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Plesio-geostrophy for Earth's core: I. Basic equations, inertial modes and induction

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An approximation is developed that lends itself to accurate description of the physics of fluid motions and motional induction on short time scales (e.g. decades), appropriate for planetary cores and in the geophysically relevant limit of very rapid rotation. Adopting a representation of the flow to be columnar (horizontal motions are invariant along the rotation axis), our characterisation of the equations leads to the approximation we call *plesio-geostrophy*, which arises from dedicated forms of integration along the rotation axis of the equations of motion and of motional induction. Neglecting magnetic diffusion, our self-consistent equations collapse all 3-D quantities into 2-D scalars in an exact manner. For the isothermal magnetic case, a series of fifteen partial differential equations is developed that fully characterises the evolution of the system. In the case of no forcing and absent viscous damping, we solve for the normal modes of the system, called inertial modes. A comparison with a subset of the known 3-D modes that are of the least complexity along the rotation axis shows that the approximation accurately captures the eigenfunctions and associated eigenfrequencies.

1. Introduction

Convection in the Earth's core is believed to be responsible for the generation of the planet's magnetic field. Unravelling the details of this process has been thwarted by the extremity of the dynamical environment: the combination of rapid rotation with low viscosity leads to a parameter regime that is impossible to simulate numerically, and, as such, demands that suitable approximations be made in order for realistic calculations to be carried out.

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The purpose of this paper is to present a new theoretical basis for representing the short-timescale dynamics of the core that takes into account at zeroth order the overwhelming effect of the rapid rotation of the planet. In gearing our theory towards short timescales, we neglect the effects of finite electrical conductivity and adopt an ideal theory of the secular variation, as championed by [1].

We build on the foundations laid down by Hough [2], Taylor [3] and Proudman [4] who showed, both theoretically and experimentally, the tendency for slow inviscid motions of a rapidly-rotating fluid to be invariant along the rotation axis. The physics of this approximation, which has been central to much development in oceanography and meteorology, is admirably summarised in [5,6]. This tendency for two-dimensionality is the starting point of a theoretical treatment that allows us to collapse the three-dimensional motions into compact forms such that they can be represented by scalars in two-dimensions (2-D), some of which arise from vertical integrations along the rotation axis. Although the ansatz of columnar flow may appear to be strong, a series of comparisons have proven its applicability and accuracy over the last years [7–9]. Theories that take into account the tendency for two-dimensionality have their beginnings in [10], and have proven their utility in many areas of oceanography.

Unlike in non-conducting fluids, the presence of electrical currents and magnetic fields **B** in the Earth's core give rise to Lorentz forces that play a potentially important role in the force balance represented by the Navier-Stokes equation. Additionally, a vital difference between planetary cores and Earth's oceans is the depth of the fluid: existing shallow water theories that have proved so effective in the oceanic realm are inapplicable to the dynamics of the deep Earth. It is these features that necessitate the development of a theory that can self-consistently represent the dynamics and inductive effects in a deep conducting fluid, which we term *plesio-geostrophy*, hereinafter PG.

As adumbrated by [11], "*Nearly all theoretical work in geophysical fluid dynamics is based on approximate forms of the equations of motion, but the best ground-rules for deriving such approximate forms are not clear*". We agree with this sentiment, and our work in developing a new form of the PG equations is based on an axial integration of the equations of motion that collapses all quantities of interest into 2-D. We also perform axial integration of the induction equation, and find a system of equations that is closed. In cylindrical coordinates (s, ϕ, z) the magnetic forces are represented in terms of certain "moments", an example of which is $\overline{B_\phi^2}$, the vertically integrated squared azimuthal magnetic field component, cross-moments such as $\overline{B_\phi B_s}$, together with specific values of **B** evaluated on the equatorial plane. In total thirteen interior magnetic quantities are required to characterise the system, along with a stream-function for the flow and a representation of the magnetic field at the core-mantle boundary that is "stirred" by the velocity field and which serves as a link between the fluid motions and the behaviour of the magnetic field in the vacuum exterior. Two further scalar fields represent the effects of buoyancy through the temperature T. All of these fields have their own evolution equation, and are self-consistent, requiring no other knowledge.

Our work is heavily influenced by the innovative work instigated by Canet and collaborators [12] and summarised in §8.09.2 of [13] (though see the discussion in [14, p1775]). Central to the methodology we implement is the idea that the Coriolis force provides a strong rigidity to fluid motions in the axial direction [10]. The central premise is a representation of horizontal fluid motions that are independent of the z coordinate, complemented by a vertical flow which, in the presence of boundaries that deviate only slightly from horizontal, is weak. Such a model was described in an oceanographic context by [15] and in a geomagnetic context by [16]. Despite the strict inapplicability of the model in the presence of steep slopes, evidence shows [17] that such an approximation works remarkably well.

We have so-far deferred from attaching specific names to theories, to avoid confusion. We find, for example, the term "quasi-geostrophic" may be interpreted differently by different practitioners, and for that reason we avoid its use. In the asymptotic theory of planar convection it represents a specific theory [18], whereas conversely §5.3 of [6] attaches the moniker to a shallow-water oceanographic setting. The geomagnetic context tends to attach the name to a representation of the fluid flow, but even this representation is not unique, existing in both divergenceless and non-divergenceless forms. In our work we follow the incompressible representation first introduced by Schaeffer & Cardin [16], and will often refer to it as the columnar representation.

Despite its novelty, the work of [12] has seen little use. An application of the theory that is specialised to magnetic fields that are invariant along the rotation axis and which vanish at the core-mantle boundary has been used to good effect by [19]. The original general equations are approximate in the sense that certain boundary terms must be neglected, and a study of this system of equations by [20] highlighted the danger of neglecting these terms. The present work is exact in the sense of having no terms that cannot be evaluated self-consistently, and allows for general magnetic fields on the core surface.

A crucial facet of theory for the present purposes was the discovery by [21,22] that the columnar representation remains a good approximation even in the presence of strong magnetic fields of amplitude $\mathcal{B}$, provided that the Lehnert number (Le) remains small. Denoting the Alfvén speed by $V_A = \mathcal{B}/\sqrt{\mu_0 \rho}$ where μ_0 is the permeability of free space and ρ is the density, the Lehnert number is

$$Le = \frac{V_A}{\Omega l}, \tag{1.1}$$

where l is a length scale and Ω is the rotation rate. Note that the Lehnert number is equivalent to a conventional Rossby number based on an Alfvén speed rather than a fluid velocity. Jault [21] found that when $Le < 10^{-2}$ the columnar assumption remains an accurate representation of fluid behaviour on short time scales. In the interior of the Earth's core $r \leq r_o$ we believe $\mathcal{B} \sim$ 3mT and, taking $l = r_o \sim$3000km , the Lehnert number is $O(10^{-4})$. The observable secular variation at the core surface might be resolvable to spherical harmonic degree 20, for which the Lehnert number is still comfortably in the columnar dynamics range. However, the theory we present is a diffusionless theory, and therefore can only apply on time scales for which magnetic diffusion can be neglected. The fundamental decay time for a length scale of the size of the core is approximately 100 000 years, and so at a length scale of $\frac{1}{20} r_o$, the decay time is roughly 250 years, placing a limit on the validity of the columnar assumption.

One of the ultimate aims of the present development is to put in place a theoretical framework that can be used specifically for the purposes of geomagnetic data assimilation. The short time scale dynamics of the core, as exhibited through the secular variation (SV) of the magnetic field, offers a way of deducing key properties of the internal structure of the Earth's fluid core. Ingredients that can plausibly be detected in the core include interior field strengths and geometries, and possibly features of the buoyancy field that drives the flow. The lofty goal of deducing these properties can most plausibly be pursued using data assimilation, that is by using observed changes in the geomagnetic field, as recorded at the Earth's surface and above by observatories, surveys and satellites, together with a model of the interior dynamics of the core; such a model of the dynamics in meteorology is traditionally called a dynamical core (where "core" has a sense of "heart" in this context). Initial studies using both sequential [23–26] and variational [27,28] formulations have given reason for optimism, though the attendant proviso concerns the appropriateness of the dynamical regime that is represented by the "dynamical core" of such schemes.

A notable implementation of a data assimilation scheme in a very specific context has already taken place through the work of [29], who detected the SV signal of a class of waves in the core called *torsional oscillations*. These oscillations, whose signature lies in the period range around 5 years, are the only class of fluid motions for which a calculable prognostic dynamical evolution equation exists. Because the motions are purely axisymmetric, their evolution equation (in the absence of viscosity) is known [30]. The motions experience a restoring force due to the stretching of magnetic field lines, which act as elastic strings. The analysis of such waves was able to deliver a measurement of the interior field strength in the Earth's core in terms of a cylindrically averaged quantity, namely the squared s-component of the magnetic field B_s. To be specific, the quantity recovered was the equatorially-symmetric average $< B_s^2 > /(4\pi H)$ where

$$< \cdot > = \int_{-H}^{H} \int_{0}^{2\pi} \cdot \, d\phi \, dz \tag{1.2}$$

in a cylindrical coordinate system (s, ϕ, z) with origin at the center of the Earth and s being the distance from the rotation axis. The height of the cylinder over which the average takes place is $H = \sqrt{r_o^2 - s^2}$ with r_o the radius of the core-mantle boundary. Gillet and co-workers [29,31] deduced that this moment constrained the interior magnetic field to minimum values of $\sim 2 - 3$ mT, the first determination of an interior field strength, and of interest for dynamo theories in that it constraints the interior field to be at least ten times the surface value. It is the ultimate goal of the geomagnetic community to complement such estimates with other estimates that are derived from non-axisymmetric components of the field.

Although the complexity of the system we derive is daunting, we suspect that the data assimilation problem may be reasonably well-constrained. The overall goal for these equations will be to use them over several decades, such as the post-1980 satellite era of geomagnetism during which changes in the Earth's magnetic field are tightly constrained through vector measurements in space. We note that the present era of geomagnetism is particularly well served through the presence of three ESA Swarm satellites in low Earth orbit, measuring the magnetic field to unprecedented accuracy [32]. The prospect of the mission lasting into the late 2020s is particularly welcome, and will be valuable for the types of data assimilation activities envisaged here.

The paper is structured as follows. In §2 we present a new version of the z-integrated equations of fluid motion in the presence of strong Lorentz forces. In §3 we use the theory to solve the normal mode problem for the class of inertial waves with the least complexity along the z axis, and compare the solutions to existing theories. The magnetic effects and their self-consistent evolution equations are presented in §4. §5 summarises the model. Details on the implementation of non-linear terms and viscous terms are presented in the Appendices, along with a short discussion of possible avenues through which magnetic diffusion can be re-introduced into the problem.

2. The fundamental axially-integrated equations

We work in the whole sphere of radius r_o, denoted V. The motivation for this originates with the canonical nature of the whole sphere problem, and because we are particularly interested in applications to both the viscous and the inviscid scenarios. In the latter case there is an absence of definitive interior boundary conditions that must apply on the so-called tangent cylinder in the case of a spherical shell, defined as the cylinder coaxial with the rotation axis that divides the core into three regions: the possibility of discontinuities on the tangent cylinder is an open question. Despite the interest in the case of no viscosity, we develop the viscous terms that can be used in the application of the equations to the case of thermal convection [9]. When an inner core is present, the projection of the tangent cylinder onto the boundary of the core covers an area of a few percent of the core surface at each pole, and we believe that our approximation is probably quite appropriate; however, future applications may well be able to take the inner core into account.

The sphere is surrounded by an electrically-insulating exterior, $\hat{V}$, and magnetic diffusion is neglected at the outset. We shall return to the question of magnetic diffusion in the Appendix. The boundary between V and $\hat{V}$ is ∂V on which

$$[\mathbf{B}] = 0, \tag{2.1}$$

where $[]$ signifies the jump across ∂V. We use cylindrical coordinates (s, ϕ, z) with origin at the center of the unit sphere V and vertical unit vector $\hat{\mathbf{z}}$ parallel to the rotation axis.

Before deriving our version of the z-integrated equations we note that there are at least three options available in order to introduce the columnar ansatz. The common method is to take the z-component of the vorticity equation before vertical integration [12,16]. A second idea has been implemented in [19], who use a projection method whereby the equations of motion are projected onto the columnar representation. This methodology shows a slight advantage over the vertical vorticity equation in recovering the slow 3-D inertial modes in a sphere [19,33], probably because the projection involves all components of the flow whereas the vertical vorticity method rests on only the horizontal components. Here we suggest the use of a third approach: vertical integration before taking the vertical vorticity. This approach allows for full control of the surface terms arising from the integration of the Lorentz force in the momentum equation.

We begin with some notation. Any function $\phi(z) \in [-H, H]$ can be split into symmetric (even) and anti-symmetric (odd) parts:

$$\phi = \phi_s + \phi_a, \tag{2.2}$$

where

$$\phi_s(z) = \phi_s(-z) \quad ; \quad \phi_a(z) = -\phi_a(-z).$$

Then the even part contributes to the symmetric integral

$$\overline{\phi} = \int_{-H}^{H} \phi \, dz = \int_{-H}^{H} \phi_s \, dz \tag{2.3}$$

whereas its odd part affects only the antisymmetric integral

$$\widetilde{\phi} = \int_{-H}^{H} \mathrm{sgn}(z) \phi \, dz = \int_{0}^{H} \phi_a \, dz - \int_{-H}^{0} \phi_a \, dz = 2 \int_{0}^{H} \phi_a \, dz. \tag{2.4}$$

We consider that the sphere is filled with a Boussinesq, electrically conducting fluid with constant values of thermal expansion coefficient α, kinematic viscosity ν, magnetic diffusivity η, thermal diffusivity κ and reference density ρ_0. We consider a gravitational field $\mathbf{g} = -\gamma \mathbf{r}$, where γ is a positive constant, and a reference state characterized by a conducting temperature profile of the form $\nabla T_s = -\chi \mathbf{r}$. We choose units of Ω^{-1} for time, sphere radius r_o for length, $\rho_0 r_o \Omega^2$ for pressure, $r_o \Omega \sqrt{\rho_0 \mu_0}$ for magnetic fields and $\mathcal{T} = \chi r_o^2 \nu / \kappa$ for temperature. Then the non-dimensional governing equations are

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2\hat{\mathbf{z}} \times \mathbf{u} = -\nabla p + \boldsymbol{\mathcal{L}} + \frac{RaE^2}{Pr} T\mathbf{r} + E\nabla^2 \mathbf{u}, \tag{2.5}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{E}{Pm} \nabla^2 \mathbf{B}, \tag{2.6}$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \frac{1}{Pr} \mathbf{u} \cdot \mathbf{r} + \frac{E}{Pr} \nabla^2 T, \tag{2.7}$$

$$\nabla \cdot \mathbf{u} = \nabla \cdot \mathbf{B} = 0, \tag{2.8}$$

where $\boldsymbol{\mathcal{L}} = (\nabla \times \mathbf{B}) \times \mathbf{B}$ is the non-dimensional Lorentz force, $Ra = \gamma \alpha \mathcal{T} r_o^4 / (\nu \kappa)$ is the traditional Rayleigh number, $E = \nu / (r_o^2 \Omega)$ is the Ekman number, $Pr = \nu / \kappa$ is the Prandtl number and $Pm = \nu / \eta$ is the magnetic Prandtl number.

An alternative non-dimensionalisation that is common in columnar-flow studies is obtained introducing a characteristic magnetic field intensity $\mathcal{B}_0$ and measuring velocities in units of the Alfvén velocity $V_a = \mathcal{B}_0 / \sqrt{\mu_0 \rho}$. This non-dimensionalisation is appropriate, for example, to study hydromagnetic wave propagation in the presence of an externally imposed magnetic field [12,14,19,34–37]. It is however inconvenient if the governing equations are to be tested in non-magnetic contexts to reproduce results from purely thermal columnar convection [38] and inertial wave propagation [33] studies. Equations (2.5)-(2.8) are of more general character, as non-magnetic cases can be considered by simply ignoring the Lorentz term $\boldsymbol{\mathcal{L}}$, without affecting the dimensional temporal units. Under this approximation the system of equations considered in [39] are recovered. The choice of magnetic field intensity units (which is commonplace in dynamo studies [16,40]) is a consequence of our desire to keep the derivation presented below of general nature, without making a-priori assumptions on the intensity of the magnetic field inside the core, and without assuming the presence of an externally imposed magnetic field. It is trivial, however, to specialise our calculations for this latter case as will be done in future studies focusing on hydromagnetic normal-modes calculations.

In the derivation below we write the Lorentz and buoyancy forces on the right hand side of (2.5) as

$$\mathbf{f} = \boldsymbol{\mathcal{L}} + \frac{RaE^2}{Pr} T\mathbf{r}. \tag{2.9}$$

At this point we introduce the most important constraint on the class of flows that we permit, namely the columnar flows of [16]. The columnar approximation is invoked by assuming

$$\mathbf{u}(s,\phi,z) = \frac{1}{H}\nabla\times(\Psi\hat{\mathbf{z}}) - \frac{sz}{H^2}u_s\hat{\mathbf{z}} = \frac{1}{sH}\frac{\partial\Psi}{\partial\phi}\hat{\mathbf{s}} - \frac{1}{H}\frac{\partial\Psi}{\partial s}\hat{\boldsymbol{\phi}} - \frac{z}{H^3}\frac{\partial\Psi}{\partial\phi}\hat{\mathbf{z}}, \tag{2.10}$$

where $\Psi(s,\phi)$ is a pseudo stream-function and $H=\sqrt{1-s^2}$ [16]. We emphasise that (2.10) is an approximation; in the absence of any rigorous asymptotic derivation of this representation, only comparisons with 3-D calculations can serve to convince the reader of the reasonableness of this ansatz. A comparison between 3-D convection calculations and calculations employing (2.10) was carried out by [9]. At an Ekman number $E=10^{-8}$, Rayleigh number $Ra=2\times10^{10}$ and Prandtl number $Pr=10^{-2}$, it was found that 99.8% of the kinetic energy of the 3-D calculation was contained in columnar modes of the form of (2.10). Very few comparisons of this type have been carried out, but we find this result to be an important cornerstone in building confidence in the veracity of (2.10). We refer the reader also to [17].

From the z-component of (2.10) we anticipate, and indeed verify (see §3 and [33]), that Ψ will be $H^3 f(s)$ (with f regular) in order that $\mathbf{u}$ remain regular as $s\to1$. The above formulation automatically satisfies the incompressibility condition $\nabla\cdot\mathbf{u}=0$ and, together with the condition

$$\Psi|_{s=1} = 0, \tag{2.11}$$

non-penetration at the spherical boundary

$$\mathbf{u}\cdot\hat{\mathbf{r}}|_{r=1} = 0, \tag{2.12}$$

where $\hat{\mathbf{r}}$ is the normal to the spherical boundary.

We turn our attention to the Navier-Stokes equation. In our development we temporarily drop the nonlinear advection terms as these pose no conceptual difficulties and are treated in Appendix A. We begin by considering the components perpendicular to the rotation axis (denoted by subscript e for "equatorial") and which we integrate over z from $(-H,H)$ to extract the symmetric component. We make extensive use of Leibniz's rule:

$$\int_{a(s)}^{b(s)}\frac{\partial}{\partial s}\zeta(s,z)\,dz = \frac{\partial}{\partial s}\int_{a(s)}^{b(s)}\zeta(s,z)\,dz - \zeta(s,b(s))\frac{\partial b}{\partial s} + \zeta(s,a(s))\frac{\partial a}{\partial s}. \tag{2.13}$$

Since the columnar ansatz has its equatorial component $\mathbf{u}_e = H^{-1}\nabla\times(\Psi\hat{\mathbf{z}})$ independent of z we find

$$2H\partial_t\mathbf{u}_e + 2\hat{\mathbf{z}}\times(2H\mathbf{u}_e) = -\nabla_e\Pi + \frac{dH}{ds}\left(p(H)+p(-H)\right)\hat{\mathbf{s}} + \overline{\mathbf{f}_e} + E\overline{(\nabla^2\mathbf{u})}_e, \tag{2.14}$$

where Π is now a 2-D pressure field, derived from the vertical integral of p, and $p(\pm H)\equiv p(z=\pm H)$. Note the presence of the original pressure, a result of the use of Leibniz' theorem. Our notation has

$$\frac{dH}{ds} = -\frac{s}{H} \tag{2.15}$$

as the decrease in northern hemisphere boundary height with radius; the southern hemisphere has an oppositely-signed derivative.

To remove the absolute pressure from (2.14) we need to consider the vertical component of (2.5). Taking the antisymmetric integral over z gives

$$\partial_t \widetilde{u}_z = -\widetilde{\nabla_z p} + \widetilde{f}_z + E\widetilde{(\nabla^2 \mathbf{u})}_z. \tag{2.16}$$

We use the form of the vertical velocity to find the required pressure:

$$-\frac{\widetilde{sz}}{H^2}\partial_t u_s = -\left(p(H) + p(-H)\right) + 2p(0) + \widetilde{f}_z + E\widetilde{(\nabla^2 \mathbf{u})}_z \tag{2.17}$$

or namely

$$-s\partial_t u_s = -\left(p(H) + p(-H)\right) + 2p(0) + \widetilde{f}_z + E(\widetilde{\nabla^2 u_z}), \tag{2.18}$$

where $p(0) \equiv p(z=0)$ and the form of the viscous friction will be derived in Appendix B.

Finally we use the azimuthal component of (2.5) on the equator:

$$2su_s + \partial_t(su_\phi) = -\partial_\phi p(0) + sf_\phi(0) + sE(\nabla^2 \mathbf{u})_\phi(0). \tag{2.19}$$

We now look at the vertical vorticity equation, obtained as the vertical component of the curl of (2.14):

$$-2\nabla_e^2 \partial_t \Psi = -\frac{1}{s}\frac{\partial}{\partial \phi}\left[\frac{dH}{ds}\left(s\partial_t(u_s) + \widetilde{f}_z + E(\widetilde{\nabla_e^2 u_z}) + 2p(0)\right)\right] + \hat{\mathbf{z}} \cdot \nabla \times \overline{\mathbf{f}_e}, \tag{2.20}$$

where ∇_e^2 is the component of the Laplacian perpendicular to the rotation axis; we note that the contribution from the Coriolis force in (2.14) vanishes identically. We made use of (2.18) to treat the term $(p(H) + p(-H))$. The term $\partial_\phi p(0)$ can be treated via (2.19). This then leads to

$$\begin{aligned}-2\nabla_e^2 \partial_t \Psi = &-\frac{1}{s}\frac{\partial}{\partial \phi}\left[\frac{dH}{ds}\left(s\partial_t(u_s) + \widetilde{f}_z + E(\widetilde{\nabla^2 u_z})\right)\right] \\ &+ \frac{\partial H}{\partial s}\left(4u_s + 2\partial_t u_\phi - 2f_\phi(0) - 2E(\nabla^2 \mathbf{u})_\phi(0)\right) + \hat{\mathbf{z}} \cdot \nabla \times [\overline{\mathbf{f}_e} + E\overline{(\nabla^2 \mathbf{u})_e}].\end{aligned} \tag{2.21}$$

Rearranging, we have the final form

$$-2\nabla_e^2 \partial_t \Psi = \frac{\partial H}{\partial s}\left[4u_s + 2\partial_t u_\phi - \frac{1}{s}\frac{\partial}{\partial \phi}s\partial_t(u_s)\right] + F, \tag{2.22}$$

where

$$\begin{aligned}F \quad = \quad &\frac{dH}{ds}\left[-2f_\phi(0) - \frac{1}{s}\frac{\partial}{\partial \phi}\widetilde{f}_z - \frac{1}{s}\frac{\partial}{\partial \phi}E(\widetilde{\nabla^2 u_z}) - 2E(\nabla^2 \mathbf{u})_\phi(0)\right] \\ &+ \hat{\mathbf{z}} \cdot \nabla \times [\overline{\mathbf{f}_e} + E\overline{(\nabla^2 \mathbf{u})_e}].\end{aligned} \tag{2.23}$$

This completes the derivation of the velocity part of the PG equations.

3. Inertial modes

In the non-magnetic, isothermal, inviscid case $\mathcal{L} = E = Ra = 0$ we consider the infinitesimal disturbances that can be supported as normal modes. To do so, we write the contributions of the velocity in terms of Ψ. Also, to remain entirely general, so that the formulae can be applied to any axisymmetric shape of container, we define

$$\beta = \frac{1}{H}\frac{dH}{ds}. \tag{3.1}$$

We assume that all azimuthal variations can be described by $\Psi \sim e^{im\phi}$ so that, in terms of Ψ, the differential equation is

$$-2\frac{1}{s}\frac{\partial}{\partial s}\left[s\frac{\partial}{\partial s}\partial_t \Psi\right] + \frac{2m^2}{s^2}\partial_t \Psi - \beta\frac{m^2}{s}\partial_t \Psi + 2\beta\frac{\partial}{\partial s}\partial_t \Psi = \frac{4}{s}\beta\frac{\partial}{\partial \phi}\Psi + F. \tag{3.2}$$

We put the equation into Sturm-Liouville self-adjoint form so that the orthogonality of the eigenfunctions in the case $F=0$ can be determined [33] using an integrating factor $\exp\int -\beta ds$, to find

$$\left[\frac{\partial}{\partial s}\left(\frac{s}{H}\frac{\partial}{\partial s}\right)+\frac{m^2}{H}\left(\frac{\beta}{2}-\frac{1}{s}\right)\right]\partial_t\Psi=-\frac{2}{H}\beta\frac{\partial}{\partial\phi}\Psi-\frac{sF}{2H}. \tag{3.3}$$

For the case of no forcing/dissipation ($F=0$) and a spherical domain (so that $\beta=-s/H^2$), under the ansatz $\Psi\sim e^{i\omega t}$ we have

$$\frac{\partial}{\partial s}\left[\frac{s}{H}\frac{\partial\Psi}{\partial s}\right]-\frac{m^2}{sH}\Psi-\frac{m^2 s}{2H^3}\Psi=\frac{2s}{H^3}\lambda\Psi, \tag{3.4}$$

where $\lambda=m/\omega$. Using the results of [33], we can see that this equation has eigenfunctions $\Psi=s^m H^3 J_{n-1}^{(3/2,m)}(2s^2-1)$, where $J_n^{(\alpha,\beta)}$ is a Jacobi polynomial, with associated eigenfrequencies

$$\omega=-\frac{m}{n(2n+2m+1)+m/2+m^2/4}. \tag{3.5}$$

A plot of these eigenfunctions, fortuitously coincident with a previous theoretical formulation, along with a comparison to true 3-D modes, can be found in Figures 3 and 4 of [33].

A completely different test-function method was developed by [19], which was used to numerically calculate the frequencies of inertial eigenmodes under the columnar ansatz. These eigenmodes were shown to have analytic forms and associated eigenfrequencies by [33], the latter being of the form

$$\omega=-\frac{m}{n(2n+2m+1)+m/2+m^2/6}, \tag{3.6}$$

which one can see is very close in form to that of (3.5). The generality of the present approach will allow magnetohydromagnetic modes supported by a general background magnetic field to be calculated in the future.

The 3-D modes inertial mode frequencies are known from the work of [41–44]. The subset of modes with the longest periods are known to have frequencies

$$\omega_{3\mathrm{D}}=-\frac{2}{m+2}\left(\sqrt{\frac{m(m+2)}{n(2m+2n+1)}+1}-1\right), \tag{3.7}$$

where n is the radial complexity and m is the azimuthal wavenumber. Figure 1 compares the periods $T=2\pi/\omega$ from (3.5) with the exact periods (3.7) for the gravest non-axisymmetric inertial modes from [45]. The agreement is excellent for large n with some significant deviation for the simplest $n=1$ mode as m increases, a behaviour seen previously in [19,33,36].

4. Evolution equations for the magnetic field

We now turn to the more complex problem of the evolution equations in the presence of a magnetic field, but absent buoyancy force. Evolution equations for $\mathcal{L}_\phi(0), \overline{\boldsymbol{\mathcal{L}}_e}$ and $\widetilde{\mathcal{L}_z}$ lead to a closed system.

(a) Vertically integrated quantities

To derive the form of the magnetic terms in the Navier-Stokes equation we need the form of the Lorentz force, which is most easily stated as the divergence of the Maxwell stress tensor:

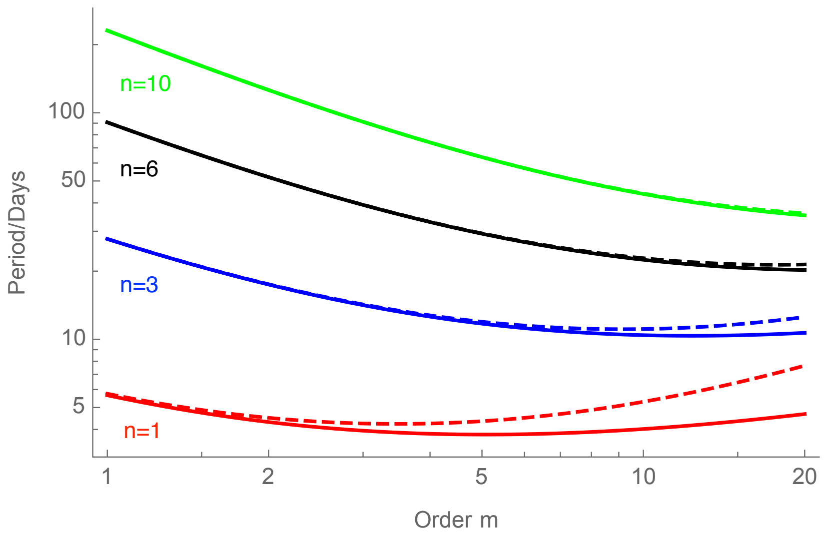

Figure 1. The periods of the PG inertial modes from [45] (solid) and from equation (3.5) (dashed) as a function of the azimuthal wave number m, for different modes n.

$$\begin{aligned}\boldsymbol{\mathcal{L}} = \nabla\cdot\mathsf{M} = &\left(\frac{1}{s}\frac{\partial}{\partial s}(s\,M_{ss}) + \frac{1}{s}\frac{\partial}{\partial \phi}M_{s\phi} - \frac{M_{\phi\phi}}{s} + \frac{\partial}{\partial z}M_{sz}\right)\hat{\mathbf{s}}\\ &+\left(\frac{1}{s}\frac{\partial}{\partial s}(s\,M_{s\phi}) + \frac{1}{s}\frac{\partial}{\partial \phi}M_{\phi\phi} + \frac{M_{s\phi}}{s} + \frac{\partial}{\partial z}M_{\phi z}\right)\hat{\boldsymbol{\phi}}\\ &+\left(\frac{1}{s}\frac{\partial}{\partial s}(s\,M_{sz}) + \frac{1}{s}\frac{\partial}{\partial \phi}M_{\phi z} + \frac{\partial}{\partial z}M_{zz}\right)\hat{\mathbf{z}},\end{aligned} \tag{4.1}$$

where $M_{ij} = M_{ji} = B_iB_j - \frac{1}{2}B^2\delta_{ij}$. We consider $\overline{\boldsymbol{\mathcal{L}}_e}$, and integrate from $-H$ to H, making use of (2.13):

$$\begin{aligned}&\overline{\boldsymbol{\mathcal{L}}_e} =\\ &\left(\frac{1}{s}\frac{\partial}{\partial s}(s\overline{M_{ss}}) + \frac{1}{s}\frac{\partial}{\partial \phi}\overline{M_{s\phi}} + \frac{\overline{M_{\phi\phi}}}{s} - \frac{dH}{ds}(M_{ss}(H)+M_{ss}(-H)) + M_{sz}(H) - M_{sz}(-H)\right)\hat{\mathbf{s}}\\ &+\left(\frac{1}{s}\frac{\partial}{\partial s}(s\overline{M_{s\phi}}) + \frac{1}{s}\frac{\partial}{\partial \phi}\overline{M_{\phi\phi}} - \frac{\overline{M_{s\phi}}}{s} - \frac{dH}{ds}(M_{s\phi}(H)+M_{s\phi}(-H)) + M_{\phi z}(H) - M_{\phi z}(-H)\right)\hat{\boldsymbol{\phi}}.\end{aligned} \tag{4.2}$$

We shall need to evaluate three quadratic moments of the magnetic field, an example of which is

$$\overline{M_{\phi\phi}} = \int_{-H}^{H} B_\phi^2\,dz, \tag{4.3}$$

which are reminiscent of the quadratic quantities originally presented in [12]. In addition we need to control the values of the boundary terms, and these will be given by the exterior potential magnetic field. This matching requires an understanding of the continuity of $\mathbf{B}$ across the viscous boundary layer in the ideal limit, which was provided by [46] and [1] (see also the clear discussion in [13]). We can thus be confident that all components of $\mathbf{B}$ are continuous between mantle and core. Terms such as

$$M_{ss}(H) = (B_s(H))^2 = (\sin\theta B_r + \cos\theta B_\theta)^2 \tag{4.4}$$

can be evaluated in spherical coordinates on ∂V. It is worth returning to (2.22) and noting that the vertical vorticity of $\overline{\boldsymbol{\mathcal{L}}_e}$, namely $\hat{\mathbf{z}} \cdot \nabla \times \overline{\boldsymbol{\mathcal{L}}_e}$, contains s and ϕ derivatives of $\overline{\boldsymbol{\mathcal{L}}_e}$. All required derivatives can be determined since

$$\frac{\partial}{\partial s} = \cos\theta \frac{\partial}{\partial \theta} \quad \text{on } \partial V. \tag{4.5}$$

Turning to $\mathcal{L}_z$ we have

$$\widetilde{\mathcal{L}_z} = \frac{1}{s}\frac{\partial}{\partial s}(s\widetilde{M_{sz}}) + \frac{1}{s}\frac{\partial}{\partial \phi}\widetilde{M_{\phi z}} - \frac{dH}{ds}\left(M_{sz}(H) - M_{sz}(-H)\right) + M_{zz}(H) + M_{zz}(-H) - 2M_{zz}(0) \tag{4.6}$$

and thus two further antisymmetric quadratic moments of the magnetic field must be evaluated.

The evolution of the magnetic field in the core can be derived from the induction equation in a fashion analogous to that used in [14]. Written in cylindrical coordinates, taking into account the fact that $\partial_z(u_s, u_\phi) = 0$, the diffusion free version of these equations reads as follows[1]:

$$\frac{\partial B_s}{\partial t} = B_s \frac{\partial u_s}{\partial s} + \frac{B_\phi}{s}\frac{\partial u_s}{\partial \phi} - u_s \frac{\partial B_s}{\partial s} - \frac{u_\phi}{s}\frac{\partial B_s}{\partial \phi} - u_z \frac{\partial B_s}{\partial z}, \tag{4.7}$$

$$\frac{\partial B_\phi}{\partial t} = B_s\left(\frac{\partial u_\phi}{\partial s} - \frac{u_\phi}{s}\right) + B_\phi\left(\frac{u_s}{s} + \frac{1}{s}\frac{\partial u_\phi}{\partial \phi}\right) - u_s\frac{\partial B_\phi}{\partial s} - \frac{u_\phi}{s}\frac{\partial B_\phi}{\partial \phi} - u_z\frac{\partial B_\phi}{\partial z}, \tag{4.8}$$

$$\frac{\partial B_z}{\partial t} = B_s\frac{\partial u_z}{\partial s} + \frac{B_\phi}{s}\frac{\partial u_z}{\partial \phi} + B_z\frac{\partial u_z}{\partial z} - u_s\frac{\partial B_z}{\partial s} - \frac{u_\phi}{s}\frac{\partial B_z}{\partial \phi} - u_z\frac{\partial B_z}{\partial z}. \tag{4.9}$$

Following [12,14] we derive evolution equations for $\overline{B_s^2}$, $\overline{B_\phi^2}$, $\overline{B_s B_\phi}$, $\widetilde{B_s B_z}$, and $\widetilde{B_\phi B_z}$ by integrating in the vertical the following quantities:

$$\frac{\partial B_s^2}{\partial t} = 2B_s\frac{\partial B_s}{\partial t}, \tag{4.10}$$

$$\frac{\partial B_\phi^2}{\partial t} = 2B_\phi\frac{\partial B_\phi}{\partial t}, \tag{4.11}$$

$$\frac{\partial B_s B_\phi}{\partial t} = B_s\frac{\partial B_\phi}{\partial t} + B_\phi\frac{\partial B_s}{\partial t}, \tag{4.12}$$

$$\frac{\partial B_s B_z}{\partial t} = B_s\frac{\partial B_z}{\partial t} + B_z\frac{\partial B_s}{\partial t}, \tag{4.13}$$

$$\frac{\partial B_\phi B_z}{\partial t} = B_\phi\frac{\partial B_z}{\partial t} + B_z\frac{\partial B_\phi}{\partial t}. \tag{4.14}$$

The derivation involves the use of the following result

$$-\int_a^b u_s z^n \frac{\partial X}{\partial s}dz - \int_a^b u_z z^n \frac{\partial X}{\partial z}dz = -u_s\frac{\partial}{\partial s}\int_a^b z^n X dz + (n+1)\frac{\partial u_z}{\partial z}\int_a^b z^n X dz, \tag{4.15}$$

valid for $X = X(s, \phi, z)$ a generic function of all cylindrical coordinates and when the integration limits (a, b) are either $(-H, H), (-H, 0)$ or $(0, H)$. (4.15) is thus applicable to all integrals of interest. To prove this result we apply Leibniz' integration rule to the first integral on the left-hand

[1]Formulae for the material derivative in cylindrical coordinates, needed in the derivation can be found in, for example, [47].

side of (4.15) and integration by parts on the second:

$$
\begin{aligned}
&-\int_a^b u_s z^n \frac{\partial X}{\partial s} dz - \int_a^b u_z z^n \frac{\partial X}{\partial z} dz \\
&= -\, u_s \left[\frac{\partial}{\partial s} \int_a^b z^n X dz - \frac{\partial b}{\partial s}\, z^n X\big|_{z=b} + \frac{\partial a}{\partial s}\, z^n X\big|_{z=a} \right] - \left[u_z z^n X \right]_{z=a}^{b} + \int_a^b X \frac{\partial}{\partial z}(z^n u_z) dz \\
&= -\, u_s \frac{\partial}{\partial s} \int_a^b z^n X dz + (n+1) \frac{\partial u_z}{\partial z} \int_a^b z^n X dz \\
&\quad + u_s \Bigg[\underbrace{\left(-\frac{\partial a}{\partial s}\, z^n X\big|_{z=a} + \frac{1}{H} \frac{\partial H}{\partial s}\, z^{n+1} X\Big|_{z=a} \right)}_{S_a} + \underbrace{\left(\frac{\partial b}{\partial s}\, z^n X\big|_{z=b} - \frac{1}{H} \frac{\partial H}{\partial s}\, z^{n+1} X\Big|_{z=b} \right)}_{S_b} \Bigg].
\end{aligned}
\tag{4.16}
$$

The surface terms S_a and S_b vanish in the cases of interest since it can be shown that $S_a(a=0)$, $S_a(a=-H)$, $S_b(b=0)$ and $S_b(b=H)$ all vanish. The final form of the second integral on the right-hand side of (4.15) and (4.16) follows from

$$
\frac{\partial}{\partial z}(z^n u_z) = -\frac{s}{H^2} u_s \frac{\partial z^{n+1}}{\partial z} = -(n+1) \frac{s}{H^2} u_s z^n = (n+1) \frac{\partial u_z}{\partial z} z^n \tag{4.17}
$$

and from the fact that $\partial_z u_z$ is independent of z.

Evolution equations for $\overline{B_s^2}$, $\overline{B_\phi^2}$, $\overline{B_s B_\phi}$ in this formalism are presented in [14] and are derived by integrating respectively (4.10), (4.11) and (4.12) between $z=-H$ and $z=H$, substituting $\partial_t B_s$ and $\partial_t B_\phi$ with the formulae (4.7) and (4.8) and making use of (4.15) with $n=0$, $a=-H$, $b=H$ and $X=B_s^2$, $X=B_\phi^2$ and $X=B_s B_\phi$. The derivation of $\partial_t \widetilde{B_s B_z}$ and $\partial_t \widetilde{B_\phi B_z}$ proceeds in a similar fashion. For example:

$$
\begin{aligned}
\frac{\partial}{\partial t} \widetilde{B_s B_z} = &-u_s \frac{\partial}{\partial s} \widetilde{B_s B_z} - \frac{u_\phi}{s} \frac{\partial}{\partial \phi} \widetilde{B_s B_z} + \widetilde{B_s B_z} \left(2 \frac{\partial u_z}{\partial z} + \frac{\partial u_s}{\partial s} \right) \\
&+ \widetilde{B_z B_\phi} \frac{1}{s} \frac{\partial u_s}{\partial \phi} + \widetilde{B_s^2 \frac{\partial u_z}{\partial s}} + \widetilde{B_s B_\phi \frac{1}{s} \frac{\partial u_z}{\partial \phi}},
\end{aligned}
\tag{4.18}
$$

where we made use of (4.15) with $n=0$, $a=0$, $b=H$ and $X=B_s B_z$. However, this procedure introduced the last two terms on the right-hand side of (4.18) that involve horizontal derivatives of u_z, linear in z. Therefore (4.18) contains $\widetilde{zB_s^2}$ and $\widetilde{zB_s B_\phi}$ and, similarly, an evolution equation for $\widetilde{B_\phi B_z}$ contains $\widetilde{zB_\phi^2}$ and $\widetilde{zB_s B_\phi}$. To close the system of equations we need evolution equations for these new terms. They can be derived in much the same way as the previous ones, considering

$$
\frac{\partial (z B_s^2)}{\partial t} = 2 z B_s \frac{\partial B_s}{\partial t}, \tag{4.19}
$$

$$
\frac{\partial (z B_\phi^2)}{\partial t} = 2 z B_\phi \frac{\partial B_\phi}{\partial t}, \tag{4.20}
$$

$$
\frac{\partial (z B_s B_\phi)}{\partial t} = z B_s \frac{\partial B_\phi}{\partial t} + z B_\phi \frac{\partial B_s}{\partial t}. \tag{4.21}
$$

Taking the integrals of these quantities, employing (4.15) with $n=1$, $a=0$, $b=H$ and $X=zB_s^2$, $X=zB_\phi^2$, $X=zB_sB_\phi$ we finally obtain a closed system of equations for the magnetic moments:

$$
\frac{\partial \overline{B_s^2}}{\partial t} = -H\left(\mathbf{u}\cdot\nabla_e\right)\left(\frac{\overline{B_s^2}}{H}\right) + 2\frac{\partial u_s}{\partial s}\overline{B_s^2} + \frac{2}{s}\frac{\partial u_s}{\partial \phi}\overline{B_s B_\phi}, \tag{4.22}
$$

$$
\frac{\partial \overline{B_\phi^2}}{\partial t} = -\frac{1}{H}\left(\mathbf{u}\cdot\nabla_e\right)\left(H\overline{B_\phi^2}\right) + 2s\overline{B_s B_\phi}\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right) - 2\overline{B_\phi^2}\frac{\partial u_s}{\partial s}, \tag{4.23}
$$

$$\frac{\partial \overline{B_s B_\phi}}{\partial t} = -\left(\mathbf{u}\cdot\nabla_e\right)\overline{B_s B_\phi} + s\overline{B_s^2}\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right) + \frac{\overline{B_\phi^2}}{s}\frac{\partial u_s}{\partial \phi}, \tag{4.24}$$

$$\begin{aligned}\frac{\partial \widetilde{B_s B_z}}{\partial t} = &-\left(\mathbf{u}\cdot\nabla_e\right)\widetilde{B_s B_z} - \frac{\partial}{\partial s}\left(\frac{s u_s}{H^2}\right)(\widetilde{zB_s^2}) - \frac{1}{H^2}\frac{\partial u_s}{\partial \phi}(\widetilde{zB_s B_\phi})\\ &+\left(2\frac{\partial u_z}{\partial z} + \frac{\partial u_s}{\partial s}\right)\widetilde{B_s B_z} + \frac{1}{s}\frac{\partial u_s}{\partial \phi}\widetilde{B_\phi B_z},\end{aligned} \tag{4.25}$$

$$\begin{aligned}\frac{\partial \widetilde{B_\phi B_z}}{\partial t} = &-\left(\mathbf{u}\cdot\nabla_e\right)\widetilde{B_\phi B_z} - \frac{\partial}{\partial s}\left(\frac{s u_s}{H^2}\right)(\widetilde{zB_s B_\phi}) - \frac{1}{H^2}\frac{\partial u_s}{\partial \phi}(\widetilde{zB_\phi^2})\\ &+\left(\frac{\partial u_z}{\partial z} - \frac{\partial u_s}{\partial s}\right)\widetilde{B_\phi B_z} + s\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right)\widetilde{B_s B_z},\end{aligned} \tag{4.26}$$

$$\frac{\partial (\widetilde{zB_s^2})}{\partial t} = -\left(\mathbf{u}\cdot\nabla_e\right)(\widetilde{zB_s^2}) + 2\left(\frac{\partial u_s}{\partial s} + \frac{\partial u_z}{\partial z}\right)(\widetilde{zB_s^2}) + 2\frac{1}{s}\frac{\partial u_s}{\partial \phi}(\widetilde{zB_s B_\phi}), \tag{4.27}$$

$$\frac{\partial (\widetilde{zB_\phi^2})}{\partial t} = -\left(\mathbf{u}\cdot\nabla_e\right)(\widetilde{zB_\phi^2}) + 2s\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right)(\widetilde{zB_s B_\phi}) - 2\frac{\partial u_s}{\partial s}(\widetilde{zB_\phi^2}), \tag{4.28}$$

$$\frac{\partial (\widetilde{zB_s B_\phi})}{\partial t} = -\left(\mathbf{u}\cdot\nabla_e\right)(\widetilde{zB_s B_\phi}) + s(\widetilde{zB_s^2})\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right) + \frac{(\widetilde{zB_\phi^2})}{s}\frac{\partial u_s}{\partial \phi} + (\widetilde{zB_s B_\phi})\frac{\partial u_z}{\partial z}. \tag{4.29}$$

(b) Evolution equations in the equatorial plane

To complete the forcing of the Navier-Stokes equation, we need to be able to evaluate $\mathcal{L}_\phi(0)$ and $M_{zz}(0)$ (which arises in (4.6)). We have

$$\mathcal{L}_\phi = \frac{1}{s}\partial_s(s\, M_{s\phi}) + \frac{1}{s}\partial_\phi M_{\phi\phi} + \frac{M_{s\phi}}{s} + \partial_z M_{\phi z} \tag{4.30}$$

and therefore evolution equations are required for $B_s|_{z=0}$, $B_\phi|_{z=0}$, $B_z|_{z=0}$ and $\partial_z B_\phi|_{z=0}$. For brevity, we indicate these (and other) quantities simply by $B_s(0)$, $B_\phi(0)$, $B_z(0)$and $\partial_z B_\phi(0)$. The induction equations evaluated on the equatorial plane are greatly simplified since $u_z(0) = 0$. In compact notation, this allows us to write the following:

$$\begin{aligned}\frac{\partial \mathbf{B}(0)}{\partial t} &= (\mathbf{B}_e(0)\cdot\nabla_e)\mathbf{u}_e - (\mathbf{u}_e\cdot\nabla_e)\mathbf{B}(0) + \mathbf{e}_z B_z(0)\frac{\partial u_z}{\partial z}, &(4.31)\\ \frac{\partial}{\partial t}\frac{\partial \mathbf{B}_e(0)}{\partial z} &= \left(\frac{\partial \mathbf{B}_e}{\partial z}(0)\cdot\nabla_e\right)\mathbf{u}_e - (\mathbf{u}_e\cdot\nabla_e)\frac{\partial \mathbf{B}_e}{\partial z}(0) - \frac{\partial u_z}{\partial z}\frac{\partial \mathbf{B}_e}{\partial z}(0). &(4.32)\end{aligned}$$

Note that the horizontal gradient operators need to be applied to the unit vectors as well as to the component of the vectors they are applied to. The above system is closed in $B_s(0)$, $B_\phi(0)$, $B_z(0)$, $\partial_z B_\phi(0)$ and $\partial_z B_s(0)$, the latter being required by the evolution equation of $\partial_z B_\phi(0)$.

(c) Boundary values and secular variation

Equations (4.2) and (4.6) involve quantities evaluated in cylindrical coordinates at $z = \pm H$; these are boundary values and we alluded to their treatment in (4.4). On the surface of the sphere ∂V the induction equation reads

$$\frac{\partial B_r}{\partial t} = -\nabla_h\cdot(\mathbf{u}_h B_r) \tag{4.33}$$

and this equation governs the stirring of the radial field by the flow scalar Ψ. The subscript h indicates an horizontal component, tangential to ∂V. Since the radial field determines the exterior potential, all components of $\mathbf{B}$ are known at the surface of the sphere, and $[\mathbf{B}]=0$ means all components are known inside ∂V. The scalar quantity B_r is the only quantity that is expressed in spherical coordinates $[\theta,\phi]\in[0,\pi]\times[0,2\pi]$. The link between Ψ, which is expressed in cylindrical coordinates, and B_r has been studied by [48], and exact band-limited relations can be found there.

(d) Summary of the evolution equations

A closed system of equations involves the eight moment equations (4.22)–(4.29), the five equations in the equatorial plane (4.31) –(4.32) , and the radial induction equation (4.33) on ∂V, augmented by the evolution equation for Ψ. These 15 equations would be augmented by two further equations for the odd and even integrals of the buoyancy field if thermal effects were to be considered.

(e) The energy equations

The total magnetic energy of the system is

$$E_M = \frac{1}{2}\int\!\!\int\left(\overline{B_s^2}+\overline{B_\phi^2}+\overline{B_z^2}\right)\, s\, ds\, d\phi, \tag{4.34}$$

and we have expressions for the first two terms. Although $\overline{B_z^2}$ does not enter the dynamical equations, we can keep track of its evolution from the initial conditions as follows.

$$\frac{\partial\overline{B_z^2}}{\partial t} = -\left(\mathbf{u}\cdot\nabla_e\right)\overline{(B_z^2)} + 3\overline{B_z^2}\frac{\partial u_z}{\partial z} + 2\overline{zB_sB_z}\frac{\partial}{\partial s}\frac{\partial u_z}{\partial z} + \frac{2}{s}\overline{zB_\phi B_z}\frac{\partial}{\partial\phi}\frac{\partial u_z}{\partial z}, \tag{4.35}$$

which requires:

$$\begin{aligned}\frac{\partial\overline{zB_sB_z}}{\partial t} = &-\left(\mathbf{u}\cdot\nabla_e\right)\overline{zB_sB_z} + \frac{\partial}{\partial s}\frac{\partial u_z}{\partial z}\overline{(z^2B_s^2)} + \frac{1}{s}\frac{\partial}{\partial\phi}\frac{\partial u_z}{\partial z}\overline{(z^2B_sB_\phi)}\\ &+\left(3\frac{\partial u_z}{\partial z}+\frac{\partial u_s}{\partial s}\right)\overline{zB_sB_z} + \frac{1}{s}\frac{\partial u_s}{\partial\phi}\overline{zB_\phi B_z},\end{aligned} \tag{4.36}$$

$$\begin{aligned}\frac{\partial\overline{zB_\phi B_z}}{\partial t} = &-\left(\mathbf{u}\cdot\nabla_e\right)\overline{zB_\phi B_z} + \frac{\partial}{\partial s}\frac{\partial u_z}{\partial z}\overline{(z^2B_sB_\phi)} + \frac{1}{s}\frac{\partial}{\partial\phi}\frac{\partial u_z}{\partial z}\overline{(z^2B_\phi^2)}\\ &+\left(2\frac{\partial u_z}{\partial z}-\frac{\partial u_s}{\partial s}\right)\overline{zB_\phi B_z} + s\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right)\overline{zB_sB_z},\end{aligned} \tag{4.37}$$

$$\frac{\partial}{\partial t}\overline{(z^2B_s^2)} = -\left(\mathbf{u}\cdot\nabla_e\right)\overline{(z^2B_s^2)} + \left(2\frac{\partial u_s}{\partial s}+3\frac{\partial u_z}{\partial z}\right)\overline{(z^2B_s^2)} + 2\frac{1}{s}\frac{\partial u_s}{\partial\phi}\overline{(z^2B_sB_\phi)}, \tag{4.38}$$

$$\frac{\partial}{\partial t}\overline{(z^2B_\phi^2)} = -\left(\mathbf{u}\cdot\nabla_e\right)\overline{(z^2B_\phi^2)} + \frac{\partial u_z}{\partial z}\overline{(z^2B_\phi^2)} + 2s\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right)\overline{(z^2B_sB_\phi)} - 2\frac{\partial u_s}{\partial s}\overline{(z^2B_\phi^2)}, \tag{4.39}$$

$$\frac{\partial\overline{(z^2B_sB_\phi)}}{\partial t} = -\left(\mathbf{u}\cdot\nabla_e\right)\overline{(z^2B_sB_\phi)} + s\overline{(z^2B_s^2)}\frac{\partial}{\partial s}\left(\frac{u_\phi}{s}\right) + \frac{\overline{(z^2B_\phi^2)}}{s}\frac{\partial u_s}{\partial\phi} + 2\overline{(z^2B_sB_\phi)}\frac{\partial u_z}{\partial z}. \tag{4.40}$$

We emphasise that these equations are not relevant for determining the dynamics of the system, but only for calculating prognostic quantities that can give insight into the energy evolution of the system.

(f) Geometric equality constraints

The geometry of the problem leads to duplication of the values at $s=1$ on the equatorial disc and $\theta=\pi/2$ on the unit sphere (the equator). In other words, the same points are represented twice, once by the spherical coordinate system on the boundary and once by the cylindrical coordinate system in the equatorial plane. We write down for completeness the equalities that must apply.

Let $\mathbf{B} = -\nabla\Phi$ in the exterior where $\Phi(r, \theta, \phi)$ is the magnetic potential, determined uniquely from B_r on ∂V as a solution to Laplace's equation with Neumann boundary conditions.

At $s = 1$ we have

$$\begin{aligned}
B_s(0)|_{s=1} &= -\partial_r\Phi(1, \pi/2, \phi), && (4.41)\\
B_\phi(0)|_{s=1} &= -\frac{1}{s}\partial_\phi\Phi(1, \pi/2, \phi), && (4.42)\\
B_z(0)|_{s=1} &= +\frac{1}{s}\partial_\theta\Phi(1, \pi/2, \phi), && (4.43)\\
\partial_z B_s(0)|_{s=1} &= +\frac{1}{s}\partial_\theta\partial_r\Phi(1, \pi/2, \phi), && (4.44)\\
\partial_z B_\phi(0)|_{s=1} &= +\frac{1}{s^2}\partial_\theta\partial_\phi\Phi(1, \pi/2, \phi), && (4.45)
\end{aligned}$$

where the shorthand is

$$\partial_r\Phi(1, \pi/2, \phi) = \lim_{r\to 1} \partial_r\Phi(r, \pi/2, \phi). \qquad (4.46)$$

Such equality constraints can be implemented gracefully by using a tailored expansion for the equatorial quantities.

5. Conclusion

The problem of deriving a suitable dynamical core for geomagnetic data assimilation on timescales of decades has remained in abeyance as a result of the difficulties attendant to inexactness of the initial theory. This obstacle is now removed with the theory of plesio-geostrophy described herein. In the case of unforced infinitesimal inertial waves the comparison between the PG modes and the true 3D modes is very good. This simple exercise provided the first validity test for the newly developed PG equations.

For the magnetic case all of the quadratic quantities required for the Lorentz force term in the equation of motion can be calculated exactly by monitoring the evolution of thirteen separate partial differential equations. A separate system of six partial differential equations computes the vertical magnetic energy that is needed to discover the evolution of the full magnetic energy. The resulting system may seem daunting, but we wish to remark that the mathematical obstacles that have hindered previous efforts and pointed out in previous work [20], have now been completely removed and physically acceptable solutions are within reach.

It is our aspiration to apply the equations to the study of thermal convection at very small Ekman number (cf. [9]), and principally to the problem of geomagnetic data assimilation. The equations lend themselves to a variational formulation, which has shown some promise on highly viscous version of the full 3D equations [27,28], and we foresee no obstacles in the derivation of the adjoint system of equations required for the calculation of derivatives. These developments will be presented in future publications.

A. The nonlinear advection terms

The contribution of the nonlinear terms to the evolution equation for Ψ (2.22) can be accounted for by adding $-(\mathbf{u}\cdot\nabla)\mathbf{u}$ to the forcing term $\mathbf{f}$ (2.9). The columnar version of $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is:

$$
\begin{aligned}
(\mathbf{u}\cdot\nabla)\mathbf{u} = & \left(u_s\frac{\partial u_s}{\partial s} + \frac{u_\phi}{s}\frac{\partial u_s}{\partial \phi} - \frac{u_\phi^2}{s}\right)\hat{\mathbf{s}} \\
& + \left(u_s\frac{\partial u_\phi}{\partial s} + \frac{u_\phi}{s}\frac{\partial u_\phi}{\partial \phi} + \frac{u_s u_\phi}{s}\right)\hat{\boldsymbol{\phi}} \\
& + \left(u_s\frac{\partial u_z}{\partial s} + \frac{u_\phi}{s}\frac{\partial u_z}{\partial \phi} + u_z\frac{\partial u_z}{\partial z}\right)\hat{\mathbf{z}},
\end{aligned}
$$

which can succinctly be decomposed as $(\mathbf{u}\cdot\nabla)\mathbf{u} = (\mathbf{u}_e\cdot\nabla)\mathbf{u}_e + (\mathbf{u}_e\cdot\nabla)u_z\hat{\mathbf{z}} + (u_z\partial_z)u_z\hat{\mathbf{z}}$. The following integral terms are needed:

a) From the horizontal momentum equation:

$$\int_{-H}^{H}(\mathbf{u}_e\cdot\nabla)\mathbf{u}_e\,dz = 2H(\mathbf{u}_e\cdot\nabla)\mathbf{u}_e, \tag{A 1}$$

whereas $(u_z\partial_z)\mathbf{u}_e = 0$.

b) From the vertical NS equation, the integral in the upper half of V:

$$\int_{0}^{H}(\mathbf{u}_e\cdot\nabla)u_z\,dz = (\mathbf{u}_e\cdot\nabla)\left(-\frac{s u_s}{2}\right), \tag{A 2}$$

and a similar term in the lower half of V, whereas

$$\int_{0}^{H}\left(u_z\frac{\partial}{\partial z}\right)u_z\,dz = \frac{\partial}{\partial z}\left(\frac{u_z^2}{2}\right) = \frac{s^2 u_s^2}{2H^2}. \tag{A 3}$$

All of the above terms can all be written in terms of the pseudo stream-function Ψ. The details are omitted.

B. The viscous terms

Let

$$\mathbf{A} = -\nabla\left(\frac{1}{H}\right)\times\Psi\hat{\mathbf{z}} = -\left(\frac{s}{H^3}\hat{\mathbf{s}}\right)\times\Psi\hat{\mathbf{z}} = +\frac{s}{H^3}\Psi\hat{\boldsymbol{\phi}} \equiv A\hat{\boldsymbol{\phi}}. \tag{B 1}$$

Then we write the columnar velocity as

$$\mathbf{u} = \nabla\times\left(\frac{\Psi}{H}\hat{\mathbf{z}}\right) + A\hat{\boldsymbol{\phi}} + u_z\hat{\mathbf{z}}, \tag{B 2}$$

which we call, respectively, $\mathbf{u}_1$, $\mathbf{u}_2$ and $u_z\hat{\mathbf{z}}$ (obviously).

Three terms are needed:

(a) Symmetric integral

The relevant viscous contribution can then be written as

$$\hat{\mathbf{z}}\cdot\nabla\times\nabla^2\mathbf{u}_e = -\hat{\mathbf{z}}\cdot(\nabla\times\nabla\times\nabla\times\mathbf{u}).$$

Then

$$\hat{\mathbf{z}}\cdot(-\nabla\times\nabla\times\nabla\times\mathbf{u}_1) = -\nabla_e^2\nabla^2\left(\frac{\Psi}{H}\right) = -\nabla_e^4\left(\frac{\Psi}{H}\right). \tag{B 3}$$

The second contribution is

$$\hat{\mathbf{z}} \cdot (-\nabla \times \nabla \times \nabla \times \mathbf{u}_2) = \hat{\mathbf{z}} \cdot \left(-\nabla \times \nabla \times \left[\frac{1}{s} \frac{\partial}{\partial s}(sA) \hat{\mathbf{z}} \right] \right) = \nabla_e^2 \left[\frac{1}{s} \frac{\partial}{\partial s}(sA) \right] \hat{\mathbf{z}}. \tag{B 4}$$

The third contribution

$$\hat{\mathbf{z}} \cdot (-\nabla \times \nabla \times \nabla \times u_z \hat{\mathbf{z}}) = 0. \tag{B 5}$$

Finally taking the symmetric integral contributes a factor $2H$.

(b) Antisymmetric integral

Only the $z-$component of $\widetilde{\nabla^2 \mathbf{u}}$ is needed. Neither $\mathbf{u}_1$ nor $\mathbf{u}_2$ contribute. Then:

$$\widetilde{\nabla^2 u_z} = -H^2 \nabla_e^2 \frac{s u_s}{H^2} = -H^2 \nabla_e^2 \frac{1}{H^3} \frac{\partial \Psi}{\partial \phi}. \tag{B 6}$$

(c) Azimuthal equatorial value

The azimuthal equatorial component can be written as:

$$(\nabla^2 \mathbf{u})_\phi(0) = -(\nabla \times \nabla \times \mathbf{u}_1)_\phi(0) - (\nabla \times \nabla \times \mathbf{u}_2)_\phi(0) - (\nabla \times \nabla \times (u_z \hat{\mathbf{z}}))_\phi(0),$$

where each term can be evaluated as:

$$-(\nabla \times \nabla \times \mathbf{u}_1)_\phi(0) = -\left(\nabla \times \nabla \times \nabla \times \left(\frac{\Psi}{H} \hat{\mathbf{z}} \right) \right)_\phi = -\frac{\partial}{\partial s} \left[\nabla^2 \left(\frac{\Psi}{H} \right) \right] = -\frac{\partial}{\partial s} \left[\nabla_e^2 \left(\frac{\Psi}{H} \right) \right], \tag{B 7}$$

$$-(\nabla \times \nabla \times \mathbf{u}_2)_\phi(0) = \nabla^2 A - \frac{A}{s^2} - \frac{1}{s^2} \frac{\partial^2 A}{\partial \phi^2} = \frac{1}{s} \frac{\partial}{\partial s} \left[s \frac{\partial A}{\partial s} \right] - \frac{A}{s^2}, \tag{B 8}$$

$$-(\nabla \times \nabla \times (u_z \hat{\mathbf{z}}))_\phi(0) = -\frac{1}{s} \frac{\partial}{\partial \phi} \frac{\partial u_z}{\partial z} = \frac{1}{s H^3} \frac{\partial^2 \Psi}{\partial \phi^2}. \tag{B 9}$$

C. Re-introduction of magnetic diffusion

We have introduced an ideal theory of fluid motion valid on time scales of years to decades. The neglect of diffusion is motivated by the fact that the longest free-decay timescale in the Earth's core is $\tau_\eta \sim 100\,000$ years. Nevertheless, lengthscales l short compared to the radius of the core r_o decay on time scales of $l^2/r_o^2 \tau_\eta$ and for $l/r_o \sim 0.01$ diffusion will be important. The theory described does not allow for an exact treatment of diffusion, however an approximate theory can be developed that we believe may be useful for capturing the basic physics.

Two length scales present themselves in the velocity field in the PG theory, away from the equatorial region: the long lengthscale $L \sim r_o$ parallel to the rotation axis, and $l \sim r_o/m$ (or r_o/n) perpendicular to this axis. As is well known, the magnetic field responds to this velocity field by arranging itself in similar structures. This motivates the neglect of vertical derivatives, while retaining horizontal derivatives. Thus in our own work we propose to replace ∇^2 by ∇_e^2 in the induction equation. With this approximation the evolution equations remain closed.

A shortcoming is an inability to represent the diffusion in the boundary layer that may form close to the core-mantle boundary. The diffusion in the radial induction equation takes the form

$$\frac{1}{r} \nabla^2 (r B_r) = \nabla^2 (s B_s + z B_z) \tag{C 1}$$

and the neglect of vertical (radial) derivatives in polar regions seems less sound. Ultimately we must drop the unknown radial second derivatives in the diffusion term, and retain the other calculable terms. A similar device was used in [49].

Ethics. No relevant considerations.

Data Accessibility. “This article has no additional data”.

Authors’ Contributions. Both authors jointly developed the theory and contributed to the manuscript.

Competing Interests. There are no competing interests.

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