On the dynamics of stochastic nonlinear dispersive partial differential equations

Kelvin Cheung

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Heriot-Watt University

Department of Mathematics,
School of Mathematical and Computer Sciences.

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Abstract

This thesis contributes towards the well-posedness theory of stochastic dispersive partial differential equations. Our investigation focuses on initial value problems associated with the stochastic nonlinear Schrödinger (SNLS) and stochastic Korteweg-de Vries (SKdV) equations. We divide this thesis into four main topics, which are the contents of Chapters 2–5.

Chapter 2 is concerned with the SNLS posed on the d-dimensional tori with either additive or multiplicative stochastic forcing. In particular, we prove local-in-time well-posedness for initial data and noise at subcritical regularities. We are also able to extend this to global-in-time well-posedness at energy subcritical regularity for certain cases. In the next two chapters, we focus on SNLS posed on the d-dimensional Euclidean space with additive noise. In Chapter 3, we prove local well-posedness with the noise at supercritical regularity while the initial data stays at critical regularity. In Chapter 4, we restrict our attention to dimension 4 and study SNLS with non-vanishing boundary conditions. In particular, we use perturbative techniques to prove global well-posedness with data in $H^1(\mathbb{R}^4) + 1$.

In Chapter 5, we move on from SNLS to SKdV, where we prove $L^2(\mathbb{T})$-global well-posedness of SKdV with multiplicative noise on the circle. We also verify that a result on the stabilisation of noise by Tsutsumi [84] continues to hold in our low regularity setting.

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Research Thesis Submission

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Contents

Notations

Given $A, B \in \mathbb{R}$, we use the notation $A \lesssim B$ to mean $A \le CB$ for some constant $C \in (0, \infty)$ and write $A \sim B$ to mean $A \lesssim B$ and $B \lesssim A$. We sometimes emphasize any dependencies of the implicit constant as subscripts on $\lesssim$, $\gtrsim$, and $\sim$; e.g. $A \lesssim_p B$ means $A \le CB$ for some constant $C = C(p) \in (0, \infty)$ that depends on the parameter p. We denote by $A \wedge B$ and $A \vee B$ the minimum and maximum between the two quantities respectively. Also, $\lceil A \rceil$ denotes the smallest integer greater or equal to A, while $\lfloor A \rfloor$ denotes the largest integer less than or equal to A.

Given a function $g : U \to \mathbb{C}$, where U is either $\mathbb{T}^d$ or $\mathbb{R}$, our convention of the Fourier transform of g is given by

$$\widehat{g}(\xi) = \int_U e^{2\pi i \xi \cdot x} g(x)\, dx\,,$$

where ξ is either an element of $\mathbb{Z}^d$ (if $U = \mathbb{T}^d$) or an element of $\mathbb{R}$ (if $U = \mathbb{R}$). For the sake of convenience, we shall omit the 2π from our writing since it does not play any role in our arguments. Note that we will also use ξ to denote space-time white noise as commonly seen in literature; this should not cause any confusion.

For $c \in \mathbb{R}$, we sometimes write $c+$ to denote $c + \varepsilon$ for sufficiently small $\varepsilon > 0$, and write $c-$ for the analogous meaning. For example, the statement '$u \in X^{s,\frac{1}{2}-}$' should be read as '$u \in X^{s,\frac{1}{2}-\varepsilon}$ for sufficiently small $\varepsilon > 0$'.

For the sake of readability, in the proofs we sometimes omit the underlying domain $\mathbb{T}^d$ from various norms, e.g. we write $\|f\|_{H^s}$ instead of $\|f\|_{H^s(\mathbb{T}^d)}$ and $\|\phi\|_{\mathrm{HS}(L^2;H^s)}$ instead of $\|\phi\|_{\mathrm{HS}(L^2(\mathbb{T}^d);H^s(\mathbb{T}^d))}$.

Chapter 1

Introduction

This thesis is dedicated to studying well-posedness properties of stochastic dispersive partial differential equations (PDEs). One of the most important examples of dispersive PDEs is the nonlinear Schrödinger equation (NLS)

$$i\partial_t u - \Delta u \pm |u|^{2k} u = 0,$$

where $u : M \times \mathbb{R} \to \mathbb{C}$ with M being either the d-dimensional torus $\mathbb{T}^d$ or Euclidean space $\mathbb{R}^d$, and $k \geq 1$. This equation is known as *focusing* in the $(-)$ case and *defocusing* in the $(+)$ case. Another example of a dispersive PDE is the Korteweg–de Vries (KdV) equation

$$i\partial_t u + \partial_x^3 u + u\partial_x u = 0,$$

where $u : M \times \mathbb{R} \to \mathbb{R}$ with M being either $\mathbb{T}$ or$\mathbb{R}$.

Nonlinear dispersive PDEs such as the NLS and KdV are canonical model equations that arise from physics and engineering. They appear ubiquitously in diverse fields including nonlinear optics, plasma physics, water waves and telecommunication systems. On the other hand, random noise is inevitable in physical experiments and applications. For example, random noise may appear as an external factor, affecting evolution processes in experiments and applications. It is therefore natural to study these equations with a random forcing. Our interest lies in the rigorous mathematical analysis on stochastic dispersive PDEs. This is important for applied sciences as it has provided solid foundations for the verification and applicability of these models. Moreover, this theoretical research has proven to be very valuable for mathematics itself. Indeed, over the last thirty years, nonlinear dispersive PDEs have presented very difficult and interesting challenges, motivating the development of many new ideas and techniques in mathematical analysis. One of the sources of richness of nonlinear dispersive PDEs is that each subclass of equations poses its own difficulties, thus requiring the elaboration of specific tools.

In this thesis, we focus our study on various models of the stochastic nonlinear Schrödinger equations (SNLS) and stochastic Korteweg–de Vries equation (SKdV), either with additive or multiplicative noise. We divide the content into four main chapters, the first three of which discuss the SNLS, while the last chapter focuses on SKdV. More specifically, the content of this thesis is laid out as follows:

- In Chapter 2, we discuss well-posedness issues for SNLS on $\mathbb{T}^d$ for both additive and multiplicative noise. This is based on the following joint work with Razvan Mosincat:

  [20] K. Cheung, R. Mosincat, *Stochastic nonlinear Schrödinger equations on tori*, Stoch. Partial Differ. Equ. Anal. Comput. 7 (2019), no. 2, 169–208.
- In Chapter 3, we prove a local well-posedness result for the additive SNLS on $\mathbb{R}^d$ with rough noise. This is based on the following joint work with Oana Pocovnicu:

  [22] K. Cheung, O. Pocovnicu, *On the local well-posedness of the stochastic cubic nonlinear Schrödinger equations on* $\mathbb{R}^d, d \geq 3$, *with supercritical noise*, preprint.
- In Chapter 4, we study the Cauchy problem for the additive SNLS on $\mathbb{R}^4$ with non-vanishing boundary condition. This is based on the following joint work with Guopeng Li:

  [19] K. Cheung, G. Li, *On the energy critical SNLS with non-vanishing boundary condition*, preprint.
- In Chapter 5, we prove global well-posedness for SKdV with multiplicative noise on $\mathbb{T}$. This is based on the following joint work with Tadahiro Oh:

  [21] K. Cheung, T. Oh, *Global well-posedness of the periodic stochastic KdV equation with multiplicative noise*, preprint.

In the remainder of this introduction, we shall briefly state the mathematical problems under consideration and the main results.

1.1 Stochastic nonlinear Schrödinger equations

The Cauchy problem associated to the stochastic nonlinear Schrödinger equation (SNLS) can be formally stated as follows:

$$\begin{cases} i\partial_t u - \Delta u \pm |u|^{2k}u = F(u, \phi\xi) \\ u|_{t=0} = u_0 \end{cases} \qquad (t,x) \in [0,\infty) \times M, \tag{1.1}$$

where $k \geq 1$, M denotes either the d-dimensional torus $\mathbb{T}^d$ or Euclidean space $\mathbb{R}^d$, $u : [0,\infty) \times M \to \mathbb{C}$ is the unknown stochastic process, ξ is a space-time white noise, and ϕ is a linear operator between some function spaces of M. The random term $F(u, \phi\xi)$ is either an *additive noise* of the form $\phi\xi$ or a *multiplicative noise* of the form $u\phi\xi$. We note that the white noise ξ is a very rough object, and this roughness often presents a serious obstruction to well-posedness. The operator ϕ acts as a smoothing operator to counteract the roughness of ξ. Most of the time, ϕ will be a Hilbert-Schmidt operator from $L^2(M)$ to some Sobolev space $H^s(M)$, where the value of s is considered to be an indication of the roughness of the noise. The precise nature of the ϕ will be stated for each of the problems we will consider.

Let us define our notion of a solution. In dispersive PDEs literature, we often consider the so-called *strong* solutions. More specifically, for the deterministic NLS (i.e. $F(u, \phi\xi) = 0$ in (1.1)), we say that u is a solution if u satisfies the Duhamel formulation

$$u(t) = S(t)u_0 \pm i \int_0^t S(t-t')(|u|^{2k}u)(t')\,dt' \tag{1.2}$$

in some function space (usually a Sobolev space) of M, where $S(t) := e^{-it\Delta}$ is the linear Schrödinger propagator. Analogously, our notion of a solution for SNLS (1.1) will be that of a *mild* solution, that is, we say that u is a solution to (1.1) if u satisfies the Duhamel formulation

$$u(t) = S(t)u_0 \pm i \int_0^t S(t-t')(|u|^{2k}u)(t')\,dt' - i\Psi(u,t)\,, \tag{1.3}$$

where the additional term $\Psi(u,t)$ is a *stochastic convolution* corresponding to $F(u,\phi\xi)$. In the case of additive noise where $F(u,\phi\xi) = \phi\xi$, we define

$$\Psi(u,t) = \int_0^t S(t-t')\phi\xi(dt'); \tag{1.4}$$

while in the multiplicative case $F(u,\phi\xi) = u\phi\xi$, we define

$$\Psi(u,t) = \int_0^t S(t-t')u\phi\xi(dt'). \tag{1.5}$$

Before moving further on SNLS, we give a brief discussion on some symmetries of the deterministic NLS, as well as their impact on the well-posedness theory.

1.1.1 On scaling and Galilean symmetries

Let us consider the NLS on $\mathbb{R}^d$:

$$\begin{cases} i\partial_t u - \Delta u \pm |u|^{2k}u = 0 \\ u|_{t=0} = u_0 \in H^s(\mathbb{R}^d) \end{cases} \quad (t,x) \in [0,\infty) \times \mathbb{R}^d. \tag{1.6}$$

The NLS enjoys a scaling symmetry: if u is a solution to NLS, then so is the function $u^\lambda(t,x) := \lambda^{-\frac{1}{k}} u\big(\frac{t}{\lambda^2}, \frac{x}{\lambda}\big)$ to NLS for any $\lambda > 0$. The *scaling-critical regularity,*

$$s_{\text{crit}} := \frac{d}{2} - \frac{1}{k}, \tag{1.7}$$

is the unique number for which $\|u^\lambda\|_{\dot{H}^{s_{\text{crit}}}} = \|u\|_{\dot{H}^{s_{\text{crit}}}}$ for any $\lambda > 0$. The Cauchy problem (1.6) is categorised as

- subcritical if $s > s_{\text{crit}}$,
- critical if $s = s_{\text{crit}}$,
- supercritical if $s < s_{\text{crit}}$.

It is worth noting that the NLS has a conserved quantity (among others) known as the energy[1], given by

$$E[u](t) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 \, dx \pm \frac{1}{2k+2}\int_{\mathbb{R}^d} |u(t,x)| \, dx. \tag{1.8}$$

[1]This is also the Hamiltonian for NLS.

In particular, the energy is also invariant under scaling whenever $s_{\text{crit}} = 1$. For this reason, the NLS is called *energy-subcritical*, *energy-critical* and *energy-supercritical* if $s_{\text{crit}} > 1$, $s_{\text{crit}} = 1$ and $s_{\text{crit}} < 1$ respectively.

The NLS has another symmetry, the Galilean symmetry: if u is a solution, then so is $u^y(t,x) := e^{\frac{i}{2}y\cdot x}e^{\frac{|y|^2}{4}t}u(t, x+ty)$ for any $y \in \mathbb{R}^d$. In particular, this symmetry leaves the $L^2(\mathbb{R}^d)$-norm invariant, inducing another critical regularity at 0. It is commonly conjectured that the NLS is well-posed in the Sobolev space $H^s(\mathbb{R}^d)$ if $s > s_{\text{crit}}$ and $s \geq 0$, and ill-posed if $s < \max\{s_{\text{crit}}, 0\}$. See [82, Section 3.1] for a more detailed discussion. In the periodic setting, the Galilean symmetry continues to hold for $y \in 2\mathbb{Z}^d$, though there is no scaling symmetry. However, the same heuristic still plays an important role in the periodic setting. In fact, this heuristic is backed up by many well-posedness and ill-posedness results in the literature for both $\mathbb{R}^d$ and $\mathbb{T}^d$, see for example [18, 24, 49, 64, 72].

Moving back to the stochastic PDE, it is therefore natural ask the following question: regarding the regularity of the initial data and the noise, how well does the same heuristic hold for SNLS? There are already some partial answers to this question in the literature. Previously, de Bouard and Debussche [35, 36] studied SNLS (1.1) on $\mathbb{R}^d$ with both additive noise $\phi\xi$ and multiplicative noise $u\phi\xi$ in the energy-subcritical setting, that is, with $s_{\text{crit}} < 1$ and ϕ being a Hilbert-Schmidt operator from $L^2(\mathbb{R}^d)$ to $H^1(\mathbb{R}^d)$. They proved global existence and uniqueness of mild solutions in (i) $L^2(\mathbb{R})$ for the one-dimensional cubic SNLS and (ii) $H^1(\mathbb{R}^d)$ for defocusing energy-subcritical SNLS. As noted in [77], a slight modification of the argument in [36] allows one to prove that (1.1) is locally well-posed in $H^s(\mathbb{R}^d)$ provided that ϕ is Hilbert-Schmidt from $L^2(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ and that $s \geq \max(s_{\text{crit}}, 0)$. More recently, by using the dispersive estimate for the linear Schrödinger operator, Oh, Pocovnicu and Wang [77] proved local well-posedness for additive SNLS equation in $\mathbb{R}^d$ with subcritical initial data and supercritical noise.

Our first contribution in this thesis is to give some more partial answers to the question mentioned above beyond the existing literature. We first study SNLS in the periodic setting in Chapter 2, where we prove local-in-time well-posedness when the initial data and the noise have subcritical regularity, we then proceed to extend this solution globally-in-time for certain cases. We move onto the Euclidean setting in Chapter 3 and focus on the cubic SNLS, where we prove local well-posedness with supercritical noise and critical data.

1.1.2 SNLS on $\mathbb{T}^d$ with subcritical noise and data

We now state the main results in Chapter 2. Consider the SNLS (1.1) on $\mathbb{T}^d$, $d \geq 1$, with either additive noise $F(u, \phi\xi) = \phi\xi$ or multiplicative noise $F(u, \phi\xi) = u\phi\xi$. We consider initial data $u_0 \in H^s(\mathbb{T}^d)$, $s > s_{\text{crit}}$ being non-negative, and the smoothing operator ϕ being Hilbert-Schmidt from $L^2(\mathbb{T}^d)$ to $H^s(\mathbb{T}^d)$. The main results can be summarised in the following two theorems:

Theorem 1.1 (Well-posedness for additive SNLS on $\mathbb{T}^d$)**.** Let $F(u, \phi\xi) = \phi\xi$. There exist a stopping time T that is almost surely positive, and a unique solution u in a subspace of $C([0, T]; H^s(\mathbb{T}^d))$ to (1.1) on $\mathbb{T}^d$. Moreover, we can extend this solution globally in time almost surely in the following cases:

(i) the (focusing or defocusing) one-dimensional cubic SNLS for all $s \geq 0$;

(ii) the defocusing (i.e. + sign) energy-subcritical SNLS for all $s \geq 1$.

Theorem 1.2 (Well-posedness for multiplicative SNLS on $\mathbb{T}^d$)**.** Let $F(u, \phi\xi) = \phi\xi$. Suppose that ϕ is also translation invariant. Then there exist a stopping time T that is almost surely positive, and a unique solution u in a subspace of $C([0, T]; H^s(\mathbb{T}^d))$ to (1.1). Moreover, we can extend this solution globally in time almost surely in the following cases:

(i) the (focusing or defocusing) one-dimensional cubic SNLS for all $s \geq 0$;

(ii) the defocusing (i.e. + sign) energy-subcritical SNLS for all $s \geq 1$.

Let us briefly describe our method of proof. Our local-in-time argument uses the Fourier restriction norm method introduced by Bourgain [8] and the periodic Strichartz estimates proved by Bourgain and Demeter [13]. In particular, the function space in which the solutions reside from the above theorems can be stated more precisely as

$$C([0, T]; H^s(\mathbb{T}^d)) \cap X^{s, \frac{1}{2}-\varepsilon},$$

for some small $\varepsilon > 0$, Here, $X^{s,\frac{1}{2}-\varepsilon}$ is the Fourier restriction norm space of space-time functions v such that $S(t)v$ has spatial regularity s and temporal regularity

$\frac{1}{2} - \varepsilon$; see Chapter 2 for the precise definition. In establishing local well-posedness for the multiplicative SNLS, we also have to combine these tools with the truncation method used by de Bouard and Debussche [34–36]. Note that the extra technical assumption of ϕ in Theorem 1.2 means that the noise under consideration is spatially-homogeneous. Finally, we establish probabilistic a priori bounds on the mass and energy of solutions (which are conserved quantities for the deterministic NLS) to extend the local solutions globally.

1.1.3 SNLS on $\mathbb{R}^d$ with supercritical noise and critical data

In Chapter 3, we consider the Cauchy problem for the SNLS (1.1) with additive noise $F(u, \phi\xi) = \phi\xi$ on $\mathbb{R}^d$. As noted previously, it is not difficult to modify the argument in de Bouard and Debussche [36] to prove local well-posedness for subcritical noise and initial data. In view of this, we shall in fact study the problem with supercritical noise and critical data. Consider the cubic SNLS with additive noise:

$$\begin{cases} i\partial_t u - \Delta u \pm |u|^{2k} u = \phi\xi \\ u|_{t=0} = u_0 \in H^{s_{\text{crit}}}(\mathbb{R}^d) \end{cases} \qquad (t,x) \in [0,\infty) \times \mathbb{R}^d, \tag{1.9}$$

where $d \geq 3$ and ϕ is Hilbert-Schmidt from $L^2(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$ for $s < s_{\text{crit}}$. We set

$$s_d := \begin{cases} \dfrac{1}{4}, & \text{if} \quad d = 3\,, \\ s_{\text{crit}} - \dfrac{2}{5}, & \text{if} \quad d \geq 4\,. \end{cases} \tag{1.10}$$

Our main result is the following theorem:

Theorem 1.3. The Cauchy problem (1.9) is locally well-posed in the following sense: there exists a unique local-in-time solution u of (1.9) that lies almost surely in a subspace of $C([0,T]; H^s(\mathbb{R}^d))$, where $T = T_\omega$ is a stopping time that is almost surely positive.

Theorem 1.3 is inspired by [6]. In this work, the authors studied the deterministic NLS (1.6) with random initial data: $u(0) = f^\omega$, where f^ω is the Wiener randomisation of some function $f \in H^s(\mathbb{R}^d)$. They proved local well-posedness of

(1.6) in $H^s(\mathbb{R}^d)$ for a range of s below s_{crit}, with respect to this randomisation. See also [5, 14, 40]. In [6], the authors decomposed a solution as $u = z^\omega + v$, where $z^\omega(t) := S(t)f^\omega$ is linear and random, and solved the fixed point problem for v. In our work, we follow a similar argument where we use the so called Da Prato-Debussche trick and decompose our solution as $u = v + \Psi$, where Ψ is the stochastic convolution, and solve the fixed point problem for v.

Our main tools for proving Theorem 1.3 are similar to those in [6]. In particular, we have the Fourier restriction norm method adapted to the spaces V^p of functions of bounded p-variation and their preduals U^p (these spaces were introduced by Koch, Tataru and their collaborators, see [8, 51, 54]). The precise definitions of these spaces will be presented in Chapter 3.

1.1.4 SNLS with non-vanishing boundary conditions

In Chapter 4, we digress from the Cauchy problem (1.1) and move onto the following (defocusing) energy-critical stochastic nonlinear Schrödinger equation on $\mathbb{R}^4$:

$$\begin{cases} i\partial_t u + \Delta u = (|u|^2 - 1)u + \phi\xi \\ u|_{t=0} = u_0, \end{cases} \qquad (t,x) \in [0,\infty) \times \mathbb{R}^4, \tag{1.11}$$

with the non-vanishing boundary condition:

$$\lim_{|x|\to\infty} |u(x)| = 1. \tag{1.12}$$

As before, u is a complex-valued function, ξ denotes a space-time white noise and ϕ is a Hilbert-Schmidt operator from $L^2(\mathbb{R}^4)$ to $H^1(\mathbb{R}^4)$. We note that for the deterministic case $\phi = 0$, (1.11) is sometimes referred to as the *Gross-Pitaevskii equation* (GP). It is related to the NLS in the following way: if u is a solution to (GP), then the function $\tilde{u}(t,x) = e^{-it}u(t,x)$ is a solution the cubic NLS with the non-vanishing boundary condition $\lim_{|x|\to\infty} |\tilde{u}(x)| = 1$. We note that (GP) constitutes the Hamiltonian evolution corresponding to the *Ginzburg-Landau energy*:

$$E[u](t) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^d} \left(|u|^2 - 1\right)^2 dx. \tag{1.13}$$

Our main result is the following theorem:

Theorem 1.4 (Unconditional global well-poseness for (1.11))**.** The Cauchy problem (1.11) with the non-vanishing boundary condition (1.1.4) is globally well-posed in the energy space[2]

$$\mathcal{E}(\mathbb{R}^4) := \{u = 1 + v : v \in H^1_{real}(\mathbb{R}^4) + i\dot{H}^1_{real}(\mathbb{R}^4)\}$$

almost surely. In particular, $u(t)$ is unique in the class $\Psi + C_t\big(\mathbb{R}; \mathcal{E}(\mathbb{R}^4)\big)$ almost surely.

Theorem 1.4 is inspired by the work of Killip, Oh, Pocovnicu and Vişan in [61], where the authors established unconditional global well-posedness of (1.11) with $\phi = 0$ under the non-vanishing boundary condition (1.1.4). Their strategy was to treat the equation as the energy critical NLS with a subcritical perturbation, and then apply the perturbative approach introduced by Tao, Vişan and Zhang in [83] together with the conservation of the Ginzburg-Landau energy (1.13) to iterate local well-posedness. As it turns out, one can adapt this method to the SNLS (1.11). Although the energy $E[u]$ is no longer conserved in this setting, but one can still establish a probabilistic a priori bound on $E[u]$. This allows us to adapt the method from [61] to prove Theorem 1.4.

1.2 Stochastic KdV equation with multiplicative noise

In the final chapter of this thesis, we study the periodic stochastic Korteweg-de Vries equation (SKdV) with multiplicative noise:

$$\begin{cases} \partial_t u + \partial_x^3 u + u\partial_x u = u\phi\xi \\ u(x,0) = u_0(x) \in L^2(\mathbb{T}) \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R}^+, \tag{1.14}$$

where u is a real-valued function, ξ is again a space-time white noise, and the smoothing operator ϕ is Hilbert-Schmidt from $L^2(\mathbb{T})$ to $L^2(\mathbb{T})$. As in the case of

[2] $\mathcal{E}(\mathbb{R}^4)$ is precisely the space of functions u such that $E[u] < \infty$.

SNLS, we consider mild solutions to (1.14), which are functions u that satisfy the Duhamel formulation

$$u(t) = U(t)u_0 - \frac{1}{2}\int_0^t U(t-t')\partial_x u^2(t')dt' + \int_0^t U(t-t')u(t')\phi\xi$$

where $U(t) = e^{-t\partial_x^3}$.

Let us give some related background from the literature. In [37], de Bouard-Debussche considered the non-periodic version of the problem with homogeneous multiplicative noise and proved global well-posedness of (1.14) in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$. More specifically, they proved the result for $u_0 \in H^s(\mathbb{R})$ when ϕ has the convolution kernel in $H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s = 0$ or 1. There are also several results on SKdV with additive noise:

$$\begin{cases} \partial_t u + (\partial_x^3 u + u\partial_x u)dt = \phi\xi \\ u(x,0) = u_0(x). \end{cases} \tag{1.15}$$

In [39], de Bouard-Debussche-Tsutsumi showed that (1.15) is locally well-posed when ϕ is a Hilbert-Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$. More recently, Oh [75] proved local well-posedness of (5.4) even when $\phi = \text{Id}$, thus handling the case of the space-time white noise. See [39] and the references therein for the previous works in the periodic and non-periodic settings as well as some of its physical background. Also, see [3], [44], [53]. Note that we often see $\partial_x u\phi\xi$ as multiplicative noise in SKdV rather than $u\phi\xi$ as in (5.1), and one can regard our study of (1.14) as the first step toward understanding more difficult multiplicative noises such as $u_x\phi\xi$.

Our main result in Chapter 5 is the following theorem:

Theorem 1.5. Let ϕ be Hilbert-Schmidt from $L^2(\mathbb{T})$ to itself. Let $u_0 \in L^2(\mathbb{T})$. The stochastic KdV (1.14) with multiplicative space-time white noise is globally well-posed almost surely.

Theorem 1.5 is proved using the Fourier restriction norm method, which was briefly mentioned in Section 1.1.2. This method was employed on the deterministic KdV equation in [9] by Bourgain to yield global well-posedness in $L^2(\mathbb{T})$, and later the well-posedness regime was improved to $H^{-\frac{1}{2}}(\mathbb{T})$ by Kenig-Ponce-Vega [59] (also see [26]). The tools used in the deterministic analysis rely heavily on the fact that solutions to the KdV equation are mean-zero. Unfortunately, this is no longer the case for SKdV (1.14). To overcome this issue, we first reduce the SKdV equation to a coupled system of mean-zero SKdV-type equation and a stochastic differential equation for the mean. Then, we follow the argument in [10, 75] and perform a nonlinear analysis on the second iteration in an appropriate Fourier restriction norm space to construct local solutions. Finally, we appeal to an a priori L^2-bound to extend our local solutions to global ones.

A secondary result in Chapter 5 is the following theorem on the stabilization by noise, where we verify that the result of Tsutsumi [84] continues to hold in our low regularity setting. It says that under certain assumptions, the mass of a solution almost surely decays to zero as time goes to infinity.

Theorem 1.6. Let ϕ and u_0 satisfy the same assumptions as in Theorem 1.5. Suppose further that there exists a constant $\alpha > \frac{1}{2}\|\phi\|_{\mathrm{HS}(L^2;L^2)}$ such that for all $v \in L^2(\mathbb{T})$, one has

$$\sum_{k=-\infty}^{\infty} \left[\int_{\mathbb{T}} \mathrm{Re}\,\big(\phi e_k(x)\big) |v(x)|^2 dx \right]^2 \geq \alpha^2 \|v\|_{L^2(\mathbb{T})}^2. \tag{1.16}$$

Then the solution u of the stochastic KdV (5.1) given by Theorem 1.5 decays in mass, that is, as $t \to 0$, we have $\|u(t)\|_{L^2(\mathbb{T})} \to 0$ almost surely.

If the decay property of the $L^2(\mathbb{T})$-norm of the solutions in Theorem 1.6 hold, then the zero solution is referred to as *pseudo-asymptotic stable*. A significance of the above result is the following. On the one hand, by Itô's formula on the equation, we obtain for $t > 0$,

$$\mathbb{E}\big[\|u(t)\|_{L^2(\mathbb{T})}^2\big] = \mathbb{E}\big[\|u(0)\|_{L^2(\mathbb{T})}^2\big] + \int_0^t \sum_{k=-\infty}^{\infty} \mathbb{E}\big[\|u(t')\phi e_n\|_{L^2(\mathbb{T})}^2 \, dt'\big],$$

which infers that the second moment of $\|u(t)\|_{L^2(\mathbb{T})}$ is non-decreasing. On the other hand, Theorem (1.6) tells us that $\|u(t)\|_{L^2(\mathbb{T})}$ itself decays to 0 almost surely.

Chapter 2

SNLS on tori

In this chapter, we study the following Cauchy problem associated to a stochastic nonlinear Schrödinger equation of the form:

$$\begin{cases} i\partial_t u - \Delta u \pm |u|^{2k}u = F(u,\phi\xi) \\ u|_{t=0} = u_0 \in H^s(\mathbb{T}^d) \end{cases} \qquad (t,x) \in (0,\infty)\times\mathbb{T}^d, \tag{2.1}$$

where $k, d \geq 1$ are integers, $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$, $u : [0,\infty)\times\mathbb{T}^d \to \mathbb{C}$ is the unknown stochastic process, and $F(u,\phi\xi)$ is either an additive noise of the form

$$F(u,\phi\xi) = \phi\xi \tag{2.2}$$

or multiplicative noise of the form

$$F(u,\phi\xi) = u \cdot \phi\xi, \tag{2.3}$$

where the right-hand side of (2.3) is understood as an Itô product[1].

We now give a more detailed description of our problem than in the introduction. Let $(\Omega, \mathcal{A}, \{\mathcal{A}_t\}_{t\geq 0}, \mathbb{P})$ be a filtrated probability space. Let W be the $L^2(\mathbb{T}^d)$-

[1] The multiplicative noise given by the Stratonovich product $u\circ\phi\xi$ with real-valued ξ is relevant in physical applications, as it conserves the mass of u (i.e. $t \mapsto \|u(t)\|^2_{L^2_x(\mathbb{T}^d)}$ is constant) almost surely. Our analysis can handle either the Itô or the Stratonovich product, and we choose to work with the former for the sake of simpler exposition.

cylindrical Wiener process given by

$$W(t, x, \omega) := \sum_{n \in \mathbb{Z}^d} \beta_n(t, \omega) e_n(x), \tag{2.4}$$

where $\{\beta_n\}_{n \in \mathbb{Z}^d}$ is a family of independent complex-valued Brownian motions associated with the filtration $\{\mathcal{A}_t\}_{t \geq 0}$ and $e_n(x) := \exp(2\pi i n \cdot x)$, $n \in \mathbb{Z}^d$. The space-time white noise ξ is given by the (distributional) time derivative of W, i.e. $\xi = \frac{\partial W}{\partial t}$. Since the spatial regularity of W is too low (more precisely, for each fixed $t \geq 0$, $W(t) \in H^{-\frac{d}{2}-\varepsilon}(\mathbb{T}^d)$ almost surely for any $\varepsilon > 0$), we consider a smoothed out version ϕW as follows. We recall the definition of a Hilbert-Schmidt operator, which is a notion used throughout this thesis: let H, K be Hilbert spaces, an operator $\phi : H \to K$ is Hilbert-Schmidt if

$$\|\phi\|^2_{\mathrm{HS}(H;K)} := \sum_{n \in \mathbb{Z}^d} \|\phi h_n\|^2_K < \infty\,, \tag{2.5}$$

where $\{h_n\}_{n \in \mathbb{Z}^d}$ is an orthonormal basis of H (recall that $\|\cdot\|_{\mathrm{HS}(H;K)}$ does not depend on the choice of $\{h_n\}_{n \in \mathbb{Z}^d}$). In many instances, we assume $\phi \in \mathrm{HS}(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$ for appropriate $s \geq 0$. In this case, ϕW is a Wiener process with sample paths in $H^s(\mathbb{T}^d)$ and its time derivative $\phi\xi$ corresponds to a noise which is white in time and correlated in space (with correlation function depending on ϕ).

As mentioned in the introduction, we use the notion of a mild solution. More precisely, we say that u is a solution if it is an adapted process u in $H^s(\mathbb{T}^d)$ that is continuous in time and satisfies the mild formulation

$$u(t) = S(t)u_0 \pm i \int_0^t S(t-t')(|u|^{2k}u)(t')\,dt' - i\Psi(u,t)\ ,\ t \geq 0\,, \tag{2.6}$$

almost surely, where $S(t) := e^{-it\Delta}$ is the linear Schrödinger propagator, and $\Psi(u,t)$ is a *stochastic convolution* as in (1.4) and (1.5) corresponding to the stochastic forcing $F(u, \phi\xi)$. By the definition for W, we can express Ψ as an orthonormal expansion in the following way: (i) for the additive noise (2.2):

$$\Psi(u,t) = \Psi(t) := \int_0^t S(t-t')\phi\,dW(t') = \sum_{n \in \mathbb{Z}^d} \int_0^t S(t-t')\phi e_n\,d\beta_n(t') \tag{2.7}$$

and (ii) for the multiplicative noise (2.3):

$$\Psi(u,t) := \int_0^t S(t-t')u(t')\phi\, dW(t') = \sum_{n\in\mathbb{Z}^d}\int_0^t S(t-t')u(t')\phi e_n\, d\beta_n(t')\,. \tag{2.8}$$

We have mentioned before in the introduction on the works by de Bouard and Debussche [35, 36] on SNLS on $\mathbb{R}^d$. Their arguments given in [35, 36] use fixed point arguments in the space $C_tH_x^1 \cap L_t^pW_x^{1,q}([0,T]\times\mathbb{R}^d)$, for some $T>0$ and some suitable $p,q\geq 1$. [2] In particular, they use the (deterministic) Strichartz estimates:

$$\|S(t)f\|_{L_t^pL_x^q(\mathbb{R}\times\mathbb{R}^d)} \leq C_{p,q}\|f\|_{L_x^2(\mathbb{R}^d)}, \tag{2.9}$$

where the pair (p,q) is admissible, i.e. $\frac{2}{p}+\frac{d}{q}=\frac{d}{2}$, $2\leq p,q,\leq\infty$, and $(p,q,d)\neq(2,\infty,2)$. On $\mathbb{T}^d$, Bourgain and Demeter [13] proved the ℓ^2-decoupling conjecture, and as a corollary, the following periodic Strichartz estimates:

$$\big\|S(t)P_{\leq N}f\big\|_{L^p_{t,x}([0,T]\times\mathbb{T}^d)} \leq C_{p,T,\varepsilon}N^{\frac{d}{2}-\frac{d+2}{p}+\varepsilon}\|f\|_{L_x^2(\mathbb{T}^d)}\,. \tag{2.10}$$

Here, $P_{\leq N}$ is the Littlewood-Paley projection onto frequencies $\{n\in\mathbb{Z}^d : |n|\leq N\}$, $p\geq\frac{2(d+2)}{d}$, and $\varepsilon>0$ is an arbitrarily small quantity [3]. However, such Strichartz estimates are not strong enough for a fixed point argument in mixed Lebesgue spaces for the deterministic NLS on $\mathbb{T}^d$. To overcome this problem, we shall employ the Fourier restriction norm method by means of $X^{s,b}$-spaces defined via the norms

$$\|u\|_{X^{s,b}} := \big\|\langle n\rangle^s\langle\tau-|n|^2\rangle^b\mathcal{F}_{t,x}(u)(\tau,n)\big\|_{L^2_\tau\mathrm{HS}_n(\mathbb{R}\times\mathbb{Z}^d)}\,. \tag{2.11}$$

The indices $s,b\in\mathbb{R}$ measure the spatial and temporal regularities of functions $u\in X^{s,b}$, and $\mathcal{F}_{t,x}$ denotes Fourier transform of functions defined on $\mathbb{R}\times\mathbb{T}^d$. This harmonic analytic method was introduced by Bourgain [8] for the deterministic

[2] Here, $W^{s,r}(\mathbb{T}^d)$ denotes the L^r-based Sobolev space defined by the Bessel potential norm:

$$\|u\|_{W^{s,r}(\mathbb{T}^d)} := \|\langle\nabla\rangle^s u\|_{L^r(\mathbb{T}^d)} = \big\|\mathcal{F}^{-1}(\langle n\rangle^s\widehat{u}(n))\big\|_{\ell^r_n(\mathbb{Z}^d)},$$

where $\langle n\rangle := \sqrt{1+|n|^2}$. When $r=2$, we have $H^s(\mathbb{T}^d)=W^{s,2}(\mathbb{T}^d)$.

[3] More recently, Killip and Vişan [63] removed the arbitrarily small loss of ε derivatives in (2.10) when $p>\frac{2(d+2)}{d}$. However, we do not need this scale-invariant improvement in our results.

nonlinear Schrödinger equation (NLS):

$$i\partial_t u - \Delta u \pm |u|^{2k} u = 0\,. \tag{2.12}$$

We now break down Theorem 1.1 and 1.2 in more details into four separate theorems below. The first one is on the local well-posedness of additive SNLS.

Theorem 2.1 (Local well-posedness for additive SNLS)**.** Given $s > s_{\text{crit}}$ non-negative, let $\phi \in \text{HS}(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$ and $F(u, \phi) = \phi\xi$. Then for any $u_0 \in H^s(\mathbb{T}^d)$, there exist a stopping time T that is almost surely positive, and a unique solution $u \in C([0,T]; H^s(\mathbb{T}^d)) \cap X^{s,\frac{1}{2}-\varepsilon}([0,T])$ to SNLS with additive noise, for some $\varepsilon > 0$.

Remark 2.2. We point out that s_{crit} is negative only for the one-dimensional cubic NLS, i.e. $(d, k) = (1, 1)$ for which $s_{\text{crit}} = -\frac{1}{2}$. Below $L^2(\mathbb{T})$, the deterministic cubic NLS on $\mathbb{T}$ was shown to be ill-posed. Indeed, Christ, Colliander and Tao [25] and Molinet [72] showed that the solution map $u_0 \in H^s(\mathbb{T}) \mapsto u(t) \in H^s(\mathbb{T})$ is discontinuous whenever $s < 0$. More recently, Guo and Oh [49] showed an even stronger ill-posedness result, in the sense that for any $u_0 \in H^s(\mathbb{T})$, $s \in (-\frac{1}{8}, 0)$, there is no distributional solution u that is also a limit of smooth solutions in $C([-T,T]; H^s(\mathbb{T}))$. In the (super)critical regime, i.e. for $s \leq -\frac{1}{2} = s_{\text{crit}}$, Kishimoto [64] showed a norm inflation phenomenon at any $u_0 \in H^s(\mathbb{T})$: for any $\varepsilon > 0$ and $u_0 \in H^s(\mathbb{T})$, there exists a solution u^ε to NLS such that $\|u^\varepsilon(0) - u_0\|_{H^s(\mathbb{T})} < \varepsilon$ and $\|u^\varepsilon(t)\|_{H^s(\mathbb{T})} > \varepsilon^{-1}$ for some $t \in (0, \varepsilon)$. See also [76, 78].

Regarding the one-dimensional cubic SNLS on $\mathbb{T}$, we point out that recently Forlano, Oh and Wang [42] studied a *renormalized* (Wick ordered, see also [30]) additive SNLS with a weaker assumption than that of Theorem 2.1 above. While we assume that $\phi \in \text{HS}(L^2(\mathbb{T}); L^2(\mathbb{T}))$, the work of [42] assumes that ϕ is γ-radonifying from $L^2(\mathbb{T})$ into the Fourier-Lebesgue space $\mathcal{F}L^{s,p}(\mathbb{T})$ with $s > 0$ and $1 < p < \infty$. In particular, this allows them to handle almost space-time white noise, namely $\phi = \langle \partial_x \rangle^{-\alpha}$ with $\alpha > 0$ arbitrarily small.

Remark 2.3. Although we present our results for SNLS on the standard torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$, our arguments hold on any torus $\mathbb{T}^d_{\boldsymbol{\alpha}} = \prod_{j=1}^d \mathbb{R}/\alpha_j\mathbb{Z}$, where $\boldsymbol{\alpha} = (\alpha_1, ..., \alpha_d) \in [0,\infty)^d$. This is because the periodic Strichartz estimates (2.10) of Bourgain and Demeter [13] hold for irrational tori ($\mathbb{T}^d_{\boldsymbol{\alpha}}$ is irrational if there is no

$\boldsymbol{\gamma} \in \mathbb{Q}^d$ such that $\boldsymbol{\gamma} \cdot \boldsymbol{\alpha} = 0$). Prior to [13], Strichartz estimates were harder to establish on irrational tori – see [50] and references therein.

Now let us recall the following conservation laws for the deterministic NLS:

$$M(u(t)) := \frac{1}{2} \int_{\mathbb{T}^d} |u(t,x)|^2 \, dx \tag{2.13}$$

$$E(u(t)) := \frac{1}{2} \int_{\mathbb{T}^d} |\nabla_x u(t,x)|^2 \pm \frac{1}{2k+2} \int_{\mathbb{T}^d} |u(t,x)|^{2k+2} \, dx, \tag{2.14}$$

where the sign $\pm$ in (2.14) matches that in (2.1) and (2.6). Recall that SNLS (2.1) with the $+$ sign is called defocusing (and focusing for the $-$ sign). We say that SNLS is energy-subcritical if $s_{\text{crit}} < 1$ (i.e. for any $k \geq 1$ when $d = 1, 2$ and for $k = 1$ when $d = 3$).

For solutions of SNLS these quantities are no longer necessarily conserved. However, Itô's lemma allows us to bound these in a probabilistic manner similarly to de Bouard and Debussche [35, 36]. Therefore, we obtain the following:

Theorem 2.4 (Global well-posedness for additive SNLS)**.** Let $s \geq 0$. Given $\phi \in \mathrm{HS}(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$, let $F(u,\phi) = \phi\xi$ and $u_0 \in H^s(\mathbb{T}^d)$. Then the H^s-valued solutions of Theorem 2.1 extend globally in time almost surely in the following cases:

(i) the (focusing or defocusing) one-dimensional cubic SNLS for all $s \geq 0$;

(ii) the defocusing energy-subcritical SNLS for all $s \geq 1$.

We now move onto the problem with multiplicative noise, i.e. SNLS with (2.3). For this case, we need a stronger assumption on ϕ. By a slight abuse of notation, for a bounded linear operator ϕ from $L^2(\mathbb{T}^d)$ to a Banach space B, we say that $\phi \in \mathrm{HS}(L^2(\mathbb{T}^d); B)$ if[4]

$$\|\phi\|^2_{\mathrm{HS}(L^2(\mathbb{T}^d);B)} := \sum_{n \in \mathbb{Z}^d} \|\phi e_n\|^2_B < \infty \, .$$

For $s \in \mathbb{R}$ and $r \geq 1$, we also define the Fourier-Lebesgue space $\mathcal{F}L^{s,r}(\mathbb{T}^d)$ via the norm

$$\|f\|_{\mathcal{F}L^{s,r}(\mathbb{T}^d)} := \big\| \langle n \rangle^s \widehat{f}(n) \big\|_{\ell^r_n(\mathbb{Z}^d)} \, .$$

[4] In fact, such operators are known as nuclear operators of order 2 and their introduction goes back to the work of A. Grothendieck on nuclear locally convex spaces.

Clearly, when $r = 2$ we have $\mathcal{F}L^{s,r}(\mathbb{T}^d) = H^s(\mathbb{T}^d)$ and for $s_1 \leq s_2$ and $r_1 \leq r_2$ we have $\mathcal{F}L^{s_2,r_1}(\mathbb{T}^d) \subset \mathcal{F}L^{s_1,r_2}(\mathbb{T}^d)$. We now state our local well-posedness result for the multiplicative SNLS.

Theorem 2.5 (Local well-posedness for multiplicative SNLS)**.** Given $s > s_{\text{crit}}$ non-negative, let $\phi \in \text{HS}(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$. If $s \leq \frac{d}{2}$, we further impose that

$$\phi \in \text{HS}(L^2(\mathbb{T}^d); \mathcal{F}L^{s,r}(\mathbb{T}^d)) \tag{2.15}$$

for some $r \in \big[1, \frac{d}{d-s}\big)$ when $s > 0$ and $r = 1$ when $s = 0$. Let $F(u, \phi) = u \cdot \phi\xi$. Then for any $u_0 \in H^s(\mathbb{T}^d)$, there exist a stopping time T that is almost surely positive, and a unique solution $u \in C([0,T]; H^s(\mathbb{T}^d)) \cap X^{s,\frac{1}{2}-\varepsilon}([0,T])$ to SNLS with multiplicative noise, for some $\varepsilon > 0$.

Remark 2.6. The assumption on ϕ here in Theorem 2.5 is slightly more general than in Theorem 1.2 from the introduction; this is because if $\phi\xi$ is a spatially homogeneous noise, i.e. ϕ is translation invariant, then the extra assumption (2.15) is superfluous. Indeed, if $\widehat{\phi e_n}(m) = 0$, for all $m, n \in \mathbb{Z}^d$, $m \neq n$ and $\phi \in \text{HS}(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$, then $\phi \in \text{HS}(L^2(\mathbb{T}^d); \mathcal{F}L^{s,r}(\mathbb{T}^d))$ for any $r \geq 1$.

We point out that an extra condition in the multiplicative case was also used by de Bouard and Debussche [36] in their study of SNLS in $H^1(\mathbb{R}^d)$, namely they required that ϕ is a γ-radonifying operator from $L^2(\mathbb{R}^d)$ into $W^{1,\alpha}(\mathbb{R}^d)$ for some appropriate α, as compared to the requirement that ϕ is Hilbert-Schmidt from $L^2(\mathbb{R}^d)$ into $H^s(\mathbb{R}^d)$ in the additive case.

In the multiplicative case, the stochastic convolution depends on the solution u and this forces us to work in the space in $L^2\big(\Omega; C([0,T]; H^s(\mathbb{T}^d)) \cap X^{s,\frac{1}{2}-\varepsilon}([0,T])\big)$. In order to control the nonlinearity in this space, we use a truncation method which has been used for SNLS on $\mathbb{R}^d$ by de Bouard and Debussche [35, 36]. Moreover, we combine this method with the use of $X^{s,b}$-spaces in a similar manner as in [34], where the same authors studied the stochastic KdV equation with low regularity initial data on $\mathbb{R}$. This introduces some technical difficulties which did not appear when using the more classical Strichartz spaces as those used in [35, 36].

Next, we prove global well-posedness of SNLS (2.1) with multiplicative noise. Similarly to the additive case, the main ingredient is the probabilistic a priori bound on the mass and energy of a local solution u. However, we also need to obtain uniform control on the $X^{s,b}$-norms for solutions to truncated versions of (2.6).

Theorem 2.7 (Global well-posedness for multiplicative SNLS)**.** Let $s \geq 0$. Given ϕ with the same assumptions as in Theorem 2.5, let $F(u, \phi) = u \cdot \phi\xi$ and $u_0 \in H^s(\mathbb{T}^d)$. Then the H^s-valued solutions of Theorem 2.5 extend globally in time in the following cases:

(i) the (focusing or defocusing) one-dimensional cubic SNLS for all $s \geq 0$;

(ii) the defocusing energy-subcritical SNLS for all $s \geq 1$.

Before concluding this introduction let us state two remarks.

Remark 2.8. Theorem 2.1 and Theorem 2.5 are almost optimal for handling the regularity of initial data since the deterministic NLS is ill-posed for $s < s_{\text{crit}}$ (see Remark 2.2). In terms of the regularity of the noise, at least in the additive noise case, it is possible to consider rougher noise by employing the Da Prato-Debussche trick, namely by writing a solution u to (2.6) as $u = v + \Psi$ and considering the equation for the residual part v. In general, this procedure allows one to treat rougher noise, see for example [5, 6, 30] where they treat NLS with rough random initial data and more recently [77] where they handled supercritical noise for the additive SNLS on $\mathbb{R}^d$. In the periodic setting however, the argument gets more complicated (see for example [5, 6] on $\mathbb{R}^d$ versus [30, 73] on $\mathbb{T}^d$). The actual implementation of the aforementioned trick requires cumbersome case-by-case analysis where the number of cases grows exponentially in k. Even for the cubic case on $\mathbb{T}^d$ the analysis is involved, whereas on $\mathbb{R}^d$ one can use bilinear Strichartz estimates which are not available on $\mathbb{T}^d$.

Remark 2.9. In the multiplicative noise case, there are well-posedness results on a general compact Riemannian manifold M without boundaries. In [16], Brzeźniak and Milllet use the Strichartz estimates of [17] and the standard space-time Lebesgue spaces (i.e. without the Fourier restriction norm method). For $M = \mathbb{T}^d$, Theorem 2.5 improves the result in [16] since it requires less regularity on the noise and initial data. In [15], Brzeźniak, Hornung, and Weiss construct martingale solutions in $H^1(M)$ for the multiplicative SNLS with energy-subcritical defocusing nonlinearities and mass-subcritical focusing nonlinearities.

Our local-in-time argument uses the Fourier restriction norm method introduced by Bourgain [8] and the periodic Strichartz estimates proved by Bourgain and Demeter [13]. In establishing local well-posedness for the multiplicative SNLS, we also have to combine these tools with the truncation method used by de Bouard and Debussche [34–36]. Moreover, by proving probabilistic a priori bounds on the mass and energy of solutions, we establish global well-posedness in (i) $L^2(\mathbb{T})$ for cubic nonlinearities (i.e. $k = 1$) when $d = 1$, and (ii) $H^1(\mathbb{T}^d)$ for all defocusing energy-subcritical nonlinearities – see Theorem 2.4 and the preceding discussion for more details.

The remainder of this chapter is organised as follows. In Section 2.1, we provide some preliminaries for the Fourier restriction norm method and prove the multilinear estimates necessary for the local well-posedness results. In Section 2.2, we prove some properties of the stochastic convolutions Ψ and $\Psi[u]$ given respectively by (2.7) and (2.8). We prove Theorems 2.1 and 2.5 in Section 2.3. Finally, in Section 2.4 we prove the global results Theorems 2.4 and 2.7.

2.1 Fourier restriction norm method

Let $s, b \in \mathbb{R}$. The Fourier restriction norm space $X^{s,b}$ adapted to the Schrödinger equation on $\mathbb{T}^d$ is the space of tempered distributions u on $\mathbb{R} \times \mathbb{T}^d$ such that the norm

$$\|u\|_{X^{s,b}} := \left\| \langle n \rangle^s \langle \tau - |n|^2 \rangle^b \mathcal{F}_{t,x}(u)(\tau, n) \right\|_{\ell^2_n L^2_\tau(\mathbb{Z}^d \times \mathbb{R})}$$

is finite. Equivalently, the $X^{s,b}$-norm can be written in its interaction representation form:

$$\|u\|_{X^{s,b}} = \left\| \langle n \rangle^s \langle \tau \rangle^b \mathcal{F}_{t,x} \left(S(-t)u(t) \right) (n, \tau) \right\|_{\ell^2_n L^2_\tau(\mathbb{Z}^d \times \mathbb{R})}, \tag{2.16}$$

where $S(t) = e^{-it\Delta}$ is the linear Schrödinger propagator. This equivalence is useful for estimating the $X^{s,b}$-norm of stochastic convolutions, see for example Lemma 2.18 below.

We now state some facts on $X^{s,b}$-spaces. The interested reader can find the proof of these and further properties in [82]. Firstly, we have the following continuous embeddings

$$X^{s,b} \hookrightarrow C(\mathbb{R}; H^s_x(\mathbb{T}^d)) \text{ , for } b > \frac{1}{2}\,, \tag{2.17}$$

$$X^{s',b'} \hookrightarrow X^{s,b} \text{ , for } s' \geq s \text{ and } b' \geq b\,. \tag{2.18}$$

We have the duality relation

$$\|u\|_{X^{s,b}} = \sup_{\|v\|_{X^{-s,-b}} \leq 1} \left| \int_{\mathbb{R}\times\mathbb{T}^d} u(t,x)\overline{v(t,x)}\, dt\, dx \right| . \tag{2.19}$$

Lemma 2.10 (Transference principle, [82, Lemma 2.9])**.** Let Y be a Banach space of functions on $\mathbb{R} \times \mathbb{T}^d$ such that

$$\|e^{it\lambda} e^{\pm it\Delta} f\|_Y \lesssim \|f\|_{H^s(\mathbb{T}^d)}$$

for all $\lambda \in \mathbb{R}$ and all $f \in H^s(\mathbb{T}^d)$. Then, for any $b > \frac{1}{2}$,

$$\|u\|_Y \lesssim \|u\|_{X^{s,b}}$$

for all $u \in X^{s,b}$.

Lemma 2.10 is useful for transferring Strichartz estimates on standard Sobolev spaces to Strichartz estimates on Fourier restriction norm spaces, see for example (2.28) below.

Given a time interval $I \subseteq \mathbb{R}$, one defines the time restricted space $X^{s,b}(I)$ via the norm

$$\|u\|_{X^{s,b}(I)} := \inf\left\{\|\tilde{u}\|_{X^{s,b}} : \tilde{u}|_I = u\right\}. \tag{2.20}$$

Lemma 2.11. Let $s \geq 0$ and $0 \leq b < \frac{1}{2}$. Then

$$\|u\|_{X^{s,b}(I)} \sim \|\mathbb{1}_I(t)u(t)\|_{X^{s,b}} \,. \tag{2.21}$$

This relation is useful for establishing local-in-time estimates, see for example in Lemma 2.18 later on. The proof of (2.21) is almost identical to [34, Lemma 2.1] for $X^{s,b}$ spaces adapted to the KdV equation, we only sketch the proof below.

Sketch proof of Lemma 2.11. Clearly, by the definition, we have

$$\|u\|_{X^{s,b}(I)} \leq \|\mathbb{1}_I(t)u\|_{X^{s,b}}\,.$$

We are required to show the inverse inequality $\|\mathbb{1}_I(t)u(t)\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}(I)}$. To this end, let $g(t) = \mathbb{1}_I(t)S(-t)u(t)$, so that

$$\begin{aligned}\|\mathbb{1}_I(t)u\|_{X^{s,b}}^2 &= \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \langle n\rangle^{2s}\langle\tau\rangle^{2b}|\widehat{g}(n,\tau)|^2\,d\tau \\ &= \sum_{n\in\mathbb{Z}} \langle n\rangle^{2s}\|\mathcal{F}_x g(n)\|_{H_t^b(\mathbb{R})}^2.\end{aligned}$$

The claim then follows from the following inequality

$$\|\mathbb{1}_I h\|_{H_t^b(\mathbb{R})} \lesssim \|h\|_{H_t^b(\mathbb{R})}. \tag{2.22}$$

The proof of (2.22) is contained in [34, Lemma 2.1] and so we omit it here. □

Lemma 2.12 (Linear estimates, [82, Proposition 2.12])**.** Let $s \in \mathbb{R}$ and suppose η is smooth and compactly supported. Then, we have

$$\|\eta(t)S(t)f\|_{X^{s,b}} \lesssim \|f\|_{H^s(\mathbb{T}^d)}\ ,\ \text{for } b \in \mathbb{R}\,; \tag{2.23}$$

$$\left\|\eta(t)\int_0^t S(t-t')F(t')dt'\right\|_{X^{s,b}} \lesssim \|F\|_{X^{s,b-1}}\ ,\ \text{for } b > \frac{1}{2}\,. \tag{2.24}$$

By localizing in time, we can gain a smallness factor, as per lemma below.

Lemma 2.13 (Time localization property, [82, Lemma 2.11])**.** Let $s \in \mathbb{R}$ and $-\frac{1}{2} < b' < b < \frac{1}{2}$. For any $T \in (0,1)$, we have

$$\|f\|_{X^{s,b'}([0,T])} \lesssim_{b,b'} T^{b-b'}\, \|f\|_{X^{s,b}([0,T])}\ .$$

We now give the proofs of the multilinear estimates necessary to control the nonlinearity $|u|^{2k}u$. Recall the L^4-Strichartz estimate due to Bourgain [8] (see also [82, Proposition 2.13]):

$$\|u\|_{L^4_{t,x}(\mathbb{R}\times\mathbb{T})} \lesssim \|u\|_{X^{0,\frac{3}{8}}}. \tag{2.25}$$

Lemma 2.14. Let $d = 1$, $s \geq 0$, $b \geq \frac{3}{8}$, and $b' \leq \frac{5}{8}$. Then, for any time interval $I \subset \mathbb{R}$, we have

$$\|u_1\overline{u_2}u_3\|_{X^{s,b'-1}(I)} \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}(I)}. \tag{2.26}$$

Proof. First assume that $I = \mathbb{R}$. By the duality relation (2.19), it suffices to show that

$$\left| \int_{\mathbb{R}\times\mathbb{T}^d} \langle\nabla\rangle^s (u_1\overline{u_2}u_3)\overline{v}\, dxdt \right| \lesssim \prod_{j=1}^{3} \|u_j\|_{X^{s,b}} \|v\|_{X^{0,1-b'}}$$

for any factors u_1, u_2, u_3, v. By Parseval's Theorem, we have

$$\left| \int_{\mathbb{R}\times\mathbb{T}^d} \langle\nabla\rangle^s (u_1\overline{u_2}u_3)\overline{v}\, dxdt \right| = \left| \sum_{n\in\mathbb{Z}} \int_{\mathbb{R}} \langle n\rangle^s \mathcal{F}_x(u_1\overline{u_2}u_3)(n)\overline{\widehat{v}(n)}\, dxdt \right|.$$

Now

$$\begin{aligned} \langle n\rangle^s \mathcal{F}_x(u_1\overline{u_2}u_3)(n) &= \langle n\rangle^s \sum_{n_1+n_2+n_3=n} \widehat{u_1}(n_1)\widehat{u_2}(n_2)\widehat{u_3}(n_3) \\ &\leq \sum_{n_1+n_2+n_3=n} \langle n_1\rangle^s \widehat{u_1}(n_1) \cdot \langle n_2\rangle^s \widehat{u_2}(n_2) \cdot \langle n_3\rangle^s \widehat{u_3}(n_3) \\ &= \mathcal{F}_x\big(\langle\nabla\rangle^s u_1 \cdot \langle\nabla\rangle^s u_2 \cdot \langle\nabla\rangle^s u_3\big) \end{aligned}$$

Thus by Parseval's Theorem again, we have

$$\left| \int_{\mathbb{R}\times\mathbb{T}^d} \langle\nabla\rangle^s (u_1\overline{u_2}u_3)\overline{v}\, dxdt \right| \leq \left| \int_{\mathbb{R}\times\mathbb{T}^d} \langle\nabla\rangle^s u_1 \cdot \overline{\langle\nabla\rangle^s u_2} \cdot \langle\nabla\rangle^s u_3 \cdot \overline{v}\, dxdt \right|.$$

The claim then follows from Hölder inequality and (2.25) for each of the four factors (hence the restrictions $b, 1 - b' \geq \frac{3}{8}$).

For an arbitrary time interval I, if $\tilde{u}_j$ is an extension of u_j, $j = 1, 2, 3$, then $\tilde{u}_1\overline{\tilde{u}_2}\tilde{u}_3$ is an extension of $u_1\overline{u_2}u_3$. We use the previous step to get

$$\|u_1\overline{u_2}u_3\|_{X^{s,b'-1}(I)} \leq \left\|\tilde{u}_1\overline{\tilde{u}_2}\tilde{u}_3\right\|_{X^{s,b'-1}} \lesssim \prod_{j=1}^{3} \|\tilde{u}_j\|_{X^{s,b}}$$

and then we take infimum over all extensions $\tilde{u}_j$'s and (2.26) follows. □

Due to the scaling and Galilean symmetries of the linear Schrödinger equation, the periodic Strichartz estimate (2.10) of Bourgain and Demeter [13] is equivalent with

$$\|S(t)P_Q f\|_{L^p_{t,x}(I\times\mathbb{T}^d)} \lesssim_{|I|} |Q|^{\frac{1}{2}-\frac{d+2}{pd}+}\|f\|_{L^2_x(\mathbb{T}^d)}, \tag{2.27}$$

for any $d \geq 1$, $p \geq \frac{2(d+2)}{d}$, $I \subset \mathbb{R}$ finite time interval, and $Q \subset \mathbb{R}^d$ dyadic cube. Here, P_Q denotes the frequency projection onto Q, i.e. $\widehat{P_Q f}(n) = \mathbf{1}_Q(n)\widehat{f}(n)$. By the transference principle (Lemma 2.10), we get

$$\|P_Q u\|_{L^p_{t,x}(I\times\mathbb{T}^d)} \lesssim_{|I|} |Q|^{\frac{1}{2}-\frac{d+2}{pd}+}\|u\|_{X^{0,b}(I)}, \tag{2.28}$$

for any $b > \frac{1}{2}$. By interpolating (2.28) with

$$\|P_Q u\|_{L^p_{t,x}(I\times\mathbb{T}^d)} \lesssim |Q|^{\frac{1}{2}-\frac{1}{p}}\|u\|_{X^{0,\frac{1}{2}-\frac{1}{p}}(I)}, \tag{2.29}$$

(which follows immediately from Sobolev inequalities, (2.16), and the $H^s(\mathbb{T}^d)$-isometry of $S(-t)$), we can lower the time regularity from $b = \frac{1}{2}+\delta$ to $\tilde{b} = \frac{1}{2}-\delta$, for sufficiently small $\delta > 0$. Thus, we also have

$$\|P_Q u\|_{L^p_{t,x}(I\times\mathbb{T}^d)} \lesssim_{|I|,\delta} |Q|^{\frac{1}{2}-\frac{d+2}{pd}+o(\delta)}\|u\|_{X^{0,\frac{1}{2}-\delta}(I)} \tag{2.30}$$

Lemma 2.14 only treats the cubic nonlinearity when $d = 1$. We now prove the following general multilinear estimates to treat other cases. The proof borrows techniques from [50].

Lemma 2.15. Let $d, k \geq 1$ such that $dk \geq 2$ and let $I \subset \mathbb{R}$ be a finite time interval. Then for any $s > s_c$, there exist $b = \frac{1}{2}-$ and $b' = \frac{1}{2}+$ such that

$$\|u_1\overline{u_2}\cdots\overline{u_{2k}}u_{2k+1}\|_{X^{s,b'-1}(I)} \lesssim_{|I|} \prod_{j=1}^{2k+1}\|u_j\|_{X^{s,b}(I)}. \tag{2.31}$$

Proof. In view of (2.21), we can assume that $u_j(t) = \mathbb{1}_I(t)u_j(t)$ and thus by the duality relation (2.19), it suffices to show

$$\left|\int_{\mathbb{R}\times\mathbb{T}^d} \big(\langle\nabla\rangle^s(u_1\overline{u_2}\cdots u_{2k+1})\big)\overline{v}\,dxdt\right| \lesssim \|v\|_{X^{0,1-b'}}\prod_{j=1}^{2k+1}\|u_j\|_{X^{s,b}}. \tag{2.32}$$

We use Littlewood-Paley decomposition: we estimate the left-hand side of (2.32) when $v = P_N v$, $u_j = P_{N_j} u_j$ for some dyadic numbers $N, N_j \in 2^{\mathbb{Z}}$, $1 \le j \le 2k+1$. Then the claim follows by triangle inequality and performing the summation

$$\sum_{N_1} \sum_{\substack{N \\ N \lesssim N_1}} \sum_{\substack{N_2 \\ N_2 \le N_1}} \cdots \sum_{\substack{N_{2k+1} \\ N_{2k+1} \le N_{2k}}} . \tag{2.33}$$

Notice that without loss of generality, we may assume that $N_1 \ge N_2 \ge \ldots \ge N_{2k+1}$, in which case we also have $N \lesssim N_1$, and that the factors v and u_j are real-valued and non-negative.

Let $\varepsilon := s - s_{\mathrm{c}}$, and we distinguish two cases.

Case 1: $N_1 \sim N_2$. By Hölder inequality,

$$N^s \int_{\mathbb{R}\times\mathbb{T}^d} u_1 u_2 \cdots u_{2k+1} v \, dxdt \lesssim N_1^{\frac{s}{2}} \|u_1\|_{L^q_{t,x}} N_2^{\frac{s}{2}} \|u_2\|_{L^q_{t,x}} \prod_{j=3}^{2k+1} \|u_j\|_{L^p_{t,x}} \|v\|_{L^r_{t,x}}, \tag{2.34}$$

with p, q, r chosen such that $\frac{2k-1}{p} + \frac{2}{q} + \frac{1}{r} = 1$. We take p, q such that $\frac{d}{2} - \frac{d+2}{p} = s_{\mathrm{crit}}$ and $\frac{d}{2} - \frac{d+2}{q} = \frac{1}{2} s_{\mathrm{crit}}$, or equivalently $p = k(d+2)$ and $q = \frac{4k(d+2)}{dk+2}$. These give the Hölder exponent $r = \frac{2(d+2)}{d}$. By (2.30) and (2.28), we get

$$N_j^{\frac{s}{2}} \|u_j\|_{L^q_{t,x}} \lesssim N_j^{-\frac{\varepsilon}{2}+} \|u_j\|_{X^{s,b}}, \quad j = 1, 2 \tag{2.35}$$

$$\|u_j\|_{L^p_{t,x}} \lesssim N_j^{-\varepsilon+} \|u_j\|_{X^{s,b}}, \quad 3 \le j \le 2k+1, \tag{2.36}$$

$$\|v\|_{L^r_{t,x}} \lesssim N^{0+} \|v\|_{X^{0,1-b'}}. \tag{2.37}$$

By choosing $\delta, \delta' \ll \varepsilon$ in $b := \frac{1}{2} - \delta$ and in $1 - b' = \frac{1}{2} - \delta'$, respectively, we get

$$\text{RHS of (2.34)} \lesssim N^{-\frac{\varepsilon}{4}} \|v\|_{X^{0,1-b'}} \prod_{j=1}^{2k+1} N_j^{-\frac{\varepsilon}{4}} \|u_j\|_{X^{s,b}}. \tag{2.38}$$

The factors $N^{-\frac{\varepsilon}{4}}$, $N_j^{-\frac{\varepsilon}{4}}$ guarantee that we can perform (2.33).

Case 2: $N_1 \gg N_2$. Then, we necessarily have $N_1 \sim N$ or else the left hand side of

(2.32) vanishes. By Hölder inequality,

$$N^s \int_{\mathbb{R}\times\mathbb{T}^d} u_1 u_2 \cdots u_{2k+1} v \, dxdt \lesssim N_1^s \|u_1\|_{L^q_{t,x}} \prod_{j=2}^{2k+1} \|u_j\|_{L^p_{t,x}} \|v\|_{L^r_{t,x}}, \tag{2.39}$$

with $\frac{2k}{p}+\frac{1}{q}+\frac{1}{r}=1$. As in Case 1, we would like to have p such that $\frac{d}{2}-\frac{d+2}{p}=s_{\text{crit}}$, or equivalently $p=k(d+2)$. However, the best we can do with the Strichartz estimate for the remaining factors is to choose $q=r=\frac{2(d+2)}{d}$, so that we have

$$N_1^s \|u_1\|_{L^q_{t,x}} \lesssim N_1^{0+} \|u_1\|_{X^{s,b}}, \tag{2.40}$$

$$\|u_j\|_{L^p_{t,x}} \lesssim N_j^{-\varepsilon+} \|u_j\|_{X^{s,b}}, \quad 2 \leq j \leq 2k+1, \tag{2.41}$$

$$\|v\|_{L^r_{t,x}} \lesssim N_1^{0+} \|v\|_{X^{0,1-b'}}. \tag{2.42}$$

Notice that we can overcome the loss of derivative N_1^s only up to a logarithmic factor. We need a slightly refined analysis.

We cover the dyadic frequency annuli of u_1 and of v with dyadic cubes of side-length N_2, i.e.

$$\{\xi_1 : |\xi_1| \sim N_1\} \subset \bigcup_\ell Q_\ell \quad , \quad \{\xi : |\xi| \sim N\} \subset \bigcup_j R_j \, .$$

There are approximately $\left(\frac{N_1}{N_2}\right)^d$-many cubes needed, and so

$$u_1 = \sum_\ell P_{Q_\ell} u_1 =: \sum_\ell u_{1,\ell} \quad , \quad v = \sum_j P_{R_j} v =: \sum_j v_j$$

are decompositions into finitely many terms. Since $|\xi_1 - \xi| \lesssim N_2$ for $\xi_1 \in \text{supp}(\widehat{u_1}), \xi \in \text{supp}(\widehat{v})$ on the convolution hyperplane, there exists a constant K such that if $\text{dist}(Q_\ell, Q_j) > KN_2$, then the integral in (2.32) vanishes. Hence the summation (2.33) is replaced by

$$\sum_{N_1} \sum_{\substack{N_2 \\ N_2 \ll N_1}} \cdots \sum_{\substack{N_{2k+1} \\ N_{2k+1} \leq N_{2k}}} \sum_{\substack{\ell,j \\ j \approx \ell}} . \tag{2.43}$$

Also, in place of (2.40)-(2.41), we now have

$$N_1^s \|u_{1,\ell}\|_{L^q_{t,x}} \lesssim N_2^{0+} \|u_{1,\ell}\|_{X^{s,b}}, \tag{2.44}$$

$$\|u_i\|_{L^p_{t,x}} \lesssim N_i^{-\varepsilon+} \|u_i\|_{X^{s,b}}, \quad 2 \le i \le 2k+1, \tag{2.45}$$

$$\|v_j\|_{L^q_{t,x}} \lesssim N_2^{0+} \|v_j\|_{X^{0,1-b'}}, \tag{2.46}$$

Therefore, by Cauchy-Schwarz inequality and Plancherel identity,

$$\begin{aligned}
\text{LHS of (2.32)} &\lesssim \sum_{N_2} \sum_{\substack{N_1 \\ N_1 \gg N_2}} \sum_{\substack{\ell, j \\ \ell \approx j}} N_2^{-\varepsilon+} \|u_{1,\ell}\|_{X^{s,b}} \|v_j\|_{X^{0,1-b'}} \prod_{i=2}^{2k+1} \|u_i\|_{X^{s,b}} \\
&\lesssim \sum_{N_2} N_2^{-\varepsilon+} \Bigg(\sum_{\substack{N_1 \\ N_1 \gg N_2}} \sum_{\ell} \|u_{1,\ell}\|^2_{X^{s,b}} \Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{N \\ N \gg N_2}} \sum_{j} \|v_j\|^2_{X^{0,1-b'}} \Bigg)^{\frac{1}{2}} \prod_{i=2}^{2k+1} \|u_i\|_{X^{s,b}} \\
&\lesssim \sum_{N_2} N_2^{-\varepsilon+} \|u_1\|_{X^{s,b}} \|v\|_{X^{0,1-b'}} \prod_{i=2}^{2k+1} \|u_i\|_{X^{s,b}} \\
&\lesssim \prod_{i=1}^{2k+1} \|u_i\|_{X^{s,b}} \|v\|_{X^{0,1-b'}}
\end{aligned}$$

and the proof is complete. □

2.2 The stochastic convolution

In this section, we prove some $X^{s,b}$-estimates on the stochastic convolution $\Psi(t)$ given either by (2.7) or (2.8). We first record the following Burkholder-Davis-Gundy inequality, which is a consequence of [71, Theorem 1.1].

Lemma 2.16 (Burkholder-Davis-Gundy inequality)**.** Let H, K be separable Hilbert spaces, $T > 0$, and W is an H-valued Wiener process on $[0, T]$. Suppose that $\{\psi(t)\}_{t \in [0,T]}$ is a process taking values in $\mathrm{HS}(H; K)$. Then there is a modification of

ψ (which we continue to denote as ψ) such that for $p \geq 1$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\int_0^t \psi(t')\,dW(t')\right\|_K^p\right] \lesssim_p \mathbb{E}\left[\left(\int_0^T \|\psi(t')\|^2_{\mathrm{HS}(H;K)}\,dt'\right)^{\frac{p}{2}}\right].$$

In addition, we prove that $\Psi(t)$ is pathwise continuous in both cases. To this end, we employ the factorization method of Da Prato [31, Lemma 2.7], i.e. we make use of the following lemma and (2.49) below.

Lemma 2.17. Let H be a Hilbert space, $T > 0$, $\alpha \in (0,1)$, and $\sigma > \left(\frac{1}{\alpha}, \infty\right)$. Suppose that $f \in L^\sigma([0,T];H)$. Then the function

$$F(t) := \int_0^t S(t-t')(t-t')^{\alpha-1} f(t')\,dt', \quad t \in [0,T] \tag{2.47}$$

belongs to $C([0,T];H)$. Moreover,

$$\sup_{t\in[0,T]} \|F(t)\|_H \lesssim_{\sigma,T} \|f\|_{L^\sigma([0,T];H)}. \tag{2.48}$$

We make use of the above lemma in conjunction with the following fact:

$$\int_\mu^t (t-t')^{\alpha-1}(t'-\mu)^{-\alpha}\,dt' = \frac{\pi}{\sin(\pi\alpha)}, \tag{2.49}$$

for all $0 < \alpha < 1$ and all $0 \leq \mu < t$. This can be seen via considerations with Euler-Beta functions, see [31].

We now treat the additive and multiplicative cases separately below in Subsection 2.2.1 and 2.2.2 respectively. The arguments for the two cases are similar, albeit with some extra technicalities in the multiplicative case.

2.2.1 The additive stochastic convolution

By Fourier expansion, the stochastic convolution (2.7) for the additive noise problem can be written as

$$\Psi(t) = \sum_{n\in\mathbb{Z}^d} e_n \sum_{j\in\mathbb{Z}^d} \widehat{(\phi e_j)}(n) \int_0^t e^{i(t-t')|n|^2} d\beta_j(t'). \tag{2.50}$$

We first state the following $X^{s,b}$-estimate on Ψ:

Lemma 2.18. Let $s \geq 0$, $0 \leq b < \frac{1}{2}$, $T > 0$, and $\sigma \in [2, \infty)$. Let Ψ be given as in (2.50). Assume that $\phi \in \mathrm{HS}(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$. Then there is a modification of Ψ (which we continue to denote as Ψ) such that

$$\mathbb{E}\left[\|\Psi\|^{\sigma}_{X^{s,b}([0,T])}\right] \lesssim T^{\frac{\sigma}{2}}(1+T^2)^{\frac{\sigma}{2}}\|\phi\|^{\sigma}_{\mathrm{HS}(L^2(\mathbb{T}^d);H^s(\mathbb{T}^d))}\,. \tag{2.51}$$

Proof. Since $\mathbb{1}_{[0,T]}(t)\mathbb{1}_{[0,T]}(t') = \mathbb{1}_{[0,T]}(t) = 1$ whenever $0 \leq t' \leq t \leq T$, we have

$$\mathbb{1}_{[0,T]}(t)\Psi(t)(x) = \sum_{n\in\mathbb{Z}^d} e_n \sum_{j\in\mathbb{Z}^d} \widehat{\phi e_j}(n)\mathbb{1}_{[0,T]}(t)e^{it|n|^2}\int_0^t \mathbb{1}_{[0,T]}(t')e^{-it'|n|^2}d\beta_j(t')$$

By (2.21), we have

$$\begin{aligned}\|\Psi(t)\|_{X^{s,b}([0,T])} &\sim \left\|\mathbb{1}_{[0,T]}(t)\Psi(t)\right\|_{X^{s,b}}\\ &= \|\langle n\rangle^s\langle\tau\rangle^b\mathcal{F}_{t,x}\left(S(-t)\mathbb{1}_{[0,T]}(t)\Psi(t)\right)(\tau,n)\|_{L^2_\tau \mathrm{HS}_n}\\ &= \left\|\langle n\rangle^s\langle\tau\rangle^b\mathcal{F}_t\big[g_n(t)\big](\tau)\right\|_{L^2_\tau\ell^2_n}\,,\end{aligned} \tag{2.52}$$

where

$$g_n(t) := \sum_{j\in\mathbb{Z}^d}\mathbb{1}_{[0,T]}(t)\int_0^t \mathbb{1}_{[0,T]}(t')e^{-it'|n|^2}\widehat{\phi e_j}(n)d\beta_j(t')\,.$$

By the stochastic Fubini theorem (see [32, Theorem 4.33]), we have

$$\begin{aligned}\mathcal{F}_t[g_n(t)](\tau) &= \int_{\mathbb{R}} e^{-it\tau}g_n(t)dt\\ &= \sum_{j\in\mathbb{Z}^d}\int_{-\infty}^{\infty}\mathbb{1}_{[0,T]}(t')e^{-it'|n|^2}\widehat{\phi e_j}(n)\int_{t'}^{\infty}\mathbb{1}_{[0,T]}(t)e^{-it\tau}\,dt\,d\beta_j(t').\end{aligned}$$

Since

$$\left|\int_{t'}^{\infty}\mathbb{1}_{[0,T]}(t)e^{-it\tau}\,dt\right| \lesssim \min\{T, |\tau|^{-1}\}\,, \tag{2.53}$$

by Burkholder-Davis-Gundy inequality (Lemma 2.16), we get

$$\mathbb{E}\Big[|\mathcal{F}_t[g_n(t)](\tau)|^\sigma\Big] \lesssim \left[\int_0^T \sum_{j\in\mathbb{Z}^d} \left|\widehat{\phi e_j}(n)\int_{t'}^\infty \mathbb{1}_{[0,T]}(t)e^{-it\tau}\,dt\right|^2 dt'\right]^{\frac{\sigma}{2}} \\ \lesssim \left[T\sum_{j\in\mathbb{Z}^d}|\widehat{\phi e_j}(n)|^2\min\{T^2,|\tau|^{-2}\}\right]^{\frac{\sigma}{2}}. \tag{2.54}$$

By (2.52), (2.54), and Minkowski inequality, we get

$$\begin{aligned}\|\Psi\|_{L^\sigma(\Omega;X^{s,b}([0,T]))} &\leq \left(\sum_{n\in\mathbb{Z}^d}\int_{-\infty}^\infty \langle n\rangle^{2s}\langle\tau\rangle^{2b}\left(\mathbb{E}\left[|\mathcal{F}[g_n](\tau)|^\sigma\right]\right)^{\frac{2}{\sigma}}d\tau\right)^{\frac{1}{2}} \\ &\lesssim T^{\frac{1}{2}}\left(\sum_{n,j\in\mathbb{Z}^d}\langle n\rangle^{2s}|\widehat{\phi e_j}(n)|^2\int_{-\infty}^\infty \langle\tau\rangle^{2b}\min\{T^2,|\tau|^{-2}\}\,d\tau\right)^{\frac{1}{2}} \\ &\lesssim T^{\frac{1}{2}}\,\|\phi\|_{\mathrm{HS}(L^2;H^s)}\left(T^2\int_{|\tau|<1}d\tau+\int_{|\tau|\geq 1}\langle\tau\rangle^{2b-2}\,d\tau\right)^{\frac{1}{2}}.\end{aligned}$$

This completes the proof of Lemma 2.18. □

The next lemma infers that Ψ has a continuous modification taking values in $H^s(\mathbb{T}^d)$. This lemma is known and can be found in, for example, [52, Proposition 1.1]. Nonetheless, we provide a proof using the factorization method mentioned before.

Lemma 2.19 (Continuity of the additive noise)**.** Let $s \geq 0$, $T > 0$, and $2 \leq \sigma < \infty$. Assume that $\phi \in \mathrm{HS}(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$. Then $\Psi(\cdot)$ belongs to $C([0,T]; H^s(\mathbb{T}^d))$ almost surely and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\Psi(t)\|^\sigma_{H^s(\mathbb{T}^d)}\right] \lesssim_T \|\phi\|^\sigma_{\mathrm{HS}(L^2(\mathbb{T}^d);H^s(\mathbb{T}^d))}. \tag{2.55}$$

Proof. We fix $\alpha \in \big(0, \frac{1}{2}\big)$ and we write the stochastic convolution as follows:

$$\begin{aligned}\Psi(t) &= \frac{\sin(\pi\alpha)}{\pi}\int_0^t \left[\int_\mu^t (t-t')^{\alpha-1}(t'-\mu)^{-\alpha}\,dt'\right] S(t-\mu)\phi\, dW(\mu) \\ &= \frac{\sin(\pi\alpha)}{\pi}\int_0^t S(t-t')(t-t')^{\alpha-1}\int_0^{t'} S(t'-\mu)(t'-\mu)^{-\alpha}\phi\, dW(\mu)\, dt',\end{aligned} \tag{2.56}$$

where we used the stochastic Fubini theorem [32, Theorem 4.33] and the group property of $S(\cdot)$. By Lemma 2.17 and (2.56) it suffices to show that the process

$$f(t') := \int_0^{t'} S(t'-\mu)(t'-\mu)^{-\alpha}\phi\, dW(\mu)$$

satisfies

$$\mathbb{E}\bigg[\int_0^T \|f(t')\|_{H^s_x}^\sigma\, dt'\bigg] \le C\big(T, \sigma, \|\phi\|_{\mathrm{HS}(L^2;H^s)}\big) < \infty, \tag{2.57}$$

for some $\sigma > \frac{1}{\alpha}$.

By Burkholder-Davis-Gundy inequality (Lemma 2.16), for any $\sigma \ge 2$ and any $t' \in [0,T]$, we get

$$\begin{aligned}\mathbb{E}\Big[\|f(t')\|_{H^s_x}^\sigma\Big] &\lesssim \left(\int_0^{t'} \|S(t'-\mu)(t'-\mu)^{-\alpha}\phi\|_{\mathrm{HS}(L^2;H^s)}^2 d\mu\right)^{\frac{\sigma}{2}} \\ &= \left(\int_0^{t'} (t'-\mu)^{-2\alpha}\sum_{j\in\mathbb{Z}^d} \|S(t'-\mu)\phi e_j\|_{H^s}^2 d\mu\right)^{\frac{\sigma}{2}} \\ &\le \|\phi\|_{\mathrm{HS}(L^2;H^s)}^\sigma \left(\frac{T^{1-2\alpha}}{1-2\alpha}\right)^{\frac{\sigma}{2}},\end{aligned}$$

where in the last step we used $2\alpha \in (0,1)$ and the $H^s(\mathbb{T}^d)$-isometry property of $S(t'-\mu)$. Hence

$$\text{LHS of (2.57)} = \int_0^T \mathbb{E}\Big[\|f(t')\|_{H^s_x}^\sigma\Big]\, dt' \lesssim \|\phi\|_{\mathrm{HS}(L^2;H^s)}^\sigma T^{\frac{\sigma}{2}(1-2\alpha)+1} < \infty.$$

The estimate (2.55) follows from (2.48). □

2.2.2 The multiplicative stochastic convolution

The multiplicative stochastic convolution $\Psi = \Psi[u]$ from (2.8) can be written as

$$\Psi[u](t) = \sum_{n\in\mathbb{Z}^d} e_n \sum_{j\in\mathbb{Z}^d} \int_0^t e^{i(t-t')|n|^2} (\widehat{u(t')\phi e_j})(n) d\beta_j(t'). \tag{2.58}$$

Recall that if $s > \frac{d}{2}$, then we have access to the algebra property of $H^s(\mathbb{T}^d)$:

$$\|fg\|_{H^s(\mathbb{T}^d)} \lesssim \|f\|_{H^s(\mathbb{T}^d)} \|g\|_{H^s(\mathbb{T}^d)} \tag{2.59}$$

which is an easy consequence of the Cauchy-Schwarz inequality. This simple fact is useful for our analysis in the multiplicative case. On the other hand, (2.59) is not available to us for regularities below $\frac{d}{2}$, but we use the following inequalities.

Lemma 2.20. Let $0 < s \leq \frac{d}{2}$ and $1 \leq r < \frac{d}{d-s}$. Then

$$\|fu\|_{H^s(\mathbb{T}^d)} \lesssim \|f\|_{\mathcal{F}L^{s,r}(\mathbb{T}^d)} \|u\|_{H^s(\mathbb{T}^d)}. \tag{2.60}$$

Also, for $s = 0$, we have

$$\|fu\|_{L^2(\mathbb{T}^d)} \lesssim \|f\|_{\mathcal{F}L^{0,1}(\mathbb{T}^d)} \|u\|_{L^2(\mathbb{T}^d)}. \tag{2.61}$$

Proof. Assume that $0 < s \leq \frac{d}{2}$ and let n_1 and n_2 denote the spatial frequencies of f and u respectively. By separating the regions $\{|n_1| \gtrsim |n_2|\}$ and $\{|n_1| \ll |n_2|\}$, and then applying Young's inequality, we have

$$\begin{aligned}\|fu\|_{H^s(\mathbb{T}^d)} &\lesssim \left\| \big(\widehat{\langle\nabla\rangle^s f} * \widehat{u}\big)(n) \right\|_{\ell^2_n} + \left\| \big(\widehat{f} * \widehat{\langle\nabla\rangle^s u}\big)(n) \right\|_{\ell^2_n} \\ &\lesssim \|f\|_{\mathcal{F}L^{s,r}} \|\widehat{u}\|_{\ell^p} + \|\widehat{f}\|_{\ell^1} \|u\|_{H^s},\end{aligned}$$

where p is chosen such that $\frac{1}{r} + \frac{1}{p} = \frac{3}{2}$. By Hölder inequality, for r' and q such that $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{q} + \frac{1}{2} = \frac{1}{p}$,

$$\|\widehat{f}\|_{\ell^1} \lesssim \|\langle n\rangle^{-s}\|_{\ell^{r'}} \|f\|_{\mathcal{F}L^{s,r}},$$

$$\|\widehat{u}\|_{\ell^p} \lesssim \|\langle n\rangle^{-s}\|_{\ell^q} \|u\|_{H^s}.$$

Since $sr' > d$ and $sq > d$ provided that $r < \frac{d}{d-s}$, the conclusion (2.60) follows.

If $s = 0$, (2.61) follows easily from Young's inequality:

$$\|fu\|_{L^2(\mathbb{T}^d)} = \|\widehat{f} * \widehat{u}\|_{\ell^2} \lesssim \|\widehat{f}\|_{\ell^1} \|\widehat{u}\|_{\ell^2} = \|f\|_{\mathcal{F}L^{0,1}} \|u\|_{L^2}.$$

□

Given ϕ as in Theorem 2.5, let us denote

$$C(\phi) := \|\phi\|_{\mathrm{HS}(L^2(\mathbb{T}^d);\mathcal{F}L^{s,r}(\mathbb{T}^d))} < \infty \,, \tag{2.62}$$

for $r = 2$ when $s > \frac{d}{2}$, for some $r \in \left[1, \frac{d}{d-s}\right)$ when $0 < s \leq \frac{d}{2}$, and for $r = 1$ when $s = 0$. Recall that if ϕ is translation invariant, then it is sufficient to assume that $C(\phi) < \infty$ with $r = 2$, for all $s \geq 0$. We now proceed to prove the following $X^{s,b}$-estimate of $\Psi[u]$.

Lemma 2.21. Let $s \geq 0$, $0 \leq b < \frac{1}{2}$, $T > 0$, and $2 \leq \sigma < \infty$. Suppose that ϕ satisfies the assumptions of Theorem 2.5. Then, for $\Psi[u]$ given by (2.8) we have the estimate

$$\mathbb{E}\left[\|\Psi[u]\|^{\sigma}_{X^{s,b}([0,T])}\right] \lesssim (T^2+1)^{\frac{\sigma}{2}} C(\phi)^{\sigma}\, \mathbb{E}\left[\|u\|^{\sigma}_{L^2([0,T];H^s(\mathbb{T}^d))}\right] . \tag{2.63}$$

Proof. We first prove (2.63). Let $g(t) := \mathbb{1}_{[0,T]}(t) S(-t)\Psi(t)$. By the stochastic Fubini theorem [32, Theorem 4.33],

$$\begin{aligned} \mathcal{F}_{t,x}(g)(\tau,n) &= \int_{\mathbb{R}} e^{-it\tau} \mathbb{1}_{[0,T]}(t) \sum_{j \in \mathbb{Z}^d} \int_0^t e^{-it'n^2} (\widehat{u(t')\phi e_j})(n)\, d\beta_j(t')\, dt \\ &= \sum_{j\in\mathbb{Z}^d} \int_0^T \int_{t'}^{\infty} \mathbb{1}_{[0,T]}(t) e^{-it\tau} e^{-it'n^2} (\widehat{u(t')\phi e_j})(n)\, dt\, d\beta_j(t') \,. \end{aligned}$$

Then by (2.21) and the assumption $0 \leq b < \frac{1}{2}$, the Burkholder-Davis-Gundy inequality (Lemma 2.16), and (2.53), we have

$$\text{LHS of (2.63)} \sim \mathbb{E}\left[\left\|\langle n\rangle^s \langle\tau\rangle^b \mathcal{F}[g](n,\tau)\right\|^{\sigma}_{L^2_\tau \ell^2_n}\right]$$

$$\lesssim \mathbb{E}\left[\left(\sum_{j,n\in\mathbb{Z}^d}\int_{\mathbb{R}}\int_0^T \langle n\rangle^{2s}\langle\tau\rangle^{2b}\left|\int_{t'}^{\infty}\mathbb{1}_{[0,T]}(t)e^{-it\tau}\,dt\right|^2\left|(\widehat{u(t')\phi e_j})(n)\right|^2 dt'\,d\tau\right)^{\frac{\sigma}{2}}\right]$$

$$\lesssim (T^2+1)^{\frac{\sigma}{2}}\,\mathbb{E}\left[\left(\int_0^T\sum_{j,n\in\mathbb{Z}^d}\langle n\rangle^{2s}\left|(\widehat{u(t')\phi e_j})(n)\right|^2 dt'\right)^{\frac{\sigma}{2}}\right].$$

If $s>\frac{d}{2}$, we apply the algebra property of $H^s(\mathbb{T}^d)$ to get

$$\|u(t')\phi e_j\|_{\ell_j^2H^s}\lesssim\|\phi\|_{\mathrm{HS}(L^2;H^s)}\|u(t')\|_{H^s}.$$

If $0\le s\le\frac{d}{2}$, we have

$$\|u(t')\phi e_j\|_{\ell_j^2H^s}\lesssim C(\phi)\|u(t')\|_{H^s}. \tag{2.64}$$

and thus (2.63) follows. □

Next, we prove the continuity of $\Psi[u](t)$ in the same way as in Lemma 2.19, i.e. by using Lemma 2.17.

Lemma 2.22 (Continuity of the multiplicative noise)**.** Let $T>0$, $s\ge 0$, $0\le b<\frac{1}{2}$, and $2\le\sigma<\infty$. Suppose that $u\in L^\sigma\big(\Omega;X^{s,b}([0,T])\big)$ and that ϕ satisfies the assumptions of Theorem 2.5. Then $\Psi[u](\cdot)$ given by (2.58) belongs to $C([0,T];H^s(\mathbb{T}^d))$ almost surely. Moreover,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\Psi[u](t)\|^\sigma_{H^s(\mathbb{T}^d)}\right]\lesssim C(\phi)^\sigma\,\mathbb{E}\left[\|u\|^\sigma_{X^{s,b}([0,T])}\right]. \tag{2.65}$$

Proof. Applying the same factorisation procedure as in the proof of Lemma 2.19 reduces the problem to proving that the process

$$f(t'):=\int_0^{t'}(t'-\mu)^{-\alpha}S(t'-\mu)\big[u(\mu)\phi\big]\,dW(\mu)$$

satisfies

$$\mathbb{E}\left[\int_0^T\|f(t')\|^\sigma_{H^s_x}\,dt'\right]\le C'\left(T,\sigma,C(\phi)\right)<\infty \tag{2.66}$$

for some $0<\alpha<1$ satisfying $\alpha>\frac{1}{\sigma}$. By the Burkholder-Davis-Gundy inequality

(Lemma 2.16) and Lemma 2.20, we have

$$\begin{aligned}\mathbb{E}\left[\|f(t')\|_{H^s_x}^{\sigma}\right] &\lesssim \mathbb{E}\left[\left(\int_0^{t'} \|(t'-\mu)^{-\alpha}S(t'-\mu)[u(\mu)\phi]\|^2_{\mathrm{HS}(L^2;H^s)}d\mu\right)^{\frac{\sigma}{2}}\right]\\ &= \mathbb{E}\left[\left(\int_0^{t'} (t'-\mu)^{-2\alpha}\sum_{j\in\mathbb{Z}^d}\|S(t'-\mu)u(\mu)\phi e_j\|^2_{H^s}d\mu\right)^{\frac{\sigma}{2}}\right]\\ &\lesssim \mathbb{E}\left[\left(\sum_{j\in\mathbb{Z}^d}\|\phi e_j\|^2_{\mathcal{F}L^{s,r}}\int_0^T (t'-\mu)^{-2\alpha}\|u(\mu)\|^2_{H^s}d\mu\right)^{\frac{\sigma}{2}}\right].\end{aligned}$$

Then, by Fubini theorem and Minkowski inequality, we obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^T \|f(t')\|^{\sigma}_{H^s_x}dt'\right] &= \Big\|\,\|f\|_{H^s_x}\Big\|^{\sigma}_{L^{\sigma}(\Omega;L^{\sigma}_{t'}[0,T])}\\ &\lesssim C(\phi)^{\sigma}\Big\|\,\Big\|(t'-\mu)^{-\alpha}\|u(\mu)\|_{H^s_x}\Big\|_{L^2_\mu(0,T])}\Big\|^{\sigma}_{L^{\sigma}(\Omega;L^{\sigma}_{t'}[0,T])}\\ &\le C(\phi)^{\sigma}\,\mathbb{E}\left[\Big\|\,\Big\|(t'-\mu)^{-\alpha}\|u(\mu)\|_{H^s_x}\Big\|_{L^{\sigma}_{t'}(0,T])}\Big\|^{\sigma}_{L^2_\mu([0,T])}\right]\\ &\lesssim C(\phi)^{\sigma}\,\mathbb{E}\left[\left(\int_0^T (T-\mu)^{2(\frac{1}{\sigma}-\alpha)}\|u(\mu)\|^2_{H^s_x}d\mu\right)^{\frac{\sigma}{2}}\right]\end{aligned}$$

By Hölder and Sobolev inequalities and (2.21), we have

$$\begin{aligned}\left(\int_0^T (T-\mu)^{2(\frac{1}{\sigma}-\alpha)}\|u(\mu)\|^2_{H^s_x}d\mu\right)^{\frac{1}{2}} &\le \Big\|(T-\mu)^{\frac{1}{\sigma}-\alpha}\Big\|_{L_\mu^{\frac{4}{1+2b}}([0,T])}\Big\|\|u(\mu)\|_{H^s_x}\Big\|_{L_\mu^{\frac{4}{1-2b}}([0,T])}\\ &\lesssim T^{1+\frac{4}{1+2b}(\frac{1}{\sigma}-\alpha)}\Big\|\mathbb{1}_{[0,T]}(\mu)\|S(-\mu)u(\mu)\|_{H^s_x}\Big\|_{L_\mu^{\frac{4}{1-2b}}}.\end{aligned}$$

There exists $\alpha=\alpha(\sigma):=\frac{1}{\sigma}+\frac{1}{4}$ for which we have

$$\mathbb{E}\left[\int_0^T \|f(t')\|^{\sigma}_{H^s_x}dt'\right] \lesssim \mathbb{E}\Big[T^{\frac{2b\sigma}{1+2b}}\|u\|^{\sigma}_{X^{s,b}([0,T])}\Big] < \infty.$$

$\square$

2.3 Local well-posedness

2.3.1 SNLS with additive noise

In this subsection, we prove Theorem 2.1. Let $b = b(k) = \frac{1}{2}-$ be given by Lemma 2.14 (in the case $d = k = 1$) or by Lemma 2.15 (in the case $dk \geq 2$). By Lemma 2.18, for any $T > 0$, there is an event Ω' of full probability such that the stochastic convolution Ψ has finite $X^{s,b}([0,T])$-norm on Ω'.

Now fix $\omega \in \Omega'$ and $u_0 \in H^s(\mathbb{T}^d)$. Consider the ball

$$B_R := \left\{u \in X^{s,b}([0,T]) : \|u\|_{X^{s,b}([0,T])} \leq R\right\}$$

where $0 < T < 1$ and $R > 0$ are to be determined later. We aim to show that the operator Λ given by

$$\Lambda u(t) = S(t)u_0 \pm i\int_0^t S(t-t')\big(|u|^{2k}u\big)(t')dt' - i\Psi(t)\ ,\ t \geq 0,$$

where Ψ is the additive stochastic convolution given by (2.50), is a contraction on B_R. To this end, it remains to estimate the $X^{s,b}([0,T])$-norm of

$$D(u) := \int_0^t S(t-t')\big(|u|^{2k}u\big)(t')\,dt'\,.$$

For any $\delta > 0$ sufficiently small (such that $b + \delta < \frac{1}{2}$), by Lemma 2.13 and (2.21):

$$\|D(u)\|_{X^{s,b}([0,T])} \lesssim T^\delta \|D(u)\|_{X^{s,b+\delta}([0,T])} \lesssim T^\delta \left\|\mathbb{1}_{[0,T]}(t)D(u)(t)\right\|_{X^{s,\frac{1}{2}+\delta}}.$$

Let η be a smooth cut-off function, supported on $[-1, T+1]$, with $\eta(t) = 1$ for all $t \in [0,T]$. For any $w \in X^{s,-\frac{1}{2}+\delta}$ that agrees with $|u|^{2k}u$ on $[0,T]$, by Lemma 2.12, we obtain

$$\left\|\mathbb{1}_{[0,T]}(t)D(u)(t)\right\|_{X^{s,\frac{1}{2}+\delta}} \lesssim \left\|\eta(t)\int_0^t S(t-t')w(t')dt'\right\|_{X^{s,\frac{1}{2}+\delta}} \lesssim \|w\|_{X^{s,-\frac{1}{2}+\delta}} \tag{2.67}$$

Then after taking the infimum over all such w, we use Lemma 2.14 or 2.15 and we get

$$\|D(u)\|_{X^{s,b}([0,T])} \lesssim T^\delta \|(u\overline{u})^k u\|_{X^{s,-\frac{1}{2}+\delta}([0,T])} \lesssim T^\delta \|u\|^{2k+1}_{X^{s,b}([0,T])} . \tag{2.68}$$

It follows that

$$\|\Lambda u\|_{X^{s,b}([0,T])} \leq c\|u_0\|_{H^s_x} + cT^\delta \|u\|^{2k+1}_{X^{s,b}([0,T])} + \|\Psi(t)\|_{X^{s,b}([0,T])} , \tag{2.69}$$

for some $c > 0$. Similarly, we obtain

$$\|\Lambda u - \Lambda v\|_{X^{s,b}([0,T])} \leq cT^\delta \left(\|u\|^{2k}_{X^{s,b}([0,T])} + \|v\|^{2k}_{X^{s,b}([0,T])}\right) \|u-v\|_{X^{s,b}([0,T])} . \tag{2.70}$$

Let $R := 2c\|u_0\|_{H^s_x} + 2\|\Psi(t)\|_{X^{s,b}([0,T])}$. From (2.69) and (2.70), we see that Λ is a contraction from B_R to B_R provided

$$cT^\delta R^{2k+1} \leq \frac{1}{2}R \ \text{ and } \ cT^\delta \left(2R^{2k}\right) \leq \frac{1}{2} . \tag{2.71}$$

This is always possible if we choose $T \ll 1$ sufficiently small. This shows the existence of a unique solution $u \in X^{s,b}([0,T])$ to (2.6) on Ω'.

Finally, we check that $u \in C([0,T];H^s)$ on the set of full probability $\Omega'' \cap \Omega'$, where Ω'' is given by Lemma 2.19, that is $\Psi \in C([0,T];H^s)$ on Ω''. By (2.21), (2.67) and Lemma 2.14 or 2.15, we also get

$$\|D(u)\|_{X^{s,\frac{1}{2}+\delta}([0,T])} \lesssim \left\|\mathbb{1}_{[0,T]}(t)D(u)(t)\right\|_{X^{s,\frac{1}{2}+\delta}} \lesssim \|u\|^{2k+1}_{X^{s,b}([0,T])} . \tag{2.72}$$

By the embedding $X^{s,\frac{1}{2}+\delta}([0,T]) \hookrightarrow C([0,T];H^s(\mathbb{T}^d))$, we have $D(u) \in C([0,T];H^s(\mathbb{T}^d))$. Since the linear term $S(t)u_0$ also belongs to $C([0,T];H^s(\mathbb{T}^d))$, we conclude that

$$u = \Lambda u \in C\big([0,T];H^s(\mathbb{T}^d)\big) \text{ on } \Omega'' \cap \Omega'.$$

Remark 2.23. From (2.71), we obtain the time of existence

$$T_{\max} := \max\left\{\tilde{T} > 0 : \tilde{T} \leq c\Big(\|u_0\|_{H^s} + \|\Psi\|_{X^{s,b}([0,\tilde{T}])}\Big)^{-\theta}\right\}, \tag{2.73}$$

where $\theta = \frac{2k}{\delta}$. Note that (2.73) will be useful in our global argument.

2.3.2 SNLS with multiplicative noise

In this subsection, we prove Theorem 2.5. Following [34], we use a truncated version of (2.6). The main idea is to apply an appropriate cut-off function on the nonlinearity to obtain a family of truncated SNLS, and then prove global well-posedness of these truncated equations. Since solutions started with the same initial data coincide up to suitable stopping times, we obtain a solution to the original SNLS in the limit.

Let $\eta : \mathbb{R} \to [0,1]$ be a smooth cut-off function such that $\eta \equiv 1$ on $[0,1]$ and $\eta \equiv 0$ outside $[-1,2]$. Set $\eta_R := \eta\left(\frac{\cdot}{R}\right)$ and consider the equation

$$i\partial_t u_R - \Delta u_R \pm \eta_R\big(\|u_R\|_{X^{s,b}([0,t])} \big)^{2k+1} |u_R|^{2k} u_R = u_R \cdot \phi\xi \,, \tag{2.74}$$

with initial data $u_R|_{t=0} = u_0$.

Remark 2.24. As in the case for the original multiplicative SNLS, we say that u_R is a solution to (2.74) if it is an adapted process in $H^s(\mathbb{T}^d)$ that is continuous in time, such that $u_R = \Lambda_R u_R$ almost surely, where Λ_R is given by

$$\Lambda_R u_R := S(t)u_0 \pm i \int_0^t S(t-t')\eta_R\left(\|u_R\|_{X^{s,b}([0,t'])}\right)^{2k+1} |u_R|^{2k} u_R(t')\, dt' - i\Psi[u_R](t)\,. \tag{2.75}$$

The key ingredient for Theorem 2.5 is the following proposition.

Proposition 2.25 (Global well-posedness for (2.74))**.** Let $s > s_{\text{crit}}$, $s \geq 0$, and $T, R > 0$. Suppose that ϕ is as in Theorem 2.5. Given $u_0 \in H^s(\mathbb{T}^d)$, there exists $b \in [0, \frac{1}{2})$ and a unique process

$$u_R \in L^2\Big(\Omega; C\big([0,T]; H^s(\mathbb{T}^d)\big) \cap X^{s,b}([0,T])\Big)$$

solving (2.74) on $[0,T]$.

Before proving this result, we state and prove the following lemma.

Lemma 2.26 (Boundedness of cut-off)**.** Let $s \geq 0$, $b \in [0, \frac{1}{2})$, $R > 0$ and $T > 0$. There exist constants $C_1, C_2(R) > 0$ such that

$$\left\|\eta_R\left(\|u\|_{X^{s,b}([0,t])}\right) u(t)\right\|_{X^{s,b}([0,T])} \leq \min\left\{C_1 \|u\|_{X^{s,b}([0,T])}\,, C_2(R)\right\}\,; \tag{2.76}$$

$$\left\|\eta_R\left(\|u\|_{X^{s,b}([0,t])}\right)u(t)-\eta_R\left(\|v\|_{X^{s,b}([0,t])}\right)v(t)\right\|_{X^{s,b}([0,T])}\leq C_2(R)\,\|u-v\|_{X^{s,b}([0,T])}\,. \tag{2.77}$$

Proof. We first prove (2.76). Let $w(t,n)=\mathcal{F}_x[S(-t)u(t)](n)$, $\kappa_R(t)=\eta_R\left(\|u\|_{X^{s,b}([0,t])}\right)$ and

$$\tau_R:=\inf\left\{t\geq 0:\|u\|_{X^{s,b}([0,t])}\geq 2R\right\}\,. \tag{2.78}$$

Then $\kappa_R(t)=0$ when $t>\tau_R$. By (2.21) and (2.16),

$$\begin{aligned}\|\kappa_R(t)u(t)\|^2_{X^{s,b}([0,T])}&\sim\left\|\mathbb{1}_{[0,T\wedge\tau_R]}\kappa_R(t)u(t)\right\|^2_{X^{s,b}}\sim\|\kappa_R(t)u(t)\|^2_{X^{s,b}([0,T\wedge\tau_R])}\\&\sim\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2s}\,\|\kappa_R(t)w(t,n)\|^2_{H^b(0,T\wedge\tau_R)}\,.\end{aligned} \tag{2.79}$$

We now estimate the $H^b(0,T\wedge\tau_R)$-norm, for which we use the following characterization (see for example [81]):

$$\|f\|^2_{H^b(a_1,a_2)}\sim\|f\|^2_{L^2(a_1,a_2)}+\int_{a_1}^{a_2}\int_{a_1}^{a_2}\frac{|f(x)-f(y)|^2}{|x-y|^{1+2b}}\,dx\,dy\,,\quad 0<b<1. \tag{2.80}$$

For the inhomogeneous contribution (i.e. coming from the L^2-norm above), we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}^d}\langle n\rangle^{2s}\,\|\kappa_R(t)w(t,n)\|^2_{L^2_t(0,T\wedge\tau_R)}&\leq\min\left\{\|u\|^2_{X^{s,b}([0,\tau_R])}\,,\|u\|^2_{X^{s,b}([0,T])}\right\}\\&\leq\min\left\{(2R)^2\,,\|u\|^2_{X^{s,b}([0,T])}\right\}.\end{aligned}$$

The remaining part of (2.79) needs a bit more work. Fix $n\in\mathbb{Z}^d$, then

$$\begin{aligned}&\int_0^{T\wedge\tau_R}\int_0^{T\wedge\tau_R}\frac{|\kappa_R(t)w(t,n)-\kappa_R(t')w(t',n)|^2}{|t-t'|^{1+2b}}\,dt'\,dt\\&\lesssim\int_0^{T\wedge\tau_R}\int_0^t\frac{|\kappa_R(t)(w(t,n)-w(t',n))|^2}{|t-t'|^{1+2b}}\,dt'\,dt\\&\qquad+\int_0^{T\wedge\tau_R}\int_0^t\frac{|(\kappa_R(t)-\kappa_R(t'))w(t',n)|^2}{|t-t'|^{1+2b}}\,dt'\,dt\\&=:\mathrm{I}(n)+\mathrm{II}(n)\,.\end{aligned}$$

It is clear that

$$\mathrm{I}(n) \lesssim \min\left\{ \|w(n)\|^2_{H^b((0,\tau_R))} \, , \|w(n)\|^2_{H^b((0,T))} \right\} ,$$

and hence

$$\sum_{n\in\mathbb{Z}^d} \mathrm{I}(n) \lesssim \min\left\{ (2R)^2 \, , \|u\|^2_{X^{s,b}([0,T])} \right\} .$$

For $\mathrm{II}(n)$, the mean value theorem infers that

$$\begin{aligned}
|\kappa_R(t) - \kappa_R(t')|^2 &\lesssim \frac{\Big(\|u\|_{X^{s,b}([0,t])} - \|u\|_{X^{s,b}([0,t'])} \Big)^2}{R^2} \left(\sup_{r\in\mathbb{R}} \eta'(r) \right)^2 \\
&\lesssim \frac{\big\| \mathbb{1}_{[t',t]} u \big\|^2_{X^{s,b}}}{R^2} \\
&\lesssim \frac{1}{R^2} \sum_{n'\in\mathbb{Z}^d} \langle n' \rangle^{2s} \|w(\cdot, n')\|^2_{H^b(t',t)} .
\end{aligned}$$

Again, we split $\|w(\cdot, n')\|^2_{H^b(t',t)}$ using (2.80) into the inhomogeneous contribution (the L^2-norm squared part) and the homogeneous contribution (the second term of (2.80)). We control here only the homogeneous contributions for $\mathrm{II}(n)$ as the inhomogeneous contributions are easier. The homogeneous part of $\mathrm{II}(n)$ is controlled by

$$\frac{1}{R^2} \sum_{n'\in\mathbb{Z}^d} \langle n' \rangle^{2s} \int_0^{T\wedge\tau_R} \int_0^t \int_{t'}^t \int_{t'}^{\lambda} \frac{|w(t',n)|^2}{|t-t'|^{1+2b}} \cdot \frac{|w(\lambda,n') - w(\lambda',n')|^2}{|\lambda-\lambda'|^{1+2b}} \, d\lambda' \, d\lambda \, dt' \, dt \tag{2.81}$$

$$\begin{aligned}
= \frac{1}{R^2} \sum_{n'\in\mathbb{Z}^d} \langle n' \rangle^{2s} \int_0^{T\wedge\tau_R} \int_0^{\lambda} \int_0^{\lambda'} \left(\int_{\lambda}^{T\wedge\tau_R} \frac{1}{|t-t'|^{1+2b}} \, dt \right) |w(t',n)|^2 \\
\times \frac{|w(\lambda,n') - w(\lambda',n')|^2}{|\lambda-\lambda'|^{1+2b}} \, dt' \, d\lambda' \, d\lambda ,
\end{aligned} \tag{2.82}$$

where we used $0 \le t' \le \lambda' \le \lambda \le t \le T \wedge \tau_R$ to switch the integrals. Now, the integral with respect to t is equal to $|T\wedge\tau_R - t'|^{-2b} - |\lambda - t'|^{-2b}$, which is bounded by

$$|T\wedge\tau_R - t'|^{-2b} \le |\lambda' - t'|^{-2b} .$$

Thus (2.82) is controlled by

$$\frac{1}{R^2}\sum_{n'\in\mathbb{Z}^d}\langle n'\rangle^{2s}\int_0^{T\wedge\tau_R}\int_0^{\lambda}\left(\int_0^{\lambda'}|\lambda'-t'|^{-2b}|w(t',n)|^2\,dt'\right)$$
$$\times\frac{|w(\lambda,n')-w(\lambda',n')|^2}{|\lambda-\lambda'|^{1+2b}}\,d\lambda'\,d\lambda\,. \tag{2.83}$$

Since $b\in\left[0,\frac{1}{2}\right)$, by Hardy's inequality (see for example [82, Lemma A.2]) the t'-integral is $\lesssim\|w(\cdot,n)\|^2_{H^b(0,\lambda')}\leq\|w(\cdot,n)\|^2_{H^b(0,T\wedge\tau_R)}$. After multiplying by $\langle n\rangle^{2s}$ and summing over $n\in\mathbb{Z}^d$, we see that (2.83) is controlled by

$$\begin{aligned}&\frac{1}{R^2}\sum_{n,n'\in\mathbb{Z}^d}\langle n\rangle^{2s}\langle n'\rangle^{2s}\,\|w(\cdot,n)\|^2_{H^b(0,T\wedge\tau_R)}\,\|w(\cdot,n)\|^2_{H^b_\lambda(0,T\wedge\tau_R)}\\&\lesssim\frac{1}{R^2}\,\|u\|^2_{X^{s,b}([0,T\wedge\tau_R])}\,\|u\|^2_{X^{s,b}([0,T\wedge\tau_R])}\\&\leq\min\left\{4\,\|u\|^2_{X^{s,b}([0,T])}\,,16R^2\right\}\,.\end{aligned}$$

We now prove (2.77). Let τ_R^u and τ_R^v be defined as in (2.78). Assume without loss of generality that $\tau_R^u\leq\tau_R^v$. We decompose

$$\begin{aligned}\text{LHS of (2.77)}\lesssim&\left\|\left(\eta_R\left(\|u\|_{X^{s,b}([0,t])}\right)-\eta_R\left(\|v\|_{X^{s,b}([0,t])}\right)\right)v(t)\right\|_{X^{s,b}([0,T])}\\&+\left\|\eta_R\left(\|u\|_{X^{s,b}([0,t])}\right)(u(t)-v(t))\right\|_{X^{s,b}([0,T])}\\=:&\,A+B\,.\end{aligned}$$

By the mean value theorem,

$$\begin{aligned}A&=\left\|\left(\eta_R\left(\|u\|_{X^{s,b}([0,t])}\right)-\eta_R\left(\|v\|_{X^{s,b}([0,t])}\right)\right)v(t)\right\|_{X^{s,b}([0,T\wedge\tau_R^v])}\\&\lesssim\frac{1}{R}\,\|v\|_{X^{s,b}([0,T\wedge\tau_R^v])}\,\|u-v\|_{X^{s,b}([0,T])}\\&\lesssim\|u-v\|_{X^{s,b}([0,T])}\,.\end{aligned}$$

For B, one runs through the same argument as for (2.76) but with $w(t,n)$ replaced by $\mathcal{F}_x\big[S(-t)\big(u(t)-v(t)\big)\big](n)$, which yields

$$B \lesssim C(R)\,\|u-v\|_{X^{s,b}([0,T])}\,.$$

□

We now conclude the proof of Proposition 2.25.

Proof of Proposition 2.25. Let $T, R > 0$. In view of the mild formulation (2.75), we consider the following Picard iteration: for $t \in [0,T]$, define

$$u_1(t) := S(t)u_0;$$

$$u_n(t) := \Lambda_R u_{n-1}(t) \qquad \forall n \geq 2.$$

To see that $\{u_n\}_{n\in\mathbb{N}}$ is well-defined, we note that the stochastic convolution $\Psi[u]$ is defined provided u is an adapted process in $H^s(\mathbb{T}^d)$. Clearly, $u_1(t)$ is adapted, and hence all terms of $u_2(t)$ is also adapted. By induction, we see that each $u_n(t)$ is also adapted and hence u_{n+1} is well-defined. To see that u_n converges to a fixed point of Λ_R, it suffices to prove that Λ_R is a contraction in some complete metric space containing $\{u_n\}_{n\in\mathbb{N}}$. To this end, we let $E_T := L^2\big(\Omega; X^{s,b}([0,T])\big)$. Arguing as in the additive case, and using Lemmata 2.26 and 2.21, we have for any $u \in E_T$ that is an adapted process in $H^s(\mathbb{T}^d)$, we have

$$\|\Lambda_R u\|_{E_T} \leq C_1\,\|u_0\|_{H^s} + C_2(R)T^\delta + C_3T^b\,\|u\|_{E_T}\,; \tag{2.84}$$

$$\|\Lambda_R u - \Lambda_R v\|_{E_T} \leq C_4(R)T^\delta\,\|u-v\|_{E_T} + C_5T^b\,\|u-v\|_{E_T}\,. \tag{2.85}$$

The estimate (2.84) above shows that Λ_R maps E_T to E_T. Moreover, since $u_1 \in E_T$ (by Lemma 2.12), (2.84) also infers (iteratively) that $u_n \in E_T$ for each n. By (2.85), Λ_R is a contraction from E_T to E_T provided we choose $T = T(R)$ sufficiently small. Thus $\{u_n\}_{n\in\mathbb{N}}$ is Cauchy in E_T and so it admits a limit $u_R \in E_T$ that is a fixed point of Λ_R. Note that T does not depend on $\|u_0\|_{H^s}$, hence we may iterate this argument to extend $u_R(t)$ to all $t \in [0,\infty)$.

Finally, to see that $u_R \in F_T := L^2\big(\Omega; C([0,T]; H^s(\mathbb{T}^d))\big)$, we first note that since

$u_R \in E_T$, Lemma 2.22 infers that $\Psi[u_R] \in F_T$. Then, by similar argument as in the end of Subsection 2.3.1, we have that $D(u_R) \in L^2(\Omega; X^{s,b}([0,T]))$, where

$$D(u_R)(t) := \int_0^t S(t-t')\big(|u_R|^{2k}u_R\big)\,dt'\,.$$

Since $L^2\big(\Omega; X^{s,\tilde{b}}([0,T])\big) \hookrightarrow F_T$ for any $\tilde{b} > \frac{1}{2}$, we have $D(u_R) \in F_T$. Also, it is clear that $S(t)u_0 \in F_T$. Hence $u_R \in F_T$. □

Proof of Theorem 2.5. Let

$$\tau_R := \inf\big\{t > 0 : \|u_R\|_{X^{s,b}([0,t])} \geq R\big\}. \tag{2.86}$$

Then, $\eta_R(\|u_R\|_{X^{s,b}([0,t])}) = 1$ if and only if $t \leq \tau_R$. Hence u_R is a solution of (2.6) on $[0,\tau_R]$. For any $\delta > 0$, we have $u_R(t) = u_{R+\delta}(t)$ whenever $t \in [0,\tau_R]$. Consequently, τ_R is increasing in R. Indeed, if $\tau_R > \tau_{R+\delta}$ for some $R > 0$ and some $\delta > 0$, then for $t \in [\tau_{R+\delta}, \tau_R]$, we have $\eta_{R+\delta}\big(\,\|u_{R+\delta}\|_{X^{s,b}([0,t])}\big) < 1$ which implies that $u_R(t) \neq u_{R+\delta}(t)$, a contradiction. Therefore,

$$\tau^* := \lim_{R\to\infty} \tau_R \tag{2.87}$$

is a well-defined stopping time that is either positive or infinite almost surely. By defining $u(t) := u_R(t)$ for each $t \in [0,\tau_R]$, we see that u is a solution of (2.6) on $[0,\tau^*)$ almost surely. □

2.4 Global well-posedness

In this section, we prove Theorems 2.4 and 2.7. Recall that the *mass* and *energy* of a solution $u(t)$ of the defocusing (2.1) are given respectively by

$$M(u(t)) = \int_{\mathbb{T}^d} \frac{1}{2}|u(t,x)|^2\,dx, \tag{2.88}$$

$$E(u(t)) = \int_{\mathbb{T}^d} \frac{1}{2}|\nabla u(t,x)|^2 + \frac{1}{2(k+1)}|u(t,x)|^{2(k+1)}dx. \tag{2.89}$$

It is well-known that these are conserved quantities for (smooth enough) solutions of the deterministic NLS equation.

For SNLS, we prove probabilistic a priori control as per Propositions 2.27 and 2.29 below. To this purpose, the idea is to compute the stochastic differentials of (2.88) and (2.89) and use the stochastic equation for u. We shall work with the following frequency truncated version of (2.1):

$$\begin{cases} i\partial_t u^N - \Delta u^N \pm P_{\leq N}|u^N|^{2k}u^N = F(u^N, \phi^N dW^N), \\ u^N|_{t=0} = P_{\leq N}u_0 =: u_0^N \end{cases} \tag{2.90}$$

where $P_{\leq N}$ is the projection onto the frequency set $\{n \in \mathbb{Z}^d : |n| \leq N\}$, that is,

$$\widehat{P_{\leq N} f}(n) = \sum_{|n|\leq N} \widehat{f}(n),$$

and that

$$\phi^N := P_{\leq N} \circ \phi \quad \text{and} \quad W^N(t) := \sum_{|n|\leq N} \beta_n(t)e_n.$$

By repeating the arguments in Section 2.3, one obtains local well-posedness for (2.90) with initial data $P_{\leq N}u_0$ at least with the same time of existence as for the untruncated SNLS.

2.4.1 SNLS with additive noise

We treat the additive SNLS in this subsection. We first prove probabilistic a priori bounds on (2.88) and (2.89) of a solution u^N of the truncated equation.

Proposition 2.27. Let $s \geq 0$. Let $m \in \mathbb{N}$, $T_0 > 0$, $u_0 \in H^s(\mathbb{T}^d)$ and $\phi \in \text{HS}(L^2(\mathbb{T}^d); H^s(\mathbb{T}^d))$, and $F(u, \phi\xi) = \phi\xi$. Suppose that $u^N(t)$ is a solution to (2.90) for $t \in [0, T]$, for some stopping time $T \in [0, T_0]$. Then there exists a constant $C_1 = C_1(m, M(u_0), T_0, \|\phi\|_{\text{HS}(L^2;L^2)}) > 0$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} M(u^N(t))^m\right] \leq C_1\,. \tag{2.91}$$

Furthermore, if $s \geq 1$ and (2.90) is defocusing, there exists a constant $C_2 =$

$C_2(m, E(u_0), T_0, \|\phi\|_{\mathrm{HS}(L^2;H^1)}) > 0$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T} E(u^N(t))^m\right] \le C_2\,. \tag{2.92}$$

The constants C_1 and C_2 are independent of N.

Proof. We first claim that

$$M(u^N(t))^m = M(u_0^N)^m$$
$$+ m\ \mathrm{Im}\left(\sum_{|j|\le N}\int_0^t M(u^N(t'))^{m-1}\int_{\mathbb{T}^d}\overline{u^N(t')}\phi^N e_j\,dx\,d\beta_j(t')\right) \tag{2.93}$$
$$+ m(m-1)\sum_{|j|\le N}\int_0^t M(u^N(t'))^{m-2}\left|\int_{\mathbb{T}^d}u^N(t')\phi^N e_j\,dx\right|^2 dt' \tag{2.94}$$
$$+ m\left\|\phi^N\right\|^2_{\mathrm{HS}(L^2;L^2)}\int_0^t M(u^N(t'))^{m-1}\,dt'. \tag{2.95}$$

To see this, let $a_n^N := \mathrm{Re}\left(\widehat{u^N}(n)\right)$ and $b_n^N := \mathrm{Im}\left(\widehat{u^N}(n)\right)$. In view of (2.90), we have

$$da_n^N = -\Big(|n|^2 b_n^N \pm \mathrm{Im}\left\{\mathcal{F}_x\big(|u^N|^{2k}u^N\big)(n)\right\}\Big)\,dt + \sum_{|j|\le N}\mathrm{Im}\,\widehat{\phi e_j}(n)\,d\beta_j^{(i)},$$

$$db_n^N = \Big(|n|^2 a_n^N \pm \mathrm{Re}\left\{\mathcal{F}_x\big(|u^N|^{2k}u^N\big)(n)\right\}\Big)\,dt - \sum_{|j|\le N}\mathrm{Re}\,\widehat{\phi e_j}(n)\,d\beta_j^{(r)},$$

where $\beta_j^{(r)} = \mathrm{Re}\,\beta_j$ and $\beta_j^{(i)} = \mathrm{Im}\,\beta_j$. By applying Itô's Lemma to the above expressions to obtain $(a_n^N)^2$ and $(b_n^N)^2$, and then summing them, we obtain

$$|\widehat{u}(n,t)|^2 = 2\,\mathrm{Im}\sum_{|j|\le N}\int_0^t \overline{\widehat{u}(n,t')}\widehat{\phi e_j}(n)\,d\beta_j(t') + 2t\sum_{|j|\le N}|\widehat{\phi e_j}(n)|^2.$$

We sum over $|n| \le N$ and then apply Plancherel and Parseval theorems to yield (2.93)–(2.95) for $m = 1$. A further application of Itô's Lemma yields (2.93)–(2.95) for general m.

We now control (2.93). By Burkholder-Davis-Gundy inequality (Lemma 2.16), Hölder and Young inequalities, we get

$$\mathbb{E}\left[\sup_{t\in[0,T]}(2.93)\right] \lesssim_m \mathbb{E}\left[\left\{\sum_{|j|\le N}\int_0^T M(u^N(t'))^{2(m-1)}\|u^N(t')\|_{L^2}^2\|\phi^N e_j\|_{L^2}^2 dt'\right\}^{\frac{1}{2}}\right]$$

$$\lesssim \|\phi^N\|_{\mathrm{HS}(L^2;L^2)}\,\mathbb{E}\left[\left\{\int_0^T M(u^N(t))^{2m-1}dt\right\}^{\frac{1}{2}}\right]$$

$$\lesssim \|\phi\|_{\mathrm{HS}(L^2;L^2)}T^{\frac{1}{2}}\,\mathbb{E}\left[\left\{\sup_{t\in[0,T]}M(u^N(t))^{m-1}\right\}^{\frac{1}{2}}\left\{\sup_{t\in[0,T]}M(u^N(t))^{m}\right\}^{\frac{1}{2}}\right]$$

$$\lesssim \|\phi\|_{\mathrm{HS}(L^2;L^2)}T_0^{\frac{1}{2}}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}M(u^N(t))^{m-1}\right]\right\}^{\frac{1}{2}}\left\{\mathbb{E}\left[\sup_{t\in[0,T]}M(u^N(t))^{m}\right]\right\}^{\frac{1}{2}}$$

Hence by Young's inequality, we infer that

$$\mathbb{E}\left[\sup_{t\in[0,T]}(2.93)\right] \le C_m\|\phi\|_{\mathrm{HS}(L^2;L^2)}^2 T_0\,\mathbb{E}\left[\sup_{t\in[0,T]}M(u^N(t))^{m-1}\right] + \frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]}M(u^N(t))^{m}\right].$$

In a straightforward way, we also have

$$\mathbb{E}\left[\sup_{t\in[0,T]}(2.94)\right] \le m(m-1)\|\phi\|_{\mathrm{HS}(L^2;L^2)}^2 T_0\,\mathbb{E}\left[\sup_{t\in[0,T]}M(u^N(t))^{m-1}\right],$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}(2.95)\right] \le 2m\|\phi\|_{\mathrm{HS}(L^2;L^2)}^2 T_0\,\mathbb{E}\left[\sup_{t\in[0,T]}M(u^N(t))^{m-1}\right].$$

Therefore, there is some $C_m > 0$ such that

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}M(u^N(t))^{m}\right] &\le M(u_0)^m + C_m T_0\,\mathbb{E}\left[\sup_{t\in[0,T]}M(u^N(t))^{m-1}\right]\\ &\quad+\frac{1}{2}\mathbb{E}\left[\sup_{t\in[0,T]}M(u^N(t))^{m}\right].\end{aligned} \tag{2.96}$$

We now wish to move the last term of (2.96) to the left-hand side. However, we do not know a priori that the moments of $\sup_{t\in[0,T]}M(u^N(t))$ are finite. To justify this, we note that (2.96) holds with T replaced by T_R, where

$$T_R := \sup\left\{t \in [0,T] : M(u^N(t)) \leq R\right\}, \quad R > 0.$$

Now the terms that would be appearing in (2.96) are finite and hence the formal manipulation is justified. Note that $T_R \to T$ almost surely as $R \to \infty$ because u (and hence u^N) belongs in $C([0,T]; H^s(\mathbb{T}^d))$ almost surely. Hence by letting $R \to \infty$ and invoking the monotone convergence theorem, one finds

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(u^N(t))^m\right] \leq 2M(u_0)^m + 2C_m T_0\, \mathbb{E}\left[\sup_{t\in[0,T]} M(u^N(t))^{m-1}\right]. \tag{2.97}$$

Hence, by induction on m, we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} M(u^N(t))^m\right] \lesssim 1\,, \tag{2.98}$$

where we note that the implicit constant is independent of N.

We now turn to estimating the energy. Applying Itô's Lemma again, we find that $E(u^N(t))^m$ equals

$$E(u_0^N)^m \tag{2.99}$$

$$+\, m\,\mathrm{Im}\left(\sum_{|j|\leq N}\int_0^t E(u^N(t'))^{m-1}\int_{\mathbb{T}^d} |u^N|^{2k} u^N \phi^N e_j\, dx\, d\beta_j(t')\right) \tag{2.100}$$

$$-\, m\,\mathrm{Im}\left(\sum_{|j|\leq N}\int_0^t E(u^N(t'))^{m-1}\int_{\mathbb{T}^d} \Delta\overline{u^N}\phi^N e_j\, dx\, d\beta_j(t')\right) \tag{2.101}$$

$$+\,(k+1)m\sum_{|j|\leq N}\int_0^t E(u^N(t'))^{m-1}\int_{\mathbb{T}^d} |u^N|^{2k}|\phi^N e_j|^2\, dx\, dt' \tag{2.102}$$

$$+\, m\left\|\nabla\phi^N\right\|^2_{\mathrm{HS}(L^2;L^2)}\int_0^t E(u^N(t'))^{m-1}\, dt' \tag{2.103}$$

$$+\,\frac{m(m-1)}{2}\sum_{|j|\leq N}\int_0^t E(u^N(t'))^{m-2}\left|\int_{\mathbb{T}^d}(-\Delta\overline{u^N} + |u^N|^{2k}\overline{u^N})\phi e_j dx\right|^2 dt'. \tag{2.104}$$

We shall control here only the difficult term (2.100) as the other terms are bounded by similar lines of argument. Firstly, by Burkholder-Davis-Gundy inequality (Lemma 2.16), we deduce

$$\mathbb{E}\left[\sup_{t\in[0,T]}(2.100)\right] \leq C_m\mathbb{E}\left[\left\{\sum_{|j|\leq N}\int_0^T E(u^N(t'))^{2(m-1)}\left|\int_{\mathbb{T}^d}|u^N|^{2k}u^N\phi^N e_j\,dx\right|^2 dt'\right\}^{\frac{1}{2}}\right].$$

Then, by duality and the (dual of the) Sobolev embedding $H^1(\mathbb{T}^d)\hookrightarrow L^{2k+2}(\mathbb{T}^d)$, we have

$$\begin{aligned}\left|\int_{\mathbb{T}^d}|u^N|^{2k}u^N\phi^N e_j\,dx\right| &\leq \left\||u^N|^{2k}u^N\right\|_{H^{-1}(\mathbb{T}^d)}\|\phi^N e_j\|_{H^1(\mathbb{T}^d)}\\ &\lesssim \left\||u^N|^{2k}u^N\right\|_{L^{\frac{2k+2}{2k+1}}(\mathbb{T}^d)}\|\phi e_j\|_{H^1(\mathbb{T}^d)}\\ &\lesssim E(u^N)^{\frac{2k+1}{2k+2}}\|\phi e_j\|_{H^1(\mathbb{T}^d)},\end{aligned}$$

provided that $1+\frac{1}{k}\geq\frac{d}{2}$. Therefore, by Hölder and Young inequalities, and similarly to the control of (2.93), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{t\in[0,T]}(2.100)\right] &\leq C_m\|\phi\|^2_{\mathrm{HS}(L^2;H^1)}T_0\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))^{m-1}\right]+\frac{1}{8}\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))^{m-\frac{1}{2k+2}}\right]\\ &\leq \tilde{C}_m\|\phi\|^2_{\mathrm{HS}(L^2;H^1)}T_0\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))^{m-1}\right]+\frac{1}{8}\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))^{m}\right],\end{aligned}$$

where in the last step we used interpolation.

We also have

$$\mathbb{E}\left[\sup_{t\in[0,T]}(2.101)\right] \leq C_m\|\phi\|_{\mathrm{HS}(L^2;H^1)}\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))^{m-1}\right]+\frac{1}{8}\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))^{m}\right]$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}(2.102)\right] \leq C_m\|\phi\|^2_{\mathrm{HS}(L^2;H^1)}+\frac{1}{8}\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N)^{m}\right]$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}(2.103)\right] \leq C_m\|\phi\|^2_{\mathrm{HS}(L^2;H^1)}\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))^{m-1}\right],$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}(2.104)\right]\leq C\,\|\phi\|^2_{\mathrm{HS}(L^2;H^1)}+\mathbb{E}\left[\sup_{t\in[0,T]}H(u^N(t))^{m-1}\right]+\frac{1}{8}\mathbb{E}\left[\sup_{t\in[0,T]}H(u^N(t))^{m}\right].$$

Gathering all the estimates, there exists $C_m>0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))\right]\leq E(u_0)^m+C_mT_0\,\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))^{m-1}\right]+\frac{1}{2}\,\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))^{m}\right].$$

Similarly to passing from (2.96) to (2.97) and by induction on m, we deduce that

$$\mathbb{E}\left[\sup_{t\in[0,T]}E(u^N(t))^m\right]\lesssim 1, \tag{2.105}$$

with constant independent of N.

□

We now argue that the probabilistic a priori bounds in fact hold for solutions of the original SNLS.

Corollary 2.28. Let s,m,T,T_0,ϕ be as in Proposition 2.27. Suppose that $u\in L^2(\Omega;C([0,T];H^s_x))$ is a solution to (2.1) with (2.2), then the estimates (2.91) and (2.92) hold with u in place of u^N under the same assumptions as Proposition 2.27.

Proof. We assume for simplicity that $s=1$. Let Λ^N be the mild formulation of (2.90), more precisely,

$$\Lambda^N(v):=S(t)u_0^N\pm i\int_0^t S(t-t')P_{\leq N}\left(|v|^{2k}v\right)(t')\,dt'-i\int_0^t S(t-t')\phi^N\,dW^N(t')\,. \tag{2.106}$$

Let

$$K(\omega):=\sup_{t\in[0,T]}\|u(t)\|_{H^1(\mathbb{T}^d)}.$$

Let $T_\omega>0$ be a stopping time satisfying

$$T_\omega\leq c\Big(2K(\omega)+\|\Psi\|_{X^{1,\frac{1}{2}-}([0,T_\omega])}\Big)^{-\theta}. \tag{2.107}$$

Then (as seen in (2.73)), Λ^N is a contraction on a ball in $X^{1,\frac{1}{2}-}([0,T_\omega])$ and has a unique fixed point u^N that satisfies the bounds in Proposition 2.27. We shall show that u^N converges to u in $F_{T_\omega} := L^2(\Omega; C([0,T_\omega]; H^1_x))$. To this end, we consider the mild formulations of u^N and u and show that each piece of u^N converges to the corresponding piece in u. Clearly, $S(t)u_0^N \to S(t)u_0$ in F_{T_ω}. For the noise, let $\Psi^N(t)$ denote the stochastic convolution in (2.106). Then

$$\Psi(t) - \Psi^N(t) = \left(\sum_{|n|>N} \sum_{j\in\mathbb{Z}^d} + \sum_{|n|\leq N} \sum_{|j|>N} \right) e_n \int_0^t e^{i(t-t')|n|^2} \widehat{\phi e_j}(n) d\beta_j(t')$$

$$= \int_0^t S(t-t')P_{>N}\phi \, dW(t') + \int_0^t S(t-t')\pi_N P_{\leq N}\phi \, dW(t') \, ,$$

where π_N denotes the projection onto the linear span of the orthonormal vectors $\{e_j : |j| > N\}$. By Lemma 2.19, the above is controlled by

$$\|P_{>N} \circ \phi\|^2_{\mathrm{HS}(L^2;H^1)} + \|\pi_N P_{\leq N}\phi\|^2_{\mathrm{HS}(L^2;H^1)} \, ,$$

which tends to 0 as $N \to \infty$ because both norms are tails of convergent series.

Finally we treat the nonlinear terms

$$Du(t) := \int_0^t S(t-t')|u|^{2k}u(t') \, dt' \, ,$$

$$D^{\leq N}u(t) := \int_0^t S(t-t')P_{\leq N} \left(|u^N|^{2k}u^N \right)(t') \, dt' \, .$$

We first fix a path for which local well-posedness holds, and prove that $Du - D^{\leq N}u \to 0$ in $X^{1,\frac{1}{2}+}$. Firstly, by Lemmata 2.12, 2.14 and 2.15, we have

$$\|Du\|_{X^{1,\frac{1}{2}+}([0,T_\omega])} \lesssim \|u\|^{2k+1}_{X^{1,\frac{1}{2}-}([0,T_\omega])}$$

and hence $Du \in X^{1,\frac{1}{2}+}([0,T_\omega])$. Now,

$$\begin{aligned} \left\|Du - D^{\leq N}u\right\|_{X^{1,\frac{1}{2}+}([0,T_\omega])} &\leq \left\| \int_0^t S(t-t')P_{\leq N}(|u|^{2k}u - |u^N|^{2k}u^N)(t') \, dt' \right\|_{X^{1,\frac{1}{2}+}([0,T_\omega])} \\ &\quad + \|P_{>N}Du\|_{X^{1,\frac{1}{2}+}([0,T_\omega])} \\ &=: \mathrm{I} + \mathrm{II} \, . \end{aligned}$$

By the definition of $P_{>N}$, we have $\mathrm{II} \to 0$ in $X^{1,\frac{1}{2}+}([0,T])$ as $N \to \infty$. By Lemmata 2.12, 2.14 and 2.15 again, we have

$$\mathrm{I} \lesssim \left(\|u\|^{2k}_{X^{1,\frac{1}{2}-}([0,T_\omega])} + \left\|u^N\right\|^{2k}_{X^{1,\frac{1}{2}-}([0,T_\omega])} \right) \left\|u - u^N\right\|_{X^{1,\frac{1}{2}-}([0,T_\omega])} \tag{2.108}$$

We claim that $\mathrm{I} \to 0$ as $N \to \infty$ as well. Indeed, Λ^N and Λ are $\frac{1}{2}$-contractions with fixed points u^N and u respectively, hence

$$\begin{aligned} \left\|u - u^N\right\|_{X^{1,\frac{1}{2}-}([0,T_\omega])} &\leq \left\|\Lambda(u) - \Lambda^N(u)\right\|_{X^{1,\frac{1}{2}-}([0,T_\omega])} + \left\|\Lambda^N(u) - \Lambda^N(u^N)\right\|_{X^{1,\frac{1}{2}-}([0,T_\omega])} \\ &\leq \left\|\Lambda(u) - \Lambda^N(u)\right\|_{X^{1,\frac{1}{2}-}([0,T_\omega])} + \frac{1}{2}\left\|u - u^N\right\|_{X^{1,\frac{1}{2}-}([0,T])} . \end{aligned}$$

By rearranging, it suffices to show that the first term on the right-hand side above tends to 0 as $N \to \infty$. Now

$$\begin{aligned} \left\|\Lambda(u) - \Lambda^N(u)\right\|_{X^{1,\frac{1}{2}-}([0,T_\omega])} \leq{}& \|P_{>N} S(t) u_0\|_{X^{1,\frac{1}{2}-}([0,T_\omega])} \\ &+ \left\| P_{>N} \int_0^t S(t-t')|u|^{2k} u(t')\, dt' \right\|_{X^{1,\frac{1}{2}-}([0,T_\omega])} \\ &+ \left\|\Psi^{>N}\right\|_{X^{1,\frac{1}{2}-}([0,T_\omega])} . \end{aligned}$$

By similar arguments as above, all the terms on the right go to 0 as $N \to \infty$. This proves our claim. By the embedding $X^{1,\frac{1}{2}+}([0,T_\omega]) \subset C([0,T]; H^1(\mathbb{T}^d))$, we have that

$$\left\|Du - D^{\leq N} u\right\|_{C([0,T_\omega];H^1)} \to 0 \tag{2.109}$$

almost surely as $N \to \infty$. By the dominated convergence theorem[5], we have $D^{\leq N} u \to Du$ in F_{T_ω}. This concludes our argument that $u^N \to u$ in F_{T_ω}. By (2.107), we can iterate this argument on $[jT_\omega, (j+1)T_\omega]$ for $j \geq 1$ until we cover the entire interval $[0,T]$, it follows that $u^N \to u$ on the whole F_T. Hence it follows that the estimates from Proposition 2.27 also hold for u. □

[5] We recall that in the local theory, u^N is obtained as a fixed point from a ball $B_R := \{v \in X^{s,\frac{1}{2}-}([0,T_\omega]) : \|v\|_{X^{s,\frac{1}{2}-}([0,T_\omega])} \leq R_\omega\}$ where $R_\omega = 2c\|u_0^N\|_{H^s_x} + 2\|\Psi^N\|_{X^{s,\frac{1}{2}-}[0,T_\omega]}$. Hence one can take the dominating function to be (for example) $2c\|u_0\|_{H^s_x} + 2\|\Psi\|_{X^{s,\frac{1}{2}-}[0,T_\omega]} + \|Du\|_{X^{s,\frac{1}{2}-}_{[0,T_\omega]}}$.

Finally, we conclude the proof of global well-posedness for the additive case.

Proof of Theorem 2.4. Let $s \in \{0,1\}$ be the regularity of u_0 from Theorem 2.4. Let $\varepsilon > 0$ and $T > 0$ be given. We claim that there exists an event Ω_ε such that a solution $u \in X^{s,b}([0,T]) \cap C([0,T]; H^s(\mathbb{T}^d))$ exists on $[0,T]$ in Ω_ε and $\mathbb{P}(\Omega \setminus \Omega_\varepsilon) < \varepsilon$. If this claim holds, then by setting

$$\Omega^* = \bigcup_{n=1}^{\infty} \Omega_{\frac{1}{n}},$$

we have that $\mathbb{P}(\Omega^*) = 1$ and u exists on $[0,T]$, proving the theorem. Let $\delta \in (0,1)$ be a small quantity chosen later. We subdivide $[0,T]$ into $M = \lceil \frac{T}{\delta} \rceil$ subintervals $I_k = [(k-1)\delta, k\delta]$. Let

$$\Omega_0 = \bigcap_{k=1}^{M} \left\{ \omega \in \Omega : \left\| \int_{(k-1)\delta}^{t} S(t-t')\phi \, dW(t') \right\|_{X^{s,b}(I_k)} \leq L \right\},$$

where $L > 0$ is some large quantity determined later. Now by Chebyshev's inequality and Lemma 2.18,

$$\begin{aligned}
\mathbb{P}(\Omega \setminus \Omega_0) &= \sum_{k=1}^{M} \mathbb{P}\left(\left\| \int_{(k-1)\delta}^{t} S(t-t')\phi \, dW(t') \right\|_{X^{s,b}(I_k)} > L \right) \\
&\leq \sum_{k=1}^{M} \frac{1}{L^2} \mathbb{E}\left[\left\| \int_{(k-1)\delta}^{t} S(t-t')\phi \, dW(t') \right\|_{X^{s,b}(I_k)}^2 \right] \\
&\lesssim \sum_{k=1}^{M} \frac{\delta(\delta^2+1)}{L^2} \left\| \phi \right\|_{\mathrm{HS}(L^2;L^2)}^2 \\
&\leq \frac{2M\delta}{L^2} \left\| \phi \right\|_{\mathrm{HS}(L^2;L^2)}^2 \\
&\lesssim \frac{T}{L^2} \left\| \phi \right\|_{\mathrm{HS}(L^2;L^2)}^2 .
\end{aligned}$$

By choosing $L = L(\varepsilon, T, \phi)$ sufficiently large, we may therefore bound $\mathbb{P}(\Omega_0^c)$ above by $\frac{\varepsilon}{2}$. Now let

$$R = \max\left\{ \left\| u_0 \right\|_{H^s}, L \right\} .$$

By local theory, there exists a unique solution $u(t)$ to (2.1) with time of existence

$T_{\max}$ given in (2.73). In particular, we note that for $\omega \in \Omega_0$,

$$c\Big(\|u_0\|_{H^s} + \|\Psi\|_{X^{s,b}_{[0,\delta]}}\Big)^{-\theta} \geq c\big(R+L\big)^{-\theta} \geq c\big(2R\big)^{-\theta}, \tag{2.110}$$

where c is as in (2.73). By choosing $\delta = \delta(R) := c(2R)^{-\theta}$, we see that $u(t)$ exists for $t \in [0,\delta]$ for all $\omega \in \Omega_0$. Now define

$$\Omega_1 = \{\omega \in \Omega_0 : \|u(\delta)\|_{H^s} \leq R\} .$$

By the same argument, $u(t)$ exists for $t \in (\delta, 2\delta)$ for all $\omega \in \Omega_1$. Iterating this argument, we have a chain of events $\Omega_0 \supseteq \Omega_1 \supseteq \cdots \supseteq \Omega_{M-1}$ where

$$\Omega_k = \{\omega \in \Omega_{k-1} : \|u(k\delta)\|_{H^s} \leq R\}$$

and $u(t)$ exists for all $t \in [0,(k+1)\delta]$ on Ω_k. Setting $\Omega_\varepsilon := \Omega_{M-1}$, $u(t)$ exists on the full interval $[0,T]$ on Ω_ε. It remains to check that $\Omega \setminus \Omega_\varepsilon$ remains small. By Corollary 2.28, we have

$$\begin{aligned}
\mathbb{P}(\Omega_\varepsilon) &\leq \mathbb{P}(\Omega \setminus \Omega_0) + \sum_{k=0}^{M-1} \mathbb{P}(\Omega_{k+1}^c \cap \Omega_k) \\
&\leq \frac{\varepsilon}{2} + \sum_{k=0}^{M-1} \mathbb{P}\left(\{\|u((k+1)\delta)\|_{H^s} > R\} \cap \Omega_k\right) \\
&\leq \frac{\varepsilon}{2} + \sum_{k=0}^{M-1} \frac{1}{R^p}\mathbb{E}\left[\mathbb{1}_{\Omega_k} \|u((k+1)\delta)\|_{H^s}^p\right] \\
&\leq \frac{\varepsilon}{2} + \frac{MC_1}{R^p} \\
&\leq \frac{\varepsilon}{2} + \frac{2TC_1(2R)^\theta}{cR^p},
\end{aligned}$$

for any $p \in \mathbb{N}$. We further enlarge R if necessary by setting

$$R = \max\left\{\frac{2TC_1}{c} + 1, L, \|u_0\|_{H^s}\right\} ,$$

and so we have that

$$\mathbb{P}(\Omega_\varepsilon) \leq \frac{\varepsilon}{2} + 2^\theta R^{\theta-p+1}\,.$$

This is smaller than ε provided we choose $p = p(\varepsilon, \theta) > 0$ sufficiently large. Thus Ω_ε satisfies our claim. $\square$

2.4.2 SNLS with multiplicative noise

In order to globalize solutions of SNLS, for the multiplicative noise case, we need to prove probabilistic control of the $X^{s,b}$-norm of the solutions of the truncated SNLS uniformly in the truncation parameter (Lemma 2.30). This requires a priori bounds on mass and energy of solutions.

From Subsection 2.3.2, we obtained a local solution of the multiplicative (2.1) with time of existence

$$\tau^* = \lim_{R\to\infty} \tau_R\,.$$

Under the hypotheses of Theorem 2.7, we shall prove global well-posedness by showing that $\tau^* = \infty$ almost surely.

Proposition 2.29. Let $T_0 > 0$ and ϕ be as in Theorem 2.7. Suppose that $u(t)$ is a solution for (2.1) with $F(u, \phi\xi) = u \cdot \phi\xi$ on $t \in [0, T]$ for some stopping time $T \in [0, T_0 \wedge \tau^*)$. Let $C(\phi)$ be as in (2.62). Then for any $m \in \mathbb{N}$, there exists $C_1 = C_1(m, M(u_0), T_0, C(\phi)) > 0$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} M(u(t))^m\right] \leq C_1\,. \tag{2.111}$$

Furthermore, if(2.1) is defocusing, there exists $C_2 = C_2(m, E(u_0), T_0, C(\phi)) > 0$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} E(u(t))^m\right] \leq C_2\,. \tag{2.112}$$

Proof. We consider the frequency truncated equation (2.90) and apply Itô's Lemma to obtain

$$M(u^N(t))^m = M(u_0^N)^m$$

$$+ m \operatorname{Im} \left(\sum_{|j| \leq N} \int_0^t M(u^N(t'))^{m-1} \int_{\mathbb{T}^d} |u^N(t')|^2 \phi^N e_j \, dx \, d\beta_j(t') \right) \tag{2.113}$$

$$+ m(m-1) \sum_{|j| \leq N} \int_0^t M(u^N(t'))^{m-2} \left| \int_{\mathbb{T}^d} |u^N(t')|^2 \phi^N e_j \, dx \right|^2 dt' \tag{2.114}$$

$$+ m(m-1) \sum_{|j| \leq N} \int_0^t M(u^N(t'))^{m-1} \int_{\mathbb{T}^d} |u(t')\phi e_j|^2 \, dx \, dt' \, . \tag{2.115}$$

To bound (2.113), we use Burkholder-Davis-Gundy inequality (Lemma 2.16) and use a similar argument as in the proof of Lemma 2.21 to get

$$\begin{aligned}
\mathbb{E}\left[\sup_{t \in [0,T]} (2.113) \right] &\lesssim \mathbb{E}\left[\sum_{|j| \leq N} \left(\int_0^T M(u^N(t'))^{2(m-1)} \left| \int_{\mathbb{T}^d} |u^N(t')\phi e_j|^2 \, dx \right|^2 dt' \right)^{\frac{1}{2}} \right] \\
&\leq C(\phi)^2 \mathbb{E}\left[\left(\int_0^T M(u^N(t'))^{2m} \right)^{\frac{1}{2}} \right] \\
&\leq C(\phi)^2 \left(\mathbb{E}\left[\sup_{t \in [0,T]} M(u^N(t))^m \right] \right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^T M(u^N(t'))^m \, dt' \right] \right)^{\frac{1}{2}}
\end{aligned}$$

Similarly, one obtains

$$\mathbb{E}\left[\sup_{t \in [0,T]} \{(2.114) + (2.115)\} \right] \lesssim C(\phi) \mathbb{E}\left[\int_0^T M(u^N(t'))^m \, dt' \right]$$

Hence there is a constant $C_1 = C_1(m, M(u_0), T, C(\phi))$ such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t \in [0,T]} M(u^N(t))^m \right] &\leq C_1 + C_1 \, \mathbb{E}\left[\int_0^T M(u^N(t'))^m \, dt' \right] \\
&\quad + C(\phi)^2 \left(\mathbb{E}\left[\sup_{t \in [0,T]} M(u^N(t))^m \right] \right)^{\frac{1}{2}} \left(\mathbb{E}\left[\int_0^T M(u^N(t'))^m \, dt' \right] \right)^{\frac{1}{2}}
\end{aligned}$$

The left-hand side is bounded above by $3\mathcal{M}$, where $\mathcal{M}$ is maximum of the three terms of the right-hand side. In any of the three cases, we may conclude the proof via simple rearrangement arguments and Gronwall's inequality.

Turning to the energy, we use Itô's Lemma and the defocusing equation to obtain that $E(u^N(t))^m$ equals

$$E(u_0^N)^m \tag{2.116}$$

$$+ m \ \mathrm{Im} \left(\sum_{|j|\le N} \int_0^t E(u^N(t'))^{m-1} \int_{\mathbb{T}^d} |u^N|^{2(k+1)} \phi^N e_j \, dx \, d\beta_j(t') \right) \tag{2.117}$$

$$- m \ \mathrm{Im} \left(\sum_{|j|\le N} \int_0^t E(u^N(t'))^{m-1} \int_{\mathbb{T}^d} (\Delta \overline{u^N}) u^N \phi^N e_j \, dx \, d\beta_j(t') \right) \tag{2.118}$$

$$+ m(k+1) \sum_{|j|\le N} \int_0^t E(u^N(t'))^{m-1} \int_{\mathbb{T}^d} |u^N|^{2(k+1)} |\phi^N e_j|^2 \, dx \, dt' \tag{2.119}$$

$$+ m \sum_{|j|\le N} \int_0^t E(u^N(t'))^{m-1} \int_{\mathbb{T}^d} |\nabla (u^N \phi^N e_j)(n)|^2 \, dx \, dt' \tag{2.120}$$

$$+ \frac{m(m-1)}{2} \left(\sum_{|j|\le N} \int_0^t E(u^N(t'))^{m-2} \left| \int_{\mathbb{T}^d} \left(-u^N \Delta \overline{u^N} + |u^N|^{2k+1} \right) \phi^N e_j \, dx \right|^2 dt' \right) \tag{2.121}$$

For (2.117), we use Burkholder-Davis-Gundy inequality (Lemma 2.16) to get

$$\mathbb{E} \left[\sup_{t\in[0,T]} (2.117) \right] \lesssim \mathbb{E} \left[\left(\sum_{|j|\le N} \int_0^T E(u^N(t'))^{2(m-1)} \left| \int_{\mathbb{T}^d} |u^N|^{2k+2} \phi^N e_j \, dx \right|^2 dt' \right)^{\frac{1}{2}} \right].$$

Now, with r as in Theorem 2.5,

$$\begin{aligned} \left| \int_{\mathbb{T}^d} |u^N|^{2k+2} \phi^N e_j \, dx \right|^2 &\le \left\| u^N \right\|_{L_x^{2k+2}}^{2(2k+2)} \left\| \phi^N e_j \right\|_{L_x^\infty}^2 \le E(u)^2 \left\| \widehat{\phi^N e_j} \right\|_{\ell^1}^2 \\ &\lesssim E(u)^2 \| \phi^N e_j \|_{\mathcal{F}L^{s,r}}^2 , \end{aligned}$$

where the last step follows from Hölder inequality as in the proof of Lemma 2.20.

Therefore, by Hölder's inequality and (2.62),

$$\mathbb{E}\left[\sup_{t\in[0,T]} (2.117)\right] \lesssim C(\phi)\,\mathbb{E}\left[\left(\int_0^T E((u^N(t')))^{2m}\,dt'\right)^{\frac{1}{2}}\right]$$

$$\leq C(\phi)\left(\mathbb{E}\left[\sup_{t\in[0,T]} E(u^N(t))^m\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T E(u^N(t'))^m\,dt'\right]\right)^{\frac{1}{2}}.$$

Similarly, we bound the other terms as follows:

$$\mathbb{E}\left[\sup_{t\in[0,T]} (2.118)\right] \lesssim C(\phi)\left(\mathbb{E}\left[\sup_{t\in[0,T]} E(u^N(t))^m\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T E(u^N(t'))^m\,dt'\right]\right)^{\frac{1}{2}}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]} \{(2.119)+(2.120)+(2.121)\}\right] \lesssim C(\phi)^2\mathbb{E}\left[\int_0^T E(u^N(t'))^m\,dt'\right]$$

It follows that there is a constant $C_2 = C_2(m, E(u_0), T, C(\phi))$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} E(u^N(t))^m\right] \leq C_2 + C_2\,\mathbb{E}\left[\int_0^T E(u^N(t'))^m\,dt'\right]$$

$$+ C_2\left(\mathbb{E}\left[\sup_{t\in[0,T]} E(u^N(t))^m\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\int_0^T E(u^N(t'))^m\,dt'\right]\right)^{\frac{1}{2}}.$$

Arguing in the same way as for the mass of u^N yields the estimate for the energy of u^N. This proves the proposition for u^N in place of u. The proposition then follows by letting $N \to \infty$. □

We now prove the following probabilistic a priori bound on the $X^{s,b}$-norm of a solution.

Lemma 2.30. Let $T, R > 0$. Let u_R be the unique solution of (2.74) on $[0,T]$. There exists $C_1 = C_1(\|u_0\|_{L^2}, T, C(\phi))$ such that

$$\mathbb{E}\left[\|u_R\|_{X^{0,b}([0,T\wedge\tau_R])}\right] \leq C_1\,.$$

Moreover, if (2.74) is defocusing, there also exists $C_2 = C_2(\|u_0\|_{H^1}, T, C(\phi))$ such that

$$\mathbb{E}\left[\|u_R\|_{X^{1,b}([0,T\wedge\tau_R])}\right] \leq C_2 \,.$$

The constants C_1 and C_2 are independent of R.

Proof. Let τ be a stopping time so that $0 < \tau \leq T \wedge \tau_R$. By a similar argument used in local theory, we have

$$\begin{aligned}\|u_R\|_{X^{s,b}([0,\tau])} &\leq C_1 \|u_R(0)\|_{H^s} + C_2\tau^\delta \|u_R\|^{2k+1}_{X^{s,b}([0,\tau])} + \|\Psi\|_{X^{s,b}([0,\tau])} \\ &\leq C_1 \|u_R\|_{C([T\wedge\tau_R];H^s)} + C_2\tau^\delta \|u_R\|^{2k+1}_{X^{s,b}([0,\tau])} + \|\Psi\|_{X^{s,b}([0,T\wedge\tau_R])} \,. \end{aligned} \tag{2.122}$$

Let $K = C_1 \|u_R\|_{C([T\wedge\tau_R];H^s)} + \|\Psi(t)\|_{X^{s,b}([0,T\wedge\tau_R])}$ and assume $K > 0$ (otherwise we are done). We claim that there exist constants $c, C > 0$ such that if $\tau = cK^{-\frac{2k}{\delta}}$

$$\|u_R\|_{X^{s,b}([0,\tau])} \leq CK \,. \tag{2.123}$$

To see this, we note that the polynomial

$$p_\tau(x) = C_2\tau^\delta x^{2k+1} - x + K \tag{2.124}$$

has exactly one positive turning point at

$$x'_+ = \big((2k+1)C_2\tau^\delta\big)^{-\frac{1}{2k}} \,.$$

Now

$$\begin{aligned} p_\tau(x'_+) &= C_2^{-\frac{1}{2k}}\left[(2k+1)^{-\frac{2k+1}{2k}} - (2k+1)^{-\frac{1}{2k}}\right]\tau^{-\frac{\delta}{2k}} + K \\ &=: -C_3\tau^{-\frac{\delta}{2k}} + K. \end{aligned}$$

The right-hand side is negative if we choose $\tau = (\frac{2}{C_3}K)^{-\frac{2k}{\delta}}$. Since $p_\tau(0) = K > 0$, we have $p_\tau(x) > 0$ for $0 \leq x < x_+$ where x_+ is the unique positive root below x'_+. Now (2.122) is equivalent to $p_\tau\big(\|u_R\|_{X^{s,b}([0,\tau])}\big) \geq 0$. But since $g(\,\cdot\,) := \|u_R\|_{X^{s,b}([0,\,\cdot\,])}$

is continuous and $g(0) = 0$, we must have

$$g(\tau) < x'_+ = ((2k+1)C_2)^{-\frac{1}{2k}} \cdot \frac{2}{C_3} K := CK\,,$$

which proves (2.123). Iterating this argument, we find that

$$\|u_R\|_{X^{s,b}([(j-1)\tau, j\tau])} \leq C\Big(\|u_R\|_{C([0,T\wedge\tau_R];H^s)} + \|\Psi(t)\|_{X^{s,b}([0,T\wedge\tau_R])} \Big) \tag{2.125}$$

for all integer $1 \leq j \leq \lceil \frac{T\wedge\tau_R}{\tau} \rceil =: M$. Putting everything together, we have

$$\begin{aligned}
\|u_R\|_{X^{s,b}([0,T\wedge\tau_R])} &\leq \sum_{j=1}^{M} \|u_R\|_{X^{s,b}([(j-1)\tau, j\tau])} \\
&\leq C\frac{T\wedge\tau_R}{\tau}\left(\|u_R\|_{C([0,T\wedge\tau_R];H^s)} + \|\Psi\|_{X^{s,b}([0,T\wedge\tau_R])}\right) \\
&\leq CT\left(\|u_R\|_{C([0,T\wedge\tau_R];H^s)} + \|\Psi\|_{X^{s,b}([0,T\wedge\tau_R])}\right)^{\frac{2k}{\delta}+1}\,.
\end{aligned}$$

By Proposition 2.29 and Lemma 2.21, all moments of the last two terms above are finite. This proves Lemma 2.30. □

We can now conclude the proof of Theorem 2.7.

Proof of Theorem 2.7. Fix $T > 0$. Since τ_R is increasing in R,

$$\begin{aligned}
\mathbb{P}(\tau^* < T) = \lim_{R\to\infty} \mathbb{P}(\tau_R < T) &= \lim_{R\to\infty} \mathbb{P}\left(\|u_R\|_{X^{s,b}([0,T\wedge\tau_R])} \geq R\right) \\
&\leq \lim_{R\to\infty} \frac{1}{R}\mathbb{E}\left[\|u_R\|_{X^{s,b}([0,T\wedge\tau_R])}\right]\,.
\end{aligned}$$

But then the right-hand side equals 0 by Lemma 2.30. It follows that $\tau^* = \infty$ almost surely. □

Chapter 3

SNLS on $\mathbb{R}^d$ with supercritical noise

In this chapter, we consider the Cauchy problem for the stochastic cubic nonlinear Schrödinger equation with additive noise (SNLS) on $\mathbb{R}^d$ for $d \geq 3$:

$$\begin{cases} i\partial_t u + \Delta u \pm \mathcal{N}(u) + \phi\xi = 0 \\ u|_{t=0} = u_0, \end{cases} \qquad (x,t) \in \mathbb{R}^d \times \mathbb{R}, \tag{SNLS}$$

where $u : \mathbb{R}^d \times \mathbb{R} \to \mathbb{C}$, $\mathcal{N}(u) = |u|^2 u$ is the cubic nonlinearity, ξ is a space-time white noise, and ϕ is a smoothing operator. As seen in Chapter 1, we can express ξ as $\frac{dW}{dt}$, where W is a cylindrical Wiener process on $L^2(\mathbb{R}^d)$, given by

$$W(t) = \sum_{n\in\mathbb{N}} \beta_n(t) e_n \,,$$

where $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal basis of $L^2(\mathbb{R}^d)$ and $\{\beta_n\}_{n\in\mathbb{N}}$ is a sequence of independent complex-valued Brownian motions on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We assume that the operator ϕ belongs to $\mathrm{HS}^s := \mathrm{HS}(L^2(\mathbb{R}^d), H^s(\mathbb{R}^d))$ for appropriate values of $s \geq 0$, namely, it is a Hilbert-Schmidt operator from $L^2(\mathbb{R}^d)$ to $H^s(\mathbb{R}^d)$.

As before, we consider solutions of (SNLS) in the mild sense, that is, functions u that satisfy

$$u(t) = S(t)u_0 \mp i \int_0^t S(t-t')\mathcal{N}(u)(t')\,dt' - i\int_0^t S(t-t')\phi\,dW_{t'}\,, \tag{3.1}$$

and we again use Ψ to denote the stochastic convolution:

$$\Psi(t) := -i \int_0^t S(t-t')\phi \, dW_{t'} .$$

The assumption that $\phi \in \mathrm{HS}^s$ insures that $\Psi \in C(\mathbb{R}; H^s(\mathbb{R}^d))$ almost surely; see Lemma 3.15 below (see also Lemma 2.19 from Chapter 1). Our goal in this chapter is to construct a unique local-in-time solution of (SNLS) in the case of very rough stochastic forcing, namely, for values of s below the regularity threshold dictated by scaling.

We now restate Theorem 1.3 from the introduction. For $d \geq 3$, we set

$$s_d := \begin{cases} \dfrac{1}{4}, & \text{if} \quad d = 3 \,, \\ s_{\text{crit}} - \dfrac{2}{5}, & \text{if} \quad d \geq 4 \,. \end{cases} \tag{3.2}$$

Theorem 3.1. Let $d \geq 3$, $s \in (s_d, s_{\text{crit}})$, and $\phi \in HS^s$. Then, given $u_0 \in H^{s_{\text{crit}}}(\mathbb{R}^d)$, there exists a stopping time T that is almost surely positive and a solution u on $[0,T]$ of (SNLS) in the sense of (3.1). Moreover, the solution lies in the class

$$\Psi + Y_2^{s_{\text{crit}}}([0,T]) \subset \Psi + C([0,T]; H^{s_{\text{crit}}}(\mathbb{R}^d)) \subset C([0,T]; H^s(\mathbb{R}^d)),$$

where $T = T_\omega$ is almost surely positive and $Y_2^{s_{\text{crit}}}$ is defined in Section 2 below (see Definition 3.6).

Theorem 3.1 is inspired by [6], where the authors studied the deterministic cubic NLS with random initial data: $u(0) = f^\omega$, where f^ω is the Wiener randomisation of some function $f \in H^s(\mathbb{R}^d)$. They proved local well-posedness of in $H^s(\mathbb{R}^d)$ for a range of s below s_{crit}, with respect to this randomisation. See also [5, 14, 40]. In [6], the authors decomposed a solution as $u = z^\omega + v$, where $z^\omega(t) := S(t)f^\omega$ is linear and random, and solved the fixed point problem for v, which satisfies:

$$\begin{cases} i\partial_t v + \Delta v = \mathcal{N}(v + z^\omega) \\ v(0) = 0. \end{cases}$$

Similarly, we use the so called Da Prato-Debussche trick and decompose our solution in (3.1) as $u = v + \Psi$. In view of the mild formulation (3.1), v satisfies

$$v(t) = S(t)u_0 \mp i \int_0^t S(t-t')\mathcal{N}(v+\Psi)(t')\, dt' .$$

In other words, v solves the equation

$$\begin{cases} i\partial_t v + \Delta v = \mathcal{N}(v+\Psi) \\ v(0) = u_0. \end{cases}$$

Our main tools for proving Theorem 3.1 are similar to those in [6]: the Fourier restriction norm method adapted to the spaces V^p of functions of bounded p-variation and their preduals U^p (these spaces were introduced by Koch, Tataru and their collaborators, see [8, 51, 54]), the bilinear refinement of the Strichartz estimate and a case-by-case analysis for estimating the terms $v\bar{v}v$, $v\bar{v}\Psi$, $v\bar{\Psi}\Psi$, $\Psi\bar{\Psi}\Psi$, etc.

There are two main differences between our arguments in this chapter and those in [6]. In [6], the analysis used Y_2^s-spaces that are constructed using the V^2 space. Since the Brownian motion does not belong to V^2 almost surely, we cannot measure the stochastic convolution in Y_2^s. The Brownian motion, however, does belong to V^p with $p > 2$, almost surely. So, given $p > 2$, we show that $\Psi \in Y_p^s$ almost surely (see Proposition 3.17 below) and use the Y_p^s-spaces in our case-by-case analysis. Interestingly, the Y^s spaces are useful both because of the critical nature of our problem (i.e. $u_0 \in H^{s_{\text{crit}}}(\mathbb{R}^d)$) and because of the above mentioned property of the stochastic convolution.

Another key ingredient in the proof of Theorem 3.1 is the space-time integrability of the stochastic convolution. Namely, the stochastic convolution belongs almost surely to $L^q([0,T]; W^{s,r}(\mathbb{R}^d))$ for any $T > 0$, $1 \le q < \infty$ and $2 \le r \le \frac{2d}{d-2}$, provided that $\phi \in HS^s$. In comparison to this, the linear solution z^ω enjoys more space-time integrability. More precisely, there is no upper bound on the values of r; compare Proposition 3.14 below and Lemma 2.2 in [6]. This limitation in the space-time integrability of the stochastic convolution forces us to adopt a different scheme from [6] in the case-by-case analysis for $d \ge 4$. In particular, we require a slightly higher regularity $s > \frac{d-2}{2} - \frac{2}{5}$ compared to [6]. (We note that for $d = 3$ we can apply an almost identical analysis to that in [6], so the regularity that we require $s > \frac{1}{4}$

matches the one in [6].)

Remark 3.2. It is possible to extend Theorem 3.1 to the case of SNLS with higher power nonlinearities $\mathcal{N}(u) = |u|^{p-1}u$, where $p \geq 5$ is an odd integer. However, the case-by-case analysis becomes exceedingly tedious, where the number of cases is at least $O(p)$. For this reason, we restrict our exposition to the cubic case.

Remark 3.3. In [77], the second author with Oh and Wang considered the SNLS with a generic power-type nonlinearity $\mathcal{N}(u) = |u|^{p-1}u$, $p > 1$. By using a simple argument based on the dispersive estimate, they proved local well-posedness in the case of subcritical initial data and supercritical noise. More precisely, in the energy-(super)critical regime, they considered $u_0 \in H^{s_0}(\mathbb{R}^d)$ and $\phi \in H^s(\mathbb{R}^d)$ where

$$s_0 > s_{\text{crit}} \qquad \text{and} \qquad s > s_{\text{crit}} - 1$$

and showed that the residual part $v = u - \Psi$ lies in $C([0,T]; W^{s_1, \frac{2d}{d-2}-}(\mathbb{R}^d)) \cap C([0,T]; H^{s_1}(\mathbb{R}^d))$ for $s_1 = \min(s_0 - 1, s)$. While the noise considered in [77] can be rougher than s_d, our initial data lies in the critical space $H^{s_{\text{crit}}}(\mathbb{R}^d)$ and we constructed $v \in C([0,T]; H^{s_{\text{crit}}}(\mathbb{R}^d))$.

3.1 Function spaces

In this section we sumarise some properties of U^p- and V^p-spaces, developed by Tataru, Koch and their collaborators [51, 54, 69]. Let H be a Hilbert space over $\mathbb{C}$. Let $\mathcal{Z}$ be the set of finite partitions $\{t_k\}_{k=0}^K$, $-\infty < t_0 < \cdots < t_K \leq \infty$ of $\mathbb{R}$. We make the convention that all functions $u : \mathbb{R} \to H$ satisfy $u(\infty) = 0$.

Definition 3.4. Let $1 \leq p < \infty$.

(i) a $U^p(\mathbb{R}; H)$-atom is a step function $a : \mathbb{R} \to H$ of the form

$$a(t) = \sum_{k=1}^{K} \phi_{k-1} \mathbb{1}_{[t_{k-1}, t_k)}(t)$$

where $\{t_k\}_{k=0}^K \in \mathcal{Z}$ and $\{\phi_k\}_{k=0}^{K-1} \subset H$ are such that $\sum_{k=0}^{K-1} \|\phi_k\|_H^p = 1$. Then we define $U^p = U^p(\mathbb{R}; H)$ to be the space of functions $u : \mathbb{R} \to H$ of the form

$$u = \sum_{j=1}^{\infty} \lambda_j a_j \qquad \text{where each } a_j \text{ is a } U^p\text{-atom and } \{\lambda_j\} \in \ell^1(\mathbb{N}, \mathbb{C}) \tag{3.3}$$

with the norm

$$\|u\|_{U^p(\mathbb{R};H)} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : u = \sum_{j=1}^{\infty} \lambda_j a_j \text{ for some } U^p\text{-atoms } a_j, \{\lambda_j\}_{j\in\mathbb{N}} \in \ell^1(\mathbb{N}; \mathbb{C}) \right\}.$$

where the infimum is taken over all possible representations of u.

(ii) We define $V^p = V^p(\mathbb{R}; H)$ to be the space of functions $u : \mathbb{R} \to H$ such that the norm

$$\|u\|_{V^p(\mathbb{R};H)} := \sup_{\{t_k\}_{k=0}^K \in \mathcal{Z}} \left(\sum_{k=1}^{K} \|u(t_k) - u(t_{k-1})\|_H^p \right)^{\frac{1}{p}}, \tag{3.4}$$

is finite. We also define $V^p_{\mathrm{rc}}(\mathbb{R}; H)$ to be the closed subspace of $V^p(\mathbb{R}; H)$ of all right-continuous functions $u \in V^p(\mathbb{R}; H)$ such that $\lim_{t\to-\infty} u(t) = 0$.

(iii) We define $U^p_\Delta H$ (and $V^p_\Delta H$, respectively) to be the space of functions $u : \mathbb{R} \to H$ such that $\|u\|_{U^p_\Delta H} < \infty$ (and $\|u\|_{V^p_\Delta H} < \infty$, respectively), where

$$\|u\|_{U^p_\Delta H} := \|S(-t)u\|_{U^p(\mathbb{R};H)} \qquad \text{and} \qquad \|u\|_{V^p_\Delta H} := \|S(-t)u\|_{V^p(\mathbb{R};H)}. \tag{3.5}$$

We denote by $V^p_{\mathrm{rc},\Delta} H$ be the subspace of all right-continuous functions in $V^p_\Delta H$.

Note that the spaces $U^p(\mathbb{R}; H)$, $V^p(\mathbb{R}; H)$ and $V^p_{\mathrm{rc}}(\mathbb{R}; H)$ are Banach spaces. Given $1 \le p < q < \infty$, the elementary embedding $\ell^p \hookrightarrow \ell^q$ implies that

$$V^p(\mathbb{R}; H) \hookrightarrow V^q(\mathbb{R}; H). \tag{3.6}$$

We also have the following continuous embeddings: for $1 \le p < q < \infty$, we have

$$U^p(\mathbb{R}; H) \hookrightarrow V^p_{\mathrm{rc}}(\mathbb{R}; H) \hookrightarrow U^q(\mathbb{R}; H) \hookrightarrow L^\infty(\mathbb{R}; H). \tag{3.7}$$

The same embeddings hold for U^p_Δ- and V^p_Δ-spaces.

We state the following transference principle for V^p_Δ-spaces.

Lemma 3.5 (Transference principle)**.** Suppose that T is a k-linear operator that satisfies

$$\|T(S(t)\phi_1, ..., S(t)\phi_k)\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)} \lesssim \prod_{j=1}^{k} \|\phi_j\|_{L_x^2}$$

for some $2 \le p, q \le \infty$. Then

$$\|T(u_1, ..., u_k)\|_{L_t^p L_x^q(\mathbb{R}\times\mathbb{R}^d)} \lesssim \prod_{j=1}^{n} \|u_j\|_{V^{p-}_{\Delta,\text{rc}} L_x^2} .$$

This follows from the transference principle for U^p_Δ spaces (see [51, Proposition 2.19 (i)]) and that $V^{p-}_{\Delta,\text{rc}}(\mathbb{R}; H) \hookrightarrow U^p_\Delta(\mathbb{R}; H)$ for $p > 2$.

Let $\eta : \mathbb{R} \to [0, 1]$ be a smooth even function such that η is supported on $[-\frac{8}{5}, \frac{8}{5}]$ and $\eta \equiv 1$ on $[-\frac{5}{4}, \frac{5}{4}]$. Let $\eta_1(\xi) := \eta(|\xi|)$ and, given a dyadic number $N > 1$, let

$$\eta_N(\xi) := \eta\left(\frac{|\xi|}{N}\right) - \eta\left(\frac{2|\xi|}{N}\right) .$$

The Littlewood-Paley projection $\mathbb{P}_N$ is the Fourier multiplier operator with symbol η_N. We also define the operators $\mathbb{P}_{\le N} := \sum_{1\le M\le N} \mathbb{P}_M$ and $\mathbb{P}_{>N} := \sum_{M>N} \mathbb{P}_M$.

Definition 3.6. Let $2 \le p < \infty$ and $s \in \mathbb{R}$.

(i) We define $X^s_p(\mathbb{R})$ to be the closure of $C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap U^2_\Delta L^2$ with respect to the norm

$$\|u\|_{X^s_p(\mathbb{R})} := \left(\sum_{\substack{N\ge 1\\ \text{dyadic}}} N^{2s} \|\mathbb{P}_N u\|^2_{U^p_\Delta L^2}\right)^{\frac{1}{2}} .$$

(ii) We define $Y^s_p(\mathbb{R})$ to be the closure of $C(\mathbb{R}; H^s(\mathbb{R}^d)) \cap V^p_{\text{rc},\Delta} L^2$ with respect to the norm

$$\|u\|_{Y^s_p(\mathbb{R})} := \left(\sum_{\substack{N\ge 1\\ \text{dyadic}}} N^{2s} \|\mathbb{P}_N u\|^2_{V^p_\Delta L^2}\right)^{\frac{1}{2}} .$$

By (3.6) and (3.7), we immediately have the embeddings

$$X^s_p(\mathbb{R}) \hookrightarrow X^s_q(\mathbb{R}) \hookrightarrow Y^s_q(\mathbb{R}) \hookrightarrow Y^s_r(\mathbb{R}) \tag{3.8}$$

for $1 \le p \le q \le r < \infty$.

The spaces defined so far are all on the whole real line $\mathbb{R}$. More generally, given any space $K(\mathbb{R})$ defined above and an interval $I \subset \mathbb{R}$, we define the local-in-time version $K(I)$ as the space of all functions $u : I \to H$ such that the norm

$$\|u\|_{K(I)} := \inf \left\{ \|\tilde{u}\|_{K(\mathbb{R})} : \tilde{u}|_I = u \right\} \tag{3.9}$$

is finite.

We now state some basic estimates regarding the function spaces Y_p^s introduced above.

Lemma 3.7 (Linear estimates)**.** Let $s \geq 0$, $p \geq 2$, and $T \in (0, \infty]$. Then

$$\|S(t)\phi\|_{Y_p^s([0,T))} \lesssim \|\phi\|_{H^s} \tag{3.10}$$

$$\left\| \int_{t_0}^t S(t-t')F\, dt' \right\|_{Y_p^s([0,T))} \lesssim \sup_{\substack{v \in Y_2^{-s}([0,T)) \\ \|v\|_{Y_2^{-s}}=1}} \left| \int_0^T \int_{\mathbb{R}^d} F(t,x)\overline{v(t,x)}\, dx\, dt \right| \tag{3.11}$$

for all $\phi \in H^s(\mathbb{R}^d)$ and $F \in L^1([0,T); H^s(\mathbb{R}^d))$.

See [51, 54] for the proof of (3.10) and (3.11) with $X_2^s(I)$-norms on the left in place of the $Y_p^s(I)$-norms. Then Lemma 3.7 follows from the embedding (3.8).

Next we state the Strichartz estimates on $\mathbb{R}^d$. We say that a pair (q, r) is Schrödinger admissible if it satisfies

$$\frac{2}{q} + \frac{d}{r} = \frac{d}{2}$$

with $2 \leq q, r \leq \infty$ and $(q, r, d) \neq (2, \infty, 2)$.

Lemma 3.8 (Strichartz estimates)**.** Let (q, r) be Schrödinger admissible with $q > 2$ and let $p \geq \frac{2(d+2)}{d}$. For $T \in (0, \infty]$, we have

$$\|u\|_{L_t^q L_x^r([0,T)\times\mathbb{R}^d)} \lesssim \|u\|_{Y_{q-}^0([0,T))} \tag{3.12}$$

$$\|u\|_{L_{t,x}^p([0,T)\times\mathbb{R}^d)} \lesssim \left\| |\nabla|^{\frac{d}{2}-\frac{d+2}{p}} u \right\|_{Y_{p-}^0([0,T))} \tag{3.13}$$

Proof. For an admissible pair (q, r), the classical Strichartz estimate states that

$$\|S(t)\phi\|_{L_t^q L_x^r([0,T)\times\mathbb{R}^d)} \lesssim \|\phi\|_{L^2} . \tag{3.14}$$

Applying the transference principle in Lemma 3.5 and the embedding

$$Y^0_{q-} \hookrightarrow V^{q-}_{\Delta} L^2_x \tag{3.15}$$

yields (3.12). As for (3.13), we use Sobolev inequality and (3.14) to get

$$\|S(t)\phi\|_{L^p_{t,x}([0,T)\times\mathbb{R}^d)} \lesssim \left\||\nabla|^{\frac{d}{2}-\frac{d+2}{p}}\phi\right\|_{L^2(\mathbb{R}^d)}.$$

The same argument used to prove (3.12) then yields (3.13). □

It will be convenient to introduce the following norm and quasinorm:

Definition 3.9. Let $d \geq 3$ and $\theta \in (0,1)$. Given an interval $I \subset \mathbb{R}$, define the $Z(I)$-norm and $Z_\theta(I)$-quasinorm as follows:

$$\|u\|_{Z(I)} := \Bigg(\sum_{\substack{N\geq 1 \\ N \text{ dyadic}}} N^{d-2} \|\mathbb{P}_N u\|^4_{L^4_{t,x}(I\times\mathbb{R}^d)} \Bigg)^{\frac{1}{4}} \tag{3.16}$$

$$\|u\|_{Z_\theta(I)} := \|u\|^\theta_{Z(I)} \|u\|^{1-\theta}_{Y_2^{\frac{d-2}{2}}(I)} . \tag{3.17}$$

From Littlewood-Paley theory and Lemma 3.8, we have

$$\|u\|_{Z(I)} \lesssim \left\|\langle\nabla\rangle^{\frac{d-2}{4}} u\right\|_{L^4_{t,x}(I\times\mathbb{R}^d)} \lesssim \|u\|_{Y_{4-}^{\frac{d-2}{2}}(I)} \lesssim \|u\|_{Y_2^{\frac{d-2}{2}}(I)} , \tag{3.18}$$

and hence we also have

$$\|u\|_{Z_\theta(I)} \lesssim \|u\|_{Y_2^{\frac{d-2}{2}}(I)} . \tag{3.19}$$

The following bilinear estimates will be useful to us:

Lemma 3.10 (Bilinear Strichartz estimates)**.** Let $1 \leq M \leq N$ be dyadic numbers. Let $\theta \in (0,1)$. For space-time functions u, v, let $u_N := \mathbb{P}_N u$ and $v_M := \mathbb{P}_M v$. Then the following estimates hold:

$$\|u_N v_M\|_{L^2_{t,x}([0,T)\times\mathbb{R}^d)} \lesssim T^{0+} \left(\frac{M}{N}\right)^{\frac{1}{2}} M^{\frac{d-2}{2}} N^{0+} \|u_N\|_{Y^0_{2+}([0,T))} \|v_M\|_{Y^0_{2+}([0,T))} \tag{3.20}$$

$$\|u_N v_M\|_{L^2_{t,x}([0,T)\times\mathbb{R}^d)} \lesssim \left(\frac{M}{N}\right)^{\frac{1}{2}-} M^{\frac{d-2}{2}} \|u_N\|_{Y^0_2([0,T))} \|v_M\|_{Y^0_2([0,T))} \tag{3.21}$$

$$\|u_N v_M\|_{L^2_{t,x}([0,T)\times\mathbb{R}^d)} \lesssim \left(\frac{M}{N}\right)^{\frac{1}{2}(1-\theta)-} \|u_N\|_{Y^0_2([0,T))} \|v_M\|_{Z_\theta([0,T))} \tag{3.22}$$

Proof. We only prove (3.20) since the proof of (3.21) can be found in [6, Lemma 3.5] and that of (3.22) can be found in [6, Lemma 6.1].

Given $\phi_1, \phi_2 \in L^2(\mathbb{R}^d)$, let $w := S(t)P_M\phi_1 \cdot S(t)P_N\phi_2$. We first use Hölder's inequality to get

$$\begin{aligned}\|w\|_{L^2_{t,x}([0,T)\times\mathbb{R}^d)} &= \|\|w\|_{L^2_x}\|_{L^2_t([0,T))} \leq T^{\frac{\varepsilon}{2+2\varepsilon}} \|\|w\|_{L^2_x}\|_{L^{2+2\varepsilon}_t(\mathbb{R})} \\ &= T^{\frac{\varepsilon}{2+2\varepsilon}} \|w\|_{L^{2+2\varepsilon}_t(\mathbb{R};L^2_x(\mathbb{R}^d))}\end{aligned} \tag{3.23}$$

Then, by Minkowski's inequality and Sobolev's inequality, we have

$$\begin{aligned}\|w\|_{L^{2+2\varepsilon}_t(\mathbb{R};L^2_x(\mathbb{R}^d))} &\leq \|\|w\|_{L^{2+2\varepsilon}_t(\mathbb{R})}\|_{L^2_x(\mathbb{R}^d)} \leq \|\|w\|_{H_t^{\frac{\varepsilon}{2+2\varepsilon}}(\mathbb{R})}\|_{L^2_x(\mathbb{R}^d)} \\ &= \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}} |\tau|^{\frac{2\varepsilon}{2+2\varepsilon}} |\hat{w}(\tau,\xi)|^2 d\tau d\xi\right)^{\frac{1}{2}}.\end{aligned} \tag{3.24}$$

Now,

$$\widehat{w}(\tau,\xi) = \int_{\xi_1+\xi_2=\xi} \delta(\tau - |\xi_1|^2 - |\xi_2|^2)\widehat{P_N\phi_1}(\xi_1)\widehat{P_M\phi_2}(\xi_2)\, d\xi_2\,,$$

and so we must have $|\tau| \sim N^2 + M^2 \lesssim N^2$. Hence, (3.24) yields

$$\|w\|_{L^{2+2\varepsilon}_t(\mathbb{R};L^2_x(\mathbb{R}^d))} \leq N^{\frac{\varepsilon}{2+2\varepsilon}} \|w\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^d)}. \tag{3.25}$$

From Bourgain [11] and Ozawa-Tsutsumi [79], we have the following bilinear refinement of the Strichartz estimate:

$$\|w\|_{L^2_{t,x}(\mathbb{R}\times\mathbb{R}^d)} \lesssim M^{\frac{d-2}{2}} \left(\frac{M}{N}\right)^{\frac{1}{2}} \|P_M\phi_1\|_{L^2_x} \|P_N\phi_2\|_{L^2_x}\,. \tag{3.26}$$

By (3.25) and (3.26), we have

$$\|w\|_{L^{2+2\varepsilon}_t(\mathbb{R};L^2_x(\mathbb{R}^d))} \lesssim M^{\frac{d-1}{2}} N^{-\frac{1}{2}+\frac{\varepsilon}{1+\varepsilon}} \|P_M\phi_1\|_{L^2_{t,x}} \|P_N\phi_2\|_{L^2_{t,x}}\,.$$

Then applying the transference principle in Lemma 3.5 and using (3.15) gives:

$$\begin{aligned}\|u_N v_M\|_{L_t^{2+2\varepsilon}(\mathbb{R};L_x^2(\mathbb{R}^d))} &\lesssim M^{\frac{d-1}{2}} N^{-\frac{1}{2}+\frac{\varepsilon}{1+\varepsilon}} \|u_N\|_{V_\Delta^{2+\varepsilon}} \|v_M\|_{V_\Delta^{2+\varepsilon}} \\ &\lesssim M^{\frac{d-1}{2}} N^{-\frac{1}{2}+\frac{\varepsilon}{1+\varepsilon}} \|u_N\|_{Y^0_{2+\varepsilon}} \|v_M\|_{Y^0_{2+\varepsilon}}.\end{aligned}$$

The inequality (3.20) then follows from (3.23). □

We also recall the classical Hölder spaces:

Definition 3.11. Let $\gamma > 0$, the γ-Hölder space $\mathcal{C}^\gamma(\mathbb{R}; H)$ is the collection of functions $u : \mathbb{R} \to H$ such that the norm

$$\|u\|_{\mathcal{C}^\gamma(\mathbb{R};H)} := \sup_{\substack{t_1\neq t_2\\ t_1,t_2\in\mathbb{R}}} \frac{\|u(t_2)-u(t_1)\|_H}{|t_2-t_1|^\gamma} + \|u\|_{L^\infty(\mathbb{R};H)} \tag{3.27}$$

is finite.

We note that Hölder spaces and V^p-spaces are related in the following way.

Lemma 3.12. Let $p \geq 2$ and let I be a bounded interval of $\mathbb{R}$. The following inequality holds:

$$\|u\|_{V^p(I;H)} \lesssim (1+|I|)^{\frac{1}{p}} \|u\|_{\mathcal{C}^{\frac{1}{p}}(I;H)}. \tag{3.28}$$

Proof. Given $u \in V^p(I;H)$, we take the trivial extension $\tilde{u}$ such that on $\tilde{u} = u$ on I and $\tilde{u} = 0$ outside I. Then

$$\|u\|_{V^p(I;H)} \leq \|\tilde{u}\|_{V^p(\mathbb{R};H)} = \sup_{\substack{\{t_k\}_{k=0}^K\in\mathcal{Z}\\ t_1,\dots,t_{K-1}\in I}} \left(\sum_{k=1}^K \left(\frac{\|\tilde{u}(t_k)-\tilde{u}(t_{k-1})\|_H}{|\tilde{t}_k-\tilde{t}_{k-1}|^{\frac{1}{p}}}\right)^p \cdot |\tilde{t}_k-\tilde{t}_{k-1}|\right)^{\frac{1}{p}}$$

where $\tilde{t}_j = t_j$ if $t_j \in I$, $\tilde{t}_0 = t_1 - 1$ if $t_0 \notin I$ and $\tilde{t}_K = t_{K-1}+1$ if $t_K \notin I$. Note that

$$\frac{\|\tilde{u}(t_k)-\tilde{u}(t_{k-1})\|_H}{|\tilde{t}_k-\tilde{t}_{k-1}|^{\frac{1}{p}}} \leq \sup_{\substack{t_1\neq t_2\\ t_1,t_2\in\mathbb{R}}} \frac{\|u^*(t_2)-u^*(t_1)\|_H}{|t_2-t_1|^{\frac{1}{p}}}$$

for any extension u^* of u to $\mathbb{R}$ and all k except possibly the cases when $k=1$ and $k=K$ when t_k might not be in I. In these latter cases we have $\|\tilde{u}(t_k)-\tilde{u}(t_{k-1})\|_H \leq$

$\|u^*\|_{L^\infty(\mathbb{R};H)}$. Hence

$$\|u\|_{V^p(I;H)} \le (|I|+2)^{\frac{1}{p}} \|u^*\|_{\mathcal{C}^{\frac{1}{p}}(\mathbb{R};H)}.$$

Taking infimum over all u^* that extend u to $\mathbb{R}$ then gives (3.28). □

Finally, we recall for convenience Schur's test, which is used several times in Section 3.3.

Lemma 3.13 (Schur's test)**.** Let $2^{\mathbb{N}_0}$ denotes the set of dyadic numbers $N \ge 1$. Suppose that $K(M,N) : 2^{\mathbb{N}_0} \times 2^{\mathbb{N}_0} \to \mathbb{C}$ such that

$$\sup_{N\in 2^{\mathbb{N}_0}} \sum_{M\in 2^{\mathbb{N}_0}} |K(M,N)| + \sup_{M\in 2^{\mathbb{N}_0}} \sum_{N\in 2^{\mathbb{N}_0}} |K(M,N)| < \infty .$$

Then

$$\sum_{N\in 2^{\mathbb{N}_0}} \sum_{M\in 2^{\mathbb{N}_0}} |K(M,N)a_M b_N| \lesssim \|a_M\|_{\ell^2_M(2^{\mathbb{N}_0})} \|a_N\|_{\ell^2_N(2^{\mathbb{N}_0})} .$$

3.2 The stochastic convolution

In this Section, we state and prove some estimates on the stochastic convolution

$$\Psi(t) = -i \int_0^t S(t-t')\phi \, dW_{t'}$$

that appeared in the mild formulation (3.1). Symbolically, Ψ is the solution of the equation

$$\Psi = -(i\partial_t + \Delta)^{-1}\phi\xi.$$

Here, we assume that $\phi \in \mathrm{HS}^s$ for some $s \in \mathbb{R}$ (though we will eventually make the restriction $s \in (s_d, s_{\mathrm{crit}}]$ in the proof of local well-posedness). Firstly, we have the following estimate on space-time norms for the stochastic convolution. This appears in [77, Lemma 2.1], where the proof is a slight modification of an argument of de Bouard and Debussche [36]. We shall give the proof here for the convenience of the reader.

Proposition 3.14 (Space-time integrability of Ψ)**.** Let $d \ge 3$, $T > 0$, $q, r \in [1, \infty)$

and $\sigma \geq \max\{q, r\}$. If $2 \leq r \leq \frac{2d}{d-2}$, then there exists $C > 0$ such that

$$\mathbb{E}\left[\|\Psi\|^{\sigma}_{L^q_t L^r_x([0,T)\times\mathbb{R}^d)}\right]^{\frac{1}{\sigma}} \leq C\sqrt{\sigma}\, T^{\theta}\, \|\phi\|_{\mathrm{HS}^0} \tag{3.29}$$

for some $\theta > 0$. Moreover, there exist constant $c, C' > 0$ such that for any $R > 0$

$$\mathbb{P}\left(\|\Psi\|_{L^q_t L^r_x([0,T)\times\mathbb{R}^d)} > R\right) \leq C' e^{-cRT^{-\theta}\|\phi\|^{-1}_{\mathrm{HS}^0}} \tag{3.30}$$

Proof. Since $\sigma \geq \max\{q, r\}$, by applying Minkowski integral inequality twice and the fact that Ψ is Gaussian, we have

$$\begin{aligned}
\mathbb{E}\left[\|\Psi\|^{\sigma}_{L^q_t L^r_x([0,T)\times\mathbb{R}^d)}\right]^{\frac{1}{\sigma}} &\leq \left\|\left\|\|\Psi\|_{L^r_x(\mathbb{R}^d)}\right\|_{L^\sigma(\Omega)}\right\|_{L^q([0,T])} \leq \left\|\|\Psi\|_{L^\sigma(\Omega)}\right\|_{L^q_t L^r_x([0,T)\times\mathbb{R}^d)} \\
&\leq C\sqrt{\sigma}\left\|\|\Psi\|_{L^2(\Omega)}\right\|_{L^q_t L^r_x([0,T)\times\mathbb{R}^d)} \\
&= C\sqrt{\sigma}\left\|\left(\sum_{k\in\mathbb{N}}\int_0^t |S(t-t')\phi e_k|^2\, dt'\right)^{\frac{1}{2}}\right\|_{L^q_t L^r_x([0,T)\times\mathbb{R}^d)}
\end{aligned}$$

Therefore, applying Minkowski's inequality once more (using $r \geq 2$),

$$\begin{aligned}
&\leq C\sqrt{\sigma}\left\|\|S(t-t')\phi e_k\|_{L^2_{t'} L^r_x([0,t)\times\mathbb{R}^d)}\right\|_{L^q_t \ell^2_k([0,T)\times\mathbb{N})} \\
&\leq C\sqrt{\sigma}\left\|\|S(\tau)\phi e_k\|_{L^2_\tau L^r_x([0,t)\times\mathbb{N})}\right\|_{L^q_t \ell^2_k([0,T)\times\mathbb{R}^d)}
\end{aligned} \tag{3.31}$$

We use Hölder's inequality to get

$$\|S(\tau)\phi e_k\|_{L^2_\tau L^r_x([0,T)\times\mathbb{R}^d)} \leq CT^{\frac{1}{2}-\frac{1}{\tilde{q}}}\sqrt{\sigma}\,\|S(\tau)\phi e_k\|_{L^{\tilde{q}}_\tau L^r_x([0,T)\times\mathbb{R}^d)}$$

for some $\tilde{q} \geq 2$. We also want $(\tilde{q}, r)$ to be admissible, that is $\frac{2}{\tilde{q}} + \frac{d}{r} = \frac{d}{2}$. Such $\tilde{q}$ exists provided $r \leq \frac{2d}{d-2}$, which is our assumption. Hence we may apply Stichartz and Hölder inequalities to get

$$\mathrm{RHS}(3.31) \lesssim T^{\frac{1}{2}-\frac{1}{\tilde{q}}+\frac{1}{q}}\left\|\|\phi e_k\|_{L^2_x}\right\|_{\ell^2_k} = T^{\frac{1}{2}-\frac{1}{\tilde{q}}+\frac{1}{q}}\,\|\phi\|_{\mathrm{HS}^0}\ ,$$

which proves (3.29). To prove (3.30), we merely need to apply Chebychev's inequality. Indeed, for any $R > 0$, we have

$$\mathbb{P}\left(\|\Psi\|_{L_t^q L_x^r([0,T)\times\mathbb{R}^d)} > R\right) \leq \frac{1}{R^\sigma}\mathbb{E}\left[\|\Psi\|^k_{L_t^q L_x^r([0,T)\times\mathbb{R}^d)}\right]$$

Choose $k = \left(\frac{R}{eCT^\theta\|\phi\|_{\mathrm{HS}^0}}\right)^2$. If $k \geq \max\{q,r\}$, then we have

$$\mathbb{P}\left(\|\Psi\|_{L_t^q L_x^r([0,T)\times\mathbb{R}^d)} > R\right) \leq e^{-k} = e^{-Re^{-1}C^{-1}T^{-\theta}\|\phi\|^{-1}_{\mathrm{HS}^0}}.$$

If $k < \max\{q,r\}$, we choose $C' > 0$ such that $C'e^{-\max\{q,r\}} > 1$. Then

$$\mathbb{P}\left(\|\Psi\|_{L_t^q L_x^r([0,T)\times\mathbb{R}^d)} > R\right) \leq 1 < C'e^{-\max\{q,r\}} < C'e^{-k}.$$

□

Recall that we proved the continuity of the stochastic convolution in the periodic setting in Lemma 2.19. The same argument can be used to prove an analogous statement in the Euclidean setting. We record this in the lemma below.

Lemma 3.15 (Continuity of Ψ)**.** Let $s \geq 0$ and $T > 0$. Suppose that $\phi \in \mathrm{HS}^s$. Then $\Psi(\cdot)$ belongs to $C([0,T];H^s(\mathbb{R}^d))$ almost surely. Moreover, for any $\sigma \in [2,\infty)$,

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\Psi(t)\|^\sigma_{H^s(\mathbb{R}^d)}\right] \lesssim_T \|\phi\|^\sigma_{\mathrm{HS}^s}.$$

Our next goal is to prove that Ψ belongs to $Y^s_p((0,T])$ with $p > 2$, almost surely. To do so, we need to obtain a uniform moment bound on the $V^p_\Delta H^s$-norm of each dyadic piece $\mathbb{P}_N\Psi$. We apply Lemma 3.12 and get control on the $\mathcal{C}^{\frac{1}{p}}H^s$-norm of $S(-t)\mathbb{P}_N\Psi$. In particular, we use the following Kolmogorov type inequality, which we quote from [43, Theorem A.10].

Lemma 3.16 (Kolmogorov)**.** Let (S,d) be a Polish space and $T > 0$. Let $K > 0$, $k > r \geq 1$. Suppose that $\Phi : \Omega\times[0,T] \to S$ is a stochastic process such that

$$\|d(\Phi_t,\Phi_s)\|_{L^\sigma(\Omega)} \leq K|t-s|^{\frac{1}{r}}$$

for every $s, t \in [0, T]$. Then Φ has a continuous version (which we denote by Φ again) such that for any $\gamma \in [0, \frac{1}{r} - \frac{1}{k})$, there exists a constant $C(r, \gamma, T) > 0$ such that

$$\mathbb{E}\left[\|\Phi\|^k_{\mathcal{C}^\gamma([0,T])}\right]^{\frac{1}{k}} \leq C(r, \gamma, T)K\,. \tag{3.32}$$

Moreover, $C(r, \gamma, T) \to 0$ as $T \to 0$.

Note that the first part of the statement is simply the classical Kolmogorov continuity theorem. On the other hand, the bound (3.32) is obtained by applying the so-called Garsia–Rodemich–Rumsay inequality. See [43, Apendix A] for more details.

Proposition 3.17 (Ψ belongs to Y^s_p with $p > 2$)**.** Let $s \in \mathbb{R}$, $\phi \in \mathrm{HS}^s$, $p > 2$ and $T > 0$. Let $\sigma > 2$ be such that $\frac{1}{p} < \frac{1}{2} - \frac{1}{k}$. Then there exists a constant $C(p, T) > 0$ such that

$$\mathbb{E}\left[\|\Psi\|^k_{Y^s_p([0,T))}\right]^{\frac{1}{k}} \leq C(p, T)\sqrt{k}\,\|\phi\|_{\mathrm{HS}^s} \tag{3.33}$$

with $C(p, T) \to 0$ as $T \to 0$. In particular, the stochastic convolution Ψ belongs to $Y^s_p([0, T))$ almost surely. Moreover, there exist constants $K(p), c(p, T) > 0$ such that

$$\mathbb{P}\left(\|\Psi\|_{Y^s_p([0,T))} > R\right) \leq K(p)e^{-c(p,T)R^2\|\phi\|^{-2}_{\mathrm{HS}^s}}\,. \tag{3.34}$$

with $c(p, T) \to \infty$ as $T \to 0$.

Proof. In this proof, all spatial norms are on $\mathbb{R}^d$ and all temporal norms are on $[0, T)$, hence we omit the domains in our writing.

First, note that by Lemma 3.15 we have that $\Psi \in C([0, T]; H^s(\mathbb{R}^d))$. For a dyadic number $N \geq 1$, let $\Psi_N = \mathbb{P}_N\Psi$ and $\phi_N = \mathbb{P}_N\phi$. By (3.28), we have

$$\|\Psi_N\|_{V^p_\Delta H^s} \lesssim (T + 1)^{\frac{1}{p}}\,\|S(-t)\Psi_N\|_{\mathcal{C}^{\frac{1}{p}}H^s} \tag{3.35}$$

Now, for $0 \leq t_1 \leq t_2 \leq T$ and any $k > 2$,

$$\|S(-t_2)\Psi_N(t_2) - S(-t_1)\Psi_N(t_1)\|_{L^k(\Omega; H^s_x)} = \left\|\int_{t_1}^{t_2} S(-t')\phi_N\, dW_{t'}\right\|_{L^k(\Omega; H^s_x)}$$

$$= \left\| \sum_{j\in\mathbb{N}} \int_{t_1}^{t_2} e^{it'|\xi|^2} \langle\xi\rangle^s \widehat{\phi_N e_j}(\xi)\, d\beta_j(t') \right\|_{L^k(\Omega;L^2_\xi)}$$

$$\lesssim \left(k \int_{t_1}^{t_2} \|\phi_N\|_{\mathrm{HS}^s}^2 \, dt' \right)^{\frac{1}{2}}$$

$$= k^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}} \|\phi_N\|_{\mathrm{HS}^s} .$$

Since $\frac{1}{p} \in \big[0, \frac{1}{2} - \frac{1}{k}\big)$, Theorem 3.16 infers that

$$\mathbb{E}\bigg[\|S(-t)\Psi_N\|^k_{\mathcal{C}_t^{\frac{1}{p}} H_x^s} \bigg]^{\frac{1}{k}} \leq C k^{\frac{1}{2}} \|\phi_N\|_{\mathrm{HS}^s} \tag{3.36}$$

for some constant $C = C(p,T) > 0$ that tends to 0 as $T \to 0$. By Minkowski's inequality, (3.35) and (3.36), we have

$$\|\Psi\|_{L^k(\Omega;Y_p^s)} \lesssim \left(\sum_{\substack{N\geq 1\\ \text{dyadic}}} \|\Psi_N\|^2_{L^k(\Omega;V^p_\Delta H^s)} \right)^{\frac{1}{2}}$$

$$\lesssim (T+1)^{\frac{1}{p}} \left(\sum_N \|S(-t)\Psi_N\|^2_{L^k(\Omega;\ \mathcal{C}_t^{\frac{1}{p}} H_x^s)} \right)^{\frac{1}{2}}$$

$$\lesssim (T+1)^{\frac{1}{p}} C(p,T) \sqrt{k} \left(\sum_N \|\phi_N\|^2_{\mathrm{HS}^s} \right)^{\frac{1}{2}}$$

$$= (T+1)^{\frac{1}{p}} C(p,T) \sqrt{k}\, \|\phi\|_{\mathrm{HS}^s} .$$

This proves (3.33). To prove (3.34), we use Chebyshev inequality to get

$$\mathbb{P}\Big(\|\Psi\|_{Y_p^s} > R \Big) \leq \frac{\mathbb{E}\Big[\|\Psi\|^k_{Y_p^s} \Big]}{R^k} . \tag{3.37}$$

Set $k = \Big(\frac{R}{eC\|\phi\|_{\mathrm{HS}^s}} \Big)^2$. If $\frac{1}{p} < \frac{1}{2} - \frac{1}{k}$, then by (3.33), the above is

$$\leq \left(\frac{C(p,T)\sqrt{k}\, \|\phi\|_{\mathrm{HS}^s}}{R} \right)^k \leq e^{-R^2 e^{-2} C(p,T)^{-2} \|\phi\|^{-2}_{\mathrm{HS}^s}}$$

If $\frac{1}{p} \geq \frac{1}{2} - \frac{1}{k}$, i.e. $k \leq \frac{2p}{p-2}$, then

$$\mathbb{P}\left(\|\Psi\|_{Y_p^s} > R\right) \leq 1 = e^{\frac{2p}{p-2}} e^{-\frac{2p}{p-2}} \leq e^{\frac{2p}{p-2}} e^{-k}.$$

This proves (3.34). □

3.3 Local well-posedness

We proceed to prove Theorem 3.1 in this section. We are required to find a fixed point of the Duhamel formula

$$u(t) = S(t)u_0 \mp i \int_0^t S(t-t')\mathcal{N}(u)(t')\,dt' + \Psi(t),$$

where

$$\Psi(t) = -i \int_0^t S(t-t')\phi\,dW_{t'}. \tag{3.38}$$

We apply the Da Prato-Debussche trick in the following way: we set $v = u - \Psi$ and $v_0 = u_0$, then solving (SNLS) is equivalent to solving the fixed point problem

$$v(t) = \Gamma(v)(t) := S(t)v_0 \mp i \int_0^t S(t-t')\mathcal{N}(v+\Psi)(t')\,dt'. \tag{3.39}$$

To this end, we prove that Γ is a contraction in an appropriate closed subset of $Y_2^s([0,T))$, where $s > s_d$.

Before we continue, we first note down some common Strichartz norms we will use. Fix a small $\varepsilon > 0$. For any space-time functions u_1 and u_2, we have by Hölder's inequality that

$$\|u_1 u_2\|_{L^2_{t,x}} \leq \|u_1\|_{L_t^{2+\varepsilon} L_x^{\frac{2d(2+\varepsilon)}{d(2+\varepsilon)-4}}} \|u_2\|_{L_t^{\frac{2(2+\varepsilon)}{\varepsilon}} L_x^{\frac{d(2+\varepsilon)}{2}}}.$$

Now

$$\left(2+\varepsilon, \frac{2d(2+\varepsilon)}{d(2+\varepsilon)-4}\right)$$

is a Schrödinger admissible pair. Hence we may apply (3.12) to deduce that

$$\|u_1 u_2\|_{L^2_{t,x}} \le \|u_1\|_{Y^0_{2+}} \|u_2\|_{L_t^{\frac{2(2+\varepsilon)}{\varepsilon}} L_x^{\frac{d(2+\varepsilon)}{2}}} .$$

Note that the above mentioned pair tends to $\left(2, \frac{2d}{d-2}\right)$ as $\varepsilon \to 0$. Also, $\left(\frac{2(2+\varepsilon)}{\varepsilon}, \frac{d(2+\varepsilon)}{2}\right) \to (\infty, d)$ as $\varepsilon \to 0$. By an abuse of notation, we shall denote

$$\left(2+, \frac{2d}{d-2}-\right) := \left(2+\varepsilon, \frac{2d(2+\varepsilon)}{d(2+\varepsilon)-4}\right)$$

$$(\infty-, d+) := \left(\frac{2(2+\varepsilon)}{\varepsilon}, \frac{d(2+\varepsilon)}{2}\right) .$$

We will use a similar notation to denote other Lebesgue norms as well.

For $I \subset \mathbb{R}$ bounded, we now define $W^s(I)$ to be the space of functions such that the norm

$$\|u\|_{W^s(I)} := \begin{cases} \|u\|_{L_t^{\infty-}(I; L_x^{\frac{2d}{d-2}}(\mathbb{R}^d))}, & \text{if } d \ge 5 \\ \max\left(\|u\|_{L_t^{\infty-}(I; L_x^{\frac{2d}{d-2}}(\mathbb{R}^d))}, \|u\|_{L_t^{\frac{4(d+2)}{d}}(I; L_x^{\frac{4d(d+2)}{d^2+2d+4}}(\mathbb{R}^d))}\right), & \text{if } d = 4 \\ \max\left(\|u\|_{L_t^{\infty-}(I; L_x^{\frac{2d}{d-2}}(\mathbb{R}^d))}, \|u\|_{L_t^{\frac{4(d+2)}{d}}(I; L_x^{\frac{4d(d+2)}{d^2+2d+4}}(\mathbb{R}^d))}, \right. & \\ \qquad\qquad \left. \|u\|_{L_t^4(I, L_x^3(\mathbb{R}^d))}, \|u\|_{L_t^{\infty-}(I, L_x^{3+}(\mathbb{R}^d))}\right), & \text{if } d = 3 \end{cases}$$

is finite. For any $1 \le q < \infty$, $\quad 2 \le r \le \frac{2d}{d-2}$ we recall that, by Proposition 3.14, the $L_t^q L_x^r([0,T) \times \mathbb{R}^d)$-norm of Ψ is almost surely finite. All the pairs (q, r) appearing in the definition of $W^s(I)$ satisfy the above mentioned property, so the W^s-norm will be a convenient norm for measuring the contribution of Ψ.

We first prove a lemma on how we control the nonlinear term of Γ. Consider

$$\Lambda(v)(t) := \mp i \int_0^t S(t-t') \mathcal{N}(v+\Psi)(t')\, dt' , \tag{3.40}$$

so that $\Gamma(v) = S(\cdot)v_0 + \Lambda(v)$.

Lemma 3.18. Let $d \ge 3$, $s \in (s_d, s_{\text{crit}}]$, $\theta \in (0,1)$, and $I = [0,T)$ be a bounded interval. Let $\phi \in \mathrm{HS}^s$. Then there exists $\delta > 0$ such that

$$\|\Lambda(v)\|_{Y_2^{\frac{d-2}{2}}(I)} \lesssim \|v\|_{Y_2^{\frac{d-2}{2}}(I)} \|v\|_{Z_\theta(I)}^2 + \|\Psi\|_{W^s(I)}^3 + T^\delta \|\Psi\|_{Y_{2+}^s(I)}^3 + \|v\|_{Y_2^{\frac{d-2}{2}}(I)}^2 \|\Psi\|_{W^s(I)} + T^\delta \|\Psi\|_{Y_{2+}^s(I)}^2 \|v\|_{Y_2^{\frac{d-2}{2}}(I)} \tag{3.41}$$

and

$$\|\Lambda(v_2) - \Lambda(v_1)\|_{Y_2^{\frac{d-2}{2}}(I)} \lesssim \left(\sum_{j=1}^{2} \Theta(v_j, \Psi) \right) \|v_2 - v_1\|_{Y_2^{\frac{d-2}{2}}(I)},$$

where

$$\Theta(v_j, \Psi) := \|v_j\|_{Y_2^{\frac{d-2}{2}}(I)} \|v_j\|_{Z_\theta(I)} + \|v_j\|_{Y_2^{\frac{d-2}{2}}(I)} \|\Psi\|_{W^s(I)} + \|\Psi\|_{W^s(I)}^2 + T^\delta \|\Psi\|_{Y_{2+}^s(I)}^2,$$

for all $v, v_1, v_2 \in Y_2^{\frac{d-2}{2}}(I)$.

Proof. In this proof, all space-time norms will be on the domain $[0,T) \times \mathbb{R}^d$ and hence we often omit this to simplify the writing. Let $\Lambda_N(v) := \mathbb{P}_{\leq N}(\Lambda v)$. We first claim that $\mathbb{P}_{\leq N}\mathcal{N}(v+\Psi) \in L^1([0,T); H^{\frac{d-2}{2}}(\mathbb{R}^d))$ almost surely. Indeed, by Bernstein, Hölder and Sobolev inequalities and (3.12), we have

$$\begin{aligned} \|\mathbb{P}_{\leq N}\mathcal{N}(v+\Psi)\|_{L_t^1 H_x^{\frac{d-2}{2}}} &\lesssim N^{\frac{d-2}{2}} \|\mathbb{P}_{\leq N}\mathcal{N}(v+\Psi)\|_{L_t^1 L_x^2} \\ &\lesssim N^{\frac{d-2}{2}} \left(\|v\|_{L_t^3 L_x^6}^3 + \|\Psi\|_{L_t^3 L_x^6}^3 \right) \\ &\lesssim N^{\frac{d-2}{2}} \left(\left\| \langle \nabla \rangle^{\frac{d-2}{3}} v \right\|_{L_t^3 L_x^{\frac{6d}{3d-4}}}^3 + \left\| \langle \nabla \rangle^{\frac{d-3}{3}} \Psi \right\|_{L_t^3 L_x^{\frac{2d}{d-2}}}^3 \right) \\ &\lesssim N^{\frac{d-2}{2}} \left(\|v\|_{Y_2^{\frac{d-2}{3}}}^3 + \left\| \langle \nabla \rangle^{\frac{d-3}{3}} \Psi \right\|_{L_t^3 L_x^{\frac{2d}{d-2}}}^3 \right). \end{aligned} \tag{3.42}$$

By Proposition 3.14, the second term is finite almost surely, hence the claim. It follows by Lemma 3.7 that we have

$$\|\Lambda_N(v)\|_{Y_2^{\frac{d-2}{2}}} \lesssim \sup_{\substack{v_4 \in Y^0([0,T)) \\ \|v_4\|_{Y_2^0} = 1}} \left| \int_0^T \int_{\mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} \mathbb{P}_{\leq N}\mathcal{N}(v+\Psi)(t,x) \overline{v_4(t,x)} \, dx \, dt \right|$$

almost surely. We estimate the right-hand side above independently of the cutoff size N. As a result, by taking $N \to \infty$, the same inequality holds for $\Lambda(v)$, thus yielding (3.41). This leads us to analyse expressions of the form

$$\left| \int_0^T \int_{\mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} \mathbb{P}_{\leq N}(w_1 w_2 w_3) v_4 \, dx \, dt \right| , \tag{3.43}$$

where $w_1, w_2, w_3 \in \{v, \Psi\}$, and $\|v_4\|_{Y_2^0} = 1$. Note that we ignored any conjugate signs above since we will always use Hölder inequality to put each term in an appropriate mixed Lebesgue norm. In most cases, we dyadically decompose each term in (3.43), in the sense that

$$(3.43) = \sum_{\substack{N_1, \dots, N_4 \geq 1 \\ N_j \text{ dyadic}}} \left| \int_0^T \int_{\mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} \prod_{j=1}^{3} \mathbb{P}_{N_j} w_j \mathbb{P}_{N_4} v_4 \, dx \, dt \right| ,$$

but we shall continue to denote $\mathbb{P}_{N_j} w_j$ as w_j, and $\mathbb{P}_{N_4} v_4$ as v_4 to simplify the writing. Note that if we can afford a small derivative loss in the highest frequency, there is no problem in summing over the dyadic blocks.

Case 1: vvv case. This case is exactly the same as in [6, Proposition 6.3, Case 1], but we repeat the argument here for the sake of completeness. Without loss of generality, we may assume $N_1 \geq N_2, N_3$.

Subcase 1.a: $N_1 \sim N_4$. By Hölder inequality and (3.22), we have

$$\begin{aligned} \left| \int_0^T \int_{\mathbb{R}^d} \langle \nabla \rangle^{\frac{d-2}{2}} v_1 v_2 v_3 v_4 \, dx \, dt \right| &\lesssim N_1^{\frac{d-2}{2}} \|v_1 v_3\|_{L^2_{t,x}} \|v_2 v_4\|_{L^2_{t,x}} \\ &\lesssim \left(\frac{N_2 N_3}{N_1 N_4} \right)^{\frac{1}{2}(1-\theta)-} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|v_2\|_{Z_\theta} \|v_3\|_{Z_\theta} \|v_4\|_{Y_2^0} . \end{aligned}$$

By summing over $N_2 \leq N_1$ and $N_3 \leq N_1$ and then using Cauchy-Schwarz inequality in summing over $N_1 \sim N_4$, the contribution coming from this case is

$$\lesssim \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|v_2\|_{Z_\theta} \|v_3\|_{Z_\theta} .$$

Subcase 1.b: $N_1 \sim N_2 \gg N_4$. This time we apply Hölder's inequality again but only use (3.22) on the first factor, and then apply (3.13) on the v_4-factor to get

$$\begin{aligned}\left|\int_0^T \int_{\mathbb{R}^d} \langle\nabla\rangle^{\frac{d-2}{2}} v_1 v_2 v_3 v_4 \, dx\, dt\right| &\lesssim N_1^{\frac{d-2}{2}} \|v_1 v_3\|_{L^2_{t,x}} \|v_2\|_{L^4_{t,x}} \|v_4\|_{L^4_{t,x}} \\ &\lesssim \left(\frac{N_3}{N_1}\right)^{\frac{1}{2}(1-\theta)-} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|v_3\|_{Z_\theta} \|v_2\|_{L^4_{t,x}} \|v_4\|_{L^4_{t,x}} \\ &\lesssim \left(\frac{N_3}{N_1}\right)^{\frac{1}{2}(1-\theta)-} \left(\frac{N_4}{N_2}\right)^{\frac{d-2}{4}} N_2^{\frac{d-2}{4}} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|\mathbb{P}_{N_3} v_3\|_{Z_\theta} \|v_2\|_{L^4_{t,x}} \|v_4\|_{Y_2^0}\,.\end{aligned}$$

Summing over N_3 and taking supremum over N_2 yield

$$\lesssim \|v_2\|_Z \|v_3\|_{Z_\theta} \left(\frac{N_4}{N_1}\right)^{\frac{d-2}{4}} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|v_4\|_{Y_2^0}\,.$$

By Schur's test, we have

$$\lesssim \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|v_2\|_{Z_\theta} \|v_3\|_{Z_\theta}\,.$$

Case 2: $\Psi\Psi\Psi$ *case.* By symmetry, we may assume $N_3 \geq N_2 \geq N_1$.

Subcase 2.a: $N_2 \sim N_3$. We consider three cases in estimating

$$\left|\int_0^T \int_{\mathbb{R}^d} \Psi_1 \langle\nabla\rangle^{\frac{d-2}{4}} \Psi_2 \langle\nabla\rangle^{\frac{d-2}{4}} \Psi_3 v_4 \, dx\, dt\right|,$$

namely $d = 3$, $d = 4$ and $d \geq 5$. For $d = 3$, by Hölder's inequality we have:

$$\begin{aligned}&\left|\int_0^T \int_{\mathbb{R}^d} \Psi_1 \langle\nabla\rangle^{\frac{1}{4}} \Psi_2 \langle\nabla\rangle^{\frac{1}{4}} \Psi_3 v_4 \, dx\, dt\right| \\ &\leq N_1^{-s} N_2^{\frac{1}{4}-s} N_3^{\frac{1}{4}-s} \|\langle\nabla\rangle^s \Psi_1\|_{L_t^{6-} L_x^6} \|\langle\nabla\rangle^s \Psi_2\|_{L_t^6 L_x^{3+}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^6 L_x^{3+}} \|v_4\|_{L_t^{2+} L_x^{6-}}\end{aligned}$$

whose contribution is bounded by $\|\Psi\|_{W^s}^3$ provided $s > \frac{1}{4} = s_3$.

For $d = 4$, by Hölder's inequality and Sobolev's inequality we have:

$$\begin{aligned}&\left|\int_0^T \int_{\mathbb{R}^d} \Psi_1 \langle\nabla\rangle^{\frac{1}{2}} \Psi_2 \langle\nabla\rangle^{\frac{1}{2}} \Psi_3 v_4 \, dx\, dt\right| \\ &\leq \|\Psi_1\|_{L_t^{6-} L_x^{4+}} \|\langle\nabla\rangle^{\frac{1}{2}} \Psi_2\|_{L_t^6 L_x^4} \|\langle\nabla\rangle^{\frac{1}{2}} \Psi_3\|_{L_t^6 L_x^4} \|v_4\|_{L_t^{2+} L_x^{4-}}\end{aligned}$$

$$\leq N_1^{-s+} N_2^{\frac{1}{2}-s} N_3^{\frac{1}{2}-s} \|\langle\nabla\rangle^s \Psi_1\|_{L_t^{6-} L_x^4} \|\langle\nabla\rangle^s \Psi_2\|_{L_t^6 L_x^4} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^6 L_x^4} \|v_4\|_{Y_2^0}$$

whose contribution is bounded by $\|\Psi\|_{W^s}^3$ provided $s > \frac{1}{2}$ (which is less restrictive than $s > s_4 = \frac{3}{5}$).

Finally, for $d \geq 5$, by Hölder's inequality and Sobolev's inequalties we have:

$$\left| \int_0^T \int_{\mathbb{R}^d} \Psi_1 \langle\nabla\rangle^{\frac{d-2}{4}} \Psi_2 \langle\nabla\rangle^{\frac{d-2}{4}} \Psi_3 v_4 \, dx \, dt \right|$$

$$\leq \|\Psi_1\|_{L_t^{6-} L_x^{\frac{3d}{2}}} \|\langle\nabla\rangle^{\frac{d-2}{4}} \Psi_2\|_{L_t^6 L_x^{\frac{12d}{3d+2}}} \|\langle\nabla\rangle^{\frac{d-2}{4}} \Psi_3\|_{L_t^6 L_x^{\frac{12d}{3d+2}}} \|v_4\|_{L_t^{2+} L_x^{\frac{2d}{d-2}}}$$

$$\leq \|\langle\nabla\rangle^{\frac{3d-10}{6}} \Psi_1\|_{L_t^{6-} L_x^{\frac{2d}{d-2}}} \|\langle\nabla\rangle^{\frac{d-2}{4}+\frac{3d-14}{12}} \Psi_2\|_{L_t^6 L_x^{\frac{2d}{d-2}}} \|\langle\nabla\rangle^{\frac{d-2}{4}+\frac{3d-14}{12}} \Psi_3\|_{L_t^6 L_x^{\frac{2d}{d-2}}} \|v_4\|_{L_t^{2+} L_x^{\frac{2d}{d-2}}}$$

$$\leq (N_1 N_2 N_3)^{\frac{3d-10}{6}-s} \|\langle\nabla\rangle^s \Psi_1\|_{L_t^{6-} L_x^{\frac{2d}{d-2}}} \|\langle\nabla\rangle^s \Psi_2\|_{L_t^6 L_x^{\frac{2d}{d-2}}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^6 L_x^{\frac{2d}{d-2}}} \|v_4\|_{Y_2^0}$$

whose contribution is bounded by $\|\Psi\|_{W^s}^3$ provided $s > \frac{3d-10}{6}$. Note that this is less restrictive than $s > s_d$.

Subcase 2.b: $N_2 \ll N_3 \sim N_4$. Let $\beta = \frac{1}{2}$ if $d = 3$ and $\beta = \frac{1}{3}$ if $d \geq 4$.

Subsubcase 2.b.i: $N_1, N_2 \ll N_3^\beta$. By Hölder inequality, (3.20) and using that $s < s_{\text{crit}} = \frac{d-2}{2}$, we have

$$\begin{aligned}\left| \int_0^T \int_{\mathbb{R}^d} \Psi_1 \Psi_2 \langle\nabla\rangle^{\frac{d-2}{2}} \Psi_3 v_4 \, dx \, dt \right| &\leq N_3^{\frac{d-2}{2}} \|\Psi_2 \Psi_3\|_{L^2_{t,x}} \|\Psi_1 v_4\|_{L^2_{t,x}} \\ &\lesssim T^{0+} N_2^{\frac{d-1}{2}-s} N_3^{\frac{d-3}{2}-s+} N_1^{\frac{d-1}{2}-s} N_4^{-\frac{1}{2}+} \prod_{j=1}^3 \|\Psi_j\|_{Y_{2+}^s} \|v_4\|_{Y_2^0} \\ &\lesssim T^{0+} N_3^{\frac{d-4}{2}-s+\beta(d-1-2s)} \prod_{j=1}^3 \|\Psi_j\|_{Y_{2+}^s} .\end{aligned}$$

The exponent on N_3 is negative provided $s > s_d$. Hence the contribution to (3.43) in this case is

$$\lesssim T^{0+} \|\Psi\|_{Y_{2+}^s}^3 .$$

Subsubcase 2.b.ii: $N_1 \ll N_3^\beta \lesssim N_2$. By Hölder's inequality, (3.20) and using that

$s < s_{\text{crit}} = \frac{d-2}{2}$, we have

$$\left| \int_0^T \int_{\mathbb{R}^d} \Psi_1 \Psi_2 \langle \nabla \rangle^{\frac{d-2}{2}} \Psi_3 v_4 \, dx \, dt \right| \leq N_3^{\frac{d-2}{2}} \|\Psi_2\|_{L_t^4 L_x^d} \|\Psi_3\|_{L_t^4 L_x^{\frac{2d}{d-2}}} \|\Psi_1 v_4\|_{L^2_{t,x}}$$

$$\lesssim T^{0+} N_2^{-s} N_3^{\frac{d-2}{2}-s} N_1^{\frac{d-1}{2}-s} N_4^{-\frac{1}{2}+} \|\langle \nabla \rangle^s \Psi_2\|_{L_t^4 L_x^d} \|\langle \nabla \rangle^s \Psi_3\|_{L_t^4 L_x^{\frac{2d}{d-2}}} \|\Psi_1\|_{Y^s_{2+}} \|v_4\|_{Y^0_2}$$

$$\lesssim T^{0+} N_3^{\frac{d-3}{2}-s+\beta\left(\frac{d-1}{2}-2s\right)+} \|\langle \nabla \rangle^s \Psi_2\|_{L_t^4 L_x^d} \|\langle \nabla \rangle^s \Psi_3\|_{L_t^4 L_x^{\frac{2d}{d-2}}} \|\Psi_1\|_{Y^s_{2+}} .$$

If $d = 3$, the exponent of N_3 is negative provided $s > \frac{1}{4} = s_3$. If $d = 4$, it is negative for $s > \frac{3}{5} = s_4$. If $d \geq 5$, we further apply Sobolev inequality to get

$$\lesssim T^{0+} N_3^{\frac{d-3}{2}-s+\beta\left(\frac{2d-5}{2}-2s\right)+} \prod_{j=2}^{3} \|\langle \nabla \rangle^s \Psi_j\|_{L_t^4 L_x^{\frac{2d}{d-2}}} \|\Psi_1\|_{Y^s_{2+}} .$$

In this case, we need $s > \frac{5d-14}{10} = s_d$ to hold. Hence the contribution to (3.43) in this case is

$$\lesssim T^{0+} \|\Psi\|^2_{W^s} \|\Psi\|_{Y^s_{2+}}$$

Subsubcase 2.b.iii: $N_1, N_2 \gtrsim N_3^{\beta}$. By Hölder's inequality,

$$\left| \int_0^T \int_{\mathbb{R}^d} \Psi_1 \Psi_2 \langle \nabla \rangle^{\frac{d-2}{2}} \Psi_3 v_4 \, dx \, dt \right| \lesssim N_3^{\frac{d-2}{2}} \prod_{j=1}^{2} \|\Psi_j\|_{L_t^{6-} L_x^{d+}} \|\Psi_3\|_{L_t^{6-} L_x^{\frac{2d}{d-2}}} \|v_4\|_{L_t^{2+} L_x^{\frac{2d}{d-2}-}}$$

$$\lesssim N_3^{\frac{d-2}{2}-s-2\beta s} \prod_{j=1}^{2} \|\langle \nabla \rangle^s \Psi_j\|_{L_t^{6-} L_x^{d+}} \|\langle \nabla \rangle^s \Psi_3\|_{L_t^{6-} L_x^{\frac{2d}{d-2}}} \|v_4\|_{Y^0_2} .$$

If $d = 3$, the exponent over N_3 is negative if $s > \frac{1}{4} = s_3$. If $d \geq 4$, we further apply Sobolev inequality to get

$$\lesssim N_3^{\frac{d-2}{2}-s+2\beta\left(\frac{d-4}{2}-s\right)+} \prod_{j=1}^{3} \|\langle \nabla \rangle^s \Psi_j\|_{L_t^{6-} L_x^{\frac{2d}{d-2}}} \|v_4\|_{Y^0_2} .$$

In this case, the exponent of N_3 is negative if $s > \frac{5d-14}{10} = s_d$. Hence the contribution to (3.43) from this case is

$$\lesssim \|\Psi\|^3_{W^s} .$$

Case 3: $vv\Psi$ case. By symmetry, we may assume $N_1 \geq N_2$.

Subcase 3.a: $N_1 \gtrsim N_3$. We only apply dyadic decomposition to v_1, v_2 and Ψ_3. In this case, we have $N_1 \sim \max\{N_2, N_3, |\xi_4|\}$ where ξ_4 is the spatial frequency of v_4. By Hölder inequality, (3.21) and Sobolev inequality,

$$\left|\int_0^T \int_{\mathbb{R}^d} \langle\nabla\rangle^{\frac{d-2}{2}} v_1 v_2 \Psi_3 v_4 \, dx\, dt\right| \lesssim \left\|\langle\nabla\rangle^{\frac{d-2}{2}} v_1 v_2\right\|_{L^2_{t,x}} \|\Psi_3 v_4\|_{L^2_{t,x}}$$

$$\lesssim \left(\frac{N_2}{N_1}\right)^{\frac{1}{2}-} \prod_{j=1}^2 \|v_j\|_{Y_2^{\frac{d-2}{2}}} \|\Psi_3\|_{L_t^{\infty-} L_x^{d+}} \|v_4\|_{L_t^{2+} L_x^{\frac{2d}{d-2}-}}$$

$$\lesssim \left(\frac{N_2}{N_1}\right)^{\frac{1}{2}-} \prod_{j=1}^2 \|v_j\|_{Y_2^{\frac{d-2}{2}}} N_3^{-s+} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-} L_x^{d+}} \|v_4\|_{Y_2^0}\,.$$

If $d \geq 4$, we further apply Sobolev inequality to get

$$\lesssim \left(\frac{N_2}{N_1}\right)^{\frac{1}{2}-} \prod_{j=1}^2 \|v_j\|_{Y_2^{\frac{d-2}{2}}} N_3^{\frac{d-4}{2}-s+} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-} L_x^{\frac{2d}{d-2}}} \|v_4\|_{Y_2^0}\,.$$

By Schur's test, summing in N_3 and using that $s > \frac{d-4}{2}$, the contribution to (3.43) in this case is

$$\lesssim \|v\|^2_{Y_2^{\frac{d-2}{2}}} \|\Psi\|_{W^s}\,.$$

Subcase 3.b: $N_1 \ll N_3$. For $d = 3$, by Hölder's inequality, (3.21) and the fact that $(4, 3)$ is a Schrödinger admissible pair, we have

$$\left|\int_0^T \int_{\mathbb{R}^d} v_1 v_2 \langle\nabla\rangle^{\frac{1}{2}} \Psi_3 v_4 \, dx\, dt\right| \leq \|v_1 \langle\nabla\rangle^{\frac{1}{2}} \Psi_3\|_{L^2_{t,x}} \|v_2 v_4\|_{L^2_{t,x}}$$

$$\leq N_1^{-\frac{d-2}{2}} N_2^{\frac{1}{2}-} N_3^{\frac{1}{2}-s} N_4^{-\frac{1}{2}+} \|\langle\nabla\rangle^{\frac{d-2}{2}} v_1\|_{L_t^4 L_x^3} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^4 L_x^6} \|v_2\|_{Y_2^{\frac{d-2}{2}}} \|v_4\|_{Y_2^0}$$

$$\leq N_1^{-\frac{d-3}{2}} N_3^{-s+} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^4 L_x^6} \|v_2\|_{Y_2^{\frac{d-2}{2}}} \|v_4\|_{Y_2^0}$$

whose contribution is $\|v\|^2_{Y_2^{\frac{d-2}{2}}} \|\Psi\|_{W^s}$ provided $s > 0$.

For $d \geq 4$, by Hölder's inequality, (3.21) and Sobolev's inequality, we have

$$\left|\int_0^T \int_{\mathbb{R}^d} v_1 v_2 \langle\nabla\rangle^{\frac{d-2}{2}} \Psi_3 v_4 \, dx\, dt\right| \leq \left\|v_1 \langle\nabla\rangle^{\frac{d-2}{2}} \Psi_3\right\|_{L^2_{t,x}} \|v_2 v_4\|_{L^2_{t,x}}$$

$$\lesssim N_3^{\frac{d-2}{2}-s} N_2^{\frac{1}{2}-} N_4^{-\frac{1}{2}+} \|v_1\|_{L_t^{2+} L_x^d} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-} L_x^{\frac{2d}{d-2}}} \|v_2\|_{Y_2^{\frac{d-2}{2}}} \|v_4\|_{Y_2^0}$$

$$\lesssim N_3^{\frac{d-2}{2}-s} N_2^{\frac{1}{2}-} N_4^{-\frac{1}{2}+} \left\|\langle\nabla\rangle^{\frac{d-4}{2}-} v_1\right\|_{L_t^{2+} L_x^{\frac{2d}{d-2}-}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-} L_x^{\frac{2d}{d-2}}} \|v_2\|_{Y_2^{\frac{d-2}{2}}} \|v_4\|_{Y_2^0}$$

$$\lesssim N_3^{\frac{d-3}{2}-s+} N_1^{-\frac{1}{2}+} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-} L_x^{\frac{2d}{d-2}}} \|v_2\|_{Y_2^{\frac{d-2}{2}}} \|v_4\|_{Y_2^0}.$$

The exponent over N_3 is negative provided $s > \frac{d-3}{2}$, which is less restrictive than $s > s_d$. Hence the contribution coming from this case is

$$\lesssim \|v\|^2_{Y_2^{\frac{d-2}{2}}} \|\Psi\|_{W^s}.$$

Case 4: $v\Psi\Psi$ case. By symmetry, we may assume $N_3 \geq N_2$.

Subcase 4.a: $N_1 \gtrsim N_3$. By Hölder's inequality and (3.12),

$$\left|\int_0^T \int_{\mathbb{R}^d} \langle\nabla\rangle^{\frac{d-2}{2}} v_1 \Psi_2 \Psi_3 v_4 \, dx\, dt\right| \lesssim \left\|\langle\nabla\rangle^{\frac{d-2}{2}} v_1\right\|_{L_t^{2+} L_x^{\frac{2d}{d-2}-}} \prod_{j=2}^{3} \|\Psi_j\|_{L_t^{\infty-} L_x^{d+}} \|v_4\|_{L_t^{2+} L_x^{\frac{2d}{d-2}-}}$$

$$\lesssim N_2^{-s} N_3^{-s} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \prod_{j=2}^{3} \|\langle\nabla\rangle^s \Psi_j\|_{L_t^{\infty-} L_x^{d+}} \|v_4\|_{Y_2^0}.$$

If $d \geq 4$, we further apply Sobolev inequality to get

$$\lesssim N_2^{\frac{d-4}{2}-s+} N_3^{\frac{d-4}{2}-s+} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \prod_{j=2}^{3} \|\Psi_j\|_{L_t^{\infty-} L_x^{\frac{2d}{d-2}}} \|v_4\|_{Y_2^0},$$

where the exponent over N_2 and N_3 are negative provided $s > \frac{d-4}{2}$, which is less restrictive than $s > s_d$. If $N_3 \gtrsim \max(N_1, N_4)$, then this allows us to sum over N_1 and N_4. Otherwise, we have $N_1 \sim N_4 \gg N_3$, in which case we can use Cauchy-Schwarz inequality to sum over $N_1 \sim N_4$. Hence the contribution to (3.43) in this case is

$$\lesssim \|v\|_{Y_2^{\frac{d-2}{2}}} \|\Psi\|^2_{W^s}.$$

Subcase 4.b: $N_3 \gg N_1$. Suppose first that $N_2 \sim N_3$, then we must have $N_4 \lesssim N_3$. By Hölder's inequality and (3.13),

$$\left|\int_0^T \int_{\mathbb{R}^d} v_1 \Psi_2 \langle\nabla\rangle^{\frac{d-2}{2}} \Psi_3 v_4 \, dx\, dt\right| \lesssim N_3^{\frac{d-2}{2}} \|v_1\|_{L^{d+2}_{t,x}} \prod_{j=2}^{3} \|\Psi_j\|_{L_t^{\frac{4(d+2)}{d}-} L_x^{\frac{4d(d+2)}{d^2+2d+4}+}} \|v_4\|_{L_t^{2+} L_x^{\frac{2d}{d-2}-}}$$

$$\lesssim N_3^{\frac{d-2}{2}-2s} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \prod_{j=2}^{3} \|\langle\nabla\rangle^s \Psi_j\|_{L_t^{\frac{4(d+2)}{d}-} L_x^{\frac{4d(d+2)}{d^2+2d+4}+}} \|v_4\|_{Y_2^0}$$

Note that $\frac{4d(d+2)}{d^2+2d+4} < \frac{2d}{d-2}$ if $d \leq 4$. If $d \geq 5$, we apply Sobolev inequality to get

$$\lesssim N_3^{\frac{d-2}{2}-2s+\frac{d^2-2d-12}{2(d+2)}+} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \prod_{j=2}^{3} \|\langle\nabla\rangle^s \Psi_j\|_{L_t^{\frac{4(d+2)}{d}-} L_x^{\frac{2d}{d-2}}} \|v_4\|_{Y_2^0}$$

The condition for the exponent of N_3 to be negative is $s > \frac{d-2}{4}$ if $d \leq 4$ and $s > \frac{d^2-d-8}{2(d+2)}$ if $d \geq 5$. These are less restrictive than $s > s_d$ for $d \geq 4$. Hence the contribution from this case is

$$\lesssim \|v\|_{Y_2^{\frac{d-2}{2}}} \|\Psi\|_{W^s}^2 .$$

Thus it remains to consider the case $N_4 \sim N_3 \gg N_2, N_1$.

Subsubcase 4.b.i: $N_1, N_2 \ll N_3^{\frac{1}{3}}$. By Hölder's inequality, (3.20) and (3.21), we have that

$$\left|\int_0^T \int_{\mathbb{R}^d} v_1 \Psi_2 \langle\nabla\rangle^{\frac{d-2}{2}} \Psi_3 v_4 \, dx\, dt\right| \leq \left\|\Psi_2 \langle\nabla\rangle^{\frac{d-2}{2}} \Psi_3\right\|_{L^2_{t,x}} \|v_1 v_4\|_{L^2_{t,x}}$$

$$\lesssim T^{0+} N_2^{\frac{d-1}{2}-s} N_3^{\frac{d-3}{2}-s+} N_1^{\frac{1}{2}} N_4^{-\frac{1}{2}+} \prod_{j=2}^{3} \|\Psi_j\|_{Y^s_{2+}} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|v_4\|_{Y_2^0}$$

$$\lesssim T^{0+} N_3^{\frac{2d-6-4s}{3}+} \prod_{j=2}^{3} \|\Psi_j\|_{Y^s_{2+}} \|v_1\|_{Y_2^{\frac{d-2}{2}}} .$$

The exponent of N_3 is negative provided

$$s > \frac{d-3}{2}, \tag{3.44}$$

which is less restrictive than $s > s_d$. Hence the contribution coming from this case is

$$\lesssim T^{0+} \|\Psi\|_{Y^s_{2+}}^2 \|v\|_{Y_2^{\frac{d-2}{2}}} .$$

Subsubcase 4.b.ii: $N_1 \ll N_3^{\frac{1}{3}} \lesssim N_2$. By Hölder's inequality,

$$\left|\int_0^T \int_{\mathbb{R}^d} v_1 \Psi_2 \langle\nabla\rangle^{\frac{d-2}{2}} \Psi_3 v_4 \, dx\, dt\right| \leq \|\Psi_2\|_{L_t^4 L_x^d} \left\|\langle\nabla\rangle^{\frac{d-2}{2}}\Psi_3\right\|_{L_t^4 L^{\frac{2d}{d-2}}} \|v_1 v_4\|_{L^2_{t,x}}$$

$$\lesssim N_1^{\frac{1}{2}} N_4^{-\frac{1}{2}+} N_2^{-s} N_3^{\frac{d-2}{2}-s} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|v_4\|_{Y_2^0} \|\langle\nabla\rangle^s \Psi_2\|_{L_t^4 L_x^d} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^4 L_x^{\frac{2d}{d-2}}}$$

$$\lesssim N_3^{\frac{3d-8-8s}{6}+} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|\langle\nabla\rangle^s \Psi_2\|_{L_t^4 L_x^d} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^4 L_x^{\frac{2d}{d-2}}} \|v_4\|_{Y_2^0} .$$

If $d = 3$, the exponent of N_3 is negative provided $s > \frac{1}{8}$. If $d \geq 4$, we further apply Sobolev inequality to get

$$\lesssim N_3^{\frac{2d-6-4s}{3}+} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|\langle\nabla\rangle^s \Psi_2\|_{L_t^4 L_x^{\frac{2d}{d-2}}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^4 L_x^{\frac{2d}{d-2}}} .$$

In this case, the exponent of N_3 remains negative if the condition (3.44) holds, which is less restrictive than $s > s_d$. Hence the contribution from this case is

$$\lesssim \|v\|_{Y_2^{\frac{d-2}{2}}} \|\Psi\|_{W^s}^2 .$$

Subsubcase 4.b.iii: $N_2 \ll N_3^{\frac{1}{3}} \lesssim N_1$. For $d = 3$, by Hölder's inequality, (3.20), and the fact that $(4, 3)$ is a Schrödinger admissible pair, we have

$$\left|\int_0^T \int_{\mathbb{R}^d} v_1 \Psi_2 \langle\nabla\rangle^{\frac{1}{2}} \Psi_3 v_4 \, dx\, dt\right| \leq N_3^{\frac{1}{2}} \|v_1 \Psi_3\|_{L^2_{t,x}} \|\Psi_2 v_4\|_{L^2_{t,x}}$$

$$\lesssim T^{0+} N_1^{-\frac{1}{2}} N_3^{\frac{1}{2}-s+} N_2^{1-s} N_4^{-\frac{1}{2}+} \left\|\langle\nabla\rangle^{\frac{1}{2}} v_1\right\|_{L_t^4 L_x^3} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^4 L_x^6} \|\Psi_2\|_{Y^s_{2+}} \|v_4\|_{Y_2^0}$$

$$\lesssim T^{0+} N_3^{\frac{1-8s}{6}+} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-} L_x^{\frac{2d}{d-2}}} \|\Psi_2\|_{Y^s_{2+}} \|v_4\|_{Y_2^0} .$$

The exponent of N_3 is negative if $s > \frac{1}{8}$, which is less restrictive than $s > s_3$.

For $d \geq 4$, by Hölder's inequality, (3.20), Sobolev inequality, and the fact that

$s < s_{\text{crit}}$, we have

$$\left|\int_0^T \int_{\mathbb{R}^d} v_1 \Psi_2 \langle\nabla\rangle^{\frac{d-2}{2}} \Psi_3 v_4 \, dx\, dt\right| \leq N_3^{\frac{d-2}{2}} \|v_1 \Psi_3\|_{L^2_{t,x}} \|\Psi_2 v_4\|_{L^2_{t,x}}$$

$$\lesssim T^{0+} N_3^{\frac{d-2}{2}-s+} N_2^{\frac{d-1}{2}-s} N_4^{-\frac{1}{2}+} \|v_1\|_{L_t^{2+}L_x^d} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}}} \|\Psi_2\|_{Y^s_{2+}} \|v_4\|_{Y^0_2}$$

$$\lesssim T^{0+} N_1^{-\frac{d-2}{2}+\frac{d-4}{2}+} N_3^{\frac{d-2}{2}-s+} N_2^{\frac{d-1}{2}-s} N_4^{-\frac{1}{2}+}$$

$$\times \left\|\langle\nabla\rangle^{\frac{d-2}{2}} v_1\right\|_{L_t^{2+}L_x^{\frac{2d}{d-2}-}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}}} \|\Psi_2\|_{Y^s_{2+}} \|v_4\|_{Y^0_2}$$

$$\lesssim T^{0+} N_3^{\frac{2d-6-4s}{3}+} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}}} \|\Psi_2\|_{Y^s_{2+}} \|v_4\|_{Y^0_2} .$$

Here, the exponent of N_3 is negative if (3.44) holds, which is less restrictive than $s > s_d$. Hence the contribution coming from this case is

$$\lesssim T^{0+} \|v\|_{Y_2^{\frac{d-2}{2}}} \|\Psi\|_{W^s} \|\Psi\|_{Y^s_{2+}} .$$

Subsubcase 4.b.iv: $N_3^{\frac{1}{3}} \lesssim N_1, N_2$. For $d = 3$, by Hölder's inequality, using the fact that $(4,3)$ is Schrödinger admissible, and (3.12) we have

$$\left|\int_0^T \int_{\mathbb{R}^d} v_1 \Psi_2 \langle\nabla\rangle^{\frac{1}{2}} \Psi_3 v_4 \, dx\, dt\right|$$

$$\lesssim N_3^{\frac{1}{2}-s} N_1^{-\frac{1}{2}} N_2^{-s} \left\|\langle\nabla\rangle^{\frac{1}{2}} v_1\right\|_{L_t^4 L_x^3} \|\langle\nabla\rangle^s \Psi_2\|_{L_t^4 L_x^{3+}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-}L_x^6} \|v_4\|_{L_t^{2+}L_x^{6-}}$$

$$\lesssim N_3^{\frac{1-4s}{3}} \|v_1\|_{Y_2^{\frac{1}{2}}} \|\langle\nabla\rangle^s \Psi_2\|_{L_t^4 L_x^{3+}} \|\langle\nabla\rangle^s \Psi_3\|_{L_t^{\infty-}L_x^6} \|v_4\|_{Y^0_2}.$$

The contribution in this case is $\|v\|_{Y_2^{\frac{d-2}{2}}} \|\Psi\|^2_{W^s}$ provided that $s > \frac{1}{4} = s_3$.

For $d \geq 4$, by Hölder's inequality and Sobolev inequality, we have

$$\left|\int_0^T \int_{\mathbb{R}^d} v_1 \Psi_2 \langle\nabla\rangle^{\frac{d-2}{2}} \Psi_3 v_4 \, dx\, dt\right|$$

$$\leq N_3^{\frac{d-2}{2}} \|v_1\|_{L_t^{2+}L_x^d} \|\Psi_2\|_{L_t^{\infty-}L_x^{d+}} \|\Psi_3\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}}} \|v_4\|_{L_t^{2+}L_x^{\frac{2d}{d-2}-}}$$

$$\lesssim N_3^{\frac{d-2}{2}-s} N_1^{-1+} N_2^{\frac{d-4}{2}-s+} \left\|\langle\nabla\rangle^{\frac{d-2}{2}} v_1\right\|_{L_t^{2+}L_x^{\frac{2d}{d-2}-}} \|\langle\nabla\rangle^s \Psi_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}}}$$

$$\times \|\langle\nabla\rangle^s\Psi_3\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}}} \|v_4\|_{Y_2^0}$$

$$\lesssim N_3^{\frac{2d-6-4s}{3}+} \|v_1\|_{Y_2^{\frac{d-2}{2}}} \|\langle\nabla\rangle^s\Psi_2\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}}} \|\langle\nabla\rangle^s\Psi_3\|_{L_t^{\infty-}L_x^{\frac{2d}{d-2}}} \|v_4\|_{Y_2^0} .$$

Thus we need the condition (3.44) to hold, which is less restrictive that $s > s_d$. Hence the contribution from this case is

$$\lesssim \|v\|_{Y_2^{\frac{d-2}{2}}} \|\Psi\|_{W^s}^2 .$$

The conclusion then follows by putting together the above four cases and by using Young's inequality. □

The following proposition concludes the proof of Theorem 3.1.

Proposition 3.19. Let $d \geq 3$, $s \in (s_d, s_{\text{crit}}]$ and $\theta \in (\frac{1}{2}, 1]$. Suppose that $u_0 \in H^{\frac{d-2}{2}}(\mathbb{R}^d)$. Then for any $\varepsilon > 0$, there exist a stopping time $T > 0$ and an event $\Omega' \subseteq \Omega$ such that $\mathbb{P}(\Omega \setminus \Omega') < \varepsilon$, and that for each $\omega \in \Omega'$, there is a unique solution $u = v + \Psi$ to (SNLS) such that v belongs to

$$B_{R,\eta,T} := \left\{ v \in Y_2^{\frac{d-2}{2}}([0,T)) \cap C([0,T); H^{\frac{d-2}{2}}) : \|v\|_{Y_2^{\frac{d-2}{2}}([0,T))} \leq 2R, \ \|v\|_{Z_\theta([0,T))} \leq 2\eta \right\}.$$

for some sufficiently large R and small η.

Proof. Set

$$\tilde{R} := \|v_0\|_{H^{\frac{d-2}{2}}(\mathbb{R}^d)} . \tag{3.45}$$

For $M > 0$, we define

$$\Omega_M := \left\{ \|\Psi\|_{Y_{2+}^s([0,1))} + \|\Psi\|_{W^s([0,1))} \leq M \right\}.$$

Then by Propositions 3.14 and 3.17, we may choose $M = M(\varepsilon)$ so that $\mathbb{P}(\Omega \setminus \Omega_M) < \varepsilon$. Let $R = \max\{\tilde{R}, M\}$, and let $\eta > 0$ be such that

$$\eta < R^{-\frac{1}{2\theta-1}}. \tag{3.46}$$

We now fix $\omega \in \Omega_M$ and proceed to show that the map Γ defined in (3.39) is a contraction on $B_{R,\eta,T}$ for some suitable T.

Recall that, by (3.40), $\Gamma v = S(t)v_0 + \Lambda v$. We first claim that Λ is an operator maping $C([0,T); H^{\frac{d-2}{2}}) \cap Y_2^{\frac{d-2}{2}}([0,T))$ to itself. Let v be a function in this space. By Proposition 3.18, $\Lambda v \in Y_2^{\frac{d-2}{2}}([0,T))$, and hence by the embedding $Y_2^{\frac{d-2}{2}} \hookrightarrow L_t^\infty H_x^{\frac{d-2}{2}}$, we have $\Lambda v \in L_t^\infty([0,T); H_x^{\frac{d-2}{2}})$. To prove continuity for Λv, it suffices to show continuity for $\mathbb{P}_{\le N}(\Lambda v)$. Indeed, continuity of Λv then follows from the uniform bound on $\|\mathbb{P}_{\le N}(\Lambda v)\|_{Y_2^{\frac{d-2}{2}}}$ independent of N as in the proof of Proposition 3.18. To this end, let $h > 0$. We have

$$
\begin{aligned}
&\|\mathbb{P}_{\le N}(\Lambda v)(t+h) - \mathbb{P}_{\le N}(\Lambda v)(t)\|_{H_x^{\frac{d-2}{2}}} \\
&\quad \le \left\| \mathbb{P}_{\le N}\left(\int_t^{t+h} S(t+h-t')\mathcal{N}(v+\Psi)\, dt' \right) \right\|_{H_x^{\frac{d-2}{2}}} \\
&\qquad + \left\| \mathbb{P}_{\le N}\left(\int_0^{t} S(t-t')[S(h)-\mathrm{Id}]\mathcal{N}(v+\Psi)\, dt' \right) \right\|_{H_x^{\frac{d-2}{2}}} \\
&\quad \le \int_t^{t+h} \|\mathbb{P}_{\le N}\mathcal{N}(v+\Psi)\|_{H_x^{\frac{d-2}{2}}}\, dt' + \int_0^t \|[S(h)-\mathrm{Id}]\mathbb{P}_{\le N}\mathcal{N}(v+\Psi)\|_{H_x^{\frac{d-2}{2}}}\, dt'
\end{aligned}
$$

By (3.42), we have $\mathbb{P}_{\le N}\mathcal{N}(v+\Psi) \in L_t^1 H_x^{\frac{d-2}{2}}$. Since $\{S(t)\}_{t\in[0,T)}$ is strongly continuous over $H^{\frac{d-2}{2}}$, the above tends to 0 as $h \to 0$. It follows that $\Lambda_N v \in C([0,T); H^{\frac{d-2}{2}}(\mathbb{R}^d))$. This concludes the proof of the claim.

With $\delta > 0$ as in Lemma 3.18, we choose $T = T(\omega) < \eta^{\frac{3}{\delta}}$ such that

$$
\begin{aligned}
\|\Psi\|_{W^s([0,T))} &\le \eta^2, \\
\|S(t)v_0\|_{Z_\theta([0,T))} &\le \eta. \qquad (3.47)
\end{aligned}
$$

Consider a function $v \in B_{R,\eta,T}$. By Lemma 3.18, there exists a constant $C > 0$ such that

$$
\|\Lambda v\|_{Y_2^{\frac{d-2}{2}}([0,T))} \le \frac{C}{4}(\eta^2 R + \eta^6 + T^\delta R^3 + \eta^2 R^2). \qquad (3.48)
$$

Since $\eta < R^{-1}$ (due to the assumption in (3.46)) and that $T < \eta^{\frac{3}{\delta}}$, we have

$$
\|\Lambda v\|_{Y_2^{\frac{d-2}{2}}([0,T))} \le C\eta R. \qquad (3.49)
$$

Similarly, Lemma 3.18 implies that

$$\|\Lambda v_2 - \Lambda v_1\|_{Y_2^{\frac{d-2}{2}}} \leq C\eta \|v_2 - v_1\|_{Y_2^{\frac{d-2}{2}}} . \tag{3.50}$$

Since $\Gamma v = S(t)v_0 + \Lambda v$, by (3.46), (3.10) and decreasing η (and hence T) if necessary, we obtain

$$\|\Gamma v\|_{Y_2^{\frac{d-2}{2}}([0,T))} \leq 2R ,$$

$$\|\Gamma v_2 - \Gamma v_1\|_{Y_2^{\frac{d-2}{2}}([0,T))} \leq \frac{1}{2} \|v_2 - v_1\|_{Y_2^{\frac{d-2}{2}}([0,T))} .$$

Lastly, by $\| \cdot \|_{Z([0,T))} \lesssim \| \cdot \|_{Y_2^{\frac{d-2}{2}}([0,T))}$, (3.47), (3.10), (3.45), and (3.46) we have

$$\begin{aligned}
\|\Gamma v\|_{Z_\theta([0,T))} &\leq \left(\|S(t)v_0\|_{Z([0,T))} + \|\Lambda v\|_{Z([0,T))}\right)^\theta \left(\|S(t)v_0\|_{Y_2^{\frac{d-2}{2}}([0,T))} + \|\Lambda v\|_{Y_2^{\frac{d-2}{2}}([0,T))}\right)^{1-\theta} \\
&\leq \left(\|S(t)v_0\|_{Z([0,T))} + C\eta^2 R\right)^\theta \left(\|S(t)v_0\|_{Y_2^{\frac{d-2}{2}}([0,T))} + C\eta^2 R\right)^{1-\theta} \\
&\leq \eta + C\eta^{2\theta} R + C\eta^{2-\theta} R^{1-\theta} + C\eta^2 R \leq 2\eta .
\end{aligned}$$

Hence Γ is a contraction on $B_{R,\eta,T}$. $\square$

Chapter 4

The 4-D energy–critical SNLS with non–vanishing boundary condition

In this chapter, we consider the Cauchy problem for the following energy-critical stochastic nonlinear Schrödinger equation on $\mathbb{R}^4$:

$$\begin{cases} i\partial_t u + \Delta u = (|u|^2 - 1)u + \phi\xi \\ u|_{t=0} = u_0, \end{cases} \qquad (t,x) \in [0,\infty) \times \mathbb{R}^4, \tag{4.1}$$

with the non-vanishing boundary condition:

$$\lim_{|x|\to\infty} |u(x)| = 1, \tag{4.2}$$

where u is a complex-valued function, where ξ denotes a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^d$ and ϕ is a bounded operator on $L^2(\mathbb{R}^d)$. The mild formulation for this equation is given by

$$u(t) = S(t)u_0 - i\int_0^t S(t-t')\big[(|u|^2-1)u\big](t')dt' - i\int_0^t S(t-t')\phi\xi(t'),$$

where $S(t) := e^{it\Delta}$ denotes the linear Schrödinger operator. Our main goal is to construct global-in-time dynamics for (4.6) in the energy-critical cases. As per usual, we denote by

$$\Psi(t) := \int_0^t S(t-t')\phi\xi(dt') \tag{4.3}$$

the stochastic convolution. We will impose that $\phi \in \mathrm{HS}(L^2; H^1)$, which ensures the regularity of Ψ to be 1 (see Lemma 3.15 in the preceding chapter). Let us recall the Ginzburg-Landau energy mentioned in the introduction:

$$E[u](t) = \frac{1}{2}\int_{\mathbb{R}^d} |\nabla u|^2 dx + \frac{1}{4}\int_{\mathbb{R}^d} \big(|u|^2 - 1\big)^2 dx.$$

We shall prove global well-posedness in the energy space $\mathcal{E}(\mathbb{R}^4)$ of functions u such that $E[u] < \infty$. More explicitly, $\mathcal{E}(\mathbb{R}^4)$ can be expressed as

$$\mathcal{E}(\mathbb{R}^4) := \{u = 1 + v : v \in H^1_{real}(\mathbb{R}^4) + i\dot{H}^1_{real}(\mathbb{R}^4)\}.$$

We now restate the main theorem of this chapter from the introduction:

Theorem 4.1 (Unconditional global well-poseness for the SNLS)**.** Let $d = 4$ and $\phi \in \mathrm{HS}(L^2; H^1)$. Assume $u_0 \in \mathcal{E}(\mathbb{R}^4)$. Then, the SNLS (4.1) with condition (4.2) is gobally well-posed in the energy space $\mathcal{E}(\mathbb{R}^4)$, almost surely. In particular, $u(t)$ is unique in the class $\Psi + C_t\big(\mathbb{R}; \mathcal{E}(\mathbb{R}^4)\big)$, almost surely.

As mentioned in the introduction, Theorem (4.1) will be proved by treating (4.1) as as the energy-critical NLS with a perturbation. To this end, we rewrite the equation as follows. Suppose that u is a solution to (4.1). If $u = 1 + v^*$, then v^* satisfies

$$\begin{cases} i\partial_t v^* + \Delta v^* = |v^*|^2 v^* + 2\,\mathrm{Re}(v^*)v^* + |v^*|^2 + 2\,\mathrm{Re}(v^*) + \phi\xi \\ v^*|_{t=0} =: u_0 - 1. \end{cases} \tag{4.4}$$

In terms of v^*, the Hamiltonian then takes the form

$$E[u](t) = E[v^* + 1] = \frac{1}{2}\int_{\mathbb{R}^4} |\nabla v^*|^2 dx + \frac{1}{4}\int_{\mathbb{R}^4} \big(|v^*|^2 + 2\,\mathrm{Re}(v^*)\big)^2 dx, \tag{4.5}$$

where we continue to denote $E[v^* + 1]$ by $E[v^*]$ for simplicity.

We now go one step further and subtract Ψ from v^*, that is, we define $v :=$

$u - 1 - \Psi$. Then v satisfies

$$\begin{cases} i\partial_t v + \Delta v = |v|^2 v + g(v, \Psi) \\ v|_{t=0} = v_0 := u_0 - 1, \end{cases} \tag{4.6}$$

where

$$\begin{aligned} g(v, \Psi) &:= (|v + 1 + \Psi|^2 - 1)(v + 1 + \Psi) - |v|^2 v \\ &= 2\,\mathrm{Re}(v)v + 2\,\mathrm{Re}(\Psi)v + 2\,\mathrm{Re}(\bar{v}\Psi)v \\ &\quad + \Psi^2 v + |v|^2 + 2\,\mathrm{Re}(v) + 2\,\mathrm{Re}(\Psi) + 2\,\mathrm{Re}(\bar{v}\Psi) + \Psi^2 \\ &\quad + \Psi|v|^2 + 2\,\mathrm{Re}(v)\Psi + 2\,\mathrm{Re}(\Psi)\Psi + 2\,\mathrm{Re}(\bar{v}\Psi)\Psi + \Psi^3. \end{aligned} \tag{4.7}$$

It is, however, more convenient to view this complicated expression as

$$\mathcal{O}\Big(\sum_{j=1}^{3} (\Psi + v)^j - v^3\Big)$$

as the real parts and the conjugate signs play little to no role in our arguments.

This chapter is organized as follows. In the preliminaries, we record the classical Strichartz estimates and the perturbation lemma from [85]. We prove the local wellposedness in Section 4.2 and lists the key perturbation lemma, Sections 4.3 splits into three parts, first we obtain the bounded Hamiltonian in Section 4.3.1; then by applying the perturbation lemma we prove the global wellposedness in Section 4.3.2; finally we prove the unconditional uniqueness.

4.1 Preliminaries

4.1.1 Strichartz estimates

We now recall the classical Strichartz estimates. Given $0 \leq q, r \leq \infty$ and a time interval $I \subseteq [0, \infty)$, we consider the mixed Lebesgue spaces $L_t^q L_x^r(I \times \mathbb{R}^4)$ of space-time functions $u(t,x)$, endowed with the norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} = \left(\int_I \left(\int_{\mathbb{R}^4} |u(x,t)|^r \, dx\right)^{\frac{q}{r}} dt\right)^{\frac{1}{q}}.$$

If the domain is clear, we often shorten this space to $L_t^q L_x^r$, as well as $L_{t,x}^r$ when $q = r$.

We say that a pair of exponents (q,r) is admissible if $\frac{2}{q}+\frac{4}{r} = 2$ with $2 \le q, r \le \infty$ and $(q,r,d) \neq (2,\infty,2)$. It is convenient to introduce the following norms. Given a space-time slab $I \times \mathbb{R}^4$, and $j \in \{0,1\}$, we define the $\dot{S}^j(I)$-norm by

$$\|u\|_{\dot{S}^j(I)} := \sup\left\{\|\nabla^j u\|_{L_t^q L_x^r(I\times\mathbb{R}^d)} : (q,r) \text{ is admissible}\right\}.$$

We use $\dot{N}^j(I)$ to denote the dual space of $\dot{S}^0(I)$. More precisely, we define

$$\|u\|_{\dot{N}^j(I)} := \inf\left\{\|\nabla^j u\|_{L_t^{q'} L_x^{r'}(I\times\mathbb{R}^d)} : (q,r) \text{ is admissible}\right\},$$

where (q',r') denotes the pair of Hölder conjugates of (q,r). We can now state the Strichartz estimates in terms of these norms; see [47, 58, 80, 87].

Lemma 4.2 (Strichartz estimates)**.** We have the following homogeneous estimate

$$\|S(t)u_0\|_{\dot{S}^j(I)} \lesssim \|u_0\|_{\dot{H}_x^j}$$

and the inhomogeneous Strichartz estimate

$$\left\|\int_{t_0}^t S(t-t')F(t')dt'\right\|_{\dot{S}^j(I)} \lesssim \|F\|_{\dot{N}^j(I)}.$$

We note down some common admissible pairs that will be used throughout this chapter:

$$(2,4), \left(6, \frac{12}{5}\right), (\infty, 2).$$

In particular, we shall define the space $\dot{X}^1(I)$ endowed with the norm

$$\|u\|_{\dot{X}(I)} := \|\nabla u\|_{L_t^6 L_x^{\frac{12}{5}}(I\times\mathbb{R}^4)}, \tag{4.8}$$

which serves as an auxiliary space in where we shall establish local well-posedness.

Finally, we recall Proposition 3.14 and Lemma 3.15, for which we restate as the following lemma for the reader's convenience:

Lemma 4.3. Suppose that $\phi \in \mathrm{HS}(L^2; H^s)$ for some $s \in \mathbb{R}$.

(i) $\Psi \in C_t\big(R_+; H^s(\mathbb{R}^d)\big)$ almost surely. In particular, for $p \geq 2$, there exists $C = C(T, \rho) > 0$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \|\Psi(t)\|_{H^s}^p\right] \leq C\|\phi\|_{\mathrm{HS}(L^2(\mathbb{R}^d);H^s(\mathbb{R}^d))}^p.$$

(ii) Given any $1 \leq q < \infty$ and $2 \leq r \leq 4$, we have $\Psi \in L_t^q([0, T]; W^{s,r}\big(\mathbb{R}^d)\big)$ almost surely for any $T > 0$. In particular, for $p \geq \max(q, r)$, there exists $C = C(T, p, q, r) > 0$ such that

$$\mathbb{E}\left[\|\Psi\|_{L^q([0,T];W^{s,r}(\mathbb{R}^d))}^p\right] \leq C\|\phi\|_{\mathrm{HS}(L^2(\mathbb{R}^d);H^s(\mathbb{R}^d))}^p.$$

4.1.2 Perturbation lemma

Consider the energy critical NLS equation

$$i\partial_t w + \Delta w = |w|^2 w. \tag{4.9}$$

Global well-posedness and scattering for (4.9) was proved by Vişan in [85]. Importantly, the following space-time bound on a global solution u to (4.9) holds:

$$\|w\|_{\dot{S}^1(\mathbb{R}\times\mathbb{R}^4)} \leq C(\|w_0\|_{\dot{H}^1}). \tag{4.10}$$

The idea of proving Theorem 4.1 is to view (4.1) as a energy-critical NLS (4.9) with a perturbation, more details are addressed in Section 4.2. The key perturbation results we will use is Lemma 4.4, which is Theorem 3.8 in [62].

Lemma 4.4 (Perturbation lemma)**.** Let $w_0 \in \dot{H}_1(\mathbb{R}^4)$, I be a compact time interval with $|I| \leq 1$. Let $\tilde{w}$ be a solution on $I \times \mathbb{R}^d$ to the perturbed equation:

$$i\partial_t\widetilde{w} + \Delta\widetilde{w} = |\widetilde{w}|^2\widetilde{w} + e \tag{4.11}$$

for some function e. There exist functions ε_0 and $\bar{C}$ mapping from $\mathbb{R}_+^3$ to $\mathbb{R}_+$, that are non-increasing in each argument, such that if

$$\|\widetilde{w}\|_{L^6_{t,x}(I\times\mathbb{R}^4)} \leq L, \tag{4.12}$$

$$\|\widetilde{w}\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^4)} \leq E_0, \tag{4.13}$$

$$\|\widetilde{w}(t_0) - w_0\|_{\dot{H}^1_x(\mathbb{R}^4)} \leq E', \tag{4.14}$$

for some $t_0 \in I$ and positive quantities L, E_0, E', and that

$$\|S(t-t_0)(\tilde{w}(t_0) - w_0)\|_{\dot{X}^1(I)} \leq \varepsilon, \tag{4.15}$$

$$\|\nabla e\|_{\dot{N}^0(I)} \leq \varepsilon, \tag{4.16}$$

for some $0 < \varepsilon < \varepsilon_0$, then there exists a solution w to (4.9) with initial data w_0 satisfying

$$\|w - \tilde{w}\|_{L^6_{t,x}(I\times\mathbb{R}^4)} \leq \bar{C}(E_0, E', L)\varepsilon, \tag{4.17}$$

$$\|w - \tilde{w}\|_{\dot{S}^1(I)} \leq \bar{C}(E_0, E', L)\varepsilon, \tag{4.18}$$

$$\|w\|_{\dot{S}^1(I)} \leq \bar{C}(E_0, E', L). \tag{4.19}$$

Remark 4.5. By the Strichartz estimate, condition (4.15) is redundant if $E' = O(\varepsilon)$.

4.2 Energy-critical NLS with a perturbation

In this section, we consider the defocusing energy-critical NLS with a perturbation:

$$\begin{cases} \partial_t v + \Delta v = (|v+f+1|^2 - 1)(v+f+1) \\ v|_{t=0} = v_0, \end{cases} \tag{4.20}$$

where f is a given deterministic function, satisfying certain regularity conditions. By applying the perturbation lemma, we prove global existence for (4.20), assuming an a priori energy bound of a solution v to (4.20). See Proposition 4.10. In Section 4.3.2, we then present the proof of Theorem 4.1 by writing (4.1) in the form (4.20) (with $f = \Psi$) and verifying the hypotheses in Proposition 4.10.

By the standard Strichartz theory, we have the following local well-posedness of the perturbed NLS (4.20).

Proposition 4.6 (Local well-posedness of the perturbed NLS)**.** Let $I_0 = [t_0, t_0 + T] \subseteq [0, \infty)$ be an interval. Let f be as in (4.20). Suppose that

$$\|v_0\|_{\dot{H}^1(\mathbb{R}^4)} \leq R \quad \text{and} \quad \|f\|_{L_t^\infty \dot{H}_x^1(I_0 \times \mathbb{R}^4)} \leq M,$$

for some $R, M \geq 1$. Then there exists some small $\eta_0 = \eta_0(R, M) > 0$ and a compact interval $I \subseteq I_0$ containing t_0 such that if

$$\|S(t - t_0)v_0\|_{\dot{X}^1(I)} + \|f\|_{\dot{X}^1(I)} \leq \eta,$$

for some $\eta \leq \eta_0$, then there exists a solution $v \in C_t\big(I; \dot{H}_x^1(\mathbb{R}^4)\big) \cap \dot{X}^1(I)$ to (4.6) with $v(t_0) = v_0$. Moreover, v satisfies

$$\|v - S(t - t_0)v_0\|_{\dot{X}^1(I)} \leq \eta \tag{4.21}$$

Remark 4.7. Note that the above proposition considers data from $\dot{H}^1(\mathbb{R}^4)$. This is fine for us because $\mathcal{E}(\mathbb{R}^4) \subset \dot{H}^1(\mathbb{R}^4)$. Indeed, we have

$$\begin{aligned} \|v_0\|_{\dot{H}^1(\mathbb{R}^4)} &\leq \|\operatorname{Re}(v_0)\|_{\dot{H}^1(\mathbb{R}^4)} + \|\operatorname{Im}(v_0)\|_{\dot{H}^1(\mathbb{R}^4)} \\ &\leq \|\operatorname{Re}(v_0)\|_{H^1(\mathbb{R}^4)} + \|\operatorname{Im}(v_0)\|_{\dot{H}^1(\mathbb{R}^4)}. \end{aligned}$$

Proof. We show that the map Γ defined by

$$\Gamma v(t) := S(t - t_0)v_0 - i \int_{t_0}^t S(t - t')\big[(|v + f + 1|^2 - 1)(v + f + 1)\big](t')dt'$$

is a contraction on

$$B_{R,M,\eta} = \{v \in \dot{X}^1(I) \cap C_t(I; \dot{H}_x^1(\mathbb{R}^4)) : \|v\|_{L_t^\infty \dot{H}^1(I \times \mathbb{R}^4)} \leq 2\widetilde{R}, \ \|v\|_{\dot{X}^1(I)} \leq 2\eta\}$$

with respect to the $\dot{X}^1(I) \cap C_t(I; \dot{H}^1_x(\mathbb{R}^4))$-metric, where $\widetilde{R} := \max(R, M)$. Let v_1, v_2, v_3 be functions in $\dot{X}_1(I)$. Then by Strichartz, Hölder and Sobolev inequalities, we have

$$\begin{aligned}
&\left\| \int_0^t S(t-t')\big(v_1\overline{v}_2 v_3 + v_1\overline{v}_2 + v_1\big)(t')dt' \right\|_{\dot{X}^1(I)} \\
&\lesssim \sum_{\{i,j,k\}=\{1,2,3\}} \|v_i v_j \nabla v_k\|_{L^2_t L^{\frac{4}{3}}_x (I\times\mathbb{R}^4)} + \sum_{\{i,j\}=\{1,2\}} \|v_i \nabla v_j\|_{L^{\frac{6}{5}}_t L^{\frac{12}{7}}_x (I\times\mathbb{R}^4)} \\
&\quad + \|\nabla v_1\|_{L^1_t L^2_x(I\times\mathbb{R}^4)} \qquad (4.22)\\
&\lesssim \|v_1\|_{\dot{X}_1(I)}\|v_2\|_{\dot{X}_1(I)}\|v_3\|_{\dot{X}_1(I)} + |I|^{\frac{1}{2}}\|v_1\|_{\dot{X}_1(I)}\|v_2\|_{\dot{X}_1(I)} \\
&\quad + |I|\|v_1\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^4)}.
\end{aligned}$$

Now, by (4.7), we have

$$\begin{aligned}
(|v+f+1|^2-1)(v+f+1) &= |v|^2 v + |f|^2 v + 2\,\mathrm{Re}\,(\overline{v}f)v + 2\,\mathrm{Re}\,(f+v)v \qquad (4.23)\\
&\quad + |v|^2 f + |f|^2 f + 2\,\mathrm{Re}\,(\overline{v}f)f + 2\,\mathrm{Re}\,(f+v)f \qquad (4.24)\\
&\quad + |v|^2 + |f|^2 + 2\,\mathrm{Re}\,(\overline{v}f) + 2\,\mathrm{Re}\,(f+v) \qquad (4.25)
\end{aligned}$$

We choose $\eta_0 \ll \widetilde{R}^{-1} \leq 1$ and $|I| \leq \min\left\{1, \eta^3 \widetilde{R}^{-1}\right\}$. Fix $\eta \leq \eta_0$ in the following. Then (4.22) and (4.23) infer that

$$\begin{aligned}
\|\Gamma v\|_{\dot{X}^1(I)} &\leq \|S(t-t_0)v_0\|_{\dot{X}^1(I)} + \|\Gamma v - S(t-t_0)v_0\|_{\dot{X}^1(I)} \\
&\lesssim \|S(t-t_0)v_0\|_{\dot{X}^1(I)} + C\Big(\|v\|^3_{\dot{X}^1(I)} + |I|^{\frac{1}{2}}\|v\|^2_{\dot{X}^1(I)} + |I|\cdot\|v\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^4)} \\
&\qquad + \|f\|^3_{\dot{X}^1(I)} + |I|^{\frac{1}{2}}\|f\|^2_{\dot{X}^1(I)} + |I|\cdot\|f\|_{L^\infty_t \dot{H}^1_x(I\times\mathbb{R}^4)}\Big) \\
&\leq \eta + C\eta^3 \leq 2\eta, \qquad (4.26)
\end{aligned}$$

provided η_0 is sufficiently small. Similarly, we have

$$\begin{aligned}\|\Gamma v\|_{L_t^\infty \dot{H}_x^1(I\times\mathbb{R}^4)} &\leq \|S(t-t_0)v_0\|_{L_t^\infty \dot{H}_x^1(I\times\mathbb{R}^4)} + \|\Gamma v - S(t-t_0)v_0\|_{L_t^\infty \dot{H}_x^1(I\times\mathbb{R}^4)} \\ &\leq R + C\eta^3 \leq 2\widetilde{R}.\end{aligned}$$

Hence Γ maps $B_{R,M,\eta}$ to $B_{R,M,\eta}$. Finally, the difference estimate follows analogously. Indeed, for $v_1, v_2 \in B_{R,M,\eta}$, we have

$$\|\Gamma v_1 - \Gamma v_2\|_{L_t^\infty \dot{H}_x^1(I\times\mathbb{R}^4)\cap \dot{X}^1(I)} \leq \frac{1}{2}\|v_1 - v_2\|_{L_t^\infty \dot{H}_x^1(I\times\mathbb{R}^4)\cap \dot{X}^1(I)}.$$

Therefore, Γ is a contraction on $B_{R,M,\eta}$. The estimate (4.21) is a direct consequence of the above estimates. □

As a consequence of Proposition 4.6, we have the following local well-posedness for the SNLS (4.1).

Lemma 4.8 (Local well-poseness for the SNLS)**.** Let $\phi \in \mathrm{HS}(L^2(\mathbb{R}^4); H^1(\mathbb{R}^4))$. Then, given any $u_0 \in \mathcal{E}(\mathbb{R}^4)$, there exists an almost surely positive stopping time $T = T_\omega(u_0)$ and a unique local-in-time solution $u = 1 + v^* \in \mathcal{E}(\mathbb{R}^4)$ to the energy-critical SNLS (4.1). Furthermore, the following blowup alternative holds; let $T^* = T^*_\omega(u_0)$ be the forward maximal time of existence. Then, either

$$T^* = \infty \quad \text{or} \quad \lim_{T\nearrow T^*} \|u\|_{\dot{X}^1(0,T)} = \infty.$$

4.3 Proof of the main theorem

We present the proof of Theorem 4.1 in this section. The first objective is to obtain an a priori bound for the energy of the solution. Armed with this bound as well as tools from the previous sections, we prove global existence by an iterative application of the perturbation lemma (Lemma 4.4). Finally, we conclude the argument by proving unconditional uniqueness.

4.3.1 Bound on the energy

Recall the definition of the energy $E[u](t)$ from (4.5). Our goal in this subsection is to state and prove a priori bound on the energy.These a priori bounds follow from Ito's lemma and the Burkholder-Davis-Gundy inequality. In order to justify an application of Ito's lemma, one needs to go through a certain approximation argument. See Proposition 3.2 in [36] for details.

Proposition 4.9. Assume the hypotheses in Lemma 4.8. Then,

(i) for any $t \in [0,T]$, the energy $E[u](t)$ defined in (4.5) can be expressed as

$$E[u](t) = E[u_0] + t\Big(\|\nabla\phi\|^2_{\mathrm{HS}(L^2;\dot{H}^1)} + \|\phi\|^2_{\mathrm{HS}(L^2;L^2)}\Big) \tag{4.27}$$

$$+ \sum_{n\in\mathbb{N}} \iint_{[0,t]\times\mathbb{R}^4} \Big(|v^*|^2 + \mathrm{Im}(\overline{v^*})^2 + 4\,\mathrm{Re}(v^*)\Big)|\phi_n|^2\, dt'\, dx \tag{4.28}$$

$$+ \mathrm{Im} \iint_{[0,t]\times\mathbb{R}^4} \Big(|v^*|^2\overline{v^*} - \Delta\overline{v^*} + |v^*|^2 + 2\,\mathrm{Re}(v^*)\overline{v^*} + 2\,\mathrm{Re}(v^*)\Big)\phi\, dW\, dx. \tag{4.29}$$

(ii) Moreover, given $T_0 > 0$, there exists a constant

$$C_E = C\big(E(u_0), T_0, \|\phi\|_{\mathrm{HS}(L^2;H^1)}\big) > 0$$

such that for any stopping time T with $0 < T < \min(T^*, T_0)$ almost surely, we have

$$\mathbb{E}\Big[\sup_{0\le t\le T} E[u](t)\Big] \le C_E. \tag{4.30}$$

where u is the solution to the defocusing energy-critical SNLS (4.1) with $u|_{t=0} = u_0$ and $T^* = T^*_\omega(u_0)$ is the forward maximal time of existence.

Proof. The expression on $E[u](t)$ follows from a similar computation as in Proposition 2.27 and hence we omit the details. We turn to prove (4.30). The term (4.27) is easily bounded:

$$\mathbb{E}\Big[\sup_{t0\le t\le T} (4.27)\Big] \lesssim T\|\phi\|_{\mathrm{HS}(L^2;\dot{H}^1)}. \tag{4.31}$$

Turning our attention to (4.28), by Hölder, Sobolev and Young inequalities, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le t\le T}(4.28)\Big] &\le C\mathbb{E}\Big[\sum_{n\in\mathbb{N}}\int_{[0,T]}\Big(\|v^*\|^2_{L^4_x(\mathbb{R}^4)}+\|v^*\|_{L^2_x(\mathbb{R}^4)}\Big)\|\phi_n\|^2_{L^4_x}\,dt'\Big]\\ &\le 2T\|\phi\|^2_{\mathrm{HS}(L^2;H^1)}\mathbb{E}\Big[\sup_{0\le t\le T}\Big(E[u](t)\Big)^{\frac12}\Big] \qquad (4.32)\\ &\le CT^2\|\phi\|^4_{\mathrm{HS}(L^2;\dot H^1)}+\frac18\mathbb{E}\Big[\sup_{0\le t\le T}E[u](t)\Big].\end{aligned}$$

Finally, we bound (4.29). By Burkholder-Gundy-Davis, Hölder, Sobolev and Young inequalities, we have

$$\begin{aligned}&\mathbb{E}\Big[\sup_{0\le t\le T}\mathrm{Im}\iint_{[0,t]\times\mathbb{R}^4}|v^*|^2\overline{v^*}\phi dW dx\Big]\\ &\le C\mathbb{E}\Big[\Big(\sum_{n\in\mathbb{N}}\int_0^T\Big|\int_{\mathbb{R}^4}|v^*|^2\overline{v^*}\phi_n dx\Big|^2dt'\Big)^{\frac12}\Big]\\ &\le C\mathbb{E}\Big[\Big(\sum_{n\in\mathbb{N}}\int_0^T\||v^*|^2v^*\|^2_{\dot H_x^{-1}}\|\phi_n\|^2_{\dot H^1_x}\,dt'\Big)^{\frac12}\Big]\\ &\le C\mathbb{E}\Big[\Big(\sum_{n\in\mathbb{N}}\int_0^T\|v^*\|^3_{L^4_x}\|\phi_n\|^2_{\dot H^1_x}dt'\Big)^{\frac12}\Big]\\ &\le C\|\phi\|_{\mathrm{HS}(L^2;H^1)}\mathbb{E}\Big[\sup_{0\le t\le T}E[u](t)^{\frac38}\Big]\\ &\le C\|\phi\|_{\mathrm{HS}(L^2;H^1)}\mathbb{E}\Big[1+\sup_{0\le t\le T}E[u](t)^{\frac12}\Big]\\ &\le C\|\phi\|_{\mathrm{HS}(L^2;H^1)}+C\|\phi\|^2_{\mathrm{HS}(L^2;H^1)}+\frac1{32}\mathbb{E}\Big[\sup_{0\le t\le T}E[u](t)\Big].\end{aligned}$$

where we used the elementary fact $A^{\frac38}\le 1\wedge A^{\frac12}\le 1+A^{\frac12}$ in the penultimate inequality. The rest of the contributions from (4.29) are controlled in a similar manner and we omit the details. Ultimately, we obtain

$$\mathbb{E}\Big[\sup_{0\le t\le T}(4.29)\Big]\le C(\phi)+\frac18\mathbb{E}\Big[\sup_{0\le t\le T}E[u](t)\Big] \qquad (4.33)$$

Combining (4.31)–(4.33) concludes the proof. □

4.3.2 Global existence

We now prove the existence part of Theorem 4.1. Let v_0 be such that $1+v_0 \in \mathcal{E}(\mathbb{R}^4)$. The bulk of the argument is contained in the following proposition on the perturbed NLS (4.20).

Proposition 4.10. Let $T > 0$ be given. Let f be as in (4.20). Assume the following conditions hold:

1. There exists $\theta > 0$ such that for any interval $I \subseteq [0, T]$, we have

$$\|f\|_{L_t^\infty \dot{H}_x^1(I\times\mathbb{R}^4)} + \|f\|_{\dot{X}^1(I)} \leq C\|\phi\|_{\mathrm{HS}(L^2;H^1)}|I|^\theta;$$

2. Given a solution v, we have the following a priori bound

$$\|v\|_{L_t^\infty \dot{H}_x^1([0,T]\times\mathbb{R}^d)} \leq R.$$

Then, there exists a time $\tau = \tau(R, \theta) > 0$ such that given any $t_0 \in [0, T)$, a solution v exists on $[t_0, t_0 + \tau] \cap [0, T]$ for this particular path. This implies that v in fact exists in the entire interval $[0, T]$, as t_0 is arbitrary.

Proof. Let v be the local solution on SNLS obtained from Proposition 4.6. The main idea is to view (4.6) as a perturbation to the energy-critical cubic NLS (4.9), that is, regard v as $\widetilde{w}$ in Lemma 4.4 with

$$e = g(v, f),$$

where $g(v, f)$ as in (4.7). The argument follows closely in [61].

Let w be the global solution to the energy-critical cubic NLS (4.9) with initial data $w(t_0) = v_0$. Then, by assumption $\|w(t_0)\|_{\dot{H}^1} \leq R$, and so by (4.10)

$$\|w\|_{\dot{X}^1(\mathbb{R})} \lesssim R.$$

This, together with assumption (2), infer that we can divide the interval $[t_0, T]$ into

$J = J(R, \phi, \theta, \eta)$ many subintervals $I_j = [t_j, t_{j+1}]$ so that

$$\|w\|_{\dot{X}^1(I_j)} + \|f\|_{\dot{X}^1(I_j)} \leq \eta \tag{4.34}$$

for some $\eta \ll \eta_0$, where η_0 is dictated by Lemma 4.6. We also write $[t_0, t_0 + \tau] = \bigcup_{j=0}^{J'} [0, t_0 + \tau] \cap I_j)$ for some $J' \leq J$, where $[t_0, t_0 + \tau] \cap I_j \neq \emptyset$ for $0 \leq j \leq J'$.

We would like to apply Proposition 4.4 on each interval I_j with $e = g(v\Psi)$. Starting with $j = 0$, we see that (4.13) is automatically satisfied with $E_0 = R$ by assumption and (4.14) holds trivially with, say, $E' = 1$ since $v(t_0) = w(t_0)$; this also infers that Condition (4.15) holds (for any ε) by Strichartz estimate. We now turn to (4.12). Since the nonlinear evolution w is small on I_j, the linear evolution $S(t - t_j)w(t_j)$ is also small on I_j. Indeed, by rearranging the Duhamel formula, we have

$$S(t - t_j)w(t_j) = w(t) + i \int_{t_j}^{t} S(t - t')\big[|w|^2 w\big](t')dt'$$

for any $t \in I_j$; together with Strichartz, Hölder and Sobolev inequalities, we obtain

$$\begin{aligned} \|S(t - t_j)w(t_j)\|_{\dot{X}^1(I_j)} &\leq \|w\|_{\dot{X}^1(I_j)} + C\|w^2 \nabla w\|_{L_t^2 L_x^{\frac{4}{3}}(I_j \times \mathbb{R}^4)} \\ &\leq \eta + C\|\nabla w\|_{L_t^6 L^{\frac{12}{5}}{}_x(I_j \times \mathbb{R}^4)} \|w\|^2_{L^6_{t,x}(I_j \times \mathbb{R}^4)} \\ &\leq \eta + C\eta^3 \\ &\leq 2\eta, \end{aligned} \tag{4.35}$$

since $\eta \ll \eta_0 \leq 1$. By Lemma 4.6 together with (4.34) and (4.35) for $j = 0$, v exists on the interval I_0, moreover,

$$\|v\|_{\dot{X}^1(I_0)} \leq \|S(t - t_0)v\|_{\dot{X}^1(I_0)} + \|v - S(t - t_0)v_0\|_{\dot{X}^1(I_0)} \leq 6\eta$$

Thus by the Sobolev embedding $\dot{W}^{1,\frac{12}{5}}(\mathbb{R}^4) \subset L^6(\mathbb{R}^4)$, we have $\|v\|_{L^6_{t,x}(I_0 \times \mathbb{R}^4)} \leq C\eta$ for some absolute constant C. Therefore, Condition (4.12) in Lemma 4.4 is satisfied with $L = C'\eta$.

Let us now verify (4.16), that is, we need to estimate $\|\nabla e\|_{\dot{N}^0(I_0)} = \|\nabla g(v, f)\|_{\dot{N}^0(I_0)}$. In view of (4.7) and (4.37), we distribute the derivative to each term and apply

Strichartz estimate to each contribution, and put the cubic, square and linear terms in $L_t^2 L^{\frac{4}{3}}$, $L_t^{\frac{6}{5}} L^{\frac{12}{7}}$ and $L_t^1 L_x^2$ respectively. We then use Hölder and Sobolev inequalities to put each term in $\dot{X}^1(I)$ (as seen in (4.22)). This gives

$$\begin{aligned}\|\nabla e\|_{\dot{N}^0(I_0)} &\lesssim \|f\|^3_{\dot{X}_1(I_0)} + |I_0|^{\frac{1}{2}}\Big(\|v\|^2_{\dot{X}_1(I_0)} + \|f\|^2_{\dot{X}_1(I_0)}\Big) \\ &\quad + |I_0|\Big(\|v\|_{L_t^\infty \dot{H}_x^1(I_0\times\mathbb{R}^4)} + \|f\|_{L_t^\infty \dot{H}_x^1(I_0\times\mathbb{R}^4)}\Big) \\ &\lesssim_\phi |I_0|^{3\theta} + |I_0|^{\frac{1}{2}}(\eta^2 + |I_0|^{2\theta}) + |I_0|(R + |I_0|^\theta) \\ &\lesssim_R \tau^{\theta'}\end{aligned} \tag{4.36}$$

for some $\theta' = \theta'(\theta) > 0$. Let $\varepsilon \in (0, \varepsilon_0)$ to be chosen later, where $\varepsilon_0 = \varepsilon_0(R, 1, C'\eta)$ is dictated by Lemma 4.4. We choose $\tau = \tau(\varepsilon, \phi, \theta, R)$ sufficiently small so that

$$\|\nabla e\|_{\dot{N}^0(I_0)} \le \varepsilon. \tag{4.37}$$

This verifies (4.16). Therefore, all hypotheses of Lemma 4.4 are satisfied on the interval I_0, with $L = C'\eta$, $E_0 = R$ and $E' = 1$. Hence we obtain

$$\|w - v\|_{\dot{S}^1(I_0)} \le \bar{C}(R, 1, C'\eta)\varepsilon =: C_0(R, \eta)\varepsilon. \tag{4.38}$$

Consider now the second interval I_1. Again, Condition (4.13) is satisfied automatically with $E_0 = R$ by assumption. Since the pair $(\infty, 2)$ is admissible, (4.38) infers that

$$\|w(t_1) - v(t_1)\|_{\dot{H}^1} \le C_0\varepsilon.$$

By choosing $\varepsilon = \varepsilon(R, \eta)$ sufficiently small, Condition (4.13) holds with $E' = 1$.

Turning to (4.12), by Strichartz, (4.38) and 4.35, we have

$$\begin{aligned}\|S(t-t_1)v(t_1)\|_{\dot{X}^1(I_1)} &\leq \big\|S(t-t_1)\big[v(t_1)-w(t_1)\big]\big\|_{\dot{X}^1(I_1)} + \|S(t-t_1)w(t_1)\|_{\dot{X}^1(I_1)} \\ &\leq \|w(t_1)-v(t_1)\|_{\dot{H}^1_x} + 2\eta \\ &\leq C_0(R,\eta)\varepsilon + 2\eta \\ &\leq 3\eta\end{aligned} \tag{4.39}$$

provided

$$C_0\varepsilon < \eta. \tag{4.40}$$

If this holds, then by Lemma 4.6 and (4.34), v exists on the interval I_0, moreover,

$$\|v\|_{\dot{X}^1(I_1)} \leq \|S(t-t_0)v(t_1)\|_{\dot{X}^1(I_1)} + \|v - S(t-t_0)v(t_1)\|_{\dot{X}^1(I_1)} \leq 8\eta$$

By Sobolev inequality, we see that Condition (4.12) is satisfied with $L = C\eta$ as before. Now, for Condition (4.15), by Strichartz estimate and (4.38), we have

$$\|S(t-t_0)(\tilde{w}(t_0)-w_0)\|_{\dot{X}^1(I)} \leq \widetilde{C}C_0\varepsilon$$

where $\widetilde{C}$ is the absolute constant coming from Strichartz estimate. Then Condition (4.15) is satisfied as long as

$$\widetilde{C}C_0\varepsilon < \varepsilon_0(R,1,C'\eta). \tag{4.41}$$

Lastly, we argue as in (4.36) to obtain

$$\|\nabla e\|_{\dot{N}^0(I_1)} \leq \varepsilon \leq \widetilde{C}C_0\varepsilon,$$

without needing to change $\tau = T(\varepsilon,\phi,\theta,R)$. Hence Condition (4.16) is satisfied provided (4.40) and (4.41) hold, which can be done by shrinking $\varepsilon = \varepsilon(R,\eta)$ if necessary. Hence Lemma 4.4 infers that

$$\|v - w\|_{\dot{S}^1(I_1)} \le \bar{C}(R, 1, C'\eta)\widetilde{C}C_0\varepsilon =: C_1(R, \eta)\varepsilon.$$

We now recursively define $C_j(R, \eta) := \bar{C}(R, 1, C'\eta)\widetilde{C}C_{j-1}$ for $1 \le j \le J'$. In other words, $C_j(R, \eta) = \bar{C}(R, 1, C'\eta)^{j+1}\widetilde{C}^j$. Arguing iteratively, we have

$$\|v - w\|_{\dot{S}^1(I_j)} \le C_j\varepsilon$$

as long as

$$\begin{aligned} C_{j-1}\varepsilon &< \eta, \\ C_j\varepsilon &< \varepsilon_0(R, 1, C'\eta). \end{aligned} \tag{4.42}$$

Since C_j is increasing in j, we just need to ensure that (4.42) holds for $j = J'$. Recalling that $J' \le J = J(R, \eta)$, we see that (4.42) holds for all j provided that ε is chosen sufficiently small, depending only on R and η. In particular, we have constructed a solution v in the entire interval $[t_0, t_0 + \tau]$, where $\tau = \tau(R, \eta, \varepsilon)$. This proves the proposition. □

4.3.3 Global existence

We are now ready to prove the existence part of Theorem 4.1. Let $T > 0$ and $\varepsilon > 0$. We claim that there exists an event $\Omega_\varepsilon \subseteq \Omega$ such that $\mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon$ and that in Ω_ε, there exist $\theta > 0$ and $R = R(T, \phi)$ such that

$$\|\Psi\|_{L^\infty_t \dot{H}^1_x([t_0, t_0+\tau] \times \mathbb{R}^4)} + \|\Psi\|_{\dot{X}^1(I)} \le C(\phi)|I|^\theta \qquad \text{for any } I \subseteq [0, T] \tag{4.43}$$

and that the a priori bound

$$\|v\|_{L^\infty_t \dot{H}^1([0,T]) \times \mathbb{R}^4} \le R \tag{4.44}$$

holds. Indeed, by Lemma 4.3 and Markov inequality, one can easily find an event of arbitrarily large probability in which the bound (4.43) holds. As for (4.44), recalling that $v = u - 1 - \Psi = v^* - \Psi$ and also the definition of $E[u]$, we have

$$\begin{aligned}\|v\|_{L_t^\infty \dot{H}^1([0,T]\times\mathbb{R}^4)} &\le \|v^*\|_{L_t^\infty \dot{H}^1([0,T]\times\mathbb{R}^4)} + \|\Psi\|_{L_t^\infty \dot{H}_x^1([0,T]\times\mathbb{R}^4)} \\ &\le \sup_{0\le t\le T} \big[E(u)(t)\big]^{\frac{1}{2}} + \|\Psi\|_{L_t^\infty \dot{H}_x^1([0,T]\times\mathbb{R}^4)}.\end{aligned}$$

Then by Lemma 4.3, Proposition 4.9 and Chebyshev's inequality, one can again find an event of arbitrarily large probability in which (4.44). Hence the claim holds and we can invoke Proposition 4.10 to extend the solution v to all times in $[0,T]$ for each $\omega \in \Omega_\varepsilon$. This completes the existence part of the proof.

4.3.4 Unconditional uniqueness

We turn now to showing that the global solutions constructed above are unique among those that are continuous (in time) with values in the energy space. We mimic the arguments in [28] and [61]. To this end, let v_0 be such that $1+v_0 \in \mathcal{E}(\mathbb{R}^4)$ and let v be the global solution to (4.6) constructed above. In particular, $v \in \dot{S}^1(I)$ for any compact time interval I. Let $\widetilde{v} : [0,t'] \times \mathbb{R}^4 \to \mathbb{C}$ be a second solution to (4.6) with the same initial data such that $1+\widetilde{v} \in C([0,t'];\mathcal{E}(\mathbb{R}^4))$ almost surely and write $z := v - \widetilde{v}$. In what follows, we fix an $\omega \in \Omega$ for which both v and $\widetilde{v} \in C([0,t'];\mathcal{E}(\mathbb{R}^4))$. As $z(0) = 0$ and ω is continuous in time, shrinking t' if necessary, we may assume

$$\|\operatorname{Re}(z)\|_{L_t^\infty \dot{H}_x^1([0,t']\times\mathbb{R}^4)} + \|\operatorname{Im}(z)\|_{L_t^\infty \dot{H}_x^1([0,t']\times\mathbb{R}^4)} \le \eta \tag{4.45}$$

for a small $\eta > 0$ to be chosen shortly. By Sobolev embedding $\dot{H}^1(\mathbb{R}^4) \subset L^4(\mathbb{R}^4)$, this yields

$$\|z\|_{L_t^\infty L_x^4([0,t']\times\mathbb{R}^4)} \lesssim \eta, \tag{4.46}$$

in particular, we have $z \in L_t^2 L_x^4([0,t'] \times \mathbb{R}^4)$. Recalling that $v \in \dot{S}^1(I)$ almost surely for any compact time interval I and further shrinking t' if necessary, we may also assume (by Sobolev embedding $W^{1,\frac{12}{5}}(\mathbb{R}^4) \subset L^6(\mathbb{R}^4)$) that

$$\|v\|_{L_{t,x}^6([0,t']\times\mathbb{R}^4)} \le \eta. \tag{4.47}$$

On the other hand, as seen at the end of the previous subsection, one can find an event of arbitrarily large probability such that (4.43) holds. Hence we may assume ω lies in this event. By Sobolev embeddings $\dot{H}^1(\mathbb{R}^4) \subset L^4(\mathbb{R}^d)$ and $\dot{W}^{1,\frac{12}{5}} \subset L^6(\mathbb{R}^4)$, as well as shrinking t' if necessary, we have

$$\|\Psi\|_{L_t^\infty L_x^4([0,t']\times\mathbb{R}^4)} \leq \eta, \tag{4.48}$$

$$\|\Psi\|_{L_{t,x}^6([0,t']\times\mathbb{R}^4)} \leq \eta. \tag{4.49}$$

Now,

$$\begin{aligned}
&\big[|v|^2 v + g(v,\Psi)\big] - \big[|\widetilde{v}|^2 v + g(\widetilde{v},\Psi)\big] \\
&\quad \sim \big(\operatorname{Re}(\Psi)|z| + |\operatorname{Re}(z)| + |\Psi||\operatorname{Re}(z)| + |\Psi|^2|z| \\
&\qquad + |\operatorname{Re}(\Psi)||\Psi||\operatorname{Re}(z)| + |\operatorname{Re}(\Psi)||\operatorname{Re}(\overline{z})| + |z|^2 \\
&\qquad + |z||v| + |\Psi||z|^2 + |\Psi||z||v| + |\operatorname{Re}(\Psi)||z|^2 \\
&\qquad + |\operatorname{Re}(\Psi)||z||v| + |z|^3 + |z||v|^2\big) \\
&\quad = O\big(|z|^3 + |z||v|^2 + |\Psi||z|^2 + |z|^2 + |\Psi||z||v| + |\Psi||z| + |\Psi|^2|z| + |z||v| + |\operatorname{Re}(z)|\big).
\end{aligned}$$

By Strichartz and Hölder inequalities together with (4.45)–(4.49), we have

$$\begin{aligned}
&\|z\|_{L_t^2 L_x^4} + \|\operatorname{Re}(z)\|_{L_t^\infty L_x^2} \\
&\quad \lesssim \|z^3\|_{L_t^2 L_x^{\frac{4}{3}}} + \|zv^2\|_{L_t^{\frac{6}{5}} L_x^{\frac{12}{7}}} + \|z^2\|_{L_t^1 L_x^2} + \|zv\|_{L_t^1 L_x^2} + \|\Psi z^2\|_{L_t^2 L_x^{\frac{4}{3}}} \\
&\qquad + \|\Psi z v\|_{L_t^{\frac{6}{5}} L_x^{\frac{12}{7}}} + \|\Psi z\|_{L_t^1 L_x^2} + \|\Psi^2 z\|_{L_t^2 L_x^{\frac{4}{3}}} + \|\operatorname{Re}(z)\|_{L_t^1 L_x^2} \\
&\quad \lesssim \|z\|_{L_t^2 L_x^4}\big[\|z\|^2_{L_t^\infty L_x^4} + \|v\|^2_{L_{t,x}^6} + t'^{\frac{1}{2}}\|z\|_{L_t^\infty L_x^4} \\
&\qquad + t'^{\frac{1}{2}}\|v\|_{L_t^\infty L_x^4} + \|\Psi\|_{L_t^\infty L_x^4}\|z\|_{L_t^\infty L_x^4} + \|\Psi\|_{L_{t,x}^6}\|v\|_{L_{t,x}^6} \\
&\qquad + t'^{\frac{1}{2}}\|\Psi\|_{L_t^\infty L_x^4} + \|\Psi\|^2_{L_{t,x}^6}\big] + t'\|\operatorname{Re}(z)\|_{L_t^\infty L_x^2}
\end{aligned}$$

$$\lesssim \|z\|_{L_t^2 L_x^4}(\eta^2 + \eta t'^{\frac{1}{2}} + t'^{\frac{1}{2}}) + t' \| \operatorname{Re}(z)\|_{L_t^\infty L_x^2}.$$

where we omitted the domain $[0,t'] \times \mathbb{R}^4$ above for the sake of readability. Taking η sufficiently small and shrinking t' further if necessary, we obtain

$$\|z\|_{L_t^2 L_x^4([0,t'] \times \mathbb{R}^4)} + \| \operatorname{Re}(z)\|_{L_t^\infty L_x^2([0,t'] \times \mathbb{R}^4)} = 0,$$

which proves $v = \widetilde{v}$ almost surely on $[0,t'] \times \mathbb{R}^4$.

By time translation invariance, this argument can be applied to any sufficiently short time interval, which yields global unconditional uniqueness. This completes the proof of Theorem 4.1.

Chapter 5

Global well-posedness of the periodic Stochastic KdV with multiplicative noise

In this chapter, we consider the Cauchy problem for the periodic stochastic Korteweg-de Vries equation (SKdV) with multiplicative noise:

$$\begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = u\phi dW \\ u|_{t=0} = u_0 \in L^2(\mathbb{T}) \end{cases} \qquad (x,t) \in \mathbb{T} \times \mathbb{R}^+, \tag{5.1}$$

where $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, u is a real-valued function, and $W(t) = \frac{\partial B}{\partial x}$ is a cylindrical Wiener process on $L^2(\mathbb{T})$. With $e_n(x) = \frac{1}{\sqrt{2\pi}}e^{inx}$, we can write

$$W(t) = \sum_n \beta_n(t) e_n(x),$$

where $\{\beta_n\}_{n\geq 0}$ is a family of mutually independent complex-valued Brownian motions (here we take β_0 to be real-valued) in a fixed probability space $(\Omega, \mathcal{F}, P)$ associated with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and $\beta_{-n}(t) = \overline{\beta_n}(t)$ for $n \geq 1$. We normalize β_n such that $\mathrm{Var}(\beta_n(1)) = 1$ for $n \geq 0$. The covariance operator ϕ is a Hilbert-Schmidt operator on $L^2(\mathbb{T})$ that maps real-valued functions to real-valued functions. Moreover, we assume that the noise is homogeneous, i.e. we assume that ϕ is a convolution operator. Abusing the notation, we also use ϕ to denote the kernel of ϕ, and write the Fourier coefficient of ϕ as ϕ_n. These assumptions mean that we have

$$\phi f(x) = \int_{\mathbb{T}} \phi(x-y) f(y) dy = \sum_{n=-\infty}^{\infty} \phi_n \widehat{f}(n) e_n(x), \tag{5.2}$$

and that $\phi_{-n} = \overline{\phi_n}$ with $\phi_0 \in \mathbb{R}$. Note that this implies that $\phi e_n = \phi_n e_n$ for all $n \in \mathbb{Z}$. The Hilbert-Schmidt assumption implies that the norm

$$\|\phi\|_{\mathcal{L}^2(L^2(\mathbb{T}))} := \Big(\sum_{n\in\mathbb{Z}} \|\phi e_n\|^2_{L^2(\mathbb{T})} \Big)^{\frac{1}{2}} = \Big(\sum_{n\in\mathbb{Z}} |\phi_n|^2 \Big)^{\frac{1}{2}} \tag{5.3}$$

is finite.

In [37], de Bouard-Debussche considered the non-periodic version of the problem with homogeneous multiplicative noise and proved global well-posedness of (5.1) in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$. More specifically, they proved the result for $u_0 \in H^s(\mathbb{R})$ when ϕ has the convolution kernel in $H^s(\mathbb{R}) \cap L^1(\mathbb{R})$ with $s = 0$ or 1.

There are also several results on SKdV with additive noise:

$$\begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = \phi dW \\ u(x,0) = u_0(x), \end{cases} \tag{5.4}$$

where ϕ is a bounded linear operator on $L^2(\mathbb{T})$. In [39], de Bouard-Debussche-Tsutsumi showed that (5.4) is locally well-posed when ϕ is a Hilbert-Schmidt operator from $L^2(\mathbb{T})$ to $H^s(\mathbb{T})$ for $s > -\frac{1}{2}$. More recently, the second author [75] proved local well-posedness of (5.4) even when $\phi = \mathrm{Id}$, thus handling the case of the space-time white noise. See [39] and the references therein for the previous works in the periodic and non-periodic settings as well as some of its physical background. Also, see [3], [44], [53]. Note that we often see $u_x \phi dW$ as multiplicative noise in SKdV rather than $u\phi dW$ as in (5.1), and one can regard our study of (5.1) as the first step toward understanding more difficult multiplicative noises such as $u_x \phi dW$.

Our first goal in this chapter is to show that (5.1) is locally well-posed when $u_0 \in L^2(\mathbb{T})$ and we take ϕ to be Hilbert-Schmidt from $L^2(\mathbb{T})$ into itself. First, we briefly review recent well-posedness results of the periodic (deterministic) KdV:

$$\begin{cases} u_t + u_{xxx} + uu_x = 0 \\ u\big|_{t=0} = u_0, \end{cases} \quad (x,t) \in \mathbb{T} \times \mathbb{R}. \tag{5.5}$$

In [9], Bourgain introduced a weighted space-time Sobolev space $X^{s,b}$ whose norm is given by

$$\|u\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} = \big\|\langle n\rangle^s \langle \tau - n^3\rangle^b \widehat{u}(n,\tau)\big\|_{L^2_{n,\tau}(\mathbb{Z}\times\mathbb{R})}, \tag{5.6}$$

where $\langle \cdot \rangle = 1 + |\cdot|$. He proved local well-posedness of (5.5) in $L^2(\mathbb{T})$ via the fixed point argument, immediately yielding global well-posedness in $L^2(\mathbb{T})$ thanks to the conservation of the L^2-norm. Kenig-Ponce-Vega [59] (also see [26]) improved Bourgain's result and established local well-posedness of (5.5) in $H^{-\frac{1}{2}}(\mathbb{T})$ by establishing the bilinear estimate

$$\|\partial_x(uv)\|_{X^{s,b-1}} \lesssim \|u\|_{X^{s,b}} \|v\|_{X^{s,b}}, \tag{5.7}$$

for $s \geq -\frac{1}{2}$ and $b = \frac{1}{2}$ under the mean-zero assumption on u and v. Colliander-Keel-Staffilani-Takaoka-Tao [26] proved the corresponding global well-posedness result in $H^{-\frac{1}{2}}(\mathbb{T})$ via the I-method.

There are also results on (5.5) which exploit the complete integrability of (5.5). In [10], Bourgain proved global well-posedness of (5.5) in the class $M(\mathbb{T})$ of measures λ, assuming that its total variation $\|\lambda\|$ is sufficiently small. His proof is based on the trilinear estimate on the second iteration of the integral formulation of (5.5), assuming an a priori uniform bound on the Fourier coefficients of the solution u of the form

$$\sup_{n\in\mathbb{Z}} |\widehat{u}(n,t)| < C \tag{5.8}$$

for all $t \in \mathbb{R}$. Then, he established the a priori estimate (5.8) using the complete integrability. More recently, Kappeler-Topalov [57] proved global well-posedness of (5.5) in $H^{-1}(\mathbb{T})$ via the inverse spectral method. For (5.1), the integrability structure is destroyed due to the noise, and thus these results are not directly applicable.

We point out that all the nonlinear estimates above were established under the assumption that *the spatial mean is zero* for all $t \in \mathbb{R}$. Firstly, let us consider the deterministic KdV (5.5). Recall that the (spatial) mean of a solution $u(t)$ is preserved under the flow. Suppose that the mean α_0 of the initial condition u_0 is not zero. Then, we can transform KdV with non-zero mean into mean-zero KdV by a Galilean transformation

$$u(x,t) \to u(x+\alpha_0 t, t) - \alpha_0$$

as in [27]. One can then proceed to use the nonlinear estimates in [9, 10, 26, 57, 59] to prove well-posedness for mean-zero KdV, which can be converted into well-posedness of the original non-zero mean with the *prescribed* mean α_0. This was in particular simple for the deterministic KdV thanks to the conservation of the mean under the flow.

In [39, 75], a similar argument was employed to reduce SKdV (5.4) with additive noise to the mean-zero case. The transformation in this case depends not only on the mean of the initial condition but also on the Brownian motion β_0 at the zeroth frequency since (5.4) does not preserve the mean of the solution. See [39, 75] for details.

In establishing nonlinear estimates for SKdV (5.1) with multiplicative noise, we also need to assume the mean-zero condition. It turns out that this is not so simple due to the multiplicative structure of the noise.

(a) Mean-zero projection: One way to handle this issue is to simply make the equation mean-zero by introducing the Dirichlet projection onto the non-zero (spatial) frequencies to the noise, i.e. we consider

$$\begin{cases} du + (\partial_x^3 u + u\partial_x u)dt = \mathbb{P}_{\neq 0}\big[u\phi dW\big] \\ u(x,0) = u_0(x) \in L^2(\mathbb{T}), \end{cases} \tag{5.9}$$

where $\mathbb{P}_{\neq 0} f(x) = \sum_{n\neq 0} \widehat{f}(n) e_n(x)$. It is easy to verify that (5.9) preserves the mean of a solution. Suppose that the mean α_0 of the initial condition is not zero. We can define $v(x,t) = u(x+\alpha_0 t, t) - \alpha_0$. Then, v satisfies

$$\begin{cases} dv + (\partial_x^3 v + v\partial_x v)dt = \mathbb{P}_{\neq 0}\big[v\phi d\widehat{W}\big] + \alpha_0 \mathbb{P}_{\neq 0}\phi d\widehat{W} \\ v(x,0) = v_0(x) = u_0(x) - \alpha_0, \end{cases} \tag{5.10}$$

where $\widehat{W}$ is given by

$$\widehat{W}(x,t) := W(x+\alpha_0,t) = \sum_n \beta_n(t) e^{in\alpha_0 t} e_n(x). \tag{5.11}$$

We have introduced additive noise in (5.10). However, note that the mean of a solution to (5.10) is zero at any time (as long as it exists), since the mean of the initial condition v_0 is zero and (5.10) also preserves the mean of a solution. Once we prove local well-posedness of (5.10), we establish local well-posedness of (5.9) with *prescribed* mean α_0.

(b) Stochastic KdV-mean system: We now discuss a different way to handle the non-zero mean case without introducing an artificial projection as in (a). In the following, we apply a sequence of transformations to (5.1) and formulate the mean-zero version of (5.1). Let $v_1(x,t) = u(x+\alpha_0 t, t) - \alpha_0$, where $\alpha_0 =$ the mean of u_0. Then, v_1 satisfies

$$\begin{cases} dv_1 + (\partial_x^3 v_1 + v_1 \partial_x v_1)dt = (v_1+\alpha_0)\phi d\widehat{W} \\ v_1(x,0) = v_0 = u_0(x) - \alpha_0, \end{cases} \tag{5.12}$$

where $\widehat{W}(x,t) = W(x+\alpha_0 t, t)$ as in (5.11). As before, the mean of the initial condition v_0 is zero. Now, let $\mu(t) = \mu(t,\omega;u)$ denote the mean of v_1 at time t. i.e.

$$\begin{aligned} \mu(t) &= \frac{1}{2\pi}\int_{\mathbb{T}}\int_0^t (v_1(r)+\alpha_0)\phi d\widehat{W}(r)dx \\ &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^t \int_{\mathbb{T}} v_1(x,r)e_n(x)dx\, \phi_n e^{in\alpha_0 r} d\beta_n(r) + \int_0^t \alpha_0\phi_0 e_0 d\beta_0(r) \\ &= \frac{1}{2\pi}\sum_{n\in\mathbb{Z}}\int_0^t \overline{\widehat{v_1}(n,r)}\phi_n e^{in\alpha_0 r} d\beta_n(r) + \frac{1}{\sqrt{2\pi}}\int_0^t \alpha_0\phi_0 d\beta_0(r). \end{aligned} \tag{5.13}$$

Define $v_2 = v_1 - \mu(t)$. Note that $\widehat{v_2}(n,t) = \widehat{v_1}(n,t)$ for $n \neq 0$, and that v_2 has spatial mean zero for all t (as long as it exists) and satisfies

$$\begin{cases} dv_2 + (\partial_x^3 v_2 + (v_2 + \mu(t))\partial_x v_2)dt = \mathbb{P}_{\neq 0}\big[v_2 \phi d\widehat{W}\big] + (\alpha_0 + \mu(t))\mathbb{P}_{\neq 0}\phi d\widehat{W} \\ v_2(x,0) = u_0(x) - \alpha_0. \end{cases} \tag{5.14}$$

where $\mathbb{P}_{\neq 0}$ is the Dirichlet projection onto the nonzero frequencies. Finally, by defining $v_3(x,t) = v_2\big(x + \int_0^t \mu(r)dr, t\big)$, we see that v_3 has the spatial mean zero for all t (as long as it exists) and that it satisfies

$$\begin{cases} dv_3 + (\partial_x^3 v_3 + v_3 \partial_x v_3)dt = \mathbb{P}_{\neq 0}\big[v_3 \phi d\widetilde{W}\big] + (\alpha_0 + \mu(t))\mathbb{P}_{\neq 0}\phi d\widetilde{W} \\ v_3(x,0) = u_0(x) - \alpha_0, \end{cases} \tag{5.15}$$

where $\widetilde{W}$ is given by

$$\widetilde{W}(x,t) := \widehat{W}\Big(x + \int_0^t \mu(r)dr, t\Big) = \beta_0(t)e_0 + \sum_{n\neq 0} \frac{1}{\sqrt{2}}\beta_n(t)e^{in(\alpha_0 t + \int_0^t \mu(r)dr)}e_n(x). \tag{5.16}$$

From (5.13) (5.15) with $v_1(x,t) = v_2(x,t) + \mu(t) = v_3\big(x - \int_0^t \mu(r)dr, t\big) + \mu(t)$, we reduced (5.1) to a coupled system of mean-zero SKdV and a stochastic differential equation for the mean of a solution u to (5.1)(*SKdV-mean system*):

$$\begin{cases} dv + (\partial_x^3 v + v\partial_x v)dt = \mathbb{P}_{\neq 0}\big[v\phi d\widetilde{W}\big] + (\alpha_0 + \mu(t))\mathbb{P}_{\neq 0}\phi d\widetilde{W} \\ d\mu = \frac{1}{2\pi}\sum_{n\neq 0} \overline{\widehat{v}(n,t)}\phi_n e^{in\int_0^t \alpha_0 + \mu(r)dr} d\beta_n(t) + \frac{1}{\sqrt{2\pi}}(\alpha_0 + \mu(t))\phi_0 d\beta_0(t) \\ v(x,0) = v_0 \text{ with mean } 0, \quad \mu(0) = 0. \end{cases} \tag{5.17}$$

Note that v has spatial mean 0 (as long as it exists) since v_0 has mean 0 and (5.17) preserves the mean of v. Therefore, we concentrate on studying well-posedness of (5.17) with mean-zero initial condition v_0.

We could have defined v directly as

$$v(x,t) = u\Big(x - \int_0^t \mu(r)dr, t\Big) + \mu(t).$$

The difference between (5.10) and (5.17) is the presence of $\mu(t)$ in the additive noise.

Recall that u is called a (local-in-time) mild solution to (5.1) if u satisfies

$$u(t) = U(t)u_0 - \frac{1}{2}\int_0^t U(t-t')\partial_x u^2(t')dt' + \int_0^t U(t-t')[u(t')\phi dW(t')] \qquad (5.18)$$

at least for $t \in [0,T]$ for some $T > 0$, where $U(t) = e^{-t\partial_x^3}$. Similarly, the mild formulation of (5.17) is given by

$$v(t) = U(t)v_0 - \frac{1}{2}\int_0^t U(t-t')\partial_x v^2(t')dt' + \Phi_1(v)(t) + \Phi_2(\mu,\alpha_0)(t) \qquad (5.19)$$

$$\mu(t) = \sum_{n\neq 0}\int_0^t \overline{\widehat{v}(n,t')}\phi_n e^{in\int_0^{t'}\mu(r)dr}d\beta_n(t') + \int_0^t (\alpha_0 + \mu(t'))\phi_0 d\beta_0(t'). \qquad (5.20)$$

where the stochastic convolutions Φ_1 and Φ_2 are given by

$$\Phi_1(t) = \Phi_1(v)(t) := \int_0^t U(t-t')\mathbb{P}_{\neq 0}\big[v(t')\phi d\widetilde{W}(t')\big] \qquad (5.21)$$

$$\Phi_2(t) = \Phi_2(\mu,\alpha_0)(t) := \int_0^t U(t-t')(\mu(t')+\alpha_0)\mathbb{P}_{\neq 0}\phi d\widetilde{W} \qquad (5.22)$$

with $\widetilde{W}$ as in (5.16).

Now, let us briefly describe how we construct a (local-in-time) mild solution (v,μ) to (5.17). Due to the presence of the multiplicative noise, we need to introduce a truncation to (5.19)-(5.20) as in de Bouard-Debussche [36, 37]. In establishing estimates in $X^{s,b}$ defined in (5.6), we need to take the temporal regularity b to be less than $\frac{1}{2}$ due to the regularity of the stochastic convolutions $\Phi_1(t)$ and $\Phi_2(t)$ (which have the same temporal regularity as the Brownian motion.) This introduces additional difficulty in nonlinear analysis for estimating the second term on the right-hand side of (5.19), since the bilinear estimate (5.7) does not hold for any $s \in \mathbb{R}$ if $b \neq \frac{1}{2}$. In order to overcome this difficulty, we follow the argument in [10, 75] and perform a nonlinear analysis on the second iteration in $X^{s,b}$ with $s = 0$ and $b < \frac{1}{2}$. The high regularity $s = 0$ allows us to proceed without assuming complete integrability and the a priori bound (5.8). After establishing local well-posedness, we can use an a priori L^2-bound to extend local-in-time solutions (v,μ) to global ones.

Theorem 1. Let ϕ be as in (5.2) such that ϕ is Hilbert-Schmidt from $L^2(\mathbb{T})$ into itself. Given mean-zero $v_0 \in L^2(\mathbb{T})$ and $\alpha_0 \in \mathbb{R}$, the coupled system (5.17) is globally well-posed, in the sense that, given any time $T > 0$, there exists a unique pair (v, μ) in the space

$$L^2\big(\Omega; C([0,\infty); L^2(\mathbb{T})\big) \times L^2\big(\Omega; C([0,\infty); \mathbb{R})\big)$$

satisfying (5.17) on $[0, \infty)$ almost surely.

Given a solution (v, m) of (5.17), a solution u of (5.1) can be recovered via

$$u(x,t) = v\bigg(x - \alpha_0 t - \int_0^t \mu(r)dr, t\bigg) + \alpha_0 + \mu(t). \tag{5.23}$$

Hence, we obtain the following theorem as a corollary to Theorem 1.

Theorem 2. Let ϕ be as in (5.2) such that ϕ is Hilbert-Schmidt from $L^2(\mathbb{T})$ into itself. The stochastic KdV (5.1) with multiplicative space-time white noise is globally well-posed (with the prescribed mean on u_0).

Remark 5.1. Technically, the uniqueness of (v, μ) in Theorem 1 holds in the smaller space

$$L^2\big(\Omega; C([0,\infty); L^2(\mathbb{T}) \cap X_T^{0,\frac{1}{2}-\delta}\big) \times L^2\big(\Omega; C([0,\infty); \mathbb{R})\big)$$

for some small $\delta > 0$, where $X_T^{0,\frac{1}{2}-\delta}$ is a time restricted Fourier restriction norm space (see (5.6) above and Section 5.1 below).

Finally, we verify that the result of Tsutsumi [84] on the stabilization by noise continues to hold in our low regularity setting. Specifically, it states that the mass of a solution almost surely decays to zero as time goes to infinity.

Theorem 3. Let ϕ and u_0 satisfy the same assumptions as in Theorem 2. Suppose further that there exists a constant $\alpha > \frac{1}{2}\|\phi\|_{\mathrm{HSt}(L_x^2)}$ such that for all $v \in L^2(\mathbb{T})$, one has

$$\sum_{k=-\infty}^{\infty} \bigg[\int_{\mathbb{T}} \mathrm{Re}\,\big(\phi e_k(x)\big)|v(x)|^2 dx\bigg]^2 \geq \alpha^2 \|v\|_{L^2(\mathbb{T})}^2. \tag{5.24}$$

Then the solution u of the stochastic KdV (5.1) given by Theorem 2 decays in mass, that is, as $t \to \infty$, we have

$$\|u(t)\|_{L^2_x(\mathbb{T})} \to 0$$

almost surely.

This chapter is organized as follows: In Section 2, we introduce some notations. In Section 3, we introduce the functional framework we will be using and state deterministic linear and bilinear estimates from [74]. In Section 4, we discuss the second iteration and the truncated version of the mean-zero stochastic KdV coupled system. We then prove the required estimates to analyse the modified systems in Section 5 and 6. In section 7, we gather everything and give a proof of Theorem 1. Finally, we briefly prove Theorem 3 in Section 8.

5.1 Function spaces

The main function space we use throughout this chapter is the Fourier restriction norm spaces $X^{s,b} = X^{s,b}(\mathbb{T} \times \mathbb{R})$ (for $s, b \in \mathbb{R}$) adapted to the KdV equation as mentioned and defined in (5.6). Equivalently, the $X^{s,b}$-norm can be written in its interaction representation form:

$$\|u\|_{X^{s,b}} = \|\langle n\rangle^s \langle \tau\rangle^b \mathcal{F}_{t,x}(U(-t)u(t))(n,\tau)\|_{\ell^2_n L^2_\tau(\mathbb{R}\times\mathbb{Z})}\,, \tag{5.25}$$

where $U(t) = e^{it\partial_x^3}$ is the linear KdV propagator.

Given an interval $I \subseteq \mathbb{R}$, we define the local-in-time version $X^{s,b}_I$ on I, by

$$\|u\|_{X^{s,b}_I} = \inf\big\{\|\widetilde{u}\|_{X^{s,b}(\mathbb{T}\times\mathbb{R})} : \widetilde{u}|_I = u\big\}.$$

Most of the time, the interval I is given by $[0, T]$ for some $T > 0$. In this case, we simply use the notation $X^{s,b}_T := X^{s,b}_{[0,T]}$.

We now state some basic properties of these spaces. One can find the proofs of these facts in, for example, [82]. Firstly, we have the following continuous embeddings

$$X^{s,b} \hookrightarrow C(\mathbb{R}; H^s_x(\mathbb{T}^d)) \text{ , for } b > \frac{1}{2}, \tag{5.26}$$

$$X^{s',b'} \hookrightarrow X^{s,b} \text{ , for } s' \geq s \text{ and } b' \geq b. \tag{5.27}$$

We have the duality relation

$$\|u\|_{X^{s,b}} = \sup_{\|w\|_{X^{-s,-b}} \leq 1} \left| \int_{\mathbb{R}\times\mathbb{T}^d} u(t,x)\overline{w(t,x)}\, dt\, dx \right|. \tag{5.28}$$

For $s \geq 0$ and $0 \leq b < \frac{1}{2}$, we have the following relation between $X^{s,b}$ and its time restricted version:

$$\|u\|_{X^{s,b}_I} \sim \|\mathbb{1}_I(t)u(t)\|_{X^{s,b}}, \tag{5.29}$$

see for example [37, Lemma 2.1] for a proof.

By (5.25), we have the following linear estimate.

Lemma 5.2. Let $s, b \in \mathbb{R}$ and $T > 0$. For any $f \in H^s$, we have

$$\|U(t)f\|_{X^{s,b}_T} \lesssim \|f\|_{H^s}. \tag{5.30}$$

By localizing in time, we can gain a smallness factor, as per lemma below.

Lemma 5.3 (Time localisation property)**.** Let $s \in \mathbb{R}$ and $-\frac{1}{2} < b' < b < \frac{1}{2}$. For any $T \in (0,1)$, we have

$$\|u\|_{X^{s,b'}_T} \lesssim_{b,b'} T^{b-b'}\|u\|_{X^{s,b}_T}.$$

The following refinement of the $L^4_{t,x}$-Strichartz inequality by Bourgain [9] is important to us. For a textbook treatment, see for example [41][Theorem 3.18]

Lemma 5.4 ($L^4_{t,x}$-Strichartz inequality)**.** For any space-time function u, we have

$$\|u\|_{X^{s,b}} \lesssim \|u\|_{X^{s,b}}.$$

5.2 Second iteration and truncation

We now go over the details of our strategy of solving the Stochastic KdV-mean system discussed previously. As mentioned before, we work with the mild formulations (5.19) and (5.20). Denoting by $\mathcal{N}(\cdot,\cdot)$ the bilinear form

$$\mathcal{N}(u_1,u_2)(t) := \int_0^t U(t-t')\partial_x(u_1u_2)(t')dt', \tag{5.31}$$

(5.19) then becomes

$$v(t) = U(t)v_0 - \frac{1}{2}\mathcal{N}(v,v)(t) + \Phi_1(v)(t) + \Phi_2(\mu,\alpha_0)(t) \tag{5.32}$$

A first approach is to attempt a contraction argument in a suitable subspace of $L^2\big(\Omega; X_T^{0,b}\big) \times L^2(\Omega; L^2([0,T]))$ for some short time T. The presence of the stochastic convolutions require us to set $b = \frac{1}{2} - \delta$ for a small $\delta > 0$. This means we need to control each of the above term in $L^2\big(\Omega, X_T^{0,\frac{1}{2}-\delta}\big)$-norm. In the following analysis, we shall assume that all space-time functions u have spatial mean zero for all t, that is, $\widehat{u}(0,t) = 0$ for all t.

We now consider the $X_T^{0,\frac{1}{2}-\delta}$-norm of the nonlinear term (5.31). By taking the spatial Fourier transform of (5.31) and the temporal Fourier transform of $\partial_x(u_1u_2)$, we may rewrite

$$\begin{aligned}\mathcal{N}(u_1,u_2)(x,t) &= -\sum_{n\in\mathbb{Z}\setminus\{0\}} ne^{inx}\int_{-\infty}^{\infty}\frac{e^{it\tau}-e^{itn^3}}{\tau-n^3}\widehat{u_1u_2}(n,\tau)d\tau \\ &=: \mathcal{I}(u_1,u_2)(x,t) + \mathrm{II}(u_1,u_2)(x,t).\end{aligned} \tag{5.33}$$

In the following, we let $(n,\tau), (n_1,\tau_1), (n_2,\tau_2)$ denote the space-time Fourier variables of $\mathcal{N}(u_1,u_2)$, u_1 and u_2 in (5.31) respectively. We also use the notation

$$k_0 := \langle \tau - n^3\rangle \text{ and } k_j := \langle \tau_j - n_j^3\rangle.$$

Recall the following observation made in Bourgain [9]:

$$n^3 - n_1^3 - n_2^3 = 3nn_1n_2, \text{ for } n = n_1 + n_2, \tag{5.34}$$

which in turn implies that

$$\text{MAX} := \max(k_0, k_1, k_2) \gtrsim \langle n n_1 n_2 \rangle. \tag{5.35}$$

We define the sets A_j for $j \in \{0, 1, 2\}$ by

$$A_j := \{(n, n_1, n_2, \tau, \tau_1, \tau_2) \in \mathbb{Z}^3 \times \mathbb{R}^3 : k_j = \text{MAX}\}, \tag{5.36}$$

Denote by $\mathcal{I}_j(u_1, u_2)$, $\text{II}_j(u_1, u_2)$ and $\mathcal{N}_j(u_1, u_2)$ the contribution of $\mathcal{I}(u_1, u_2)$, $\text{II}(u_1, u_2)$ and $\mathcal{N}(u_1, u_2)$ respectively on A_j. Then

$$\mathcal{N}(u_1, u_2) = \sum_{j=0}^{2} \mathcal{N}_j(u_1, u_2) = \sum_{j=0}^{2} \mathcal{I}_j(u_1, u_2) + \text{II}_j(u_1, u_2).$$

The term $\|\mathcal{N}_0(u_1, u_2)\|_{X^{0,\frac{1}{2}-\delta,T}}$ can be controlled via the standard bilinear estimate as in [9] and [59] (see Lemma 5.6 below). However, we cannot do the same for $\mathcal{N}_1(u_1, u_2)$ (and symmetrically, $\mathcal{N}_2(u_1, u_2)$). Indeed, the bilinear estimate is known to fail for temporal regularity below $\frac{1}{2}$ for the contribution $\mathcal{N}_1(u_1, u_2)$ (see [60]). Instead, we will consider a second iteration of (5.19), more specifically, we will substitute in (5.19) for the first argument of $\mathcal{N}_1$ (respectively, the second argument of $\mathcal{N}_2$).

Note that the space-time Fourier transforms of $U(t)v_0$ and $\text{II}(v, v)$ are distributions supported on $\{\tau = n^3\}$. On the other hand, we have $k_j = \text{MAX} \gtrsim \langle n n_1 n_2 \rangle \gg 1$ on A_j, hence the terms $\mathcal{N}_1(U(t)v_0, v)$, $\mathcal{N}_1(\text{II}(v, v), v)$, $\mathcal{N}_2(v, U(t)v_0)$ and $\mathcal{N}_2(v, \text{II}(v, v))$ vanish and do not appear in (5.37). This leads us to consider the second iterated SKdV-mean system:

$$\begin{aligned} v(t) =& U(t)v_0 - \frac{1}{2}\mathcal{N}_0(v, v)(t) + \Phi_1(v)(t) + \Phi_2(\mu, \alpha_0)(t) \\ & - \frac{1}{2}\mathcal{N}_1\big(\mathcal{I}(v, v), v\big)(t) + \mathcal{N}_1\big(\Phi_1(v) + \Phi_2(\mu, \alpha_0), v\big)(t) \\ & - \frac{1}{2}\mathcal{N}_2\big(v, \mathcal{I}(v, v)\big)(t) + \mathcal{N}_2\big(v, \Phi_1(v) + \Phi_2(\mu, \alpha_0)\big)(t). \end{aligned} \tag{5.37}$$

$$\mu(t) = \sum_{n \neq 0} \int_0^t \overline{\widehat{v}(n, t')} \phi_n e^{in \int_0^{t'} \mu(r) dr} d\beta_n(t') + \int_0^t (\alpha_0 + \mu(t'))\phi_0 d\beta_0(t'). \tag{5.38}$$

Meanwhile, we consider the following truncation to the equation in order to handle the multiplicative noise. Let $\eta : \mathbb{R} \to [0,1]$ be a smooth function supported on $[-1,2]$ such that $\eta(t) = 1$ for $t \in [0,1]$. Given $R > 0$, and a mean-zero space-time function u, we define the following notation:

$$\begin{aligned} \eta_{R}(t) &:= \eta\left(\frac{t}{R}\right) \\ \widetilde{\eta_{R}}(u)(t) &:= \eta_{R}\left(\|u\|_{X_t^{0,\frac{1}{2}-\delta}}\right), \\ \mathcal{T}_{R} u &:= \eta_{R}(u)u. \end{aligned} \tag{5.39}$$

The following lemma from [37] relates the truncation $\mathcal{T}_R$ and $X^{s,b}$-norms.

Lemma 5.5 (Lemma 2.2 in [37])**.** Let $R > 0$. There exist constants $C_1(R) > 0$ and $C_2 > 0$ (the latter does not depend on R) such that for any $u, v \in X_T^{0,b}$, one has

$$\left\|\mathcal{T}_R v\right\|_{X_T^{0,b}} \leq \min\left\{C_1(R), C_2\|v\|_{X_T^{0,\frac{1}{2}-\delta}}\right\} \tag{5.40}$$

$$\left\|\mathcal{T}_R v - \mathcal{T}_R u\right\|_{X_T^{0,\frac{1}{2}-\delta}} \leq C_2\left\|v - u\right\|_{X_T^{0,\frac{1}{2}-\delta}} \tag{5.41}$$

Note that (5.41) is not explicitly stated and proved in [37] but the proof follows the same argument as for (5.40). See also [20, Lemma 4.3] for the same lemma in the context of stochastic nonlinear Schrödinger equations.

We now turn back to (5.19)–(5.20) and introduce $\mathcal{T}_R$ to various places. This leads us to consider the following R-truncated SKdV mean-zero system:

$$\begin{aligned} v(t) =& U(t)v_0 - \frac{1}{2}\left[\mathcal{N}_0(\mathcal{T}_R v, v) + \mathcal{N}_1(v, \mathcal{T}_R v) + \mathcal{N}_2(\mathcal{T}_R v, v)\right](t) \\ &\qquad + \mathcal{T}_R\Phi_1(\mathcal{T}_R v)(t) + \mathcal{T}_R\Phi_2(\mu, \alpha_0)(t) \end{aligned} \tag{5.42}$$

$$\mu(t) = \sum_{n\neq 0}\int_0^t \overline{\widehat{v}(n,t')}\phi_n e^{in\int_0^{t'}\mu(r)dr} d\beta_n(t') + \int_0^t (\alpha_0 + \mu(t'))\phi_0 d\beta_0(t'). \tag{5.43}$$

If we do the same as before and consider its second iteration, we arrive at the following second iterated R-truncated SKdV-mean system:

$$\begin{aligned} v(t) &= U(t)v_0 - \frac{1}{2}\mathcal{N}_0\big(v, \mathcal{T}_R v\big)(t) + \mathcal{T}_R\Phi_1(\mathcal{T}_R v)(t) + \mathcal{T}_R\Phi_2(\mu,\alpha_0)(t) \\ &\quad - \frac{1}{2}\mathcal{N}_1\Big(\mathcal{I}(v,\mathcal{T}_R v) + \mathbb{1}_{[0,T]}\mathcal{T}_R\big(\Phi_1(\mathcal{T}_R v)\big) + \mathbb{1}_{[0,T]}\mathcal{T}_R\big(\Phi_2(\mu,\alpha_0)\big), \mathcal{T}_R v\Big)(t) \\ &\quad - \frac{1}{2}\mathcal{N}_2\Big(\mathcal{T}_R v, \mathcal{I}(\mathcal{T}_R v, v) + \mathbb{1}_{[0,T]}\mathcal{T}_R\big(\Phi_1(\mathcal{T}_R v)\big) + \mathbb{1}_{[0,T]}\mathcal{T}_R\big(\Phi_2(\mu,\alpha_0)\big)\Big)(t) \end{aligned} \tag{5.44}$$

$$\mu(t) = \sum_{n\neq 0}\int_0^t \overline{\widehat{v}(n,t')}\phi_n e^{in\int_0^{t'}\mu(r)dr}d\beta_n(t') + \int_0^t(\alpha_0+\mu(t'))\phi_0 d\beta_0(t'). \tag{5.45}$$

The systems (5.19)–(5.20) and (5.37)–(5.38) are equivalent whenever $t\in[0,T\wedge\tau_R]$, where $\tau_R = \tau_R(v,\alpha_0,\mu)$ is the stopping time

$$\tau_R := \inf\Big\{t>0: \|v\|_{X_t^{0,\frac{1}{2}-\delta}} \vee \|\Phi_1(\mathcal{T}_R v)\|_{X_t^{0,\frac{1}{2}-\delta}} \vee \|\Phi_2(\alpha_0,\mu)\|_{X_t^{0,\frac{1}{2}-\delta}} \geq R\Big\}. \tag{5.46}$$

Our strategy of proving Theorem 1 shall be as follows. We first consider smooth approximations $\{u_0^N\}_{N\in 2^{\mathbb{Z}}}$ of the initial data $u_0\in L^2(\mathbb{T})$. By a high regularity global well-posedness result of multiplicative SKdV as stated in Tsutsumi [84], there exist global solutions u^N to the multiplicative SKdV. This correspond to solutions (v^N,μ^N) of the SKdV-mean system, and by considering second iteration and truncation discussed, we are able to perform $X^{s,b}$-analysis to show that (v^N,μ^N) is Cauchy in some $X^{s,b}$ topology. We then proceed to show that the limit (v,μ) is indeed a solution to the SKdV-mean system. Meanwhile, one can also perform a contraction on the second iterated R-truncated SKdV-mean system (5.44)–(5.45), and this secures the uniqueness of our solution (v,μ). Finally, we establish the continuity-in-time of v by analysing each term in the mild formulation separately. This turns out to be not so trivial since some of the terms do not necessarily lie in $X^{0,b}$ with $b>\frac{1}{2}$, which prevents us from directly applying the embedding (5.26).

5.3 Deterministic estimates

In this section, we state and prove various multilinear estimates of the nonlinear terms necessary for our analysis.

Lemma 5.6. Let u_1, u_2 be space-time functions with spatial mean 0. Let $\delta > 0$. Then

$$\|\mathcal{N}_0(u_1, u_2)\|_{X^{0,\frac{1}{2}-\delta}} \lesssim \|u_1\|_{X^{0,\frac{1}{3}}} \|u_2\|_{X^{0,\frac{1}{3}}}.$$

Proof. Recall that we have $k_0 = \text{MAX}$. This implies that $k_0 \gg 1$ and we may replace $|\tau - n^3|$ by k_0. We first note that II_0 is a free solution, i.e. $\text{II}_0(t) = U(t)f$ where

$$\widehat{f}(n) = \int_{k_0=\text{MAX}} \frac{n}{\tau - n^3} \widehat{u_1 u_2}(n, \tau) d\tau \,.$$

Hence by Lemma 5.2 and duality, there exists $d \in \ell^2(\mathbb{Z})$ with $\|d\|_{\ell^2_n} \leq 1$ such that for any $b \in \mathbb{R}$,

$$\begin{aligned}\|\text{II}_0(u_1, u_2)\|_{X^{0,b}} &\lesssim \|f\|_{\ell^2_n} \\ &\lesssim \sum_{\substack{n, n_1 \in \mathbb{Z}\setminus\{0\} \\ n_1+n_2=n}} \int_{\tau_1+\tau_2=\tau} \frac{\langle n \rangle}{k_0} d(n)\, \widehat{u_1}(n_1, \tau_1)\, \widehat{u_2}(n_2, \tau_2)\, d\tau_1\, d\tau.\end{aligned}$$

By using $\langle n \rangle \lesssim \langle n_1 \rangle \langle n_2 \rangle$ and (5.34), we have

$$\frac{\langle n \rangle}{k_0} \lesssim \frac{(\langle n \rangle \langle n_1 \rangle \langle n_2 \rangle)^{\frac{1}{2}}}{(\langle n \rangle \langle n_1 \rangle \langle n_2 \rangle)^{1-4\delta} k_1^{2\delta} k_2^{2\delta}} \lesssim \frac{1}{\langle n \rangle^{\frac{1}{2}-4\delta} \langle n_1 \rangle^{\frac{1}{2}-4\delta} \langle n_2 \rangle^{\frac{1}{2}-4\delta} k_1^{2\delta} k_2^{2\delta}}$$

Hence the above is controlled by

$$\sum_{\substack{n, n_1 \in \mathbb{Z}\setminus\{0\} \\ n_1+n_2=n}} \frac{d(n)}{\langle n \rangle^{\frac{1}{2}-4\delta}} \left[\int \frac{\widehat{u_1}(n_1, \tau_1)}{\langle n \rangle^{\frac{1}{2}-4\delta} k_1^{2\delta}}\, d\tau_1 \right] \left[\int \frac{\widehat{u_2}(n_2, \tau_2)}{\langle n \rangle^{\frac{1}{2}-4\delta} k_2^{2\delta}}\, d\tau_2 \right] =: \sum_{n \in \mathbb{Z}} g_0(n)\, g_1 * g_2(n).$$

By Parseval Theorem, Hölder and Hausdorff-Young inequalities, we have

$$\int \widehat{g_0}(-x) \widehat{g_1}(x) \widehat{g_2}(x) dx \leq \prod_{j=0}^{2} \|\widehat{g_j}\|_{L^3_x} \leq \prod_{j=0}^{2} \|\widehat{g_j}\|_{\ell^{\frac{3}{2}}_n}$$

$$\leq \|\langle n\rangle^{-(\frac{1}{2}-4\delta)}\|_{\ell_n^6}^3 \|d\|_{\ell_n^2} \prod_{j=1}^{2} \left\| \int \frac{\langle \tau - n^3\rangle^{\frac{1}{2}-\delta}\widehat{u_j}(n,\tau)}{\langle \tau - n^3\rangle^{\frac{1}{2}+\delta}} d\tau \right\|_{\ell_n^2}^2$$

$$\lesssim \|u_1\|_{X^{0,\frac{1}{2}-\delta}} \|u_2\|_{X^{0,\frac{1}{2}-\delta}}.$$

We now turn to bounding $\mathcal{I}_0$. By duality, there exists $d \in L^2_{\tau,n}$ with $\|d\|_{L^2_{\tau,n}} \leq 1$ such that

$$\begin{aligned}
\|\mathcal{I}_0(u_1,u_2)\|_{X^{0,\frac{1}{2}-\delta}} &\lesssim \sum_{\substack{n,n_1\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2}} \frac{\langle n\rangle}{k_0^{\frac{1}{2}+\delta}} \int_{-\infty}^{\infty} \widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2) d(n,\tau) d\tau_1 d\tau \\
&\lesssim \sum_{\substack{n,n_1\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2}} \frac{\langle n\rangle^{\frac{1}{2}+\delta}\langle n_1\rangle^{\frac{1}{2}-\delta}\langle n_2\rangle^{\frac{1}{2}-\delta}}{(\langle n n_1 n_2\rangle)^{\frac{1}{2}+\delta}} \int_{-\infty}^{\infty} \widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2) d(n,\tau) d\tau_1 d\tau \\
&\lesssim \sum_{\substack{n,n_1\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2}} \int_{-\infty}^{\infty} \widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2) d(n,\tau) d\tau_1 d\tau \\
&\leq \|d\|_{L^2_{n,\tau}} \|u_1\|_{L^4_{t,x}} \|u_2\|_{L^4_{t,x}} \\
&\lesssim \|u_1\|_{X^{0,\frac{1}{3}}} \|u_2\|_{X^{0,\frac{1}{3}}}.
\end{aligned}$$

Putting everything together gives us

$$\|\mathcal{N}_0(u_1,u_2)\|_{X^{0,\frac{1}{2}-\delta}} \lesssim \|u_1\|_{X^{0,\frac{1}{3}}} \|u_2\|_{X^{0,\frac{1}{3}}}.$$

□

We now discuss the contributions coming from $\mathcal{N}_1$ (and symmetrically, $\mathcal{N}_2$). Note that the estimates allow us to put $\mathcal{N}_1$ in the larger space $X^{0,\frac{1}{2}+\delta}$.

Lemma 5.7. Let u, v, w, z be space-time functions with spatial mean 0. Let $\delta > 0$. Then for $T > 0$ sufficiently small, the following estimates hold:

$$\|\mathcal{N}_1(z,w)\|_{X_T^{0,\frac{1}{2}+\delta}} \lesssim \|z\|_{X_T^{0,\frac{1}{2}}} \|w\|_{X_T^{0,\frac{1}{3}}} \tag{5.47}$$

$$\|\mathcal{N}_1(\mathcal{I}(u,v),w)\|_{X_T^{0,\frac{1}{2}+\delta}} \lesssim \|u\|_{X^{0,\frac{1}{3},T}} \|v\|_{X_T^{0,\frac{1}{3}}} \|w\|_{X_T^{0,\frac{1}{3}}}. \tag{5.48}$$

Proof. Suppose first that $k_1 \sim 1$, then $k_1 \sim k_1^b$ for any $b \in \mathbb{R}$. In particular, we have

$$\begin{aligned} \|\mathcal{N}_1(z,w)\|_{X_T^{0,\frac{1}{2}+\delta}} \sim \|\mathcal{N}_1(z,w)\|_{X_T^{0,\frac{1}{2}-2\delta}} &\lesssim T^\delta \|\mathcal{N}_1(z,w)\|_{X_T^{0,\frac{1}{2}-\delta}} \\ &\lesssim T^\delta \|\mathcal{N}_1(z,w)\|_{X_T^{0,\frac{1}{2}+\delta}}. \end{aligned} \tag{5.49}$$

Now suppose that $k_1 \gg 1$. By duality, there exists $d \in L^2_{n,\tau}$ with $\|d\|_{L^2_{n,\tau}} \leq 1$ such that $\|\mathcal{I}_1(z,w)\|_{X^{0,\frac{1}{2}+\delta}}$ and $\|\mathrm{II}_1(z,w)\|_{X^{0,\frac{1}{2}+\delta}}$ are controlled by

$$\lesssim \sum_{\substack{n,n_1\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{\langle n\rangle}{k_0^{\frac{1}{2}-\delta}}\, d(n,\tau)\widehat{z}(n_1,\tau_1)\widehat{w}(n_2,\tau_2)d\tau_1 d\tau. \tag{5.50}$$

Using $\langle n\rangle \lesssim \langle n\rangle\langle nn_1n_2\rangle^{-\frac{1}{2}}k_1^{\frac{1}{2}} \lesssim k_1^{\frac{1}{2}}$ and a $L^4_{t,x}L^2_{t,x}L^4_{t,x}$ Hölder inequality, the above is

$$\begin{aligned} &\lesssim \sum_{\substack{n,n_1\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{d(n,\tau)}{k_0^{\frac{1}{2}-\delta}}\, k_1^{\frac{1}{2}}\widehat{z}(n_1,\tau_1)\widehat{w}(n_2,\tau_2)d\tau_1 d\tau. \\ &\lesssim \|z\|_{X^{0,\frac{1}{2}}}\|w\|_{X^{0,\frac{1}{3}}}. \end{aligned}$$

By (5.49) and (5.48), the estimate (5.47) holds for a sufficiently small $T > 0$.

We now turn to (5.48). We first note that (5.49) and (5.50) continue to hold with $z = I(u,v)$. In particular, there exists $d \in L^2_{n,\tau}$ with $\|d\|_{L^2_{n,\tau}} \leq 1$ such that $\|\mathcal{I}_1(\mathcal{I}(u,u),u)\|_{X^{0,\frac{1}{2}+\delta}}$ and $\|\mathrm{II}_1(\mathcal{I}(u,u),u_3)\|_{X^{0,\frac{1}{2}+\delta}}$ are controlled by

$$\begin{aligned} &\lesssim \sum_{\substack{n,n_1,n_3\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2\\ n_1=n_3+n_4}} \int_{\substack{\tau=\tau_1+\tau_2\\ \tau_1=\tau_3+\tau_4}} \frac{\langle n\rangle\langle n_1\rangle}{k_0^{\frac{1}{2}-\delta}k_1}\, \widehat{w}(n_2,\tau_2)\widehat{u}(n_3,\tau_3)\widehat{v}(n_4,\tau_4)d(n,\tau)d\tau_1 d\tau_3 d\tau \\ &\lesssim \sum_{\substack{n,n_1,n_3\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2\\ n_1=n_3+n_4}} \int_{\substack{\tau=\tau_1+\tau_2\\ \tau_1=\tau_3+\tau_4}} \frac{d(n,\tau)}{k_0^{\frac{1}{2}-\delta}}\frac{\widehat{w}(n_2,\tau_2)}{\langle n_2\rangle}\widehat{u}(n_3,\tau_3)\widehat{v}(n_4,\tau_4)d\tau_1 d\tau_3 d\tau. \end{aligned}$$

We may thus conclude the proof by a $L^4_{t,x}L^4_{t,x}L^4_{t,x}L^4_{t,x}$ Hölder inequality. $\square$

5.4 Stochastic estimates

In this section, we present several estimates on the stochastic convolutions $\Phi_1(v)$ and $\Phi_2(\mu, \alpha_0)$ defined in (5.21)-(5.22). We recall that ϕ is a convolution operator (see (5.2)) that is Hilbert-Schimdt from $L^2(\mathbb{T})$ into itself. The objects of study here are the stochastic convolutions

$$\Phi_1(t) = \Phi_1(v)(t) := \int_0^t U(t-t')\mathbb{P}_{\neq 0}\big[v(t')\phi d\widetilde{W}(t')\big] \tag{5.51}$$

$$\Phi_2(t) = \Phi_2(\mu, \alpha_0)(t) := \int_0^t U(t-t')(\mu(t') + \alpha_0)\mathbb{P}_{\neq 0}\phi d\widetilde{W} \tag{5.52}$$

given in (5.21) and (5.22), as well as the mean

$$\begin{aligned} M(t) &= M(\mu, v, \alpha_0)(t) \\ &:= \sum_{n\neq 0} \int_0^t \overline{\widehat{v}(n,t')}\phi_n e^{in\int_0^{t'} \mu(r)dr} d\beta_n(t') + \int_0^t (\alpha_0 + \mu(t'))\phi_0 d\beta_0(t') \end{aligned} \tag{5.53}$$

We assume that $v, v_1, v_2 \in L^\gamma(\Omega; X^{0,\frac{1}{2}-\delta,T})$ and $\mu, \mu_1, \mu_2 \in L^\gamma(\Omega; L^2([0,T]))$ for some $\gamma \in [2,\infty)$, $\delta > 0$ and $T > 0$.

Lemma 5.8. There exists $\theta > 0$ such that the following estimates hold:

$$\|\Phi_1(v)\|_{L^\gamma(\Omega, X_T^{0,\frac{1}{2}-\delta})} \lesssim T^\theta \|\phi\|_{\mathcal{L}^2(L^2)} \|v\|_{L^\gamma(\Omega, X_T^{0,0})} \tag{5.54}$$

$$\|\Phi_1(v_2) - \Phi_1(v_1)\|_{L^\gamma(\Omega, X_T^{0,\frac{1}{2}-\delta})} \lesssim T^\theta \|\phi\|_{\mathcal{L}^2(L^2)} \|v_2 - v_1\|_{L^\gamma(\Omega, X_T^{0,0})} \tag{5.55}$$

$$\|\Phi_2(\mu, \alpha_0)\|_{L^\gamma(\Omega, X_T^{0,\frac{1}{2}-\delta})} \lesssim T^\theta \|\phi\|_{\mathcal{L}^2(L^2)} \big(\|\mu\|_{L^\gamma(\Omega, L^2([0,T]))} + T^{\frac{1}{2}}|\alpha_0|\big) \tag{5.56}$$

$$\|\Phi_2(\mu_2, \alpha_0) - \Phi_2(\mu_1, \alpha_0)\|_{L^\gamma(\Omega, X_T^{0,\frac{1}{2}-\delta})} \lesssim T^\theta \|\phi\|_{\mathcal{L}^2(L^2)} \|\mu_2 - \mu_1\|_{L^\gamma(\Omega, L^2([0,T]))}. \tag{5.57}$$

Proof. We only prove (5.54) here, since (5.55) follows from (5.54) by the linearity of Φ_1 as a function of v, and the other two estimates follow from similar arguments.

Let $g(t) = \mathbb{1}_{[0,T]}(t)U(-t)\Phi_1(t)$. Then by stochastic Fubini theorem,

$$\widehat{g}(n,\tau) \sim \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_0^T e^{-it\tau} \int_0^t e^{it'n^3} \widehat{v(t')\phi e_k}(n) e^{in\alpha_0 t'} d\beta_k(t')\, dt$$

$$= \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_0^T \left(\int_{t'}^T e^{-it\tau} dt \right) e^{it'n^3} \phi_k \widehat{v}(n-k,t') e^{in\alpha_0 t'} d\beta_k(t').$$

Hence by Burkholder–Davis–Gundy inequality, we have

$$\|\Phi_1(v)\|_{L^\gamma(\Omega, X_T^{0,\frac{1}{2}-\delta})} \sim \|\langle\tau\rangle^{\frac{1}{2}-\delta} \widehat{g}(n,\tau)\|_{L^\gamma(\Omega, L^2_{n,\tau})}$$

$$\lesssim \left\| \left(\sum_{\substack{n\in\mathbb{Z}\\ k\in\mathbb{Z}\setminus\{0\}}} \int_\infty^\infty \langle\tau\rangle^{1-2\delta} \left| \int_{t'}^T e^{-it\tau} dt \right|^2 \int_0^T |\phi_k \widehat{v}(n-k,t')|^2 dt'\, d\tau \right)^{\frac{1}{2}} \right\|_{L^\gamma(\Omega)}$$

$$\lesssim (T+T^2)^{\frac{1}{2}} \left(\int_{-\infty}^\infty \langle\tau\rangle^{-1-2\delta} d\tau \right)^{\frac{1}{2}} \|\phi\|_{\mathcal{L}^2(L^2)} \|v\|_{L^\gamma(\Omega, X^{0,0,T})}$$

$$\lesssim (T+T^2)^{\frac{1}{2}} \|\phi\|_{\mathcal{L}^2(L^2)} \|v\|_{L^\gamma(\Omega, X_T^{0,0})}$$

□

Remark 5.9. Lemma 5.8 holds for the higher regularity $s > 0$ as well.

We now present some estimates on the mean $M(m, v, \alpha_0)$.

Lemma 5.10. There exists $\theta > 0$ such that the following estimates hold:

$$\|M(\mu, v, \alpha_0)\|_{L^\gamma(\Omega; L^\infty([0,T]))}$$

$$\lesssim \|\phi\|_{\mathrm{HSt}(L^2_x)} \Big(T^\theta \alpha_0 + T^\theta \|\mu\|_{L^2(\Omega; L^\infty([0,T]))} + \|v\|_{L^2(\Omega; X_T^{0,0})} \Big)$$

$$\|M(\mu_2, v_2, \alpha_0) - M(m_1, v_1, \alpha_0)\|_{L^\gamma(\Omega; L^\infty([0,T]))}$$

$$\lesssim \|\phi\|_{\mathrm{HSt}(L^2_x)} \Big(T^\theta \|\mu_2 - \mu_1\|_{L^2(\Omega; L^\infty([0,T]))} + \|v_2 - v_1\|_{L^2(\Omega; X_T^{0,0})} \Big).$$

Proof. By Doob submartingale inequality and Ito isometry, we have

$$\left\| \int_0^t (\alpha_0 + \mu(t'))\phi_0 d\beta_0(t') \right\|_{L^\gamma(\Omega; L^\infty([0,T]))} \lesssim \left\| \int_0^T (\alpha_0 + \mu(t'))\phi_0 d\beta_0(t') \right\|_{L^\gamma(\Omega)}$$

$$\lesssim \Big\| \int_0^T (\alpha_0 + \mu(t'))\phi_0 d\beta_0(t') \Big\|_{L^2(\Omega)} = |\phi_0| \Big(\int_0^T \mathbb{E}\big[(\alpha_0 + \mu(t'))^2\big] dt' \Big)^{\frac{1}{2}}$$

$$\lesssim T^{\frac{1}{2}} \phi_0 \alpha_0 + |\phi_0| \|\mu\|_{L^2(\Omega; L^2([0,T]))}$$

$$\leq T^{\frac{1}{2}} \phi_0 \alpha_0 + T^{\frac{1}{2}} |\phi_0| \|\mu\|_{L^2(\Omega; L^\infty([0,T]))}.$$

In the same way, the first contribution of M is estimated as follows:

$$\Big\| \sum_{n\neq 0} \int_0^t \overline{\widehat{v}(n,t')} \phi_n e^{in\int_0^{t'} \mu(r)dr} d\beta_n(t') \Big\|_{L^\gamma(\Omega; L^\infty([0,T]))}$$

$$\lesssim \Big\| \sum_{n\neq 0} \int_0^T \overline{\widehat{v}(n,t')} \phi_n e^{in\int_0^{t'} \mu(r)dr} d\beta_n(t') \Big\|_{L^2(\Omega)}$$

$$= \sqrt{2} \Big(\sum_{n\neq 0} \int_0^T \mathbb{E}\big[|\widehat{v}(n,t')|^2\big] |\phi_n|^2 dt' \Big)^{\frac{1}{2}}$$

$$\leq \sqrt{2} \sup_n |\phi_n| \|v\|_{L^2(\Omega; X_T^{0,0})}$$

The difference estimate follows from a similar computation. □

Finally, we turn to the nonlinearity $\mathcal{N}_1$ where one of the entries is a stochastic convolution. We first consider the following set; given $n_1 \in \mathbb{Z}$, define

$$\Omega(n_1) := \{\eta \in \mathbb{R} : \eta = -3nn_1n_2 + o(\langle nn_1n_2\rangle^{\frac{1}{100}}) \text{ for some } n \in \mathbb{Z} \text{ with } n = n_1 + n_2\}. \tag{5.58}$$

We will need the following lemma from [26, (7.50) and Lemma 7.4]:

Lemma 5.11. For any $n_1 \in \mathbb{Z} \setminus \{0\}$, we have

$$\int \langle \tau_1 - n_1^3 \rangle^{-1} \mathbb{1}_{\Omega(n_1)}(\tau_1 - n_1^3) d\tau_1 \lesssim 1. \tag{5.59}$$

Lemma 5.12. Let $T > 0$ and let u, v be space-time functions with spatial mean 0. Then

$$\|\mathcal{N}_1(u,v)\|_{X_T^{0,\frac{1}{2}+\delta}} \lesssim \Big(\|u\|_{X_T^{0,\frac{1}{2}-\delta}} + \|\mathbb{1}_{\Omega(n_1)}(\tau_1 - n_1^3) k_1^{\frac{1}{2}} \widehat{u}\|_{L^2_{n_1,\tau_1}} \Big) \|v\|_{X_T^{0,\frac{1}{2}-\delta}}. \tag{5.60}$$

Proof. We are required to control the moments of

$$\sum_{\substack{n,n_1\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2}}\int_{\tau=\tau_1+\tau_2}\frac{\langle n\rangle}{k_0^{\frac{1}{2}-\delta}}\,d(n,\tau)\widehat{u}(n_1,\tau_1)\widehat{v}(n_2,\tau_2)d\tau_1 d\tau. \tag{5.61}$$

We split into two cases:

- **Case (i)**: $\max\{k_0,k_2\}\gtrsim\langle nn_1n_2\rangle^{\frac{1}{100}}$. Rewriting (5.61) gives

$$\sum_{\substack{n,n_1\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2}}\int_{\tau=\tau_1+\tau_2}\frac{\langle n\rangle}{(k_0k_2)^{200\delta}k_1^{\frac{1}{2}-\delta}}\frac{d(n,\tau)}{k_0^{\frac{1}{2}-201\delta}}\left[k_1^{\frac{1}{2}-\delta}\widehat{w}(n_1,\tau_1)\right]\left[k_2^{200\delta}\widehat{\widetilde{v}}(n_2,\tau_2)\right]d\tau_1 d\tau.$$

In this region,

$$\frac{\langle n\rangle}{(k_0k_2)^{200\delta}k_1^{\frac{1}{2}-\delta}}\lesssim\frac{\langle n_1n_2\rangle^{\frac{1}{2}+\delta}\langle n\rangle^{\frac{1}{2}-\delta}}{\langle nn_1n_2\rangle^{2\delta}\langle nn_1n_2\rangle^{\frac{1}{2}-\delta}}\lesssim 1.$$

By an $L^4_{t,x}L^2_{t,x}L^4_{t,x}$-Hölder inequality and Lemma 5.3, there exists $\theta>0$ such that

$$(5.61)\lesssim\|u\|_{X_T^{0,\frac{1}{2}-\delta}}\|v\|_{X_T^{0,\frac{1}{3}+200\delta}}\lesssim T^\theta\|u\|_{X_T^{0,\frac{1}{2}-\delta}}\|v\|_{X_T^{0,\frac{1}{2}-\delta}}$$

- **Case (ii)**: $\max\{k_0,k_2\}\ll\langle nn_1n_2\rangle^{\frac{1}{100}}$. Then $\tau_1-n_1^3\in\Omega(n_1)$. By using

$$\frac{\langle n\rangle}{k_1^{\frac{1}{2}}}\lesssim\frac{\langle n\rangle^{\frac{1}{2}}\langle n_1n_2\rangle^{\frac{1}{2}}}{\langle nn_1n_2\rangle^{\frac{1}{2}}}\lesssim 1,$$

and an $L^4_{t,x}L^2_{t,x}L^4_{t,x}$-Hölder inequality and Lemma 5.3, we have

$$(5.61)\lesssim\sum_{\substack{n,n_1\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2}}\int_{\tau=\tau_1+\tau_2}\frac{d(n,\tau)}{k_0^{\frac{1}{2}-\delta}}\left[\mathbb{1}_{\Omega(n_1)}(\tau_1-n_1^3)k_1^{\frac{1}{2}}\widehat{u}(n_1,\tau_1)\right]\widehat{v}(n_2,\tau_2)d\tau_1 d\tau. \tag{5.62}$$

$$\lesssim\|\mathbb{1}_{\Omega(n_1)}(\tau_1-n_1^3)k_1^{\frac{1}{2}}\widehat{u}\|_{L^2_{n_1,\tau_1}}\|v\|_{X_T^{0,\frac{1}{2}-\delta}} \tag{5.63}$$

This concludes the proof. □

Lemma 5.13. Let $T>0$ and let u,v be space-time functions with spatial mean 0.

There exist constants $\theta, C(R) > 0$ such that

$$\begin{aligned} &\big\|\mathcal{N}_1\big(\mathbb{1}_{[0,T]}\mathcal{T}_R\big(\Phi_1(\mathcal{T}_R u)\big), \mathcal{T}_R v\big)\big\|_{L^\gamma(\Omega;X_T^{0,\frac{1}{2}-\delta})} \\ &\qquad \le T^\theta C(R)\min\Big\{\|\phi\|_{\mathcal{L}^2(L^2)}\|u\|_{L^\gamma(\Omega;X_T^{0,0})}, \|v\|_{L^\gamma(\Omega;X^{0,\frac{1}{2}-\delta,T})}\Big\} \end{aligned} \tag{5.64}$$

$$\begin{aligned} &\big\|\mathcal{N}_1\big(\mathbb{1}_{[0,T]}\mathcal{T}_R\big(\Phi_2(\mu,\alpha_0)\big), \mathcal{T}_R v\big)\big\|_{L^\gamma(\Omega;X_T^{0,\frac{1}{2}-\delta})} \\ &\qquad \le T^\theta C(R)\min\Big\{\|\phi\|_{\mathcal{L}^2(L^2)}\Big(\|\mu\|_{L^\gamma(\Omega;L^2([0,T]))} + |\alpha_0|\Big), \|v\|_{L^\gamma(\Omega;X^{0,\frac{1}{2}-\delta,T})}\Big\} \end{aligned} \tag{5.65}$$

Proof. We only prove (5.64) since (5.65) is similar. We apply Lemma 5.58 to see that the $X_T^{0,\frac{1}{2}-\delta}$-norm is bounded by

$$\Big(\|\mathcal{T}_R\Phi_1(u)\|_{X_T^{0,\frac{1}{2}-\delta}} + \|\mathbb{1}_{\Omega(n_1)}(\tau_1 - n_1^3)k_1^{\frac{1}{2}}\mathcal{F}_{t,x}(\mathcal{T}_R\Phi_1(\mathcal{T}_R u))\|_{L^2_{n_1,\tau_1}}\Big)\|\mathcal{T}_R v\|_{X_T^{0,\frac{1}{2}-\delta}}$$

We now take $L^\gamma(\Omega)$-norm of the above expression and treat the two terms separately.

For the first term, (5.64) is obtained by alternately using the first bound in (5.40) to control one factor by $C(R)$ and using the second bound in (5.40) to control the other factor, and additionally using (5.54) to bound $\|\Phi_1(u)\|_{X_T^{0,\frac{1}{2}-\delta}}$.

For the second term, we use Hölder Inequality to get

$$\|\mathbb{1}_{\Omega(n_1)}(\tau_1 - n_1^3)k_1^{\frac{1}{2}}\mathcal{F}_{t,x}(\mathcal{T}_R\Phi_1(\mathcal{T}_R u))\|_{L^{\gamma_1}(\Omega;L^2_{n_1,\tau_1})}\|\mathcal{T}_R v\|_{L^{\gamma_2}(\Omega;X_T^{0,\frac{1}{2}-\delta})}. \tag{5.66}$$

where $\gamma < \gamma_1, \gamma_2 < \infty$ satisfy $\frac{1}{\gamma_1} + \frac{1}{\gamma_2} = \frac{1}{\gamma}$. By Burkholder-Davis-Gundy inequality and Lemma 5.11,

$$\begin{aligned} &\big\|\mathbb{1}_{\Omega(n_1)}(\tau_1 - n_1^3)k_1^{\frac{1}{2}}\mathcal{F}_{t,x}(\mathcal{T}_R\Phi_1(\mathcal{T}_R u))\big\|_{L^{\gamma_1}(\Omega;L^2_{n_1,\tau_1})} \\ &\qquad \lesssim \Bigg[\mathbb{E}\Bigg(\sum_{k,n_1\in\mathbb{Z}}\int_{-\infty}^{\infty}k_1^{-1}\mathbb{1}_{\Omega(n_1)}(\tau_1 - n_1^3)d\tau_1\int_0^T|\widehat{\mathcal{T}_R u}(n_1-k,t')\phi_k|^2dt'\Bigg)^{\frac{\gamma}{2}}\Bigg]^{\frac{1}{\gamma_1}} \\ &\qquad \lesssim \|\phi\|_{\mathcal{L}^2(L^2_x)}\|\mathcal{T}_R u\|_{L^{\gamma_1}(\Omega;X_T^{0,0})}. \end{aligned} \tag{5.67}$$

For the first bound in (5.64), we set $\gamma_1 = \gamma$ and $\gamma_2 = \infty$, and simply bound

$\|\mathcal{T}_R v\|_{X_T^{0,\frac{1}{2}-\delta}}$ by $C(R)$ and discard the cutoff $\eta_T(u)$ in (5.67). To get the second bound, we set $\gamma_1 = \frac{\gamma(\gamma+\varepsilon)}{\varepsilon}$ and $\gamma_2 = \gamma+\varepsilon$. Then (5.66) is bounded by

$$\|\mathcal{T}_R u\|_{L^{\frac{\gamma(\gamma+\varepsilon)}{\varepsilon}}(\Omega;X^{0,\frac{1}{2}-\delta})}\|\mathcal{T}_R v\|_{L^{\gamma+\varepsilon}(\Omega;X^{0,\frac{1}{2}-\delta})} \leq C_1(R)^{1+\frac{\varepsilon}{\gamma+\varepsilon}} C_2^{\frac{\gamma}{\gamma+\varepsilon}} \|v\|^{\frac{\gamma}{\gamma+\varepsilon}}_{L^{\gamma}(\Omega;X^{0,\frac{1}{2}-\delta})},$$

where $C_1(R)$ and C_2 are as in Lemma 5.5. Letting $\varepsilon \to 0$ concludes the proof. □

We end this section with some probabilistic a priori bounds on solutions. Firstly we have the following a priori bound on the mass of solutions of SKdV.

Lemma 5.14. Let $T > 0$. Suppose that $u \in C([0,\tau]; L^2(\mathbb{T}))$ is a solution of the Stochastic KdV (5.1) with initial data $u_0 \in L^2(\mathbb{T})$, where τ is a stopping time. Then for any $\gamma \geq 2$, there exists a constant $C_1 = C_1(\gamma,\phi,T,u_0)$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq \tau\wedge T} \|u\|^{\gamma}_{L^2(\mathbb{T})}\right] \leq C_1. \tag{5.68}$$

Moreover, if (v,μ) is the solution to the SKdV mean-system (5.19)–(5.20) with subscribed mean α_0, then there exists another constant $C_2 = C_2(\alpha_0,\gamma,\phi,T,u_0)$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq \tau\wedge T} |\mu(t)|^{\gamma}\right] + \mathbb{E}\left[\sup_{0\leq t\leq \tau\wedge T} \|v\|^{\gamma}_{L^2(\mathbb{T})}\right] \leq C_2. \tag{5.69}$$

Proof. The proof of (5.68) follows the exact same argument as in [37][Lemma 3.1] by apply Itô's Formula on (5.1) and bounding each term. Now, let (v,μ) be the solution to the SKdV mean-system (5.19)–(5.20) with subscribed mean α_0. Then

$$u(x,t) = v\left(x-\alpha_0 t - \int_0^t \mu(r)dr, t\right) + \alpha_0 + \mu(t).$$

Now, we have

$$\begin{aligned}\|v\|_{L^2_x} &\leq \left(\int_{\mathbb{T}} \left|v\left(x-\mu_0 t - \int_0^t \mu(r)dr, t\right) + \alpha_0 + \mu(t)\right|^2 dx\right)^{\frac{1}{2}} + \alpha_0 + \mu(t) \\ &= \|u\|_{L^2_x} + \alpha_0 + \mu(t).\end{aligned} \tag{5.70}$$

By Lemma 5.10, for any $0 \leq t \leq \tau \wedge T$, we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq \tau\wedge T} |\mu(t)|^\gamma\Big] \lesssim \|\phi\|_{\mathrm{HSt}(L^2_x)}\bigg(T^\theta \mathbb{E}\Big[\sup_{0\leq t\leq \tau\wedge T} |\mu(t)|^2\Big] + \mathbb{E}\Big[\|v\|^2_{X^{0,0}_{\tau\wedge T}}\Big]\bigg).$$

Now, $\|v\|_{X^{0,0}_{\tau\wedge T}} \leq T \sup_{0\leq t\leq \tau\wedge T} \|v\|_{L^2_x}$. Since $\gamma \geq 2$, by Young's inequality, we have

$$\mathbb{E}\Big[\sup_{0\leq t\leq S} |\mu(t)|^\gamma\Big] \leq C(\phi, T, \gamma) + \frac{1}{2}\mathbb{E}\Big[\sup_{0\leq t\leq \tau\wedge T} |\mu(t)|^\gamma\Big] + \frac{1}{4}\mathbb{E}\Big[\sup_{0\leq t\leq \tau\wedge T} \|v(t)\|^\gamma_{L^2_x}\Big]. \tag{5.71}$$

Then (5.69) follows from (5.70) and (5.71). □

Lemma 5.14 implies the following estimate, which tells us that the $X^{0,\frac{1}{2}-\delta}$-norm of solutions of the R-truncated system does not grow in R.

Lemma 5.15. Let $T, R > 0$. Let $(v_R, \mu_R) \in C([0,T]; L^2(\mathbb{T}))$ be the solution to the R-truncated system (5.19)–(5.20) with data v_0 and subscribed mean α_0. Then there exists $C = C(\alpha_0, \|v_0\|_{L^2_x}, \phi, T)$, independent of R, such that

$$\mathbb{E}\Big[\|v_R\|_{X^{0,\frac{1}{2}-\delta}_{T\wedge\tau_R}}\Big] \leq C.$$

Proof. Let $S \in [0, T\wedge\tau_R]$ be a stopping time. By (5.37) and using the estimates established in Sections 5.3 and 5.4 above, there are constants $C, \theta > 0$ such that

$$\begin{aligned}\|v_R\|_{X^{0,\frac{1}{2}-\delta}_S} &\leq C\|v_0\|_{L^2_x} + CT^\theta \|v_R\|^2_{X^{0,\frac{1}{2}-\delta}_S} + CT^\theta\|v_R\|^3_{X^{0,\frac{1}{2}-\delta}_S} \\ &\quad + C\|v_R\|_{X^{0,\frac{1}{2}-\delta}_S} \|\mathbb{1}_{\Omega(n)}(\tau - n^3)k^{\frac{1}{2}}\widehat{\Phi}\|_{L^2_{n,\tau}} + \|\Phi\|_{X^{0,\frac{1}{2}-\delta}_{T\wedge\tau_R}},\end{aligned}$$

where $\Omega(n)$ is as defined in 5.58, and $\Phi = \Phi_1(v_R) + \Phi_2(\mu_R, \alpha_0)$. By Young's inequality, and assuming that $\|v_R\|_{X^{0,\frac{1}{2}-\delta}_S} \geq 1$ (otherwise we simply bound it by 1), we have

$$\begin{aligned}\|v_R\|_{X^{0,\frac{1}{2}-\delta}_S} &\leq C_1\|v_R\|_{C([0,T];L^2_x)} + C_1 S^\theta \|v_R\|^3_{X^{0,\frac{1}{2}-\delta}_S} \\ &\quad + C_2(T)\|\mathbb{1}_{\Omega(n)}(\tau - n^3)k_0^{\frac{1}{2}}\widehat{\Phi}\|^2_{L^2_{n,\tau}} + \|\Phi\|_{X^{0,\frac{1}{2}-\delta}_{T\wedge\tau_R}}.\end{aligned} \tag{5.72}$$

Let

$$K = C_1 \|v_R\|_{C([0,T];L^2_x)} + C_2(T) \|\mathbb{1}_{\Omega(n)}(\tau - n^3) k_0^{\frac{1}{2}} \widehat{\Phi}\|^2_{L^2_{n,\tau}} + \|\Phi\|_{X^{0,\frac{1}{2}-\delta}_{T\wedge\tau_R}}, \tag{5.73}$$

then (5.72) is equivalent to

$$p_S\left(\|v_R\|_{X^{0,\frac{1}{2}-\delta}_S}\right) \geq 0 \tag{5.74}$$

where p_S denotes the polynomial $p_S(x) = C_1 S^\theta x^3 - x + K$. We shall show via a continuity argument that

$$\|v_R\|_{X^{0,\frac{1}{2}-\delta}_S} \lesssim K \tag{5.75}$$

if we choose $S \sim K^{-\frac{2}{\theta}}$. Indeed, we first note that p_S has a unique positive turning point at $x_+ := (3C_1 S^\theta)^{-\frac{1}{2}}$, and $p_S(x_+) = -cS^{-\frac{\theta}{2}} + K$ for some $c > 0$. Choosing $S = \left(\frac{2K}{c}\right)^{-\frac{2}{\theta}}$, we have that $P_S(x_+) < 0$. Let $0 < x_{r1} < x_+ < x_{r2}$ be the two positive roots of p_S. Then (5.72) can only hold if $0 \leq \|v_R\|_{X^{0,\frac{1}{2}-\delta}_S} \leq x_{r1}$ or $\|v_R\|_{X^{0,\frac{1}{2}-\delta}_S} \geq x_{r2}$. But the latter is impossible since the function $g(S) := \|v_R\|_{X^{0,\frac{1}{2}-\delta}_S}$ is continuous and $g(0) = 0$. Hence we must have $\|v_R\|_{X^{0,\frac{1}{2}-\delta}_S} \leq x_{r1} < x_+ \sim S^{-\frac{\theta}{2}}$, which implies (5.75).

Iterating this argument on $I_j := [jS, (j+1)S]$ for $0 \leq j \leq \left\lceil \frac{T\wedge\tau_R}{S} \right\rceil$, we get that

$$\|v_R\|_{X^{0,\frac{1}{2}-\delta}_{I_j}} \lesssim K,$$

whence

$$\|v_R\|_{X^{0,\frac{1}{2}-\delta}_{T\wedge\tau_R}} \lesssim \frac{T\wedge\tau_R}{S} K \lesssim T K^{1+\frac{2}{\theta}}. \tag{5.76}$$

Since we are on the interval $[0, T\wedge\tau_R]$ In view of (5.73), the first and third terms K have moments bounded by constants (independent of R) by Lemma 5.14 and 5.8. To bound the moments of the second term, we apply the same argument we used in (5.67) to get

$$\mathbb{E}\left[\|\mathbb{1}_{\Omega(n)}(\tau - n^3) k_0^{\frac{1}{2}} \widehat{\Phi}\|^\gamma_{L^2_{n,\tau}}\right]^{\frac{1}{\gamma}} \lesssim \|\phi\|_{\mathrm{HSt}(L^2_x)} \left(\|v_R\|_{L^\gamma(\Omega;X^{0,0}_T)} + \|\mu_R\|_{L^\gamma(\Omega;L^\infty([0,T]))}\right)$$

The right-hand-side is bounded again by Lemma 5.14. Hence the lemma is proved by taking γ-moment on (5.76). □

5.5 Proof of Theorem 1

In this section, we give a proof of Theorem 1. We fix $u_0 \in L^2(\mathbb{T})$ as the initial data to the SKdV problem (5.1) and let $v_0 = u_0 - \alpha_0 := u_0 - \int_{\mathbb{T}} u_0(x)dx$.

5.5.1 Existence and uniqueness

Our first step is to prove the local well-posedness of the iterated SKdV system (5.44)–(5.45) with initial data v_0 and subscribed mean α_0. In what follows, we define the space

$$E_T := \left\{ (v, \mu) \in L^2(\Omega; X^{0,\frac{1}{2}-\delta,T}) \times L^2(\Omega; L^\infty[0,T]) : \int_{\mathbb{T}} v = 0 \right\} \tag{5.77}$$

We first show that the second iterated SKdV-mean system (5.37)–(5.38) is locally well-posed.

Proposition 5.16. Let $v_0 \in L^2(\mathbb{T})$, $\alpha_0 \in \mathbb{R}$ and $R > 0$. Then there exists a unique global-in-time solution $(v_R, \mu_R) \in L^2(\Omega; X^{0,\frac{1}{2}-\delta,\tau^*}) \times L^2(\Omega; L^\infty([0,\tau^*]))$ to the R-truncated system (5.44)–(5.45) almost surely.

Moreover, let τ_R be as defined in (5.46). Then

$$\tau^* := \lim_{R\to\infty} \tau_R$$

is either positive or equals to ∞, and that

$$(v, \mu)(t) := (v_R, \mu_R)(t) \quad \text{for} \quad t \in [0, \tau_R]$$

is the unique solution to the system (5.19)–(5.20) over the time interval $[0, \tau^*)$.

Proof. Let

$$\Gamma_R(v,\mu) := \big(\Gamma_{1,R}(v,\mu), \Gamma_2(v,\mu)\big) := \big(\text{RHS}(5.44), \text{RHS}(5.45)\big)$$

Let $T > 0$. We shall prove that Γ_R is a contraction on E_T. Putting together the estimates in Sections 5.3 and 5.4 as well as using Lemmata 5.2 and 5.5, there exist constants $C_1(R), C_2(T, \phi, \alpha_0) > 0$ such that

$$\big\|\Gamma_{1,R}(v,\mu)\big\|_{L^2(\Omega;X_T^{0,\frac{1}{2}-\delta})} \leq C_1(T,R)\Big(\|v_0\|_{L^2_x} + \|\phi\|_{\mathrm{HSt}(L^2_x)}\|v\|_{L^2(\Omega;X_T^{0,\frac{1}{2}-\delta})} + \alpha_0\|\mu\|_{L^2(\Omega;L^\infty([0,T]))} + \alpha_0\Big)$$

$$\big\|\Gamma_{2,R}(v,\mu)\big\|_{L^2(\Omega;L^2([0,T]))} \leq C_2(T,\phi)\Big(\alpha_0 + \|\mu\|_{L^2(\Omega;L^\infty([0,T]))} + \|v\|_{L^2(\Omega;X_T^{0,\frac{1}{2}-\delta})}\Big).$$

Hence Γ maps E_T to itself. Similarly, we have

$$\big\|\Gamma_{1,R}(v_2,\mu_2) - \Gamma_{1,R}(v_1,\mu_1)\big\|_{L^2(\Omega;X_T^{0,\frac{1}{2}-\delta})} + \big\|\Gamma_{2,R}(v_2,\mu_2) - \Gamma_{2,R}(v_1,\mu_1)\big\|_{L^2(\Omega;L^\infty([0,T]))}$$
$$\leq T^\theta C(R,\phi)\bigg(\|v_2 - v_1\|_{L^2(\Omega;L^\infty([0,T]))} + \|\mu_2 - \mu_1\|_{L^2(\Omega;L^\infty([0,T]))}\bigg).$$

Hence by choosing $T = T(R,\phi)$ sufficiently small, we have

$$\|\Gamma_R(v_2,\mu_2) - \Gamma_R(v_1,\mu_1)\|_{E_T} \leq \frac{1}{2}\|(v_2 - v_1, \mu_2 - \mu_1)\|_{E_T}.$$

Hence $\Gamma_R : E_T \to E_T$ is a contraction, and so there exists a unique $(v_R, \mu_R) \in E_T$ satisfying the R-truncated system (5.44)–(5.45). Note that the time of existence of $((v_R, \mu_R))$ does not depend on $\|v_0\|_{L^2_x}$, thus we can iterate the argument to get the global-in-time solution (v_R, μ_R).

Let $\tau_R := \tau_R(v_R, \mu_R, \alpha_0)$ be defined in (5.46). Then (v_R, μ_R) is a solution of (5.37)–(5.38) on $[0, \tau_R]$. If $R' > R$, then $(v_R, \mu_R)(t) = (v_{R'}, \mu_{R'})(t)$ whenever $t \in [0, \tau_R]$. Consequently, τ_R is increasing in R, and $\tau^* := \lim_{R\to\infty} \tau_R$ is a well-defined stopping time that is either positive or infinite almost surely. By defining $(v,\mu)(t) = (v_R, \mu_R)(t)$ for each $t \in [0, \tau_R]$, we see that (v,μ) is a solution of the system (5.19)–(5.20) on $[0, \tau^*)$ almost surely. □

We now construct a global solution (v,μ) to the original SKdV-mean system, that is, given $T > 0$, we construct (v,μ) satisfying (5.37)–(5.38) on $[0,T]$. Let $N \in 2^{\mathbb{N}}$ be a dyadic number. Let $u_0^N := \mathbb{P}_N u_0$. Then $u_0^N \in H^2$ and by Theorem 1.1 in Tsutsumi

[84], there exists a global solution $u^N \in L^2\big(\Omega; L^\infty_{2T} H^2_x\big) \cap L^6\big(\Omega; C_{2T} L^2_x\big)$ to SKdV (5.1) on $[0, 2T]$. By letting α_0 be the mean of u_0^N (noting that $\alpha_0 = \widehat{u}_0(0) = \widehat{u_0^N}(0)$ is invariant in N), we can associate u^N with the solution (v^N, μ^N) to the SKdV-mean system (5.17), so that

$$u^N(x,t) = v^N\bigg(x - \alpha_0 t - \int_0^t \mu^N(r)dr, t\bigg) + \alpha_0 + \mu^N(t).$$

Given $R > 0$, (v^N, μ^N) also satisfies the R-truncated system (5.42)–(5.43) on the time interval $[0, \tau_{R,N}]$, where $\tau_{R,N} = \tau_R(v^N, \mu^N, \alpha_0)$ is as defined in (5.46). By Markov's inequality,

$$\begin{aligned} &\mathbb{P}\big(\tau_{R,N} < 2T\big) \\ &= \mathbb{P}\bigg(\|v^N\|_{X^{0,\frac12-\delta}_{2T\wedge\tau_{R,N}}} \vee \|\Phi_1(\mathcal{T}_R v^N)\|_{X^{0,\frac12-\delta}_{2T\wedge\tau_{R,N}}} \vee \|\Phi_2(\alpha_0, \mu^N)\|_{X^{0,\frac12-\delta}_{2T\wedge\tau_{R,N}}} \geq R \bigg) \\ &\lesssim \frac{1}{R}\mathbb{E}\bigg[\|v^N\|_{X^{0,\frac12-\delta}_{2T\wedge\tau_{R,N}}} + \|\Phi_1(\mathcal{T}_R v^N)\|_{X^{0,\frac12-\delta}_{2T\wedge\tau_{R,N}}} + \|\Phi_2(\alpha_0, \mu^N)\|_{X^{0,\frac12-\delta}_{2T\wedge\tau_{R,N}}} \bigg]. \end{aligned} \tag{5.78}$$

By Lemmata 5.8, 5.10 and 5.15, the expectation above is bounded by a constant not dependent[1] on R and N. By choosing N' so that

$$\tau_{R,N'} \leq 2 \inf_N \tau_{R,N},$$

we can find $R > 0$ sufficiently large so that for any $\varepsilon > 0$, we have

$$\mathbb{P}\Big(\inf_N \tau_{R,N} < T\Big) \leq \mathbb{P}\big(\tau_{R,N'} < 2T\big) < \varepsilon,$$

and hence

$$\mathbb{P}\Big(\inf_N \tau_{R,N} < T\Big) = 0.$$

We fix such R, so that almost surely, (v^N, μ^N) is a solution to the R-system (5.44)–(5.45) on $[0, T]$. In particular, we can view any appearance of $\mathcal{T}_R$ in (5.44) as the

[1]The constant does depend on the initial data in the sense that it is an increasing function of the L^2_x-norm of the initial data. However, we have $\|u_0^N\|_{L^2_x} \leq \|u_0\|_{L^2_x}$, and so we can modify the constant to not depend on N.

identity operator. This allows us to apply Lemma 5.5, and we shall do so implicitly below. We first show that $\{v^N, \mu^N)\}_N$ is Cauchy in $L^2(\Omega; X^{0,\frac{1}{2}-\delta})$. Indeed for $M, N \in 2^{\mathbb{N}}$ and any $0 < \widetilde{T} \leq T$, we may proceed as in the proof of Proposition 5.16 to get

$$\|(v^N, \mu^N) - (v^M, \mu^M)\|_{E_{\widetilde{T}}} \leq C_1 \|v_0^N - v_0^M\|_{L^2_x} + C_2(R, \phi)\widetilde{T}^\theta \|(v^N, \mu^N) - (v^M, \mu^M)\|_{E_{\widetilde{T}}}.$$

Then by choosing $\widetilde{T} \leq (2C_2)^{-\frac{1}{\theta}}$ and rearranging, we have

$$\|(v^N, \mu^N) - (v^M, \mu^M)\|_{E_{\widetilde{T}}} \leq 2C_1 \|v_0^N - v_0^M\|_{L^2_x}.$$

By definition of $\mathbb{P}_{\leq N}$, the sequence $\{v_0^N\}_{N \in 2^{\mathbb{N}}}$ is Cauchy on $L^2(\mathbb{T})$. Hence $\big\{(v^N, \mu^N)\big\}_{N \in 2^{\mathbb{N}}}$ is Cauchy in $E_{\widetilde{T}}$. The time $\widetilde{T}$ does not depend on the initial data v_0, hence by repeating this argument on $[\widetilde{T}, 2\widetilde{T}]$, we see that $\big\{(v^N, \mu^N)\big\}_{N \in 2^{\mathbb{N}}}$ is Cauchy in $E_{[\widetilde{T}, 2\widetilde{T}]}$. By iterating $\lceil T/\widetilde{T} \rceil$ times, we have that $\big\{(v^N, \mu^N)\big\}_{N \in 2^{\mathbb{N}}}$ is Cauchy in E_T. Let $(v, \mu) \in E_T$ denote the limit. We now show that (v, μ) solves the SKdV-mean system (5.17).

In view of (5.44)–(5.45), by Lemmata 5.2, 5.8 and 5.6, we have, as $N \to \infty$,

$$U(\cdot)v_0^N \to U(\cdot)v_0,$$

$$\Psi_1(v^N) + \Psi_2(\mu^N, \alpha_0) \to \Psi_1(v) + \Psi_2(\mu, \alpha_0),$$

$$\mathcal{N}_0(v^N, v^N) \to \mathcal{N}_0(v, v)$$

in $L^2(\Omega; X_T^{0,\frac{1}{2}-\delta})$, as well as $\mu^N \to \mu$ in $L^2(\Omega; L^\infty([0,T]))$. On the other hand, we observe that (by Lemma 5.7) $\mathcal{N}_1(v^N, v^N)$ and $\mathcal{N}_2(v^N, v^N)$ are in fact Cauchy in the stronger space $L^2(\Omega; X_T^{0,\frac{1}{2}+\delta})$ and hence respectively admit some limits $\mathcal{N}_1^*$ and $\mathcal{N}_2^*$ in said space. We can then write

$$v = U(\cdot)v_0 + \mathcal{N}_0(v, v) + \mathcal{N}_1^* + \mathcal{N}_2^* + \Psi_1(v^*) + \Psi_2(\mu^*, \alpha_0).$$

We are thus required to show that $\mathcal{N}_1^* = \mathcal{N}_1(v^N, v^N)$ (the argument for $\mathcal{N}_2^* = \mathcal{N}_2(v, v)$ is symmetric), that is, we show that $\mathcal{N}_1(v^N, v^N) \to \mathcal{N}_1(v, v)$. By (5.41), it suffices to show that $\mathcal{N}_1(v^N, v^N) \to \mathcal{N}_1(v, v)$. To this end, we simply substitute the equations of v^N and v into the first arguments, so that

$$\mathcal{N}_1(v^N, v^N) = \mathcal{N}_1\big(\mathcal{N}(v^N, v^N) + \Psi_1(v^N) + \Psi_2(\mu^N, \alpha_0), v^N\big),$$

$$\mathcal{N}_1(v, v) = \mathcal{N}_1\big(\mathcal{N}_0(v, v) + \mathcal{N}_1^* + \mathcal{N}_2^* + \Psi_1(v) + \Psi_2(\mu, \alpha_0), v\big).$$

By Lemmata 5.13 and 5.8, we have

$$\mathcal{N}_1\big(\Psi_1(v^N) + \Psi_2(\mu^N, \alpha_0), v^N\big) \to \mathcal{N}_1\big(\Psi_1(v) + \Psi_2(\mu, \alpha_0), v\big).$$

Moving onto the nonlinear piece, we write

$$\mathcal{N}_1\big(\mathcal{N}(v^N, v^N)\big) = \sum_{j=0}^{2} \mathcal{N}_1\big(\mathcal{N}_j(v^N, v^N)\big)$$

and treat each piece separately. By (5.47) and using a telescoping sum, for $j \in \{1, 2\}$, we have

$$\begin{aligned}&\big\|\mathcal{N}_1\big(\mathcal{N}_j(v^N, v^N), v^N\big) - \mathcal{N}_1\big(\mathcal{N}_j^*, v\big)\big\|_{X_T^{0,\frac{1}{2}+\delta}} \\ &\qquad \lesssim \big\|\mathcal{N}_j(v^N, v^N) - \mathcal{N}_j^*\big\|_{X_T^{0,\frac{1}{2}+\delta}} \big\|v^N\big\|_{X_T^{0,\frac{1}{2}-\delta}} + \big\|\mathcal{N}_j^*\big\|_{X_T^{0,\frac{1}{2}+\delta}} \big\|v^N - v\big\|_{X_T^{0,\frac{1}{2}-\delta}}.\end{aligned}$$

Since $\tau_R > T$ almost surely, $\|v^N\|_{X_T^{0,\frac{1}{2}-\delta}} \le C(R)$ for some constant $C(R) > 0$. The same applies to $\|\mathcal{N}_j(v^N, v^N)\|_{X_T^{0,\frac{1}{2}+\delta}}$ after substituting the equation for v^N and apply the same arguments as seen above. By possibly passing to a subsequence, we have that $\mathcal{N}_j(v^N, v^N)$ converges to $\mathcal{N}_j^*$ in $X_T^{0,\frac{1}{2}+\delta}$ almost surely. It follows that $\|\mathcal{N}_j^*\|_{X_T^{0,\frac{1}{2}+\delta}} \le C(R)$ almost surely. Therefore,

$$\begin{aligned}&\big\|\mathcal{N}_1\big(\mathcal{N}_j(v^N, v^N), v^N\big) - \mathcal{N}_1\big(\mathcal{N}_j^*, v\big)\big\|_{L^2(\Omega; X_T^{0,\frac{1}{2}+\delta})} \\ &\qquad \lesssim C(R)\bigg(\big\|\mathcal{N}_j(v^N, v^N) - \mathcal{N}_j^*\big\|_{L^2(\Omega; X_T^{0,\frac{1}{2}+\delta})} + \big\|v^N - v\big\|_{L^2(\Omega; X_T^{0,\frac{1}{2}-\delta})}\bigg) \\ &\qquad \to 0\end{aligned}$$

as $N \to \infty$. Finally, it remains to show that

$$\mathcal{N}_1\big(\mathcal{N}_0(v^N, v^N), v^N\big) \to N_1\big(\mathcal{N}_0(v, v), v\big)$$

in $L^2(\Omega; X_T^{0,\frac{1}{2}+\delta})$. To see this, we write, as in Section 5.2,

$$\mathcal{N}_1\big(\mathcal{N}_0(v^N, v^N), v^N\big) = \mathcal{N}_1\big(\mathcal{I}_0(v^N, v^N), v^N\big)$$

$$N_1\big(\mathcal{N}_0(v, v), v\big) = \mathcal{N}_1\big(\mathcal{I}_0(v, v), v\big).$$

The claimed convergence then follows by applying the estimate (5.48). This concludes the proof that $\mathcal{N}_1\big(v^N, v^N\big) \to \mathcal{N}_1\big(v, v\big)$ in $L^2(\Omega; X^{0,\frac{1}{2}+\delta})$.

In summary, we have proved that $(v, \mu) \in L^2(\Omega; X_T^{0,\frac{1}{2}-\delta}) \times L^2(\Omega; L^\infty([0, T]))$ is a solution to the SKdV-mean system (5.17). This solution is also unique in said space. Indeed, any solution to (5.17) is also a solution to the iterated system (5.37)–(5.38), which is unique by Proposition 5.16 up to time τ^*. But as seen in (5.78), $\tau^* = \infty$ almost surely. Hence uniqueness holds up to the whole interval $[0, T]$. It remains to verify that v lies in $L^2(\Omega; C([0, T]; L^2(\mathbb{T})))$, and we shall do so in the next subsection.

5.5.2 Continuity in time

We constructed the solution (v, μ) to (5.17) in the previous section, and we are now required to show that $v \in L^2(\Omega; C([0, T]; L^2(\mathbb{T})))$. We write $v(t)$ as

$$U(t)v_0 - \frac{1}{2}\sum_{j=0}^{2} \mathcal{N}_j(v, v)(t) + \Phi_1(v)(t) + \Phi_2(\mu, \alpha_0)(t).$$

The linear part $U(t)v_0$ clearly lies in $C([0, T]; L^2_x)$. As seen in the previous section, we have $\mathcal{N}_j(v, v) \in L^2\big(\Omega; X_T^{0,\frac{1}{2}+\delta}\big)$ and hence $\mathcal{N}_j(v, v) \in L^2(\Omega; C([0, T]; L^2(\mathbb{T})))$ by the embedding (5.26). The continuity of the stochastic convolutions $\Phi_1(v)$ and $\Phi_2(\mu, \alpha_0)$ follow in a similar manner as in the case for SNLS (as in Lemma (2.19)).

It remains to verify that $\mathcal{N}_0(v, v) \in L^2(\Omega; C([0, T]; L^2(\mathbb{T})))$. We shall follow [9, 26] and prove that $\mathcal{N}_0(v, v) \in L^2\big(\Omega; \mathcal{F}\ell^2_n L^1_\tau\big)$, where $\mathcal{F}\ell^2_n L^1_\tau$ is the space-time

Fourier-Lebesgue space endowed with the norm

$$\|u\|_{\mathcal{F}\ell^2_n L^\infty_\tau} = \|\widehat{u}(n,\tau)\|_{\ell^2_n L^1_\tau(\mathbb{Z}\times\mathbb{R})}.$$

We have the embedding

$$\mathcal{F}\ell^2_n L^1_\tau \hookrightarrow C([0,T];L^2). \tag{5.79}$$

Indeed, by Plancherel's Theorem and Minkowski's inequality, we have

$$\|u\|_{L^\infty_t L^2_x([0,T]\times\mathbb{T})} \leq \left\| \int_{\mathbb{R}} e^{it\tau}\widehat{u}(n,\tau)d\tau \right\|_{\ell^2_n L^\infty_t} \leq \|u\|_{\mathcal{F}\ell^2_n L^1_\tau}.$$

Moreover, the Fourier transform of an L^1 function is continuous. Hence the embedding (5.79). The content of the next lemma then concludes our proof.

Lemma 5.17. Let u_1, u_2 be mean zero space-time functions. Then

$$\big\|\eta_T \mathcal{N}_0(u_1,u_2)\big\|_{\mathcal{F}\ell^2_n L^1_\tau} \lesssim \|u_1\|_{X^{0,\frac{1}{2}-\delta}}\|u_2\|_{X^{0,\frac{1}{2}-\delta}}.$$

Proof. This proof is similar in flavour to the proof of Lemma 5.13. Appealing to (5.33) and that $|\tau - n^3| \gg |nn_1n_2| > 0$, we have that

$$\big\|\eta_T \mathcal{N}_0(u_1,u_2)\big\|_{\mathcal{F}\ell^2_n L^1_\tau} \lesssim_\eta \left\| \frac{\langle n\rangle}{k_0} \widehat{u_1 u_2}(n,\tau) \right\|_{\ell^2_n L^1_\tau}. \tag{5.80}$$

We split into two cases.

• **Case (i)**: $k_1 \gtrsim \langle nn_1n_2\rangle^{\frac{1}{100}}$. By Cauchy-Schwartz and duality, there exists $d \in L^2_{n,\tau}$ with $\|d\|_{L^2_{n,\tau}} \leq 1$ such that

$$\begin{aligned}\text{RHS}(5.80) &\leq \left\| \big\|\langle \tau - n^3\rangle^{-(\frac{1}{2}+\delta)}\big\|_{L^2_\tau} \left\| \frac{\langle n\rangle}{k_0^{\frac{1}{2}-\delta}} \widehat{u_1 u_2}(n,\tau)\right\|_{L^2_\tau} \right\|_{\ell^2_n} \\ &\lesssim \sum_{\substack{n,n_1\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2}} \int_{\tau_1+\tau_2=\tau} \frac{\langle n\rangle}{k_0^{\frac{1}{2}-\delta}} d(n,\tau)\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)d\tau_1 d\tau.\end{aligned}$$

Now,

$$\frac{\langle n\rangle}{k_0^{\frac{1}{2}-\delta}} \lesssim \frac{(\langle n\rangle\langle n_1\rangle\langle n_2\rangle)^{\frac{1}{2}-\delta}}{k_0^{\frac{1}{2}-\delta}} \cdot \frac{(\langle n\rangle\langle n_1\rangle\langle n_2\rangle)^{\delta}}{k_1^{100\delta}} k_1^{100\delta} \lesssim k_1^{100\delta}.$$

The result then follows by an application of $L^2_{t,x}L^4_{t,x}L^4_{t,x}$-Hölder inequality and the L^4-Strichartiz inequality.

• **Case (ii)**: $k_1 \ll \langle nn_1n_2\rangle^{\frac{1}{100}}$. We consider the set $\Omega(n)$ defined as in (5.58). By Cauchy-Schwartz, Lemma 5.11 and duality, we have

$$\begin{aligned}
\text{RHS(5.80)} &= \left\| \mathbb{1}_{\Omega(n)}(\tau - n^3)\frac{\langle n\rangle}{k_0}\widehat{u_1u_2}\right\|_{\ell^2_n L^1_\tau} \\
&\lesssim \left\| \left(\int_{\mathbb{R}} \mathbb{1}_{\Omega(n)}(\tau - n^3)\langle \tau - n^3\rangle^{-1} d\tau\right)^{\frac{1}{2}} \left\| \frac{\langle n\rangle}{k_0^{\frac{1}{2}}}\widehat{u_1u_2}\right\|_{L^2_\tau}\right\|_{\ell^2_n} \\
&\lesssim \sum_{\substack{n_1\in\mathbb{Z}\setminus\{0\}\\ n=n_1+n_2}} \int_{\tau=\tau_1+\tau_2} \frac{\langle n\rangle}{k_0^{\frac{1}{2}}} d(n,\tau)\widehat{u_1}(n_1,\tau_1)\widehat{u_2}(n_2,\tau_2)d\tau_1 d\tau.
\end{aligned}$$

Since $\frac{\langle n\rangle}{k_0^{\frac{1}{2}}} \lesssim 1$, the result then follows again by an application of $L^2_{t,x}L^4_{t,x}L^4_{t,x}$-Hölder inequality and the L^4-Strichartiz inequality. □

5.6 Time decay of solutions

We briefly prove Theorem 3 in this last section. We can repeat the argument of the proof of Theorem 1.2 in [84] almost verbatim. We only need to verify that in our setting, one can continue to use the exponential martingale inequality to bound the second term on the right-hand side of (3.1) in [84]:

$$\sum_{k=-\infty}^{\infty} \int_0^t \frac{\text{Re }\langle u(t)\phi e_k, u(t)\rangle}{\|u(t)\|^2_{L^2_x(\mathbb{T})}} d\beta_k^{(r)}(t),$$

where $\beta_k^{(r)} = \text{Re } \beta_k$. More specifically, we need to check that the Novikov condition

$$\mathbb{E}\left[\exp\left(\frac{b^2}{2}\int_0^T \sum_{k=-\infty}^{\infty}\left|\frac{\text{Re }\langle u(t)\phi e_k, u(t)\rangle}{\|u(t)\|^2_{L^2_x(\mathbb{T})}}\right|^2 dt\right)\right] < \infty. \tag{5.81}$$

Tsutsumi used Sobolev embedding to bound the quantity inside the exponential by

$$\frac{b^2T^2}{2}\sum_{k=-\infty}^{\infty}\|\phi e_k\|_{L_x^\infty}^2 \lesssim \frac{b^2T^2}{2}\sum_{k=-\infty}^{\infty}\|\phi e_k\|_{H_x^2}^2 = \frac{b^2T^2}{2}\|\phi\|_{\mathcal{L}^2(L^2;H^2)}^2.$$

This is not available to us since ϕ is not necessarily in $\mathcal{L}^2(L^2;H^2)$. However, since we assumed ϕ to be a convolution operator, we have that

$$\mathrm{Re}\,\big\langle u(t)\phi e_k, u(t)\big\rangle \leq |\phi_k|\big|\langle u(t)e_k, u(t)\rangle\big| \leq |\phi_k|\|u(t)\|_{L_x^2(\mathbb{T})}^2.$$

Hence

$$\mathrm{LHS}(5.81) \lesssim \exp\left(\frac{b^2T^2}{2}\|\phi\|_{\mathrm{HSt}(L_x^2)}^2\right) < \infty.$$

The rest of the proof then follows as in [84].

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