\frac{\partial h_1(0.454, 0.0228718)}{\partial \zeta} = h_1(0.454, 0.0228718) \cdot \frac{\partial \ln(h_1(0.454, 0.0228718))}{\partial \zeta} \leq 1 \cdot 10^{-7},$$

which implies that

$$h_1(0.454, \zeta_1) \leq h_1(0.454, 0.0228718) + 1 \cdot 10^{-7} \cdot 0.0000001 \leq 0.999983. \quad \square$$

The proof of the next lemma is similar but significantly simpler. It is mostly a routine bounding some expressions. So, we present the proof of Lemma 3.40 in Section 3.2.8.

Lemma 3.40. *For every*

$$(\chi,\zeta,\xi) \in \Omega = \{(\chi,\zeta,\xi) : 0.454 < \chi \leq 0.45537,\ 0 \leq \zeta \leq \frac{1}{2} - \chi,\ 0 \leq \xi \leq 1 - 2\chi - 2\zeta\},$$

we have

$$\frac{\partial \ln(h(\chi,\zeta,\xi))}{\partial \chi} < 0. \quad \square \tag{3.60}$$

Since $h(\chi,\zeta,\xi) > 0$ for each $(\chi,\zeta,\xi) \in \Omega$, Lemma 3.40 yields that for each fixed ζ and ξ, the maximum of $h(\chi,\zeta,\xi)$ over $(\chi,\zeta,\xi) \in \Omega$ is attained at $\chi = 0.454$. By Lemma 3.32, this maximum is at most 0.999983. This yields (3.46).

3.2.6 Completion of the proof of Theorem 3.29

By (3.46) and Lemma 3.31, for all positive integers n and x such that n is even and $0.454n < x \leq 0.45537n$,

$$\frac{q(x,n)}{(3n-1)!!} \leq O(n^6) \cdot 0.999983^n.$$

It follows that

$$\frac{1}{(3n-1)!!} \sum_{x=\lceil 0.454n \rceil}^{\lfloor 0.45537n \rfloor} q(x,n) \leq O(n^7) \cdot 0.999983^n \to 0 \quad \text{as } n \to \infty. \tag{3.61}$$

Thus by Lemma 3.30, the number of pairings $F \in \mathcal{G}'_{16}(n)$ with $0.454n < \alpha(F) \leq 0.45537n$ is $o\left((3n-1)!!\right)$. Together with Theorem 3.28, this means that almost no pairings have independence ratio larger than 0.454. Thus by Corollary 3.4 we conclude that almost no n-vertex 3-regular graphs of girth at least 16 have independence ratio larger than 0.454. This proves Theorem 3.29 and thus also Theorem 3.17.

3.2.7 Tables for Claim 3.39

See Tables 3.1 and 3.2.

3.2.8 Proof of Lemma 3.40

By definition, the boundary, $\partial\Omega$, of Ω is

$$\partial\Omega = \{(\chi,\zeta,\xi) : \xi = 0, 2\chi + 2\zeta \leq 1, 0.454 \leq \chi \leq 0.45537, \zeta \geq 0\} \cup$$

$$\{(\chi,\zeta,\xi) : \zeta = 0, 2\chi + \xi \leq 1, 0.454 \leq \chi \leq 0.45537, \xi \geq 0\} \cup$$

$$\{(\chi,\zeta,\xi) : \chi = 0.454, 2\zeta + \xi \leq 0.092, \zeta \geq 0, \xi \geq 0\} \cup$$

$$\{(\chi,\zeta,\xi) : \chi = 0.45537, 2\zeta + \xi \leq 0.08926, \zeta \geq 0, \xi \geq 0\}.$$

We also will consider the 2-dimensional set

$$\Omega_1 = \{(\chi,\zeta) : 0.454 \leq \chi \leq 0.45537, 0 \leq \zeta \leq 0.5 - \chi\}.$$

Then the boundary of Ω_1 is

$$\partial\Omega_1 = \{(\chi,\zeta) : 0.454 \leq \chi \leq 0.45537, \zeta = 0\} \cup \{(\chi,\zeta) : 0 \leq \zeta \leq 0.046, \chi = 0.454\}$$

$$\cup \{(\chi,\zeta) : 0 \leq \zeta \leq 0.04463, \chi = 0.45537\} \cup \{(\chi,\zeta) : 0.454 \leq \chi \leq 0.45537, \chi + \zeta = \frac{1}{2}\}.$$

k	$A_4(0.001k)$	$-\ln(z_1(0.001k))$	$\ln(z_2(0.001k))$	$-\frac{(\xi_1'(0.001k)-4)^2}{z_1(0.001(k+1))}$	$-\frac{(\xi_1'(0.001k)-5)^2}{z_2(0.001k)}$	$M_1(k)$
0	135.9762	2.553562	0.05277836	-166.7356	-20.83335	19.7
1	132.6679	2.507105	0.04831009	-161.790	-21.1686	19.6
2	129.6543	2.462392	0.04379588	-157.2333	-21.50947	19.5
3	126.903	2.419288	0.03923514	-153.0194	-21.85574	19.3
4	124.384	2.377674	0.03462729	-149.1122	-22.20758	19.1
5	122.0728	2.33745	0.02997174	-145.4794	-22.56505	18.8
6	119.9504	2.298492	0.02526786	-142.0935	-22.92824	18.5
7	117.9977	2.260743	0.02051507	-138.9302	-23.29722	18.2
8	116.1989	2.224115	0.01571273	-135.9683	-23.67205	17.8
9	114.5404	2.188538	0.01086022	-133.1892	-24.05282	17.5
10	113.0099	2.153948	0.005956888	-130.5765	-24.43961	17.1
11	111.5969	2.120286	0.001002109	-128.1157	-24.83248	16.7
12	110.292	2.0876	-0.004004782	-125.793	-25.23152	16.3
13	109.0867	2.055542	-0.009064451	-123.5994	-25.63682	15.8
14	107.9738	2.024367	-0.01417756	-121.5220	-26.04845	15.4
15	106.9466	1.993934	-0.01934483	-119.5525	-26.4664	15.0
16	106.000	1.964204	-0.02456692	-117.6823	-26.89104	14.5
17	105.1262	1.935144	-0.02984456	-115.9041	-27.32218	14.0
18	104.3229	1.90673	-0.03517846	-114.2111	-27.76000	13.6
19	103.585	1.878905	-0.04056935	-112.5970	-28.20460	13.1
20	102.9088	1.851668	-0.04601797	-111.0562	-28.65606	12.6
21	102.2906	1.824985	-0.05152507	-109.5836	-29.1144	12.1
22	101.7273	1.798832	-0.05709143	-108.1746	-29.57998	11.6
23	101.217	1.773185	-0.06271781	-106.8248	-30.05263	11.1
24	100.7543	1.748024	-0.06840502	-105.5304	-30.53255	10.7
25	100.3398	1.723329	-0.07415386	-104.2877	-31.01984	10.2
26	99.97009	1.699082	-0.07996514	-103.0934	-31.51462	9.7
27	99.64358	1.675265	-0.08583972	-101.9446	-32.016	9.2
28	99.3585	1.651862	-0.09177843	-100.8383	-32.52708	8.7
29	99.11297	1.628858	-0.09778215	-99.77206	-33.04501	8.2
30	98.90584	1.606239	-0.1038517	-98.74333	-33.5708	7.7
31	98.7358	1.583989	-0.1099881	-97.74996	-34.10486	7.2
32	98.6015	1.562098	-0.1161922	-96.78986	-34.64704	6.7
33	98.50187	1.54056	-0.122464	-95.86113	-35.19758	6.3
34	98.43615	1.519339	-0.1288073	-94.96198	-35.75661	5.8

Table 3.1: Upper bounds for expressions in $M_1(k)$.

k	$A_4(0.001(k+1))$	$-\ln(z_1(0.001k))$	$\ln(z_2(0.001k))$	$-\frac{(\xi_1'(0.001k)-4)^2}{z_1(0.001(k+1))}$	$-\frac{(\xi_1'(0.001k)-5)^2}{z_2(0.001k)}$	$M_3(k)$
36	98.43379	1.477873	-0.1417047	-93.24588	-36.90074	4.9
37	98.49569	1.457599	-0.1482617	-92.42593	-37.48615	4.4
38	98.58802	1.437619	-0.1548924	-91.62957	-38.08066	4.0
39	98.71033	1.417923	-0.1615978	-90.85551	-38.68445	3.6
40	98.86225	1.398503	-0.1683790	-90.1025	-39.29768	3.1
41	99.04347	1.379352	-0.1752370	-89.36971	-39.92054	2.7
42	99.25376	1.36046	-0.1821731	-88.65582	-40.55321	2.3
43	99.49293	1.341822	-0.1891883	-87.95995	-41.1958	1.9
44	99.76085	1.323429	-0.1962838	-87.28121	-41.84877	1.5
45	100.0576	1.305276	-0.2034610	-86.61873	-42.51206	1.1

Table 3.2: Upper bounds for expressions in $M_3(k)$.

By the definition of h,

$$\frac{\partial \ln(h(\chi,\zeta,\xi))}{\partial\chi} = 4\ln(2\chi-3\zeta-\frac{1}{2}) + 4\ln(2\chi+\zeta-\frac{1}{2}) - \ln(1-2\chi-2\zeta)$$

$$-\ln(1-2\chi+4\zeta) + 2\ln(1-2\chi-2\zeta-\xi) + 2\ln(1-2\chi+4\zeta-\xi) - 10\ln(10\chi-5\zeta+\xi-\frac{7}{2}).$$

Similarly to the proof of Lemma 3.32, we present $\frac{\partial \ln(h(\chi,\zeta,\xi))}{\partial\chi}$ in the form $\sum_{j=1}^{6} B_j$, where

$$B_1(\chi,\zeta) := 4\ln(2\chi-3\zeta-\frac{1}{2}), \quad B_2(\chi,\zeta) := 4\ln(2\chi+\zeta-\frac{1}{2}), \tag{3.62}$$

$$B_3(\chi,\zeta,\xi) := 2\ln(1-2\chi-2\zeta-\xi) - \ln(1-2\chi-2\zeta), \quad B_4(\chi,\zeta) := \ln(1-2\chi+4\zeta), \tag{3.63}$$

$$B_5(\chi,\zeta,\xi) := 2\ln(1-2\chi+4\zeta-\xi), \text{ and } B_6(\chi,\zeta,\xi) := \ln(10\chi-5\zeta+\xi-\frac{7}{2}), \tag{3.64}$$

and then bound each of the terms separately.

Claim 3.41. *For all* $(\chi,\zeta)\in\Omega_1$, $B_1(\chi,\zeta) < -3.55$.

Proof. For each $(\chi,\zeta)\in\Omega_1$, we have $\chi-\frac{3}{2}\zeta-0.25>0$, since $\chi\geq 0.454$ and $\zeta\leq 0.046$. As for each $(\chi,\zeta)\in\Omega_1$,

$$\frac{\partial B_1(\chi,\zeta)}{\partial\chi} = \frac{4}{\chi-\frac{3}{2}\zeta-0.25} > 0 \qquad \textit{and} \qquad \frac{\partial B_1(\chi,\zeta)}{\partial\zeta} = \frac{-6}{\chi-\frac{3}{2}\zeta-0.25} < 0,$$

the maximum is attained at a corner on the boundary $\partial\Omega_1$. Comparing the values of B_1 at the four corners of $\partial\Omega_1$, we see that the maximum is attained at $(\chi,\zeta)=(0.45537,0)$ and $B_1(0.45537,0) < -3.55$. □

Claim 3.42. *For all* $(\chi,\zeta)\in\Omega_1$, $B_2(\chi,\zeta) < -3.14$.

Proof. For each $(\chi,\zeta)\in\Omega_1$, we have $2\chi+\zeta-\frac{1}{2}>0$. As for each $(\chi,\zeta)\in\Omega_1$,

$$\frac{\partial B_2(\chi,\zeta)}{\partial\chi}=\frac{8}{2\chi+\zeta-\frac{1}{2}}>0,\qquad \textit{and}\qquad \frac{\partial B_2(\chi,\zeta)}{\partial\zeta}=\frac{4}{2\chi+\zeta-0.5}>0,$$

the maximum is attained at a corner of the boundary $\partial\Omega_1$. Comparing the values of B_2 at the four corners of $\partial\Omega_1$, we see that the maximum is attained at $(\chi,\zeta)=(0.45537,0.04463)$, and $B_2(0.45537,0.04463)<-3.14$. $\square$

Claim 3.43. *For all* $(\chi,\zeta,\xi)\in\Omega,\ B_3(\chi,\zeta,\xi)<0$.

Proof. We can write $B_3(\chi,\zeta,\xi)$ in the form

$$B_3(\chi,\zeta,\xi)=\ln(1-2\chi-2\zeta-\xi)+\ln\left(\frac{1-2\chi-2\zeta-\xi}{1-2\chi-2\zeta}\right),$$

and observe that $\ln(1-2\chi-2\zeta-\xi)<0$ (since $2\chi+2\zeta+\xi>0$) and $\ln(\frac{1-2\chi-2\zeta-\xi}{1-2\chi-2\zeta})\le 0$ (since $1-2\chi-2\zeta-\xi\le 1-2\chi-2\zeta$) . $\square$

Claim 3.44. *For all* $(\chi,\zeta)\in\Omega_1,\ B_4(\chi,\zeta)<-1.28$.

Proof. For each $(\chi,\zeta)\in\Omega_1$, $-2\chi+4\zeta+1>0$. As for each $(\chi,\zeta)\in\Omega_1$,

$$\frac{\partial B_4(\chi,\zeta)}{\partial\chi}=\frac{-2}{-2\chi+4\zeta+1}<0,\qquad \textit{and}\qquad \frac{\partial B_4(\chi,\zeta)}{\partial\zeta}=\frac{4}{-2\chi+4\zeta+1}>0,$$

the maximum of B_4 is attained at a corner of the boundary $\partial\Omega_1$. Comparing the values of B_4 at the four corners of $\partial\Omega_1$, we see that the maximum is attained at $(\chi,\zeta)=(0.454,0.046)$, and $B_4(0.454,0.046)<-1.28$. $\square$

Claim 3.45. *For all* $(\chi,\zeta,\xi)\in\Omega,\ B_5(\chi,\zeta,\xi)<-2.57$.

Proof. For each $(\chi,\zeta,\xi)\in\Omega-\partial\Omega$, we have $2\chi-4\zeta+\xi-1<0$ since $2\chi+2\zeta+\xi<1\le 1+4\zeta+2\zeta$. Since

$$\lim_{2\chi+\xi\to 1}B_5(\chi,0,\xi)=-\infty,$$

the maximum of B_5 is not attained at $\zeta=0, 2\chi+\xi=1$. As for each $(\chi,\zeta,\xi)\in\Omega$,

$$\frac{\partial B_5(\chi,\zeta,\xi)}{\partial\chi}=\frac{4}{2\chi-4\zeta+\xi-1}<0,\qquad \frac{\partial B_5(\chi,\zeta,\xi)}{\partial\zeta}=\frac{-8}{2\chi-4\zeta+\xi-1}>0,$$

and

$$\frac{\partial B_5(\chi,\zeta,\xi)}{\partial\xi}=\frac{2}{2\chi-4\zeta+\xi-1}<0,$$

the maximum of B_5 is attained at a corner of the boundary $\partial\Omega$. Comparing the values of B_5 at the corners of $\partial\Omega$, we see that the maximum is attained at $(\chi,\zeta,\xi)=(0.454,0.046,0)$ and $B_5(0.454,0.046,0)<-2.57$. $\square$

Claim 3.46. *For all* $(\chi, \zeta, \xi) \in \Omega$, $B_6(\chi, \zeta, \xi) < 0.14$.

Proof. For each $(\chi, \zeta, \xi) \in \Omega$, we have $10\chi - 5\zeta + \xi - \frac{7}{2} > 0$ since $10\chi - \frac{7}{2} \geq 1.04$ and $5\zeta \leq 0.23$. As for each $(\chi, \zeta, \xi) \in \Omega$,

$$\frac{\partial B_6(\chi,\zeta,\xi)}{\partial \chi} = \frac{10}{10\chi - 5\zeta + \xi - \frac{7}{2}} > 0, \qquad \frac{\partial B_6(\chi,\zeta,\xi)}{\partial \zeta} = \frac{-5}{10\chi - 5\zeta + \xi - \frac{7}{2}} < 0,$$

and

$$\frac{\partial B_6(\chi,\zeta,\xi)}{\partial \xi} = \frac{1}{10\chi - 5\zeta + \xi - \frac{7}{2}} > 0,$$

the maximum of B_5 is attained at a corner of the boundary $\partial\Omega$. Comparing the values of B_6 at the corners of $\partial\Omega$, we see that the maximum is attained at $(\chi, \zeta, \xi) = (0.45537, 0, 0.08926)$ and $B_6(0.454, 0, 0.08926) < 0.14$. ☐

By Claims 3.41–3.46, for each $(\chi, \zeta, \xi) \in \Omega$,

$$\begin{aligned}\frac{\partial \ln(h(\chi,\zeta,\xi))}{\partial \chi} &= B_1(\chi,\zeta) + B_2(\chi,\zeta) + B_3(\chi,\zeta,\xi) + B_4(\chi,\zeta) + B_5(\chi,\zeta,\xi) + B_6(\chi,\zeta,\xi) \\ &< -3.55 - 3.14 + 0 - 1.28 - 2.57 + 0.14 < 0. \quad \square\end{aligned}$$

3.3 Packing chromatic number of 1-subdivisions of cubic graphs

In Section 3.3, we give the first upper bound on $\chi_p(D(G))$ for subcubic G: we show that $\chi_p(D(G))$ is bounded by 8 in this class. We will prove the following slightly stronger result.

Theorem 3.47. *For every connected subcubic graph G, the graph $D(G)$ has a packing 8-coloring such that color 8 is used at most once.*

The theorem will be proved in the language of packing S-colorings introduced in [52] and used in [49, 53].

We will use the following observation of Gastineau and Togni [49].

Proposition 3.48 ([49] Proposition 1)**.** *Let G be a graph and $S = (s_1, ..., s_k)$ be a non-decreasing sequence of integers. If G is packing S-colorable then $D(G)$ is packing $(1, 2s_1+1, ..., 2s_k+1)$-colorable.*

In particular, if G is packing $(1,1,2,2,3,3)$-colorable, then $D(G)$ has a packing 7-coloring. In view of this, by a *feasible* coloring of G we call a coloring of G with colors $1_a, 1_b, 2_a, 2_b, 3_a, 3_b$ such that the distance between two vertices of color i_x is at least $i+1$ for all $1 \leq i \leq 3$ and $x \in \{a, b\}$.

In Section 3.3.1, we will show that if a 2-degenerate subcubic graph G has a feasible coloring f and v, u are vertices of G with degree at most 2, then we can change f to another feasible coloring with some control on the colors of v and u. The long proof of one of the lemmas, Lemma 3.52, is postponed till Section 3.3.4. Based on the lemmas of Section 3.3.1, in Section 3.3.2 we prove the following theorem (that gives a better bound than Theorem 3.47 but for a narrower class of graphs).

Theorem 3.49. *Every 2-degenerate subcubic graph G has a feasible coloring. In particular, $D(G)$ has a packing 7-coloring.*

In Section 3.3.3 we use Theorem 3.49 and the lemmas in Section 3.3.1 to derive Theorem 3.47. In Section 3.3.4 we present a proof of Lemma 3.52.

3.3.1 Lemmas on feasible coloring

Definition 3.50. *For a positive integer s and a vertex a in a graph G, the* ball $B_G(a,s)$ in G of radius s with center a *is $\{v \in V(G) : d_G(v,a) \le s\}$, where $d_G(v,a)$ denotes the distance in G from v to a.*

Lemma 3.51. *Let G be a subcubic graph and f be a feasible coloring of G. Suppose there are 2-vertices $u, v \in V(G)$ with $f(u) = f(v) = 2_a$. Let $N(u) = \{u_1, u_2\}$ and $N(v) = \{v_1, v_2\}$. Then G has a feasible coloring g satisfying one of the following:*
(a) $g(u) = 2_a$ and $g(v) \in \{1_a, 1_b\}$ or $g(v) = 2_a$ and $g(u) \in \{1_a, 1_b\}$;
(b) $\{g(u), g(v)\} = \{2_a, 2_b\}$;
(c) $\{g(u_1), g(u_2)\} = \{g(v_1), g(v_2)\} = \{1_a, 1_b\}$, and exactly one of u, v has color 2_a.

Proof. If $\{f(u_1), f(u_2)\} \neq \{1_a, 1_b\}$, then we recolor u with a color $\alpha \in \{1_a, 1_b\} - \{f(u_1), f(u_2)\}$, and (a) holds. Thus by the symmetry between u and v we may assume

$$f(u_1) = f(v_1) = 1_a \quad \text{and} \quad f(u_2) = f(v_2) = 1_b. \tag{3.65}$$

Since $f(u) = f(v) = 2_a$, $N(u) \cap N(v) = \emptyset$. In other words,

$$\text{all vertices } u_1, u_2, v_1 \text{ and } v_2 \text{ are distinct.} \tag{3.66}$$

Let G_1 denote the subgraph of G induced by the vertices of colors 1_a and 1_b. If u_1 and u_2 are in distinct components of G_1, then after switching the colors in the component of G_1 containing u_2, we obtain a coloring contradicting (3.65). Thus we may assume

$$G \text{ has a } 1_a, 1_b\text{-colored } u_1, u_2\text{-path } P_u \text{ and a } 1_a, 1_b\text{-colored } v_1, v_2\text{-path } P_v. \tag{3.67}$$

Case 1: $u_1u_2 \in E(G)$. If $|N(u_1)| = 3$, then let $u_3 \in N(u_1) - \{u, u_2\}$. Similarly, if $|N(u_2)| = 3$, then let $u_4 \in N(u_2) - \{u, u_1\}$. If $2_b \notin f(N(u_1) \cup N(u_2))$, then after recoloring u with 2_b we get a coloring satisfying (b). Thus we may assume

$$|N(u_1)| = 3 \text{ and } f(u_3) = 2_b. \tag{3.68}$$

Let $N(u_3) \subseteq \{u_1, u_5, u_6\}$. If $2_a \notin f(N(u_3))$, then since $f(u_4) \neq 2_a$ (because $d(u, u_4) = 2$) after switching the colors of u and u_1 we obtain a coloring satisfying (a). So we may assume $f(u_5) = 2_a$.

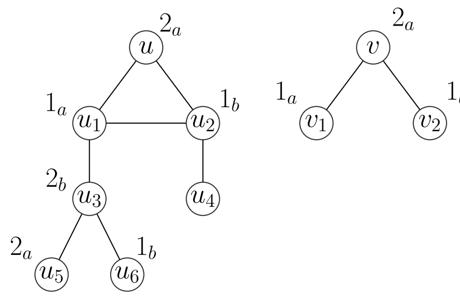


Figure 3.4: Case 1.1.

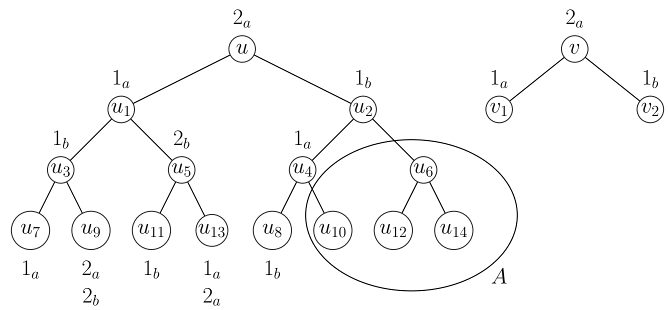


Figure 3.5: Case 2.1.

Case 1.1: $|N(u_2)| < 3$ or $f(u_4) \neq 2_b$. If $1_b \notin f(N(u_3))$, then we can recolor u_3 with 1_b. By the case, we can recolor u with 2_b to obtain a coloring satisfying (b). So we may assume $f(u_6) = 1_b$. Then the coloring g obtained from f by recoloring u and u_3 with 1_a and u_1 with 2_b satisfies (a).

Case 1.2: $|N(u_2)| = 3$ and $f(u_4) = 2_b$. If $u_4 = u_3$, then $N(u_3) = \{u_1, u_2, u_5\}$. Then u has no vertices of color 3_a at distance at most 3, so after recoloring u with 3_a, we obtain a coloring g satisfying (c). Thus, $u_4 \neq u_3$.

Case 1.2.1: $1_b \notin f(N(u_3))$. We recolor u_3 with 1_b. If $2_a \notin f(N(u_4) - u_2)$, then we recolor u_2 with 2_a and u with 1_b to obtain a coloring satisfying (a). If $1_a \notin f(N(u_4) - u_2)$, then we recolor u_4 with 1_a, u_2 with 2_b, and u with 1_b to obtain a coloring satisfying (a). Thus, we may assume

$$f(N(u_4) - u_2) = \{1_a, 2_a\}.$$

Then recoloring u_4 with 1_b, u_2 with 2_b, and u with 1_b, we obtain a coloring satisfying (a).

Case 1.2.2: $1_b \in f(N(u_3))$. Since $f(u_5) = 2_a$, this means u_6 exists and $f(u_6) = 1_b$. Then we recolor u_3 and u_2 with 1_a and u_1 with 1_b. If $2_a \notin f(N(u_4) - u_2)$, then we recolor u_2 with 2_a and u with 1_a to obtain a coloring satisfying (a). If $1_b \notin f(N(u_4) - u_2)$, then we recolor u_4 with 1_b and u with 2_b to obtain a coloring satisfying (b). Thus, we may assume

$$f(N(u_4) - u_2) = \{1_b, 2_a\}.$$

Then we recolor u_4 with 1_a, u_2 with 2_b, and u with 1_a to obtain a coloring satisfying (a).

Case 2: $u_1u_2 \notin E(G)$. Then we may assume that $N(u_1) \subseteq \{u, u_3, u_5\}$, $N(u_2) \subseteq \{u, u_4, u_6\}$ and by (3.67), $f(u_3) = 1_b$ and $f(u_4) = 1_a$. Furthermore, since by the case, $u_3 \neq u_2$, we may assume that $N(u_3) \subseteq \{u_1, u_7, u_9\}$ and $f(u_7) = 1_a$. It is possible that $u_7 = u_4$, but this will not affect the proof below. Similarly, we will assume that $N(u_4) \subseteq \{u_2, u_8, u_{10}\}$ and $f(u_8) = 1_b$. As in Case 1, $2_b \in f(N(u_1) \cup N(u_2))$, since otherwise we can recolor u with 2_b and (b) will hold. In our notation, this means $2_b \in \{f(u_5), f(u_6)\}$. By symmetry, we will assume $f(u_5) = 2_b$. We also will assume $N(u_5) \subseteq \{u_1, u_{11}, u_{13}\}$ and $N(u_6) \subseteq \{u_2, u_{12}, u_{14}\}$, where some vertices can coincide.

Case 2.1: $|N(u_2)| < 3$ or $f(u_6) \neq 2_b$. If $1_b \notin f(N(u_5))$, then we can recolor u_5 with 1_b, and then u with 2_b. The resulting coloring satisfies (b). So we may assume $f(u_{11}) = 1_b$. If $2_a \notin \{f(u_9), f(u_{13})\}$, then by switching the colors of u and u_1, we obtain a coloring satisfying (a). Thus $2_a \in \{f(u_9), f(u_{13})\}$. If $f(u_9) = 2_a$ and $f(u_{13}) \neq 1_a$ or if $f(u_{13}) = 2_a$ and $f(u_9) \neq 2_b$, then after switching the colors of u_1 and u_5 and recoloring u with 1_a, we again get a coloring satisfying (a). So,

$$\textit{either } f(u_9) = 2_a \textit{ and } f(u_{13}) = 1_a \textit{ or } f(u_{13}) = 2_a \textit{ and } f(u_9) = 2_b. \tag{3.69}$$

If u_6 does not exist, then by (3.69), the only vertex in $N(N(u)) \cup N(N(N(u)))$ that can be colored with 3_a or 3_b is u_{10}. Thus after recoloring u with a color in $\{3_a, 3_b\} - f(u_{10})$ we obtain a coloring satisfying (c). So suppose u_6 exists. Let $A = \{u_6, u_{10}, u_{12}, u_{14}\} \cap V(G)$. If $1_a \notin \{f(u_{12}), f(u_{14})\}$, then we can recolor u_6 with 1_a without changing color of any other vertex. Thus we may assume

$$1_a \in f(A). \tag{3.70}$$

If a color $x \in \{2_a, 2_b\}$ is not in $f(A)$, then after recoloring u_2 with x and u with 1_b, we get a coloring satisfying (a). Thus

$$2_a, 2_b \in f(A). \tag{3.71}$$

By the argument above, in particular, by (3.69), colors 3_a and 3_b are not used on vertices in $B = \{u_1, u_2, u_3, u_4, u_5, u_7, u_8, u_9, u_{11}, u_{13}\}$. If at least one of them, say 3_a, is also not used on A, then after recoloring u with 3_a, we obtain a coloring satisfying (c). Thus

$$3_a, 3_b \in f(A). \tag{3.72}$$

Since $|A| \leq 4$, relations (3.70), (3.71) and (3.72) cannot hold at the same time, a contradiction.

Case 2.2: $|N(u_2)| = 3$ and $f(u_6) = 2_b$. Suppose first that $u_6 = u_5$ and that $N(u_5) = \{u_1, u_2, u_{11}\}$. If $f(u_9) \neq 2_b$ and $f(u_{11}) \neq 1_a$, then after switching the colors of u_1 and u_5 and recoloring u with 1_a, we get a coloring satisfying (a). So, $f(u_9) = 2_b$ or $f(u_{11}) = 1_a$. Similarly, considering switching colors of u_2 and u_5, we obtain that $f(u_{10}) = 2_b$ or $f(u_{11}) = 1_b$. Together, this means

$$\textit{the colors of at least two vertices in } \{u_9, u_{10}, u_{11}\} \textit{ are in } \{1_a, 1_b, 2_b\}. \tag{3.73}$$

By (3.73), some color $y \in \{3_a, 3_b\}$ is not used on $B(u, 3)$. Then after recoloring u with y, we obtain a coloring satisfying (c).

Now we assume $u_6 \neq u_5$. If $1_a \notin \{f(u_{12}), f(u_{14})\}$, then after recoloring u_6 with 1_a, we get Case 2.1. Thus below we assume $f(u_{12}) = 1_a$. If $2_a \notin \{f(u_{10}), f(u_{14})\}$, then we obtain a coloring satisfying (a) by switching the colors of u and u_2. Thus, $2_a \in \{f(u_{10}), f(u_{14})\}$. If $f(u_{14}) \neq 1_b$ and $f(u_{10}) \neq 2_b$, then after switching the colors of u_2 and u_6 and recoloring u with 1_b, we again get a coloring satisfying (a). So,

$$\textit{either } f(u_{10}) = 2_a \textit{ and } f(u_{14}) = 1_b \textit{ or } f(u_{10}) = 2_b \textit{ and } f(u_{14}) = 2_a. \tag{3.74}$$

Let $A = \{u_9, u_{11}, u_{13}\} \cap V(G)$. If $2_a \notin f(A)$, then we obtain a coloring satisfying (a) by switching the colors of u and u_1. Thus,

$$2_a \in f(A). \tag{3.75}$$

If $1_a \notin f(\{u_{11}, u_{13}\})$ and $f(u_9) \neq 2_b$, then after switching the colors of u_1 and u_5 and recoloring u with 1_a, we again get a coloring satisfying (a). Therefore,

$$1_a \in f(\{u_{11}, u_{13}\}) \text{ or } f(u_9) = 2_b. \tag{3.76}$$

By the argument above, in particular, by (3.74), colors 3_a and 3_b are not used on vertices in $B = \{u_1, u_2, u_3, u_4, u_5, u_7, u_8, u_{10}, u_{12}, u_{14}\}$. If at least one of them, say 3_a, is also not used on A, then after recoloring u with 3_a, we obtain a coloring satisfying (c). Thus,

$$3_a, 3_b \in f(A). \tag{3.77}$$

But $|A| \leq 3$, relations (3.75), (3.76), and (3.77) cannot hold at the same time, a contradiction. □

Our second lemma is:

Lemma 3.52. *Let G be a subcubic graph and f be a feasible coloring of G. Suppose there is a 2-vertex $u \in V(G)$ with $N(u) = \{u_1, u_2\}$. If $f(u) \in \{3_a, 3_b\}$, then G has a feasible coloring g satisfying the following:*
(a) $g(u) \notin \{3_a, 3_b\}$, and
(b) at most one vertex is recolored into 3_a or 3_b, and this vertex (if there is one such vertex) is at distance at most 3 from u and has degree 3 in G, and at most one vertex of f-color 3_a or 3_b apart from u is recolored into some other color, and this vertex (if there is one such vertex) has new color in $\{1_a, 1_b\}$.

The proof of this lemma is a long case analysis, so we postpone it to Section 3.3.4.

3.3.2 Proof of Theorem 3.49

We prove the theorem by induction on the number of vertices. When $n \leq 6$, the claim holds obviously, since we have 6 colors. When $n > 6$, we assume the argument holds for every graph with fewer than n vertices. Let G be any 2-degenerate subcubic graph with n vertices. We may assume G is connected. Let w be a vertex in G such that $d(w) \leq 2$.

Case 1: $d(w) = 1$ and let $N(w) = w'$. Since $G - w$ is an $(n-1)$-vertex connected subcubic graph with $d_{G-w}(w') \leq 2$, by the induction hypothesis, $G - w$ has a packing $(1, 1, 2, 2, 3, 3)$-coloring f. We color w with a color $x \in \{1_a, 1_b\} - f(w')$ to extend f to G.

Case 2: $d(w) = 2$ and let $N(w) = \{w_1, w_2\}$. Note that $G-w$ has at most two connected components and each connected component is a connected subcubic graph with less than n vertices and is 2-degenerate. By the induction hypothesis, $G - w$ has a feasible coloring f. We may assume that $|N_{G-w}(w_1)| = |N_{G-w}(w_2)| = 2$. Otherwise we can first apply the induction hypothesis to obtain a packing $(1, 1, 2, 2, 3, 3)$-coloring f on $G - w$, then add leaves to w_1 and w_2 to obtain a new graph G' with $|N_{G'-w}(w_1)| = |N_{G'-w}(w_2)| = 2$, then assign proper colors to those leaves we just added to obtain a packing $(1, 1, 2, 2, 3, 3)$-coloring f' on $G' - w$, then prove that G' has a packing $(1, 1, 2, 2, 3, 3)$-coloring, which can be used to get our desired coloring on G. So below we assume $N(w_1) = \{w, w_3, w_4\}$ and $N(w_2) = \{w, w_5, w_6\}$

By Lemma 3.52, $G - w$ has a feasible coloring f_1 such that $f_1(w_1) \notin \{3_a, 3_b\}$. Then by Lemma 3.52 again, $G - w$ also has a feasible coloring f_2 such that $f_2(w_2) \notin \{3_a, 3_b\}$ and no vertex of degree 2 in $G - w$ changed its color to 3_a or 3_b. Thus we also have $f_2(w_1) \notin \{3_a, 3_b\}$.

Case 2.1: Either $f_2(w_1) \neq f_2(w_2)$ or $f_2(w_1) = f_2(w_2) \in \{1_a, 1_b\}$. If $\{f_2(w_1), f_2(w_2)\} \neq \{1_a, 1_b\}$, then we extend f_2 to G by assigning $f_2(w) = \alpha \in \{1_a, 1_b\} - \{f_2(w_1), f_2(w_2)\}$. By the case, if $f_2(w_1) = f_2(w_2)$, then $f_2(w_1) = f_2(w_2) \in \{1_a, 1_b\}$. Therefore, the extension of f_2 to G is feasible since we do not introduce new conflicts between w_1 and w_2 by adding w. Thus, we may assume

$$f_2(w_1) = 1_a \quad \text{and} \quad f_2(w_2) = 1_b. \tag{3.78}$$

If w_1 and w_2 are in distinct components of the subgraph G_2 of $G - w$ formed by the vertices of colors 1_a and 1_b in f_2, then after switching the colors 1_a and 1_b with each other in the component of G_2 containing w_2, we obtain a coloring contradicting (3.78). Thus we may assume

$$G - w \textit{ has a } 1_a, 1_b\textit{-colored } w_1, w_2\textit{-path } P_w. \tag{3.79}$$

In particular, we may assume $f_2(w_3) = 1_b$ and $f_2(w_5) = 1_a$ (possibly, $w_3 = w_2$ and then $w_5 = w_1$).

If $\{2_a, 2_b\} \not\subseteq f_2(N(w_1) \cup N(w_2) - \{w\})$, then we can extend f_2 to G by assigning $f_2(w) = \beta \in \{2_a, 2_b\} - f_2(N(w_1) \cup N(w_2) - \{w\})$. Thus, we may assume

$$|N(w_1)| = |N(w_2)| = 3, \{2_a, 2_b\} \subset f_2(N(w_1) \cup N(w_2) - \{w\}), \text{ and by symmetry} \tag{3.80}$$

$$f_2(w_4) = 2_a \quad \text{and} \quad f_2(w_6) = 2_b. \tag{3.81}$$

If $1_b \notin f_2(N(w_4) - w_2)$, then we can extend f_2 to a feasible coloring of G by recoloring w_4 with 1_b and letting $f_2(w) = 2_a$. By this and the symmetric statement for w_6 we can assume that

$$w_4 \textit{ has a neighbor } w_7 \textit{ with } f_2(w_7) = 1_b \textit{ and } w_6 \textit{ has a neighbor } w_8 \textit{ with } f_2(w_8) = 1_a. \tag{3.82}$$

Case 2.1.1: $w_1 w_2 \in E(G)$ (i.e., $w_3 = w_2$ and $w_5 = w_1$). If $1_a \notin f_2(N(w_4) - w_1)$, then we obtain a feasible coloring on G by switching colors of w_1 and w_4, assigning 1_a to w, and using f_2 on other vertices. Therefore, by (3.82), we may assume $f_2(N(w_4) - w_1) = \{1_a, 1_b\}$. Similarly, by (3.82), we may assume $f_2(N(w_6) - w_2) = \{1_a, 1_b\}$. With (3.78), (3.81), and the case, $3_a \notin f_2(B(w,3) - \{w\})$ and we can extend f_2 to G by assigning $f_2(w) = 3_a$.

Case 2.1.2: $w_1 w_2 \notin E(G)$. If $N(w_3) \cup N(w_4)$ does not contain a vertex w_9 of color 2_b, then we can recolor w_1 with 2_b and color w with 1_a. So we may assume that $N(w_3) \cup N(w_4)$ contains a vertex w_9 of color 2_b and symmetrically $N(w_5) \cup N(w_6)$ contains a vertex w_{10} of color 2_a. Furthermore, if $1_a \notin f_2(N(w_4) - w_1)$ and $2_a \notin f_2(N(w_3) - w_1)$, then we can recolor w_1 with 2_a and color w and w_4 with 1_a. With (3.79) and (3.82), all vertices in $B(w_1, 2) - w$ have colors in $\{1_a, 1_b, 2_a, 2_b\}$. Symmetrically, we can assume all vertices in $B(w_2, 2) - w$ have colors in $\{1_a, 1_b, 2_a, 2_b\}$. Then we can color w with 3_a.

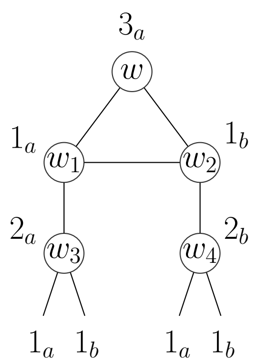


Figure 3.6: Case 2.1.1.

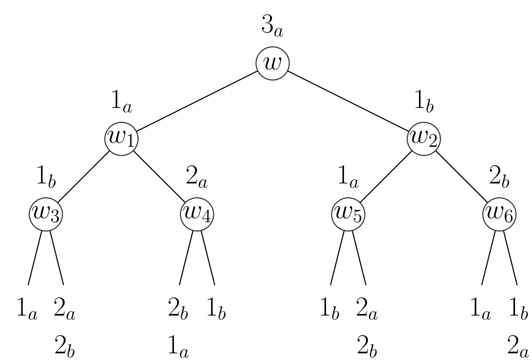


Figure 3.7: Case 2.1.2.

By the choice of f_2 and the symmetry of 2_a and 2_b, the remaining case is:

Case 2.2: $f_2(w_1) = f_2(w_2) = 2_a$. In particular, this means $w_1 w_2 \notin E(G)$. By Lemma 3.51, $G - w$ has a coloring g satisfying one of the following:
(a) $g(w_1) = 2_a$ and $g(w_2) \in \{1_a, 1_b\}$ or $g(w_2) = 2_a$ and $g(w_1) \in \{1_a, 1_b\}$;
(b) $\{g(w_1), g(w_2)\} = \{2_a, 2_b\}$;
(c) $\{g(w_3), g(w_4)\} = \{g(w_5), g(w_6)\} = \{1_a, 1_b\}$, and exactly one of w_1, w_2 has color 2_a.

If (a) or (b) occurs, then we again get Case 1. We do not get Case 1 only if (c) occurs and one of w_1, w_2 has g-color in $\{3_a, 3_b\}$. But then 2_b is not present in $B(w, 2)$ and we can color w with 2_b. □

3.3.3 Cubic graphs

A *good* coloring is a packing $(1, 1, 2, 2, 3, 3, k)$-coloring for an integer $k \geq 4$ and color k used at most once. By Proposition 3.48, Theorem 3.47 follows from the following fact.

Theorem 3.53. *Every connected cubic graph has a good coloring.*

Proof. Let G be a connected cubic graph with $n \geq 2$ vertices. Since G is connected, it has a non-cut vertex w (simply take a leaf vertex of a spanning tree of G). Let $N(w) = \{w_1, w_2, w_3\}$.

Case 1: $0 \le |E(G[\{w_1, w_2, w_3\}])| \le 1$. If $|E(G[\{w_1, w_2, w_3\}])| = 0$, then let $G' = G - w + w_2w_3$. If $|E(G[\{w_1, w_2, w_3\}])| = 1$, then by symmetry we may assume $w_2w_3 \in E(G)$. Let $G' = G - w$. Note that G' is a connected subcubic graph with vertex w_1 of degree at most two. By Theorem 3.49, G' has a feasible coloring. Hence by Lemma 3.52, G' has a feasible coloring f with

$$f(w_1) \notin \{3_a, 3_b\}. \tag{3.83}$$

Let $N_{G'}(w_1) = \{w_4, w_5\}$, $N_{G'}(w_2) = \{w_3, w_6, w_7\}$, and $N_{G'}(w_3) = \{w_2, w_8, w_9\}$. It is possible that $|\{w_4, w_5, w_6, w_7, w_8, w_9\}| < 6$, but this will not affect the proof below.

For $j \in \{1, 2, 3\}$ and $x, y \in V(G) - w$, a (j, x, y)-*conflict* in (G, f) is the situation that $f(x) = f(y) \in \{j_a, j_b\}$ and $d_G(x, y) \le j$. If (G, f) has no (j, x, y)-conflicts for any $j \in \{1, 2, 3\}$ and $x, y \in V(G) - w$, then we can extend f to a good coloring of G by letting $f(w) = k$.

Suppose now that (G, f) has a (j, x, y)-conflict for some $j \in \{1, 2, 3\}$ and $x, y \in V(G) - w$ (there could be more than one conflict). Then

$$d_G(x, y) \le j < d_{G'}(x, y). \text{ This means } \{x, y\} \cap \{w_1, w_2, w_3\} \ne \emptyset \text{ and } j \ge 2. \tag{3.84}$$

Since $w_2w_3 \in E(G')$, (3.84) yields that in each (j, x, y)-conflict, one of x and y is in $\{w_1, w_4, w_5\}$ and the other is in $\{w_2, w_3, w_6, w_7, w_8, w_9\}$. By (3.83), we have the following two cases.

Case 1.1: $f(w_1) \in \{1_a, 1_b\}$, say $f(w_1) = 1_a$. Then each conflict is a $(3, x, y)$-conflict.

Case 1.1.1: There is only one conflict. We may assume it is a $(3, w_4, w_2)$-conflict, where $f(w_4) = f(w_2) = 3_a$. If $f(N_G(w_2) - w) \ne \{1_a, 1_b\}$, then we can recolor w_2 with one of 1_a and 1_b and eliminate the conflict. If $f(w_3) \ne 1_b$, then we can recolor w_4 with k and color w with 1_b. So we may assume

$$f(N_G(w_2) - w) = \{1_a, 1_b\} \quad \textit{and} \quad f(w_3) = 1_b. \tag{3.85}$$

Furthermore, if $f(w_5) \ne 1_b$ or $1_a \notin f(N_G(w_3) - w)$, then we can recolor w_1 and w_3 with the same color $\alpha \in \{1_a, 1_b\}$, recolor w_4 with k and color w with $\beta \in \{1_a, 1_b\} - \alpha$. Otherwise, some $\gamma \in \{2_a, 2_b\}$ is not present on $N(w_3) \cup \{w_5\}$, and by (3.85) we can recolor w_4 with k and color w with γ.

Case 1.1.2: There are two conflicts. By the case and symmetry, we may assume $f(w_4) = f(w_2) = 3_a$ and $f(w_5) = f(w_3) = 3_b$. Applying Lemma 3.52 to vertex w_2 and coloring f of $G - w$, we obtain a feasible coloring g of $G - w$ such that $g(w_2) = \gamma \notin \{3_a, 3_b\}$ and at most one of w_3, w_4, w_5 changed its color.

Case 1.1.2.1: Neither w_4 nor w_5 changed its color. Then we color w_3 with color k, w with a color $\beta \in \{1_a, 1_b\} - \gamma$, w_1 with a color $\alpha \in \{1_a, 1_b\} - \beta$, and use g on other vertices.

Case 1.1.2.2: One vertex of $\{w_4, w_5\}$ changed its color. We prove the case when w_4 changed its color, say $g(w_4) = \beta \in \{1_a, 1_b\}$, the case w_5 changed its color is similar. We may assume that

$$g(w_2) = \gamma \in \{1_a, 1_b\} \quad \text{and} \quad \gamma = \beta, \tag{3.86}$$

since otherwise we color w_1 with a color $\alpha \in \{1_a, 1_b\} - \beta$, w with a color $\mu \in \{1_a, 1_b\} - \alpha$, w_3 with color k, and use g on other vertices. We may also assume that some vertex, say $w_6 \in N(w_2) - w$, have color $\delta \in \{1_a, 1_b\} - \gamma$, since otherwise we recolor w_2 with δ and it contradicts (3.86). We may also assume that $g(\{w_8, w_9\}) = \{1_a, 1_b\}$, since otherwise we color w_3 with a color $\mu \in \{1_a, 1_b\} - g(\{w_8, w_9\})$, w with color k, and use f on other vertices. Note that $|g(N(w) \cup N(N(w))) \cap \{2_a, 2_b\}| \leq 1$. Then we color w_1 with a color $\alpha \in \{1_a, 1_b\} - \beta$, w_3 with color k, w with a color $\lambda \in \{2_a, 2_b\} - g(N(w) \cup N(N(w)))$, and use g on other vertices to obtain a good coloring.

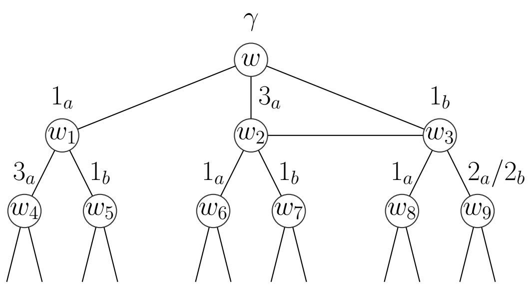


Figure 3.8: Case 1.1.1.

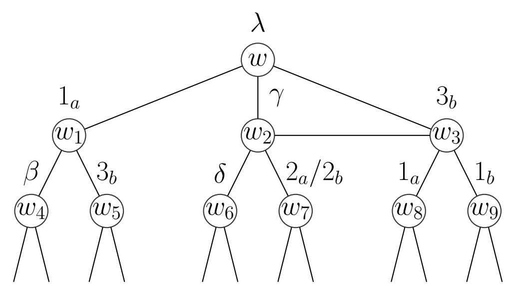


Figure 3.9: Case 1.1.2.2.

Case 1.2: $f(w_1) \in \{2_a, 2_b\}$, say $f(w_1) = 2_a$. Since we cannot switch to Case 1.1, we need $\{f(w_4), f(w_5)\} = \{1_a, 1_b\}$. So the only possible conflict is a $(2, w_1, y)$-conflict, where $y \in \{w_2, w_3\}$. We may assume $f(w_2) = 2_a$. Then we recolor w_1 with k and color w with $\alpha \in \{1_a, 1_b\} - f(w_3)$.

Case 2: $|E(G[\{w_1, w_2, w_3\}])| = 2$, say $w_1w_2 \in E(G)$ and $w_2w_3 \in E(G)$. We obtain a good coloring g of G by using f on $G - w$ and assigning color k to w. Note that adding w back will not create conflicts because the distance between any two vertices in $G - w$ remains the same.

Case 3: $G[\{w_1, w_2, w_3\}] = K_3$. Then $G = K_4$, and K_4 has a good coloring. $\square$

3.3.4 Proof of Lemma 3.52

Recall the claim of the lemma:

Lemma 3.52. *Let G be a subcubic graph and f be a feasible coloring of G. Suppose there is a 2-vertex $u \in V(G)$ with $N(u) = \{u_1, u_2\}$. If $f(u) \in \{3_a, 3_b\}$, then G has a feasible coloring g satisfying the following:*
(a) $g(u) \notin \{3_a, 3_b\}$, and
(b) at most one vertex is recolored into 3_a or 3_b, and this vertex (if there is one such vertex) is at distance at most 3 from u and has degree 3 in G, and at most one vertex of f-color 3_a or 3_b apart from u is recolored into some other color, and this vertex (if there is one such vertex) has new color in $\{1_a, 1_b\}$.

Proof. Without loss of generality, we assume that $f(u) = 3_a$. If $\{f(u_1), f(u_2)\} \neq \{1_a, 1_b\}$, then we recolor u with a color $x \in \{1_a, 1_b\} - \{f(u_1), f(u_2)\}$ to obtain a coloring satisfying (a) and (b). Thus we may assume

$$f(u_1) = 1_a \quad \text{and} \quad f(u_2) = 1_b. \tag{3.87}$$

Let G_1 denote the subgraph of G induced by the vertices of colors 1_a and 1_b. If u_1 and u_2 are in distinct components of G_1, then after switching the colors in the component of G_1 containing u_2, we obtain a coloring contradicting (3.87). Thus we may assume

$$G \textit{ has a } 1_a, 1_b\textit{-colored } u_1, u_2\textit{-path } P_u. \tag{3.88}$$

Case 1: $u_1u_2 \in E(G)$. If $|N(u_1)| = 3$, then let $u_3 \in N(u_1) - \{u, u_2\}$. Similarly, if $|N(u_2)| = 3$, then let $u_4 \in N(u_2) - \{u, u_1\}$. If $\{2_a, 2_b\} \nsubseteq f(N(u_1) \cup N(u_2))$, then after recoloring u with a color $x \in \{2_a, 2_b\} - f(N(u_1) \cup N(u_2))$ we obtain a coloring satisfying (a) and (b). By symmetry, we may assume

$$|N(u_1)| = |N(u_2)| = 3, \quad f(u_3) = 2_a \quad \textit{and} \quad f(u_4) = 2_b. \tag{3.89}$$

If $1_b \notin f(N(u_3))$, then we can recolor u_3 with 1_b and u with 2_a to obtain a coloring satisfying (a) and (b). So we may assume $1_b \in f(N(u_3))$. Similarly, we may assume $1_a \in f(N(u_4))$. If $|N(u_3)| = 2$ or $1_a \notin f(N(u_3) - \{u_1\})$, then we can recolor u_3 with 1_a, u_1 with 2_a, and u with 1_a to obtain a coloring satisfying (a) and (b). So we may assume

$$|N(u_3)| = 3 \text{ and let } u_5, u_6 \in N(u_3) - \{u_1\} \text{ with } f(u_5) = 1_a, f(u_6) = 1_b. \tag{3.90}$$

Similarly, we may assume

$$|N(u_4)| = 3 \text{ and let } u_7, u_8 \in N(u_4) - \{u_2\} \text{ with } f(u_7) = 1_a, f(u_8) = 1_b. \tag{3.91}$$

Case 1.1: $u_5 = u_7$ and $u_6 = u_8$. If $1_b \notin f(N(u_5))$, then we can recolor u_5 with 1_b, u_3 with 1_a, u_1 with 2_a, and u with 1_a to obtain a coloring satisfying (a) and (b). So we may assume $1_b \in f(N(u_5))$. Similarly, we may assume $1_a \in f(N(u_6))$. Then we can recolor u_1 with 3_a and u with 1_a to obtain a coloring satisfying (a) and (b).

Case 1.2: $u_5 = u_7$ or $u_6 = u_8$, but not both. By symmetry, we may assume $u_6 = u_8$ and $u_5 \neq u_7$. It is possible that $u_5u_6 \in E(G)$ or $u_6u_7 \in E(G)$, but this will not affect the proof below.

Similarly to Case 1.1, we may assume

$$1_b \in f(N(u_5)), \quad 1_a \in f(N(u_6)) \quad \text{and} \quad 1_b \in f(N(u_7)). \tag{3.92}$$

By $3_a \notin f(N(u_6))$, we can also assume $3_a \in f(N(u_5))$, because otherwise we recolor u_1 with 3_a and u with 1_a to obtain a coloring satisfying (a) and (b). With (3.89) and (3.92), we have $f(N(u_5)) = \{1_b, 2_a, 3_a\}$. However, we can recolor u_1 with 3_b and u with 1_a to obtain a coloring satisfying (a) and (b).

Case 1.3: $u_5 \neq u_7$ and $u_6 \neq u_8$. Then $N(u_3) \cap N(u_4) = \emptyset$ and $d(u_3, u_4) \geq 3$. Similarly to Case 1.2,

$\{1_a, 1_b, 3_a, 3_b\} \subset f(N(u_5) \cup N(u_6) - \{u_3\})$. Therefore, we can recolor u_3 with 2_b and u with 2_a to obtain a coloring satisfying (a) and (b).

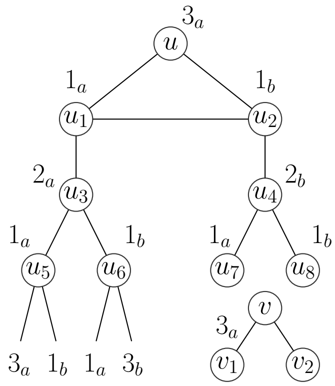


Figure 3.10: Case 1.3.

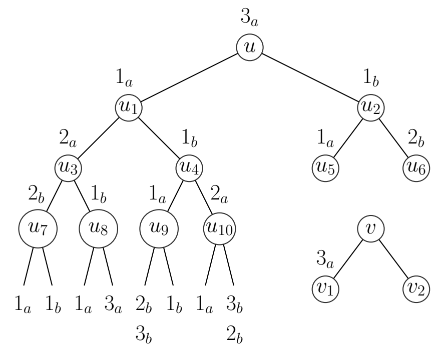


Figure 3.11: Case 2.1.

Case 2: $u_1u_2 \notin E(G)$. If $\{2_a, 2_b\} \nsubseteq f(N(u_1) \cup N(u_2))$, then after recoloring u with a color $x \in \{2_a, 2_b\} - f(N(u_1) \cup N(u_2))$ we obtain a coloring satisfying (a) and (b). With (3.88), we may assume that

$$N(u_1) = \{u, u_3, u_4\}, \quad f(u_3) = 2_a, \quad f(u_4) = 1_b, \tag{3.93}$$

$$N(u_2) = \{u, u_5, u_6\}, \quad f(u_5) = 1_a \quad \text{and} \quad f(u_6) = 2_b. \tag{3.94}$$

If $u_3u_4 \in E(G)$, then $1_a \in f(N(u_4) - \{u_1, u_3\})$ because of (3.88). We also have $2_b \in f(N(u_3) - \{u_1, u_4\})$ because otherwise we can recolor u_1 with 2_b and u with 1_a to obtain a coloring satisfying (a) and (b). Thus, we may assume $|N(u_3)| = |N(u_4)| = 3$ and let $u_7 \in N(u_3) - \{u_1, u_4\}, u_8 \in N(u_4) - \{u_1, u_3\}$, $f(u_7) = 2_b$, and $f(u_8) = 1_a$. Then, we can recolor u_1 with 2_a, u_3 with 1_a, and u with 1_a to obtain a coloring satisfying (a) and (b). Because of symmetry, we may assume

$$u_3u_4 \notin E(G) \quad \text{and} \quad u_5u_6 \notin E(G). \tag{3.95}$$

If $1_b \notin f(N(u_3))$, then we recolor u_3 with 1_b and u with 2_a to obtain a coloring satisfying (a) and (b). With (3.88), we may assume that

$$1_b \in f(N(u_3)) \quad \text{and} \quad 1_a \in f(N(u_4)). \tag{3.96}$$

If $2_b \notin f(B(u_1, 2))$, then we can recolor u_1 with 2_b and u with 1_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$2_b \in f(N(u_3)) \cup f(N(u_4)). \tag{3.97}$$

If $1_a \notin f(N(u_3) - \{u_1\})$ and $2_a \notin f(N(u_4))$, then we can recolor u_3 with 1_a, u_1 with 2_a, and u with

1_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$|N(u_3)| = |N(u_4)| = 3 \tag{3.98}$$

and

$$1_a \in f(N(u_3) - \{u_1\}) \text{ or } 2_a \in f(N(u_4)). \tag{3.99}$$

Let $\{u_7, u_8\} \in N(u_3)$, $\{u_9, u_{10}\} \in N(u_4)$. By (3.96), we may assume

$$f(u_8) = 1_b \quad \text{and} \quad f(u_9) = 1_a. \tag{3.100}$$

By (3.97) and (3.99), we have

$$\textit{either } f(u_7) = 2_b \textit{ and } f(u_{10}) = 2_a \textit{ or } f(u_7) = 1_a \textit{ and } f(u_{10}) = 2_b. \tag{3.101}$$

If $3_a \notin f(B(u_1, 3) - \{u\})$, then we can recolor u_1 with 3_a and u with 1_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$3_a \in f(B(u_1, 3) - \{u\}). \tag{3.102}$$

Similarly, we may assume

$$3_b \in f(B(u_1, 3) - \{u\}). \tag{3.103}$$

Case 2.1: $f(u_7) = 2_b$ and $f(u_{10}) = 2_a$. By (3.95) and $|N(u_2)| = 3$, we have

$$\{u_8, u_{10}\} \cap (\{u_i : i \in [6]\} \cup \{u\}) = \emptyset.$$

It is possible that $u_9 = u_5$ or $u_7 = u_6$, but this will not affect the proof below.

If $2_b \notin f(B(u_4, 2))$, then we can recolor u_4 with 2_b, u_1 with 1_b, and u with 1_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$2_b \in f(B(u_4, 2)). \tag{3.104}$$

If $1_a \notin f(N(u_{10}))$, then we can recolor u_{10} with 1_a and it contradicts (3.99). Thus, we may assume

$$1_a \in f(N(u_{10})). \tag{3.105}$$

We may also assume

$$f(N(u_7) - \{u_3\}) = \{1_a, 1_b\}, \tag{3.106}$$

because otherwise we can recolor u_7 with a color $x \in \{1_a, 1_b\} - f(N(u_7) - \{u_1\})$ and it contradicts (3.101). By (3.102) and (3.103), we know that

$$\{3_a, 3_b\} \subset f(N(u_7) \cup N(u_8) \cup N(u_9) \cup N(u_{10})). \tag{3.107}$$

If $\{3_a, 3_b\} \subset f(N(u_7) \cup N(u_8))$, then by (3.106) we have $f(N(u_8)) = \{2_a, 3_a, 3_b\}$. Then, we can recolor u_8 with 1_a, u_3 with 1_b, and u with 2_a to obtain a coloring satisfying (a) and (b). By symmetry, we may assume

$$3_b \notin f(N(u_7) \cup N(u_8)). \tag{3.108}$$

By (3.107) and (3.108), we know that $3_b \in f(N(u_9) \cup N(u_{10}))$. By (3.88), $1_b \in f(N(u_9) - \{u_4\})$. With (3.104), (3.105), and $2_b \notin f(\{u, u_1, u_3, u_9, u_{10}\})$ we know that

$$f(N(u_9) \cup N(u_{10}) - \{u_4\}) = \{1_a, 1_b, 2_b, 3_b\}, \text{ hence } 1_b \notin f(N(u_{10}) - \{u_4\}).$$

Therefore, we can recolor u_4 with 2_a, u_{10} with 1_b, u_3 with 1_a, u_1 with 1_b, and u with 1_a to obtain a coloring satisfying (a) and (b).

Case 2.2: $f(u_7) = 1_a$ and $f(u_{10}) = 2_b$. If $1_a \notin f(N(u_6))$, then we can recolor u_6 with 1_a and u with 2_b to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$1_a \in f(N(u_6) - \{u_2\}). \tag{3.109}$$

Case 2.2.1: $u_3u_5 \in E(G)$, i.e., $u_7 = u_5$. It is possible that $u_4u_6 \in E(G)$, or $u_4u_5 \in E(G)$, or $\{u_4u_5, u_4u_6\} \subset E(G)$, but this will not affect the proof below. By (3.88),

$$1_b \in f(N(u_9) - \{u_4\}), \tag{3.110}$$

and

$$1_b \in f(N(u_5) - \{u_2\}). \tag{3.111}$$

If $1_a \notin f(N(u_{10}) - \{u_4\})$, then we can recolor u_{10} with 1_a and it contradicts (3.101). Thus, we may assume

$$1_a \in f(N(u_{10}) - \{u_4\}). \tag{3.112}$$

If $1_a \notin f(N(u_8))$, then we can recolor u_8 with 1_a, u_3 with 1_b, and u with 2_a to obtain a coloring satisfying (a) and (b). If $2_b \notin f(N(u_8))$, then we can recolor u_3 with 2_b and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$f(N(u_8)) = \{1_a, 2_a, 2_b\}. \tag{3.113}$$

By (3.102), (3.103), (3.110), (3.111), (3.112), and (3.113), we have

$$\{1_a, 1_b, 3_a, 3_b\} \subset f(N(u_9) \cup N(u_{10}) - \{u_4\}). \tag{3.114}$$

By (3.114), $1_b \notin f(N(u_{10}) - \{u_4\})$, and $2_b \notin f(B(u_4, 2) - \{u_{10}\})$. Then, we can recolor u_{10} with 1_b, u_4 with 2_b, u_1 with 1_b, and u with 1_a to obtain a coloring satisfying (a) and (b).

With Case 2.2.1 handled, from now on by symmetry we may assume

$$u_3u_5 \notin E(G) \quad \text{and} \quad u_4u_6 \notin E(G). \tag{3.115}$$

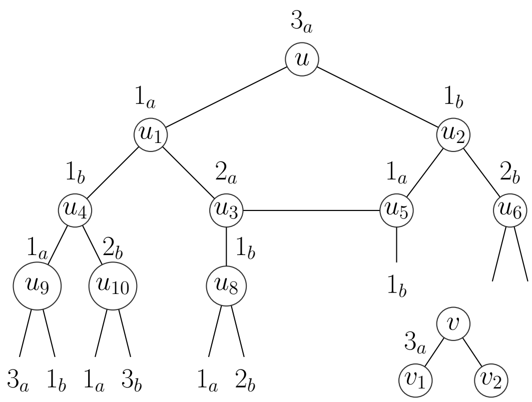


Figure 3.12: Case 2.2.1.

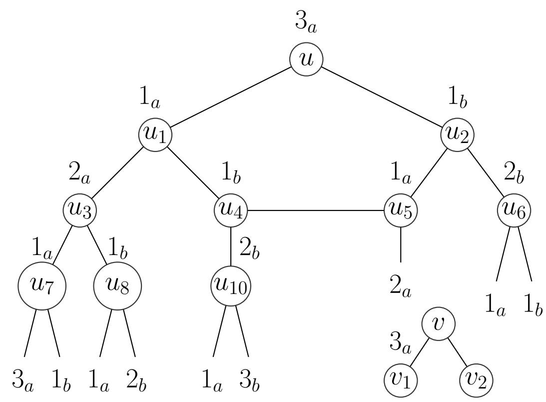


Figure 3.13: Case 2.2.2.

Case 2.2.2: $\{u_3u_5, u_4u_6\} \cap E(G) = \emptyset$ and $u_4u_5 \in E(G)$, i.e., $u_9 = u_5$. If $2_a \notin f(N(u_5) \cup N(u_6))$, then we can recolor u_2 with 2_a and u with 1_b to obtain a coloring satisfying (a) and (b). If $1_b \notin f(N(u_6) - \{u_2\})$ and $2_b \notin f(N(u_5) - \{u_2, u_4\})$, then we can recolor u_6 with 1_b, u_2 with 2_b, and u with 1_b to obtain a coloring satisfying (a) and (b). With (3.109), we know

$$f(N(u_5) - \{u_2, u_4\}) = \{2_a\} \quad \text{and} \quad f(N(u_6) - \{u_2\}) = \{1_a, 1_b\}$$

$$\text{or } f(N(u_5) - \{u_2, u_4\}) = \{2_b\} \quad \text{and} \quad f(N(u_6) - \{u_2\}) = \{1_a, 2_a\}.$$

If $f(N(u_5) - \{u_2, u_4\}) = \{2_b\}$ and $f(N(u_6) - \{u_2\}) = \{1_a, 2_a\}$, then we recolor u_5 with 2_a, u_2 with 1_a, and u with 1_b to obtain a coloring satisfying (a) and (b). Thus, we can assume that

$$f(N(u_5) - \{u_2, u_4\}) = \{2_a\} \quad \text{and} \quad f(N(u_6) - \{u_2\}) = \{1_a, 1_b\}. \tag{3.116}$$

If $1_b \notin f(N(u_7) - \{u_3\})$, then we can recolor u_7 with 1_b and it contradicts (3.101). Thus, we may assume

$$1_b \in f(N(u_7) - \{u_3\}). \tag{3.117}$$

If $1_a \notin f(N(u_8) - \{u_3\})$, then we can recolor u_8 with 1_a and it contradicts (3.100). If $1_a \notin f(N(u_{10}) - \{u_4\})$, then we can recolor u_{10} with 1_a and it contradicts (3.101). Therefore, we may assume

$$1_a \in f(N(u_{10}) - \{u_4\}) \quad \text{and} \quad 1_a \in f(N(u_8) - \{u_3\}). \tag{3.118}$$

If $2_b \notin f(N(u_7) \cup N(u_8) - \{u_3\})$, then we can recolor u_3 with 2_b and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$2_b \in f(N(u_7) \cup N(u_8) - \{u_3\}). \tag{3.119}$$

By previous arguments, we know that $\{3_a, 3_b\} \cap f(\{u_2, u_3, u_4, u_5, u_6, u_7, u_8, u_{10}\}) = \emptyset$. With (3.102), (3.103), and (3.116), we know that $\{3_a, 3_b\} \subset f(N(u_7) \cup N(u_8) \cup N(u_{10}) - \{u_3, u_4\})$. Moreover, by (3.117), (3.118), (3.119), and symmetry, we may assume that

$$f(N(u_{10}) - \{u_4\}) = \{1_a, 3_b\}.$$

But we can recolor u_{10} with 1_b, u_4 with 2_b, u_1 with 1_b, and u with 1_a to obtain a coloring satisfying (a) and (b).

Case 2.2.3: $\{u_3u_5, u_4u_6, u_4u_5\} \cap E(G) = \emptyset$ and $u_4u_7 \in E(G)$, i.e., $u_7 = u_9$. If $1_a \notin f(N(u_8) - u_3)$, then we recolor u_8 with 1_a, u_3 with 1_b, and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we may assume $1_a \in f(N(u_8) - u_3)$. If $1_a \notin f(N(u_{10}) - u_4)$, then we recolor u_{10} with 1_a, u_1 with 2_b, and u with 1_a to obtain a coloring satisfying (a) and (b). Thus, we may also assume $1_a \in f(N(u_{10}) - u_4)$. If $2_b \notin f(N(u_7) \cup N(u_8) - \{u_3, u_4\})$, then we recolor u_3 with 2_b and u with 2_a to obtain a coloring satisfying (a) and (b). With (3.102), (3.103), and symmetry, we may assume $f(N(u_7) \cup N(u_8) - \{u_3, u_4\}) = \{1_a, 2_b, 3_a\}$ and $f(N(u_{10}) - u_4) = \{1_a, 3_b\}$. We recolor u_7 with 1_b, u_4 with 1_a, u_1 with 1_b, and u with 1_a to obtain a coloring satisfying (a) and (b). Thus, we may also assume $u_4u_7 \notin E(G)$.

Below we have $\{u_3u_5, u_4u_6, u_4u_5, u_4u_7\} \cap E(G) = \emptyset$. Moreover, by the case (Case 2.2),

$$\{u_3u_6, u_4u_8, u_3u_9, u_3u_{10}\} \cap E(G) = \emptyset.$$

Therefore, we also have $|\{u_i : i \in [10]\}| = 10$.

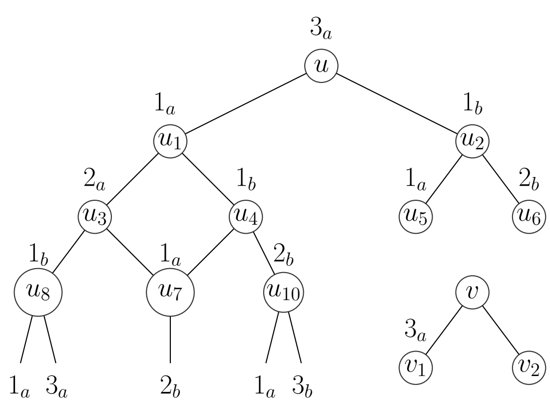


Figure 3.14: Case 2.2.3.

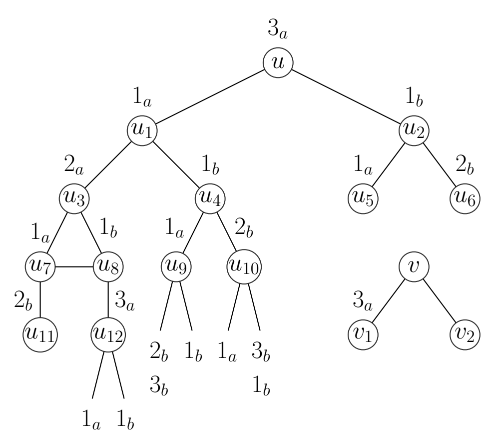


Figure 3.15: Case 2.2.4.

Case 2.2.4: $u_7u_8 \in E(G)$. By (3.88), $1_b \in f(N(u_9) - \{u_4\})$. If $1_a \notin f(N(u_{10}) - \{u_4\})$, then we recolor u_{10} with 1_a, u_1 with 2_b, and u with 1_a to obtain a coloring satisfying (a) and (b). Thus, we may assume $1_a \notin f(N(u_{10}) - \{u_4\})$. By (3.102) and (3.103), $\{3_a, 3_b\} \subset f(N(u_7) \cup N(u_8) \cup N(u_9) \cup N(u_{10}))$. If $\{3_a, 3_b\} \subset f(N(u_9) \cup N(u_{10}))$, then $f(N(u_9) \cup N(u_{10}) - \{u_4\}) = \{1_a, 1_b, 3_a, 3_b\}$, $1_b \notin f(N(u_{10}) - \{u_4\})$ and $2_b \notin f(N(u_9) - \{u_4\})$. Then, we can recolor u_{10} with 1_b, u_4 with 2_b,

u_1 with 1_b, and u with 1_a to obtain a coloring satisfying (a) and (b). Thus, by symmetry, we can assume

$$3_a \in f(N(u_7) \cup N(u_8) - \{u_3\}) \quad \text{and} \quad 3_a \notin f(N(u_9) \cup N(u_{10}) - u_4). \tag{3.120}$$

If $2_b \notin f(N(u_7) \cup N(u_8) - \{u_3\})$, then we recolor u_3 with 2_b and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we may assume $2_b \in f(N(u_7) \cup N(u_8) - \{u_3\})$. Let $u_{11} \in N(u_7) - \{u_3, u_8\}$ and $u_{12} \in N(u_8) - \{u_3, u_7\}$. We may assume

$$f(u_{11}) = 2_b \quad \text{and} \quad f(u_{12}) = 3_a, \tag{3.121}$$

since the proof for the case $f(u_{11}) = 3_a$ and $f(u_{12}) = 2_b$ is similar. Note that $3_a \notin f(B(u_1, 3) - u_{12})$. If $1_a \notin f(N(u_{12}) - \{u_8\})$, then we recolor u_{12} with 1_a, u_1 with 3_a, and u with 1_a to obtain a coloring satisfying (a) and (b). If $1_b \notin f(N(u_{12}) - \{u_8\})$, then we recolor u_{12} with 1_b, u_8 with 1_a, u_7 with 1_b, u_1 with 3_a, and u with 1_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$f(N(u_{12}) - \{u_8\}) = \{1_a, 1_b\}. \tag{3.122}$$

If $1_b \notin f(N(u_{11}) - \{u_7\})$, then we can recolor u_{11} with 1_b, u_3 with 2_b, and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$1_b \in f(N(u_{11}) - \{u_7\}). \tag{3.123}$$

Then, we can recolor u_8 with 2_a, u_3 with 1_b, and u with 2_a to obtain a coloring satisfying (a) and (b).

Case 2.2.5: $u_7u_8 \notin E(G), u_8u_9 \in E(G)$. Similarly to (3.112) and (3.117), we may assume

$$1_a \in f(N(u_{10}) - \{u_4\}) \quad \text{and} \quad 1_b \in f(N(u_7) - \{u_3\}). \tag{3.124}$$

If $2_b \notin f(N(u_7) \cup N(u_8) - \{u_3\})$, then we recolor u_3 with 2_b and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we may assume $2_b \in f(N(u_7) \cup N(u_8) - \{u_3\})$. If $1_b \notin f(N(u_{10}) - \{u_4\})$ and $2_b \notin f(N(u_9) - \{u_4\})$, then we can recolor u_{10} with 1_b, u_4 with 2_b, u_1 with 1_b, and u with 1_a to obtain a coloring satisfying (a) and (b). From (3.102) and (3.103), we know that $f(N(u_8) \cup N(u_9) - \{u_3, u_4\}) \subset \{2_b, 3_a, 3_b\}$. But it contradicts (3.88). Therefore, we may assume $u_8u_9 \notin E(G)$.

Case 2.2.6: $u_7u_8 \notin E(G), u_8u_9 \notin E(G)$. If $|N(u_7)| = |N(u_8)| = |N(u_9)| = |N(u_{10})| = 3$, then we let

$$\{u_{11}, u_{12}\} \subset N(u_7) - \{u_3\}, \quad \{u_{13}, u_{14}\} \subset N(u_8) - \{u_3\}, \quad \{u_{15}, u_{16}\} \subset N(u_9) - \{u_4\},$$

$$\text{and} \quad \{u_{17}, u_{18}\} \subset N(u_{10}) - \{u_4\}.$$

It is possible that $|\{u_i : i \in [18] - [10]\}| \neq 8$ or $\{u_5, u_6\} \cap \{u_i : i \in [18] - [10]\} \neq \emptyset$, but this will not affect the proof below.

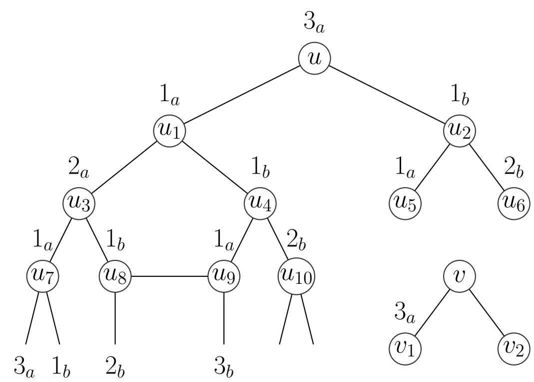


Figure 3.16: Case 2.2.5.

Similarly to (3.110), (3.111), (3.112), (3.113), we may assume

$$f(u_{12}) = f(u_{16}) = 1_b \quad \text{and} \quad f(u_{13}) = f(u_{17}) = 1_a. \tag{3.125}$$

Similarly to (3.119) and (3.120), we may assume

$$\{2_b, 3_a\} \subset f(N(u_7) \cup N(u_8) - \{u_3\}). \tag{3.126}$$

If $1_b \notin f(N(u_{10}) - \{u_4\})$ and $2_b \notin f(N(u_9) - \{u_4\})$, then we can recolor u_{10} with 1_b, u_4 with 2_b, u_1 with 1_b, and u with 1_a to obtain a coloring satisfying (a) and (b). With (3.103), we may assume

$$\textit{either } f(u_{15}) = 3_b \textit{ and } f(u_{18}) = 1_b \textit{ or } f(u_{15}) = 2_b \textit{ and } f(u_{18}) = 3_b. \tag{3.127}$$

If $|N(u_{11})| = |N(u_{12})| = |N(u_{13})| = |N(u_{14})| = 3$, then we let $\{u_{19}, u_{20}\} \subset N(u_{11})$, $\{u_{21}, u_{22}\} \subset N(u_{12})$, $\{u_{23}, u_{24}\} \subset N(u_{13})$, $\{u_{25}, u_{26}\} \subset N(u_{14})$.

By (3.126), we have

$$\textit{either } f(u_{11}) = 2_b \textit{ and } f(u_{14}) = 3_a \textit{ or } f(u_{11}) = 3_a \textit{ and } f(u_{14}) = 2_b. \tag{3.128}$$

Case 2.2.6.1: $f(u_{11}) = 2_b$ and $f(u_{14}) = 3_a$. If $1_b \notin f(N(u_{13}) - \{u_8\})$, then we can recolor u_{13} with 1_b, u_8 with 1_a, u_3 with 1_b, and u with 2_a to obtain a coloring satisfying (a) and (b). If $2_b \notin f(N(u_{13}) \cup N(u_{14}) - \{u_8\})$, then we can recolor u_8 with 2_b, u_3 with 1_b, and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$2_b \in f(N(u_{13}) \cup N(u_{14}) - \{u_8\}). \tag{3.129}$$

If $2_a \notin f(N(u_{13}) \cup N(u_{14}) - \{u_8\})$, then we can recolor u_8 with 2_a, u_3 with 1_b, and u with 2_a to

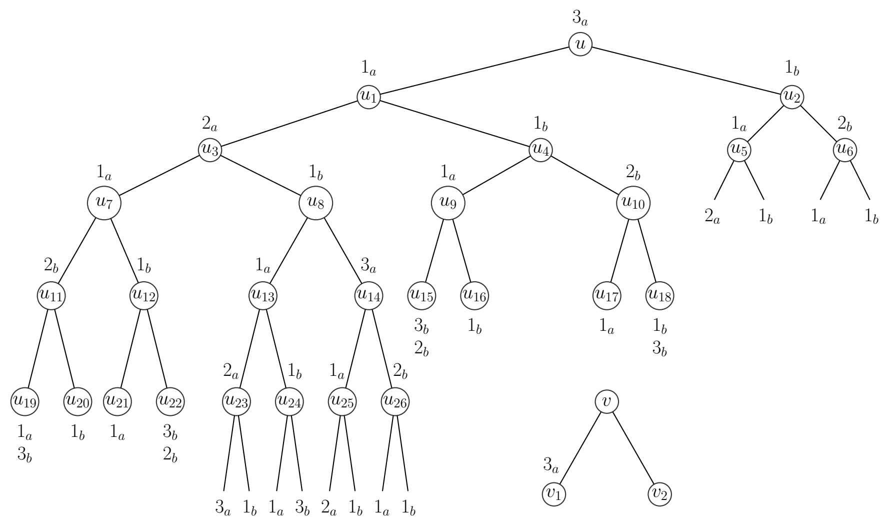


Figure 3.17: Case 2.2.6.1.

obtain a coloring satisfying (a) and (b). Thus, we may also assume

$$2_a \in f(N(u_{13}) \cup N(u_{14}) - \{u_8\}). \tag{3.130}$$

If $1_b \notin f(N(u_{11}) - \{u_7\})$, then we can recolor u_{11} with 1_b and it contradicts (3.128). Similarly, $1_a \in f(N(u_{14}) - \{u_8\})$. If $1_a \notin f(N(u_{12}) - \{u_7\})$, then we can recolor u_{12} with 1_a, u_7 with 1_b, and it contradicts (3.101). Similarly, $1_b \in f(N(u_{13}) - \{u_8\})$. Thus, we may assume

$$|N(u_{13})| = |N(u_{14})| = 3, f(u_{20}) = f(u_{24}) = 1_b, \quad \text{and} \quad f(u_{21}) = f(u_{25}) = 1_a. \tag{3.131}$$

Furthermore, by (3.129) and (3.130), we assume

$$f(u_{23}) = 2_a \quad \text{and} \quad f(u_{26}) = 2_b, \tag{3.132}$$

since the argument for $f(u_{23}) = 2_b$ and $f(u_{26}) = 2_a$ is similar. If $\{1_a, 1_b\} \neq f(N(u_{26}) - \{u_{14}\})$, then we can recolor u_{26} with a color $x \in f(N(u_{26}) - \{u_{14}\}) - \{1_a, 1_b\}$, u_8 with 2_b, u_3 with 1_b, and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$f(N(u_{26}) - \{u_{14}\}) = \{1_a, 1_b\}. \tag{3.133}$$

If $1_b \notin f(N(u_{25}) - \{u_{14}\})$, then we can recolor u_{25} with 1_b, u_{14} with 1_a, and it contradicts (3.128). Thus, we may assume

$$1_b \in f(N(u_{25}) - \{u_{14}\}). \tag{3.134}$$

If $f(u_{19}) \neq 1_a$ and $f(u_{22}) \neq 2_b$, then we can recolor u_{11} with 1_a, u_7 with 2_b, u_3 with 1_a, u_1 with 2_a, and u with 1_a to obtain a coloring satisfying (a) and (b). If $3_b \notin f(N(u_{11}) \cup N(u_{12}) - \{u_7\})$, then we can recolor u_3 with 3_b and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we can assume

$$\textit{either } f(u_{19}) = 1_a \textit{ and } f(u_{22}) = 3_b \textit{ or } f(u_{19}) = 3_b \textit{ and } f(u_{22}) = 2_b. \tag{3.135}$$

If $2_a \notin f(N(u_{25}) \cup N(u_{26}) - \{u_{14}\})$, then by (3.135), we can recolor u_{14} with 2_a, u_3 with 3_a, and u with 2_a to obtain a coloring satisfying (a) and (b). With (3.133), we may assume

$$2_a \in f(N(u_{25}) - \{u_{14}\}). \tag{3.136}$$

Similarly to (3.131), we may assume

$$1_a \in f(N(u_{24}) - \{u_{13}\}) \quad \text{and} \quad 1_b \in f(N(u_{23}) - \{u_{13}\}). \tag{3.137}$$

If $\{3_a, 3_b\} \not\subseteq f(N(u_{23}) \cup N(u_{24}))$, then we can recolor u_8 with a color $x \in f(N(u_{23}) \cup N(u_{24})) - \{3_a, 3_b\}$, u_{14} with 1_b, u_3 with 1_b, and u with 2_a to obtain a coloring satisfying (a) and (b). Therefore,

$$f(N(u_{23}) \cup N(u_{24}) - \{u_{13}\}) = \{1_a, 1_b, 3_a, 3_b\} \quad \text{and} \quad 2_b \notin f(B(u_{13})).$$

We recolor u_{13} with 2_b, u_8 with 1_a, u_3 with 1_b, and u with 2_a to obtain a coloring satisfying (a) and (b).

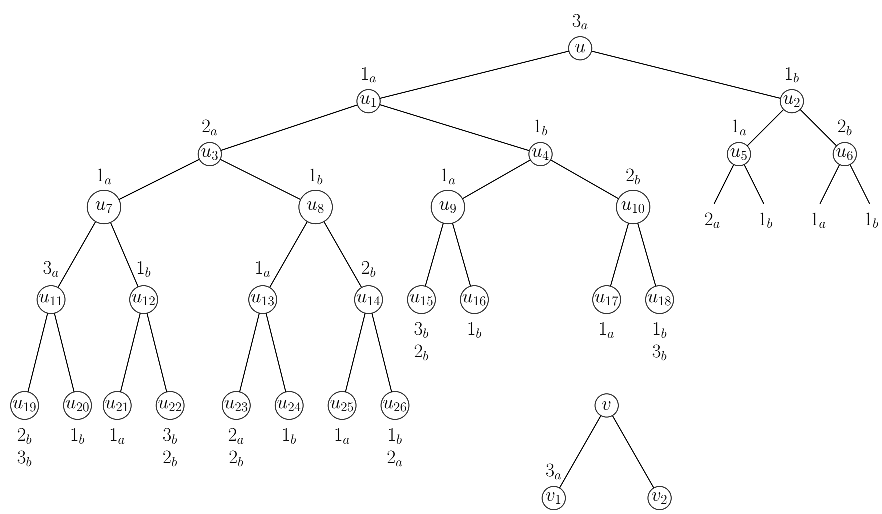


Figure 3.18: Case 2.2.6.2.

Case 2.2.6.2: $f(u_{11}) = 3_a$ and $f(u_{14}) = 2_b$. Similarly to (3.131), we may assume

$$f(u_{20}) = f(u_{24}) = 1_b \quad \text{and} \quad f(u_{21}) = f(u_{25}) = 1_a. \tag{3.138}$$

Similarly to (3.130), we may assume

$$2_a \in f(N(u_{13}) \cup N(u_{14}) - \{u_8\}). \tag{3.139}$$

If $1_b \notin f(N(u_{14}) - \{u_8\})$ and $2_b \notin f(N(u_{13}))$, then we can recolor u_8 with 2_b, u_{14} with 1_b, u_3 with 1_b, and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$\begin{aligned} & f(N(u_{13}) - \{u_8\}) = \{1_b, 2_a\} \quad \text{and} \quad f(N(u_{14}) - \{u_8\}) = \{1_a, 1_b\} \\ \text{or} \quad & f(N(u_{13}) - \{u_8\}) = \{1_b, 2_b\} \quad \text{and} \quad f(N(u_{14}) - \{u_8\}) = \{1_a, 2_a\}. \end{aligned} \tag{3.140}$$

If $2_b \notin f(N(u_{11}) \cup N(u_{12}))$, then we can recolor u_7 with 2_b and it contradicts (3.101). If $3_b \notin f(N(u_{11}) \cup N(u_{12}))$, then we can recolor u_3 with 3_b and u with 2_a to obtain a coloring satisfying (a) and (b). Thus, we may assume

$$f(N(u_{11}) \cup N(u_{12}) - \{u_7\}) = \{1_a, 1_b, 2_b, 3_b\}. \tag{3.141}$$

Specifically, we know that $1_a \notin f(N(u_{11}) - \{u_7\})$ and $2_a \notin f(B(u_7, 2) - \{u_3\})$. Therefore, we recolor u_{11} with 1_a, u_7 with 2_a, u_3 with 3_a, and u with 2_a to obtain a coloring satisfying (a) and (b). $\square$

3.4 Packing $(1, 1, 2, 2)$-coloring of some subcubic graphs

3.4.1 Introduction

Brešar, Klavžar, Rall, and Wash [22] later conjectured this.

In Section 3.4, we consider packing $(1, 1, 2, 2)$-coloring of subcubic graphs with bounded maximum average degree, *mad(G)*, which is defined to be $\max\{\frac{2|E(H)|}{|V(H)|} : H \subset G\}$.

Theorem 3.54. *Every subcubic graph G with $mad(G) < \frac{30}{11}$ is packing $(1, 1, 2, 2)$-colorable.*

Since planar graphs with girth at least g have maximum average degree less than $\frac{2g}{g-2}$, we obtain the following corollary, which extends the result of Borodin and Ivanova [16] on packing $(1, 1, 2, 2)$-coloring of subcubic planar graphs.

Corollary 3.55. *Every subcubic planar graph with girth at least 8 is packing $(1, 1, 2, 2)$-colorable.*

By Proposition 1.18, we also have the following immediate corollary, which confirms Conjecture 1.15 for subcubic graphs with maximum average degree less than $\frac{30}{11}$.

Corollary 3.56. *If G is a subcubic graph with $mad(G) < \frac{30}{11}$, then $\chi_p(D(G)) \leq 5$.*

Proof. Proposition 1.18 implies that if G is packing $(1, 1, 2, 2)$-colorable then $D(G)$ is packing

$(1,3,3,5,5)$-colorable, which implies a packing $(1,2,3,4,5)$-coloring of $D(G)$ and thus $\chi_p(D(G)) \leq 5$. $\square$

We introduce some notation used in Section 3.4. A k-vertex (k^+-vertex, k^--vertex) is a vertex of degree k (at least k, at most k). For each $u \in V(G)$, call v a k-neighbor of u if v is a neighbor of u and has degree k. $N_G^d(u)$ denotes the set of all vertices that are at distance d from u.

3.4.2 Proof of Theorem 3.54

Let G be a minimum counterexample to Theorem 3.54 with fewest number of vertices. For simplicity, we use $(1,1,2,2)$-coloring instead of packing $(1,1,2,2)$-coloring in the rest of Section 3.4. We assume that the colors are $\{1_a, 1_b, 2_a, 2_b\}$ such that vertices with color 1_a (or 1_b) are not adjacent and vertices with color 2_a (or 2_b) must have distance at least two.

Lemma 3.57. $\delta(G) \geq 2$.

Proof. Suppose otherwise that v is a 1-vertex in G with $uv \in E(G)$. By the minimality of G, $G \setminus \{v\}$ has a $(1,1,2,2)$-coloring f. Then we can extend f to G by coloring v with a color in $\{1_a, 1_b\} \setminus \{f(u)\}$, which contradicts the assumption that G is a minimum counterexample. $\square$

Lemma 3.58. *There are no adjacent 2-vertices in G.*

Proof. Suppose otherwise that u, v are adjacent 2-vertices in G. Let $N_G^1(u) = \{u', v\}$ and $N_G^1(v) = \{u, v'\}$. By the minimality of G, $G \setminus \{u, v\}$ has a $(1,1,2,2)$-coloring f. We color u (respectively v) with a color in $\{1_a, 1_b\} \setminus \{f(u')\}$ (respectively $\{1_a, 1_b\} \setminus \{f(v')\}$). We obtain a $(1,1,2,2)$-coloring of G unless u, v receive the same color. Thus, we may assume $f(u') = f(v') = 1_b$ and $f(u) = f(v) = 1_a$. Moreover, we may assume $d(u') = 3$ and $f(u') = \{1_a, 2_a, 2_b\}$, since otherwise we recolor u with a color $x \in \{2_a, 2_b\} \setminus f(u')$ and obtain a $(1,1,2,2)$-coloring of G. We obtain a $(1,1,2,2)$-coloring of G by recoloring u' with 1_a and u with 1_b, which is a contradiction. $\square$

We will use lemma 3.59 extensively in the rest of Section 3.4.

Lemma 3.59. *Let v be a 2-vertex in G with two neighbors u, w. Let $N_G^1(u) = \{v, u_1, u_2\}$ and $N_G^1(w) = \{v, w_1, w_2\}$. Let f be a $(1,1,2,2)$-coloring of $G - v$. Then either $\{f(u), f(w)\} = \{1_a, 1_b\}$, and $\{1_a, 1_b\} \subseteq \{f(u), f(u_1), f(u_2)\}$, $\{1_a, 1_b\} \subseteq \{f(w), f(w_1), f(w_2)\}$ and $\{2_a, 2_b\} \subseteq f(N_G^2(v))$; or $f(u) = f(w) \in \{2_a, 2_b\}$, and $\{f(u_1), f(u_2)\} = \{f(w_1), f(w_2)\} = \{1_a, 1_b\}$.*

Proof. We may color v with some $x \in \{1_a, 1_b\} \setminus \{f(u), f(w)\}$ to obtain a $(1,1,2,2)$-coloring of G, unless $\{f(u), f(w)\} = \{1_a, 1_b\}$ or $f(u) = f(w) \in \{2_a, 2_b\}$.

Case 1: $\{f(u), f(w)\} = \{1_a, 1_b\}$. By symmetry, we assume $f(u) = 1_a$ and $f(w) = 1_b$. We have $1_b \in \{f(u_1), f(u_2)\}$ since otherwise we can recolor u with 1_b and color v with 1_a to obtain a

$(1,1,2,2)$-coloring of G. Similarly, we have $1_a \in \{f(w_1), f(w_2)\}$. Moreover, if $\{2_a, 2_b\} \nsubseteq f(N_G^2(v))$ then we can color v with a color $x \in f(N_G^2(v)) \setminus \{2_a, 2_b\}$ to obtain a $(1,1,2,2)$-coloring of G. Thus, $\{2_a, 2_b\} \subseteq f(N_G^2(v))$.

Case 2: $f(u) = f(w) \in \{2_a, 2_b\}$. If $\{f(u_1), f(u_2)\} \neq \{1_a, 1_b\}$, then we recolor u with some $x \in \{1_a, 1_b\} \setminus \{f(u_1), f(u_2)\}$ and color v with $y \in \{1_a, 1_b\} \setminus \{x\}$ to obtain a $(1,1,2,2)$-coloring of G. Thus, we have $\{f(u_1), f(u_2)\} = \{1_a, 1_b\}$ and similarly $\{f(w_1), f(w_2)\} = \{1_a, 1_b\}$. $\square$

By symmetry, whenever the situation in Lemma 3.59 happens, we may assume $f(u) = 1_a$, $f(w) = 1_b$, $\{f(w_1), f(w_2)\} = \{1_a, 2_a\}$ and $\{f(u_1), f(u_2)\} = \{1_b, 2_b\}$ in the former case and $f(u) = f(w) = 2_a$ in the latter case.

Lemma 3.60. *Each 3-vertex in G has at most one 2-neighbor.*

Proof. Suppose not, i.e., u_2 is a 3-vertex in G with $N_G^1(u_2) = \{u_1, v_2, u_3\}$ and $d(u_1) = d(u_3) = 2$. Let v_i be the neighbors of u_i distinct from u_2 for each $i \in \{1,3\}$. For each $i \in [3]$, let $N_G^1(v_i) = \{u_i, v_i'\}$ if $d(v_i) = 2$ and $N_G^1(v_i) = \{u_i, v_i', v_i''\}$ if $d(v_i) = 3$. By Lemma 3.59, $G - u_1$ has a $(1,1,2,2)$-coloring f such that either $f(v_1) = 1_a, f(u_2) = 1_b$ or $f(v_1) = f(u_2) = 2_a$.

Case 1: $f(v_1) = 1_a, f(u_2) = 1_b$. By symmetry, we have $\{f(v_1'), f(v_1'')\} = \{1_b, 2_b\}$ and $\{f(v_2), f(u_3)\} = \{1_a, 2_a\}$.

Case 1.1: $f(v_2) = 1_a$ and $f(u_3) = 2_a$. If $f(v_3) \neq 1_a$, then we can recolor u_3 with 1_a and color u_1 with 2_a to obtain a $(1,1,2,2)$-coloring of G, which is a contradiction. Thus, $f(v_3) = 1_a$ and we recolor u_3 with 1_b. If $1_b \notin \{f(v_2'), f(v_2'')\}$, then we recolor v_2 with 1_b, u_2 with 1_a and color u_1 with 1_b to obtain a $(1,1,2,2)$-coloring of G. Thus, $1_b \in \{f(v_2'), f(v_2'')\}$. If $\{2_a, 2_b\} \nsubseteq \{f(v_2'), f(v_2'')\}$, then we obtain a $(1,1,2,2)$-coloring of G by recoloring u_2 with a color $x \in \{2_a, 2_b\} \setminus \{f(v_2'), f(v_2'')\}$ and coloring u_1 with 1_b. Thus, $\{1_b, 2_a, 2_b\} \subseteq \{f(v_2'), f(v_2'')\}$, which is a contradiction.

Case 1.2: $f(v_2) = 2_a$ and $f(u_3) = 1_a$. If $f(v_3) \neq 1_b$, then we can recolor u_3 with 1_b, u_2 with 1_a and color u_1 with 1_b to obtain a $(1,1,2,2)$-coloring of G, which is a contradiction. Thus, $f(v_3) = 1_b$. If $\{1_a, 1_b\} \nsubseteq \{f(v_2'), f(v_2'')\}$, then we obtain a $(1,1,2,2)$-coloring of G by recoloring v_2 with a color $x \in \{1_a, 1_b\} \setminus \{f(v_2'), f(v_2'')\}$, u_2 with 2_a and color u_1 with 1_b. Thus, $\{1_a, 1_b\} \subseteq \{f(v_2'), f(v_2'')\}$. It follows that $2_b \notin \{f(v_2'), f(v_2'')\}$, and we obtain a $(1,1,2,2)$-coloring of G by recoloring u_2 with 2_b and coloring u_1 with 1_b, which is a contradiction.

Case 2: $f(v_1) = f(u_2) = 2_a$. By symmetry, $f(v_1') = 1_a, f(v_1'') = 1_b, f(v_2) = 1_a, f(u_3) = 1_b$. If $f(v_3) \neq 1_a$, then we recolor u_3 with 1_a, u_2 with 1_b and color u_1 with 1_a. Thus, $f(v_3) = 1_a$. If $1_b \notin \{f(v_3'), f(v_3'')\}$, then we recolor v_3 with 1_b, u_3 with 1_a, u_2 with 1_b and color u_1 with 1_a. Thus, $1_b \in \{f(v_3'), f(v_3'')\}$. If $\{2_a, 2_b\} \nsubseteq \{f(v_3'), f(v_3'')\}$, then we recolor u_3 by a color $x \in \{2_a, 2_b\} \setminus \{f(v_3'), f(v_3'')\}$, u_2 with 1_b and color u_1 with 1_a to obtain a $(1,1,2,2)$-coloring of G. Therefore, $\{1_b, 2_a, 2_b\} \subseteq \{f(v_3'), f(v_3'')\}$, which is a contradiction. $\square$

For convenience, call a 3-vertex v in G *special* if all neighbors of v are 3-vertices.

Lemma 3.61. *Let u be a 2-vertex in G, then there are at least two special 3-vertices in $N^2_G(u)$.*

Proof. Suppose not, i.e., there are at most one special 3-vertices in $N^2_G(u)$. Let $N^1_G(u) = \{u_1, u_2\}$. By Lemma 3.58, both u_1 and u_2 are 3-vertices. Let $N^1_G(u_1) = \{u, v_1, v_2\}$ and $N^1_G(u_2) = \{u, v_3, v_4\}$. By Lemma 3.60, $d(v_i) = 3$ for each $i \in [4]$ and we may assume by symmetry that both v_1 and v_2 are non-special. By Lemma 3.60 again, v_1 (respectively v_2) has exactly one 2-neighbor, say w_1 (respectively w_3). Let $N^1_G(v_1) = \{u_1, w_1, w_2\}$, $N^1_G(v_2) = \{u_1, w_3, w_4\}$, $N^1_G(w_1) = \{v_1, x_1\}$, $N^1_G(w_2) = \{v_1, x_2, x_3\}$, $N^1_G(w_3) = \{v_2, x_4\}$ and $N^1_G(w_4) = \{v_2, x_5, x_6\}$ (note that it is possible that $v_1v_2 \in E(G)$). By Lemma 3.59, $G - u$ has a $(1, 1, 2, 2)$-coloring f such that either $f(u_1) = 1_a, f(u_2) = 1_b$ or $f(u_1) = f(u_2) = 2_a$.

Case 1: $f(u_1) = 1_a$ and $f(u_2) = 1_b$. By symmetry, $f(v_1) = 1_b, f(v_2) = 2_b, f(v_3) = 1_a$ and $f(v_4) = 2_a$.

Claim: $\{f(w_1), f(w_2)\} = \{1_a, 2_b\}$ and $\{f(w_3), f(w_4)\} = \{1_b, 2_a\}$.

Proof of Claim: If $1_a \notin \{f(w_1), f(w_2)\}$, then we recolor v_1 with 1_a, u_1 with 1_b and color u with 1_a to obtain a $(1, 1, 2, 2)$-coloring of G. Thus, $1_a \in \{f(w_1), f(w_2)\}$. If $1_b \notin \{f(w_3), f(w_4)\}$, then we recolor v_2 with 1_b and color u with 2_b. Thus, $1_b \in \{f(w_3), f(w_4)\}$. If $2_a \notin f(N^2_G(u_1))$, then we can recolor u_1 with 2_a and color u with 1_a. Thus, $2_a \in \{f(w_1), f(w_2), f(w_3), f(w_4)\}$. Now we may assume that $2_b \notin \{f(w_1), f(w_2)\}$, since otherwise we have $\{f(w_1), f(w_2)\} = \{1_a, 2_b\}$ and $\{f(w_3), f(w_4)\} = \{1_b, 2_a\}$ (and we are done). Then $1_a \in \{f(w_3), f(w_4)\}$, since otherwise we can recolor v_2 with 1_a, u_1 with 2_b and color u with 1_a. By symmetry, we assume that $f(w_3) = 1_a$, $f(w_4) = 1_b$ and we also have $\{f(w_1), f(w_2)\} = \{1_a, 2_a\}$.

If $f(x_4) \neq 1_b$ or $2_a \notin f(N^2_G(w_3))$, then we recolor w_3 with 1_b or 2_a, color v_2 with 1_a, u_1 with 2_b and u with 1_a to obtain a $(1, 1, 2, 2)$-coloring of G. Thus, $f(x_4) = 1_b$ and $f(N^1_G(x_4) - \{w_3\}) = \{1_a, 2_a\}$, since if $1_a \notin f(N^1_G(x_4) - \{w_3\})$ then we recolor x_4 with 1_a and it contradicts our previous conclusion that $f(x_4) = 1_b$.

Case a: $f(w_1) = 1_a$ and $f(w_2) = 2_a$. Then $f(x_1) = 1_b$, since otherwise we can recolor w_1 with 1_b, v_1 with 1_a, u_1 with 1_b and color u with 1_a to obtain a $(1, 1, 2, 2)$-coloring of G. If $\{1_a, 1_b\} \neq \{f(x_2), f(x_3)\}$, then we can recolor w_2 with a color $x \in \{1_a, 1_b\} \setminus \{f(x_2), f(x_3)\}$, v_1 with 2_a, u_1 with 1_b and color u with 1_a, which is a contradiction. Thus, $\{f(x_2), f(x_3)\} = \{1_a, 1_b\}$. Now we can recolor v_1 and w_3 with 2_b, v_2 with 1_a, u_1 with 1_b and color u with 1_a, which is a contradiction.

Case b: $f(w_1) = 2_a, f(w_2) = 1_a$. Then $f(x_1) = 1_a$, since otherwise we can recolor w_1 with 1_a, u_1 with 2_a and color u with 1_a to obtain a $(1, 1, 2, 2)$-coloring of G. If $1_b \notin \{f(x_2), f(x_3)\}$, then we can recolor w_2 with 1_b, v_1 with 1_a, u_1 with 1_b and color u with 1_a. If $2_b \notin \{f(x_2), f(x_3)\}$, then we can recolor v_1 and w_3 with 2_b, v_2 with 1_a, u_1 with 1_b and color u with 1_a. Thus, we have $\{f(x_2), f(x_3)\} = \{1_b, 2_b\}$. Now we can recolor w_1 with 1_b, v_1 with 2_a, u_1 with 1_b and color u with 1_a, which is a contradiction.

This completes the proof of the Claim.

By the Claim, we have the following two subcases.

Case 1.1: $f(w_1) = 1_a$ and $f(w_2) = 2_b$. Then $f(x_1) = 1_b$, since otherwise we can recolor w_1 with 1_b, v_1 with 1_a, u_1 with 1_b and color u with 1_a to obtain a $(1,1,2,2)$-coloring of G. Moreover, $\{f(x_2), f(x_3)\} = \{1_a, 1_b\}$, since otherwise we can recolor w_2 with 1_a or 1_b, v_1 with 2_b, v_2 with 1_a, u_1 with 1_b and color u with 1_a. Now we can recolor v_1 with 2_a, u_1 with 1_b and color u with 1_a, which is a contradiction.

Case 1.2: $f(w_1) = 2_b$ and $f(w_2) = 1_a$. Then $1_b \in \{f(x_2), f(x_3)\}$, since otherwise we can recolor w_2 with 1_b, v_1 with 1_a, u_1 with 1_b and color u with 1_a to obtain a $(1,1,2,2)$-coloring of G. Also $2_b \in \{f(x_2), f(x_3)\}$, since otherwise we can recolor w_1 with a color $x \in \{1_a, 1_b\} \setminus \{f(x_1)\}$, v_1 with 2_b, v_2 with 1_a, u_1 with 1_b and color u with 1_a. Note that $f(x_1) = 2_a$, for otherwise we can recolor v_1 with 2_a, u_1 with 1_b and color u with 1_a. Now we can recolor w_1 and v_2 with 1_a, u_1 with 2_b and color u with 1_a, which is a contradiction.

Case 2: $f(u_1) = f(u_2) = 2_a$. By symmetry, $f(v_1) = 1_a, f(v_2) = 1_b, f(v_3) = 1_a, f(v_4) = 1_b$. If $1_b \notin \{f(w_1), f(w_2)\}$, then we recolor v_1 with 1_b, u_1 with 1_a and color u with 1_b to obtain a $(1,1,2,2)$-coloring of G. Thus, $1_b \in \{f(w_1), f(w_2)\}$. Similarly, $1_a \in \{f(w_3), f(w_4)\}$. If $2_b \notin f(N_G^2(u_1))$, then we recolor u_1 with 2_b and color u with 1_a. Therefore, $2_b \in \{f(w_1), f(w_2), f(w_3), f(w_4)\}$.

Case 2.1: $f(w_2) = 1_b$, $f(w_4) = 1_a$.

Case 2.1.1: $f(w_3) \neq 2_b$. Then $f(w_3) = 1_a$ and $f(w_1) = 2_b$. If $2_b \notin \{f(x_2), f(x_3)\}$, then we can recolor w_1 with a color $x \in \{1_a, 1_b\} \setminus \{f(x_1)\}$, v_1 with 2_b, u_1 with 1_a and color u with 1_b. Thus, $2_b \in \{f(x_2), f(x_3)\}$. If $1_a \notin \{f(x_2), f(x_3)\}$, then we can recolor w_2 with 1_a, v_1 with 1_b, u_1 with 1_a and color u with 1_b. Therefore, $\{f(x_2), f(x_3)\} = \{1_a, 2_b\}$. Then $f(x_1) = 2_a$, for otherwise we can recolor v_1 with 2_a, u_1 with 1_a and color u with 1_b. We now recolor w_1 with 1_b. Then we obtain a $(1,1,2,2)$-coloring of G by recoloring u_1 with 2_b and coloring u with 1_a or 1_b, which is a contradiction.

Case 2.1.2: $f(w_1) \neq 2_b$. Then $f(w_1) = 1_b$ and $f(w_3) = 2_b$. Similarly to Case 2.1.1, we can recolor w_3 with 1_a. Then we obtain a $(1,1,2,2)$-coloring of G by recoloring u_1 with 2_b and coloring u with 1_a or 1_b, which is a contradiction.

Case 2.1.3: $f(w_1) = f(w_3) = 2_b$. Similarly to Case 2.1.1, we can recolor w_1 with 1_b and w_3 with 1_a. Then we obtain a $(1,1,2,2)$-coloring of G by recoloring u_1 with 2_b and coloring u with 1_a or 1_b, which is a contradiction.

Case 2.2: $f(w_2) \neq 1_b$ or $f(w_4) \neq 1_a$. By symmetry, we may assume that $f(w_2) = 2_b$ and $f(w_1) = 1_b$. Then $f(x_1) = 1_a$, for otherwise we can recolor w_1 with 1_a, v_1 with 1_b, u_1 with 1_a, and color u with 1_b to obtain a $(1,1,2,2)$-coloring of G. If $\{f(x_2), f(x_3)\} \neq \{1_a, 1_b\}$, then w_2 can be recolored with $x \in \{1_a, 1_b\} \setminus \{f(x_2), f(x_3)\}$, v_1 with 2_b, u_1 with 1_a, and color u with 1_b. Therefore, $\{f(x_2), f(x_3)\} = \{1_a, 1_b\}$. We now recolor v_1 with 2_a, u_1 with 1_a, and color u with 1_b, which is a contradiction. $\square$

We are now ready to complete the proof of Theorem 3.54. We use a discharging argument. Let the initial charge $\mu(v) = d(v) - \frac{30}{11}$ for each $v \in V(G)$. Since $mad(G) < \frac{30}{11}$, we have

$$\sum_{v \in V(G)} (d(v) - \frac{30}{11}) = 2|E(G)| - n \cdot \frac{30}{11} \leq mad(G) \cdot n - \frac{30}{11} \cdot n < 0.$$

To lead to a contradiction, we shall use the following discharging rules to redistribute the charges so that the final charge of every vertex v in G, denote by $\mu^*(v)$, is non-negative.

(R1) Each special 3-vertex v gives $\frac{1}{11}$ to each 2-vertex in $N_G^2(v)$.

(R2) Each non-special 3-vertex v gives $\frac{3}{11}$ to each 2-neighbor.

Let v be a vertex in G. By Lemma 3.57, $d(v) \in \{2, 3\}$. If $d(v) = 2$, then by Lemma 3.58 and (R2) v gets $\frac{3}{11}$ from each of two 3-neighbors. By Lemma 3.61 there are at least two special 3-vertices in $N_G^2(v)$ and each of which gives $\frac{1}{11}$ to v by (R1). So $\mu^*(v) \geq 2 - \frac{30}{11} + \frac{3}{11} \cdot 2 + \frac{1}{11} \cdot 2 = 0$. Let $d(v) = 3$. If v is not special, then by Lemma 3.60, v has exactly one 2-neighbor, so gives $\frac{3}{11}$ by (R2); if v is special, then v has at most three 2-vertices in $N_G^2(v)$ by Lemma 3.60, so by (R1), v gives $\frac{1}{11} \cdot 3$. So in either case, $\mu^*(v) \geq 3 - \frac{30}{11} - \max\{\frac{3}{11}, \frac{1}{11} \cdot 3\} = 0$.

Chapter 4

Directed intersection number and the information content of digraphs

Results in Chapter 4 are joint work with Machado and Milenkovic.

4.1 Introduction

In the WWW network, a number of pages are devoted to *topic or item disambiguation*; in disambiguation pages, a number of identical names of designators are used to describe different entities which are further clarified and narrowed down in context via links to more specific pages. For example, typing the word "Michael Jordan" into a search engine such as Google produces a Wikipedia page which lists athlete, actors, scientists and other persons bearing this name. From this web page, one can choose to follow a link to any one of the items sharing the same two keywords, "Michael" and "Jordan". Most of the specific pages do not link back to the disambiguation page: For example, following the link to "Michael Jordan (footballer)" does not allow for returning to the disambiguation page, and may hence be viewed as a directed link. Furthermore, disambiguation pages tend to have little content, usually in the form of lists, while the pages that linked from them tend to have significantly more information about one of the individuals.

Motivated by such directed networks of webpages, we consider the following problem, illustrated by a small-scale directed graph depicted in Figure 4.1. Assume that the vertices A, B, C, D correspond to four webpages that contain different collections of topics, files, or networks, represented by color-coded rectangles (For example, each color (shape) may correspond to a different person bearing the same name). There is a link between two webpages if they have at least one topic in common (e.g., the same name or some other shared feature). For a directed graph, in addition to the shared content assumption one needs to provide an explanation for the direction of the links, i.e., which vertex in the arc represents the tail and which vertex in the arc represents the head. In the context of the above described webpage linkages, it is reasonable to assume that a webpage links to another terminal webpage if the latter covers more topics, i.e., contains additional information compared to the source page. In Figure 4.1, the link between webpages A and B is directed from A to B, since B lists three topics, while A lists only two. This gives rise to two generative constraints for the existence of a directed edge: Shared information content and content size dominance. This is a natural generative assumption, which has been exploited in a similar form in a number of data

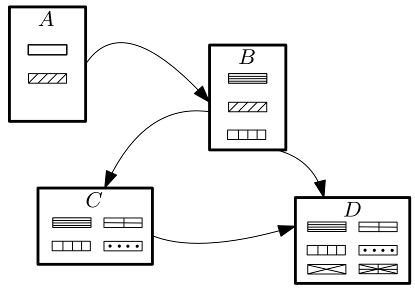
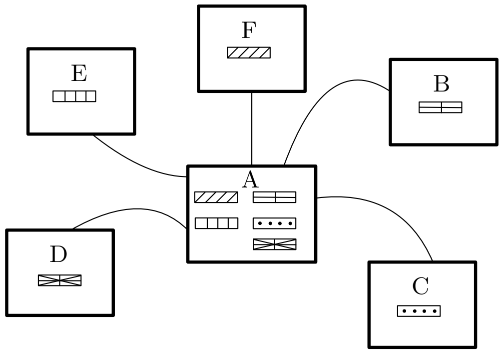


Figure 4.1: An information storage network such as the World Wide Web. Each vertex contains a list of color-coded topics or files, representing its information content (e.g., vertex B contains a topic with horizontal lines, a topic with slashes, and a topic with vertical lines). Vertices A and B are connected through an arc (A,B) since they share the topic with slashes and A lists two, while B lists three files.

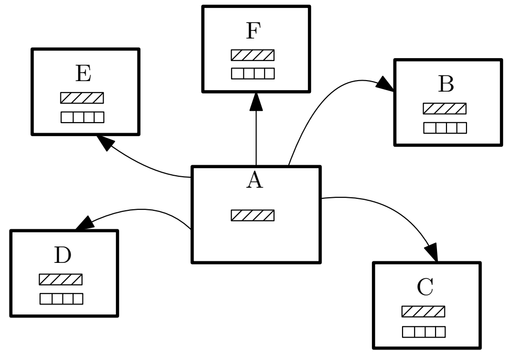
mining contexts [87, 32].

We are interested in the following question. Let D be a directed graph with vertex set V and arc set A, and assume that each vertex $v \in V$ is associated with a nonempty subset $\varphi(v)$ of a finite ground set $\mathcal{C}$, called the *color set*, such that $(u, v) \in A$ if and only if $|\varphi(u) \cap \varphi(v)| \geq 1$ and $|\varphi(u)| < |\varphi(v)|$ (i.e., two vertices share an arc if their color sets intersect and the color set of the tail is strictly smaller than the color set of the head). If such a representation is possible, we refer to it as a *directed intersection representation.* The question of interest is to determine the smallest cardinality of the ground set $\mathcal{C}$ which allows for a directed intersection representation of a digraph D with $|V| = n$ vertices, henceforth termed the directed intersection number of D. Clearly, not all digraphs allow for such a representation. For example, a directed triangle D with $V = \{1, 2, 3\}$ and $A = \{(1, 2), (2, 3), (3, 1)\}$ does not admit a directed intersecting representation, as such a representation would require $|\varphi(1)| < |\varphi(2)| < |\varphi(3)| < |\varphi(1)|$, which is impossible. The same is true of every digraph that contains cycles, but as we subsequently show, every directed acyclic graph (DAG) admits a directed intersection representation. We focus on connected DAGs, although our results apply to disconnected graphs with either no or some small modifications.

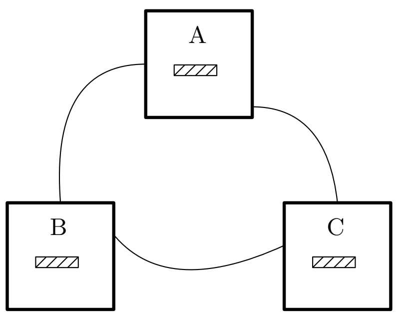
The problem of finding directed intersection representations of digraphs is closely associated with the intersection representation problem for undirected graphs. Intersection representations are of interest in many applications such as keyword conflict resolution, traffic phasing, latent feature discovery and competition graph analysis [79, 80, 33]. Formally, the vertices $v \in V$ of a graph G are associated with subsets $\varphi(v)$ of a ground set $\mathcal{C}$ so that $uv \in E$ if and only if $|\varphi(u) \cap \varphi(v)| \geq 1$. The intersection number (IN) of the graph G is the smallest size of the ground set $\mathcal{C}$ that allows for an intersection representation, and it is well-defined for all graphs. Finding the intersection number of a graph is equivalent to finding the edge clique cover number, as proved by Erdós, Goodman and Posa in [37]. Determining the edge clique cover number is NP-hard, as shown by Orlin [78]. The intersection number of an undirected graph may differ vastly (can be larger or smaller) from the DIN of some of its orientations, whenever the latter exists. This is illustrated by two examples in



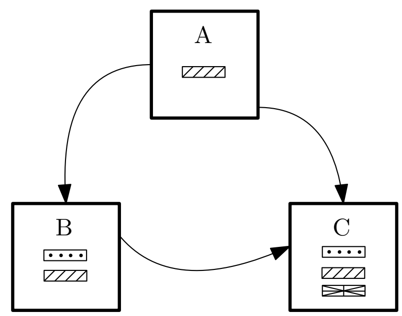
(a) The intersection number of a star (a tree with diameter at most two) is equal to $|E| = n - 1$ (e.g., 5).



(b) The DIN of any star digraph is 2.



(c) The intersection number of a complete graph is 1.



(d) The DIN of a "complete" DAG on three vertices is exactly 3.

Figure 4.2: A comparison of the intersection numbers and DINs of the star and complete graph/DAG.

Figure 4.2.

This chapter is organized as follows. Section 4.2 contains a constructive proof that all DAGs have a finite directed intersection representation and by algorithmically producing such a representation. In the same section, we inductively prove an improved upper bound which is $\frac{5n^2}{8} - \frac{3n}{4} + 1$. In Section 4.3 we introduce the notion of DIN-extremal DAGs and describe constructions of acyclic digraphs with DINs equal to

$$\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

4.2 Representations of Directed Acyclic Graphs

We use the notation and terminology described below. Whenever clear from the context, we omit the argument n.

The in-degree of a vertex v is the number of arcs for which v is the head, while the out-degree is the number of arcs for which v is the tail. The set of in-neighbors of v is the set of tails of arcs for which v is the head, and is denoted by $N^-(v)$. The set of out-neighbors $N^+(v)$ is defined similarly.

For a given acyclic digraph $D(V, A)$, let $\Gamma : V \to \mathbb{N}$ be a mapping that assigns to each vertex $v \in V$ the length of the longest directed path that terminates at v. The map Γ induces a partition of the vertex set V into *levels* $(V_0, \ldots, V_\ell)$, such that $V_i = \{v \in V : \Gamma(v) = i\}$. We refer to V_i for $i = 1, \ldots, \ell$ as the *longest path decomposition* of G. Clearly, there is no arc between any pair of vertices u and v at the same level V_i as this would violate the longest path partitioning assumption. Note that although the longest path problem is NP-hard for general graphs, it is linear time for DAGs. Finding the longest path in this case can be accomplished via topological sorting [36].

Theorem 4.1. *Every DAG $D(V, A)$ on n vertices admits a directed intersection representation. Moreover, $DIN(n) \leq \frac{5}{8}n^2 - \frac{1}{4}n$.*

Proof. We prove the existence claim and upper bound by describing a constructive color assignment algorithm.

Step 1: We order the vertices of the digraph as $V = (v_1, v_2, \ldots, v_n)$ so that if $(v_i, v_j) \in A$, then $i < j$. One such possible ordering is henceforth referred to as a left-to-right order, and it clearly exists as the digraph is acyclic. We then construct the longest path decomposition and order the vertices in the graph starting from the first level and proceeding to the last level. The order of vertices inside each level is irrelevant.

Step 2: We group vertices into pairs in order of their labels, i.e., (v_{2i-1}, v_{2i}), for $1 \leq i \leq \frac{n}{2}$, and then for $1 \leq i \leq n$ in order assign to each vertex v_i a color set distinct from the color set of all other vertices. The size of the sets assigned to v_i is $\frac{n}{2} - \lceil \frac{i}{2} \rceil$.

Remark 4.2. *In this step we used exactly*

$$2 \cdot \left(\frac{n}{2} - 1 + \frac{n}{2} - 2 + \ldots + 1\right) = 2 \cdot \frac{1 + \frac{n}{2} - 1}{2} \cdot \left(\frac{n}{2} - 1\right) = \frac{n^2}{4} - \frac{n}{2} \tag{4.1}$$

distinct colors. Those colors are going to be reused to establish arcs of the digraph.

Step 3: For $1 \leq i \leq n-2$, we assign common colors for arcs from v_i to vertices belonging to pairs that follow the pair in which v_i lies. More precisely:

- If $(v_i, v_{2j-1}) \notin A$ and $(v_i, v_{2j}) \notin A$ for some j such that $2 \cdot \lceil \frac{i}{2} \rceil < 2j - 1 \leq n - 1$, then we do nothing and move to the next step.

- If $(v_i, v_{2j-1}) \in A$ and $(v_i, v_{2j}) \notin A$ for some j such that $2 \cdot \lceil \frac{i}{2} \rceil < 2j - 1 \leq n - 1$, then we copy one color from $\varphi(v_i)$ not previously used in Step 3 and place it into the color set of v_{2j-1}, $\varphi(v_{2j-1})$.

- If $(v_i, v_{2j-1}) \notin A$ and $(v_i, v_{2j}) \in A$ for some j such that $2 \cdot \lceil \frac{i}{2} \rceil < 2j - 1 \leq n - 1$, then we copy one color from $\varphi(v_i)$ not previously used in Step 3 and place it into the color set of v_{2j}, $\varphi(v_{2j})$.

• If $(v_i, v_{2j-1}) \in A$ and $(v_i, v_{2j}) \in A$ for some j such that $2 \cdot \lceil \frac{i}{2} \rceil < 2j-1 \leq n-1$, then we copy one color from $\varphi(v_i)$ not previously used in Step 3 and place it into both $\varphi(v_{2j-1})$ and $\varphi(v_{2j})$.

Remark 4.3. *Since each vertex v_i has a color set $\varphi(v_i)$ with $\frac{n}{2} - \lceil \frac{i}{2} \rceil$ colors, and there are $\frac{n}{2} - \lceil \frac{i}{2} \rceil$ pairs following the pair that vertex v_i is located in the previously fixed left-to-right ordering, we will never run out of colors during the above color assignment process.*

The color sets obtained after the previously described procedure are denoted by φ'.

Step 4: To the color sets of each pair of vertices (v_{2i-1}, v_{2i}), we add at most $3i$ new colors. The augmented color sets, denoted by φ'', satisfy 1) if $v_{2i-1}v_{2i}$ is an arc, then $|\varphi''(v_{2i-1})| = \frac{n}{2} + 2i - 2$ and $|\varphi''(v_{2i})| = \frac{n}{2} + 2i - 1$; 2) if $v_{2i-1}v_{2i}$ is not an arc, then $|\varphi''(v_{2i-1})| = |\varphi''(v_{2i})| = \frac{n}{2} + 2i - 1$.

In **Step 4** we add at most

$$\frac{n}{2} + 2i - 1 - 1 - \left(\frac{n}{2} - i\right) = 3i - 2$$

colors to the color set of v_{2i-1} and at most

$$\frac{n}{2} + 2i - 1 - \left(\frac{n}{2} - i\right) = 3i - 1$$

colors to the color set of v_{2i} to reach the desired color-set sizes. Note that some colors may be reused so that at this step, at most $3i-1$ new colors are actually needed for a pair (v_{2i-1}, v_{2i}). Note that in **Step 3**, for each pair (v_{2i-1}, v_{2i}), we added in total at most $2i-2$ colors to both $\varphi'(v_{2i-1})$ and $\varphi'(v_{2i})$. Since $3i-2 > 2i-2$, we added at least one color in common for the pair (v_{2i-1}, v_{2i}) so that the intersection condition is satisfied when $v_{2i-1}v_{2i}$ is an arc.

Thus, the number of colors used so far is at most

$$\begin{aligned}(3 \cdot 1 - 1) + (3 \cdot 2 - 1) + \ldots + \left(3 \cdot \frac{n}{2} - 1\right) &= 3 \cdot \left(1 + 2 + \ldots + \frac{n}{2}\right) - \frac{n}{2} \\ &= 3 \cdot \frac{1 + \frac{n}{2}}{2} \cdot \frac{n}{2} - \frac{n}{2} = \frac{3}{8}n^2 + \frac{n}{4}. \end{aligned} \tag{4.2}$$

Next, we claim that φ'' is a valid representation that uses at most $\frac{5}{8}n^2 - \frac{n}{4}$ colors. From (4.1) and (4.2), we know that we used at most

$$\frac{n^2}{4} - \frac{n}{2} + \frac{3}{8}n^2 + \frac{n}{4} = \frac{5}{8}n^2 - \frac{n}{4}$$

colors.

The size condition obviously holds since $|\varphi''(v_i)| = \frac{n}{2} + i - 1$ and $(v_i, v_j \in A$ implies $|\varphi(v_i)| < |\varphi(v_j)|$. The intersection condition also holds since for each (v_i, v_j) with $i < j$, one has

• If $(v_i, v_j) \in A$, then

1) If (v_i, v_j) is a pair, then $\varphi''(v_i)$ and $\varphi''(v_j)$ have by the previous procedure at least one color in common.

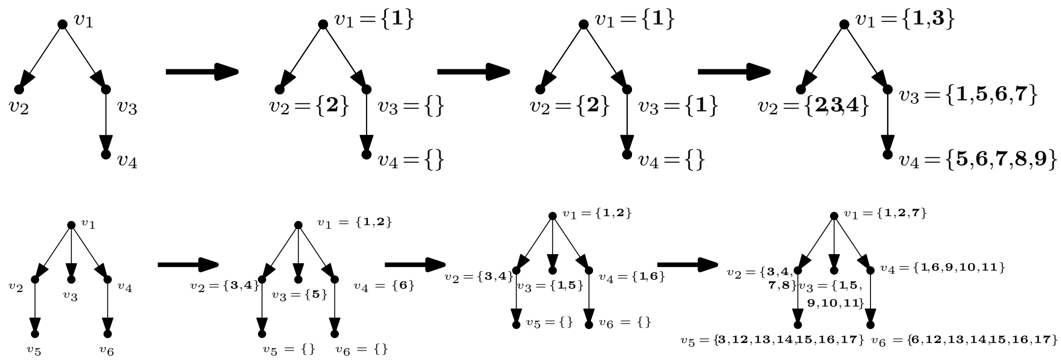


Figure 4.3: Directed intersection representations for two rooted trees with four and six vertices, respectively. The representations were obtained by using a vertex partition according to the longest terminal path and the constructive algorithm of Theorem 4.1.

2) If (v_i, v_j) is not a pair, then we added a color for this arc in **Step 3**.

• If $(v_i, v_j) \notin A$, then

1) If (v_i, v_j) is a pair, then by previous procedure $|\varphi''(v_i)| = |\varphi''(v_j)|$.

2) If (v_i, v_j) is not a pair, then $\varphi''(v_i)$ and $\varphi''(v_j)$ have no color in common based on **Step 2** and **Step 3**. □

On the example of the directed rooted tree shown in Figure 4.3, we see that more careful book-keeping and repeating of the colors used at the different levels allows one to reduce the cardinality of the representation set $\mathcal{C}$ compared to the one guaranteed by the construction of Theorem 4.1. If the vertices of the tree on the top figure are labeled according to the preorder traversal of the tree [77] as v_1, v_2, v_3, and v_4, the longest terminal path vertex partition equals $V_0 = \{v_1\}$, $V_1 = \{v_2, v_3\}$, $V_2 = \{v_4\}$. Using this decomposition and Theorem 4.1, we arrive at a bound for the DIN equal to 9. It is straightforward to see the actual DIN of the tree equals 5. Similarly, the algorithm of Theorem 4.1 assigns 17 distinct colors to the vertices of the tree depicted at the bottom of the figure, while the actual DIN of the tree equals 6. Nevertheless, as we will see in Section 4.3, a color assignment akin to the one described in Theorem 4.1 is needed to handle a number of Hamiltonian DAGs (a digraph with a spanning path).

The algorithm described in the proof of Theorem 4.1 established that every DAG has a directed intersection representation and introduced an algorithmic upper bound on the DIN number of any DAG on n vertices with a leading term $\frac{5}{8}n^2$. An improved upper bound may be obtained using inductive arguments, as described in our main result, Theorem 4.4, and its proof. For simplicity, we only present the proof for even n. If n is odd, one can add a vertex v_{n+1} to the end of the path and apply the theorem to obtain a directed intersection representation that uses at most $\frac{5}{8}(n+1)^2 - \frac{3}{4}(n+1) + 1$ colors.

Theorem 4.4. *Let $D = (V, A)$ be an acyclic digraph on n vertices. If n is even, then*

$$DIN(D) \leq \frac{5n^2}{8} - \frac{3n}{4} + 1.$$

Proof. We prove a stronger statement which asserts that for a left-to-right ordering of the vertices V of an arbitrary acyclic digraph D, there exists a representation φ such that

(a) $|\varphi(v_1)| = \frac{n}{2}$, $|\varphi(v_2)| \geq \frac{n}{2}$, and $|\varphi(v_i)| \geq \frac{n}{2} + 1$ for $3 \leq i \leq n$.

(b) For each pair (v_{2i-1}, v_{2i}), if $(v_{2i-1}, v_{2i}) \in A$ then $|\varphi(v_{2i-1})| = |\varphi(v_{2i})| - 1$, and if $(v_{2i-1}, v_{2i}) \notin A$ then $|\varphi(v_{2i-1})| = |\varphi(v_{2i})|$ for $1 \leq i \leq \frac{n}{2}$.

(c) $\cup_{i=1}^{n} \varphi(v_i)$ contains at most $\frac{5n^2}{8} - \frac{3n}{4} + 1$ colors.

The base case $n = 2$ is straightforward, as a connected DAG contains only one arc. In this case, we use $\{1\}$ for the head and $\{1, 2\}$ for the tail, and this representation clearly satisfies (a), (b), and (c).

We hence assume $n \geq 4$ and delete the arc (v_1, v_2) from D to obtain a new digraph D'; the ordering $(v_3, \ldots, v_n)$ is still a left-to-right ordering of D'. Thus, by the induction hypothesis, D' has a representation φ' satisfying

1) $|\varphi'(v_3)| = \frac{n}{2} - 1$, $|\varphi'(v_4)| \geq \frac{n}{2} - 1$, and $|\varphi'(v_i)| \geq \frac{n}{2}$ for $5 \leq i \leq n$;

2) For each pair of vertices (v_{2i-1}, v_{2i}), if $(v_{2i-1}, v_{2i}) \in A$, then $|\varphi(v_{2i-1})| = |\varphi(v_{2i})| - 1$, and if $(v_{2i-1}, v_{2i}) \notin A$, then $|\varphi(v_{2i-1})| = |\varphi(v_{2i})|$ for $2 \leq i \leq \frac{n}{2}$, and

3) The representation φ' uses at most

$$\frac{5(n-2)^2}{8} - \frac{3(n-2)}{4} + 1 = \frac{5n^2}{8} - \frac{3n}{4} + 1 - \left(\frac{5}{2}n - 4\right) \tag{4.3}$$

colors.

We initialize our procedure by letting $\varphi = \varphi'$.

Case 1: $(v_1, v_2) \notin A$.

Step 1: Assign to v_1 a set of $\frac{n}{2} - 1$ new colors, say $\{\alpha_1, \ldots, \alpha_{\frac{n}{2}-1}\}$. Let $\varphi(v_1) = \{\alpha_1, \ldots, \alpha_{\frac{n}{2}-1}\}$. Assign to v_2 a set of $\frac{n}{2} - 1$ new colors, say $\{\beta_1, \ldots, \beta_{\frac{n}{2}-1}\}$, all of which are distinct from the colors in $\{\alpha_1, \ldots, \alpha_{\frac{n}{2}-1}\}$. Let $\varphi(v_2) = \{\beta_1, \ldots, \beta_{\frac{n}{2}-1}\}$.

Step 2: Add the same color γ to both $\varphi(v_1)$ and $\varphi(v_2)$.

Step 3: For arcs including v_1, and for each $2 \leq i \leq \frac{n}{2}$, we perform the following procedure:

- If $(v_1, v_{2i-1}) \in A$ and $(v_1, v_{2i}) \in A$, then we copy a color from $\varphi(v_1)$ (say, α_{i-1}) to both $\varphi(v_{2i-1})$ and $\varphi(v_{2i})$.
- If $(v_1, v_{2i-1}) \in A$ and $(v_1, v_{2i}) \notin A$, then we copy a color from $\varphi(v_1)$ (say, α_{i-1}) to $\varphi(v_{2i-1})$.
- If $(v_1, v_{2i-1}) \notin A$ and $(v_1, v_{2i}) \in A$, then we copy a color from $\varphi(v_1)$ (say, α_{i-1}) to $\varphi(v_{2i})$.

• If $(v_1, v_{2i-1}) \notin A$ and $(v_1, v_{2i}) \notin A$, then we do nothing.

Step 4: For arcs including v_2, and for each $2 \leq i \leq \frac{n}{2}$, we perform the following procedure:

• If $(v_2, v_{2i-1}) \in A$ and $(v_2, v_{2i}) \in A$, then we copy a color from $\varphi(v_2)$ (say, β_{i-1}) to both $\varphi(v_{2i-1})$ and $\varphi(v_{2i})$.

• If $(v_2, v_{2i-1}) \in A$ and $(v_2, v_{2i}) \notin A$, then we copy a color from $\varphi(v_2)$ (say, β_{i-1}) to $\varphi(v_{2i-1})$.

• If $(v_2, v_{2i-1}) \notin A$ and $(v_2, v_{2i}) \in A$, then we copy a color from $\varphi(v_2)$ (say, β_{i-1}) to $\varphi(v_{2i})$.

• If $(v_2, v_{2i-1}) \notin A$ and $(v_2, v_{2i}) \notin A$, then we do nothing.

Next, assume that the DAG representation φ is as constructed above.

Step 5: For each $2 \leq i \leq \frac{n}{2}$, we add colors to both $\varphi(v_{2i-1})$ and $\varphi(v_{2i})$ so that the new representation φ satisfies

$$|\varphi(v_j)| - |\varphi'(v_j)| = 3.$$

In the process, we reuse colors to minimize the number of newly added colors. Since the procedures in Step 3 and Step 4 increase the color set of each vertex by at most 2, one may need to add as many as 3 new colors to a vertex representation (Note that we actually only need the difference to be 2, but for consistency with respect to Case 2 we set the value to 3). As an example, assume that we added $j \in \{0, 1, 2\}$ colors to $\varphi(v_{2i-1})$ and $k \in \{0, 1, 2\}$ colors to $\varphi(v_{2i})$ in Step 3 and Step 4. Then, we need to add $\max\{3 - j, 3 - k\}$ colors to obtain the desired representation, which for $j = 0$ or $k = 0$ results in 3 new colors. This is repeated for each pair, with at most 3 distinct added colors.

Claim 4.5. *The representation φ includes at most $\frac{5}{2}n - 4$ new colors.*

Proof. We used

$$\frac{n}{2} - 1 + \frac{n}{2} - 1 + 1 = n - 1$$

colors in Step 1 and Step 2. We used at most $3 \cdot (\frac{n}{2} - 1)$ in Step 5. Therefore, we used at most

$$n - 1 + \frac{3}{2}n - 3 = \frac{5}{2}n - 4$$

new colors in total. □

Claim 4.6. *The color assignments φ constitute a valid representation satisfying conditions **(a)**, **(b)**, and **(c)**.*

Proof. **(i):** For a pair of vertices (u, w) such that $u \in V - \{v_1, v_2\}$ and $w \in V - \{v_1, v_2\}$, we consider the following cases

1) If $(u, w) \in A$, then since φ' constituted a valid representation, we have that a) the intersection condition holds for φ because the two vertices still have representations with a color in common, and b) the size condition holds since we added three colors to both the color sets of u and w.

2) If $(u,w) \notin A$, and if u, w belong to different pairs, then since φ' is a valid representation and we added distinct colors to different pairs of vertices in Step 5, φ is a valid representation. This claim holds since if the vertices u and w have no color in common in φ', then they still have no color in common after different colors are added in Step 5. Furthermore, if the representation sets of the vertices had the same size before we added three colors to each color set, the sizes will remain the same. If u, w belong to the same pair, their color set sizes were the same in φ' and they stay the same after colors are added in Step 5. Hence, φ is still valid.

Similarly, for a pair of vertices (u,w) such that $u \in \{v_1, v_2\}$ and $w \in V - \{v_1, v_2\}$, we consider the following cases.

1) If $(u,w) \in A$, then the intersection condition holds for φ because we added a common color to the color sets of u and w in Step 3 or Step 4. Furthermore, the size condition holds since

$$|\varphi(w)| = |\varphi'(w)| + 3 \geq \frac{n}{2} - 1 + 3 > \frac{n}{2} = |\varphi(u)|.$$

Therefore, φ is a valid representation.

2) If $(u,w) \notin A$, then φ is valid since we did not add any common color to the color sets of the two vertices, and the set $\varphi'(u)$ was obtained by augmenting it with distinct colors.

Recall that under Case 1, $(v_1, v_2) \notin A$ and $|\varphi(v_1)| = |\varphi(v_2)|$. Hence, φ is a valid representation.

In addition, we have

(a): $|\varphi(v_1)| = |\varphi(v_2)| = \frac{n}{2}$ and $|\varphi(v_i)| \geq \frac{n}{2} - 1 + 3 \geq \frac{n}{2} + 1$, for $3 \leq i \leq n$.

(b): For each pair (v_{2i-1}, v_{2i}), if $(v_{2i-1}, v_{2i}) \in A$, then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})| - 1$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 3 = |\varphi'(v_{2i})| - 1 + 3 = |\varphi(v_{2i})| - 1.$$

If $(v_{2i-1}, v_{2i}) \notin A$, where $2 \leq i \leq \frac{n}{2}$, then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})|$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 3 = |\varphi'(v_{2i})| + 3 = |\varphi(v_{2i})|.$$

These properties also hold for $i = 1$, as previously established.

(c): By Claim 4.5, we used at most $\frac{5}{2}n - 4$ new colors. □

Case 2: $(v_1, v_2) \in A$.

Step 1: This step follows along the same lines as Step 1 of Case 1.

Step 2: Add a common color γ to both $\varphi(v_1)$ and $\varphi(v_2)$ to satisfy the intersection constraint, and add a new color δ to $\varphi(v_2)$ to satisfy the size constraint.

Step 3: This step follows along the same lines as Step 3 of Case 1.

Step 4: This step follows along the same lines as Step 4 of Case 1.

Step 5: This step follows along the same lines as Step 4 of Case 1.

Using the same counting arguments as before, it can be shown that the above steps introduce $\frac{5}{2}\,n-3$ new colors (see the claim below).

Claim 4.7. *We used at most* $2.5n - 3$ *new colors.*

Claim 4.8. *One can remove (save) one color from the given representation.*

Proof. **Case 1:** $(v_2, v_3) \in A$.

Case 1.1: $(v_2, v_4) \in A$. Then $\beta_1 \in \varphi(v_3) \cap \varphi(v_4)$ and we can save one color for the pair (v_3, v_4) in Step 5 as only two colors suffice.

Case 1.2: $(v_2, v_4) \notin A$.

Case 1.2.1: $(v_1, v_3) \in A$. If $(v_1, v_4) \in A$, then $\alpha_1 \in \varphi(v_3) \cap \varphi(v_4)$ and we can save one color introduced in Step 5. If $(v_1, v_4) \notin A$, then $\beta_1 \in \varphi(v_3)$ and $\alpha_1 \in \varphi(v_3)$. We replace $\beta_1 \in \varphi(v_3)$ by δ and replace $\beta_1 \in \varphi(v_2)$ by α_1 and remove β_1. This saves one color.

Case 1.2.2: $(v_1, v_4) \in A$. Since $\beta_1 \in \varphi(v_3)$ and $\alpha_1 \in \varphi(v_4)$, we can discard one color used in Step 5.

Case 1.2.3: $(v_1, v_3) \notin A$ and $(v_1, v_4) \notin A$. Then α_1 is unused and we can thus replace α_1 in $\varphi(v_1)$ by δ to save one color.

Case 2: $(v_2, v_3) \notin A$.

Case 2.1: $(v_2, v_4) \in A$. Then $\beta_1 \in \varphi(v_4)$. If $(v_1, v_3) \in A$, then $\alpha_1 \in \varphi(v_3)$ and we can save a color in Step 5. Thus, we may assume that $(v_1, v_3) \notin A$. In this case, if $(v_1, v_4) \in A$, then $\alpha_1 \in \varphi(v_4)$ and we replace $\alpha_1 \in \varphi(v_4)$ by a color we used in Step 5 for v_3 (recall that in Step 5, we added three new colors to $\varphi(v_3)$ and only reused one of them in $\varphi(v_4)$; hence, there are two colors remaining). In addition, we replace $\alpha_1 \in \varphi(v_1)$ by β_1 to save one color. Thus, we may assume $(v_1, v_4) \notin A$. Then, α_1 is not used in the second pair and we may replace $\alpha_1 \in \varphi(v_1)$ by δ to save one color.

Case 2.2: $(v_2, v_4) \notin A$.

Case 2.2.1: If $(v_1, v_3) \in A$ and $(v_1, v_4) \in A$, then $\alpha_1 \in \varphi(v_3) \cap \varphi(v_4)$ and we saved a color in Step 5.

Case 2.2.2: If $(v_1, v_3) \notin A$ and $(v_1, v_4) \notin A$, then we may replace $\beta_1 \in \varphi(v_2)$ by α_1 to save one color.

Case 2.2.3: If $(v_1, v_3) \in A$ and $(v_1, v_4) \notin A$ or $(v_1, v_3) \notin A$ and $(v_1, v_4) \in A$, then we modify Step 5 by requiring that the color sets be augmented by two rather than three colors. This allows us to save at least one color. □

Claim 4.9. *The representation φ is valid and it satisfies conditions **(a)**, **(b)**, and **(c)**.*

Proof. We separately consider two cases.

• For Case 2.2.3,

For a pair of vertices (u, w) such that $u \in V - \{v_1, v_2\}$ and $w \in V - \{v_1, v_2\}$, we consider the following cases.

1) If $(u, w) \in A$, then since φ' constituted a valid representation we have that a) the intersection condition holds for φ because the two vertices still have a representation with a color in common, and b) the size condition holds since we added two colors to both the color set of u and w.

2) If $(u, w) \notin A$, and if u, w belong to different pairs, then since φ' is a valid representation and we added distinct colors to different pairs in Step 5, φ is a valid representation. This claim holds since if the vertices u and w have no color in common in φ', then they still have no color in common after different colors are added in Step 5. Furthermore, if the color set representations of two vertices had the same size, then since we added two colors to both color sets, the color sets of the vertices will still have the same size. If u, w belong to the same pair, then their color size were the same in φ' and remain the same after colors are added in Step 5. Hence, φ is a valid representation.

Similarly, for a pair of vertices (u, w) such that $u \in \{v_1, v_2\}$ and $w \in V - \{v_1, v_2, v_3, v_4\}$, we consider the following cases.

1) If $(u, w) \in A$, then a) the intersection condition holds for φ because we added one common color in Step 3 or Step 4, and b) the size condition holds since

$$|\varphi(w)| = |\varphi'(w)| + 2 \geq \frac{n}{2} + 2 > \frac{n}{2} + 1 \geq |\varphi(u)|.$$

Therefore, φ is a valid representation.

2) If $(u, w) \notin A$, then φ is valid since

$$\varphi(w) \geq \frac{n}{2} + 2 > \frac{n}{2} + 1 \geq \varphi(u)$$

and we did not add a common color for the two vertices, and $\varphi'(u)$ was obtained by adding distinct colors to $\varphi(u)$.

For (v_1, v_3), when $(v_1, v_3) \in A$ we added α_1 to $\varphi(v_3)$ so that

$$|\varphi(v_3)| = \frac{n}{2} + 1 > \frac{n}{2} = |\varphi(v_1)|.$$

When $(v_1, v_3) \notin A$ we added distinct colors to $\varphi(v_1)$ and $\varphi(v_3)$. Thus, φ is valid.

For (v_1, v_4), when $(v_1, v_4) \in A$ we added α_1 to $\varphi(v_4)$ so that

$$|\varphi(v_4)| = \frac{n}{2} + 1 > \frac{n}{2} = |\varphi(v_1)|.$$

When $v_1 v_4 \notin A$ we added distinct colors to $\varphi(v_1)$ and $\varphi(v_4)$. Thus, φ is valid.

For (v_2, v_3), we added distinct colors to $\varphi(v_2)$ and $\varphi(v_3)$. Thus, φ is valid.

For (v_2, v_4), we added distinct colors to $\varphi(v_2)$ and $\varphi(v_4)$. Thus, φ is valid.

For (v_1, v_2), since $(v_1, v_2) \in A$, $\gamma \in \varphi(v_1) \cap \varphi(v_2)$, and $|\varphi(v_1)| = |\varphi(v_2)| - 1$ we have that φ is valid.

To verify that conditions **(a)**, **(b)** and **(c)** are satisfied, observe that:

(a): $|\varphi(v_1)| = |\varphi(v_2)| - 1 = \frac{n}{2}$ and $|\varphi(v_i)| \geq \frac{n}{2} + 1$ for $3 \leq i \leq n$.

(b): For each pair (v_{2i-1}, v_{2i}), if $(v_{2i-1}, v_{2i}) \in A$ then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})| - 1$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 2 = |\varphi'(v_{2i})| - 1 + 2 = |\varphi(v_{2i})| - 1.$$

This claim is also true for $i = 1$, which we already showed.

If $(v_{2i-1}, v_{2i}) \notin A$, where $2 \leq i \leq \frac{n}{2}$, then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})|$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 2 = |\varphi'(v_{2i})| + 2 = |\varphi(v_{2i})|.$$

(c): By Claim 4.7 and Claim 4.8, we used at most $2.5n - 4$ new colors.

• For the other cases,

For a pair of vertices (u, w) such that $u \in V - \{v_1, v_2\}$ and $w \in V - \{v_1, v_2\}$, we consider the following cases.

If $(u, w) \in A$ then since φ' was valid 1) the intersection condition still holds for φ because they still have color in common and 2) the size condition still hold since we added three colors to each of the color set of u and w.

If $(u, w) \notin A$, and the two vertices are in different pairs then since φ' was valid and we added distinct colors to different pairs in Step 5, we have that φ is valid because if u and v have no color in common in φ' then they still have no color in common after we added different colors in Step 5; if they had the same size in φ' then since we added three colors to each color set their sizes remain the same. If the two vertices are in the same pair then their color size was the same in φ' and it stays the same after adding colors in Step 5. Hence, φ is still valid.

For a pair of vertices (u, w) such that $u \in \{v_1, v_2\}$ and $w \in V - \{v_1, v_2\}$, we consider the following cases.

If $(u, w) \in A$ then 1) the intersection condition holds for φ because we added a common color in Step 3 or Step 4 to the color sets of u and w and 2) the size condition hold since

$$|\varphi(w)| = |\varphi'(w)| + 3 \geq \frac{n}{2} - 1 + 3 > \frac{n}{2} + 1 \geq |\varphi(u)|.$$

Therefore, φ is valid.

If $(u, w) \notin A$ then φ is valid since we did not add any common color for them and u uses distinct colors from φ'.

For (v_1, v_2), since $(v_1, v_2) \in A$, $\gamma \in \varphi(v_1) \cap \varphi(v_2)$, and $|\varphi(v_1)| = |\varphi(v_2)| - 1$ we have that φ is valid.

To verify that conditions **(a)**, **(b)** and **(c)** are satisfied, observe that:

(a): $|\varphi(v_1)| = |\varphi(v_2)| - 1 = \frac{n}{2}$ and $|\varphi(v_i)| \geq \frac{n}{2} - 1 + 3 \geq \frac{n}{2} + 1$ for $3 \leq i \leq n$.

(b): For each pair (v_{2i-1}, v_{2i}), if $(v_{2i-1}, v_{2i}) \in A$ then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})| - 1$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 3 = |\varphi'(v_{2i})| - 1 + 3 = |\varphi(v_{2i})| - 1.$$

This claim is also true for $i = 1$, which we already showed.

If $(v_{2i-1}, v_{2i}) \notin A$, where $2 \leq i \leq \frac{n}{2}$, then $|\varphi'(v_{2i-1})| = |\varphi'(v_{2i})|$. Thus,

$$|\varphi(v_{2i-1})| = |\varphi'(v_{2i-1})| + 3 = |\varphi'(v_{2i})| + 3 = |\varphi(v_{2i})|.$$

(c): By Claim 4.7 and Claim 4.8, we used at most $2.5n - 4$ new colors.
This proves the claim. □

This completes the proof of the theorem. □

4.3 Extremal DIN Digraphs and Lower Bounds

The derivations in Section 4.2 proved that for any DAG D on n vertices, one has

$$DIN(D) \leq \frac{5n^2}{8} - \frac{3n}{4} + 1. \tag{4.4}$$

In comparison, the intersection number of any graph on n vertices is upper bounded by $\frac{n^2}{4}$ [37]. Furthermore, the existence of undirected graphs that meet the bound $\frac{n^2}{4}$ can be established by observing that the intersection number of a graph is equivalent to its edge-clique cover number and since there is a triangle-free graph on n vertices, which has at least $\lfloor \frac{n^2}{4} \rfloor$ edges. The extremal graphs with respect to the intersection number are the well-known Turan graphs $T(n, 2)$ [88].

Consequently, the following question is of interest in the context of directed intersection representations: Do there exist DAGs that meet the upper bound in (4.4) and which DIN values are actually achievable? To this end, we introduce the notion of *DIN-extremal* DAGs: A DAG on n vertices is said to be DIN-extremal if it has the largest DIN among all DAGs with the same number of vertices.

Directed path DAGs, e.g., directed acyclic graphs $D(V, A)$ with $V = \{1, 2, \ldots, n\}$ and

$$A = \{(1, 2), (2, 3), (3, 4), \ldots, (n-1, n)\}$$

have DINs $\frac{n^2}{4} + O(n)$. The following result formalizes this observation.

Proposition 4.10. *Let $D(V,A)$ be a directed path on n vertices. If n is even, then $DIN(D) = \frac{n^2+2n}{4}$; if n is odd, then $DIN(D) = \frac{n^2+2n+1}{4}$.*

Proof. Without loss of generality, assume that the vertices in the directed path are labeled by the positive integers $1,\ldots,n$, starting from the root vertex and ending in the source vertex of the path. We start by placing one single color in $\varphi(1)$ and two colors in $\varphi(2)$ so that one color is reused from $\varphi(1)$ and one is new. We repeat this procedure for the vertices $3 \geq i \leq n$ to ensure that $\varphi(i) = i$, and so as to reuse $2,2,3,3,\ldots,\lceil\frac{n-1}{2}\rceil,\lceil\frac{n}{2}\rceil$ colors for consecutive vertices. This establishes an upper bound on the DIN, since $2\sum_{i=1}^{\lceil\frac{n-2}{2}\rceil} i + \lceil\frac{n-1}{2}\rceil + \lceil\frac{n}{2}\rceil$ equals either $\frac{n^2+2n}{4}$ or $\frac{n^2+2n+1}{4}$, depending on n being even or odd, respectively. To show that the upper bound is met, observe that a labeling that assigns colors

$$\{1\},\{1,2\},\{2,3,4\},\{3,4,5,6\},\{5,6,7,8,9\}...$$

to the vertices of the path is valid. $\square$

Figure 4 provides examples of DIN-extremal DAGs for $n \leq 7$ vertices. These digraphs were obtained by combining computer simulations and proof techniques used in establishing the upper bound of (4.4). Direct verification for large n through exhaustive search is prohibitively complex, as the number of connected/disconnected DAGs with n vertices follows a "fast growing" recurrence [81]. For example, even for $n = 6$, there exist 5984 different unlabeled DAGs. Note that all listed extremal DAGs contain a directed path visiting each of the n vertices exactly once. As such, the digraphs have a unique topological order induced by the directed path, and for the decomposition described on page 5 one has $|V_i| = 1$ for all $i \in [n]$. Note that the bound in (4.4) for $n = 2,3,4,5,6,7$ equals $2,4,8,12,19,26$, respectively. Hence, the upper bound in (4) is loose for $n \geq 6$.

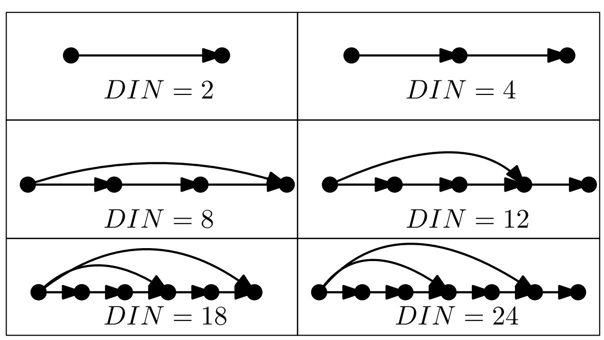


Figure 4.4: Examples of DIN-extremal digraphs for $n \leq 7$.

For all $n \leq 7$ the extremal digraphs are what we refer to as *source arc-paths*, illustrated in Figure 5 a),b). A source arc-path on n vertices has the following arc set

$$A = \{(v_1, v_{2k}) : k \in [\lfloor n/2 \rfloor]\} \cup \{(v_k, v_{k+1}) : k \in [n-1]\}.$$

It is straightforward to prove the following result.

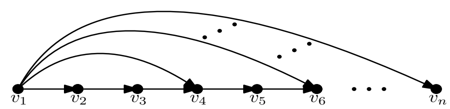


Figure 4.5: Source arc-path, n even.

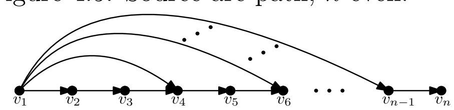


Figure 4.6: Source arc-path, n odd.

Proposition 4.11. *The DIN of a source arc-path on n vertices is equal to $\lfloor \frac{n^2}{2} \rfloor = \lfloor \frac{4n^2}{8} \rfloor$. Hence, the DIN of source arc-paths is by $\frac{n^2}{8}$ smaller than the leading term of the upper bound* (4.4).

Proof. A *directed triangle* in a digraph $D = (V, A)$ is a collection of three vertices $\{v_i, v_j, v_k\}$ such that $(v_i, v_j) \in A$, $(v_j, v_k) \in A$, and $(v_i, v_k) \in A$. Since a source arc-path avoids directed triangles and every vertex has a color set of different size than another (due to the presence of the directed Hamiltonian path), every color may be used at most twice. We need $\frac{n}{2}$ colors for $\varphi(v_1)$ to represent the arcs $v_1 v_{2i}$, where $1 \leq i \leq \frac{n}{2}$. Since the size of the color sets φ increases along the directed path, vertex v_j in the natural ordering has $\varphi(v_j) \geq \frac{n}{2} + j - 1$. Furthermore, $(v_{2i}, v_{2j}) \notin A$ for a source arc-path, for all $1 \leq i < j \leq \frac{n}{2}$. Thus, $\varphi(v_{2i}) \cap \varphi(v_{2j}) = \emptyset$, $1 \leq i < j \leq \frac{n}{2}$. This implies the number of colors needed is

$$\geq \frac{n}{2} + 1 + \frac{n}{2} + 3 + \cdots + \frac{n}{2} + n - 1 = \frac{n}{2} \cdot \frac{n}{2} + \frac{(1 + n - 1)(\frac{n}{2})}{2} = \frac{n^2}{2}.$$

To show that the above lower bound is met, we exhibit the following representation φ with $\frac{n}{2}$ colors:
1) $\varphi(v_1) = \{c_1, \ldots, c_{\frac{n}{2}}\}$, $\varphi(v_2) = \{c_1, f_1, g_{1,1}, \ldots, g_{\frac{n}{2}-1,1}\}$.
2) For $2 \leq i \leq \frac{n}{2} - 1$,

$$\varphi(v_{2i}) = \{c_i, d_i, f_i, g_{1,i}, \ldots, g_{\frac{n}{2}+2i-4,i}\},$$

$$\varphi(v_n) = \{c_{\frac{n}{2}}, d_{\frac{n}{2}}, g_{1,\frac{n}{2}}, \ldots, g_{\frac{n}{2}+n-3,\frac{n}{2}}\}.$$

3) For $2 \leq i \leq \frac{n}{2} - 1$,

$$\varphi(v_{2i-1}) = \{d_i, f_{i-1}, g_{1,i}, \ldots, g_{\frac{n}{2}+2i-4,i}\}.$$

$$\varphi(v_{n-1}) = \{f_{\frac{n}{2}-1}, d_{\frac{n}{2}}, g_{1,\frac{n}{2}}, \ldots, g_{\frac{n}{2}+n-4,\frac{n}{2}}\}.$$

□

For $n \geq 8$, there exist DAGs with DINs that exceed those of source arc-paths which are obtained by adding carefully selected additional arcs. For even integers n, the DIN of such digraphs equals

$$\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

A digraph with the above DIN has a vertex set $V = \{v_1, \ldots, v_n\}$ and arcs constructed as follows:

Step 1: Initialize the arc set as $A = \emptyset$.

Step 2: Add to A arcs of a source-arc-path, i.e.,

$$A = A \cup \{(v_1, v_{2i}) : i \in [\frac{n}{2}]\} \cup \{(v_j, v_{j+1}) : j \in [n-1]\}.$$

Step 3: Add arcs with tails and heads in the set $\{v_3, v_5, \ldots, v_{n-1}\}$ according to the following rules:

Step 3.1: If $\frac{n-2}{2}$ is even, then let $X = \{v_3, v_5, \ldots, v_{\frac{n}{2}}\}$ and $Y = \{v_{\frac{n}{2}+2}, \ldots, v_{n-1}\}$. Add all arcs between X and Y except for $(v_{\frac{n}{2}}, v_{\frac{n}{2}+2})$.

Step 3.2: If $\frac{n-2}{2}$ is odd, then let $X = \{v_3, v_5, \ldots, v_{\frac{n}{2}+1}\}$ and $Y = \{v_{\frac{n}{2}+3}, \ldots, v_{n-1}\}$. Add all arcs between X and Y except for $(v_{\frac{n}{2}+1}, v_{\frac{n}{2}+3})$.

The above described digraphs have no directed triangles and their number of arcs equals

$$\lfloor \frac{(\frac{n}{2}-1)^2}{4} \rfloor - 1 = \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

We start with the following lower bound on the DIN number of the augmented source-arc-path digraphs.

Proposition 4.12. *The DIN of the above family of digraphs is at least*

$$\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

Proof. Due to the presence of the arc of a source-arc-path, v_1 requires at least $\frac{n}{2}$ colors. Furthermore, since the graph has a spanning directed path, the size of the color sets increases along the path. Based on the previous two observations, one can see that v_i requires at least $\frac{n}{2} + i - 1$ colors for all $i \in [n]$.

Since there are no arcs in the digraph induced by the vertex set $\{v_2, v_4, \ldots, v_n\}$ with even labels, the color sets of these vertices have to be mutually disjoint. Thus, the number of colors needed to color vertices with even indices is at least

$$\frac{n}{2} + 1 + \frac{n}{2} + 3 + \ldots + \frac{n}{2} + n - 1 = \frac{n^2}{2}.$$

Since the digraphs avoid directed triangles and every pair of vertices has a different color set sizes, we require one additional color to represent each of the arcs added in Step 3. Due to the absence of directed triangle, we need at least $\lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1$ colors. Furthermore, the color sets used for the two previously described vertex sets are disjoint. Thus, the number of colors required is at least

$$\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

$\square$

To show that the above number of colors suffices to represent the digraphs under consideration, we provide next a representation φ using $\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1$ colors.

Claim 4.13. *There exists a representation φ using $\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1$ colors.*

Proof. We start by exhibiting a representation φ' of the source-arc-path that uses $\frac{n^2}{2}$ colors and then change the color assignments accordingly:

1) Set $\varphi(v_1) = \{c_1, \ldots, c_{\frac{n}{2}}\}$ and $\varphi(v_2) = \{c_1, f_1, g_{1,1}, \ldots, g_{\frac{n}{2}-1,1}\}$.

2) For $2 \leq i \leq \frac{n}{2} - 1$, set

$$\varphi(v_{2i}) = \{c_i, d_i, f_i, g_{1,i}, \ldots, g_{\frac{n}{2}+2i-4,i}\} \quad \text{and} \quad \varphi(v_n) = \{c_{\frac{n}{2}}, d_{\frac{n}{2}}, g_{1,\frac{n}{2}}, \ldots, g_{\frac{n}{2}+n-3,\frac{n}{2}}\}.$$

3) For $2 \leq i \leq \frac{n}{2} - 1$, set

$$\varphi(v_{2i-1}) = \{d_i, f_{i-1}, g_{1,i}, \ldots, g_{\frac{n}{2}+2i-4,i}\} \quad \text{and} \quad \varphi(v_{n-1}) = \{f_{\frac{n}{2}-1}, d_{\frac{n}{2}}, g_{1,\frac{n}{2}}, \ldots, g_{\frac{n}{2}+n-4,\frac{n}{2}}\}.$$

Let $m := \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1$.

1') Set $\Gamma_{2i-1} = \{g_{1,i}, \ldots, g_{\frac{n}{2}+2i-4,i}\}$.

2') Order the m arcs in the graph induced by $\{v_3, v_5, \ldots, v_{n-1}\}$ in an arbitrary fashion, say $\{e_1, \ldots, e_m\}$. Set a counter variable to $k = 1$.

3') For $e_k = (v_{2i-1}, v_{2j-1})$, assign a previously unused color h_k to both $\varphi(v_{2i-1})$ and $\varphi(v_{2j-1})$. Pick one color g' from Γ_{2i-1} and a color g'' from Γ_{2j-1} not previously used in the procedure. Set

$$\varphi(v_{2i-1}) = \varphi(v_{2i-1}) \cup h_k - g' \qquad \text{and} \qquad \Gamma_{2i-1} = \Gamma_{2i-1} - g',$$

$$\varphi(v_{2j-1}) = \varphi(v_{2j-1}) \cup h_k - g'' \qquad \text{and} \qquad \Gamma_{2j-1} = \Gamma_{2j-1} - g''.$$

Let $k = k + 1$. If $k \leq m$, go to Step 3'), otherwise stop.

4') Since each v_{2i-1} has degree at most $\frac{n}{4}$ on the digraph induced by $\{v_3, \ldots, v_{n-1}\}$ and at step $k = 1$ we had $|\Gamma_{2i-1}| = \frac{n}{2} + 2i - 4$, we do not run out of colors to replace. This follows since when we choose g' from Γ_{2i-1} we always have $\geq \frac{n}{2} + 2i - 4 - \frac{n}{4}$ colors available.

5') Since g', g'' were used twice in φ' and deleted only once in the processing steps (and thus remain in the union of the colors), each iteration of the procedure in 3) introduces exactly one new color (e.g., h_k) to φ. Therefore, the number of colors used is

$$\frac{n^2}{2} + m = \frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1.$$

□

This completes the construction of digraphs on n vertices with DIN values $\frac{n^2}{2} + \lfloor \frac{n^2}{16} - \frac{n}{4} + \frac{1}{4} \rfloor - 1$.

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