ON THE ANALYSIS AND DESIGN OF PENSION CONTRACTS FROM THE CUSTOMER'S PERSPECTIVE

by

Zhaoxun Mei

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Department of Actuarial Mathematics and Statistics
School of Mathematical and Computer Sciences
Heriot-Watt University

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Abstract

This thesis aims to design new long term investment structured products that can be used by insurance companies to smooth investment returns for their customers. These products are widely known as pension contracts as they are mainly used in the accumulation part of a pension scheme. Two popular pension schemes in UK are the Defined Benefit (DB) scheme and the Defined Contribution (DC) scheme. Recently, with the transition from the DB scheme to the DC scheme, more individuals must provide for their own retirement without the security of an employer-backed pension promise. Thus, it is of importance to provide the customer with suitable long term investment products. The thesis, consisting of three research papers, aims to show how to design a new pension contract that best meet the demand from the customers.

In order to better understand the pension contracts, our first paper (Chapter 3) carefully examines a traditional with-profits contract in the market. This paper gives a closed form solution for the pricing of this contract and shows that it is overvalued to the customers because of its embedded guarantees. In addition, the smoothing method of this contract exposes the insurer to a risk that cannot be hedged. Moreover, the inter-generation risk sharing has been studied for this contract.

The smoothing method, which is a typical feature of the with-profits products, is examined in detail in the second paper (Chapter 4). This paper compares three common smoothing methods of with-profits contracts in UK and see how the smoothing method performs. We not only compare the absolute terminal smoothed value, but also take the interim utility, customers' satisfaction within the investment horizons, in to account. This has been done by using Multi-Cumulative Prospect the-

ory(MCPT).

The third paper (Chapter 5) propose a new pension contract with the features of guarantees and bonuses. It has transparent structure and clear distribution rule. Under Cumulative Prospect thoery (CPT), the new contract generates higher utility than the contract introduced in Guillén et al. (2006). The result provides the evidence why the guarantees should be included in the pension contract. In addition, our result shows with the increase of policyholder's investment horizons, the proportion of risky asset in the underlying investment portfolio increases while the proportion of risk free asset decreases. This result conforms to the traditional life cycle pension investment advice.

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Publications

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Chapter 1

Motivation and introduction

Saving and investment for retirement is an important issue in almost every country. Samuelson (1958) proposed that people tend to consume less than they produce during their working years so that they can consume something in retirement when they produce nothing. In the real world, a pension contract is just one of the saving products in which people invest when they are young for their life after retirement. which plays an important role in individuals' lifetime wealth portfolio.

As investment products, a large part of pension products are invested in the stock market. The returns of these products are highly dependent on the performance of the underlying assets. Hence, the payout of these contracts are very volatile. They may enjoy high returns in some good years while making a great loss in some bad years. In this sense, smoothed returns are beneficial to both customers and pension companies. On one hand, smoothed returns are helpful for customers to make their life investment plan. On the other hand, payoffs to customers are liabilities for pension companies. Smoothed returns indicate less volatility of the insurers' liabilities which help insurance companies to better estimate their liabilities.

With the transition from Defined Benefit pension schemes to Defined Contribution pension schemes, more individuals must provide for their own retirement without the protection of an employer-backed pension promise. In order to protect the customers from adverse market performance, the research on designing innovative pension products is of significance.

The purpose of this thesis is to design a long term structured investment product for insurance companies to provide the customer's retirement with the financial support. We present three chapters to complete this target. Specifically, we start with reviewing existing pension contracts, especially, the with-profits contracts. A detailed analysis of an existing with-profits contracts is given. In addition, we carefully study one important feature of with-profits contracts, the smoothing mechanism. Lastly, we propose a new contract which provides the lump sum capital for a customer's retirement. In order to help the readers understand these three chapters, we provide a brief introduction to the background of this topic at the beginning. The thesis is organised as follows.

Chapter 2 gives the background material on the thesis. At first, we give a general overview of pension products in the market. The second part of this chapter is introducing how to price a pension contract and manage financial risk. In order to design a pension contract which meets customers' demand, the theoretical background of designing long-term investment products is discussed in the last part.

In Chapter 3, we study an existing pension contract in detail. A closed form solution to price this contract is provided. The result shows that this contract gives a higher value to the customers than its worth owing to the embedded guarantees. Because of a special return smoothing method used in this contract, the customers are able to choose an advantageous start date for the contract and expose the insurer to a non-hedgeable risk. The closed formula to price this speculating entry strategy is also given in this chapter. Moreover, a study of inter-generational risk sharing is given in this chapter as well.

Return smoothing method is a typical feature in traditional pension contracts and is believed to be a virtue of them. Chapter 4 compares three smoothing methods used in current with-profits contracts in terms of fairness and smoothing effects. Both the analytical formulae and numerical methods are used to study how the smoothing methods perform. We not only study the absolute terminal value of each smoothing methods, but also focus on the interim utility of the customers within the investment horizons. A behaviour economics model, Multi Cumulative Prospect Theory (MCPT) is used to study the interim utility of each method.

Chapter 5 proposes a new pension contract with the features of guarantees and bonuses. It has a transparent structure and clear distribution rule. We compare this new contract to an existing contract in the market by using a behavioural model, Cumulative Prospect Theory (CPT). The result shows that the new contract is more attractive to a CPT-maximising customer. In addition, guarantees are proved helpful to improve customers' utility and thus it should be included in the pension contract. Under CPT, we notice that the dynamic optimal investment strategy is a lifestyle investment strategy. With the increase of the investment horizon, more money should be invested in the risky asset. This conforms to traditional pension investment advice.

Chapter 2

Background of pension contracts design

2.1 An overview of pension contracts

For a typical individual, providing a retirement income consists of the accumulation phase and the decumulation phase. The accumulation phase is the period that people use a portion of their monthly salary to build up a pension pot. After retirement, people start to withdraw money from the pension pot and this period is called the decumulation phase. This thesis is more interested in the investment element of a pension contract. Thus, we focus on the accumulation phase of a pension scheme.

There are various types of life insurance products in the market. Based on how to share investment risk and return, life insurance contracts can be generally grouped into three types: with-profits contracts, without-profits contracts and unit-linked contracts. In the rest of the section, an overview of typical pension contracts in the accumulation phase is provided.

2.1.1 With-profits contract

With-profits contract, or participating policy in US, was historically a significant part of the UK life insurance product, but recent years have faced challenges and changes to the business. With-profits business accounted for 42% of total new business in 1985 (measured as Annual Premium Equivalent premium income), but the share reduced to 5% in 2013[1].

The idea of with-profits business can be traced back to mid-eighteenth century. It origins from James Dodson's envision of mutual life assurer. Dodson (1756) suggests there would be two classes of policies for a mutual life assurer: one would underwrite the guarantees and participate in the profits and risks of the insurer while the other only enjoy the fixed amount of assured without sharing life office's experience (profit or loss). The revolutionary idea was implemented by the Equitable Life after the death of Dodson. Since then, with-profit business becomes the paradigm of the profession and prospered in the next 200 years.

With-profits contract generally consists of a term life insurance policy and a savings vehicle, called an endowment policy. Term insurance provides a benefit if the life insured dies within the term. The payoff of the endowment is a lump sum to the policyholder who survives to a known date. This allows the customers to build up funds for a specific purpose, like an income in retirement. To this extent, with-profits contracts works the same as a without-profits contract. However, the significant difference is with-profits contracts will give the policyholder with additional periodic return distributions which are not decided at inception but are determined within the term of the contract to show policyholder's participation in life office's profits.

There are two widely used return distributions strategies followed by the insurance companies, the UK style and the US style. The UK method is known as uniform reversionary bonus method, which was first used by the Equitable Life in the late 18th century. The reversionary bonus is generally expressed as proportion of the sum assured and applied to all contracts in force. Once the bonus is declared, it increases the amount of sum of assured and can never be taken away. The US distribution rule is called the contribution method which was first proposed by Homans (1863) in his paper "On the equitable distribution of surplus". Under the US style, office's profits are usually returned to the policyholders in the form of an annual

[1]Source: "The Management of With-Profits Funds in Run-off", Working party report of Institute and faculty of Actuaries

dividend. The dividend is calculated for each policy to be in proportion to their contributions to the office's surplus, i.e., the excess of his payment to the cost of insurance. In contrast to UK's distribution rule, the dividend can be cash payment, deductions for the later premium, reinvested in the insurance or used to increase the sum assured.

With-profits contracts falls into two main types, the conventional with-profits contracts and the unitised with-profits contracts. Unitised with-profits contracts are operated in a similar way to the conventional contracts except the policy value of the unitised with-profits contracts is expressed by a number of units and its unit value. The change of the value of unitised with-profits contracts can be implemented either by fixing the unit value and changing the number of units to reflect the change of the policy value or changing the unit value while fixing the number of units. Unitised with-profits contract is more popular than conventional with-profits products because it provides a more transparent structure and return distribution rule.

As this chapter mainly studies the investment element of UK's with-profits products, the term life part of the contract is not considered. In this sense, a with-profits contracts works as a pure investment product which provides a lump sum at the maturity in return for the previous payment of premiums. It is a medium to long-term investment vehicle, typically with 20 - 30 years investment horizons. The premiums of the with-profits contracts are pooled by the insurer into a life fund or their investment portfolio which allows the cost of managing with-profits policies and meeting claims to be shared. To have a high overall return, a significant proportion of the fund is invested in high risk assets, like equities and property. The payout to the policyholder of a with-profits contract depends on the underlying performance of the investment fund which may rise significantly in some good years and slump sharply in bad years. In order to mitigate the effects of extreme short term price movements of the investment fund, with-profits contract generally provides a smoothed return which makes the policy value increase stably. In addition, the downside risk borne by the policyholder is limited by the embedded guarantees. Specificaly, the aim of with-profits contracts is to protect the customers against the short term fluctuation in investment returns and to provide a competitive long term rate of return, which is achieved by the following important features of with-profits contracts.

2.1.1.1 Smoothing

The aim of smoothing is to remove short-term volatility from the customers' payout value. The short-term fluctuation is anticipated to cancel itself out over the long term. The asset share of a policy is the underlying value of the investment, which is the accumulation of the premium paid less the expenses at the realised return of the underlying fund. The insurer holds back some profits in good years and covers the amounts needed in bad years. On average, the policy payout of with-profits contracts should equal the asset share. Smoothing is a good way to avoid the dramatic short term price movement. However, it is not able to protect the policyholder from long term market falls. When the value of asset share decreases, the value of the payout falls as well.

2.1.1.2 Guarantee

In order to protect the policyholder from long term adverse price movements, the insurer of with-profits contracts provides guarantees to the policyholder. The guarantees can be a guaranteed amount at the maturity date of the contract or an annually guaranteed interest rate applied to the policy value. If the payout is less than the guaranteed value, the guaranteed value is paid to the policyholder by the insurer. Thus, guarantees are valuable when the underlying investment returns are poor and volatile.

Because of the existence of guarantee, the value of the policy will be invariant if the market falls even though the value of the backing assets will meet a reduction. Policyholders can benefit from these guarantees by having a steady stream of income regardless of how their invested assets perform. But on the other hand, guarantees becomes a burden for the insurers. The cost of hedging rises in such a low interest rate environment. In addition, the new regulation, Solvency II requires the insurance companies hold more capital for the guarantees they issue. Because the risk return profile, insurance companies seldom provides the guarantees for free.

2.1.1.3 Bonus

Bonus is another valuable part of with-profits contracts. If the underlying fund performs well, the insurer may increase the guarantees through the declaration of a bonus. Once a bonus is added, it can never be taken away. The bonus is distributed in the form of regular bonus, declared during the policy term or terminal bonus, which is declared at the maturity date. Regular bonus is generally determined every year based on the investment performance of the insurer. If the underlying assets perform badly, the regular bonus can be zero, i.e. no bonus is declared. Terminal bonus is declared at the maturity date to make sure the payout is fair to the policyholder. In UK, the bonus is generally declared as the form of rate of return of the benefit or sum assured. Cash payments and a reduction in future premiums are also possible forms of bonus in some other countries.

With-profits contracts used to be the most popular pension contract in UK's pension market. However, since the beginning of this century, a few insurance companies, such as Equitable Life, Eagle Star, Royal and Sun Alliance failed to meet their high value guarantees provided with the with-profits contracts. This can be attributed to the decline of interest rates since 1990 and the poor performance of stock market after the burst of the Dot-Com bubble. But more importantly, their failure arose from the mis-pricing of the complex guarantees underlying the with-profits contracts. The insurers neglected the risk that those guaranteed would bite. Further more, for customers, these embedded guarantees and the profit distribution rule are complex and difficult to understand. As (Guillén et al., 2006) stated, with-profits contracts received much criticism from the public because of its opaqueness and complexity. All the above reasons make the with-profits contracts less attractive and demanding. On the other hand, insurers have less motivation to provide with-profits contracts to the customers. This is because cost of hedging seems high in a low interest environment. In addition, under the new regulation Solvency II, guarantees embedded with-profits contracts subject to much higher capital requirements.

Even though with-profits contracts are becoming less popular, its merits and the embedded actuarial thought are valuable in the design of innovative pension products. As the terminal payout of the with profits contracts equals to the sum of assured plus a participation in the profit of the insurance company, the risk of with-profits contracts are shared by the customers and the insurer.

2.1.2 Without-profits contracts

Without-profits contract is similar to with-profits contract. The only difference is that, for without-profits contract, the sum assured is fixed at the inception of the contract and the customer has no participation in the profit of the insurer. Term life insurance product is a common form of with-profits contracts. As the payout to a customer is a guaranteed amount of money, the risks of without-profits contracts are fully borne by the insurers.

2.1.3 Unit-linked contracts

Compared to with-profits contracts, policyholders' premiums of unit-linked contracts are invested into some investment funds. The policyholder buys units of the investment fund with the premium. The value of the units is determined by the performance of the underlying investment fund. The payout of a unit-linked contract is easy to calculate and understand. As there are no guaranteed maturity benefits and the value of the terminal payout only depends on the value and units allocated to the contract, the investment risk is only assumed by the policyholders. In order to protect the downside risk of the policyholders, some guarantees are provided as a rider to the unit-linked contracts.

Goecke (2013) suggests there is a tendency that the market share of with-profits contract is overtaken by the unit-linked contract. The transparency and simple structure make unit-linked contracts the dominant player in UK and other European pension markets.

2.2 The pricing and risk management of pension products

2.2.1 Fairness

Because of the embedded guarantees, the terminal payoff of the with-profits contracts are heavily dependent on the realisation path of the underlying investment performance. The fair price of with-profits are usually determined by risk-neutral pricing which arises from the fundamental theory of asset pricing (FTAP). The FTAP originated from Harrison and Kreps (1979) and Harrison and Pliska (1981, 1983), which established the mathematical theory for Black-Scholes-Merton formulae in option pricing.

Fair pricing is one of the most important part in designing new pension and insurance products. The concept of fairness can be traced back to *Aristotles Nicomachean Ethics* which suggests that fairness means the equality and reciprocity in exchange (see (Broadie and Rowe, 2002), Judson (1997) and Johnson (2015)). Johnson (2015) further proposed that the ethical concept of reciprocity built the foundation for FTAP. By assuming FTAP, the value of the payoff is fair to both the insurers and the customers.

In some cases, true fairness is not likely to exist. When it comes to the fairness in a pooled portfolio, Donnelly (2015) gives a definition of actuarial fairness which is defined as the expected benefits equalling the contributions for each member. Similarly, it is unlikely to have true fairness in the pricing of with-profits contracts which include multiple generations. Milevsky and Salisbury (2016) suggest that the closest concept of truly fairness is equitable which means the expected payout for each generation is the same.

2.2.2 Risk-neutral pricing

Policyholders of pension contracts generally pay premiums to their accounts every month. For with-profits and without-profits contracts, policyholders' premiums are pooled together in a pot and then invested by the insurance company. For unit-linked contracts, on the other hand, the policyholders tend to choose the underlying investment funds themselves. As we stressed earlier, the term insurance feature of the unit-linked contract is not considered here. In this sense, unit-linked contracts look like a mutual fund.

The pension contracts are viewed as pure investment products in this chapter. The pricing methodology is applied to make sure that the contracts are financially fair to both the insurer and the customers. If we view the customers' premiums as an asset of an insurance company, the terminal payout to the policyholder at the end of the contract is a liability of the insurer. The financial fairness for the insurer is that the expected present value of the liability equals the expected present value of the asset. For a policyholder, a financially fair contract means the present value of what he or she will receive at the terminal date of the contract is equal to the present value of the premiums. The financial fairness is based on the notion of no arbitrage. That is to say the expectation is taken under a risk neutral measure $\mathbb{Q}$. Mathematically, a contract with single premium P and investment horizon $T \in N^+$ in a constant interest rate world is said to be financially fair if

$$P = e^{-rT} E^Q[A_T], \tag{2.1}$$

where A_T is the terminal value of customer payout and the r is the constant discount rate.

The aim of pricing is to find the expected present value of the terminal payout. Ideally, a closed form solution is achieved. Then it will bring much convenience to the pricing team. However, in most cases, a closed form formula for the pricing function of pension contracts is not available, especially when mortality, lapse and expenses are considered. In practice, numerical methods, especially Monte Carlo simulation, are widely used in pricing pension contracts.

As a unit-linked contract is like a mutual fund, what the customer will receive is the terminal value of the investment from the underlying investment fund. In the next section, we discuss more about the pricing of with-profits contracts because of its complexity. A few examples about with-profits contracts will be discussed.

2.2.3 Typical structure of with-profits contracts

With-profits contracts typically have three features: smoothing, guarantees and bonus. However, the calculation of the terminal value of each contract can be quite different. To make this clear, in this part, we introduce some typical contracts from the literature and show the payout of these contracts. For easy comparison, all the contracts are assumed to be single premium and no lapse and expense are considered. The terminal value of each contract depends on the performance of a reference portfolio managed by the insurance company. The price process of this reference portfolio is assumed to follow a geometric Brownian motion:

$$\begin{cases} X_0 & = x_0 \\ dX_t & = \mu X_t dt + \sigma X_t dW_t, \end{cases} \tag{2.2}$$

where the expected rate growth rate μ, and volatility σ are strictly positive constants. W is a Wiener process defined on the filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t)_{t\leq 0}, \mathbb{P})$ where $(\mathcal{F}_t)$ is the sigma algebra containing the historical information up to time t of this process. The maturity of the contract is a positive integer T and t is within the finite horizon $[0, T]$. Then the annual return of the reference portfolio is

$$R_t = \frac{X_t}{X_{t-1}} - 1, \text{ for } t = 1, 2, ..., T, \tag{2.3}$$

so that $\{1 + R_t\}_{t=1,2,...,T}$ are independent and log-normally distributed random variables.

Using geometric Brownian motion to model asset price implies strong assumptions like a constant expected return and volatility as well as independent returns in disjoint time intervals, which is not true in practice. However, for long investment horizons, the geometric Brownian motion is not a bad model for illustrating the contract structure. The reason why we choose the geometric Brownian motion to represent the price process is because of its popularity in previous works. Some examples include Grosen and Jørgensen (2000), Bacinello (2001) and Bacinello

(2001). In the following, we summarise the pensions contracts introduced in their paper below to give a broad idea of the structure of with-profits contracts and how to determine their payoffs. The details of each contract are given as follows.

2.2.3.1 Grosen and Jørgensen (2000)

We first introduce the contract studied by Grosen and Jørgensen (2000). At each time $t \in 1, 2, ..., T$, the value of the customer account A_t increases at the policy interest rate r_t^p which is the sum of bonus rate $r_t^b \geq 0$ and the constant guaranteed rate of return $g \geq 0$. That is

$$r_t^p = g + r_t^b, \qquad \text{for} \quad t = 1, 2, ..., T. \tag{2.4}$$

The guarantee rate of return is set by the insurance company. Then the value of the customer account increases at least at the rate of g each year. Bonus rate is an additional compensation to the guaranteed rate. The explicit formula of the bonus rate is presented by the equation (2.5):

$$\begin{cases} r_t^b = \max\left[0, \alpha\left(\frac{B_{t-1}}{A_{t-1}} - \gamma\right) - g\right], & \text{for} \quad t = 1, 2, ..., T, \\ B_t = X_t - A_t, \quad \text{for} \quad t = 1, 2, ..., T, \end{cases} \tag{2.5}$$

where γ is a constant target for the ratio of the value of the bonus account to the customer account $(\frac{B_t}{A_t})$ and α is the participation ratio.

The above equation shows $r_b(t)$ has the form of a call option and the t th year bonus rate $r_b(t)$ is determined at time $t-1$. The value of customer account at the end of each year is

$$\begin{cases} A_0 = P, \\ A_t = A_{t-1} \times (1 + r_p) = A_0 \prod_{j=1}^{t}\left[1 + \max(\alpha\left(\frac{B_{j-1}}{A_{j-1}} - \gamma\right), g)\right], & \text{for} \quad t = 1, 2, ..., T. \end{cases} \tag{2.6}$$

Now the formula to calculate the terminal payout A_T is derived.

2.2.3.2 Bacinello (2001)

Bacinello (2001) introduce a contract whose customer account increases at the policy rate of r_t^p every year. Mathematically,

$$r_t^p = \max(\alpha R_t, g) = g + \max(\alpha R_t - g, 0), \quad t = 1, 2, ..., T, \tag{2.7}$$

where $\alpha \in [0, 1]$ represents the constant participation ratio and $R_t \in \mathbb{R}$ is the random annual return of the underlying asset in year t. The constant $g \geq 0$ is the guaranteed rate of return. Thus the value of customer account A_t in year t is:

$$A_t = A_{t-1}(1 + r_p(t)) = A_0 \prod_{j=1}^{t} \left[1 + \max(\alpha R_j, g)\right], \text{ for } t = 1, 2, ..., T, \tag{2.8}$$

where A_0 is the initial premium P. The terminal value in the customer account A_T is the payout.

2.2.3.3 Døskeland and Nordahl (2008b)

Døskeland and Nordahl (2008b) introduce a more complex contract by including an insurer account C_t. Let A_t and B_t denote the customer account and the bonus account, respectively, at time t. When $t = 0$, the balance sheet of the insurance company is shown in Table 2.1. The parameter ψ is the capital structure parameter, which determines how the risk and return are shared between the insurer and the customers.

Assets	Liabilities
X_0	$C_0 = (1 - \psi)X_0$
	$B_0 = 0$
	$A_0 = \psi X_0 = \text{P}$

Table 2.1: Balance sheet at time $t = 0$.

The mathematical evolution of the value of the customer account A_t, the bonus account B_t and the insurer account C_t for $t = 1, 2, ..., T$ are given as

$$A_t = \begin{cases} X_t & \text{if } X_t \le A_{t-1}e^{\mathrm{g}} \\ A_{t-1}e^{g}, & \text{if } A_{t-1}e^{\mathrm{g}} < X_t \le (A_{t-1} + C_{t-1})e^{\mathrm{g}} + B_{t-1} \\ A_{t-1}e^{\mathrm{g}} + \psi\eta(1-b) & \\ \times(X_t - ((A_{t-1} + C_{t-1})e^{\mathrm{g}} + B_{t-1})), & \text{if } X_t > (A_{t-1} + C_{t-1})e^{\mathrm{g}} + B_{t-1}, \end{cases} \tag{2.9}$$

$$B_t = \begin{cases} 0, & \text{if } A_t \le L_1 \\ X_t - A_{t-1}e^{\mathrm{g}} - C_{t-1}, & \text{if } L_1 < X_t \le L_2 \\ B_{t-1}, & \text{if } L_3 < X_t \le L_3 \\ B_{t-1} + \psi\eta b(X_t - ((A_{t-1} + C_{t-1})e^{\mathrm{g}} + B_{t-1})), & \text{if } X_t > L_4, \end{cases} \tag{2.10}$$

for

$$L_1 = A_{t-1}e^{\mathrm{g}} + C_{t-1}, \tag{2.11}$$

$$L_2 = A_{t-1}e^{\mathrm{g}} + C_{t-1} + B_{t-1}, \tag{2.12}$$

$$L_3 = (A_{t-1} + C_{t-1})e^{\mathrm{g}} + B_{t-1}, \tag{2.13}$$

$$L_4 = A_{t-1}e^{\mathrm{g}} + C_{t-1}e^{\mathrm{g}} + B_{t-1} \tag{2.14}$$

and

$$C_t = X_t - A_t - B_t, \tag{2.15}$$

where $\psi \in (0, 1)$ is the capital structure parameter, η is the customer share of the profits and b is the proportion of declared bonuses credited to the bonus account. In any year, if the event $X_t < A_{t-1}e^g$ happens, the company is said to be bankrupt. Then the customer receives the amount $A_T = X_\tau e^{r(T-\tau)}$ where τ is the random time of default in $[0, T]$. Otherwise, the terminal value A_T is determined by Equation (2.9).

2.2.4 Risk management

2.2.4.1 Estimating Greeks

With-profits products generally include guarantees which can be very valuable. Estimating the risk of guarantees is of great importance to insurance and pensions companies. Thus, risk management is also an important part of the design of pension contracts. In this section, we review some important methods used to measure the risk of embedded guarantees.

Price sensitivities, commonly referred as the Greeks, are the derivatives of the option price with respect to the parameters of the model. In other words, the Greeks present the change of the option price with regard to the change of the parameters of the option price. In the Black-Scholes model, the parameters of the option price includes volatility, risk free rate, maturity, current underlying price and strike price. The first derivative of option price with respect to the underlying price is called Delta(Δ). It gives the number of units of underlying asset to hold in the hedging portfolio. The second derivative of option price with respect to the underlying price is Gamma(Γ) which shows the optimal time interval to re-balance the hedging portfolio.

The Greeks play an important role in risk management of financial derivatives. The embedded guarantees are essentially some forms of options. How to efficiently estimate Greeks is of great importance to life and pension companies. As most of the exotic options do not have closed form formula, it is more difficult to calculate their Greeks. In this section, we are following closely with Glasserman (2003) in introducing three methods that estimate price sensitivities using Monte Carlo methods.

2.2.4.2 Finite Difference Approximations

Finite difference approximation estimates the Greeks by approximating the continuous time differential equation with discrete time difference equation. To be specific, let θ ranging over an open interval in $\mathbb{R}$ denote a parameter of a model. For each value of θ, we can use this model to generate a random variable $Y(\theta)$ which is within some interval of real line. In the context of option pricing, $Y(\theta)$ is the simulation result of discounted payoff of a option. Then the price of the option $\alpha(\theta)$ is given as

$$\alpha(\theta) = E[Y(\theta)]. \tag{2.16}$$

If the parameter θ represents the current price of underlying asset, then $\alpha'(\theta)$ is the option's Delta while $\alpha''(\theta)$ is the option's Gamma. If θ denotes time to expiration, then $\alpha'(\theta)$ is the option's Theta. We can simulate n independent replications $Y_1(\theta), ..., Y_n(\theta)$ and the average of all independent result is denoted as $\overline{Y}(\theta)$. Let $\epsilon \sim N(0, v^2)$ represent the residual of observed sample mean of discounted payoff from the theoretical value $\alpha(\theta)$. That is to say

$$\overline{Y}(\theta) = \alpha(\theta) + \epsilon. \tag{2.17}$$

In order to estimate $\alpha'(\theta)$, the forward difference estimator is often used because it is simple and easy to implement. The forward difference estimator of $\alpha'(\theta)$ is presented as

$$\hat{\Delta}_F = \frac{\overline{Y}(\theta + h) - \overline{Y}(\theta)}{h}, \tag{2.18}$$

where h is a small real value larger than 0.

By Taylor's expansion, assuming α is twice differentiable at θ, we can get

$$\alpha(\theta + h) = \alpha(\theta) + \alpha'(\theta)h + \frac{1}{2}\alpha''(\theta)h^2 + o(h^2). \tag{2.19}$$

From equation (2.17), (2.18) and (2.19), the bias of forward difference estimator is

$$\text{Bias}(\hat{\Delta}_F) = E[\hat{\Delta}_F - \alpha'(\theta)] = \frac{1}{2}\alpha''(\theta)h + o(h). \tag{2.20}$$

It is clear from equation (2.20) that smaller value of h leads to less bias. On the other hand, we know the variance of the estimator is

$$\mathrm{Var}(\hat{\Delta}_F) = h^{-2}[\overline{Y}(\theta + h) - \overline{Y}(\theta)]. \tag{2.21}$$

In this sense, smaller value of h leads to larger variance. Hence, how to decide the value of h is a trade off between variance and the bias of the estimates. The Mean Squared Error (MSE) is defined as

$$\mathrm{MSE}(\hat{\Delta}_F) = \mathrm{Var}(\hat{\Delta}_F) + \mathrm{Bias}(\hat{\Delta}_F)^2. \tag{2.22}$$

Hence, minimising the mean square error could be a possible objective. Let $\theta_1 = \theta_0 + h$, we have $\overline{Y}(\theta_0) = \alpha(\theta_0) + \epsilon_0$ and $\overline{Y}(\theta_1) = \alpha(\theta_0 + h) + \epsilon_1$ where ϵ_0, ϵ_1 are residuals. Now we can rewrite equation (2.18) as:

$$\hat{\Delta}_F = \frac{\alpha(\theta_0 + h) + \epsilon_1 - \alpha(\theta_0) - \epsilon_0}{h} = \Delta + bh + \frac{\epsilon_1 - \epsilon_0}{h}, \tag{2.23}$$

where Δ is $\alpha'(\theta_0)$ and b is $\frac{1}{2}\alpha''(\theta_0)$. Thus, the mean squared error is:

$$\mathrm{MSE}(\hat{\Delta}_F) = E[\hat{\Delta_F} - \Delta]^2 = E\left[\left(bh + \frac{\epsilon_1 - \epsilon_0}{h}\right)^2\right]. \tag{2.24}$$

Let the correlation coefficient ρ of ϵ_1 and ϵ_0 is independent of h. By differentiating equation (2.24) with respect to h and setting first derivative equal to 0, we find the optimal value of h:

$$h^* = \sqrt[4]{\frac{2v^2(1-\rho)}{b^2}}, \tag{2.25}$$

which minimises the MSE. The second derivative of equation (2.24) is positive, which supports the above result. Another popular estimator of the finite difference method is the central difference estimator. It is preferred to the forward difference estimator because the central difference estimator has less bias. The central difference estimator is given as

$$\hat{\Delta}_C = \frac{\overline{Y}(\theta + h) - \overline{Y}(\theta - h)}{2h}. \tag{2.26}$$

However, it requires one more simulation than forward difference estimator (one for $\overline{Y}(\theta+h)$, one for $\overline{Y}(\theta)$ and one for $\overline{Y}(\theta-h)$), thus needing more computing resource.

2.2.4.3 Pathwise Derivative Estimates

Instead of simulating at multiple parameter values, pathwise derivative method estimates the derivatives directly. Hence, this method has increased computing speed and requires less resource. Here we outline the pathwise derivative method.
We know that

$$\alpha'(\theta) = \frac{d}{d\theta} E^Q[Y(\theta)]. \tag{2.27}$$

One important assumption of pathwise derivative estimate method is the interchangeability of expectation and differentiation. That is to say

$$\alpha'(\theta) = \frac{d}{d\theta} E^Q[Y(\theta)] = E^Q[\frac{d}{d\theta} Y(\theta)], \tag{2.28}$$

or

$$\alpha'(\theta) = \lim_{h \to 0} E^Q \left[\frac{Y(\theta + h) - Y(\theta)}{h} \right] = E^Q \left[\lim_{h \to 0} \frac{Y(\theta + h) - Y(\theta)}{h} \right]. \tag{2.29}$$

(Glasserman, 2003, p.393) points that one necessary and sufficient condition for this assumption is uniform integrability of $\frac{Y(\theta+h)-Y(\theta)}{h}$. In other words, for any θ and h there exists a random variable C such that

$$\frac{|Y(\theta + h) - Y(\theta)|}{h} \leq C \text{ a.s.} \tag{2.30}$$

(Glasserman, 2003, p.396) also suggests that the pathwise derivative method typically produces consistent estimators if the payoff function is continuous but is usually not applicable if the payoff function is discontinuous. (Glasserman, 2003, p.394) points out that the payoffs of standard European options, Asian options and look back option are all continuous.

An example of how pathwise derivative method work for European call option is given below. It is known that the discounted payoff Y of a call option is given as

$$Y = e^{-rT}\max(S_T - K, 0) \tag{2.31}$$

with underlying price of

$$S_T = S_0 e^{(r-\frac{1}{2}\sigma^2)T+\sigma\sqrt{T}Z}, \tag{2.32}$$

where Z is a standard normal random variable.

From equation(2.32), we know that

$$\frac{dS_T}{dS_0} = \frac{S_T}{S_0}. \tag{2.33}$$

Equation (2.31) shows us

$$Y = \begin{cases} e^{-rT}(S_T - K), & \text{if} \quad S_T > K; \\ \\ 0, & \text{if} \quad S_T \leq K. \end{cases}$$

So the derivative $\frac{dY}{dS_T}$ is given as

$$\frac{dY}{dS_T} = e^{-rT}\mathbb{1}_{\{S_T > K\}}, \tag{2.34}$$

where $\mathbb{1}$ is the zero-one indicator function.

Combining equation (2.33) and equation(2.34) gives the form of Delta:

$$\hat{\Delta}_{FD} = E^Q[\frac{dY}{dS_0}] = E^Q[\frac{dY}{dS_T}\frac{dS_T}{dS_0}] = e^{-rT}E^Q[\frac{S_T}{S_0}\mathbb{1}_{\{S_T \geq K\}}]. \tag{2.35}$$

Then it is easy to use the simulation result to calculate the Delta. One advantage of pathwise derivative estimate against finite difference approximation is that the pathwise method provides unbiased estimates of derivatives. The other is that the pathwise method requires less computing time since re-sampling is not necessary. The reason is the output generated from the initial simulation includes considerable information which could be used to estimate the price sensitivities directly.

2.2.4.4 Likelihood ratio method

As explained in the last section, the pathwise method requires that the payoff function is continuous. It is primarily this requirement that limits its scope and application. Instead of differentiating the payoff function in respect of the parameter of interest, the likelihood ratio method gives another approach by using the relationship between the parameter of interest and probability density of the price of the underlying asset. The discounted payoff Y can be written as a function of a random vector $X = (X_1, ..., X_m)$ where X_i denotes the underlying asset price at time i. The probability density function of X is denoted as $g_\theta(X)$ where θ is a parameter of the density function. Then the expected discounted payoff is given by

$$E^Q[Y] = \int_{\mathcal{R}^m} f(x) g_\theta(x) dx. \tag{2.36}$$

By assuming the interchangeability of differentiation and integration, we have

$$\begin{aligned} \frac{d}{d\theta} E^Q[Y(\theta)] = & \int_{\mathcal{R}^m} f(x) \frac{d}{d\theta} g_\theta(x) dx = \int_{\mathcal{R}^m} f(x) \frac{dg_\theta(x)/d\theta}{g_\theta(x)} g_\theta(x) dx \\ = & E^Q\left[f(x) \frac{d \ln g_\theta(x)}{d\theta} \right]. \end{aligned} \tag{2.37}$$

Now the likelihood ratio estimator of $\frac{d}{d\theta} E^Q[Y]$ is given as:

$$f(x) \frac{d \ln g_\theta(x)}{d\theta}. \tag{2.38}$$

An example of the calculation the delta of European call option using likelihood ratio method is also given as follows. The payoff function of an European call option only depends on the underlying asset price at time T which is the expiration date of the option. In order to conform previous notation, we use S_T and S_0 to denote the underlying asset price at time T and time 0 respectively. The probability density of S_T is

$$g_{S_0}(x) = \frac{1}{x\sigma\sqrt{T}}\phi(d(x)), x \geq 0, \tag{2.39}$$

where

$$\phi(Z) = \frac{1}{\sqrt{2\pi}}e^{-z^2/2}, \tag{2.40}$$

$$d(x) = \frac{\ln(x/S_0) - (r - \sigma^2/2)T}{\sigma\sqrt{T}} \tag{2.41}$$

and Z is a standard normal random variable. Thus, the expected discounted payoff is

$$E^Q[Y(\theta)] = \int_0^\infty e^{-rT} \max(x - K, 0)g(x)dx \tag{2.42}$$

and the estimator of Δ is given as

$$\begin{aligned} \hat{\Delta}_{LR} = \frac{dE^Q[Y(S_0)]}{dS_0} &= E^Q\left[e^{-rT}\max(S_T - K, 0)\frac{d\ln g_{S_0}(S_T)}{dS_0}\right] \\ &= E^Q\left[e^{-rT}\max(S_T - K, 0)\frac{Z}{S_0\sigma\sqrt{T}}\right]. \end{aligned} \tag{2.43}$$

In this section, we broadly introduce three widely used methods to calculate the Greeks. Different methods have their own advantages. Finite difference method is easy to understand. Pathwise derivative estimates is more accurate and efficient. Likelihood ratio method provides relatively good estimation and requires less assumptions. Which one to use in calculating the Greeks mainly depends on the specific situation. In this thesis, finite difference method is used in Chapter 5 to calculate the risky asset in the optimal portfolio.

2.3 Utility theory and pension contract design

Nowadays, there are so many pension products in the market. In order to sell more contracts to the market, from the perspective of a pension company, the question of whether a new product will be popular is of significance. Whether the product meets investors' requirements is one question they need to answer. On the other hand, for individuals, they are keen on which product is the best one for them in terms of giving them the highest satisfaction or which one is the most aligned with their preference. It is not always true to choose the product generating the highest expected payoff. Buying a pension contract is not only able to generate a profit for the customers but also possible for the customers to end up with a loss. This is a typical problem of making decision under uncertainty. In order to answer these questions, it is necessary to have a tool or standard which could tell if one product is preferred to another one.

There is an elaborate theory, utility theory, which helps us to understand people's decisions under uncertainty. Utility is a psychological term to express how people perceive goods or wealth. For example, if you prefer an apple to an orange, then you give the apple a higher utility. In the rest of this chapter, we introduce two widely used utility theories in the context of economics and finance. In addition, the application of utility theory in the design process of pension contracts is discussed.

2.3.1 Expected utility theory

Expected Utility Theory (EUT) has been widely used to explain people's choice under uncertainty. In the eighteenth century, Daniel Bernoulli suggested using utility function to solve the famous St. Petersburg paradox which is based on an interesting game.

The game is to toss a fair coin until the first head appears. If you get a head in the first toss, you will receive \$2. If you get a tail on the first throw, you continue to throw the coin until you get a head. You will receive $\$2^n$ if the first head appears at the n^{th} coin toss. How much are you willing to pay to play this game?

Playing this game is like doing an investment or gamble. People need to make a decision (pay an amount of money) under uncertainty (without knowing the payoff). A common way to describe people's investment behaviour is the mathematical expectation. The expected payoff of one investment is the sum of the product of the probability of the outcomes and their responding payoffs. If the discounted expected terminal value is larger than the initial investment, then people should accept this investment. If not, then people should reject this investment. Expectation maximisation is a widely accepted view before the proposal of EUT.

However, using expected value of payoff violates reality. This is because the expected payoff of this game is infinite: if X is the payoff, then

$$E(X) = \frac{1}{2} \times 2 + \frac{1}{2^2} \times 2^2 + ... + \frac{1}{2^n} \times 2^n + ... = \sum_{n=1}^{\infty} 1 = \infty. \tag{2.44}$$

This result means no matter how much the game costs, the player is still willing to play. However, this is inconsistent with the practice. For example, it is hard to imagine one would spend 1 million pounds to play such a game.

Daniel Bernoulli believed that people might maximise the expected utility of an investment when they make decisions. Specifically, people's view of uncertain events can be modelled as a utility function which is a concave function. In this sense, the value of St. Petersburg paradox becomes finite. For instance, if the utility function is assumed to be $\log_2(X)$ where X is the random payoff, then

$$E(x) = \frac{1}{2} \times \log_2(2) + \frac{1}{2^2} \times \log_2(2^2) + ... + \frac{1}{2^n} \times \log_2(2^n) + ... = 2. \tag{2.45}$$

Based on Bernoulli's work, Von Neumann and Morgenstern (1944) give four axioms that support Expected Utility Theory. Before introducing the axioms, some notations which will be used later are given here. Let $U(x)$ denote the utility function of wealth x which represents the happiness or satisfaction that and amount of money x could bring to someone. $\succsim$ represents "is preferred to", $\succ$ means "is strictly preferred to",$\sim$ shows "indifferent to". First, four axioms under which expected utility holds are presented.

1. Completeness. For any gamble A and B, we have either $A \prec B$, $A \succ B$ or $A \backsim B$.

2. Transitivity. If gamble $A \succsim B$ and $B \succsim C$, then $A \succsim C$.

3. Continuity. If $A \succsim B$ and $B \succsim C$, then there exist a $p \in [0,1]$ so that $B \sim pA + (1-p)C$.

4. Independence. For any $p \in (0,1]$, if $A \succsim B$, then $pA + (1-p)C \succsim pB + (1-p)C$ always holds.

A gamble consists of a series of possible mutually exclusive outcomes $x_1, x_2, x_3, ..., x_n$ with corresponding probabilities $p_1, p_2, p_3, ..., p_n$. Expected utility theory states if an individual's preferences satisfy the above four axioms, then there exists a function to assign to each outcome x_i a real number $U(x_i)$ such that for any gamble A and B:

$$A \succ B \;\text{ iff }\; E[U(A)] > E[U(B)], \tag{2.46}$$

where $E[U(x_1, p_1; x_2, p_2; x_3, p_3; ...x_n, p_n)] = p_1U(x_1) + p_2U(x_2) + p_3U(x_3) + ... + p_nU(x_n)$. In other words, the value of one gamble equals the utility of each outcome of the gamble multiplied by the corresponding probability.

In the following, we introduce two characteristics of EUT and present one widely used family of EUT functions.

- Non-satiation

  For rational investors, they always prefer more money to less money. This is called non-satiation in utility theory and it can be expressed as:

  $$\text{if } \; x_1 > x_2, \;\; \text{then } \; U(x_1) > U(x_2). \tag{2.47}$$

  In other words, $U(x)$ is an increasing function, that is $U'(x) > 0$.

- Risk aversion

  Another property of utility function is risk aversion. People tend to favor an investment with certain payoff rather than another investment with the same but uncertain expected payoff.

$$U(\frac{x_1 + x_2}{2}) > \frac{1}{2}U(x_1) + \frac{1}{2}U(x_2). \tag{2.48}$$

  That is to say $U''(x) < 0$, which means utility functions are concave under EUT.

Hyperbolic absolute risk aversion (HARA) utility functions constitute an important class of utility functions because of the mathematical tractability. A utility function $U(x)$ is called HARA if the reciprocal of its absolute risk aversion is a linear function of wealth X. That is to say,

$$\frac{1}{ARA(X)} = \frac{X}{1-\gamma} + \frac{b}{a}, \tag{2.49}$$

where $ARA(X) = -\frac{U''(X)}{U'(X)}$ with $a > 0$, $X > 0$ and $\frac{aX}{1-\gamma} + b > 0$. Thus, a HARA utility function has the form:

$$U(X) = \frac{1-\gamma}{\gamma}\left(\frac{aX}{1-\gamma} + b\right)^{\gamma}. \tag{2.50}$$

Two popular forms of HARA utility functions are Constant Absolute Risk Aversion (CARA) utility function and Constant Relative Risk Aversion (CRRA) utility function. CARA utility function shows people invest the same amount of money in risky asset with the increase of their portfolio while CRRA utility function assume people are willing to keep a specific percentage of their portfolio in risky asset. To be specific, CARA utility function requires ARA is a constant. This means γ in equation (2.49) goes to positive or negative infinity. Exponential utility function is the unique example of CARA. It has the form

$$U(X) = 1 - \exp(-\alpha X), \quad \alpha \neq 0. \tag{2.51}$$

A CRRA utility function should have a constant Relative Risk Aversion (RRA) which is defined as

$$RRA(X) = -X\frac{U''(X)}{U'(X)}. \tag{2.52}$$

Some forms of CRRA utility function can be expressed as:

$$u(X) = \begin{cases} \frac{X^{1-\eta}-1}{1-\eta}, & \eta \neq 1, \\ \ln(X), & \eta = 1. \end{cases} \tag{2.53}$$

2.3.1.1 Merton's solution

As we explained above, people's preference can be modelled as concave utility function. In order to see how to employ EUT in the context of lifetime investment and pension products, we examine a classical asset-allocation problem which was first solved by Merton (1969). Specifically, suppose a world where investors can only invest money into one risky asset and one risk free asset. How should people distribute their wealth into these two assets to give them the largest expected utility, if their utility function is CRRA? This can be formulated as a stochastic optimal control problem. In the following, we give a solution using a method from dynamic programming. The idea is to construct a partial differential equation, the Hamilton-Jacobi-Bellman (HJB) equation. The solution to the HJB equation is the solution to the above problem. We follow (Björk, 2004, Chapter 19) closely in deriving the solution to this problem.

Specifically, investors start to invest their money into the two available assets at time 0 and receive all the money at time T. The dynamic process of the price of risky asset S_t is:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad S_0 = s_0, \quad a.s. \tag{2.54}$$

where μ is assumed to be constant expected return, $\sigma > 0$ is the constant volatility of the underlying risky asset and W is a standard Brownian motion process $W := \{W_t : t \in [0,T]\}$ defined on a filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t), \mathbb{P})$ on the finite interval $[0,T]$. The process of risky free-asset B_t is given as:

$$dB_t = rB_t dt, \tag{2.55}$$

where r is the risk free interest rate which is assumed constant.

Define a $\mathcal{F}_t$-progressively measurable stochastic process $\pi \in [0,1]$, where $E\int_0^T |\pi(t,X_t)|^2 dt < \infty$. $\pi(t,X_t)$ is the proportion of wealth invested into the risky asset at time t. Then $1-\pi(t,X_t)$ is the fraction of the investor's wealth invested into the risk-free asset at time t. Here, borrowing and short selling are not allowed, i.e. $\pi(t,X_t) \in [0,1]$ a.e. Thus, the dynamics of the investor's wealth are

$$dX_t^\pi = X_t^\pi[(1-\pi(t,X_t^\pi))r + \pi(t,X_t^\pi)\mu]dt + \pi(t,X_t^\pi)\sigma X_t^\pi dW_t \tag{2.56}$$

$$X_0^\pi = x_0 \quad a.s. \tag{2.57}$$

The wealth process X_t is called the state process and the portfolio process $\pi(t,X_t)$ is the control process. The utility of terminal wealth is $U(X_T)$. U is a CRRA utility function of the form:

$$U(x) = \frac{x^{1-\lambda}}{1-\lambda}, \quad x > 0, \tag{2.58}$$

where $0 < \lambda < 1$. A higher λ indicates a higher degree of risk aversion.

Our aim is to maximize the discounted expected utility of terminal wealth given the initial wealth $x_0 > 0$. That is to say, we are maximising the value function $\mathcal{J} : [0,T] \times \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ below.

$$\mathcal{J}(t,x,\pi) = E\left(U(X_T^\pi)|X_t^\pi = x_0\right). \tag{2.59}$$

The X_T^π is the wealth process obtained by following a specific admissible portfolio π. Björk (2004) gives the definition of an admissible control as:

A portfolio π is called admissible if

- $\pi(t, x) \in \Pi$ for $t \in \mathbb{R}_+$ and $x \in \mathbb{R}_+$.
- for any given begin state (t, x), the SDE

$$dX_s = \mu(s, X_s, \pi(s, X_s))ds + \sigma(s, X_s, \pi(s, X_s))dW_s \tag{2.60}$$

$$X_t = x \tag{2.61}$$

  has a unique solution.

Let $\mathcal{A}(x)$ denote the class of admissible portfolio with

$$\mathcal{A}(x) := \{\pi : [0, T] \times R \to R : \pi(t, y) \in \Pi,\ \forall\ t \in [0, T], y \in R\}\,. \tag{2.62}$$

Our aim now is to find an admissible control rule π_o so that

$$\mathcal{J}(t, x, \pi_o) = \sup_{\pi \in \mathcal{A}(x)} E[U(X_T^\pi)|X_t^\pi = x]. \tag{2.63}$$

The optimal value function is defined as

$$V(t, x) = \mathcal{J}(t, x, \pi_o),\ \forall (t, x) \in [0, T] \times \mathbb{R}_+. \tag{2.64}$$

Now we need to derive the HJB partial differential equation. Assume investors use two strategies to allocate their money:

- Strategy 1. Use optimal control rule π_o. The optimal discounted expected utility given by this strategy is

$$V(t, x) = E[U(X_T^{\pi_o})|X_t = x]. \tag{2.65}$$

- Strategy 2. Use an arbitrary strategy π first over time period $[t, t+h]$ for a quite short time period $h > 0$ and then choose the optimal strategy for the rest of time. The utility generated by this strategy can be expressed as

$$E[V(t+h, X^{\pi}_{t+h})|X^{\pi}_t = x]. \tag{2.66}$$

As strategy 1 is always the optimal one, the following inequality holds:

$$V(t,x) \geq E[V(t+h, X^{\pi}_{t+h})|X_t = x]. \tag{2.67}$$

It is assumed that V is a smooth function. According to Itô formula, we have

$$\begin{aligned}
&V(t+h, X^{\pi}_{t+h}) = V(t, X^{\pi}_t) \\
&+ \int_t^{t+h} \left\{ \frac{\partial V(s, X^{\pi}_s)}{\partial t} + [(1-\pi(s,X_s))r + \pi(s,X_s)\mu]X^{\pi}_s \frac{\partial V(s, X^{\pi}_s)}{\partial x} \right. \\
&\left. + \frac{1}{2}\pi^2(s,X_s)\sigma^2 (X^{\pi}_s)^2 \frac{\partial^2 V(s, X^{\pi}_s)}{\partial x^2} \right\} ds + \int_t^{t+h} \sigma \frac{\partial V(s, X^{\pi}_s)}{\partial x} dW_s.
\end{aligned} \tag{2.68}$$

If enough integrability is assumed, the expectation of the stochastic integral is zero. Thus, substituting equation (2.68) into inequality (2.67) gives:

$$\begin{aligned}
&E\left[\int_t^{t+h} \left[\frac{\partial V(s, X^{\pi}_s)}{\partial t} + [(1-\pi(s,X_s))r + \pi(s,X_s)\mu]x \frac{\partial V(s, X^{\pi}_s)}{\partial x} \right.\right. \\
&\left.\left. \quad + \frac{1}{2}\pi^2(s,X_s)\sigma^2 x^2 \frac{\partial^2 V(s, X^{\pi}_s)}{\partial x^2} \right] ds | X_t = x \right] \leq 0.
\end{aligned} \tag{2.69}$$

Letting $h \to 0$, $X_t = x$ and dividing equation(2.69) by h, we can get the following partial differential equation (PDE) for all $(t,x) \in [0,T] \times \mathbb{R}_+$:

$$\frac{\partial V(t,x)}{\partial t} + [(1-\pi(t,x))r + \pi(t,x)\mu]x \frac{\partial V(t,x)}{\partial x} + \frac{1}{2}\pi^2(t,x)\sigma^2 x^2 \frac{\partial^2 V(t,x)}{\partial x^2} \leq 0. \tag{2.70}$$

The equality in the above equation only holds when the arbitrary strategy π equals to the optimal strategy π_o. For $(t,x) \in (0,T) \times \mathbb{R}_+$ and $\pi(t,x) \in [0,1]$, the HJB equation can be expressed as

$$\begin{cases}
\frac{\partial V(t,x)}{\partial t} + \sup_{\pi \in \mathcal{A}(x)} \left\{ [(1-\pi(t,x))r + \pi(t,x)\mu]x \frac{\partial V(t,x)}{\partial x} + \frac{1}{2}\pi^2(t,x)\sigma^2 x^2 \frac{\partial^2 V(t,x)}{\partial x^2} \right\} = 0, \\
V(T,x) = U(x).
\end{cases}$$

(2.71)

In the above, we have shown that if V is the optimal value function and π_o is the optimal control rule, then V and π_o is the solution and responding control rule to the HJB equation. The above result shows the HJB equation is a necessary condition of the optimal control problem. In the following we will show that the HJB equation is also a sufficient condition of the problem. That is to say if we have a sufficiently integrable function H and control rule π_g solves the HJB equation, then the function H is the optimal value function and control rule π_g is the admissible control rule. We know that

$$\begin{cases} \frac{\partial H(t,x)}{\partial t} + [(1-\pi_g(t,x))r + \pi_g(t,x)\mu]x\frac{\partial H(t,x)}{\partial x} + & \\ \qquad\qquad\qquad\qquad \frac{1}{2}\pi_g^2(t,x)\sigma^2 x^2 \frac{\partial^2 H(t,x)}{\partial x^2} & = 0, \quad \forall\ (t,x) \in (0,T)\times \mathbb{R}_+ \\ \qquad\qquad\qquad\qquad H(T,x) & = U(x), \quad \forall\ x \in \mathbb{R}_+ \end{cases} \tag{2.72}$$

where π_g satisfies

$$\begin{aligned} & [(1-\pi_g(t,x))r + \pi_g(t,x)\mu]x\frac{\partial H(t,x)}{\partial x} + \frac{1}{2}\pi_g^2(t,x)\sigma^2 x^2 \frac{\partial^2 H(t,x)}{\partial x^2} \\ = & \sup_{\pi \in \mathcal{A}(x)} \left\{ [(1-\pi(t,x))r + \pi(t,x)\mu]x\frac{\partial V(t,x)}{\partial x} + \frac{1}{2}\pi^2(t,x)\sigma^2 x^2 \frac{\partial^2 V(t,x)}{\partial x^2} \right\}. \end{aligned} \tag{2.73}$$

As H is smooth, the following can be derived by Itô formula.

$$\begin{aligned} H(T, X_T^{\pi_g}) = H(t, X_t) + \int_t^T & \left\{ \frac{\partial H(s,X_s)}{\partial t} + [(1-\pi_g(s,X_s))r + \pi_g(s,X_s)\mu]X_s \frac{\partial H(s,X_s)}{\partial x} \right. \\ & \left. + \frac{1}{2}\pi_g^2(s,X_s)\sigma^2 X_s^2 \frac{\partial^2 H(s,X_s)}{\partial x^2} \right\} ds + \int_t^T \sigma \frac{\partial H(s,X_s)}{\partial x} dW_s. \end{aligned} \tag{2.74}$$

Inserting equation(2.72) with $x := X_s$ and $t := s$ into Equation (2.74) and taking conditional expectations based on the information available at time t shows

$$H(t,x) = E\left[U(X_T^{\pi_g}) | X(t) = x\right] = \mathcal{J}(t,x,\pi_g). \tag{2.75}$$

As the optimal strategy should give the largest utility, we have

$$V(t,x) \geq \mathcal{J}(t,x,\pi_g) = H(t,x). \tag{2.76}$$

Additionally, assume an arbitrary control rule $\pi \in \mathcal{A}(x)$ was chosen. Then according to Itô formula, we have

$$\begin{aligned} H(T, X_T^{\pi}) = H(t,x) + \int_t^T \Bigg\{ \frac{\partial H(s,X_s)}{\partial t} + [(1-\pi(s,X_s))r + \pi(s,X_s)\mu]x \frac{\partial H(s,X_s)}{\partial x} \\ + \frac{1}{2}\pi^2(s,X_s)\sigma^2 x^2 \frac{\partial^2 H(s,X_s)}{\partial x^2} \Bigg\} ds + \int_t^T \sigma \frac{\partial H(s,X_s)}{\partial x} dW_s. \end{aligned} \tag{2.77}$$

As H is a solution to the HJB equation, we have:

$$\frac{\partial H(t,x)}{\partial t} + [(1-\pi(t))r + \pi(t)\mu]x \frac{\partial H(t,x)}{\partial x} + \frac{1}{2}\pi^2(t)\sigma^2 x^2 \frac{\partial^2 H(t,x)}{\partial x^2} \leq 0. \tag{2.78}$$

Substituting Equation (2.78) into Equation (2.77) gives:

$$H(t,x) \geq U(X_T^{\pi}) - \int_t^T \sigma \frac{\partial H(s,X_s)}{\partial x} dW_s. \tag{2.79}$$

Taking the conditional expectation of both sides of Equation (2.79) shows:

$$H(t,x) \geq E\left[U(X_T^{\pi}) | X(t) = x\right] = \mathcal{J}(t,x,\pi). \tag{2.80}$$

As π is an arbitrary control rule, so the following equation naturally holds.

$$H(t,x) \geq \sup_{\pi \in \mathcal{A}(x)} \mathcal{J}(t,x,\pi) = V(t,x). \tag{2.81}$$

Combining Equation (2.76) and Equation (2.81) shows

$$H(t,x) = V(t,x) = \mathcal{J}(t,x,\pi_g). \tag{2.82}$$

This is shows that H is the optimal value function and π_g is the optimal control rule. In other words, the solution to the HJB PDE is the solution to our optimal problem. In the following, the derivation of the solution to the HJB PDE (Equation (2.71)) is given. As we know the form of utility function, thus we could guess a solution has the form like:

$$V(t,x) = y(t)u(x), \tag{2.83}$$

where y is a function of t. Taking the partial derivatives gives us:

$$\frac{\partial V}{\partial t} = y'(t)u(x) = y'(t)\frac{x^{1-\lambda}}{1-\lambda}, \tag{2.84}$$

$$\frac{\partial V}{\partial x} = y(t)u'(x) = y(t)x^{-\lambda}, \tag{2.85}$$

$$\frac{\partial^2 V}{\partial x^2} = y(t)u''(x) = -y(t)\lambda x^{-\lambda-1}. \tag{2.86}$$

Substitute the equations (2.84), (2.85) and (2.86) into the HJB equation (2.71) and we have:

$$\begin{aligned}
&\frac{\partial V(t,x)}{\partial t} + \sup_{\pi\in\mathcal{A}(x)}\left\{[(1-\pi(t,x))r + \pi(t,x)\mu]x\frac{\partial V(t,x)}{\partial x} + \frac{1}{2}\pi^2(t,x)\sigma^2 x^2\frac{\partial^2 V(t,x)}{\partial x^2}\right\} = 0\\
&\Leftrightarrow y'(t)\frac{x^{1-\lambda}}{1-\lambda} + \frac{x^{1-\lambda}}{1-\lambda}(1-\lambda)y(t)\sup_{\pi\in\mathcal{A}(x)}\left\{[(1-\pi(t,x))r + \pi(t,x)\mu] - \frac{1}{2}\pi^2(t,x)\sigma^2\lambda\right\} = 0\\
&\Leftrightarrow y'(t)u(x) + u(x)(1-\lambda)y(t)\sup_{\pi\in\mathcal{A}(x)}\left\{[(1-\pi(t,x))r + \pi(t,x)\mu] - \frac{1}{2}\pi^2(t,x)\sigma^2\lambda\right\} = 0.
\end{aligned} \tag{2.87}$$

As $u(x) > 0$, we divided $u(x)$ on both sides of the above equation. Then we have

$$y'(t) + p(t)y(t) = 0; \tag{2.88}$$

$$p(t) = (1-\lambda)\sup_{\pi\in\mathcal{A}(x)}\left\{[(1-\pi(t,x))r + \pi(t,x)\mu] - \frac{1}{2}\pi^2(t,x)\sigma^2\lambda\right\}; \tag{2.89}$$

$$y(T) = 1. \tag{2.90}$$

First, we should find the π_o which satisfy the supremum in Equation (2.89). We take the first order derivative of Equation (2.89) with respect to π and let it equals to 0, then we get:

$$(\mu - r) - \pi_o(t, x)\lambda\sigma^2 = 0. \tag{2.91}$$

Transforming the above equation gives

$$\pi_o(t, x) = \frac{\mu - r}{\lambda\sigma^2}. \tag{2.92}$$

As all parameters in the right hand side of equation(2.92) are given constants, then π_o is also a constant. Substituting Equation (2.92) into Equation (2.89) shows $p(t) = p$ for all $t \in [0, T]$, with

$$p := (1 - \lambda)\left[r + \frac{(\mu - r)^2}{2\sigma^2\lambda}\right]. \tag{2.93}$$

Then our problem becomes to solve the following ODE:

$$y'(t) + py(t) = 0 \quad \Leftrightarrow \quad y(t) = \exp(-pt).$$

As $y(T) = 1$, thus the solution is given as

$$V(t, x) = y(t)u(x) = \exp(p(T - t))u(x). \tag{2.94}$$

Here, we can conclude that under a CRRA utility function, the proportion of wealth invested in the risky asset is constant.

2.3.2 Cumulative prospect theory

In this section, we introduce a behavioural model of explaining people's decision making under uncertainty: cumulative prospect theory (CPT). Expected utility theory (EUT) is a normative theory which suggests how people should behave. It assumes that people are rational. However, in practice, people sometimes make irrational decisions. CPT, on the other hand, is a descriptive theory which shows how people do behave.

Although expected utility theory was viewed as a dominant tool to describe individuals' behaviour under uncertainty for many years, many violations of EUT suggest it is not an adequate descriptive model in decision making. Tversky and Kahneman (1992) list five violations of EUT: framing effects, non-linear preferences, source dependence, risk seeking and loss aversion. Rabin (2000) also provides evidence that EUT is not a good model to explain risk aversion over modest risks. In addition, EUT utility functions depend only on the final wealth rather than the change of wealth, which violate the practice. In order to better explain people's behaviour in decision making under uncertainty, Tversky and Kahneman (1992) proposed Cumulative Prospect Theory.

Before introducing this new theory, some definitions are presented below. Let f denote a risky prospect $(x_{-m}, p_{-m}; x_{-m+1}, p_{-m+1}; ...; x_0, p_0; ...; x_{n-1}, p_{n-1}; x_n, p_n)$ where x_i is an outcome of the prospect and p_i is corresponding probability for $i = -m, -m+1, ..., n$. Under CPT, outcomes are compared to a reference point which is often the current state. In this paper, we only concern the utility of money, the reference point denotes the current wealth. Hence, positive outcome represents the profit while negative outcome indicates the loss. Outcomes are arranged in ascending order in the prospect. That is to say if $i < j$, then $x_i < x_j$. Additionally, a positive subscript denotes a positive outcome and a negative subscript denotes a negative outcome. And zero subscript denotes neutral outcome. Let f^+ denotes the positive part of f and f^- denotes the negative part of f. As CPT evaluates gains and losses separately, the overall utility is the sum of utility of positive part and the utility of negative part.

The utility of a prospect in CPT is calculated by a value function and a weight function. The value function in CPT is monotonically increasing and satisfies $v(x_0) = v(0) = 0$. The weight function is expressed as a capacity function $w : [0, 1] \to [0, 1]$ (Choquet (1954)) which is a scaling of probabilities. For positive prospects, the decision weight π_i^+ is

$$\pi_i^+ = w^+(p_i + ... + p_n) - w^+(p_{i+1} + ... + p_n). \tag{2.95}$$

Similarly, the decision weight for negative prospects is:

$$\pi_i^- = w^-(p_{-m} + ... + p_i) - w^-(p_{-m} + ... + p_{i-1}). \tag{2.96}$$

A capacity function satisfies $w(\phi) = 0$ and $w(S) = 1$ where ϕ denotes the empty set and S stands for a universal set. For any set $A, B \in S$, if $A \subset B$, $w(A) \leq w(B)$. Now, the utility function for a prospect $(x_{-m}, p_{-m}; ...; x_0, p_0; ...x_n, p_n)$ are given as:

$$V(f) = V(f^+) + V(f^-) = \sum_{i=0}^{n} \pi_i^+ v(x_i) + \sum_{i=-m}^{0} \pi_i^- v(x_i), \quad -m < i < n. \tag{2.97}$$

Based on experimental analysis and non-linear regression, Tversky and Kahneman (1992) approximate the value function with a two-part power function:

$$v(x) = \begin{cases} x^\alpha \text{ if } x \geq 0 \\ -\lambda(-x)^\beta \text{ if } x < 0, \end{cases} \tag{2.98}$$

where $\lambda = 2.25$ and $\alpha = \beta = 0.88$. They also give the form of capacity function as

$$\begin{aligned} w^+(p) &= \tfrac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{\frac{1}{\gamma}}}, \\ w^-(p) &= \tfrac{p^\delta}{(p^\delta + (1-p)^\delta)^{\frac{1}{\delta}}}, \end{aligned} \tag{2.99}$$

where $\gamma = 0.61$ and $\delta = 0.69$. The values of π_i^+ and π_i^- are calculated via equations (2.95) and (2.96).

Figure 2.1 and Figure 2.2 plot a value function and weighting function of CPT. It is seen that the shape of the value function is like an "S" and the shape of weighting function is an inverse "S". The shape of the value function shows a diminishing sensitivity from the reference point in both the positive part and negative part. The concavity in positive domain shows people are risk averse in returns while convexity in negative domain tells people are risk seeking in loss. This is supported by the

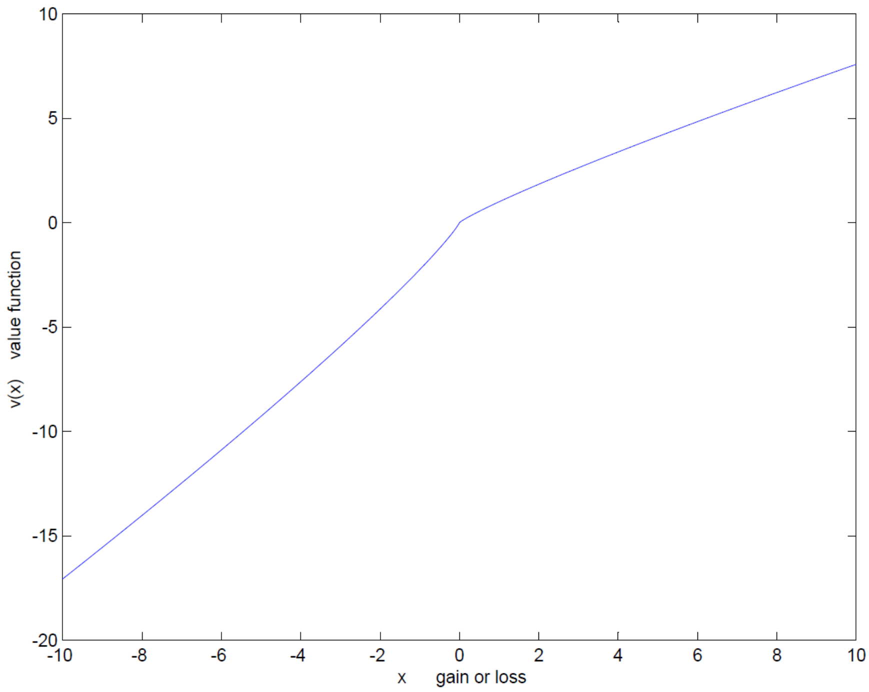


Figure 2.1: Value function based on equation (2.98) where $\lambda = 2.25$ and $\alpha = \beta = 0.88$.

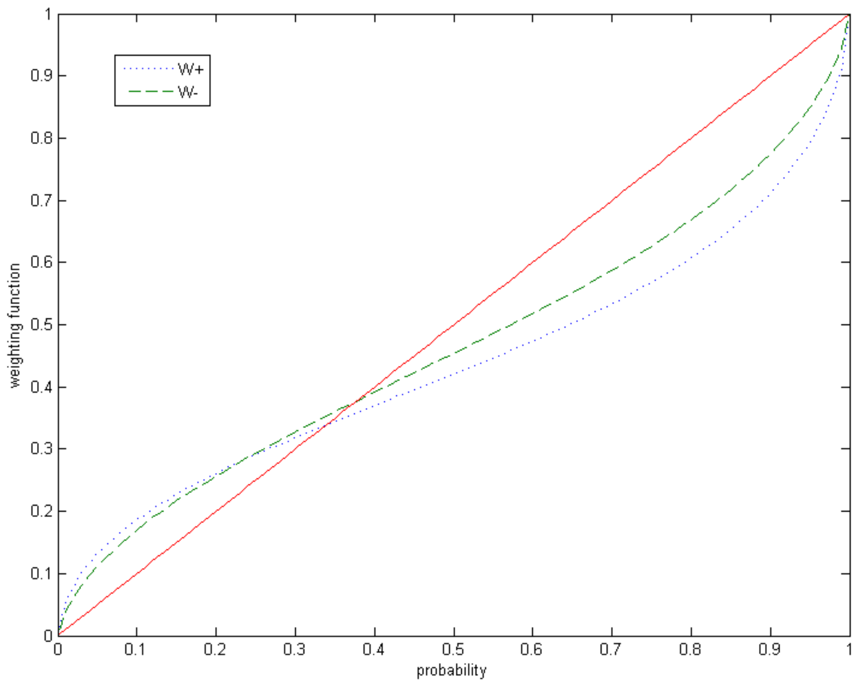


Figure 2.2: Weighting function based on equation (2.99) where $\gamma = 0.61$ and $\delta = 0.69$.

observation that people prefer a substantial probability of a loss to a sure reduced loss. Additionally, the curvature for losses is steeper than for gains.

The weighting function shows risk seeking and risk aversion for gains and losses of small probabilities, respectively. It also suggests risk aversion and risk seeking for gains and losses of high probability. In other words, the weighting function is a non-linear function of probabilities. It overvalues small probabilities and undervalues moderate and high probabilities.

CPT is viewed as a better model to capture the features of human's behaviour than EUT. However, it also has some limitations. For instance, Ingersoll (2008) suggests Tversky-Kahneman's probability weighting function is decreasing for some values of its parameters. In which case, the CPT is inconsistent with the first-order stochastic dominance. Ingersoll (2008) also lists some alternative weighting functions which could avoid this issue. These weighting functions include

- Prelec (1998)'s weighting function

$$w(p) = e^{-(-\ln p)^{\varphi}}. \tag{2.100}$$

- Lattimore et al. (1992)'s weighting function

$$w(p) = \frac{\phi p^{\eta}}{\phi p^{\eta} + (1-p)^{\eta}}. \tag{2.101}$$

In each case, $0 < \eta \leq 1$, $\phi > 0$ and $\varphi > 0$.

Døskeland and Nordahl (2008b) suggest Prelec's weighting function is based on behavioural axioms rather than mathematical convenience. In Chapter 5, the Prelec (1998)'s weighting function is used to calculate the CPT utility.

In addition, Shafir and LeBoeuf (2002) and Newell et al. (2015) argue that emotion is an important factor in people's decision process but CPT fails to take it into account.

2.3.3 The design of pension products

As we introduced above, mathematical expectation, Expected Utility Theory (EUT) and Cumulative Prospect Theory (CPT) are all used to understand how people make decisions under uncertainty. Mathematical expectations is the sum of the product of possible payoffs and their responding probabilities. In order to take account of individuals' risk preferences, EUT calculates the sum of the product of the utility of possible payoffs and the responding probabilities. As empirical evidence suggests a non-linear probability function is better to explain how people deal with probabilities, Tversky and Kahneman (1992) propose CPT by using subjective probabilities in the calculation of the utility of a prospect. In addition, the inclusion of the loss aversion in the value function is another distinctive feature of CPT.

Buying pensions is making a long-term investment for people's spending after retirement. Pension contracts, essentially, stands for the underlying investment strategy. Hence, the process of designing a pension contract is to find the optimal investment strategy based on the customers' preferences.

The optimal investment strategy under the EUT has been well studied in the academic literature. By assuming a Black-Scholes world, the optimal investment strategy is to put a constant portion in the risky asset over the investment horizon, which maximises the EUT utility (Merton, 1969). The dynamic portfolio is given in Figure 2.3. Thus, a pension contract following this constant proportion investment strategy is the optimal pension product for EUT-maximising customers.

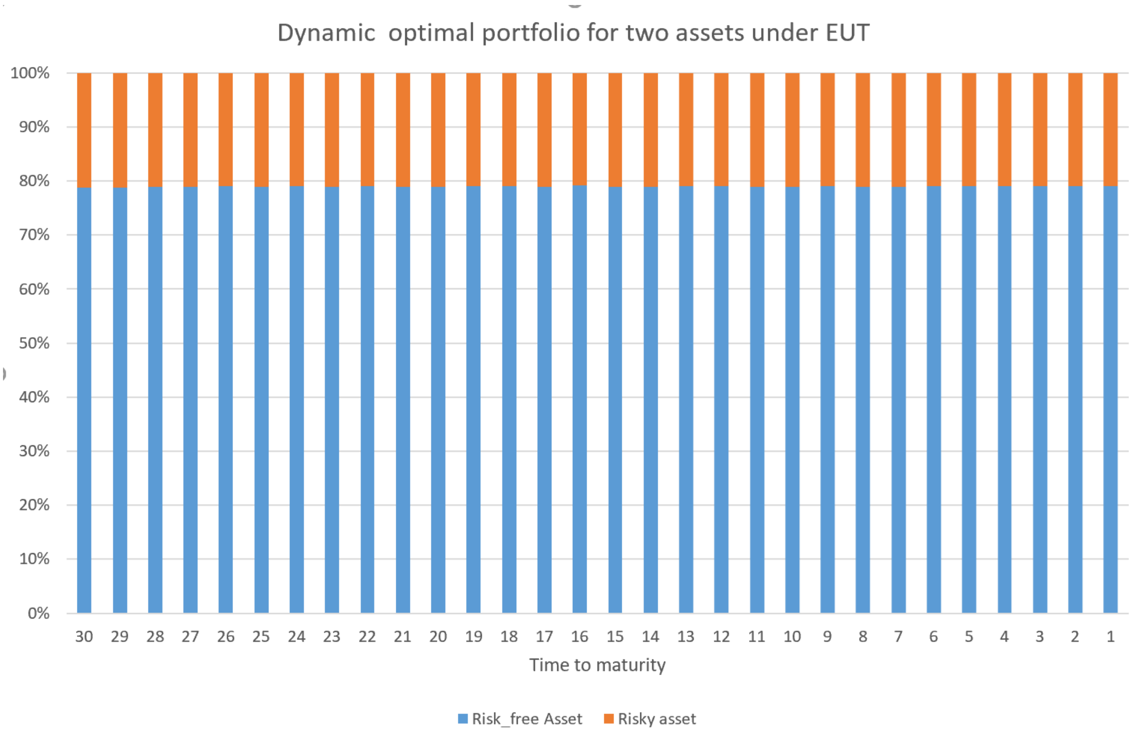


Figure 2.3: Dynamic optimal portfolio for two assets under EUT.

Due to the complexity of CPT, a closed formula for the optimal strategy is hard to obtain. However, a Monte Carlo simulation provides an approximation for the optimal dynamic investment strategy, shown in Figure 2.4. We can see that the dynamic optimal investment strategy is to decrease the exposure to the risky asset as the investment horizon shortens. This strategy conforms to the life style investment strategy, that people should invest more money in risky assets when they are young and reduce their exposure to risky assets when they are getting old. In this sense, an optimal pension contract for a CPT customer should follow this life style investment strategy.

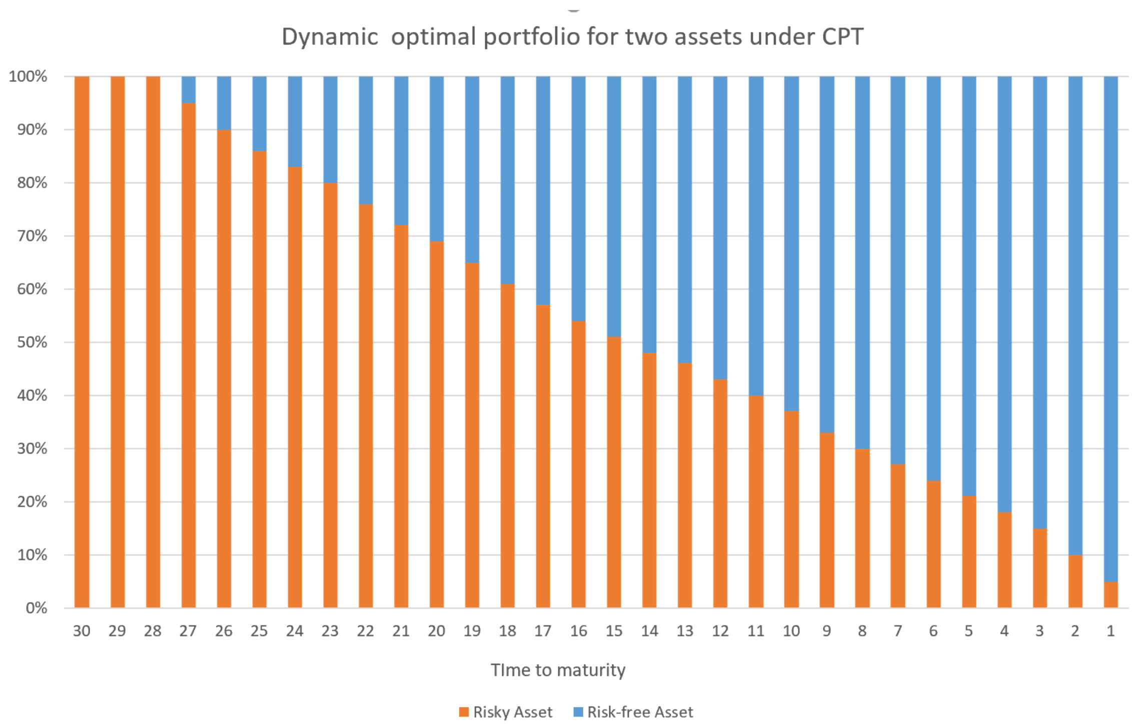


Figure 2.4: Dynamic optimal portfolio for two assets under CPT.

In this thesis, a large part of the study is carried out under the CPT. We believe CPT is able to capture more features of people's behaviours under uncertainty. As pension contracts, essentially, represent the underlying investment strategy, the optimal investment strategy under the CPT provides the intuition for the development of our new contract.

Chapter 3

Return smoothing method in a Pension contract: risk emerges

3.1 Introduction

With-profits contract, or participating policy in US, is one of the most popular products in the life insurance industry. It is an insurance contract that has access to the profit of the insurance company. With-profits contract typically consists of a term life insurance contract and a savings vehicle, or you could say an endowment assurance contract. A benefit is provided if the life insured died within the term and a lump sum is paid out if the policyholder survives to a known date. As we are keen on the investment characteristic of with-profits contract, the term insurance part is not considered in the paper. Instead, we focus on the pure investment part with-profits contracts.

With-profits contracts are designed to protect the customers against short term fluctuations in investment returns while providing them with a competitive long term rate of return. As customers' premiums are generally pooled into an investment fund which is highly exposed to the equity market, the short term return of the underlying investment is very volatile. In order to mitigate the short term adverse price movement of the underlying fund, with-profits contracts generally give the customer a smoothed return. Additionally, a customer's downside risk which arises from the possibility of a sustained decrease in the financial market is limited by the guarantees embedded in with-profits contracts. Moreover, the guaranteed amount of a customer's payout can be increased by the declaration of the bonuses by the insurer.

As the with-profits contract is generally viewed as savings-investment product with embedded interest rate guarantees, option pricing theory or contingent claim pricing is widely used in their valuation. Specifically, Wilkie (1987) was the first to apply option pricing theory to the pricing of with-profits contracts. Since then, a number of papers have explored the pricing of with-profits contracts. Briys and De Varenne (1997) propose a simple closed form solution to value the liabilities of with-profits contracts from the perspective of the insurer by using a contingent claim methodology. Grosen and Jørgensen (2000) study a with-profits contract by decomposing it into two parts, the customer account and the bonus account. As the closed form formula for the pricing of the contract is not available, Grosen and Jørgensen (2000) show us how to use numerical methods - the Monte Carlo simulation and the binomial tree - to price with-profits contracts. Jensen et al. (2001) study the same model but use a finite difference algorithm to value the contract, which provides faster and more accurate results. Grosen and Jørgensen (2000) extend their previous research by considering the lapse in which case the value of a surrender option should be calculated when pricing the contract. Bacinello (2003) study the surrender option of a particular with-profits contract which is sold in Italy. They suggest that surrender option works like an American-style put option, so that backward recursive binomial methods can be used to price the contract. Bacinello (2001) shows how to price a with-profits contract when mortality risk is considered, which makes the theoretical pricing model even closer to practice. Most conventional contracts give guarantees for free, which leads to the contracts being issued in favor of the policyholders from the perspective of risk-neutral pricing. By applying financial engineering techniques, Hansen and Miltersen (2002) show two ways to charge the fees of issuing guarantees.

The value of a with-profits contract is affected by: the guaranteed interest rate at which the premiums are accumulated, the bonus rate which is added to the guarantee rate if the underlying investment performed better than it was predicted, the participation rate in the profit of the insurer and the volatility of the underlying investment. In addition, the premium schedule (like how much and how often to pay the premium) and expense also has impact on the pricing of the with-profits contract. The guarantee rate, bonus rate and participation rate are generally determined by the insurer. By fixing all but one parameters, the last parameter can be determined to make the contract fair. Bacinello (2001) gives a detailed instructions of how to set the parameters to get a fair contract. The volatility sometimes cannot be controlled by the insurer and only can be observed from the market. However, in some other cases, if the insurer is able to change their investment portfolio, the volatility can also be affected by the insurance company. Kleinow and Willder (2007) and Kleinow (2009) study the case that the investment portfolio of the with-profits contracts is absolutely controlled by the management discretion.

Smoothing is viewed as a virtue of with-profits contracts as it reduces the fluctuation of the underlying investment. However, as the smoothing mechanism is generally complex and opaque, only a few paper touches on the smoothing mechanism of with-profits products. In order to have a deeper understanding of the with-profits contracts, this chapter studies an interesting UK with-profits contract in detail.

In this paper, we present the pricing of a with-profits contract with smoothing mechanism and discuss how the smoothing method could affect the price.

The remainder of this chapter is organised as follows. The next section introduces the structure of this with-profits contract and the market model. Section 3.3 shows the effects of smoothing and then derives the closed form solution for a fair price of this contract. Section 3.4 studies a special risk arising from the smoothing mechanism. As the contract is not a fair contract for the insurer, Section 3.5 discusses the effects of charging an annual management fee. Moreover, the risk sharing of with-profits contracts between different generations are discussed in this paper. The last section is the conclusion.

3.2 Model

3.2.1 Contract

In this chapter, we introduce an interesting with-profits contract in UK. In order to focus on the structure of this contract, we do not consider mortality and lapse. Additionally, neither expenses nor administrative costs are taken into account. The contract is assumed to be a single premium contract. The premium P is paid at start and then invested in an investment fund managed by the insurer. The underlying fund consists of one risky asset and one risk free asset. The rate of return of the fund in year n is denoted by $R_n \in \mathbb{R}$ and is independent with other years. The term of this contract is N years. Because of the embedded guarantees and the possible bonus, on the maturity date N, the policyholder receives a terminal payout which is determined by the value of the underlying investment and the guaranteed amount of money. The guarantee is provided by the insurer for free. Let A_n denote the underlying value of the policyholder's underlying investment at the end of year n for $n \in \{0, 1, 2, 3, ..., N\}$, then

$$A_n = \begin{cases} P, & n = 0 \\ A_{n-1}(1 + R_n), & n \in \{1, 2, 3, ..., N\}. \end{cases} \tag{3.1}$$

As we mentioned earlier, the return of the underlying fund is volatile due to the exposure to the equity market. To reduce the volatility on the customer's payoff, the insurer increases the policyholder's account value $\tilde{A}_n$ at a smoothed return $\tilde{R}_n$ rather than the real investment return of the underlying fund. The smoothed return $\tilde{R}_n$ over year n is defined as a geometric mean of yearly returns for five years [1], i.e.

$$\tilde{R}_n = [(1 + R_{n-2})(1 + R_{n-1})(1 + R_n)(1 + R_{n+1})(1 + R_{n+2})]^{\frac{1}{5}} - 1. \tag{3.2}$$

It is important to point out here that the return R_{n+1} and R_{n+2} are not known at the end of year n. The smoothed return $\tilde{R}_n$ is $\mathcal{F}_{n+2}$-measurable where $\mathcal{F}_{n+2}$ is the information available at time $n + 2$. And the expected smoothed return over year n

[1] This 5-year geometric average smoothing method is specified in insurer's product booklet.

is

$$\begin{aligned}
&E[\tilde{R}_n|\mathcal{F}_n] \\
=&[(1+R_{n-2})(1+R_{n-1})(1+R_n)]^{\frac{1}{5}}E[(1+R_{n+1})^{\frac{1}{5}}|\mathcal{F}_n]E[(1+R_{n+2})^{\frac{1}{5}}|\mathcal{F}_n]-1.
\end{aligned} \tag{3.3}$$

In practice, when the insurer is calculating the smoothed return at the end of year n, the insurer use their best knowledge to estimate the return in year $n+1$ and $n+2$. That is to say, for $k=1,2$, the insurer sets the conditional random variable

$$E[(1+R_{n+k})^{\frac{1}{5}}|\mathcal{F}_n] = (1+E[R_{n+k}|\mathcal{F}_n])^{\frac{1}{5}}. \tag{3.4}$$

It is worth stressing that the above formula is not mathematically correct but it is assumed in practice. Then

$$\begin{aligned}
&E[\tilde{R}_n|\mathcal{F}_n] \\
=&[(1+R_{n-2})(1+R_{n-1})(1+R_n)]^{\frac{1}{5}}E[(1+R_{n+1})^{\frac{1}{5}}|\mathcal{F}_n]E[(1+R_{n+2})^{\frac{1}{5}}|\mathcal{F}_n]-1 \\
=&[(1+R_{n-2})(1+R_{n-1})(1+R_n)(1+E[R_{n+1}|\mathcal{F}_n])(1+E[R_{n+2}|\mathcal{F}_n])]^{\frac{1}{5}}-1
\end{aligned} \tag{3.5}$$

After the policy progresses to year $n+1$, the realised return R_{n+1} becomes known. The smoothed return $E[\tilde{R}_n|\mathcal{F}_n]$ is then updated with the known fund return R_{n+1} replacing $E[R_{n+1}|\mathcal{F}_n]$. In addition, $E[R_{n+2}|\mathcal{F}_n]$ is replaced by the new best estimated return $E[R_{n+2}|\mathcal{F}_{n+1}]$. For instance, at the end of year 1, the insurer is aware of the return of the underlying fund in year 1 (current year), 0 (last year) and -1 (the year before last year). But the returns for year 2 and year 3 are unknown. When the insurer calculates the smoothed return $\tilde{R}_1$ in year 1, estimated returns are needed for year 2 and 3, respectively. At time 2, the return over year 2 is known, so the insurer needs to replace the estimated return $E[R_2|\mathcal{F}_1]$ and $E[R_3|\mathcal{F}_1]$ with the realized return R_2 and $E[R_3|\mathcal{F}_2]$, respectively, for the re-calculation of the smoothed return $E[\tilde{R}_1|\mathcal{F}_2]$. The new smoothed return in year n at time $n+1$ is

$$\begin{aligned} & E[\tilde{R}_n|\mathcal{F}_{n+1}] \\ =& [(1+R_{n-2})(1+R_{n-1})(1+R_n)(1+R_{n+1})]^{\frac{1}{5}}(1+E[R_{n+2}|\mathcal{F}_{n+1}])^{\frac{1}{5}}-1 \end{aligned} \tag{3.6}$$

At time $n+2$, all the returns in the calculation of the smoothed return $\tilde{R}_n$ are known, i.e.,

$$\begin{aligned} \tilde{R}_n &= E[\tilde{R}_n|\mathcal{F}_{n+2}] \\ &= [(1+R_{n-2})(1+R_{n-1})(1+R_n)(1+R_{n+1})(1+R_{n+2})]^{\frac{1}{5}}-1 \end{aligned} \tag{3.7}$$

And the policyholder's account value is calculated as:

$$\tilde{A}_n = P\prod_{k=1}^{n}(1+E[\tilde{R}_k|\mathcal{F}_n]) \tag{3.8}$$

Policyholder's terminal payout on maturity N is the maximum of the terminal value of the guaranteed amount and the customer account, that is

$$PO_N = \max(G_N, \tilde{A}_N). \tag{3.9}$$

Here the guaranteed value process G_n is given as

$$G_n = \begin{cases} P, & n = 0 \\ G_{n-1}(1+g+\beta_n), & n \in \{1,2,3,...,N\}. \end{cases} \tag{3.10}$$

where $g \geq 0$ denotes the guaranteed growth rate and $\beta_n \geq 0$ is the bonus rate applied at each time n.

3.2.2 Market Model

Before we derive the pricing formula of this contract, the market model is introduced. The policyholder's premium is invested in a fund which consists of only two assets, one risk free asset and one risky asset. Let r denote the constant risk free interest rate and the evolution of the value of the risk free asset is given as:

$$dB_t = rB_t dt, \qquad B_0 = b, \tag{3.11}$$

The price of the risky asset is assumed to follow a geometric Brownian motion.

$$\begin{cases} S_0 &= s \\ dS_t &= \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}, \end{cases} \tag{3.12}$$

where the expected growth rate μ, and volatility σ are positive constants. $W_t^{\mathbb{P}} := \{W_t^{\mathbb{P}} : t \in [0,T]\}$ is a Wiener process defined on the filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t), \mathbb{P})$ on the finite interval $[0,T]$.

A constant mix strategy is assumed to be followed by the underlying fund, which means that a constant proportion of the fund value is invested in the risky asset and the risk free asset at all times. The proportion of the value of the fund invested in the risky asset is called equity-backed ratio in with-profits business, which is predefined in the contract. Let a constant $\delta \in [0,1]$ denote the equity-backed ratio. Then the dynamics of the fund value F_t are

$$\begin{cases} F_0 &= P; \\ dF_t &= (r + (\mu - r)\delta)F_t dt + \sigma\delta F_t dW_t^{\mathbb{P}}. \end{cases} \tag{3.13}$$

As we are studying the financial fairness of this contract, a risk neutral measure $\mathbb{Q}$ is need to calculate the fair price of the contract. Denoting by $W_t^{\mathbb{Q}} := W_t^{\mathbb{Q}} : t \in [0,T]$ a Brownian motion under Q, the dynamics of the fund value can be written as

$$\begin{cases} F_0 &= P; \\ dF_t &= rF_t dt + vF_t dW_t^{\mathbb{Q}}. \end{cases} \tag{3.14}$$

where $v = \delta\sigma$. Then the return R_n in year n is

$$R_n = \frac{F_n}{F_{n-1}} - 1 = \exp\left[r - \frac{1}{2}v^2 + v(W_n^{\mathbb{Q}} - W_{n-1}^{\mathbb{Q}})\right] - 1 \quad \text{for} \quad n = 1, 2, 3...\text{N}. \tag{3.15}$$

3.3 Pricing

In this part, a closed form solution to the pricing of this contract is presented.

3.3.1 Smoothing mechanism

Recall that policyholder's terminal payout is the maximum of the terminal value of the guaranteed amount and the customer's account. Rewrite Equation (3.9) as

$$PO_N = G_N + \max(0, \tilde{A}_N - G_N). \tag{3.16}$$

It is observed from the above equation that the customer's payout is the sum of the terminal guaranteed value and the payoff of a call option, where the guaranteed value G_N is the strike price and the customer's account value is the price of the underlying asset. As the guaranteed value G_N can be calculated from Equation (3.10), the expected value of the terminal payout can be calculated if the payoff of this call option is known. However, the distribution of the terminal value of customer's account $\tilde{A}_N$ is not known, the existing Black-Scholes formula cannot be used directly to express the payoff. In the following, we derive the distribution of the terminal value of policyholder's account.

From Equation (3.15), we know $1+R_n$ follows the log normal distribution with location $(r-\frac{1}{2}v^2)$ and scale v. Let $Y_n = 1+R_n$, then $\{Y_n\}_{n=1}^N$ are independent copies of a log-normal distributed random variable Y, and $Y \sim \text{logN}(r-\frac{1}{2}v^2, v^2)$. Recall that the value of customer's account increases at the smoothed returns. Then the expected value of the customer's account under the risk neutral measure $\mathbb{Q}$ at the end of the contract is

$$\begin{aligned}
E^{\mathbb{Q}}[\tilde{A}_N] = E^{\mathbb{Q}}\Big[&P(1+\tilde{R}_1)(1+\tilde{R}_2)\cdots(1+E^{\mathbb{Q}}[\tilde{R}_{N-1}|\mathcal{F}_N])(1+E^{\mathbb{Q}}[\tilde{R}_N|\mathcal{F}_N])\Big] \\
=E^{\mathbb{Q}}\Big[&P[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\cdots \\
&= [(Y_{N-2})(Y_{N-1})(Y_N)(1+E^{\mathbb{Q}}[R_{N+1}|\mathcal{F}_N])(1+E^{\mathbb{Q}}[R_{N+2}|\mathcal{F}_N])]^{\frac{1}{5}}\Big].
\end{aligned} \tag{3.17}$$

When we calculate the terminal value of the policyholder's account, after the updating of estimated returns, only two estimated returns $E^{\mathbb{Q}}[R_{N+1}|\mathcal{F}_N]$ and $E^{\mathbb{Q}}[R_{N+2}|\mathcal{F}_N]$ are in the formula. In terms of pricing, the insurer use the current risk free rate of interest as the best estimation of the returns in year $N+1$ and $N+2$. In a Black-Scholes world, thus, the estimated returns are are assumed to be the constant risk free interest rate r. Thus, the expected value of the customer's account at the end of the contract is

$$E^{\mathbb{Q}}[\tilde{A}_N] = E^{\mathbb{Q}}\left[P[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\cdots[(Y_{N-2})(Y_{N-1})(Y_N)\exp(2r)]^{\frac{1}{5}}\right]. \tag{3.18}$$

For simplicity, we let the initial premium $P=1$. As Y_n are independent, then Equation (3.18) becomes

$$\begin{aligned} E^{\mathbb{Q}}[\tilde{A}_N] &= E^{\mathbb{Q}}\left[(Y_{-1})^{\frac{1}{5}}(Y_0)^{\frac{2}{5}}(Y_1)^{\frac{3}{5}}(Y_2)^{\frac{4}{5}}(\Pi_{n=3}^{N-2}Y_i)(Y_{N-1})^{\frac{4}{5}}(Y_N)^{\frac{3}{5}}\exp(2r)^{\frac{3}{5}}\right] \\ &= E^{\mathbb{Q}}[Y^{\frac{1}{5}}]E^{\mathbb{Q}}[Y^{\frac{2}{5}}]E^{\mathbb{Q}}[Y^{\frac{3}{5}}]^2E^{\mathbb{Q}}[Y^{\frac{4}{5}}]^2E^{\mathbb{Q}}[Y]^{N-4}\exp(\frac{3r}{5}). \end{aligned} \tag{3.19}$$

Equation (3.19) is the product of log normal moments. From the properties of log-normal distribution, for any real numbers k, the k-th moment of a log-normally distributed variable is

$$E^{\mathbb{Q}}[Y^k] = \exp(k(r-\frac{1}{2}v^2)+\frac{1}{2}k^2v^2). \tag{3.20}$$

Now we can calculate the expected terminal value of customer's account as

$$E^{\mathbb{Q}}[\tilde{A}_N] = \exp(Nr-\frac{3}{5}v^2) \quad \text{for} \quad N \geq 5. \tag{3.21}$$

Similarly, we calculate the variance of the terminal value of the customer by

$$\begin{aligned} V^{\mathbb{Q}}[\tilde{A}_N] =& E^{\mathbb{Q}}[(\tilde{A}_N)^2] - (E^{\mathbb{Q}}[\tilde{A}_N])^2 \\ &= \exp(2Nr+(N-3)v^2) - \exp(2Nr-\frac{6}{5}v^2). \end{aligned} \tag{3.22}$$

As $\tilde{A}_N$ is a log normal random variable and the first two moments are already obtained, the terminal customer account value can be written as

$$\begin{aligned}\tilde{A}_N &= P\exp\left[(r-\frac{1}{2}(1-\frac{3}{5N})v^2)N+\sqrt{(1-\frac{9}{5N})}v\sqrt{N}Z\right]\\ &= P\exp\left[(r-\frac{1}{2}(1-\frac{3}{5N})v^2)N+\sqrt{(1-\frac{9}{5N})}vW_N^{\mathbb{Q}}\right]\end{aligned} \tag{3.23}$$

where Z is a standard normal random variable.

3.3.2 Pricing of the contract

Recall that the payoff of the contract is the maximised value of $\tilde{A_N}$ and the guaranteed value G_N. As the distribution for the $\tilde{A_N}$ is obtained above, we can derive the payout as long as the terminal guaranteed amount is known. For simplicity, it is assumed that the annual guarantee rate is constant and no terminal bonus will be given. In this sense, the guaranteed amount increases at a deterministic rate.

Assume that the G_N is a constant and take the expectation of Equation (3.9) and discount back to the starting time, i.e.,

$$\begin{aligned}E^{\mathbb{Q}}[e^{-rN}PO_N] &= e^{-rN}E^{\mathbb{Q}}[\max(\tilde{A}_N-G_N,0)]+e^{-rN}G_N\\ &= e^{-rN}\int_{-\infty}^{\infty}\max\left(Pe^{(r-\frac{1}{2}(1-\frac{3}{5N})v^2)N+\sqrt{(1-\frac{9}{5N})}v\sqrt{N}y}-G_N,0\right)f(y)dy+e^{-rN}G_N.\end{aligned} \tag{3.24}$$

The integrand is zero when

$$Pe^{(r-\frac{1}{2}(1-\frac{3}{5N})v^2)N+\sqrt{(1-\frac{9}{5N})}v\sqrt{N}y}-G_N\leq 0, \tag{3.25}$$

i.e. when

$$y\leq -d_2=\frac{1}{\sqrt{(1-\frac{9}{5N})}v\sqrt{N}}\Big(\ln\big(\frac{G_N}{P}\big)-\big(r-\frac{1}{2}(1-\frac{3}{5N})v^2\big)N\Big). \tag{3.26}$$

Now we need only to consider the integral when the integrand is positive, i.e.

$$\begin{aligned}
E^{\mathbb{Q}}[e^{-rN}PO_N] =& e^{-rN}\int_{-d_2}^{\infty}\Big(Pe^{(r-\frac{1}{2}(1-\frac{3}{5N})v^2)N+\sqrt{(1-\frac{9}{5N})}v\sqrt{N}y}-G_N\Big)f(y)dy \\
&+e^{-rN}G_N \\
=& e^{-rN}P\int_{-d_2}^{\infty}e^{(r-\frac{1}{2}(1-\frac{3}{5N})v^2)N+\sqrt{(1-\frac{9}{5N})}v\sqrt{N}y}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\
&+e^{-rN}G_N-G_Ne^{-rN}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy. \qquad (3.27)
\end{aligned}$$

Completing the square on the exponent in the first integral

$$\begin{aligned}
&-\frac{1}{2}(1-\frac{3}{5N})v^2N+\sqrt{1-\frac{9}{5N}}v\sqrt{N}y-\frac{1}{2}y^2 \\
&-\frac{1}{2}(N-\frac{3}{5})v^2+\sqrt{N-\frac{9}{5}}vy-\frac{1}{2}y^2 \\
&=-\frac{1}{2}(y-\sqrt{N-\frac{9}{5}}v)^2-\frac{3}{5}v^2. \qquad (3.28)
\end{aligned}$$

Thus changing the variable in the first integrand to $z := y-\sqrt{N-\frac{9}{5}}v$, and using $\Phi(x)$ to denote the cumulative distribution function (CDF) of a standard normal random variable at the value $x\in\mathbb{R}$,

$$\begin{aligned}
&E^{\mathbb{Q}}[e^{-rN}PO_N] \\
&=Pe^{-\frac{3}{5}v^2}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-\sqrt{N-\frac{9}{5}}v)^2}dy+e^{-rN}G_N-G_Ne^{-rN}\int_{-d_2}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-y^2/2}dy \\
&=Pe^{-\frac{3}{5}v^2}\int_{-d_2-\sqrt{N-\frac{9}{5}}v}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}dz+e^{-rN}G_N-G_Ne^{-rN}\Phi(d_2) \\
&=Pe^{-\frac{3}{5}v^2}\Phi(d_2+\sqrt{N-\frac{9}{5}}v)+e^{-rN}G_N(1-\Phi(d_2)) \\
&=Pe^{-\frac{3}{5}v^2}\Phi(d_1)+e^{-rN}G_N(1-\Phi(d_2)) \\
&=Pe^{-\frac{3}{5}v^2}\Phi(d_1)+e^{-rN}G_N\Phi(-d_2), \qquad (3.29)
\end{aligned}$$

where

$$d_1 = \frac{1}{\sqrt{(N-\frac{9}{5})v}}\Big(\ln\big(\frac{P}{G_N}\big) + \big(r + \frac{1}{2}(1-\frac{3}{N})v^2\big)N\Big),$$
$$d_2 = \frac{1}{\sqrt{(N-\frac{9}{5})v}}\Big(\ln\big(\frac{P}{G_N}\big) + \big(r - \frac{1}{2}(1-\frac{3}{5N})v^2\big)N\Big). \tag{3.30}$$

An alternative solution which is much easier for the pricing formula of this contract is given below. Equation (3.23) can also be written in the form of SDE, that is

$$\begin{cases} \tilde{A}_0 &= P; \\ d\tilde{A}_t &= (r - \frac{3}{5N}v^2)\tilde{A}_t dt + \sqrt{1-\frac{9}{5N}}v\tilde{A}_t dW_t^{\mathbb{Q}}. \end{cases} \tag{3.31}$$

If we let $q = \frac{3}{5N}v^2$ and $u = \sqrt{1-\frac{9}{5N}}v$, then Equation (3.31) becomes

$$\begin{cases} \tilde{A}_0 &= P; \\ d\tilde{A}_t &= (r - q)\tilde{A}_t dt + u\tilde{A}_t dW_t^{\mathbb{Q}}, \end{cases} \tag{3.32}$$

which is similar to the SDE of stock price with continuous dividends. Recall that the payout of this contract has the form

$$PO_N = \max(G_N, \tilde{A}_N) = \max(\tilde{A_N} - G_N, 0) + G_N. \tag{3.33}$$

In the last part of Equation (3.33), as $\tilde{A_N}$ follows a log-normal distribution and G_N works as a strike price, we directly write the pricing formula for this contract by using Black-Scholes formula when the risky asset pays a continuous dividend. That is

$$E^{\mathbb{Q}}[e^{-rN}PO_N] = G_N e^{-rN} + e^{-qN}\tilde{A}_0'\Phi(d_1) - G_N e^{-rN}\Phi(d_2) \tag{3.34}$$

where

$$d_1 = \frac{\ln\frac{\tilde{A}_0}{G_N} + (r - q + \frac{1}{2}u^2)N}{u\sqrt{N}} = \frac{1}{\sqrt{(N-\frac{9}{5})v}}\Big(\ln\Big(\frac{P}{G_N}\Big) + \Big(r + \frac{1}{2}(1 - \frac{3}{N})v^2\Big)N\Big),$$
$$d_2 = d_1 - u\sqrt{N} = \frac{1}{\sqrt{(N-\frac{9}{5})v}}\Big(\ln\big(\frac{P}{G_N}\big) + \big(r - \frac{1}{2}(1 - \frac{3}{5N})v^2\big)N\Big), \tag{3.35}$$

which conforms to the previous result.

3.3.3 Numerical Results

In the above, the closed form solution of customer's payout is derived. In order to have a better understanding of this contract, we calculate the present value of the contract by using the following group of parameters:

$$r = 0.02,\ \sigma = 0.2,\ N = 25,\ P = 1,\ \delta = 0.8$$[2]

As we discussed before, once the regular bonus is declared, the bonus becomes one part of the guarantee. We let $d = g + \beta_n$ denote the revised guarantees. Table 3.1 shows the present value of each contract with different revised guarantees under the risk neutral measure $\mathbb{Q}$. The adjusted guarantees are assumed constant for the whole period within the contract. We can see from the result that the present values of the terminal payout are higher than the initial premium. In the case that the revised guarantee rate is 0, customer is guaranteed to receive back his initial premium. However, the expected value the customer receives is 7% higher than the premium. As the guarantee is provided for free, the insurer actually undervalues this with-profits contract. With the increase of the adjusted guarantee rate, the contract becomes more valuable.

[2] The parameters are arbitrarily chosen for illustration. From 2009 to 2018, the US 10 year bond yield was around 2.5% and the realised volatility of SP500 during the same period was 18.5%. The choice of the parameters is reasonable.

$d = g + \beta_n$	Present value of the terminal payout
0	1.0768
0.01	1.1573
0.02	1.2869

Table 3.1: Present value of the contract for different adjusted guaranteed rate d. d is defined as the sum of bonus rate and the guaranteed growth rate. The other parameter values are $r = 0.02$, $\sigma = 0.2$, $N = 25$, $P = 1$, $\delta = 0.8$.

3.4 Beating the insurance company: Speculating entry

Because of the smoothing mechanism used in this contract, the terminal payout of the contract not only depends on the performance of the underlying fund within the term of the contract, but also on the return in the two years before the start of the contract. Thus, the smart customer may choose to enter into the contract if he observes that the underlying fund experienced good returns in the last two years. In this case, we are interested in determining if the smart customer could have a higher expected terminal payout.

In order to investigate this, a simplistic one-customer model is assumed. In other words, there is no inter-generational effects in this case. The customer has two strategies to buy the contract. The first one is called random entry in which case the customer just buys the contract directly without knowing any information about previous returns. The second strategy is speculating entry, which means the customer only buys the contract if the returns in the last two years meet a specific requirement. For instance, in this paper, the requirement is a higher than the risk free rate return in the last two years, i.e. $Y_{-1}^{\frac{1}{5}} Y_0^{\frac{2}{5}} > \exp\left(\frac{r}{5}\right) \exp\left(\frac{2r}{5}\right)$. In the following, we derive the closed formula for the contract price of speculating entry.

Recall that the terminal value of the customer's account is expressed as

$$\tilde{A_N} = PY_{-1}^{\frac{1}{5}} Y_0^{\frac{2}{5}} Y_1^{\frac{3}{5}} Y_2^{\frac{4}{5}} \left(\prod_{i=3}^{N-2} Y_i\right) Y_{N-1}^{\frac{4}{5}} Y_N^{\frac{3}{5}} e^{\left(\frac{2r}{5}\right)} e^{\left(\frac{r}{5}\right)}. \tag{3.36}$$

First, we let $X = Y_{-1}^{\frac{1}{5}} Y_0^{\frac{2}{5}}$. Note that

$$
\begin{aligned}
E[X] =& E[Y_{-1}^{\frac{1}{5}}]E[Y_0^{\frac{2}{5}}] \\
=& \exp\big(\frac{1}{5}(r-\frac{1}{2}v^2)+\frac{1}{2}\frac{1}{25}v^2\big)\exp\big(\frac{2}{5}(r-\frac{1}{2}v^2)+\frac{1}{2}\frac{4}{25}v^2\big) \\
=& \exp\big(\frac{3}{5}r-\frac{1}{5}v^2\big).
\end{aligned}
\tag{3.37}
$$

And

$$
\begin{aligned}
V[X] =& E[X^2]-(E[X])^2 \\
=& E[Y_{-1}^{\frac{2}{5}}]E[Y_0^{\frac{4}{5}}]-(E[Y_{-1}^{\frac{1}{5}}])^2(E[Y_0^{\frac{2}{5}}])^2 \\
=& \exp\big(\frac{2}{5}(r-\frac{1}{2}v^2)+\frac{1}{2}\frac{4}{25}v^2\big)\exp\big(\frac{4}{5}(r-\frac{1}{2}v^2)+\frac{1}{2}\frac{16}{25}v^2\big) \\
& -\exp\big(\frac{2}{5}(r-\frac{1}{2}v^2)+\frac{1}{25}v^2\big)\exp\big(\frac{4}{5}(r-\frac{1}{2}v^2)+\frac{4}{25}v^2\big) \\
=& \exp\big(\frac{6}{5}r-\frac{2}{5}v^2\big)\Big(\exp\big(\frac{1}{5}v^2\big)-1\Big).
\end{aligned}
\tag{3.38}
$$

We conclude that $X \sim \mathrm{logN}(\frac{3}{5}(r-\frac{1}{2}v^2), \frac{1}{5}v^2)$. Similarly, letting $X' = (\prod_{i=3}^{N-2} Y_i) Y_{N-1}^{\frac{4}{5}} Y_N^{\frac{3}{5}} \exp\big(\frac{3r}{5}\big)$

$$
\begin{aligned}
E[X'] =& E[Y_1^{\frac{3}{5}}]E[Y_4^{\frac{4}{5}}]E[(\prod_{i=3}^{N-2} Y_i)]E[Y_{N-1}^{\frac{3}{5}}]E[Y_N^{\frac{4}{5}}]e^{(\frac{3r}{5})} \\
=& \exp\Big(2\big(\frac{3}{5}(r-\frac{1}{2}v^2)+\frac{1}{2}\frac{9}{25}v^2\big)\Big)\exp\Big(2\big(\frac{4}{5}(r-\frac{1}{2}v^2)+\frac{1}{2}\frac{16}{25}v^2\big)\Big)\exp\Big(Nr-\frac{17}{5}r\Big) \\
=& \exp\Big((N-\frac{3}{5})r-\frac{2}{5}v^2\Big)
\end{aligned}
\tag{3.39}
$$

and

$$
\begin{aligned}
V[X'] =& E[X'^2]-(E[X'])^2 \\
=& \exp\Big(2\big(\frac{6}{5}(r-\frac{1}{2}v^2)+\frac{1}{2}\frac{36}{25}v^2\big)\Big)\exp\Big(2\big(\frac{8}{5}(r-\frac{1}{2}v^2)+\frac{1}{2}\frac{64}{25}v^2\big)\Big) \\
& \cdot\exp\Big((N-4)\big(2(r-\frac{1}{2}v^2)+\frac{1}{2}4v^2\big)\Big)\exp\big(\frac{6}{5}r\big) \\
& -\exp\Big((2N-\frac{6}{5})r-\frac{4}{5}v^2\Big) \\
=& \exp\Big((2N-\frac{6}{5})r+(N-\frac{14}{5})v^2\Big)-\exp\Big((2N-\frac{6}{5})r-\frac{4}{5}v^2\Big).
\end{aligned}
\tag{3.40}
$$

Thus, the distribution for X' is $\mathrm{logN}((N-\frac{3}{5})r-(\frac{N}{2}-\frac{3}{5})v^2, (N-2)v^2)$. Then the analytic formula for the expected payout of the contract for speculating entry, i.e. $E^{\mathbb{Q}}[PO_N|Y_{-1}^{\frac{1}{5}}Y_0^{\frac{2}{5}} > \exp(\frac{3r}{5})]$.

$$\begin{aligned}
&E^{\mathbb{Q}}[PO_N|Y_{-1}^{\frac{1}{5}}Y_0^{\frac{2}{5}} > \exp(\frac{3r}{5})] \\
=&E^{\mathbb{Q}}[\max(\tilde{A_N}, G_N)|Y_{-1}^{\frac{1}{5}}Y_0^{\frac{2}{5}} > \exp(\frac{3r}{5})] \\
=&G_N + E^{\mathbb{Q}}\left[\max(\tilde{A_N} - G_N, 0)|Y_{-1}^{\frac{1}{5}}Y_0^{\frac{2}{5}} > \exp(\frac{3r}{5})\right].
\end{aligned} \tag{3.41}$$

As $\tilde{A_N} = XX'$, Equation(3.41) can be written as

$$\begin{aligned}
=&G_N + E^{\mathbb{Q}}\left[\max(XX' - G_N, 0)|X > \exp(\frac{3r}{5})\right] \\
=&G_N + \frac{\int\int_{x>\exp(\frac{3r}{5})} \max(xx' - G_N, 0) f_{X,X'}(x,x')dxdx'}{\mathrm{P}(x > \exp(\frac{3r}{5}))} \\
=&G_N + \frac{\int_{x>\exp(\frac{3r}{5})} \int_{x'>\frac{G_N}{x}} (xx' - G_N) f_{X'}(x')dx' f_X(x)dx}{\mathrm{P}(x > \exp(\frac{3r}{5}))} \\
=&G_N + \frac{\int_{x>\exp(\frac{3r}{5})} \left(\int_{x'>\frac{G_N}{x}} x' f_{X'}(x')dx'\right) x f_X(x)dx}{\mathrm{P}(x > \exp(\frac{3r}{5}))} \\
&- \frac{G_N \int_{x>\exp(\frac{3r}{5})} \int_{x'>\frac{G_N}{x}} f_{X'}(x')dx' f_X(x)dx}{\mathrm{P}(x > \exp(\frac{3r}{5}))}.
\end{aligned} \tag{3.42}$$

For the numerator of the second term on the right-hand side, we have

$$\begin{aligned}
&\int_{x>\exp(\frac{3r}{5})} \left(\int_{x'>\frac{G_N}{x}} x' f_{X'}(x')dx'\right) x f_X(x)dx \\
=&\int_{x>\exp(\frac{3r}{5})} \left(\exp\left((N - \frac{3}{5})r - \frac{2}{5}v^2\right)\Phi\left(\frac{(N-\frac{3}{5})r + (\frac{N}{2} - \frac{7}{5})v^2 - \ln\frac{G_N}{x}}{\sqrt{N-2}v}\right)\right) x f_X(x)dx.
\end{aligned} \tag{3.43}$$

As it is showed above that $X \sim \log\mathrm{N}(\frac{3}{5}(r - \frac{1}{2}v^2), \frac{1}{5}v^2)$, then

$$X = \exp(\frac{3}{5}(r - \frac{1}{2}v^2) + \sqrt{0.2}vH), \tag{3.44}$$

where $H \in N(0,1)$ is a standard normal random variable. Substituting Equation

(3.44) into Equation (3.43) gives

$$\exp{(Nr - \frac{7}{10}v^2)} \int_{\frac{0.3v}{\sqrt{0.2}}}^{\infty} \Phi\left(\frac{Nr + (\frac{N}{2} - \frac{17}{10})v^2 - \ln G_N + \sqrt{0.2}vh}{\sqrt{N-2}v}\right) \exp{(\sqrt{0.2}vh)} f_H(h) dh. \tag{3.45}$$

Now change the variable of integration from log-normal random variable X to standard normal random variable H. First note that

$$\begin{aligned}
&\exp{(\sqrt{0.2}vh)} f_H(h) \\
=&\exp{(\sqrt{0.2}vh)} \frac{1}{\sqrt{2\pi}} \exp{(-\frac{1}{2}h^2)} \\
=&\frac{1}{\sqrt{2\pi}} \exp{(-\frac{1}{2}(h^2 - 2\sqrt{0.2}vh + 0.2v^2))} \exp{(0.1v^2)} \\
=&\exp{(0.1v^2)} \frac{1}{\sqrt{2\pi}} \exp{(-\frac{1}{2}(h - \sqrt{0.2}v)^2)}.
\end{aligned} \tag{3.46}$$

Let $z = -(h - \sqrt{0.2}v)$, then Equation (3.45) becomes

$$\begin{aligned}
&\exp{(Nr - \frac{6}{10}v^2)} \int_{-\infty}^{-\frac{0.1}{\sqrt{0.2}}v} \Phi\left(\frac{Nr + \frac{N-3}{2}v^2 - \ln G_N - \sqrt{0.2}vz}{\sqrt{N-2}v}\right) f(z) dz \\
=&\exp{(Nr - \frac{6}{10}v^2)} \int_{-\infty}^{-\frac{0.1}{\sqrt{0.2}}v} \Phi\left(\frac{\frac{Nr + \frac{N-3}{2}v^2 - \ln G_N}{\sqrt{(N-\frac{9}{5})v^2}} - \frac{\sqrt{0.2v^2}}{\sqrt{(N-\frac{9}{5})v^2}}z}{\frac{\sqrt{(N-2)v^2}}{\sqrt{(N-\frac{9}{5})v^2}}}\right) f(z) dz \\
=&\exp{(Nr - \frac{6}{10}v^2)} \int_{-\infty}^{C_1} \Phi\left(\frac{C_2 - \rho z}{\sqrt{1-\rho^2}}\right) f(z) dz,
\end{aligned} \tag{3.47}$$

where $C_1 = -\frac{0.1}{\sqrt{0.2}}v$, $C_2 = \frac{Nr + \frac{N-3}{2}v^2 - \ln G_N}{\sqrt{(N-\frac{9}{5})v^2}}$ and $\rho = \frac{\sqrt{0.2v^2}}{\sqrt{(N-\frac{9}{5})v^2}}$. We notice that Equation (3.47) has the same form as a bi-variate standard normal cumulative distribution function (CDF). Thus, Equation (3.47) equals to

$$\begin{aligned}
&\exp{(Nr - \frac{6}{10}v^2)} \int_{-\infty}^{C_1} \Phi\left(\frac{C_2 - \rho z}{\sqrt{1-\rho^2}}\right) f(z) dz \\
=&\exp{(Nr - \frac{6}{10}v^2)} \Phi_2(C_1, C_2; \rho).
\end{aligned} \tag{3.48}$$

Similarly, if we integrate with H, the second numerator in Equation (3.42) can be expressed as

$$
\begin{aligned}
&G_N \int_{x>\exp\left(\frac{3r}{5}\right)} \int_{x'>\frac{G_N}{x}} f_{X'}(x')dx' f_X(x)dx \\
=&G_N \int_{\frac{0.3v}{\sqrt{0.2}}}^{\infty} \Phi\left(\frac{Nr-(\frac{N}{2}-\frac{3}{10})v^2-\ln G_N+\sqrt{0.2}vh}{\sqrt{N-2}v}\right) f_H(h)dh. \qquad (3.49)
\end{aligned}
$$

Let $z' = -h$, then Equation (3.49) becomes

$$
\begin{aligned}
&G_N \int_{-\infty}^{-\frac{0.3v}{\sqrt{0.2}}} \Phi\left(\frac{Nr-(\frac{N}{2}-\frac{3}{10})v^2-\ln G_N-\sqrt{0.2}vz'}{\sqrt{(N-2)v}}\right) f(z')dz' \\
=&G_N \int_{-\infty}^{-\frac{0.3v}{\sqrt{0.2}}} \Phi\left(\frac{\frac{Nr-(\frac{N}{2}-\frac{3}{10})v^2-\ln G_N}{\sqrt{(N-\frac{9}{5})v^2}}-\frac{\sqrt{0.2v^2}}{\sqrt{(N-\frac{9}{5})v^2}}z'}{\frac{\sqrt{(N-2)v^2}}{\sqrt{(N-\frac{9}{5})v^2}}}\right) f(z')dz' \\
=&G_N \int_{-\infty}^{C_3} \Phi\left(\frac{C_4-\rho z'}{\sqrt{1-\rho^2}}\right) f(z')dz' \\
=&G_N\Phi_2(C_3, C_4; \rho), \qquad (3.50)
\end{aligned}
$$

where $C_3 = -\frac{0.3v}{\sqrt{0.2}}$ and $C_4 = \frac{Nr-(\frac{N}{2}-\frac{3}{10})v^2-\ln G_N}{\sqrt{(N-\frac{9}{5})v^2}}$. The denominator in Equation (3.42) is

$$
\mathrm{P}(x > \exp(\frac{3r}{5})) = \Phi(-\frac{0.3}{\sqrt{0.2}}v). \qquad (3.51)
$$

Thus, the expression of the value of the contract for speculating entry is

$$
\begin{aligned}
&E^{\mathbb{Q}}[PO_N | Y_{-1}^{\frac{1}{5}} Y_0^{\frac{2}{5}} > \exp(\frac{3r}{5})] \\
&= G_N + \frac{\exp(25r-0.6v^2)\Phi_2(C_1, C_2; \rho) - G_N\Phi_2(C_3, C_4; \rho)}{\Phi(-\frac{0.3}{\sqrt{0.2}}v)}. \qquad (3.52)
\end{aligned}
$$

Discounting back to time 0, the closed formula of the price of the contract when the customer speculatively entering this contract is

$$
\begin{aligned}
&E^{\mathbb{Q}}[e^{-rN}PO_N|Y_{-1}^{\frac{1}{5}}Y_0^{\frac{2}{5}} > \exp{(\frac{3r}{5})}] \\
&= G_N e^{-rN} + \exp{(-0.6v^2)}\frac{\Phi_2(C_1, C_2; \rho)}{\Phi(-\frac{0.3}{\sqrt{0.2}}v)} - G_N e^{-rN}\frac{\Phi_2(C_3, C_4; \rho)}{\Phi(-\frac{0.3}{\sqrt{0.2}}v)} \\
&= G_N e^{-rN} + e^{-qN}\tilde{A}_0\frac{\Phi_2(C_1, C_2; \rho)}{\Phi(-\frac{0.3}{\sqrt{0.2}}v)} - G_N e^{-rN}\frac{\Phi_2(C_3, C_4; \rho)}{\Phi(-\frac{0.3}{\sqrt{0.2}}v)}.
\end{aligned} \tag{3.53}
$$

Comparing the above formula with Equation (3.34), we notice the two formulae look very similar, the difference lie in the CDF functions which behaviour like probabilities. As the closed formulae is obtained, the insurer is able to calculate the price in an efficient way.

Similarly, we calculate the value of the contract for speculating entry with different levels of guarantees, which are given in Table 3.2.

$d = g + \beta_n$	Random Entry	Speculating Entry
0	1.0768	1.1294
0.01	1.1573	1.2058
0.02	1.2869	1.3270

Table 3.2: Present value of terminal payout for customers adopting different strategies. The parameter values are $r = 0.02$, $\sigma = 0.2$, $N = 25$, $P = 1$, $d = 0$ and $\delta = 1$.

It is seen that speculating entry gives the customers a much higher payout than the random entry customers. Hence, the speculating entry customer is compensated by the customers purchase the contract when the return in previous two years is bad. If there are more customers purchasing this contract speculatively, only buying contracts when the return in previous two years are better than expected, the insurer is not able to hedge this risk unless the insurer limit the number of policyholder in a year. Otherwise, the insurer cannot invest enough in previous two years to pay the higher than expected return.

3.5 Making a fair contract

As we shown above, the contract is under-valued by the insurer. The insurer could face a large risk if financial market returns decline. Hence, providing free guarantees is not a good choice for the insurer. In the following, we show how much a fair annual fee the insurer should charge for this contract.

It is assumed that an annual management fee is charged as a proportion m of the customer account value at the end of each year to cover the cost of issuing the guarantees. Then the customer account value after the deduction of the management fee is given as

$$\tilde{A}_n^{AMF} = \tilde{A}_n(1-m)^n \quad \text{for} \quad n = 1, 2, 3, ..., N. \tag{3.54}$$

Now the terminal value of customers account at the end of the contract is

$$\begin{aligned}\tilde{A}_N^{AMF} &= P(1-m)^N \exp\left[(r - \frac{1}{2}(1-\frac{3}{5N})v^2)N + \sqrt{(1-\frac{9}{5N})}v\sqrt{N}Z\right] \\ &= P(1-m)^N \exp\left[(r - \frac{1}{2}(1-\frac{3}{5N})v^2)N + \sqrt{(1-\frac{9}{5N})}vW_N^{\mathbb{Q}}\right]\end{aligned} \tag{3.55}$$

The fair price of the contract is

$$E^{\mathbb{Q}}[e^{-rN}PO_N^{AMF}] = G_N e^{-rN} + e^{-qN}P(1-m)^N\Phi(d_1) - G_N e^{-rN}\Phi(d_2) \tag{3.56}$$

where

$$\begin{aligned}d_1 &= \frac{\ln\frac{P(1-m)^N}{G_N} + (r - q + \frac{1}{2}u^2)N}{u\sqrt{N}} \\ &= \frac{1}{\sqrt{(N-\frac{9}{5})}v}\Big(\ln\Big(\frac{P(1-m)^N}{G_N}\Big) + \Big(r + \frac{1}{2}(1-\frac{3}{N})v^2\Big)N\Big), \\ d_2 &= d_1 - u\sqrt{N} \\ &= \frac{1}{\sqrt{(N-\frac{9}{5})}v}\Big(\ln\Big(\frac{P(1-m)^N}{G_N}\Big) + \Big(r + \frac{1}{2}(1-\frac{3}{N})v^2\Big)N\Big)\end{aligned} \tag{3.57}$$

In Table 3.3, the fair annual fees for different revised guarantees are presented.

$d = g + \beta_n$	annual management fee m
0	0.0020
0.01	0.0102
0.02	0.0540

Table 3.3: Fair annual fees for different adjusted guaranteed rate d. d is defined as the sum of bonus rate and the guaranteed growth rate. $r = 0.02$, $\sigma = 0.2$, $N = 25$, $P = 1$, $\delta = 0.8$.

As we cannot obtain an explicit formula for the annual management fee the result is calculated by using Newton's method.

3.6 Intergenerational Risk sharing and return redistribution

In Section 3.3, we have shown that the contract is not a fair contract. There is a return distribution from the insurer to the policyholders. In practice, policyholders enter into the contract at different times and their premiums are pooled in one fund. Comparing to without-profits product and unit-linked products, with-profits product is expected to reduce the risk of their policyholders by time diversification and risk sharing between different generations. Døskeland and Nordahl (2008a) show there exist a cross subsidisation from the pearly generations to the later generations by examining one with-profits contract. In addition, Hieber et al. (2015) propose that the cohort with lower guaranteed growth rates benefiting the cohort with higher guaranteed rate. Thus, the fairness among different generations in the contract we studied above is also interesting to us.

Because of the bonus and guarantees included in the pension contract, true fairness is not likely to exist. Milevsky and Salisbury (2016) show the closest to being truly fair is equitable which means the expected payout for each generation is the same. That is, for any generation $i, j \in \{1, 2, 3, ...\}$ and $i \neq j$, the following equation always holds to justify the fairness.

$$E[PO_N^i] = E[PO_N^j]. \tag{3.58}$$

Thus, in this part, we focus on the equitableness of this contract. As the way in which bonuses are determined is subjective and opaque in practice, some assumptions to make the bonus declaration process clear are stated here. Each year, there is one and only one customer who enters into the contract. For any generation, the guaranteed amount is always met by the insurer. In addition, the bonus rate in year n is determined by the total assets A_n^{TOT} and total liabilities L_n^{TOT} at time n. The total asset is the sum of the value of the underlying investment of each generation whose policy is still valid. That is

$$A_n^{TOT} = \begin{cases} \sum_{i=1}^{n} A_n^i, & n \leq N \\ \sum_{i=n-N+1}^{n} A_n^i, & n > N. \end{cases} \tag{3.59}$$

where A_n^i denotes the underlying value of the investment for ith generation. Similarly, the total liabilities is the sum of all guaranteed amount of the payout of all the generations in force.

$$L_n^{TOT} = \begin{cases} \sum_{i=1}^{n} G_n^i, & n \leq N \\ \sum_{i=n-N+1}^{n} G_n^i, & n > N. \end{cases} \tag{3.60}$$

If the total asset value in year n is larger than the total liabilities at the end of last year accumulated at the guarantee rate and the bonus rate, i.e. if $AS_n > L_{n-1}(1 + g + \beta)$, then a bonus β is declared for that year. Otherwise, no bonus is given in that year. In other words, if there is bonus declared in that year, the adjusted guaranteed rate is $g + \beta$. If there is no bonus declared, the adjusted guaranteed rate is g.

Thus, the declared bonus rate in a year is determined by the total assets and total liabilities. Morevoer, the declared bonus rate in one year is the same for each cohort. Hence, there exists an inter-generational risk transfer and return redistribution. To study the risk sharing effects, numerical simulation is used in calculating the payout for each generation. We let $\beta = 0.02$ and $g = 0$. That is to say the adjusted guaranteed rate is either 2% or 0%, the customer can at least get their premium back.

Figure 3.1 presents the expected payout for each generation when the expected return of the underlying risky asset $\mu = 8\%$ and $\sigma = 20\%$. We can see that the payout for later generation is higher. After the payout achieves the highest point around 23rd generation, the payout decreases slightly and then becomes flat. As the expected return is higher comparing to the increase of the liability, earlier generations are more likely obtain positive return in which case leads to higher expected bonus rate. This can be proved by Figure 3.2 which is the expected guaranteed amount against the generation. We can see that Figure 3.1 and Figure 3.2 possess the same trend and have the optimal point at the same time. In addition, for the first 25 years, as there is no payout happens, the expected asset share is always increasing, which helps generate a higher bonus rate. The reason that the optimal point appears before the asset share achieve the optimal point at the end of 25 years just before giving out the payout. The reason to this earlier optimal point is because the expected bonus rates between the 23rd generation and the 25th generation are larger than the equilibrium payout, which can be observed from Figure 3.3.

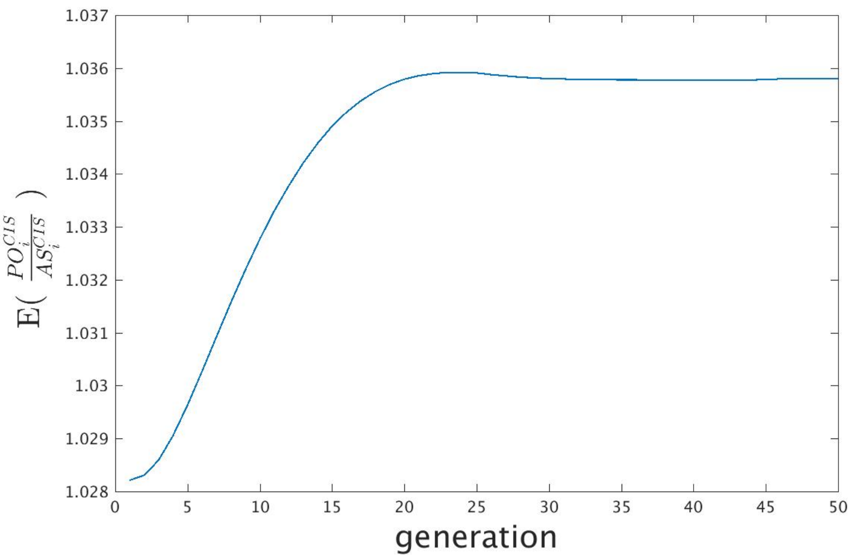


Figure 3.1: Expected terminal payout over asset share against generations. $\mu = 0.08$, $r = 0.02$, $\sigma = 0.2$, $N = 25$, $P = 1$, $\delta = 0.8$.

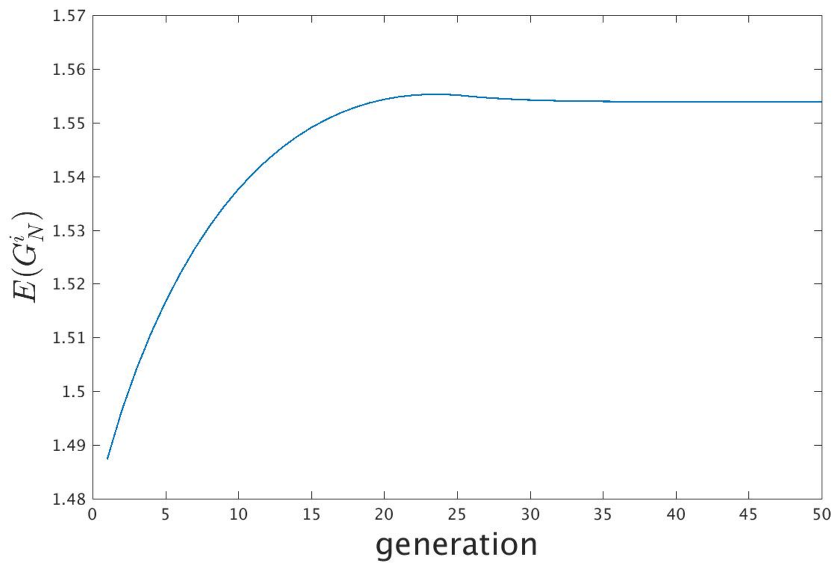


Figure 3.2: Expected terminal guaranteed amount against generations. $\mu = 0.08$, $r = 0.02$, $\sigma = 0.2$, $N = 25$, $P = 1$, $\delta = 0.8$.

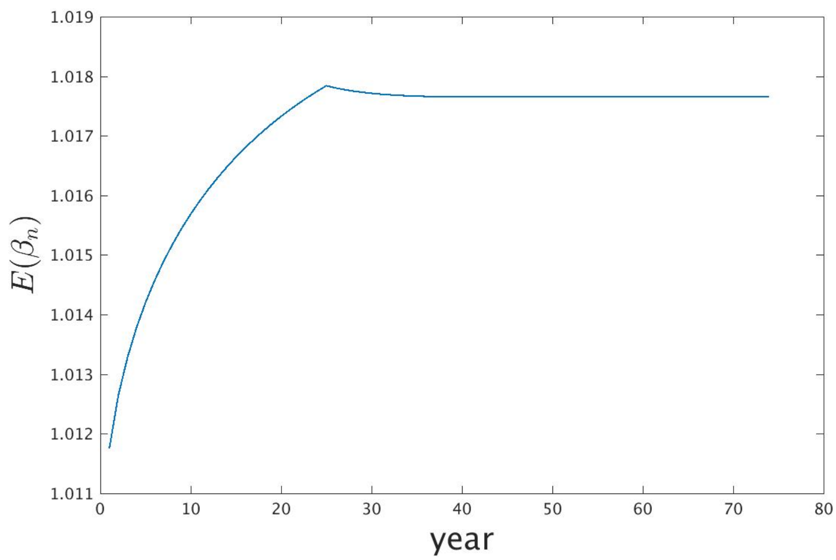


Figure 3.3: Expected bonus rate against years. $\mu = 0.08$, $r = 0.02$, $\sigma = 0.2$, $N = 25$, $P = 1$, $\delta = 0.8$.

Based on the model we used above, we can see that a return distribution from early generations to later generations. This is because later generation are more likely to share a larger portion of the bonus.

3.7 Conclusion

In this chapter, we analyse an interesting with-profits product in detail. The closed form solution for the fair pricing of this contract is obtained under the Black-Scholes model. Our result shows that the insurer undervalues this contract owing to the free embedded guarantees. Specifically, for a 0% annual guarantee, the contract worths 7% more than the price charged by the insurer. With the increase of the guarantee rate, the value of the contract is even higher.

More importantly, due to the return smoothing method depends on the return before the term of the contract, a special risk regarding to the speculating entry to this contract exposes the insurer to an unhedgeable risk. We also provide a closed form formula for the fair pricing of this unhedgeable risk, which increase the value of this contract by another 5%. In order to show how to make this contract a fair contract, we propose a solution by how to charge an annual management fee.

In the last part of this chapter, we study the inter-generation effect for this contract. We study the expected payoff for 50 generations by using a numerical simulation method. Based on the bonus distribution rule we assumed, there is a return distribution from early generations to later generations. This is because the early generations build up a reserve which is beneficial to the later generations.

Further extension to this chapter could be using a more realistic model, stochastic interest rate and stock return, to price this with-profits contract. As the independent returns is a strong assumptions, a correlated returns asset model maybe more realistic to the practice. In addition, with-profits policyholders share the same reserve, it is interesting to know if this joint product is better than an individual product.

Chapter 4

A comparison of smoothing methods in with-profits products

4.1 Introduction

Guarantees or options embedded products have become less popular in the market recently because its bad reputation of complex valuation and leading to enormous insolvency problems (see Guillén et al. 2006; Gatzert and Schmeiser 2013). Without the provision of guarantees and bonuses, a with-profits contract works like a mutual fund. A feature distinguishes it from the mutual fund is the smoothing which is used to smooth the extreme ups and downs of the markets. An example of how the smoothing method works is illustrated in Figure 4.1. We can see from the figure that the insurer holds back some gains when the market performs well and pays out the held back gains when the market performs poorly.

Recently, some traditional with-profits contracts providers launch the pension products with smoothing only and stop issuing free guarantees to the policyholders. Without the embedded guarantees and the possible bonuses, the customers receive a smoothed payout at the maturity date, which is only determined by the smoothing method. Different smoothing methods tend to give different terminal payouts.

There are many papers studying with-profits contracts. Only a few of them touch on the topic of smoothing mechanism. They mainly study the smoothing mechanism in terms of pricing. See Haberman et al. (2003), Guillén et al. (2006) and

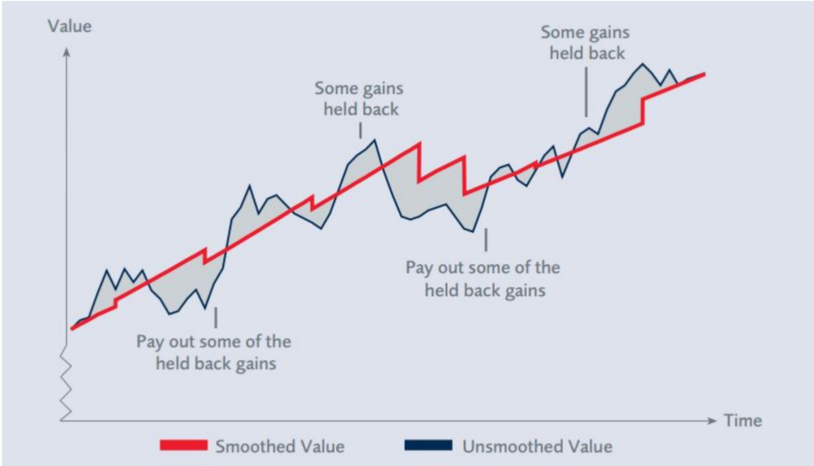


Figure 4.1: Example path of smoothed fund value by using the bandwidth smoothing method.

Løchte Jørgensen (2007). The aim of smoothing is expecting the value of customers' investment could cancel itself out over the long term to avoid short term fluctuations. To our best knowledge, no one has examined the effects of smoothing methods before.

It is interesting and also important to know how the smoothing method performs and if it is able to give a fair payout to the customers. The fairness here has a different definition from previous chapters. It is defined as the expected value of the smoothed payout equals the actual investment value. Specifically, in this chapter, we analyses three smoothing methods of with-profits contracts applied in the UK and discuss to what extent the smoothing reduce the variations of the policyholder's benefit. In addition, we use Multi-Cumulative Prospect Theory (MCPT) to calculate the interim utility in order to measure the smoothing effects year by year.

4.2 Smoothing methods

In this part, we introduce three smoothing methods which are used by UK life and pension companies: the geometric average method, the weighted sum method and the bandwidth method. Before giving the details of each method, some assumptions of the pension contract are introduced first.

The pension contract is assumed as a one-off premium contract. The premium P is paid at start and then invested in an investment fund by the insurer. The rate of return of the investment fund in year n is denoted as $R_n \in \mathbb{R}$. As the expenses are not considered in this chapter, the actual value of the fund A_n in each year n is the underlying value of policyholder's investment. Mathematically,

$$A_n = \begin{cases} P, & n = 0 \\ A_{n-1}(1 + R_n), & n \in \{1, 2, 3, ..., N\}. \end{cases} \tag{4.1}$$

The term of this contract is N years. At the maturity time N, instead of receiving the actual fund value, the policyholder receives a smoothed fund value as the terminal payout. The calculation of the smoothed fund value of each smoothing method are given in the following.

4.2.1 Geometric average (GA) smoothing method

The first smoothing method we introduce is the geometric average (GA) method. By using the GA smoothing method, the smoothed fund value increases at a smoothed return which is a geometric mean of five annual returns of the underlying fund [1]. Haberman et al. (2003) introduces a with-profits policy whose annual increase rate is based on the geometric return in the last four years.

The smoothing method we study here is different. The smoothed return depends not only on the returns in the past but also on the estimated returns in the future. Specifically, the smoothed return for year n is calculated as

$$R_n^{GA} = [(1 + R_{n-2})(1 + R_{n-1})(1 + R_n)(1 + R_{n+1})(1 + R_{n+2})]^{\frac{1}{5}} - 1 \tag{4.2}$$

where R_{n-2}, R_{n-1}, R_n are the realised returns of the underlying investment fund in each of the previous two years and the current year, respectively. R_{n+1} and R_{n+2} are the annual returns in each of the next two years.

It is important to point out that these two returns, R_{n+1} and R_{n+2} are not known in year n. That is to say, the smoothed return R_n^{GA} is $\mathcal{F}_{n+2}$ measurable, where $\mathcal{F}_{n+2}$ is the information available up to time $n + 2$. Thus, when calculating

[1] This 5-year geometric average smoothing method is specified in insurer's product booklet.

the smoothed fund value at the end of year n, the insurer needs to estimate these two future returns. The estimated return, or the expected growth rate (EGR) in practice, ${}_nR^*_{n+i} \in \mathbb{R}$ for $i = 1, 2$ is the insurer's best estimate return of the next two years given all the available information up to time n. When time progresses to the end of year $n+1$, the actual return in year $n+1$ is known. Then the EGR ${}_nR^*_{n+1}$ estimated in year n used to calculate the smoothed return R^{GA}_n is updated by the responding actual return over year $n+1$. Similarly, the future return in year $n+2$ will be updated by its realised value when it is known at time $n+2$. The mathematical formulae to express the smoothed return over year n calculating at different times are given as:
Calculating at time n,

$$ {}_nR^{GA}_n = [(1+R_{n-2})(1+R_{n-1})(1+R_n)(1+{}_nR^*_{n+i})(1+{}_nR^*_{n+2})]^{\frac{1}{5}} - 1 \tag{4.3} $$

Calculating at time $n+1$,

$$ {}_{n+1}R^{GA}_n = [(1+R_{n-2})(1+R_{n-1})(1+R_n)(1+R_{n+1})(1+{}_{n+1}R^*_{n+2})]^{\frac{1}{5}} - 1 \tag{4.4} $$

Calculating at time $n+i$ where $i \geq 2$,

$$ \begin{aligned} {}_{n+i}R^{GA}_n &= R^{GA}_n \\ &= [(1+R_{n-2})(1+R_{n-1})(1+R_n)(1+R_{n+1})(1+R_{n+2})]^{\frac{1}{5}} - 1 \end{aligned} \tag{4.5} $$

The smoothed value of customer's payout using GA method is calculated explicitly as:

$$ A^{GA}_n = P\Pi^n_{k=1}(1+{}_nR^{GA}_k) \tag{4.6} $$

The distribution of the smoothed return R^{GA}_n, has a lower variance than the variance of the actual returns, as it is a geometric average of those returns. For example, if $1 + R_{n+k} \sim logN(\mu, \sigma^2)$ for $k = -2, -1, 0, 1, 2$ and $R_{n-2}, R_{n-1}, ..., R_{n+2}$

are independent random variables, then $1 + R_n^{GA} \sim logN(\mu, \frac{1}{5}\sigma^2)$. The volatility parameter of R_n^{GA} is a fifth of σ^2 and in the sense it is a smoothed investment return.

4.2.2 Weigthed sum (WS) smoothing method

Under the weighted sum (WS) method, the smoothed fund value A_n^{WS} in year n is the weighted average of the actual fund value A_n in year n and the previous smoothed fund value A_{n-1}^{WS} accumulated at an expected growth rate (EGR) in year n but published at time $n-1$, ${}_{n-1}R_n^*$. That is to say:

$$A_n^{WS} = \begin{cases} P, & n = 0 \\ A_{n-1}^{WS}(1 + {}_{n-1}R_n^*)\kappa + A_n(1-\kappa), & n \in \{1, 2, 3, ..., N\}, \end{cases} \tag{4.7}$$

where $\kappa \in [0, 1]$ is a constant smoothing factor. After recursive substitution of Equation (4.7), the smoothed fund value becomes

$$A_n^{WS} = P\kappa^n \prod_{i=1}^{n}(1 + {}_{i-1}R_i^*) + P(1-\kappa)\sum_{j=1}^{n} \kappa^{n-j} \prod_{i=j+1}^{n} (1 + {}_{i-1}R_i^*)\Pi_{i=1}^{j}(1 + R_i) \tag{4.8}$$

It is observed that when $\kappa = 0, A_n^{WS} = A_n$. That is to say, no smoothing works in this case. While $\kappa = 1, A_n^{WS} = P\prod_{i=1}^{n}(1 + {}_{i-1}R_i^*)$, the policyholder's smoothed fund value only increases at the expected growth rate.

4.2.3 Bandwidth (BW) smoothing method

The last method we introduce is called the bandwidth (BW) smoothing method. In this method, each year the smoothed fund value increases at the expected growth rate which is set by the insurer. At the end of each year, the insurer monitors the percentage gap between the smoothed fund value and the actual underlying fund value, which is calculated by dividing the difference between these two values by the smoothed fund value. If the percentage gap is more than 5%[2], the smoothed fund

[2]This 5 % band is specified in insurer's product booklet.

Figure 4.2: Example of bandwidth smoothing method.

value is adjusted by reducing the difference in value by half, repeatedly, until the percentage gap is less than 5%.

An example of how the smoothing method works is given in Figure 4.2.[3] When the smoothed price is 100 pence and the unsmoothed price is 94 pence, a difference of 6 pence suggests the percentage gap is 6% (> 5%). The insurer reduces the smoothed fund value to 97 pence, since halving the original value difference of 6 pence implies that the new smoothed price is 97 pence. When the smoothed price is 100 pence and the unsmoothed price is 108 pence, the percentage gap is 8% (> 5%). Now the value difference should be narrowed to 4 pence by increasing the smoothed price to 104 pence.

It is important to note that this updating procedure could be done more than once if necessary, until the percentage gap is less than 5%. For example, suppose the original smoothed fund value is 100. If the actual fund value is 124, the gap is 24%. In this case, we need to reduce the gap three times to let the gap be smaller than 5%. At the first time, the difference is reduced from 24 to 12. But the new percentage gap is still larger than 5%. Then the difference should be reduced twice more, that is from 12 to 6 and from 6 to 3. Now the gap is $\frac{124}{124-3} - 1 = 2.47\%$, in which case the gap satisfies the 5% restriction.

[3] Market parameters are referred from Guillén et al. (2006).

4.3 The fairness of the smoothing methods under geometric Brownian motion

In this part, we discuss if the three smoothing mechanisms give policyholders a fair payout. As the smoothing is carried out under the real world measure, in this part, our aim is to examine if the expected smoothed terminal fund value for each smoothing method is the same as the expected terminal value of the underlying investment. That is

$$E[A_N^{\chi}|\mathcal{F}_n] = E[A_N|\mathcal{F}_n] \qquad \text{for} \quad \chi \in \{GA, WS, BW\}. \tag{4.9}$$

4.3.1 Market Model

In order to focus on comparing the smoothing effects, no expense, mortality and lapse are taken into account here. A geometric Brownian motion is assumed for the actual fund value S as

$$\begin{cases} S_0 &= s \\ dS_t &= \mu S_t dt + \sigma S_t dW_t^{\mathbb{P}}. \end{cases} \tag{4.10}$$

where the expected increase rate μ, and volatility σ are positive constants. $W^{\mathbb{P}}$ is a Wiener process defined on the filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t), \mathbb{P})$ on the finite interval $[0, T]$. Then the return R_n in year n is

$$R_n = \frac{S_n}{S_{n-1}} = \exp\left[\mu - \frac{1}{2}\sigma^2 + \sigma(W_n^{\mathbb{P}} - W_{n-1}^{\mathbb{P}})\right] - 1 \quad \text{for} \quad n = 1, 2, 3, ..., N. \tag{4.11}$$

4.3.2 GA method

It is known that the smoothed fund value of GA method increases at the geometric mean of five years' worth of returns. Let $Y_n = 1 + R_n$, then $\{Y_n\}_{n=1}^N$ are independent copies of a log-normal distributed random variable Y. In the following, we derive the expected value of the smoothed payout at each time $n \leq N$ where $N \geq 7$ [4] given the information up to N. For simplicity, we let the single premium $P = 1$. Then the expected fund value at the end of year 1 is

$$\begin{aligned} E[A_1^{GA}|\mathcal{F}_N] =& E[(1 + {}_N R_1^{GA})] = E\left[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\right] \\ =& E[Y^{\frac{1}{5}}]^5. \end{aligned} \tag{4.12}$$

Similarly, for year 2 to 4, we have

$$\begin{aligned} E[A_2^{GA}|\mathcal{F}_N] =& E[(1 + {}_N R_1^{GA})(1 + {}_N R_2^{GA})] \\ =& E\left[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}[(Y_0)(Y_1)(Y_2)(Y_3)(Y_4)]^{\frac{1}{5}}\right] \\ =& E\left[(Y_{-1})^{\frac{1}{5}}[(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{2}{5}}(Y_4)^{\frac{1}{5}}\right] \\ =& E[Y^{\frac{1}{5}}]^2 E[Y^{\frac{2}{5}}]^4. \end{aligned} \tag{4.13}$$

$$\begin{aligned} E[A_3^{GA}|\mathcal{F}_N] =& E[(1 + {}_N R_1^{GA})(1 + {}_N R_2^{GA})(1 + {}_N R_3^{GA})] \\ =& E\left[(Y_{-1})^{\frac{1}{5}}(Y_0)^{\frac{2}{5}}[(Y_1)(Y_2)(Y_3)]^{\frac{3}{5}}(Y_4)^{\frac{2}{5}}(Y_5)^{\frac{1}{5}}\right] \\ =& E[Y^{\frac{1}{5}}]^2 E[Y^{\frac{2}{5}}]^2 E[Y^{\frac{3}{5}}]^3. \end{aligned} \tag{4.14}$$

$$\begin{aligned} E[A_4^{GA}|\mathcal{F}_N] =& E[(1 + {}_N R_1^{GA})(1 + {}_N R_2^{GA})(1 + {}_N R_3^{GA})(1 + {}_N R_4^{GA})] \\ =& E\left[(Y_{-1})^{\frac{1}{5}}(Y_0)^{\frac{2}{5}}(Y_1)^{\frac{3}{5}}(Y_2)^{\frac{4}{5}}(Y_3)^{\frac{4}{5}}(Y_4)^{\frac{3}{5}}(Y_5)^{\frac{2}{5}}(Y_6)^{\frac{1}{5}}\right] \\ =& E[Y^{\frac{1}{5}}]^2 E[Y^{\frac{2}{5}}]^2 E[Y^{\frac{3}{5}}]^2 E[Y^{\frac{4}{5}}]^2. \end{aligned} \tag{4.15}$$

[4] In order to allow the smoothing method fully work, the maturity of this contract should be larger than 7 years. Additionally, it is reasonable to assume that a pension contract has an investment horizon longer than 7 years.

For the expected smoothed payout from year 5 to year $N-2$,

$$\begin{aligned}
E[A_n^{GA}|\mathcal{F}_N] =& E[(1+{}_NR_1^{GA})(1+{}_NR_2^{GA})\cdots(1+{}_NR_n^{GA})] \\
=& E\Big[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\cdots \\
& [(Y_{n-2})(Y_{n-1})(Y_n)(Y_{n+1})(Y_{n+2})]^{\frac{1}{5}}\Big] \\
=& E[Y^{\frac{1}{5}}]^2E[Y^{\frac{2}{5}}]^2E[Y^{\frac{3}{5}}]^2E[Y^{\frac{4}{5}}]^2E[Y]^{n-4}.
\end{aligned} \tag{4.16}$$

For the calculation of the expected returns of the smoothed fund value in years $N-1$ and N, we need to use estimated returns in the future. As we focus on the smoothing effects, in this chapter, it is assumed the insurer makes good estimation about the future returns in which case the estimated returns are equal to the expected return of the underlying fund, i.e., ${}_NR_{N+1}^* = {}_NR_{N+2}^* = \exp(\mu)-1$. Then

$$\begin{aligned}
E[A_{N-1}^{GA}|\mathcal{F}_N] =& E[(1+{}_NR_1^{GA})(1+{}_NR_2^{GA})\cdots(1+R_{N-1}^{GA})|\mathcal{F}_N] \\
=& E\Big[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\cdots \\
& [(Y_{N-3})(Y_{N-2})(Y_{N-1})(Y_N)\exp(\mu)]^{\frac{1}{5}}\Big] \\
=& E[Y^{\frac{1}{5}}]E[Y^{\frac{2}{5}}]^2E[Y^{\frac{3}{5}}]^2E[Y^{\frac{4}{5}}]^2E[Y]^{N-5}\exp(\frac{\mu}{5})
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
E[A_N^{GA}|\mathcal{F}_N] =& E[(1+{}_NR_1^{GA})(1+{}_NR_2^{GA})\cdots(1+{}_NR_{N-1}^{GA})(1+{}_NR_N^{GA})|\mathcal{F}_N] \\
=& E\Big[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\cdots[(Y_{N-2})(Y_{N-1})(Y_N)\exp(2\mu)]^{\frac{1}{5}}\Big],
\end{aligned} \tag{4.18}$$

i.e.,

$$E[A_N^{GA}|\mathcal{F}_N] = E[Y^{\frac{1}{5}}]E[Y^{\frac{2}{5}}]E[Y^{\frac{3}{5}}]^2E[Y^{\frac{4}{5}}]^2E[Y]^{N-4}\exp(\frac{3\mu}{5}) \tag{4.19}$$

From Jensen's inequality, we know that

$$E[Y^p] < (E[Y])^p \quad \text{for} \quad 0<p<1. \tag{4.20}$$

Then the expected terminal fund value

$$\begin{aligned} E[A_N^{GA}|\mathcal{F}_N] &< (E[Y])^{\frac{1}{5}}(E[Y])^{\frac{2}{5}}(E[Y])^{\frac{6}{5}}(E[Y])^{\frac{8}{5}}(E[Y])^{N-4}\exp(\frac{3\mu}{5}) \\ &= (E[Y])^N = E[A_N|\mathcal{F}_N]. \end{aligned} \tag{4.21}$$

That is to say,

$$E[A_N^{GA}|\mathcal{F}_N] < E[A_N|\mathcal{F}_N] \quad \text{a.s.} \tag{4.22}$$

We have shown that the expected terminal value of the smoothed fund value is smaller than the unsmoothed one. Thus, the GA smoothing mechanism is not fair for the policyholder. So far, the properties of the distribution of Y haven't been used yet. In other words, this result holds for any distribution of Y as long as annual investment returns are independent.

Equation (4.21) shows the expected value of GA smoothed payout is the product of a series of moments of the random variable Y. As Y is a log-normal random variable with location parameter $(\mu - \frac{1}{2}\sigma^2)$ and scale parameter σ, i.e., $Y \sim \text{logN}(\mu - \frac{1}{2}\sigma^2, \sigma^2)$, we are able to give the analytic formula of the expected value of the smoothed payout for the GA method. From the properties of log-normal distribution, we know that for $k = 1, 2, ..., 5$,

$$E[Y^{\frac{k}{5}}] = \exp(\frac{k}{5}(\mu - \frac{1}{2}v^2) + \frac{1}{2}(\frac{k}{5})^2 v^2) = \exp(\frac{k}{5}\mu + \frac{k^2 - 5k}{50}v^2). \tag{4.23}$$

Substituting Equation (4.23) into Equation (4.21) gives

$$\begin{aligned} E[A_N^{GA}|\mathcal{F}_N] =& E[Y^{\frac{1}{5}}]E[Y^{\frac{2}{5}}]E[Y^{\frac{3}{5}}]^2 E[Y^{\frac{4}{5}}]^2 E[Y]^{N-4}\exp(\frac{3\mu}{5}) \\ =& \exp(N\mu - \frac{3}{5}v^2). \end{aligned} \tag{4.24}$$

In order to have a deeper understanding why the expected terminal value of the smoothed fund is less than the actual one, we calculate the annual increase rate[5]

[5]This is not the same as the expected return of the smoothed fund value $E[R_k^{GA}|\mathcal{F}_n]$ in year n. The reason why we use the expected increase rate is we cannot see the dependence effect from the expected smoothed return in year n.

I_n of the expected value of the smoothed fund in each year n by using the result of Equation (4.23). For year 1, as the single premium $P = 1$,

$$1 + I_1^{GA} = \frac{E[A_1^{GA}|\mathcal{F}_N]}{P} = \frac{E[Y^{\frac{1}{5}}]^5}{1} = \exp(\mu - \frac{2}{5}v^2) < \exp(\mu). \tag{4.25}$$

In year 2,

$$1 + I_2^{GA} = \frac{E[A_2^{GA}|\mathcal{F}_N]}{E[A_1^{GA}|\mathcal{F}_N]} = \frac{E[Y^{\frac{2}{5}}]^4}{E[Y^{\frac{1}{5}}]^3} = \exp(\mu - \frac{6}{25}v^2) < \exp(\mu). \tag{4.26}$$

For year 3,

$$1 + I_3^{GA} = \frac{E[A_3^{GA}|\mathcal{F}_N]}{E[A_2^{GA}|\mathcal{F}_N]} = \frac{E[Y^{\frac{3}{5}}]^3}{E[Y^{\frac{2}{5}}]^2} = \exp(\mu - \frac{3}{25}v^2) < \exp(\mu). \tag{4.27}$$

In year 4,

$$1 + I_4^{GA} = \frac{E[A_4^{GA}|\mathcal{F}_N]}{E[A_3^{GA}|\mathcal{F}_N]} = \frac{E[Y^{\frac{4}{5}}]^2}{E[Y^{\frac{3}{5}}]^1} = \exp(\mu - \frac{1}{25}v^2) < \exp(\mu). \tag{4.28}$$

For year $i = (5, 6, ..., N-2)$,

$$1 + I_i^{GA} = \frac{E[A_i^{GA}|\mathcal{F}_N]}{E[A_{i-1}^{GA}|\mathcal{F}_N]} = \exp(\mu). \tag{4.29}$$

In year $N-1$, the estimated underlying fund return is the expected return μ. Then

$$1 + I_{N-1}^{GA} = \frac{E[A_{N-1}^{GA}|\mathcal{F}_N]}{E[A_{N-2}^{GA}|\mathcal{F}_N]} = E[Y]\frac{\exp\left(\frac{\mu}{5}\right)}{E[Y^{\frac{1}{5}}]} = \exp\left(\mu + \frac{2}{25}v^2\right) > \exp(\mu). \tag{4.30}$$

For the last year, we have

$$1 + I_N^{GA} = \frac{E[A_N^{GA}|\mathcal{F}_N]}{E[A_{N-1}^{GA}|\mathcal{F}_N]} = E[Y]\frac{\exp\left(\frac{2r}{5}\right)}{E[Y^{\frac{2}{5}}]} = \exp\left(\mu + \frac{3}{25}v^2\right) > \exp(\mu). \tag{4.31}$$

A numerical result for the increase rate of the expected value of GA method $\frac{E[A_n^{GA}|\mathcal{F}_N]}{E[A_{n-1}^{GA}|\mathcal{F}_N]}$ in year $n \in \{1, 2, ..., N\}$ is presented in Figure 4.3. For the first four years, the increase rates are lower than the expected return while the increase rate in the last two years are higher. The numerical result supports the results of our mathematical derivation above.

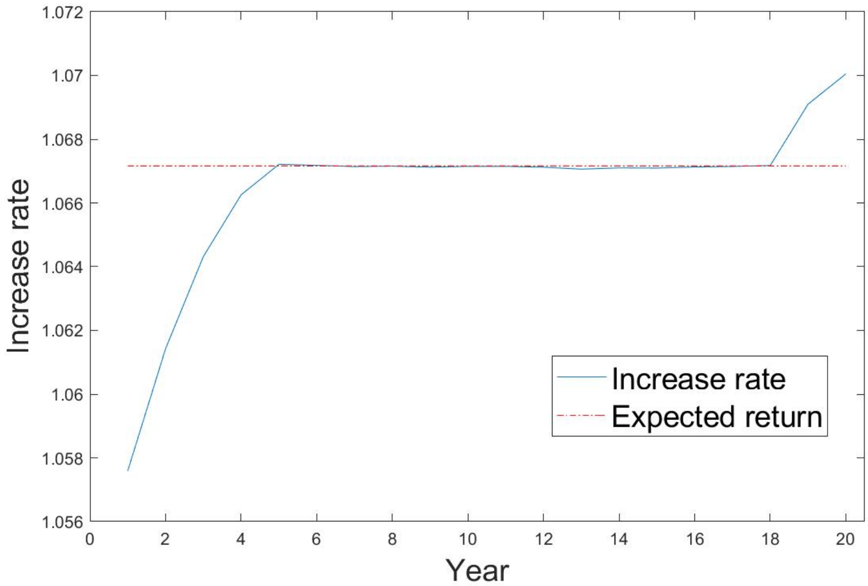


Figure 4.3: The increase rate of the expected value of the smoothed fund using GA method against the expected return of the actual fund value. $\mu = 0.065$, $\sigma = 0.15$, $N = 20$, $P = 1$.

For the GA method, the smoothing parameter is the length of the smoothing window (in practice it is 5 years). The effects of different smoothing windows on the value of the GA method is given in Table 4.1. The result shows the shorter the smoothing window, the higher the smoothed fund value and the closer to the actual fund value. When the smoothing window narrows down to 1 year, no smoothing exist. In this case, the expected smoothed fund value is just the actual fund value.

Smoothing window (in years)	GA smoothed value
9	3.5781
7	3.5947
5	3.6195
3	3.6386
1 (no smoothing)	3.6670

Table 4.1: The effects of different smoothing windows on the expected terminal value under the GA method. $\mu = 0.065$, $\sigma = 0.15$, $N = 20$, $P = 1$.

4.3.3 WS method

Recall that the WS smoothing mechanism works as:

$$A_n^{WS} = \begin{cases} P, & n = 0 \\ A_{n-1}^{WS}(1 + {}_{n-1}R_n^*)\kappa + A_n(1-\kappa), & n \in \{1, 2, 3, ..., N\}. \end{cases} \tag{4.32}$$

As we focus on the smoothing effects, we assume the expected growth rate ${}_{n-1}R_n^*$ equals the expected return $\exp(\mu) - 1$, consistent with the assumptions in the previous section. For easier notation, let $\mu' = \exp(\mu) - 1$ denote the discrete expected return. Then the smoothed fund value becomes

$$A_n^{WS} = \begin{cases} P, & n = 0 \\ A_{n-1}^{WS}(1 + \mu')\kappa + A_n(1-\kappa), & n \in \{1, 2, 3, ..., N\}. \end{cases} \tag{4.33}$$

After recursive substitution of Equation (4.33), we get

$$A_N^{WS} = P\kappa^N(1+\mu')^N + (1-\kappa)\sum_{i=1}^{N} P\prod_{j=1}^{i}(1+R_j)\kappa^{N-i}(1+\mu')^{N-i}. \tag{4.34}$$

Then the expected terminal value of the smoothed fund value AS_N^{WS} for a smoothing factor $\kappa \in [0, 1]$ is

$$\begin{aligned} E[A_N^{WS}|\mathcal{F}_N] &= P\kappa^N(1+\mu')^N + P(1-\kappa)\sum_{i=1}^{N}\prod_{j=1}^{i} E[(1+R_j)]\kappa^{N-i}(1+\mu')^{N-i} \\ &= P\kappa^N(1+\mu')^N + P(1-\kappa)\sum_{i=1}^{N}(1+\mu')^i\kappa^{N-i}(1+\mu')^{N-i} \\ &= P\kappa^N(1+\mu')^N + P(1-\kappa)(1+\mu')^N\frac{1-\kappa^N}{1-\kappa} \\ &= P(1+\mu')^N. \end{aligned} \tag{4.35}$$

Thus

$$E[A_N|\mathcal{F}_N] = E[P\prod_{n=1}^{N} Y_n] = P\prod_{n=1}^{N} E[Y_n] = P(1+\mu')^N = E[A_N^{WS}|\mathcal{F}_N], \tag{4.36}$$

and we conclude that the WS smoothing mechanism is fair for both policyholders and insurance companies. The fairness result is not affected by the value of the smoothing factor κ.

4.3.4 BW Method

By using the bandwidth smoothing method, the insurer monitors the gap between the value of the actual fund value and the smoothed fund value. If the gap is more than 5% of the smoothed fund value, the difference is repeatedly reduced by half until the gap is less than 5%. Specifically, if the percentage gap is between 10% and 20% (upside or downside), two reductions are needed. For the gap within the range of 20% and 40% (upside or downside), three reductions are required. If the gap is larger than 40% and smaller than 80% (upside or downside), the smoothed value should be updated four times.

Figure 4.4 shows how the smoothed value changes against the actual fund value. It is observed that this smoothing method generates a non-linear smoothing effect. In addition, Figure 4.5 compares the new and original smoothed fund value by fixing the actual fund value to 100. The result shows that a bigger difference between the actual fund value and the smoothed fund value may lead to a smaller difference after the smoothing, which also suggests a non-linear smoothing effect.

The smoothed fund value at maturity time N is denoted by A_N^{BW} for the BW method. Because of the highly complicated, non-linear smoothing method, we are not able to give an analytic formula for the price of a contract using the BW method. However, we are able to approximate the solution for a one year contract A_1^{BW}. Under the real world measure $\mathbb{P}$, the actual fund value is expected to accumulate at the expected return μ. As before, the expected growth rate is assumed to be this expected return.

Figure 4.6 shows the histogram of the 1 year percentage gap between the actual fund value and the smoothed fund value when the underlying asset price follows the geometric Brownian motion. We notice that the probability of the gap larger than 40% (upside or downside) in one year is relatively small. Thus, we could treat the

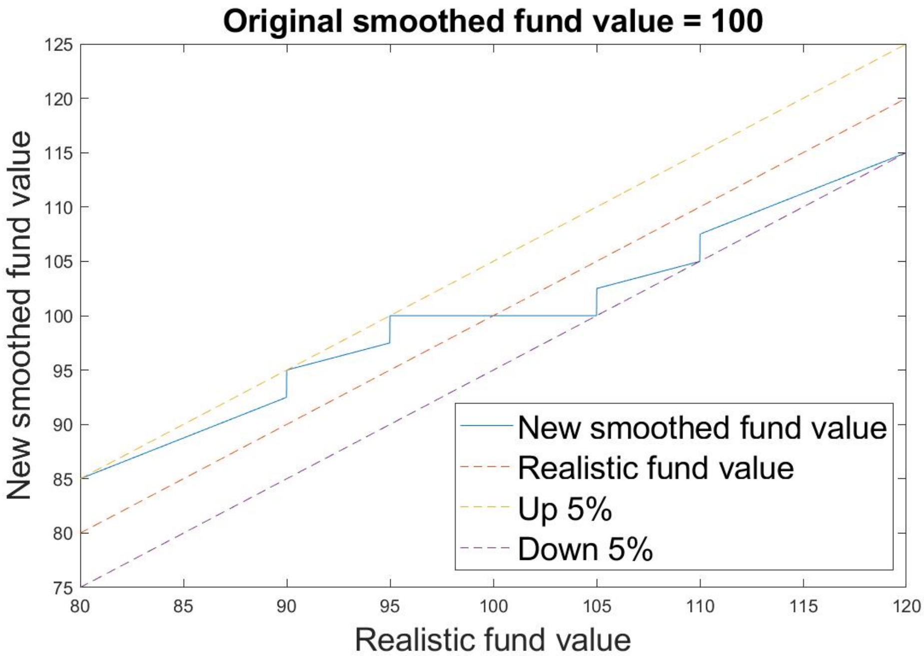


Figure 4.4: The smoothed fund value changes with actual fund value, when the original fund value equals 100.

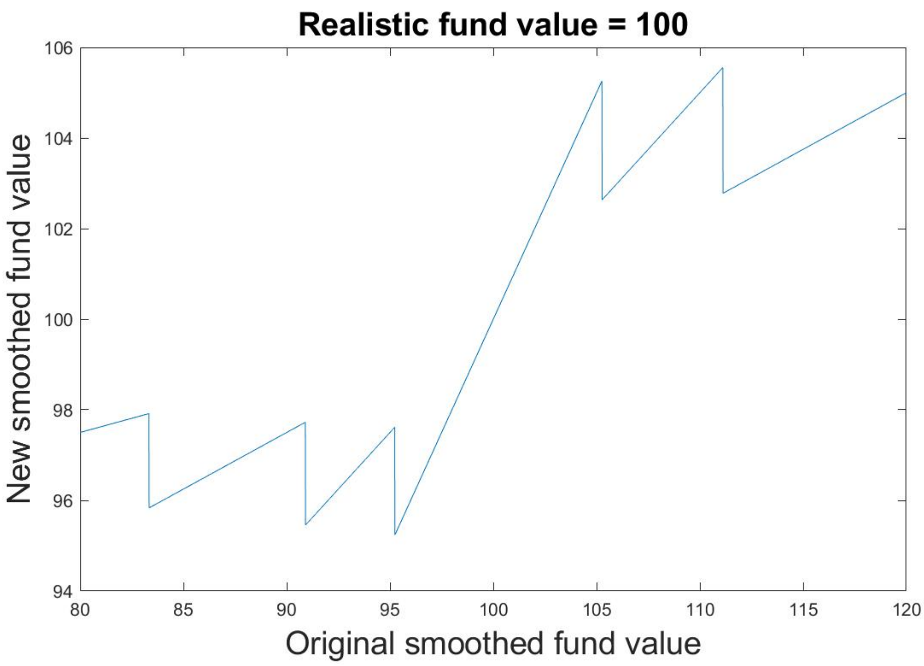


Figure 4.5: The smoothed fund value changes with the original smoothed fund value, when the actual fund value equals to 100.

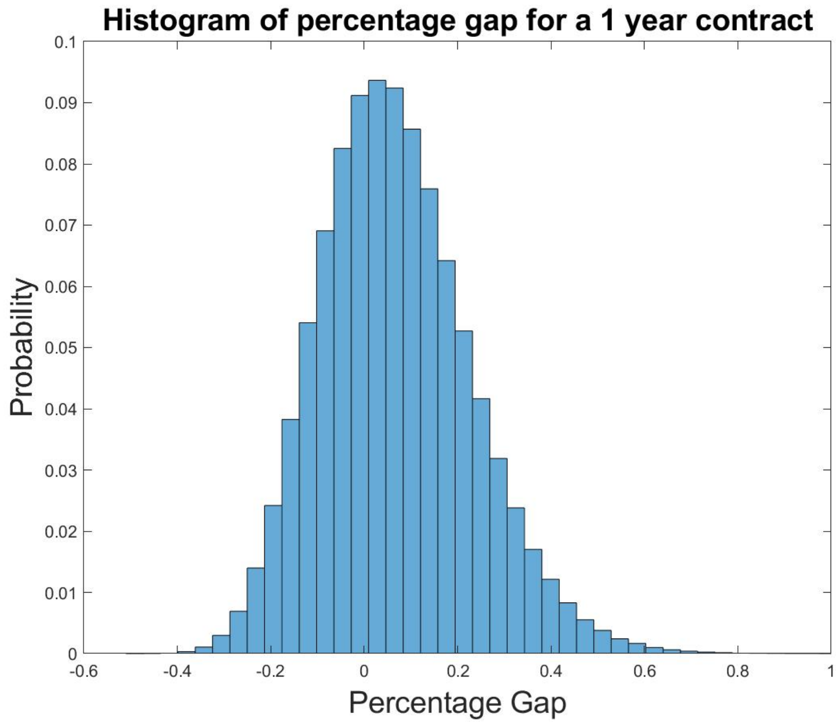


Figure 4.6: The histogram of the percentage gap based on the Monte Carlo simulation for a 1 year contract, under the BW method. The parameters are $\mu = 0.065$ and $\sigma = 0.15$.

scenarios in which the gap is larger than 40% as if they fell in the scenario with gap between 20% and 40%. By using the above approximation, we underestimate the value of the upside gap and overestimate the downside gap. As the probability of scenarios with downside gap bigger than 40% is much smaller than the upside gap. the difference between the smoothed fund value and the actual fund value mainly comes from the upside gap. Thus, the approximated value $E[A_1^{BW^*}]$ is lower than the true value $E[A_1^{BW}]$ due to the underestimation of the upside.

Specifically, we are able to approximate the expected price of this contract by considering 7 cases listed in equation (4.37),

$$A_1^{BW^*} = \begin{cases} A_1 - \frac{1}{8}[A_1 - Pe^{\mu}] = \frac{7}{8}A_1 + \frac{1}{8}Pe^{\mu}, \text{ for } A_1 \in (1.2Pe^{\mu}, +\infty) \\ A_1 - \frac{1}{4}[A_1 - Pe^{\mu}] = \frac{3}{4}A_1 + \frac{1}{4}Pe^{\mu}, \text{ for } A_1 \in (1.1Pe^{\mu}, 1.2Pe^{\mu}] \\ A_1 - \frac{1}{2}[A_1 - Pe^{\mu}] = \frac{1}{2}A_1 + \frac{1}{2}Pe^{\mu}, \text{ for } A_1 \in (1.05Pe^{\mu}, 1.1Pe^{\mu}] \\ Pe^{\mu}, \text{ for } A_1 \in (0.95Pe^{\mu}, 1.05Pe^{\mu}] \\ A_1 + \frac{1}{2}[Pe^{\mu} - A_1] = \frac{1}{2}A_1 + \frac{1}{2}Pe^{\mu}, \text{ for } A_1 \in (0.9Pe^{\mu}, 0.95Pe^{\mu}] \\ A_1 + \frac{1}{4}[Pe^{\mu} - A_1] = \frac{3}{4}A_1 + \frac{1}{4}Pe^{\mu}, \text{ for } A_1 \in (0.8Pe^{\mu}, 0.9Pe^{\mu}] \\ A_1 + \frac{1}{8}[Pe^{\mu} - A_1] = \frac{7}{8}A_1 + \frac{1}{8}Pe^{\mu}, \text{ for } A_1 \in (0, 0.8Pe^{\mu}]. \end{cases} \tag{4.37}$$

For the first case of equation (4.37), the expected value is given as

$$\begin{aligned}
&E[\frac{7}{8}A_1 + \frac{1}{8}Pe^{\mu}|A_1 > 1.2Pe^{\mu}] \\
=&\frac{7}{8}E[A_1|A_1 > 1.2Pe^{\mu}] + \frac{1}{8}Pe^{\mu}E[A_1 > 1.2Pe^{\mu}] \\
=&\frac{7}{8}\int_{A_1>1.2Pe^{\mu}} Pe^{(\mu-\frac{1}{2}\sigma^2+\sigma y)}f(y)dy + \frac{1}{8}\int_{A_1>1.2Pe^{\mu}} Pe^{\mu}f(y)dy \\
=&\frac{7}{8}Pe^{\mu}\int_{y>\frac{\ln(1.2)}{\sigma}+\frac{1}{2}\sigma} e^{(-\frac{1}{2}\sigma^2+\sigma y)}\frac{1}{\sqrt{2\pi}}e^{(-\frac{1}{2})y^2}dy + \frac{1}{8}Pe^{\mu}\int_{y>\frac{\ln(1.2)}{\sigma}+\frac{1}{2}\sigma} f(y)dy \\
=&\frac{7}{8}Pe^{\mu}\int_{y>\frac{\ln(1.2)}{\sigma}+\frac{1}{2}\sigma} \frac{1}{\sqrt{2\pi}}e^{(-\frac{1}{2})(y-\sigma)^2}dy + \frac{1}{8}Pe^{\mu}\int_{y>\frac{\ln(1.2)}{\sigma}+\frac{1}{2}\sigma} f(y)dy
\end{aligned}$$

Let $z = y - \sigma$, we have

$$\begin{aligned}
=&\frac{7}{8}Pe^{\mu}\int_{z>\frac{\ln(1.2)}{\sigma}-\frac{1}{2}\sigma} \frac{1}{\sqrt{2\pi}}e^{(-\frac{1}{2})(z)^2}dz + \frac{1}{8}Pe^{\mu}\int_{y>\frac{\ln(1.2)}{\sigma}+\frac{1}{2}\sigma} f(y)dy \\
=&\frac{7}{8}Pe^{\mu}\Phi(\frac{-\ln(1.2)}{\sigma} + \frac{1}{2}\sigma) + \frac{1}{8}Pe^{\mu}\Phi(\frac{-\ln(1.2)}{\sigma} - \frac{1}{2}\sigma)
\end{aligned} \tag{4.38}$$

Similarly, we are able to write the expression for the other cases. Summing all the cases gives the solution to a one year contract, i.e.,

$$
\begin{aligned}
&E[A_1^{BW}] > E[A_1^{BW^*}] \\
=&Pe^{\mu}\Bigg(\frac{7}{8}\Phi(\frac{-\ln(1.2)}{\sigma}+\frac{1}{2}\sigma)+\frac{3}{4}(\Phi(\frac{-\ln(1.1)}{\sigma}+\frac{1}{2}\sigma)-\Phi(\frac{-\ln(1.2)}{\sigma}+\frac{1}{2}\sigma)) \\
&+\frac{1}{2}(\Phi(\frac{-\ln(1.05)}{\sigma}+\frac{1}{2}\sigma)-\Phi(\frac{-\ln(1.1)}{\sigma}+\frac{1}{2}\sigma)) \\
&+\frac{1}{2}(\Phi(\frac{-\ln(0.9)}{\sigma}+\frac{1}{2}\sigma)-\Phi(\frac{-\ln(0.95)}{\sigma}+\frac{1}{2}\sigma)) \\
&+\frac{3}{4}(\Phi(\frac{-\ln(0.8)}{\sigma}+\frac{1}{2}\sigma)-\Phi(\frac{-\ln(0.9)}{\sigma}+\frac{1}{2}\sigma)) \\
&+\frac{7}{8}\Phi(\frac{\ln(0.8)}{\sigma}-\frac{1}{2}\sigma) \\
&+\frac{1}{8}\Phi(\frac{-\ln(1.2)}{\sigma}-\frac{1}{2}\sigma)+\frac{1}{4}(\Phi(\frac{-\ln(1.1)}{\sigma}-\frac{1}{2}\sigma)-\Phi(\frac{-\ln(1.2)}{\sigma}-\frac{1}{2}\sigma)) \\
&+\frac{1}{2}(\Phi(\frac{-\ln(1.05)}{\sigma}-\frac{1}{2}\sigma)-\Phi(\frac{-\ln(1.1)}{\sigma}-\frac{1}{2}\sigma)) \\
&+\Phi(\frac{-\ln(0.9)}{\sigma}-\frac{1}{2}\sigma)-\Phi(\frac{-\ln(1.05)}{\sigma}-\frac{1}{2}\sigma) \\
&+\frac{1}{2}(\Phi(\frac{-\ln(0.9)}{\sigma}-\frac{1}{2}\sigma)-\Phi(\frac{-\ln(0.95)}{\sigma}-\frac{1}{2}\sigma)) \\
&+\frac{1}{4}(\Phi(\frac{-\ln(0.8)}{\sigma}-\frac{1}{2}\sigma)-\Phi(\frac{-\ln(0.9)}{\sigma}-\frac{1}{2}\sigma)) \\
&+\frac{1}{8}\Phi(\frac{\ln(0.8)}{\sigma}+\frac{1}{2}\sigma)\Bigg)
\end{aligned}
\tag{4.39}
$$

After substituting the value of μ and σ into the equation (4.39), we find out that

$$
E[A_1^{BW}] > E[A_1^{BW^*}] > E[A_1]. \tag{4.40}
$$

In the above, we gave an approximated solution to a one year contract. However, for a N year contract, numerical method should be used to calculate the price. Our result shows BW method always gives a slightly higher payout than the unsmoothed payout. As the difference is small, we plot the difference between the expected value using BW smoothing method and the unsmoothed value , $E[A_n^{BW}] - E[A_n]$, at the end of each year n in the Figure 4.7.

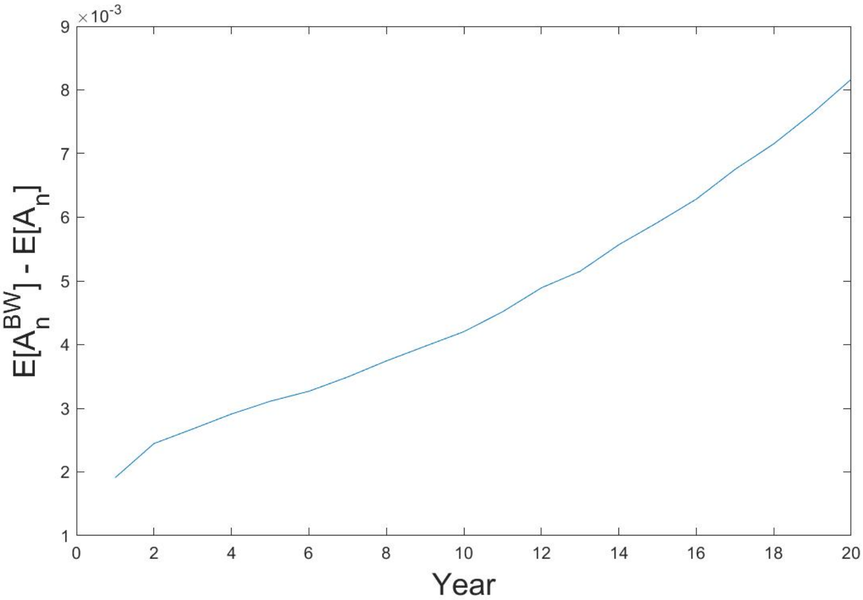


Figure 4.7: The difference between the expected value using BW smoothing method and the unsmoothed value , $E[A_n^{BW}] - E[A_n]$. The parameters are $\mu = 0.065$, $\sigma = 0.15$, $N = 20$, $P = 1$.

4.4 Further analysis of the smoothing methods

In the previous section, we have shown that, under the Black-Scholes world, the geometric average (GA) method overvalues the contract, the bandwidth (BW) undervalues the contract and the weighted sum (WS) method gives a fair payout to the customers. The Black-Scholes model assumes the returns in each year are independent, but this is not consistent with previous empirical studies. For example, Fama and French (1988) studies all New York Stock Exchange stocks data from 1926 to 1985 and show that a negative auto-correlation is observed for longer horizons (2 to 5 years). Poterba and Summers (1988) did a similar empirical study in another 17 countries and conclude the same result. On the other hand, Lo and MacKinlay (1988) and Lo and MacKinlay (1990) find out the returns of equity portfolio are typically positively auto-correlated for short horizons. Grundy (1991) and Cont (2001) use more recent data and provide consistent evidence for positively auto-correlated returns at short horizons. Thus, it is interesting to know what happens in a financial market model when investment returns are dependent.

In the following, we follow Lo and Wang (1995) to use the bivariate trending Ornstein-Uhlenbeck(OU) process, which considers both the short-term positive auto-correlation and the long term reversal, to model the price of the underlying fund. In addition, an analytical result of the fairness for GA method under auto-regress AR(1) has been given in Appendix (4.6).

4.4.1 Trending Bivariate Ornstein-Uhlenbeck(OU) model

In order to study correlated returns in financial market, Lo and Wang (1995) propose the bivariate trending Ornstein-Uhlenbeck(OU) process to model the price of the underlying asset, which is defined as

$$\begin{cases} dS_t = \Big(\mu - \kappa\big(\log \frac{S_t}{S_0} - (\mu - \frac{1}{2}\sigma^2)t\big) + \lambda H_t\Big) S_t dt + \sigma S_t dW_t^{(s)}, \\ dH_t = -\delta H_t dt + \sigma_x dW_t^{(h)}, \\ S_0 = s, H_0 = h, \end{cases} \tag{4.41}$$

where $W_t^{(s)}$ and $W_t^{(h)}$ are two independent standard Brownian motions. The process H_t is not observable and is used to make the long term mean of the underlying return process more flexible. $\kappa \geq 0$ and $\delta \geq 0$ are the mean reverting parameters of processes S_t and H_t, respectively. This is more straightforward if we let the de-trended return $q_t = \log S_t - (\mu - \frac{1}{2}\sigma^2)t$, then Equation (4.41) becomes

$$\begin{cases} dq_t = \kappa\big((q_0 + \frac{\lambda}{\kappa} H_t) - q_t\big) dt + \sigma dW_t^{(s)}, \\ dH_t = -\delta H_t dt + \sigma_h dW_t^{(h)}, \\ q_0 = \log s, H_0 = h. \end{cases} \tag{4.42}$$

In the following, we present the numerical result of the smoothed terminal value of each smoothing methods by assuming the underlying fund follows the bi-variate trending OU process. For comparison, the geometric process is also used to model the underlying fund price process. The parameters of the above processes given in Table 4.2 are referenced from the calibration result of Thierfelder (2015) which is based on 20 years data of FTSE 100.

Table 4.3 shows that when the underlying fund value follows both the geometric Brownian motion and the Bi-variate trending OU process, the GA method generates a smaller payout than the actual fund value. The BW method gives a slightly

Price Process	μ	κ	δ	σ	σ_h	λ
BiVar Trend	0.09	3.75	1.81	0.154	1.2	1
GBM	0.09	-	-	0.183	-	-

Table 4.2: Parameters for the processes of underlying fund price.

BiVar Trend	Actual	GA	WS	BW
mean of terminal value	6.186	6.105	6.186	6.199
variance of terminal value	1.77	0.70	1.25	1.58

GBM	Actual	GA	WS	BW
mean of terminal value	6.051	5.931	6.050	6.064
variance of terminal value	35.02	29.55	31.17	34.29

Table 4.3: Mean and variance of the terminal payout of a 20 years contract from different smoothing methods.

higher payout. Under both cases, the WS method tend to give a fair payout to the customers.

4.4.2 Smoothing effects

In the previous section, we only compare the terminal value of three smoothing methods. However, as the insurance companies and pension funds generally sends annual statement of the pension portfolios to their customers at the end of each year, Bellemare et al. (2005) suggest that such interim information affects the customers' behaviours. In addition, Benartzi and Thaler (1995) provides evidence that long-term investors choose their investment strategies based on short evaluation horizons. Furthermore, Arkes et al. (2008) show that investors tend to adjust their reference point within the period of the investment. As the aim of smoothing is to smooth out the short term fluctuation of the fund value over the long-term, it is not reasonable to neglect the short-term performance of the smoothing method. Thus, in this section, we follow Ruß and Schelling (2018) and use the Multi Cumulative Prospect Theory (MCPT) to evaluate the short-term performance of different smoothing methods.

In CPT, an investment or a prospect f is expressed as

$$f = (e_{-m}, p_{-m}; e_{-m+1}, p_{-m+1}; ...; e_0, p_0; ...; e_{n-1}, p_{n-1}; e_n, p_n;)$$

with possible outcomes e_i and the corresponding probabilities p_i. The outcomes are compared against a reference point (usually the initial value of investment A_0, or the premium P) and arranged in ascending order in terms of the payout value. CPT values gains and loss separately. Let $d_t = e_i - A_0$ denote the payout for each possible outcome. Then the CPT utility of an investment is the sum of utility of positive prospect f^+ and negative prospect f^-, or i.e.,

$$V(f) = V(f^+) + V(f^-) = \sum_{i=0}^{n} \pi_i^+ v(d_i) + \sum_{i=-m}^{0} \pi_i^- v(d_i) \tag{4.43}$$

where v is the value function and π_i is the decision weights associated with payout d_i. The decision weight π is a subjective probability which is calculated as the first order difference of the cumulative weighting function:

$$\pi_i^+ = w(p_i + ... + p_n) - w(p_{i+1} + ... + p_n), \; 0 \leq i \leq n-1; \tag{4.44}$$

$$\pi_i^- = w(p_{-m} + ... + p_i) - (p_{-m} + ... + p_{i-1}), \; 1-m \leq i \leq 0; \tag{4.45}$$

$$\pi_n^+ = w(p_n); \tag{4.46}$$

$$\pi_{-m}^- = w(p_{-m}). \tag{4.47}$$

Value function $v(x)$ shows how people value things (like gambling or investment). It is monotonically increasing. A positive outcome suggests a positive utility value while negative outcome leads to a negative utility value. The value function is concave in the positive part and convex in the negative part. The curvature for losses is steeper than for gains.

Weighting function presents how people deal with the probabilities. It does not have a linear relationship with the real probability, as it overweight small probability and underweight large probabilities.

Ruß and Schelling (2018)'s MCPT is an extension of CPT by taking into account multiple reference points and comparison horizons in calculating customers' subjective utility. For an investment A, the customer is assumed to evaluate the portfolio at the end of each year. In this sense, the reference point in each year is naturally the value of the portfolio last year.

Let $D_t = A_t - A_{t-1}$ denote the value change of the investor's investment in year t, then the MCPT interim utility of an investment A for investment horizon $T \in \mathbb{Z}^+$ is given by

$$MCPT(A) := \sum_{t=1}^{T} V(D_t). \tag{4.48}$$

The total utility of an investment is a linear combination of the interim utility and the terminal utility. That is,

$$CPT^{com}(A) = sMCPT(A) + (1-s)V(D), \tag{4.49}$$

where $D = A_T - A_0$ is the difference between the terminal value and the initial investment and s controls the influence from interim utility on the total utility.

Below, we present the numerical result of the MCPT utility for each smoothing method. The controlling factor s is chosen to be 0.5 by following Ruß and Schelling (2018). In addition, the value function, weighting function and the parameters used in this section are taken from Ruß and Schelling (2018). Specifically, the value function is

$$v(d) = \begin{cases} d^{\alpha} \text{ if } d \geq 0 \\ -\lambda(-d)^{\beta} \text{ if } d < 0, \end{cases} \tag{4.50}$$

where $\lambda = 2.25$ and $\alpha = \beta = 0.88$.

The weighting function w rescales the probabilities and satisfies $w(0) = 0$ and $w(1) = 1$.

$$w(p) = \frac{p^{\gamma}}{(p^{\gamma} + (1-p)^{\gamma})^{\frac{1}{\gamma}}} \tag{4.51}$$

where $\gamma = 0.65$.

Table 4.4 shows the total utility which combines the interim utility and the terminal utility of the investment following each smoothing method. The results suggest that the GA method gives the customers the highest utility. WS method

BiVar Trend	Actual	GA	WS	BW
CPT^{com}	1.0667	5.3651	3.9640	2.0974

GBM	Actual	GA	WS	BW
CPT^{com}	0.7253	3.5081	2.8066	1.4830

Table 4.4: Combined utility of the terminal payout of a 20 years contract for different smoothing methods.

and BW method generates the second highest and the third highest utility. The underlying investment has the least utility, which arises from its high variance.

4.5 Conclusion

Return smoothing method is an important feature of traditional pension contract. By using a smoothing method, The insurers is able to keep some profit in good years and give it back to the customers when the market performs poorly. As the smoothing method provides the customer a more stable return by time diversification and risk sharing, it is believed to be a virtue of it. In this chapter, we carefully compare three smoothing methods, geometric average (GA) method, weighted sum (WS) method and the Bandwidth (BW) method, which are still used in with-profits contracts nowadays.

We first examine if each smoothing method provide a equitable payout for each policyholders. By assuming the underlying investment fund follows the classic geometric Brownian motion, our results indicate that the GA smoothing mechanism underpays the customer while the BW method overpays the customer. The WS smoothing mechanism generates a fair payout. Empirical evidence shows that the annual investment returns are not independent. In order to identify if these result still hold when the returns are correlated, we use a more realistic model, the trending bivariate OU model, to model the asset returns and our finding supports the previous result.

We not only care about the fairness of the terminal value, but also interest in the utility generated by each smoothing methods to the customers. Especially, we take into account the interim utility in this chapter. Each year, the policyholders should receive an annual update about the performance of their investment. Thus, we believe the interim utility is also important part of the customers' overall utility. Specifically, Multi-Cumulative Prospect Theory (MCPT) is used to calculate the interim utility and the terminal utility of the customers for each smoothing method. To our surprise, the results suggest that GA method generate the highest overall utility even though it provides the least expected return to the customers. On the other hand, BW method overpays the customers but generates less utility to the policyholders. These results hold when the underlying fund value follow both the GBM model and trending bivariate OU process.

4.6 Appendix: the fairness of the GA smoothing method under auto-regress AR(1) model

In this section, we present the analytical result of geometric average (GA) smoothing method when the underlying returns follows an AR(1) model by relaxing the assumption of return independence. Let S_n be the price of the underlying fund at the end of year n. The log return $X_n = \log(\frac{S_n}{S_{n-1}})$ in year n follows the autoregressive AR(1) model:

$$X_{n+1} - \theta = a(X_n - \theta) + \beta\varepsilon^{\mathbb{P}}_{n+1}, \tag{4.52}$$

where a is the AR parameter which controls the speed at which such trajectories will reverse back to the long term mean θ. $\varepsilon^{\mathbb{P}}_{n+1}$ is a white noise process with mean zero and variance one under real world measure $\mathbb{P}$. β is a constant volatility parameter. In order to let the AR(1) model be stationary, $|a| < 1$ is assumed. After recursive iteration of equation (4.52), we have

$$X_n = a^n X_0 + (1 - a^n)\theta + \sum_{i=1}^{n} a^{i-1}\beta\varepsilon^{\mathbb{P}}_{n-i+1}, \tag{4.53}$$

where the initial value X_0 is θ. It is observed that the expected value for return in each year under AR model is the same, i.e.

$$E(X_n) = \theta \quad \forall \quad n \in N. \tag{4.54}$$

And for variance, we have

$$Var(X_n) = \beta^2(1 + a^2 + a^4 + \cdots + a^{2n}) = \beta^2(\frac{1 - a^{2n}}{1 - a^2}). \tag{4.55}$$

When n is larger, we have,

$$Var(X_n) \to \frac{\beta^2}{1 - a^2}. \tag{4.56}$$

Recall that we let A_N and A_N^{GA} denote the actual value and the smoothed value of investment fund. In the following, we compare the expected smoothed terminal value $E[A_N^{GA}]$ against the actual value $E[A_N]$ for GA smoothing method. For a contract of N years maturity, the actual fund value is

$$\begin{aligned} E[A_N] &= E[(1+R_1)(1+R_2)\cdots(1+R_N)] \\ &= E\left[Y_1 Y_2 \cdots Y_{N-1} Y_N\right], \end{aligned} \tag{4.57}$$

and the smoothed fund value is

$$\begin{aligned} E[A_N^{GA}] &= E[(1+R_1^{GA})(1+R_2^{GA})\cdots(1+R_N^{GA})] \\ =&E\left[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\cdots[(Y_{N-2})(Y_{N-1})(Y_N)\exp{(2\theta)}]^{\frac{1}{5}}\right]. \end{aligned} \tag{4.58}$$

For simplicity, we let the starting time of the contract be $n+2$, that is to say $Y_i = 1 + R_i = \exp{(X_{n+2+i})}$ for $i = -1, 0, 1, 2, ..., N$. Hardy (2003) suggests that if the pricing is not designed to apply at a specific starting date, the reasonable starting value would be the long term mean value of variables. Thus, we let the starting value X_{n+2+i} equal to the long term average of the return θ and the variance is $\frac{\beta^2}{1-a^2}$. Then

$$\begin{aligned} E[A_N] &= E\left[\exp{(X_{n+3} + X_{n+4} + \cdots + X_{n+N+1} + X_{n+N+2})}\right] \\ =&E\left[\exp\left(N\theta - \theta\sum_{i=1}^{N} a^{i+2} + X_n\sum_{i=1}^{N} a^{i+2} + \sum_{i=1}^{N} a^{i+1}\beta\varepsilon_{n+1}^{\mathbb{P}} + \sum_{i=1}^{N} a^{i}\beta\varepsilon_{n+2}^{\mathbb{P}}\right.\right. \\ &\left.\left. + \sum_{i=1}^{N} a^{i-1}\beta\varepsilon_{n+3}^{\mathbb{P}} + \sum_{i=1}^{N-1} a^{i-1}\beta\varepsilon_{n+4}^{\mathbb{P}} + \cdots + \sum_{i=1}^{1} a^{i-1}\beta\varepsilon_{n+N+2}^{\mathbb{P}}\right)\right]. \end{aligned} \tag{4.59}$$

The smoothed fund value is calculated as

$$
\begin{aligned}
&E[A_N^{GA}] = E[(1+R_1^{GA})(1+R_2^{GA})\cdots(1+R_N^{GA})] \\
=&E\left[[(Y_{-1})(Y_0)(Y_1)(Y_2)(Y_3)]^{\frac{1}{5}}\cdots[(Y_{N-2})(Y_{N-1})(Y_N)\exp(2\theta)]^{\frac{1}{5}}\right] \\
=&E\left[\exp\left(N\theta - \theta(\frac{1}{5}a + \frac{2}{5}a^2 + \underbrace{\frac{3}{5}a^3 + \frac{4}{5}a^4 + \sum_{i=5}^{N} a^i + \frac{4}{5}a^{N+1} + \frac{3}{5}a^{N+2})}_{a^3C}\right.\right. \\
&+\ X_n(\frac{1}{5}a + \frac{2}{5}a^2 + \underbrace{\frac{3}{5}a^3 + \frac{4}{5}a^4 + \sum_{i=5}^{N} a^i + \frac{4}{5}a^{N+1} + \frac{3}{5}a^{N+2})}_{a^3C} \\
&+\ \beta\varepsilon_{n+1}^{\mathbb{P}}(\frac{1}{5} + \frac{2}{5}a + \underbrace{\frac{3}{5}a^2 + \frac{4}{5}a^3 + \sum_{i=5}^{N} a^{i-1} + \frac{4}{5}a^{N} + \frac{3}{5}a^{N+1})}_{a^2C} \\
&+\ \beta\varepsilon_{n+2}^{\mathbb{P}}(\frac{2}{5} + \underbrace{\frac{3}{5}a + \frac{4}{5}a^2 + \sum_{i=5}^{N} a^{i-2} + \frac{4}{5}a^{N-1} + \frac{3}{5}a^{N})}_{aC} \\
&+\ \beta\varepsilon_{n+3}^{\mathbb{P}}\underbrace{(\frac{3}{5} + \frac{4}{5}a + \sum_{i=5}^{N} a^{i-3} + \frac{4}{5}a^{N-2} + \frac{3}{5}a^{N-1})}_{C} \\
&+\ \beta\varepsilon_{n+4}^{\mathbb{P}}\underbrace{(\frac{4}{5} + \sum_{i=5}^{N} a^{i-4} + \frac{4}{5}a^{N-3} + \frac{3}{5}a^{N-2})}_{C_4} \\
&+\ \beta\varepsilon_{n+5}^{\mathbb{P}}\underbrace{(\sum_{i=1}^{N-4} a^{i-1} + \frac{4}{5}a^{N-4} + \frac{3}{5}a^{N-3})}_{C_5} \\
&\qquad\vdots \\
&+\ \beta\varepsilon_{n+N}^{\mathbb{P}}\underbrace{(\sum_{i=1}^{1} a^{i-1} + \frac{4}{5}a + \frac{3}{5}a^2)}_{C_N} \\
&+\ \beta\varepsilon_{n+N+1}^{\mathbb{P}}(\frac{4}{5} + \frac{3}{5}a) \\
&\left.\left.+\ \beta\varepsilon_{n+N+2}^{\mathbb{P}}(\frac{3}{5})\right)\right].
\end{aligned}
\tag{4.60}
$$

In order to compare the expected value of the smoothed fund value and the unsmoothed fund value, we separate some items in Equation (4.59). Thus,

$$
\begin{aligned}
E[A_N] = E\Big[\exp\Big(&N\theta \\
-\theta(&\frac{1}{5}(a^4+a^{N+1})+\frac{2}{5}(a^3+a^{N+2})+\frac{3}{5}a^3+\frac{4}{5}a^4+\sum_{i=5}^{N}a^i+\frac{4}{5}a^{N+1}+\frac{3}{5}a^{N+2}) \\
+\ X_n(&\frac{1}{5}(a^4+a^{N+1})+\frac{2}{5}(a^3+a^{N+2})+\underbrace{\frac{3}{5}a^3+\frac{4}{5}a^4+\sum_{i=5}^{N}a^i+\frac{4}{5}a^{N+1}+\frac{3}{5}a^{N+2}}_{a^3C}) \\
+\ \beta\varepsilon^{\mathbb{P}}_{n+1}(&\frac{1}{5}(a^3+a^N)+\frac{2}{5}(a^2+a^{N+1})+\underbrace{\frac{3}{5}a^2+\frac{4}{5}a^3+\sum_{i=5}^{N}a^{i-1}+\frac{4}{5}a^N+\frac{3}{5}a^{N+1}}_{a^2C}) \\
+\ \beta\varepsilon^{\mathbb{P}}_{n+2}(&\frac{1}{5}(a^2+a^{N-1})+\frac{2}{5}(a+a^N)+\underbrace{\frac{3}{5}a+\frac{4}{5}a^2+\sum_{i=5}^{N}a^{i-2}+\frac{4}{5}a^{N-1}+\frac{3}{5}a^N}_{aC}) \\
+\ \beta\varepsilon^{\mathbb{P}}_{n+3}(&\frac{1}{5}(a+a^{N-2})+\frac{2}{5}(1+a^{N-1})+\underbrace{\frac{3}{5}+\frac{4}{5}a+\sum_{i=5}^{N}a^{i-3}+\frac{4}{5}a^{N-2}+\frac{3}{5}a^{N-1}}_{C}) \\
+\ \beta\varepsilon^{\mathbb{P}}_{n+4}(&\frac{1}{5}(1+a^{N-3})+\frac{2}{5}(a^{N-2})+\underbrace{\frac{4}{5}+\sum_{i=5}^{N}a^{i-4}+\frac{4}{5}a^{N-3}+\frac{3}{5}a^{N-2}}_{C_4}) \\
+\ \beta\varepsilon^{\mathbb{P}}_{n+5}(&\frac{1}{5}(a^{N-4})+\frac{2}{5}(a^{N-3})+\underbrace{\sum_{i=1}^{N-4}a^{i-1}+\frac{4}{5}a^{N-4}+\frac{3}{5}a^{N-3}}_{C_5}) \\
&\vdots \\
+\ \beta\varepsilon^{\mathbb{P}}_{N}(&\frac{1}{5}a+\frac{2}{5}a^2+\underbrace{\sum_{i=1}^{i}a^{i-1}+\frac{4}{5}a+\frac{3}{5}a^2}_{C_N}) \\
+\ \beta\varepsilon^{\mathbb{P}}_{N+1}(&\frac{1}{5}+\frac{2}{5}a+\frac{4}{5}+\frac{3}{5}a) \\
+\ \beta\varepsilon^{\mathbb{P}}_{N+2}(&\frac{2}{5}+\frac{3}{5})\Big)\Big].
\end{aligned}
\tag{4.61}
$$

We let Z^{GA} and Z denote the exponent in equation (4.59) and (4.60), respectively. That is to say

$$E[A_N] = E\left[\exp\left(Z\right)\right], \tag{4.62}$$

$$E[A_N^{GA}] = E\left[\exp\left(Z^{GA}\right)\right]. \tag{4.63}$$

From the properties of log-normal random variables, we know that

$$E[A_N] = E\left[\exp\left(Z\right)\right] = \exp\big(E(Z) + \frac{1}{2}Var(Z)\big), \tag{4.64}$$

$$E[A_N^{GA}] = E\left[\exp\left(Z^{GA}\right)\right] = \exp\left(E\big(Z^{GA}\right) + \frac{1}{2}Var(Z^{GA})\big). \tag{4.65}$$

In order to prove $E\left[A_N^{GA}\right] < E\left[A_N\right]$, we only need to show that $Var(Z) > Var(Z^{GA})$ as $E\left[Z^{GA}\right] = E\left[Z\right]$. We have

$$Var(Z^{GA}) = Var(X_n)(\frac{1}{5}a + \frac{2}{5}a^2 + a^3C)^2 \tag{4.66}$$

$$+ \beta^2(\frac{1}{5} + \frac{2}{5}a + a^2C)^2 \tag{4.67}$$

$$+ \beta^2(\frac{2}{5} + aC)^2 \tag{4.68}$$

$$+ \beta^2(C^2 + C_4^2 + C_5^2 + \cdots + C_{N-1}^2 + C_N^2) \tag{4.69}$$

$$+ \beta^2((\frac{4}{5} + \frac{3}{5}a)^2 + (\frac{3}{5})^2), \tag{4.70}$$

and we also have

$$
\begin{aligned}
Var(Z) \geq & Var(X_n)(\frac{1}{5}a^4 + \frac{2}{5}a^3 + a^3C + \frac{2}{5}a^{N+2})^2 && (4.71)\\
& + \beta^2(\frac{1}{5}a^3 + \frac{2}{5}a^2 + a^2C + \frac{2}{5}a^{N+1})^2 && (4.72)\\
& + \beta^2(\frac{1}{5}a^2 + \frac{2}{5}a + aC + \frac{2}{5}a^{N})^2 && (4.73)\\
& + \beta^2(\frac{1}{5}a + \frac{2}{5} + C + \frac{2}{5}a^{N-1})^2 && (4.74)\\
& + \beta^2(\frac{1}{5} + C_4 + \frac{2}{5}a^{N-2})^2 && (4.75)\\
& + \beta^2(C_5 + \frac{2}{5}a^{N-3})^2 && (4.76)\\
& \quad \vdots \\
& + \beta^2((\frac{1}{5}a^2 + \frac{2}{5}a^3) + C_{N-1})^2 && (4.77)\\
& + \beta^2((\frac{1}{5}a + \frac{2}{5}a^2) + C_N)^2 && (4.78)\\
& + \beta^2(\frac{1}{5} + \frac{2}{5}a + \frac{4}{5} + \frac{3}{5}a)^2 + \beta^2(\frac{2}{5} + \frac{3}{5})^2. && (4.79)
\end{aligned}
$$

As we showed before,

$$
Var(X_n) = \frac{\beta^2}{1-a^2} = \beta^2(1 + a^2 + a^4 + \cdots). \tag{4.80}
$$

We rewrite equation (4.66) as

$$
\begin{aligned}
&Var(X_n)(\frac{1}{5}a+\frac{2}{5}a^2+a^3C)^2\\
=&(\frac{1}{5}a+\frac{2}{5}a^2)^2(1+a^2)+(\frac{1}{5}a+\frac{2}{5}a^2)^2(a^4+a^6+\cdots)\\
&+2a^3C(\frac{1}{5}a+\frac{2}{5}a^2)+2a^3C(\frac{1}{5}a+\frac{2}{5}a^2)(a^2+a^4+\cdots)\\
&+(a^3C)^2(1+a^2+a^4+\cdots)\\
=&(\frac{1}{5}a+\frac{2}{5}a^2)^2+(\frac{1}{5}a^2+\frac{2}{5}a^3)^2+2C(\frac{1}{5}a^4+\frac{2}{5}a^5)\\
&+(\frac{1}{5}a^3+\frac{2}{5}a^4)^2(1+a^2+\cdots)+2a^3C(\frac{1}{5}a^3+\frac{2}{5}a^4)(1+a^2+\cdots)\\
&+(a^3C)^2(1+a^2+a^4+\cdots)\\
=&\underbrace{(\frac{1}{5}a+\frac{2}{5}a^2)^2}_{A1}+\underbrace{(\frac{1}{5}a^2+\frac{2}{5}a^3)^2}_{A2}+\underbrace{2C(\frac{1}{5}a^4+\frac{2}{5}a^5)}_{A3}\\
&+\underbrace{(\frac{1}{5}a^3+\frac{2}{5}a^4+a^3C)^2Var(X_n)}_{A4}.
\end{aligned}
\tag{4.81}
$$

$A1$ is the same as the first term in brackets of the expansion of Equation (4.78). Similarly, $A2$ can be cancelled out with the first term in brackets of of the expansion of Equation (4.77). As $|a| < 1$, $A3$ is smaller than $2C(\frac{2}{5}a^4 + \frac{1}{5}a^5)$ which is found in the expansion of the Equation (4.72). In addition, we can see that $A4$ is smaller than $Var(X_n)(\frac{1}{5}a^4+\frac{2}{5}a^3+a^3C)^2$ which is Equation (4.71). For each item in equation (4.67) - (4.70), we can always find a responding item in equation (4.72) - (4.79).

In the above, we have shown that $E\left[A_N^{GA}\right] < E\left[A_N\right]$ holds for the AR(1) model. Hence, we can conclude that the result that the GA smoothing methods underpaying the customer does not only hold when the investment returns are independent.

Chapter 5

The design of pension contracts: on the perspective of customers

5.1 Introduction

We study a new pension contract that operates in the accumulation phase of a pension scheme. It can be used by an insurance company to smooth the investment return for customers. In contrast to traditional with-profits products, it has a transparent structure and a clearly defined rule for bonus distribution. We compare the new contract to another two similar contracts, which have been studied intensively in previous papers[1]. Our results show that the new contract is a more attractive contract for a customer under the Cumulative Prospect Theory, which suggests that guarantees should be included in pension contracts.

Pension saving and investment plays an important role in an individual's lifetime wealth management. Thanks to the development of science and health care, people live much longer than before. In order to have a decent lifestyle in retirement, people should have sufficient savings before they are out of the workforce. As Samuelson (1958) suggests, people should save some of their income during their working years for their retirement.

For customers, choosing a pension contract is making a decision under uncer-

[1]The literature includes Guillén et al. (2006), Jørgensen and Linnemann (2012), Linnemann et al. (2015), Døskeland and Nordahl (2008b) and Chen et al. (2015).

tainty. Hence, decision theory is often used in designing pension and insurance contracts. One widely used decision theory is Expected Utility Theory (Von Neumann and Morgenstern, 1944) which is a dominant theory in the last century to explain individuals' behaviour under uncertainty. The calculation of expected utility is easy to understand and implement. Specifically, the expected utility of an investment is just the sum of the utility of each possible outcome weighted by its probability. However, there are critics of Expected Utility Theory (henceforth called EUT) as it fails to explain people's behaviour in some cases. Tversky and Kahneman (1992) summarise five violations of EUT in explaining people's behaviour under uncertainty. The most famous examples are Allais' paradox (Allais, 1953) and Ellsberg's paradox (Ellsberg, 1961).

A new theory to describe individual's behaviour, Cumulative Prospect Theory (and its original version Prospect Theory[2]), is becoming more popular in evaluating how people make decision under uncertainty. Cumulative Prospect Theory (henceforth called CPT) is proposed by Tversky and Kahneman (1992) to explain the choice made by people violating the standard EUT. The most distinct part of CPT is that it values outcomes based on gains or losses relative to a reference level of wealth rather than on the absolute value of the wealth at retirement. In addition, people tend to overvalue small probabilities and undervalue moderate and high probabilities. Instead of using real world probability, CPT uses a cumulative weighting function which is a distortion of the probability.

This paper focuses on the accumulation of the pension savings. In this sense, the pension contract works as a long term saving product which helps people accumulate money during the working years and its proceeds can be used to provide them with an income after their retirement. How to design an attractive pension product for customers is of significance to the product development department of life insurance companies and pension providers. Chen et al. (2015) suggests there are generally two ways to design a pension contract. One is to solve an optimization problem for

[2]Cumulative Prospect Theory is a further development of Prospect Theory. It introduces a cumulative weighting function to substitute the separate weighting function in Prospect Theory. For the details of these two theories, see Kahneman and Tversky (1979) and Tversky and Kahneman (1992).

a specific utility theory and reversely design the contract from the optimal strategy. This method has been used by Bruhn and Steffensen (2013) to reversely engineer an optimal product under EUT. Hens and Rieger (2014) also use this method to approximate the optimal payoff under the Prospect Theory. The other way is to find out the contract that delivers the highest utility value among some contracts on the perspective of the customer. This can be done by calculating the utility of different products using a chosen utility function. The product generating the highest utility is the most attractive one (this is done in Døskeland and Nordahl 2008b, Branger et al. 2010, Chen et al. 2015). Another widely used method is to find the optimal portfolio which gives the largest utility for customers. If one contract exists in the optimal portfolio while the other does not, then the former should be viewed as a more attractive product for customers (see Døskeland and Nordahl 2008a).

In this paper, the second way is chosen as we design the contract first. Specifically, we compare our new contract against the other two similar contracts as well as a risky asset and a risk-free asset in two methods, finding the contract generates the highest utility and investigating the optimal portfolio under both the CPT and EUT, by assuming customers use a buy-and-hold strategy. As individual pension investors tend not to review their investment holdings frequently, buy-and-hold is a more reasonable investment strategy. The comparison results show that our contract is an attractive contract to a CPT-maximising customer. However, we do not reach the same conclusion for an EUT-maximising customer.

For a CPT investor, Døskeland and Nordahl (2008b) and Dichtl and Drobetz (2011) provide the evidence that investment guarantees increase an investor's utility. This explains why many traditional pension contracts in the market incorporate guarantees, either in the form of interest rate guarantees or as a sum assured. Hens and Rieger (2014) suggest that an optimal structured product should not only have a downside protection but also retain the potential of upside return. Their finding is consistent with the design fashion of including a bonus feature into pension contracts. Hence, our new contract is presented as the combination of a ratchet type guarantees and a possible bonus. This kind of structure removes the risk of loss for a policyholders investment and keeps the opportunity to make a positive return. There are two contracts studied in previous literature which are similar to our new product.

There are some pension products studied in the literature with a similar structure in terms of ratchet style guarantees, a smoothing mechanism and a possible bonus. They are different from each other mainly because of their bonus determination method. As Zemp (2011) suggests, most of these products can be grouped in two categories. One is a return-based bonus distribution. For instance, Bacinello (2001) determines the bonus by comparing the annual investment return with the guaranteed rate while Haberman et al. (2003) use the past three years' average return. The other category is a reserve-based bonus distribution. These contracts generally have a target buffer ratio or a target reserve level for the insurance company, as Grosen and Jørgensen (2000) and Hansen and Miltersen (2002) show. In contrast to these approaches, the bonus determination method of our new contract is to compare the value of a specified investment account and the customer account. It has a well-defined return distribution rule and a transparent product structure.

The remainder of the paper is organised as follows. In Section 5.2, we introduce the new pension contract. The pricing and some characteristics of the new contract are also given in this part. Additionally, two similar pension contracts, one with a similar product structure but without guarantee embedded[3] and the other providing similar guarantees but with an entirely different product structure[4], are introduced in this part as well. Section 5.3 compares the new contract with these two contracts under CPT. In Section 5.4, some robustness and sensitivity test are carried out. In order to know how these two contracts perform for an EUT customer, we also provide a comparison for these three contracts under EUT in Section 5.5. The last part of this paper is the conclusion.

[3]This contract is discussed by Guillén et al. (2006), Jørgensen and Linnemann (2012) and Linnemann et al. (2015).

[4]This contract is studied by Døskeland and Nordahl (2008b) and Chen et al. (2015).

5.2 Product design

Mortality, surrender and expense are not considered in this paper[5], as we are more interested in the savings aspect of pension products. Here, the new contract is introduced in detail. A brief introduction of the other two similar contracts are also given in this part.

5.2.1 Financial market model

It is assumed that the investment takes over a finite time horizon $[0, T]$, where T is a strictly positive integer. There are only two underlying assets in the market. One is the risk free asset and the other is the risky asset. The risk free asset can be a government bond while the risky asset could be an equity index. The prices of both assets are available in the market. Pension contracts are derivatives of these two underlying assets. The evolution of the value of the risk free asset B_t is

$$dB_t = rB_t dt, \qquad B_0 = b, \tag{5.1}$$

where $r > 0$ is the constant risk-free interest rate and $b > 0$ is a constant. The price of the risky asset follows the following geometric Brownian motion.

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \qquad S_0 = s, \tag{5.2}$$

where the expected growth rate μ, volatility σ and initial value s are positive constants. W is a standard Brownian motion defined on the filtered probability space $(\Omega, \mathbb{F}, (\mathcal{F}_t), \mathbb{P})$. All the perfect market assumptions suggested by Black and Scholes (1973) hold here. The Black-Scholes model is widely used in literature for the study of insurance and pension contracts, e.g. Døskeland and Nordahl (2008b), Branger et al. (2010) and Bauer et al. (2006).

[5]There are many papers studying the pricing of life and pension contracts when mortality, surrender and expense are considered; see Albizzati and Geman (1994), Grosen and Jørgensen (1997), Bacinello (2001) and Hansen and Miltersen (2002).

5.2.2 New contract

The inspiration for the new contract arises from Guillén et al. (2006). In their paper, they study a pension product with the features of a transparent structure and no embedded guarantee. We refer to this contract as the GJN contract, although it appears to be a contract that is sold in Denmark[6].

The GJN contract enjoys the return from the underlying asset but cannot provide the protection for policyholders against the downside risk if the market performs badly. However, according to the findings of Hens and Rieger (2014), incorporating the feature of downside capital protection while keeping the exposure to the upward return makes structured products more attractive. Hence, by modifying the return distribution rule in the GJN contract, we obtain a new pension contract whose value to the customers has a lower bound. In addition, the payoff of the new contract is determined by the performance of a specified asset whose price is known to the public. Along with the clear profit distribution rule, this ensures that the return from the new contract is transparent.

5.2.2.1 Structure of new contract

Our new contract consists of three accounts: the investment account A, the customer account D and the smoothing account U. The value of the investment account is always equal to the sum of customer account D and smoothing account U. Mathematically,

$$A_t = D_t + U_t, \quad \text{for all} \quad t \in [0, T]. \tag{5.3}$$

The investment account A is a notional account which replicates the trend of the risky asset. For simplicity, the customer is assumed to pay a one-off premium $P > 0$ at the start. Then the value of the investment account of this customer is given as:

$$A_t = P \exp\left[(\mu - \frac{1}{2}\sigma^2)t + \sigma W_t\right], \quad \text{for all} \quad t \in [0, T]. \tag{5.4}$$

[6]This contract is called Tidspension. The contract introduced in Guillén et al. (2006) is the accumulation part of Tidspension.

On the expiration date T, the customer will receive all the money in the customer account D. At the end of year n, the nominal value of the customer account increases at a pre-declared guaranteed rate g_n (it is generally declared at the beginning of the year, in practice). Apart from this guaranteed return, the customer account is credited an annual bonus which is defined as a fixed proportion $\alpha \in [0, 1]$ of the excess value of the investment account over the guaranteed value of customer account. The proportion α is generally called the participation rate or distribution ratio. The other part, $1 - \alpha$ of the excess value, goes to the smoothing account U which provides the customer account with the guaranteed return when the market performance is bad. The value of customer account and smoothing account are updated only at the end of each year. The value of the customer account is:

$$D_n = \begin{cases} P, & n = 0, \\ (1 + g_n)D_{n-1} + \alpha \max\Big[A_n - (1 + g_n)D_{n-1}, 0\Big], & n \in \{1, 2, ..., T\}. \end{cases} \tag{5.5}$$

At the end of each year, the customer receives a positive return, or so-called bonus, if the value of the investment account is larger than the value of the customer account in the previous year increased at the guaranteed rate $g_n \geq 0$. Otherwise, the bonus is zero and the customer account only increases at the guaranteed rate. According to equations (5.3) - (5.5), the value of the smoothing account is

$$U_n = \begin{cases} 0, & n = 0, \\ A_n - (1 + g_n)D_{n-1} - \alpha \max\Big[A_n - (1 + g_n)D_{n-1}, 0\Big], & n \in \{1, 2, ..., T\}. \end{cases} \tag{5.6}$$

From equation (5.6), we can see that the smoothing account can be negative in which case the value of the investment account is less than the customer account.

It is important to note that the value of the customer account or smoothing account is a nominal value. It is not the market value of each account, although the nominal value and the market value have the same initial value and terminal value. There is no cash-flow to the customer account at the end of each year before the maturity date T. The only cash-flow happens at time T when the insurer pays the terminal value D_T to the policyholder. The advantage of the contract is that the customers can understand what they receive at the terminal time T. In addition, the investment strategy that replicates their terminal payoff is not, in general, to invest their premium entirely in the risky asset. Thus, the value of the replicating portfolio is not likely to be equal to the value of the customer account before the maturity date T. This is discussed in Guillén et al. (2006) in relation to the contract that they study.

5.2.2.2 Terminal value of the customer account

In the following, we give an expression for the terminal value for the customer account. The customer of our new contract receives an annual bonus which can never be negative, as can be seen from equation (5.5). The annual bonus is the payoff of a one year call option with underlying price A_n and strike price $K_n = (1+g_n)D_{n-1}$. We let C_n denote the payoff of nth year option, i.e.,

$$C_n = \max[A_n - (1+g_n)D_{n-1}, 0], \;\; n \in \{1, 2, ..., T\}. \tag{5.7}$$

For simplicity, the guaranteed rate g_n is assumed to be a constant value g, i.e., $g_n = g$ for $n \in \{1, 2, ..., T\}$. After recursive substitution of the second part in equation (5.5), we get

$$D_n = D_j(1+g)^{n-j} + \alpha \sum_{i=j+1}^{n} C_i(1+g)^{n-i}, \quad \text{for } 0 \le j < n \le T, \tag{5.8}$$

where $j \in \{0, 1, ..., T\}$. If we let $j = 0$ and $n = T$, then the value of the customer account at the maturity date is

$$D_T = D_0(1+g)^T + \alpha \sum_{i=1}^{T} C_i(1+g)^{T-i}. \tag{5.9}$$

From equation (5.9), we can separate the value of customer account into two parts. The first part is a pure risk-free bond with the interest rate g and the second part is a series of consecutive forward start one-year call options (ratchet style). All options begin from the start of the contract but only the first option's strike price is known at the start. In year n, there are $T-n+1$ contracts in force and only the strike price of nth option is known. Hence, the terminal value in the customer account depends not only on the participation rate α, guarantee rate g and the contract term T, but also on the paths of $\{A_n\}_{n=0}^T$.

5.2.2.3 Fair pricing of the contract

In this section, we show how to fairly price this contract and present some of its characteristics. We have shown the payoff of the new contract above. But it is also important for insurers to know the market value of the new contract during its lifetime.

Pricing pension contracts is generally carried out under an equivalent martingale measure (Harrison and Kreps, 1979). There are many papers using this method in the pricing of pension contracts, like Bacinello (2001), Grosen and Jørgensen (2000) and Bauer et al. (2006).

Let $r_f = e^r - 1$ be the discretely-compounded annual risk free interest rate and $V^D(t)$ denote the market value of the customer account at time t. Discounting the terminal value D_T back to time 0 under the equivalent martingale measure $\mathbb{Q}$, we get the initial price of this contract. That is

$$V^D(0) = (\tfrac{1+g}{1+r_f})^T D_0 + \tfrac{\alpha}{(1+r_f)^T}\mathbb{E}^{\mathbb{Q}}\Big(\textstyle\sum_{i=1}^T \max\big[A_i - (1+g)D_{i-1}, 0\big](1+g)^{T-i}\Big). \tag{5.10}$$

If the expected discounted value of the terminal value in the customer account $V^D(0)$, at time 0, is equal to customer's initial premium P, then this contract is viewed as a fair contract for the customer. We also can price this contract from the perspective of the insurer. The insurer does not put any money into the smoothing account at the start of the contract and will keep the terminal value in the smoothing account which can be positive or negative. For a fair contract, the expected discounted value of the smoothing account is zero. However, as Equation (5.6) shows, valuing the smoothing account is a matter of valuing the customer account first and then subtracting its value from the investment account. That is, $V^U(0) = A_0 - V^D(0)$, where $V^U(t)$ is the market value of the smoothing account at time t. Thus, we only need to show the pricing based on the customer account.

Observe from equation (5.10) that the first component, $(\frac{1+g}{1+r_f})^T D_0$, is the present value of a fixed income asset with the interest rate g. The second part is the present value of a series of the payoffs of call options. The payoff for each of these call options is worth at least zero. Thus we must have $g \leq r_f$ to make this contract fair. As we cannot get a closed form solution for the pricing of the new product due to the path-dependency of the options, Monte Carlo simulation is used to price this contract.

Under a risk neutral measure $\mathbb{Q}$, the value of the investment account can be expressed as

$$A_t = P \exp\left[(r - \frac{1}{2}\sigma^2)t + \sigma W_t^{\mathbb{Q}}\right], \quad \text{for all} \quad t \in [0, T]. \tag{5.11}$$

For a contract with fixed term T, the guarantee rate g and participation rate α are two parameters under the insurer's control which could be adjusted to make this contract a fair one. In order to identify the relationship between g and α, we fix the value of one parameter and find out the value of the other which makes the contract fair. Newton's method is used to solve this problem.

Figure 5.1 shows the relationship between these two parameters for a 20 year contract with risk free rate $r_f = 0.04$ and volatility $\sigma \in \{0.1, 0.2, 0.3\}$. The participation rate α decreases with the increase of the guarantee rate g. This is intuitive, as the customer should have a lower share of the excess return with the increase of the guaranteed return. If the guaranteed return equals to the risk free rate, then the customer only holds a pure risk free bond. The figure also shows that the higher the volatility σ, the lower the participation rate α. As the market become more volatile, the investment account is more likely to make a loss, the guarantees become more valuable and thus lead to a smaller participation rate.

As we mentioned earlier, the value of the customer account D_t tends to be different from the market value of the customer account $V^D(t)$ at any time $t \in (0, T)$,

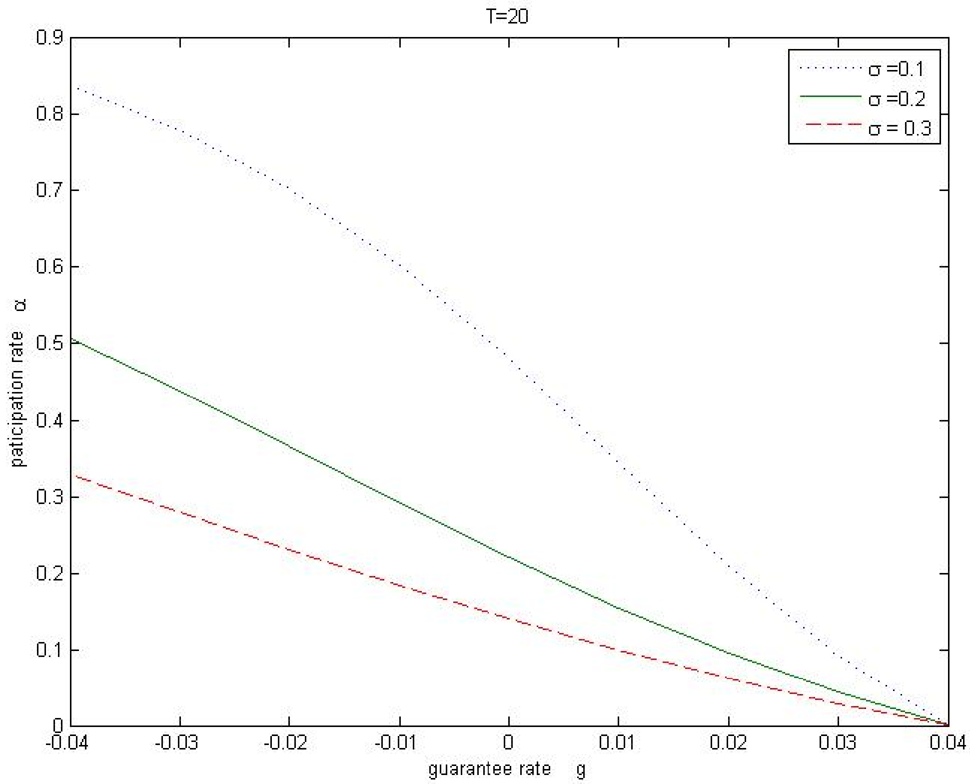


Figure 5.1: Relationship between guarantee rate g and participation rate α. $T = 20$ years, $r_f = 0.04$ and $\sigma \in \{0.1, 0.2, 0.3\}$.

though the start value and the terminal value of the two are the same. In order to show how the customer account value and the market value of customer account evolve in different market scenarios, we choose two different scenarios simulated from equation (5.4). The parameters we used in the simulation are $r_f = 0.04$, $\mu = 0.065$ and $\sigma = 0.15$.[7]

- Bull market scenario.

  The value of the risky asset mainly follows an upward trend.

- Bear market scenario.

  The value of the risky asset mainly follows a downward trend.

Figure 5.2 shows the behaviour of the nominal value of the customer account, the market value of customer account and the investment account value in a bull market scenario. The customer account value D_t rises in a step-wise fashion every year. It

[7]The parameters are used by Døskeland and Nordahl (2008b) in their paper. FTSE All Share historical return from 1986 to 2015 was 6.4 % and the volatilit was 15.8%. The yield of 10 year government bond from 1996 to 2015 was 4.35%. Thus, we believe the chosen parameters are reasonable.

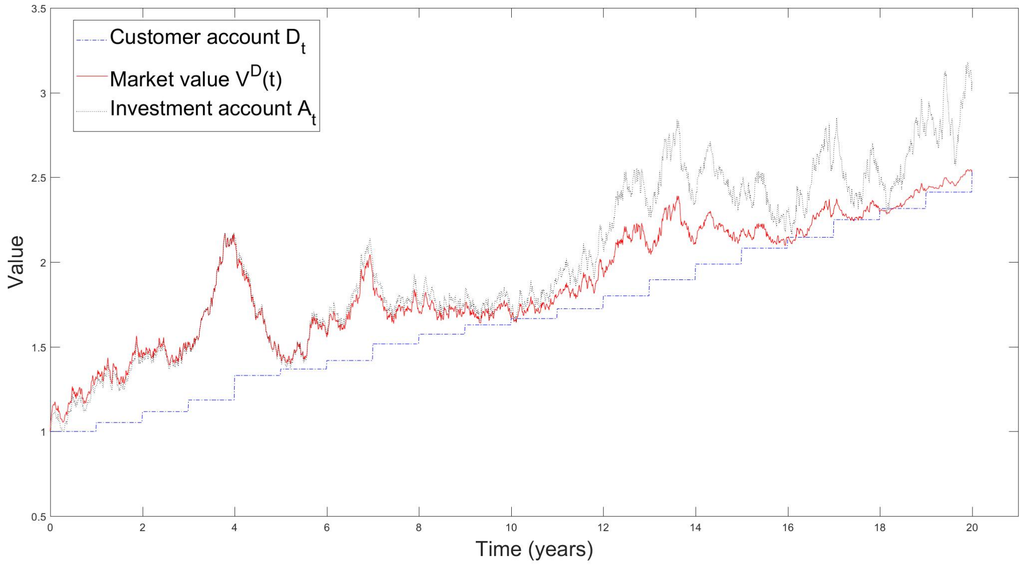


Figure 5.2: The behaviour of account balance D_t, market value of customer account $V^D(t)$ and investment account A_t in bull market. $\alpha = 0.13$. $T = 20$ years, $g = 0.02$, $r_f = 0.04$, $\mu = 0.065$ and $\sigma = 0.15$. In the simulation, there are 100 steps in each year.

never decreases due to the guarantee on the participation in the risky asset return, unlike the contract studied in Guillén et al. (2006). In the first 5 years, the market value of customer account $V^D(t)$ follows closely with the investment account A_t. In contrast, it follows closely the customer account value D_t in the later period of the contract. At the terminal time, $V^D(T) = D_T$. In this scenario, the investment account has a much higher terminal value than the customer account.

In Figure 5.3, the bear market scenario, the market value of customer account $V^D(t)$ also tracks closely the investment fund for the first few years and moves along the customer account value during the last few years. But this time the customer account has a much larger terminal value than the investment account. In other words, the customer's money still increases even when the risky asset performs badly.

5.2.2.4 Investment strategy

Rather than investing all of the customer's premium in the risky asset, the insurer could construct the replicating portfolio to replicate the market value of the customer account. This could mitigate the hedging error to which the former strategy exposes

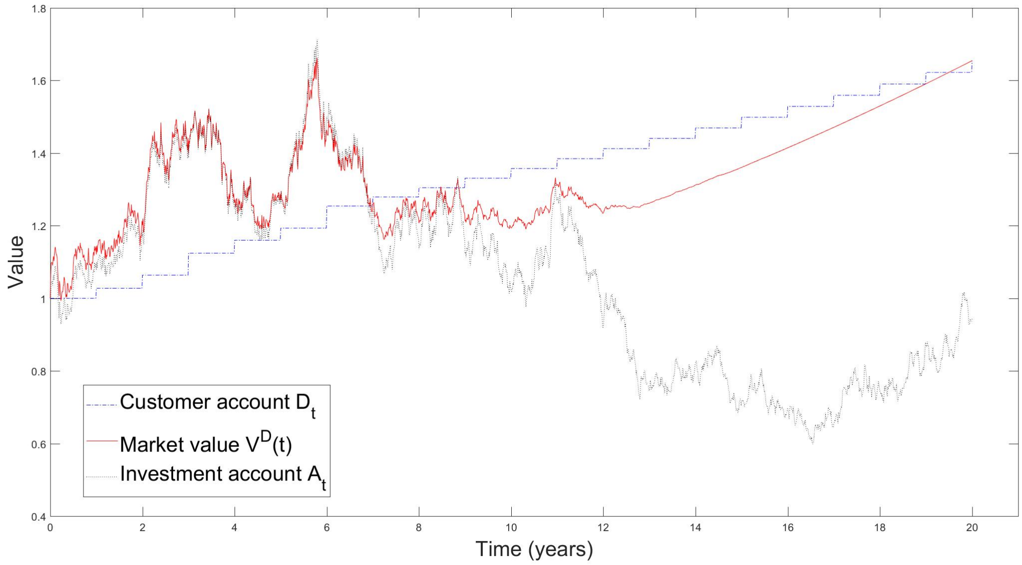


Figure 5.3: The behaviour of account balance D_t, market value of customer account $V^D(t)$ and investment account A_t in bear market. $\alpha = 0.13$. $T = 20$ years, $g = 0.02$, $r_f = 0.04$, $\mu = 0.065$ and $\sigma = 0.15$. In the simulation, there are 100 steps in each year.

the insurer. The weight in risky asset, or the Delta of the portfolio, can be calculated by using a Monte Carlo method. Figure 5.4 shows the average weight of portfolio in the risky asset, which is calculated using a finite difference approximation. We can see that as the contract approaches the maturity date, the weight of the risky asset in the portfolio is decreasing. The hedging portfolio shows a life-style investment strategy. In other words, you should put more money in risky asset if you are young and hold less in the risky asset when you are old. If the insurer follows the replicating portfolio, then at time T, the value of the replicating portfolio is $V^D(T) = D_T$, a.s.

Suppose the insurer does not follow the replicating portfolio. Instead, they invest all the premium into the risky asset at the start and hold until the maturity. This approach leads to a hedging error, $A_t - V^D(t)$, which equals the market value of the smoothing account, $V^U(t)$. The smoothing account belongs to the insurer, which makes sense as the hedging error is the responsibility of the insurer too. If the insurer chooses not to follow the replicating strategy for the contract, then it should bear the financial consequences, rather than the customers. In this sense, if we are in the bull scenario (Figure 5.2), the excess investment gains, $A_T - D_T$, goes into the smoothing account, which belongs to the insurer. If we are in the bear

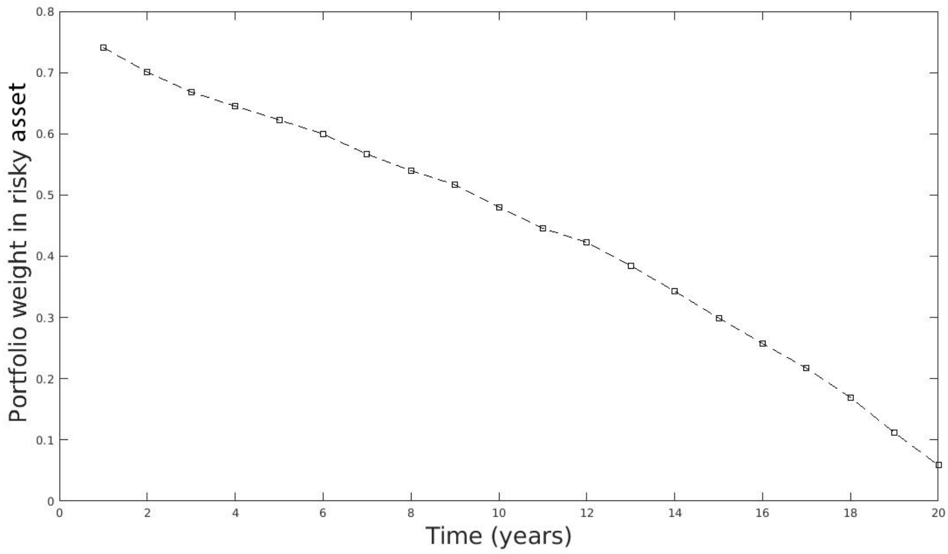


Figure 5.4: The mean of portfolio weight in risky asset. It is calculated by simulating 100 paths. $\alpha = 0.13$. $T = 20$ years, $r_f = 0.04$, $\mu = 0.065$ and $\sigma = 0.15$.

scenario as Figure 5.3 shows, the insurer needs to use its own money to meet the full terminal value of the customer account D_T by injecting the amount, $D_T - A_T$, at time T.

5.2.3 GJN contract

In this part, we briefly introduce the pension contract which is discussed in Guillén et al. (2006), as we will compare the new contract against this contract in next section. The structure of GJN contracts is similar to our new contract. The only difference is GJN contract lacks protection for the downside risk. Specifically, each year, the value of customer account increases at a constant discretely-compounded reference policy interest rate r_p. The reference policy interest rate r_p in the GJN contract is analogous to the guaranteed rate g in the new contract. In addition to this, the policyholder can receive a fixed portion of the difference between the value of the investment account A_n and the value of the previous year's customer account D'_{n-1} accumulated at the reference policy interest rate r_p. Mathematically, the development of the customer account in the GJN contract is expressed as

$$D'_n = \begin{cases} P, & n = 0, \\ (1+r_p)D'_{n-1} + \alpha'[A_n - (1+r_p)D'_{n-1}], & n \in \{1, ..., T\}, \end{cases} \tag{5.12}$$

where the constant $\alpha' \in [0,1]$ is the participation rate. Similar to the new contract, the value of the smoothing account in GJN contract is $U'_n = A_n - D'_n$.

It is worth mentioning here that if the investment account A_n has a lower value than the guaranteed value of customer account, the value of $[A_n - (1+r_p)D'_{n-1}]$ can be negative. That is to say that the customer account is credited with a negative bonus and the value of customer account decreases. This is the critical difference between the GJN contract and the new contract. In the new contract, the customer will never have a loss in the customer account.

For both contracts, the smoothing account accumulates wealth when the market performance is good and gives money back when the market performs badly. Ideally, the smoothing account is expected to even out the short-term fluctuations over a long term. In some situations when the market returns are too bad, the smoothing account in both contracts would end up with a negative value, i.e., a loss. In some other situations, the smoothing account ends up with a positive balance and generates profit for the insurance company. As we discussed before, this is only a true profit or loss if the company does not follow the replicating strategy for the terminal value of the customer account.

5.2.4 DN contract

Døskeland and Nordahl (2008b) introduce a contract which consists of four accounts: the investment account A''_t, customer account L_t, bonus account I_t and insurer account E_t. For easier comparison, we change the wording and notation of this contract. Specifically, at time 0, the customer pays the premium P into the customer account, i.e., $L_0 = P$. Different from the other two contracts, the insurer also needs to put money in at the outset for this contract and the money is deposited into the insurer account. The money in customer account and insurer account are invested into the risky asset. The market value of this investment is denoted by the investment account A''_t. That the insurer does not follow the replicating portfolio is a critical distinction for the DN contract. The proportion of the premium relates to the value of the investment account at the start, $\psi = \frac{P}{A''_0}$, is called the capital structure parameter. The initial value of the insurer account is $E_0 = \frac{1-\psi}{\psi}P$. Thus, $A''_0 = L_0 + E_0$. Additionally, the bonus account has an initial value $I_0 = 0$.

The mathematical expression of the customer account L_t, the bonus account I_t and the insurer account E_t for $t = 1, 2, ..., T$ are given as

$$L_t = \begin{cases} A''_t, & \text{if } A''_t \leq L_{t-1}(1+g''), \\ L_{t-1}(1+g''), & \text{otherwise } A''_t \leq (L_{t-1}+E_{t-1})(1+g'') + I_{t-1} \\ L_{t-1}(1+g'') + \psi\eta(1-\theta) & \\ \cdot\Big(A''_t - \big[(L_{t-1}+E_{t-1})(1+g'') + I_{t-1}\big]\Big), & \text{if } A''_t > (L_{t-1}+E_{t-1})(1+g'') + I_{t-1}, \end{cases} \tag{5.13}$$

$$I_t = \begin{cases} 0, & \text{if } A''_t \leq L_{t-1}(1+g'') + E_{t-1}, \\ A''_t - L_{t-1}(1+g'') - E_{t-1}, & \text{otherwise } A''_t \leq L_{t-1}(1+g'') + E_{t-1} + I_{t-1}, \\ I_{t-1}, & \text{otherwise } A''_t \leq (L_{t-1}+E_{t-1})(1+g'') + I_{t-1}, \\ I_{t-1} + \psi\eta\theta & \\ \cdot\Big(A''_t - \big[(L_{t-1}+E_{t-1})(1+g'') + I_{t-1}\big]\Big), & \text{if } L_t > L_{t-1}(1+g'') + E_{t-1}(1+g'') + I_{t-1}, \end{cases} \tag{5.14}$$

$$E_t = A''_t - L_t - I_t, \tag{5.15}$$

where $\eta \in [0, 1]$ is the customer share of the profits and $\theta \in [0, 1]$ is the proportion of declared bonuses credited to the bonus account.

The customer receive at the expiration date D''_T is the sum of the value of the customer account L_T and the bonus account I_T. In the paper of Døskeland and Nordahl (2008b), the insurer is allowed to go into bankruptcy ($A''_t < L_{t-1}(1 + g'')$), in which case the terminal payoff to the customers is the value of the investment account at the time of bankruptcy, accumulated at the risk-free rate to the maturity date of the contract. That is,

$$D''_T = A_\tau (1 + r_f)^{T-\tau}, \tag{5.16}$$

where $\tau \in (0, T)$ is the bankruptcy time. As the customer account is only updated at the end of each year, the bankruptcy time can only be an integer time. It is noted that in the case of bankruptcy, the annual guarantees can not be met. In other words, customers may receive less than the guaranteed value.

5.3 Comparison under CPT

In this part, we compare the new contract with the GJN contract and the DN contract in detail. The results provide the evidence that the new contract is more preferable than the other two. Cumulative Prospect Theory (CPT) is used in this section to compare these three contracts. The comparison is based on the results of Monte Carlo simulations. An alternative test using EUT is given in Section 5.5.

5.3.1 Cumulative Prospect Theory

Now we examine the performance of these three contracts under the behavioural Economics model, CPT. Compared to EUT, CPT assumes that a person values an investment by gains or losses rather than the terminal wealth. The gains and losses are calculated by comparing the terminal value of an investment to a reference point which is often the current wealth. In this paper, we choose the initial investment, the premium P, as the reference point. If the terminal value the customer received D^*_T (D_T, D'_T, D''_T) is smaller than the premium, i.e., $X = D^*_T - P < 0$, the outcome is viewed as a loss. Otherwise, it is a gain. An investment consisting of m outcomes of losses and n outcomes of gains is expressed as a risk prospect $f = (x_{-m}, p_{-m}; x_{-m+1}, p_{-m+1}; ...; x_i, p_i; ...; x_{n-1}, p_{n-1}; x_n, p_n)$ where x_i is a potential outcome of the prospect and p_i is the corresponding probability for $i = \{-m, -m+1, 0, ...n-1, n\}$. CPT evaluates gains and losses separately, and the overall utility is the sum of utility of the positive part f^+ and the negative part f^-.

$$V(f) \quad = V(f^+) + V(f^-) = \textstyle\sum_{i=0}^{n} \pi_i^+ v(x_i) + \sum_{i=-m}^{0} \pi_i^- v(x_i), \quad -m < i < n, \tag{5.17}$$

where $v(x_i)$ is the value function of outcome x_i and $\pi_i^{+(-)}$ is the decision weight of this outcome. The value function shows how the individual values the outcomes. The decision weight is calculated by weighting function $w : [0,1] \to [0,1]$ (Choquet (1954)) which is a distortion of real probability. The weighting function describes how people deal with probabilities. For a positive outcome x_j, the decision weight π_j^+ is

$$\pi_j^+ = w(p_j + ... + p_n) - w(p_{j+1} + ... + p_n). \tag{5.18}$$

On the other hand, the decision weight for a negative outcome x_i is:

$$\pi_i^- = w(p_{-m} + ... + p_i) - w(p_{-m} + ... + p_{i-1}). \tag{5.19}$$

The value function used in this paper is proposed by Tversky and Kahneman (1992),

$$v(x) = \begin{cases} x^\beta, & \text{if } x \geq 0, \\ -\lambda(-x)^\beta, & \text{if } x < 0, \end{cases} \tag{5.20}$$

where $\lambda = 2.25$ and $\beta = 0.5$. $\lambda > 1$ is the loss aversion parameter, which shows individuals are much more sensitive to losses than gains. β is the sensitivity of customers to the increasing gains or losses. Prelec (1998) proposes a one parameter weighting function, namely the Prelec's weighting function:

$$w(p) = e^{-(-\ln p)^\varphi}. \tag{5.21}$$

Døskeland and Nordahl (2008b) suggest Prelec's weighting function is based on behavioural axioms rather than mathematical convenience. In this section, we follow Døskeland and Nordahl (2008b) to use the Prelec's weighting function and their chosen parameter $\varphi = 0.75$. There are other possible choices of the weighting functions, one of which we investigate in Section 5.4.

5.3.2 Analysis of products under CPT

A pension contract is a long term investment. Once a customer purchases a pension contract, it is not likely for him to sell this policy or change his investment portfolio. Hence, a buy-and-hold strategy is a reasonable assumption which we make for all the comparisons below. We find that our new contract generates the highest certainty equivalent value of CPT utility, compared to other available investments.

It is assumed that a customer has an initial wealth of 1 unit and plans to invest the money with a horizon of T years. There are five possible investment opportunities for the customer: the new contract, the GJN contract, the DN contract, the risk-free asset and the risky asset. Each asset has an initial price of 1 and for the pension contracts the price is the premium. It is possible to buy any amount of the above five assets.

5.3.2.1 Holding exactly one asset or contract

In the first analysis, we find out holding all of the customer's wealth in a single asset or contract for T years results in the largest CPT utility at the end of the time horizon. Numerical simulation of $1,000,000$ paths is used to solve this problem. For the parameters of simulation, we follow Guillén et al. (2006) and let the investment horizon be $T = 20$ years which is a reasonable investment horizon for a pension contract. The risk-free rate is assumed as $r = 0.04$. For the risky asset, the expected return is $\mu = 0.065$ and the volatility is assumed as $\sigma = 0.15$. In order to match the properties of the new contract and the DN contract, we arbitrarily choose the same guarantee rate $g = g'' = 0.02$ for both contracts [8]. Making the new contract fair, the corresponding participation ratio can be solved by letting equation (5.10) equal

[8] Døskeland and Nordahl (2008b) use a 2% guaranteed rate in their paper.

the initial investment P under the risk neutral measure $\mathbb{Q}$. The numerical result gives $\alpha = 0.13$. We use the same method to find the fair value of the parameter ψ for the adjusted DN contract. Guillén et al. (2006) show that a fair GJN contract only requires $r_p = r_f$ if the underlying asset follows a geometric Brownian motion. Hence we can also set the participation rate of GJN model equal to 0.13 so that both contracts have the same participation rate.

Based on the simulation paths, we calculate the expected terminal wealth, the standard deviation of the terminal wealth and the certainty equivalent value (CEV) of the CPT utility for holding each of the above five investments under a buy-and-hold strategy. CEV is the amount of money which generates with absolute certainty the same expected CPT utility as a given risky asset. The results are given in Table 5.1. From the CEV row, we can see that holding only the new contract generates the highest CEV which suggests it produces the highest CPT utility for a 20-year investment horizon. The GJN contract generates slightly higher CPT utility than the risky asset. The risk-free asset produces the smallest CPT utility. In other words, if the customer can hold exactly one of the above assets or contracts for 20 years, then the new contract is the best choice. This result is consistent with the result in Døskeland and Nordahl (2008b) which shows holding a contract with annual guarantees gives higher CPT utility than holding the underlying asset itself. It is important to point out that the DN contract does not give a high CPT utility. Compared to the other two contracts, the expected terminal value is much less. This is because the insurer of DN contract is so likely to go into bankruptcy (with probability of 69%). After bankruptcy, the DN contract works as a risk-free asset. This also explains the small standard deviation of the DN contract.

5.3.2.2 Holding a combination of the assets and contracts

If the customer could choose any combination of the above five assets instead of holding only one of them, what is the optimal portfolio generates the largest CPT utility? Now the optimisation problem is to find the weights of the above five assets in the optimal portfolio. In general, individual investors will not do borrowing and

	New contract	GJN contract	DN contract	Risk-free asset	Risky asset
CEV	2.7854	2.6371	2.5892	2.1911	2.6337
$\mathbb{E}(D_T)$	2.8844 (5.3 %)	3.0623 (5.6 %)	2.5680 (4.7 %)	2.1911 (3.9 %)	3.5213 (6.3 %)
$SD(D_T)$	1.5022	1.6970	1.0559	0	2.6483

Table 5.1: Certainty equivalent value of CPT utility (CEV), expected value ($\mathbb{E}$) and standard deviation (SD) of the terminal wealth by holding exactly one of the new contract, the GJN contract, the DN contract, the risk-free asset and the risky asset. The value in the bracket is the annualised continuously-compounded return of the expected terminal wealth. The CPT utility is calculated by equation (5.20) and equation (5.21). The parameters are $g = g'' = 0.02$, $r_p = 0.04$, $P = 1$, $\alpha = 0.13$, $T = 20$, $r_f = 0.04$, $\mu = 0.065$, $\sigma = 0.15$, $\psi = 0.9$, $\theta = 0.2$, $\varphi = 0.75$ and $\beta = 0.5$.

	New contract	GJN contract	DN contract	Risk-free asset	Risky asset
weight	44 %	0	13%	0	43 %

Table 5.2: Proportion of the new contract, the GJN contract, the DN contract, risk-free asset and the risky asset in the optimal portfolio under CPT . The parameters are $g = g'' = 0.02$, $r_p = 0.04$, $P = 1$, $\alpha = 0.13$, $T = 20$, $r_f = 0.04$, $\mu = 0.065$, $\sigma = 0.15$, $\psi = 0.9$, $\theta = 0.2$, $\varphi = 0.75$ and $\beta = 0.5$. The CEV of CPT utility of this optimised portfolio is 2.9150.

short selling. Hence, short-selling and borrowing are not allowed here. Additionally, re-balancing investments is not common in practice for individual pension investors, so the buy-and-hold strategy is also assumed here.

The simulation result shows, in the optimal portfolio, the new contract takes up 44% of the optimised portfolio while the risky asset receives 43%. The DN contracts occupies the remaining weight. No weight is assigned to the GJN contract and risk-free asset. The CEV of CPT utility of this optimised portfolio is 2.9150, which is larger than holding any one asset or contract in Table 5.1. That is to say, the optimal pension investment portfolio consists of only the new contract, the DN contract and the risky asset. Holding any other asset leads to a loss of CPT utility. In this sense, pension contracts with guarantees are more attractive than those without. Based on the results from Section 5.3.2.1 and 5.3.2.2, we can conclude that under both methods, the new contract is preferred to the GJN contract and the DN contract from the perspective of a CPT-maximising customer.

5.3.3 Sensitivity analysis

In the market, pension contract customers are from different age groups. It is natural that customers tend to buy a pension contract with the term that best matches their time until retirement. In this part, we investigate how the proportion of the optimal portfolio would change for different investment horizons.

We calculate the optimised portfolio in terms of the above five investment opportunities for different investment horizons from 1 year to 30 years. The result is presented in Figure 5.5. The new contract and the risky asset dominate the optimal portfolio. Specifically, the new contract takes up the largest proportion in the optimal portfolio for horizons from 1 year to 17 years while the risky asset becomes the dominant component for investment horizon with 18 years or more. The reason is that the risky asset is much more volatile and thus is more likely to make a loss for a short horizon. As the investment horizon increases, its higher expected return makes the risky asset more attractive. DN contract emerges in year 5 and then increases with the investment horizon. For horizons longer than 25 years, the proportion of DN contract starts to decrease gradually. As before, we find that the GJN contract is not in the optimal portfolio for any investment horizon. As it neither avoids the investment losses for short horizons nor has a higher return in the long term, it is not surprising that it is not in the optimal portfolio for any investment horizon.

One important feature of DN contract worth mentioning here. For DN contract, the insurer also needs to invest money in at the outset. In addition, as the expected return of risky asset is higher than the guaranteed return of insurer account, the bonus the customer receives is higher than the other two contracts. So the expected return for DN contract is actually higher than the new contract and the GJN contract for short horizons. However, once the insurer go into bankruptcy, the guarantees cannot be met. For customers with short investment horizons, keeping the deposit safe is very important. This is proved by Figure 5.5 that only the risk-free asset and the new contract are in the optimal portfolio. For longer investment horizons, both the new contract and DN contract tend to have a good expected return if the market performs good. However, for those bear market scenarios, the DN contract

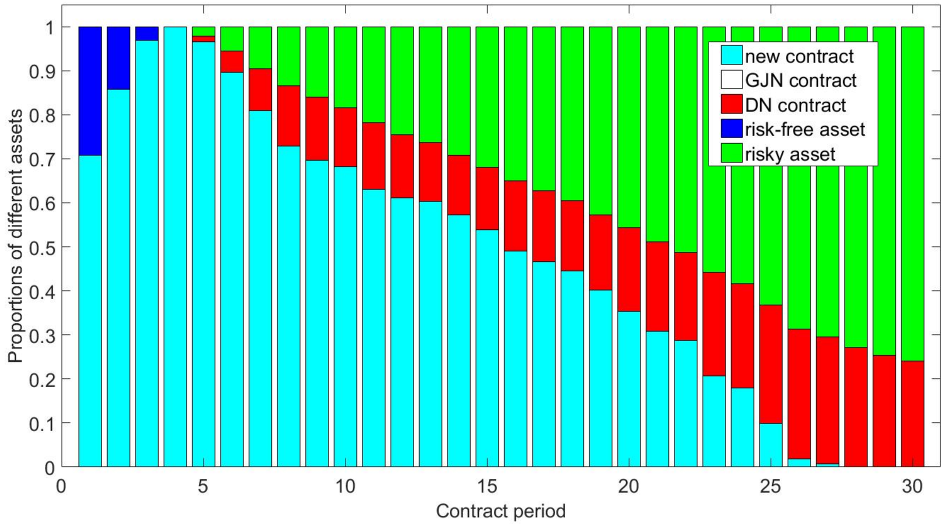


Figure 5.5: The composition of the optimal portfolio for different investment horizons. The parameters are $P = 1$, $r_f = 0.04$, $\mu = 0.065$, $\sigma = 0.15$, $\varphi = 0.75$ and $\beta = 0.5$.

is likely to go into bankruptcy. In this case, the value of DN contract accumulates at the risk free rate (4%) which is higher than the guaranteed rate (2%). As the correlation between DN contract and the risky asset is much less than the correlation between the new contract and the risky asset, the combination of risky asset and the DN contract is a better portfolio. This explains why the proportion of DN contract increases with the investment horizons for most of time.

With a higher μ and σ,[9] a broadly similar result is obtained (Figure 5.6). Both results are consistent with the well known pension investment advice that if customers are young, they should put more money in the risky asset to benefit from the higher expected returns, whereas older people should buy less risky products. There is a distinct point in Figure 5.6, which is the dramatic decline of the proportion of the DN contract when the investment horizon changes from 13 years to 14 years. The proportion slumps from 43% for a 13-year-length contract to 0% for a 14-year-length contract.

In order to show this sudden drop in detail, we examine the CPT CEV for all combinations of the risky asset and the DN contract for investment horizons of 13 years and 14 years (Figure 5.7). With the increase of the investment horizon, the

[9] We choose $\mu = 0.10$ and $\sigma = 0.20$ whose value are based on US S&P 500 historical data from 1997 to 2016. The average annual return of S&P 500 in this period is 9.27% and the 12-month realised volatility is 19.7%. The average 10-year Treasury rate during the same period is 3.97%.

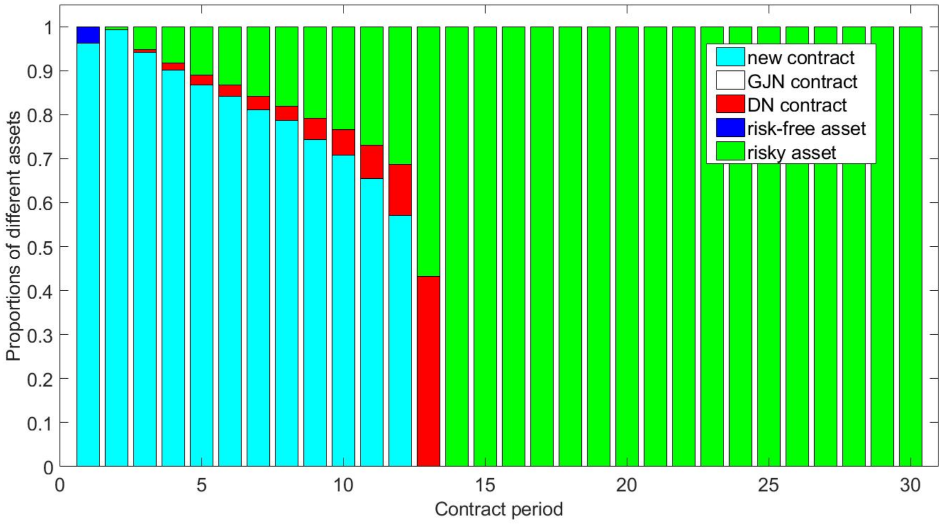
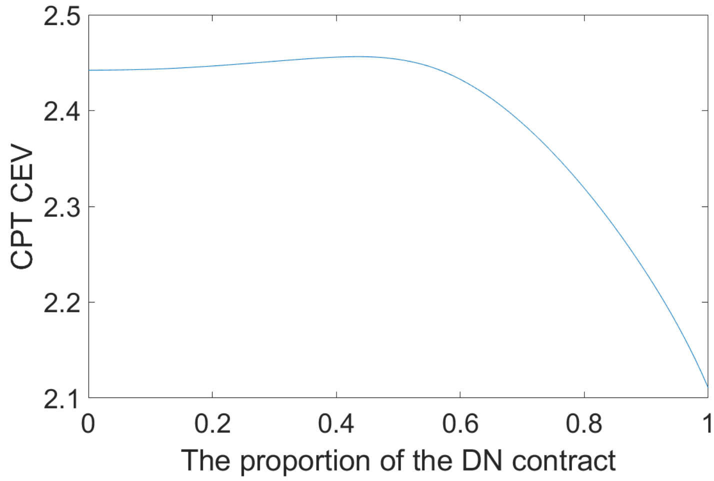


Figure 5.6: The composition of the optimal portfolio for different investment horizons. The parameters are $P = 1$, $r_f = 0.04$, $\mu = 0.1$, $\sigma = 0.2$, $\varphi = 0.75$ and $\beta = 0.5$.

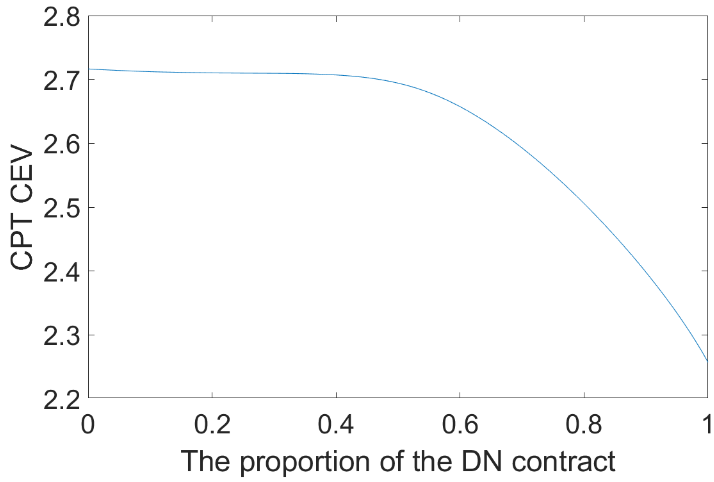
CPT CEV of holding either 100% in the risky asset or 100% in the DN contract is rising. But the CPT CEV of holding 100% in the risky asset increases much faster than holding 100% in the DN contract. This is consistent with the increased investment in the risky asset in the optimal portfolio as the investment horizon gets longer (Figure 5.5). As the risky asset giving a larger boost to the CPT CEV, and the CPT CEV being reasonably flat for initial portfolios with between 50% and 100% in the risky asset at the 14-year-time horizon (Figure 5.7b). The optimal point changes from an internal point to a left end point from 13-year horizon to 14-year horizon. This explains the plummet of the proportion of DN contract in Figure 5.6.

5.4 Robust testing of the CPT-based results

In this part, some robust tests of our result are presented. Firstly, we calculate the result using an alternative CPT weighting function that is proposed in Tversky and Kahneman (1992), but keeping the same value function, Equation (5.20). The new



(a) $T = 13$.



(b) $T = 14$.

Figure 5.7: The CPT CEV for combinations of the DN contract and the risky asset. The parameters are $r_f = 0.04$, $\mu = 0.1$, $\sigma = 0.2$.

	New contract	GJN contract	DN contract	Risk-free asset	Risky asset
CEV	2.7265	2.4183	2.5723	2.1911	2.3711
$\mathbb{E}(D_T)$	2.8837 (5.3 %)	3.0631 (5.6 %)	2.5683 (4.7 %)	2.1911 (3.9 %)	3.5261 (6.3 %)
$SD(D_T)$	1.5072	1.7021	1.0605	0	2.6667

Table 5.3: Certainty equivalent value of CPT utility (CEV), expected value ($\mathbb{E}$) and standard deviation (SD) of the terminal wealth by holding exactly one of the GJN contract, the new contract, the DN contract, the risk-free asset and the risky asset. The value in the bracket is the annualised continuously-compounded return of the expected terminal wealth. The CPT utility is calculated by Equation (5.20) and Equation (5.22). The parameters are $g = g'' = 0.02$, $r_p = 0.04$, $P = 1$, $\alpha = 0.13$, $T = 20$, $r_f = 0.04$, $\mu = 0.065$, $\sigma = 0.15$, $\psi = 0.9$, $\theta = 0.2$, $\varphi = 0.75$, $\nu = 0.61$ and $\delta = 0.69$.

weighting function is given as:

$$\begin{cases} w^+(p) &= \frac{p^\nu}{(p^\nu+(1-p)^\nu)^{\frac{1}{\nu}}}, \\ w^-(p) &= \frac{p^\delta}{(p^\delta+(1-p)^\delta)^{\frac{1}{\delta}}}, \end{cases} \tag{5.22}$$

where $\nu = 0.61$ and $\delta = 0.69$. Otherwise, we use the same parameter values as in Section 5.3.2.

The CPT CEV for holding only one of the five assets or contracts using the new weighting function is calculated (Table 5.3). Ordering by the CEV value, the same ranking is obtained as before (Table 5.1), albeit with different values. Calculating the optimal portfolio, again we find that only the new contract (28% of the initial optimal portfolio), the DN contract(34% of the initial optimal portfolio) and the risky asset (38% of the initial optimal portfolio) are in it, for a 20-year-time horizon. Even under a different weighting function, the GJN contract, which does not have guarantees embedded, is not in the optimal portfolio.

Furthermore, calculating the optimal portfolio at different time horizons, we obtain similar figures to Figure 5.5 and Figure 5.6. In Figure 5.8, the risky asset appears in the optimal portfolio for time horizons of 11 years or more, compared to 5 years or more in Figure 5.5. In Figure 5.9, the risky asset forms 100% of the optimal portfolio for time horizons of 18 years or more, compared to 15 years or

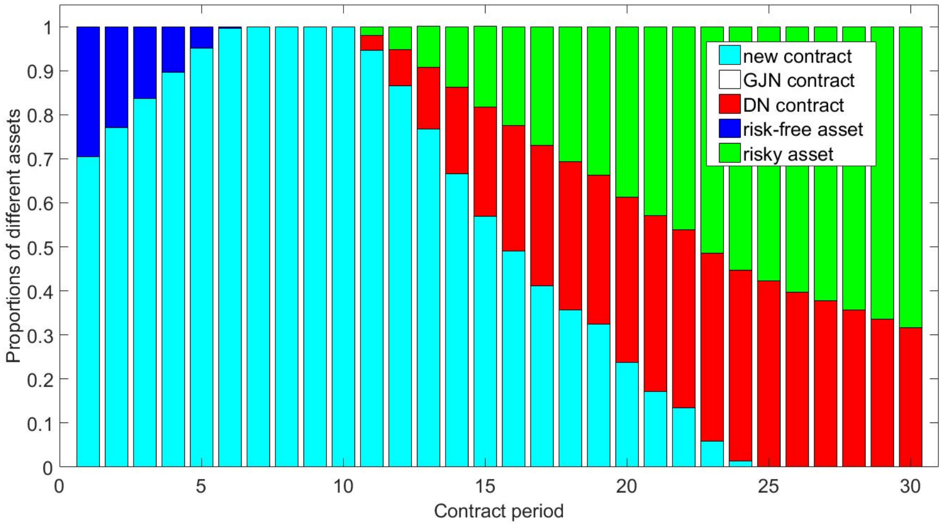


Figure 5.8: The composition of optimal portfolio for different terms using the value function and weighting function defined in Tversky and Kahneman (1992). $r_f = 0.04$, $\mu = 0.065$ and $\sigma = 0.15$.

more in Figure 5.6.

Next, we use a different parameterization of the value function (5.20) (The weighting function is given by Equation (5.21), which we used in Section 5.3.3). The parameter β controls the curvature of the value function. With the increase of β, the value function shows less risk aversion for gains and less risk seeking for losses.

The optimal portfolio for different horizons is re-calculated with $\beta = 0.88$, as used by Tversky and Kahneman (1992) (Figure 5.10, otherwise using the same parameters as Figure 5.5). The risk-free asset is not in the optimal portfolio for any investment horizon. Additionally, the risky asset has a much higher percentage of the optimal portfolio than before. This result reflects a lower aversion to loss by the customer.

5.5 Comparison under EUT

In this part, we compare the new contract against the GJN contract and the DN contract under Expected utility theory (EUT). Although the EUT has some weaknesses to explain individual's behaviour under uncertainty, it is still used widely in measuring individuals' preference for investments. For example, Døskeland and

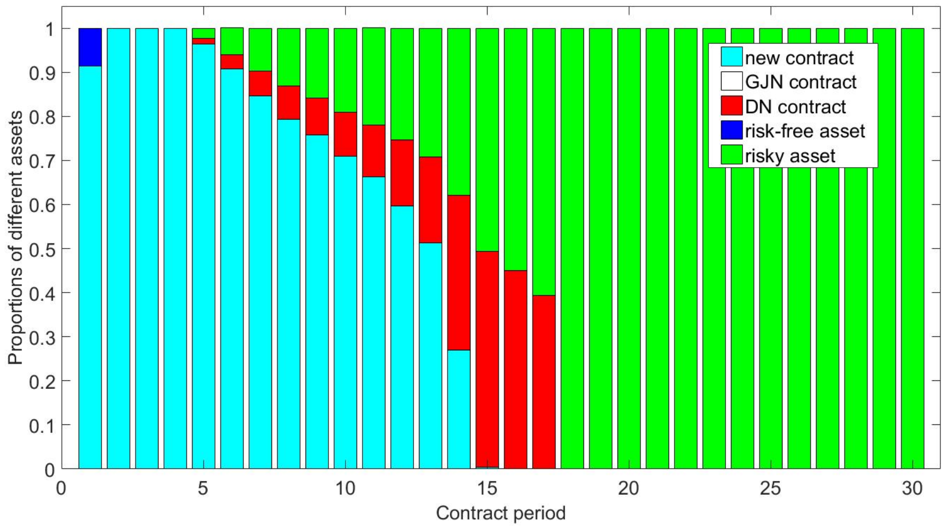


Figure 5.9: The composition of optimal portfolio for different terms using the value function and weighting function defined in Tversky and Kahneman (1992). $r_f = 0.04$, $\mu = 0.1$ and $\sigma = 0.2$.

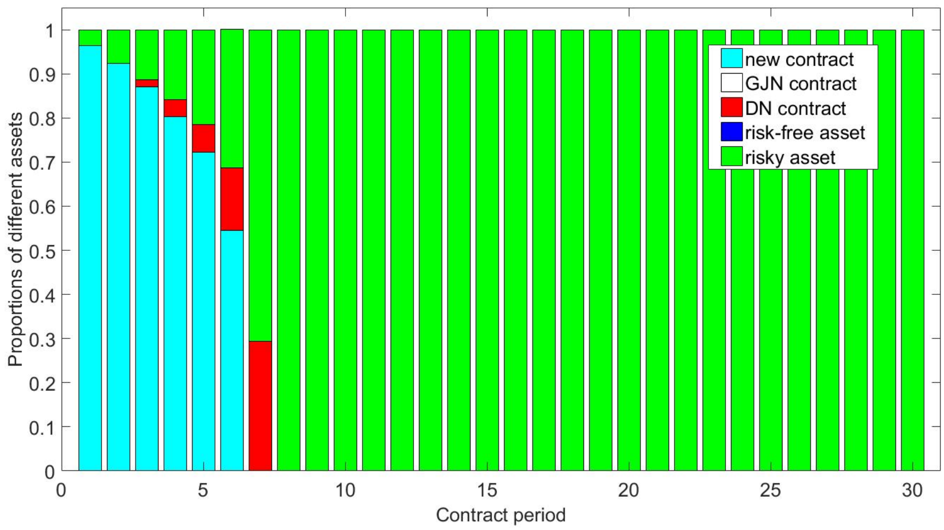


Figure 5.10: The composition of the optimal portfolio for different investment horizons. The parameters are $r_f = 0.04$, $\mu = 0.065$, $\sigma = 0.15$ and $\beta = 0.88$.

Nordahl (2008a) use a constant relative risk aversion (CRRA) utility function to represent household's preference when studying a with-profits contract. In addition, Schmeiser and Wagner (2015) maximise the participating contract policyholders' utility by employing the CRRA utility function. More recent research on measuring the utilities of pension products can be see from Chen et al. (2016) and Braun et al. (2019).

It is assumed that the policyholder's preference can be expressed by a CRRA utility function as follows:

$$u(M) = \frac{1}{1-\gamma} M^{1-\gamma}, \quad \gamma > 0, \gamma \neq 1, \tag{5.23}$$

where $M > 0$ is the terminal wealth of the customer and γ is the relative risk aversion coefficient.

Reworking with CEV results of Section 5.3.2.1, we find that the risk free bond has the highest value when $\gamma = 5$ and $\gamma = 7$ (Table 5.4). However, the DN contract is the most attractive when $\gamma = 3$ (Table 5.4). In all cases, holding 100% in the risky asset is least attractive because of its high volatility.

Similarly, we re-calculate the results of Section 5.3.2.2 with $\gamma = 5$, for a 20-year-time horizon. The optimal buy-and-hold portfolio is to invest 5% in the GJN contract, 79% in the risk-free asset and the remainder in the risky asset. The new contract and DN contract are not in the optimal portfolio despite they having a higher CEV than the risky asset (Table 5.4).

Figure 5.11 shows how the optimal porfolio changes against the investment horizons under EUT. We can see that the optimal portfolio is mainly composed by the GJN contract, the risk-free asset and the risky asset. As we have calculated the static optimal portfolio rather than the dynamic optimal portfolio, the optimal investment strategy is different to the well-known Merton's solution for the dynamic optimal portfolio. Merton (1969) shows the optimal dynamic investment strategy for a CRRA utility function is that a constant proportion of wealth should be invested in risky asset and risk-free asset, respectively. Our results show a relatively stable proportion of three different assets and contracts, the GJN contract, risky-free asset

CEV	New	GJN	DN	Risk-free asset	Risky asset
$\gamma = 3$	2.2531	2.1428	2.2538	2.1911	1.7921
$\gamma = 5$	2.0558	1.7545	2.1591	2.1911	1.1401
$\gamma = 7$	1.9365	1.4771	2.0970	2.1911	0.7339

Table 5.4: Expected utility of the terminal wealth of holding the GJN contract, the new contract, the DN contract, the risk-free asset and the risky asset under the utility function defined by Equation (5.23). The parameters are $\alpha = 0.13$. $T = 20$, $r_f = 0.04$, $\mu = 0.065$, $\sigma = 0.15$, $g = g'' = 0.02$, $r_p = 0.04$ and $P = 1$.

and risky asset, in the optimal portfolio beyond 2 years time horizon. The only exception is the DN contract emerges in the optimal portfolio for short horizons(1 year and 2 years). This is due to the special structure of DN contract. As we discussed above, because the insurer also needs to put money in at the outset, the customer of DN contract tend to receive a higher bonus and thus have a higher expected return for short investment horizons. However, once the insurer go bankrupt, the DN contract works as a risk free asset.

As the Figure 5.11 shows, GJN contract is an attractive contract under the EUT. Especially, with the increase of the investment horizon, the proportion of the GJN contract is increasing gradually while the proportion of risky asset is decreasing. As the replicating portfolio of the GJN contract consists of more risky asset (less risk-free asset) for long horizon contracts than short[10], the optimal portfolio in Figure 5.11 seems to provide an approximated strategy to the Merton's solution.

5.6 Conclusion

In this paper, we examine a new pension contract in the accumulation phase of a pension scheme. This new contract occupies the characteristics of guarantees and bonuses but has a transparent structure and clear profit distribution rule. Under Cumulative Prospect Theory, the contract gives higher utility than the contract introduced in Guillén et al. (2006) and the contract studied by Døskeland and Nordahl (2008b).

The result shows that shielding customers from poor stock market returns is attractive to the customer. By the application of financial engineering techniques

[10]This is discussed in Guillén et al. (2006).

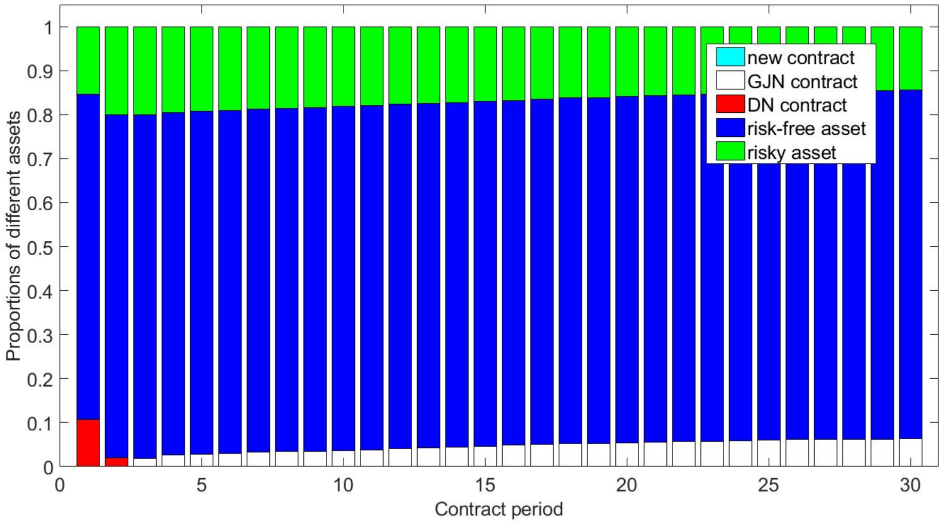


Figure 5.11: The composition of optimal portfolio for different terms using the value function and weighting function defined in Tversky and Kahneman (1992). $r_f = 0.04$, $\mu = 0.1$ and $\sigma = 0.2$.

in pricing and risk management, the insurer can manage the additional risk which it faces in issuing such guarantees. In addition, we show that with the increase of policyholder's investment horizon, the proportion of the risky asset in the optimal portfolio grows larger while the proportion of the risk-free asset decreases. This result conforms to traditional pension investment advice that young people should invest more money in the risky asset while old people should reduce their exposure to risky assets.

There are much more we can investigate to continue our research. For example, as we only find the best buy-and-hold strategy among some different sub-optimal dynamic strategies. Hence, finding the product which matches the dynamic optimal investment strategy is an interesting area for future research.

Chapter 6

Conclusion and Outlook

This thesis reviews the traditional pension contracts and propose a new pension contracts that can be used by insurance companies to smooth investment returns for their customers. Three research papers contained in this thesis show the reader how to design an innovative pension contract that best meet the demand for a CPT-maximising customers.

In order to have a deep understanding of the pension contracts, Chapter 3 carefully examines a traditional with-profits contract in the market. We start with deriving the analytical formula for the fair pricing of this contract by assuming the underlying fund prices following the geometric Brownian motion. Our finding indicates that this contract is more valuable than its price owing to its free issued guarantees. With the increase of the guarantee rate, the contract could be more valuable to the customers. In addition, the smoothing method is the geometric average of 5 years return which include 2 yearly return before the term of the contract. This special structure exposes the insurer to a risk that cannot be hedged. An analytical formula for fair pricing of unhedgeable risk is obtained in this chapter as well. Moreover, the inter-generation risk sharing has been studied for a 50 generations model. Under the assumption of the return distribution and risk sharing rule, there is an cross-subsidization between different generations. The later generations benefit from the return distributions from previous generations.

Smoothing is an important feature of the with-profits products. It is viewed as a benefit to the customers as it reduces the market risk and provides the customers more stable investment returns. In order to understand how the smoothing work, in Chapter 4, We compare three widely used smoothing methods of with-profits contracts in UK, the geometric average (GA) method, the weighted sum (WS) method and the Bandwidth (BW) method. The main contributions of this chapter are on the fairness and the smoothing effects of different smoothing methods. By comparing the smoothed and actual terminal value, our findings show that the GA method provides a less payout to the customers while the BW method overpays. The WS method generate a fair terminal payout to the customer. In addition, we are also interested in the utility or the satisfaction from different smoothing methods to the customers. Rather than only considering the terminal utility, we also take into account the interim utility within the investment horizons. This is implemented by calculating the Multi-Cumulative Prospect Theory (MCPT) utility. The result is very interesting. Even though the GA method provides least terminal value, but it generates the highest MCPT utility among the three smoothing methods. On the other hand, the BW method provides a higher than expected return but give the customers less utility.

Based on the findings in previous two chapters, Chapter 5 propose a new pension contract with the features of guarantees and smoothing. It has transparent structure and clear distribution rule. Under Cumulative Prospect thoery (CPT), the new contract generates higher utility than the contract introduced in Guillén et al. (2006). The result provides the evidence why the guarantees should be included in the pension contract. In addition, our result shows with the increase of policyholder's investment horizons, the proportion of risky asset in underlying investment portfolio also increases while the proportion of risk free asset decreases. This result conforms to the traditional life style pension investment advice.

The dot-com bubbles and 2008 financial crisis leaded to significant change to the life insurance and pension industry. With-profits contracts used to be the dominant player in the market. Due to its opaqueness and complexity, it only accounts for less than 5% market share of the new business now. The simple and transparent unit-linked contract becomes the most popular one in the market. However, we still believe that some merits, such as smoothing and guarantees, of with-profits contracts are beneficial to the customers. Thus, it is expected that making the with-profits contract more transparent while keeping its merits are the future of new life and pension products.

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