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Amplifiers for the Moran Process

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The Moran process, as studied by Lieberman, Hauert and Nowak, is a randomised algorithm modelling the spread of genetic mutations in populations. The algorithm runs on an underlying graph where individuals correspond to vertices. Initially, one vertex (chosen uniformly at random) possesses a mutation, with fitness $r > 1$. All other individuals have fitness 1. During each step of the algorithm, an individual is chosen with probability proportional to its fitness, and its state (mutant or non-mutant) is passed on to an out-neighbour which is chosen uniformly at random. If the underlying graph is strongly connected then the algorithm will eventually reach *fixation*, in which all individuals are mutants, or *extinction*, in which no individuals are mutants. An infinite family of directed graphs is said to be *strongly amplifying* if, for every $r > 1$, the extinction probability tends to 0 as the number of vertices increases. A formal definition is provided in the paper. Strong amplification is a rather surprising property — it means that in such graphs, the fixation probability of a uniformly-placed initial mutant tends to 1 even though the initial mutant only has a fixed selective advantage of $r > 1$ (independently of n). The name "strongly amplifying" comes from the fact that this selective advantage is "amplified". Strong amplifiers have received quite a bit of attention, and Lieberman et al. proposed two potentially strongly-amplifying families — superstars and metafunnels. Heuristic arguments have been published, arguing that there are infinite families of superstars that are strongly amplifying. The same has been claimed for metafunnels. In this paper, we give the first rigorous proof that there is an infinite family of directed graphs that is strongly amplifying. We call the graphs in the family "megastars". When the algorithm is run on an n-vertex graph in this family, starting with a uniformly-chosen mutant, the extinction probability is roughly $n^{-1/2}$ (up to logarithmic factors). We prove that all infinite families of superstars and metafunnels have larger extinction probabilities (as a function of n). Finally, we prove that our analysis of megastars is fairly tight — there is no infinite family of megastars such that the Moran algorithm gives a smaller extinction probability (up to logarithmic factors). Also, we provide a counter-example which clarifies the literature concerning the isothermal theorem of Lieberman et al.

CCS Concepts: •**Mathematics of computing** → **Discrete mathematics; Markov processes;**

Additional Key Words and Phrases: Moran process, amplifiers, evolutionary dynamics

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1. INTRODUCTION

This paper is about a randomised algorithm called the Moran process. This algorithm was introduced in biology [Moran 1958; Lieberman et al. 2005] to model the spread of genetic mutations in populations. Similar algorithms have been used to model the spread of epidemic diseases, the behaviour of voters, the spread of ideas in social networks, strategic interaction in evolutionary game theory, the emergence of monopolies, and cascading failures in power grids and transport networks [Asavathiratham et al. 2001; Berge 2001; Gintis 2000; Kempe et al. 2003; Liggett 1999].

There has been past work about analysing the expected convergence time of the algorithm [Díaz et al. 2014; Díaz et al. 2016]. In fact, the fast-convergence result of [Díaz et al. 2014] implies that when the algorithm is run on an undirected graph, and the "fitness" of the initial mutation is some constant $r > 1$, there is an FPRAS for the "fixation probability", which is the probability that a randomly-introduced initial mutation spreads throughout the whole graph.

This paper answers an even more basic question, originally raised in [Lieberman et al. 2005], about the long-term behaviour of the algorithm when it is run on directed graphs. In particular, the question is whether there even exists an infinite family of (directed) graphs such that, when the algorithm is run on an n-vertex graph in this family, the fixation probability is $1 - o(1)$, as a function of n. A heuristic argument that this is the case was given in [Lieberman et al. 2005], but a counter-example to the argument (and to the hypothesized bound on the fixation probability) was given in [Díaz et al. 2013]. A further heuristic argument (with a revised bound) was given in [Jamieson-Lane and Hauert 2015]. Here we give the first rigorous proof that there is indeed a family of "amplifiers" with fixation probability $1-o(1)$. Before describing this, and the other results of this paper, we describe the model.

The Moran algorithm has a parameter r which is the fitness of "mutants". All non-mutants have fitness 1. The algorithm runs on a directed graph. In the initial state, one vertex is chosen uniformly at random to become a mutant. After this, the algorithm runs in discrete steps as follows. At each step, a vertex is selected at random, with probability proportional to its fitness. Suppose that this is vertex v. Next, an out-neighbour w of v is selected uniformly at random. Finally, the state of vertex v (mutant or non-mutant) is copied to vertex w.

If the graph is finite and strongly connected then with probability 1, the process will either reach the state where there are only mutants (known as *fixation*) or it will reach the state where there are only non-mutants (*extinction*). In this paper, we are interested in the probability that fixation is reached, as a function of the mutant fitness r, given the topology of the underlying graph. If $r < 1$ then the single initial mutant has lower fitness than the non-mutants that occupy every other vertex in the initial configuration, so the mutation is overwhelmingly likely to go extinct. If $r = 1$, an easy symmetry argument shows that the fixation probability is $\frac{1}{n}$ in any strongly connected graph on n vertices [Díaz et al. 2014, Lemma 1].[1] Because of this, we restrict attention to the case $r > 1$. Perhaps surprisingly, a single advantageous mutant can have a very high probability of reaching fixation, despite being heavily outnumbered in the initial configuration.

A directed graph is said to be *regular* if there is some positive integer d so that the in-degree and out-degree of every vertex is d. In a strongly connected regular graph on n vertices, the fixation probability of a mutant with fitness $r > 1$ when the Moran

[1]The result is stated in [Díaz et al. 2014] for undirected graphs but the proof goes through unaltered for strongly connected directed graphs.

algorithm is run is given by

$$\rho_{\text{reg}}(r,n) = \frac{1-\frac{1}{r}}{1-\frac{1}{r^n}}, \tag{1}$$

so the extinction probability of such a mutant is given by

$$\zeta_{\text{reg}}(r,n) = \frac{\frac{1}{r}-\frac{1}{r^n}}{1-\frac{1}{r^n}}. \tag{2}$$

Thus, in the limit, as n tends to ∞, the extinction probability tends to $1/r$. To see why (1) and (2) hold, note that, for every configuration of mutants, the number of edges from mutants to non-mutants is the same as the number of edges from non-mutants to mutants. Suppose that the sum of the individuals' fitnesses is W and consider an edge (u,v). If u is a mutant in the current state, it is selected to reproduce with probability r/W, and, if this happens, the offspring is placed at v with probability $1/d$. Similarly, if u is not a mutant, reproduction happens along (u,v) with probability $1/(dW)$. So, in any state, the number of mutants is r times as likely to increase at the next step of the process as it is to decrease. If we observe the number of mutants every time it changes, the resulting stochastic process is a random walk on the integers, that starts at 1, absorbs at 0 and n, increases with probability $\frac{r}{r+1}$ and decreases with probability $\frac{1}{r+1}$. It is well known that this walk absorbs at n with probability (1) and at 0 with probability (2). In particular, the undirected n-vertex complete graph is regular. Thus, by (2), its extinction probability tends to $1/r$.

When the Moran process is run on non-regular graphs the extinction probability may be quite a bit lower than $1/r$. Consider the undirected $(n+1)$-vertex "star" graph, which consists of single centre vertex that is connected by edges to each of n leaves. In the limit as $n \to \infty$, the n-leaf star has extinction probability $\frac{1}{r^2}$ [Lieberman et al. 2005; Broom and Rychtár 2008]. Informally, the reason that the extinction probability is so small is that the initial mutant is likely to be placed in a leaf, and, at each step, a mutation at a leaf is relatively unlikely to be overwritten.

Lieberman et al. [2005] refer to graphs which have smaller extinction probability than (2) (and therefore have larger fixation probability than (1)) as *amplifiers*. The terminology comes from the fact that the selective advantage of the mutant is being "amplified" in such graphs.

The purpose of this paper is to explore the long-term behaviour of the Moran process by quantifying how good amplifiers can be. For this, it helps to have some more formal definitions.

Definition 1.1. Consider a function $\zeta(r,n)\colon \mathbb{R}_{>1}\times\mathbb{Z}_{\geq 1} \to \mathbb{R}_{\geq 0}$. An infinite family Υ of directed graphs is said to be *up-to-ζ fixating* if, for every $r>1$, there is an n_0 (depending on r) so that, for every graph $G \in \Upsilon$ with $n \geq n_0$ vertices, the following is true: When the Moran process is run on G, starting from a uniformly-random initial mutant, the extinction probability is at most $\zeta(r,n)$. □

Equation (2) demonstrates that the infinite family of strongly-connected regular graphs is up-to-ζ_{reg} fixating and since $\zeta_{\text{reg}} \leq 1/r$, this family is also up-to-$1/r$ fixating. Informally, an infinite family of graphs is said to be *amplifying* if it is up-to-ζ fixating for a function $\zeta(r,n)$ which is "smaller" than $\zeta_{\text{reg}}(r,n)$. Here is the formal definition.

Definition 1.2. An infinite family of directed graphs is *amplifying* if it is up-to-ζ fixating for a function $\zeta(r,n)$ which, for every $r>1$, satisfies $\lim_{n\to\infty}\zeta(r,n) < 1/r$. □

The infinite family of graphs containing all undirected stars (which can be viewed as directed graphs with edges in both directions) is up-to-$\zeta(r,n)$ fixating for a function $\zeta(r,n)$ satisfying $\lim_{n\to\infty}\zeta(r,n) = 1/r^2$, so this family of graphs is amplifying.

Lieberman et al. [2005] were interested in infinite families of digraphs for which the extinction probability tends to 0, prompting the following definition.

Definition 1.3. An infinite family of directed graphs is *strongly amplifying* if it is up-to-ζ fixating for a function $\zeta(r,n)$ which, for every $r > 1$, satisfies $\lim_{n\to\infty}\zeta(r,n) = 0$. □

Note that the infinite family of undirected stars is not strongly amplifying since the extinction probability of stars tends to $1/r^2$ rather than to 0.

Prior to this paper, there was no (rigorous) proof that a strongly amplifying family of digraphs exists (though there were heuristic arguments, as we explain later). Proving rigorously that there is an infinite family of directed graphs that is strongly amplifying for the Moran algorithm is one of our main contributions.

Lieberman et al. [2005] produced good intuition about strong amplification and defined two infinite families of graphs — superstars and metafunnels — from which it turns out that strongly amplifying families can be constructed. It is extremely difficult to analyse the Moran process on these families, due mostly to the complexity of the graphs, and the difficulty of dealing with issues of dependence and concentration. Thus, all previous arguments have been heuristic. For completeness, we discuss these heuristic arguments in Section 1.4.

In this paper, we define a new family of digraphs called megastars. The definition of megastars is heavily influenced by the superstars of Lieberman et al. Our main theorem is the following.

THEOREM 1.4. *There exists an infinite family of megastars that is strongly amplifying.*

Megastars are not easier to analyse than superstars or metafunnels. The reason for our focus on this class of graphs is that it turns out to be provably better amplifying than any of the previously-proposed families. We will present several theorems along these lines. Before doing so, we define the classes of graphs.

1.1. Metafunnels, superstars and megastars

1.1.1. Metafunnels. We start by defining the metafunnels of [Lieberman et al. 2005]. Let k, ℓ and m be positive integers. The (k,ℓ,m)-*metafunnel* is the directed graph $\mathcal{G}_{k,\ell,m}$ defined as follows. (See Figure 1.)

The vertex set $V(\mathcal{G}_{k,\ell,m})$ is the union of $k+1$ disjoint sets $V_0,\ldots,V_k$. The set V_0 contains the single vertex v^* which is called the *centre vertex*. For $i \in [k]$, V_i is the union of ℓ disjoint sets $V_{i,1},\ldots,V_{i,\ell}$, each of which has size m^i. The edge set of $\mathcal{G}_{k,\ell,m}$ is

$$(V_0 \times V_k) \cup (V_1 \times V_0) \cup \bigcup_{i\in[k-1]} \bigcup_{j\in[\ell]} (V_{i+1,j} \times V_{i,j}) .$$

Lieberman et al. refer to metafunnels with $\ell = 1$ as "funnels".

1.1.2. Superstars. We next define the superstars of [Lieberman et al. 2005]. Let k, ℓ and m be positive integers. The (k,ℓ,m)-*superstar* is the directed graph $\mathcal{S}_{k,\ell,m}$ defined as follows. (See Figure 2.) The vertex set $V(\mathcal{S}_{k,\ell,m})$ of $\mathcal{S}_{k,\ell,m}$ is the disjoint union of ℓ size-m sets $R_1,\ldots,R_\ell$ (called *reservoirs*) together with $k\ell$ vertices $v_{1,1}, v_{1,2},\ldots,v_{\ell,k}$ and a single centre vertex v^*. The edge set of $\mathcal{S}_{k,\ell,m}$ is given by

$$E(\mathcal{S}_{k,\ell,m}) = \bigcup_{i=1}^{\ell} \Big((\{v^*\} \times R_i) \cup (R_i \times \{v_{i,1}\}) \cup \{(v_{i,j}, v_{i,j+1}) \mid j \in [k-1]\} \cup \{(v_{i,k}, v^*)\}\} \Big).$$

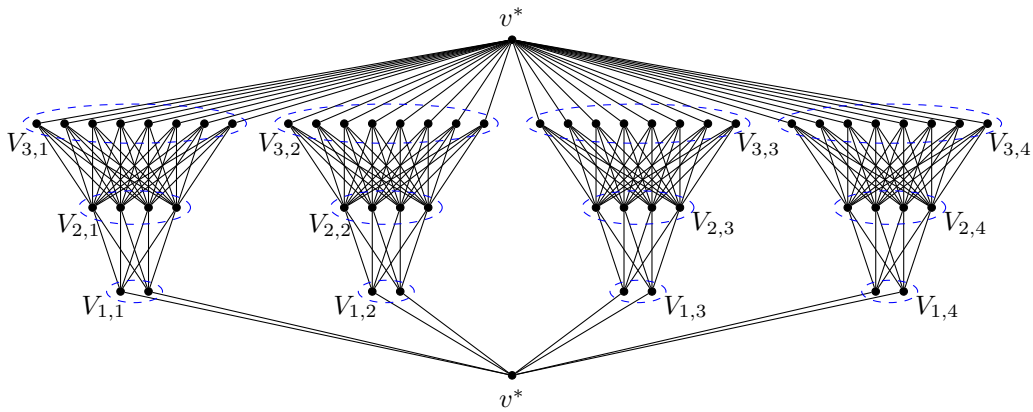

Fig. 1. The metafunnel $\mathcal{G}_{3,4,2}$. All edges are directed downwards in the diagram and the centre vertex v^* is shown twice, once at the top and once at the bottom of the diagram. There are $\ell = 4$ copies of the basic unit, each of which consists of $k = 3$ levels $V_{1,j}$, $V_{2,j}$ and $V_{3,j}$, with $|V_{i,j}| = m^i = 2^i$.

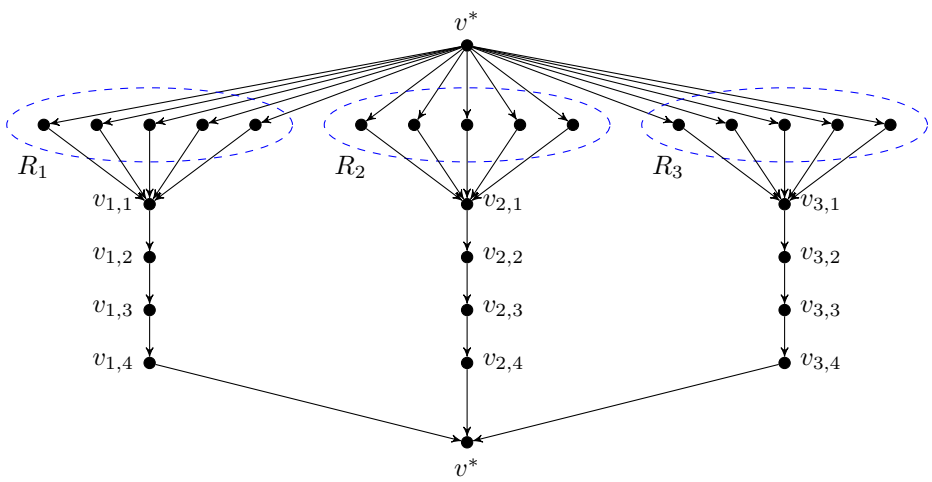

Fig. 2. The superstar $\mathcal{S}_{4,3,5}$, with $\ell = 3$ reservoirs R_1, R_2 and R_3, each of size $m = 5$, connected by a path with $k = 4$ vertices to v^*. The centre vertex v^* is shown twice, at both the top and bottom of the diagram.

1.1.3. Megastars. Finally, we define the new class of megastars, which turn out to be provably-better amplifiers than either metafunnels or superstars. The intuition behind the design of this class of graphs is that the path $v_{i,1}v_{i,2}\dots v_{i,k}$ linking the i'th reservoir R_i of a superstar to the centre vertex v^* is good for amplifying but that a clique is even better.

Let k, ℓ and m be positive integers. The (k,ℓ,m)-megastar is the directed graph $\mathcal{M}_{k,\ell,m}$ defined as follows. (See Figure 3.) The vertex set $V(\mathcal{M}_{k,\ell,m})$ of $\mathcal{M}_{k,\ell,m}$ is the disjoint union of ℓ sets $R_1,\dots,R_\ell$ of size m, called reservoirs, ℓ sets $K_1,\dots,K_\ell$ of size k, called cliques, ℓ "feeder vertices" $a_1,\dots,a_\ell$ and a single centre vertex v^*. The edge set of $\mathcal{M}_{k,\ell,m}$ consists of the following edges:

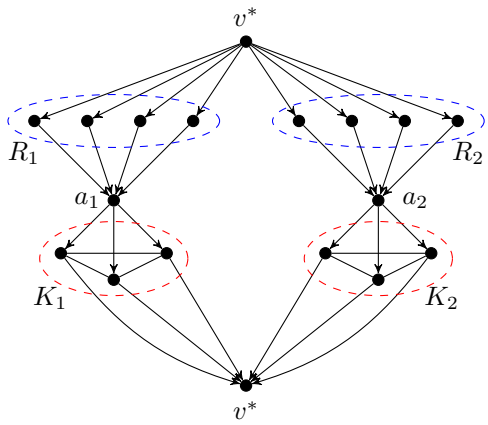

Fig. 3. The megastar $\mathcal{M}_{3,2,4}$, with $\ell = 2$ reservoirs R_1 and R_2, each of size $m = 4$. Each reservoir R_i is attached, via the feeder vertex a_i to a clique of size $k = 3$. The centre vertex v^* is shown twice, once at the top and once at the bottom of the diagram. The edges within the cliques K_1 and K_2 are bidirectional.

— an edge from v^* to every vertex in $R_1 \cup \cdots \cup R_\ell$,
— for each $i \in [\ell]$, an edge from each vertex in R_i to a_i,
— for each $i \in [\ell]$, an edge from a_i to each vertex in K_i,
— for each $i \in [\ell]$, edges in both directions between every pair of distinct vertices in K_i,
— an edge from every vertex in $K_1 \cup \cdots \cup K_\ell$ to v^*.

1.2. Our results

Our main result is that there is an infinite family of megastars that is strongly amplifying, so we start by defining this family. Although megastars are parameterised by three parameters, k, ℓ and m, the megastars in the family that we consider have a single parameter ℓ, so we define k and m to be functions of ℓ.

Definition 1.5. Let $m(\ell) = \ell$ and $k(\ell) = \lceil (\log \ell)^{23} \rceil$. Let

$$\Upsilon_{\mathcal{M}} = \{\mathcal{M}_{k(\ell),\ell,m(\ell)} \mid \ell \in \mathbb{Z}, \ell \geq 2\}. \quad \square$$

Our main result can then be stated as follows.

THEOREM 1.6. *Let $\zeta_{\mathcal{M}}(r,n) = (\log n)^{23} n^{-1/2}$. The family $\Upsilon_{\mathcal{M}}$ is up-to-$\zeta_{\mathcal{M}}$ fixating.*

COROLLARY 1.7. *The family $\Upsilon_{\mathcal{M}}$ is strongly amplifying.*

The proof of Theorem 1.6 requires a complicated analysis, accounting for dependencies and concentration. The theorem, as stated here, follows directly from Theorem 6.1 which is proved in Section 6 (see Page 52).

The reason that we studied megastars rather than the previously-introduced superstars and metafunnels is that megastars turn out to be provably better amplifying than any of the previously-proposed families. To demonstrate this, we prove the following theorem about superstars.

THEOREM 1.8. *Let $\zeta(r,n)$ be any function such that, for any $r > 1$,*

$$\lim_{n\to\infty} \zeta(r,n)(n \log n)^{1/3} = 0.$$

Then there is no infinite family of superstars that is up-to-ζ fixating.

The function $\zeta_{\mathcal{M}}$ from Theorem 1.6 certainly satisfies $\lim_{n\to\infty} \zeta_{\mathcal{M}}(r,n)(n\log n)^{1/3} = 0$, so Theorem 1.8 shows that there is no infinite family of superstars that is up-to-$\zeta_{\mathcal{M}}$ fixating. More mundanely, it shows, for example, that if $\zeta(r,n) = n^{-1/3}(\log n)^{-1}$, then no infinite family of superstars is up-to-ζ fixating. Theorem 1.8 is a direct consequence of Theorem 4.1 which is proved in Section 4 (see Page 19). It turns out that analysing superstars is a little bit easier than analysing megastars or metafunnels, so this is the first proof that we present.

Taken together, Theorems 1.6 and 1.8 show that superstars are worse amplifiers than megastars. We next show that metafunnels are *substantially* worse. We start with the following simple-to-state theorem.

THEOREM 1.9. *Fix any $\delta > 0$ and let $\zeta(r,n) = n^{-\delta}$. Then there is no infinite family of metafunnels that is up-to-ζ fixating.*

In fact, Theorem 1.9 can be strengthened by an exponential amount.

THEOREM 1.10. *Fix any $\epsilon < 1/2$ and let $\zeta(r,n) = n^{-1/(\log n)^{\epsilon}}$. Then there is no infinite family of metafunnels that is up-to-ζ fixating.*

Theorems 1.9 and 1.10 are a direct consequence of Theorem 5.1 which is proved in Section 5 (see Page 32). In fact, Theorem 5.1 provides even tighter bounds, though these are more difficult to state.

The theorems that we have already described (Theorem 1.6, Theorem 1.8 and Theorem 1.10) are the main contributions of the paper. Together, they show that there is a family of megastars that is strongly amplifying, and that there are no families of superstars or metafunnels that amplify as well. For completeness, we present a theorem showing that the analysis of Theorem 1.6 is fairly tight, in the sense that there are no infinite families of megastars that amplify substantially better than $\Upsilon_{\mathcal{M}}$ — in particular, our bound on extinction probability can only be improved by factors of $\log(n)$. It cannot be improved more substantially.

THEOREM 1.11. *Let $\zeta(r,n) = n^{-1/2}/(52r^2)$. There is no infinite family of megastars that is up-to-ζ fixating.*

Theorem 1.11 follows from Theorem 7.3, which is straightforward, and is proved in Section 7 (see Page 86). We conclude the paper with a digression which perhaps clarifies the literature. It is stated, and seems to be commonly believed, that an evolutionary graph (a weighted version of the Moran process — see Section 8 for details) is "isothermal" if and only if the fixation probability of a mutant placed uniformly at random is $\rho_{\text{reg}}(r,n)$. This belief seems to have come from an informal statement of the "isothermal theorem" in the main body of [Lieberman et al. 2005] (the formal statement in the supplementary material of [Lieberman et al. 2005] is correct, however) and it has spread, for example, as Theorem 1 of [Shakarian et al. 2012]. In the final section of our paper, we clear this up by proving the following proposition, which says that there is a counter-example.

PROPOSITION 1.12. *There is an evolutionary graph that is not isothermal, but has fixation probability $\rho_{\text{reg}}(r,n)$.*

The definitions needed to prove Proposition 1.12 are deferred to Section 8 (see Page 87).

1.3. Proof techniques

As we have seen, it is easy to study the Moran process on a d-regular graph by considering the transition matrix of the corresponding Markov chain (which looks like a

one-dimensional random walk). Highly symmetric graphs such as undirected stars can also be handled in a straightforward matter, by directly analysing the transition matrix. Superstars, metafunnels and megastars are more complicated, and the number of mutant-configurations is exponential, so instead we resort to dividing the process into phases, as is typical in the study of randomised algorithms and stochastic processes.

An essential and common trick in the area of stochastic processes (for example, in work on the voter model) is moving to continuous time. Instead of directly studying the discrete-time Moran process, one could consider the following natural continuous-time model which was studied in [Díaz et al. 2016]: Given a set of mutants at time t, each vertex waits an amount of time before reproducing. For each vertex, the period of time is chosen according to the exponential distribution with parameter equal to the vertex's fitness, independently of the other vertices. If the first vertex to reproduce is v at time $t+\tau$ then, as in the standard, discrete-time version of the process, one of its out-neighbours w is chosen uniformly at random, the individual at w is replaced by a copy of the one at v, and the time at which w will next reproduce is exponentially distributed with parameter given by its new fitness. The discrete-time process is recovered by taking the sequence of configurations each time a vertex reproduces. Thus, the fixation probability of the discrete-time process is *exactly the same* as the fixation probability of the continuous-time process. So moving to the continuous-time model causes no harm. As Díaz et al. [2016] explain, analysis can be easier in the continuous-time model because certain natural stochastic domination techniques apply in the continuous-time setting but not in the discrete-time setting.

It turns out that moving to the model of Díaz et al. [2016] does not suffice for our purposes. A major problem in our proofs is dealing with dependencies. In order to make this feasible, we instead study a continuous-time model (see "the clock process" in Section 3.1) in which every edge of the underlying graph G is equipped with two Poisson processes, one of which is called a *mutant clock* and the other of which is called a *non-mutant clock*. The clock process is a stochastic process in which all of these clocks run independently. The continuous-time Moran process (Definition 3.2) can be recovered as a function of the times at which these clocks trigger.

Having all of these clocks available still does not give us the flexibility that we need. We say that a vertex u "spawns a mutant" in the Moran process if, at some point in time, u is a mutant, and it is selected for reproduction. We wish to be able to discuss events such as the event that the vertex u does not spawn a mutant until it has already been a mutant for some particular amount of time. In order to express such events in a clean way, making all conditioning explicit, we define additional stochastic processes called "star-clocks" (see Section 3.3). All of the star-clocks run independently in the star-clock process.

In Section 3.4 we provide a coupling of the star-clock process with the Moran process. The coupling is valid in the sense that the two projections are correct — the projection onto the Moran process runs according to the correct distribution and so does the projection onto the star-clock process. The point of the coupling is that the different star-clocks can be viewed as having their own "local" times. In particular, there is a star-clock $M^*_{(u,v)}$ which controls reproductions from vertex u onto vertex v *during the time that* u *is a mutant*. The coupling enables us to focus on relevant parts of the stochastic process, making all conditioning explicit.

The processes that we have described so far are all that we need to derive our upper bound on the fixation probability of superstars (Section 4). This is the easiest of our main results.

Analysing the Moran process on metafunnels is more difficult. By design, the initial mutant x_0 is likely to be placed in the "top of a funnel" (in the set V_k). In the analysis,

it is useful to be able to create independence by considering a "strain" of mutants which contains all of the descendants of a particular mutant spawned by x_0. Like the Moran process itself, a strain can be viewed as a stochastic process depending on the triggering of the clocks. In order to facilitate the proof, we define a general notion of "mutant process" (Section 3.2) — so the Moran process is one example of a mutant process, and a strain is another. The analysis of the Moran process on metafunnels involves both of these and also a third mutant process which is essentially the bottom level of a strain (called its head). Strains and heads-of-strains share some common properties, and they are analysed together as "colonies" in Section 5.4.1. The analysis of the metafunnel is the technically most difficult of our results.

Fortunately, the analysis of the megastar in Section 6 does not require three different types of mutant processes — it only requires one. The process that is considered is not the Moran process itself. Instead, it is a modification of the Moran process called the megastar process. The megastar process is similar to the Moran process except that the feeder vertices are forced to be non-mutants, except when their corresponding cliques are completely full or completely empty. It is easy to show (see the proof of Theorem 6.1) that the fixation probability of the Moran process is at least as high as the fixation probability of the megastar process. However, the megastar process is somewhat easier to analyse because the cliques evolve somewhat independently. The proof of the key lemma (Lemma 6.3) is fairly long but it is not conceptually difficult. The point is to prove that, with high probability, the cliques fill up and cause fixation.

1.4. Comparison with previous work

The Moran process is similar to a discrete version of directed percolation known as the contact process. There is a vast literature (e.g., [Liggett 1999; Durrett 2010; Shah 2009; Durrett and Steif 1993]) on the contact process and other related infection processes such as the voter model and susceptible-infected-susceptible (SIS) epidemic models. Often, the questions that are studied in these models are different from the question that we study here. For example, in voter systems [Durrett and Steif 1993] the two states (mutant/non-mutant) are often symmetric (similar to our $r = 1$ case) and the models are often studied on infinite graphs where the question is whether the process absorbs or not (both kinds of absorption, fixation and extinction, are therefore called "fixation" in some of this work). The particular details of the Moran process are very important for us because the details of the algorithm determine the long-term behaviour. For example, unlike the Moran process, in the contact process [Bezuidenhout and Grimmett 1990], the rate at which a node becomes a non-mutant is typically taken to be 1, whereas the rate at which a node becomes a mutant is proportional to the number of mutant neighbours. In the discrete-time versions of many commonly-studied models, a node is chosen randomly at each step for replacement, rather than (as in the Moran process) for reproduction. In any case, the important point for us is that the details of the algorithm are important — results do not carry over from one algorithm to the other. Therefore, we concentrate in this section on previous work about calculating the fixation probability of the Moran process itself.

Lieberman et al. [2005] studied the fixation probability of the Moran process and introduced superstars and metafunnels. Intuitively, a superstar is a good amplifier because (as long as m is sufficiently large) the initial mutation is likely to be placed in a reservoir and (as long as ℓ is sufficiently large) this is unlikely to be killed quickly by the centre vertex. Moreover, the paths of a superstar are good for amplifying the selective advantage of mutants because, after the infection spreads from a reservoir vertex to the beginning of a path, it is likely to "pick up momentum" as it travels down the path, arriving at the centre vertex as a chain of $\Theta(k)$ mutants (which, taken together, are more likely to cause the centre to spread the infection than a single mutant arriv-

ing at the centre would be). As we have seen (Theorems 1.6 and 1.8) megastars are provably better for amplification than superstars. The reason for this is that a clique is substantially better than a path at doing this "amplification". Nevertheless, the amplifying properties of superstars strongly influenced our decision to study megastars.

Lieberman et al. [2005, Equation (2)] claimed[2] that for sufficiently large n, the fixation probability of a superstar with parameter k tends to $1 - r^{-(k+2)}$, and that "similar results hold for the funnel and metafunnel". They provided a heuristic sketch-proof for the superstar, but not for the funnel or metafunnel. Hauert [2008, Equation (5)] claims specifically that the fixation probability of funnels tends to $1 - r^{-(k+1)}$. As far as we know, no heuristic arguments have been given for funnels or metafunnels.

In any event, Díaz et al. [2013] showed that the $1 - r^{-(k+2)}$ claim for superstars is incorrect for the case $k = 3$. In particular, for this case they showed that the fixation probability is at most $1 - \frac{r+1}{2r^5+r+1}$, which is less than the originally claimed value of $1 - r^{-5}$ for all $r \geq 1.42$.

Subsequently, Jamieson-Lane and Hauert [2015, Equation (5)] made a more detailed but still heuristic[3] analysis of the fixation probability of superstars. They claim that for superstars with parameter k and with $\ell = m$, the fixation probability ρ_k has the following bounds for fixed $r > 1$,

$$1 - \frac{1}{r^4(k-1)(1-\frac{1}{r})^2} - o(1) \leq \rho_k \leq 1 - \frac{1}{r^4(k-1)} + o(1), \tag{3}$$

where the $o(1)$ terms tend to 0 as $\ell \to \infty$. They claim that their bounds are a good approximation as long as $k \ll \ell = m \sim \sqrt{n}$. It is not clear exactly what "$\ll$" means in this context. Certainly there are parameter regimes where $k = o(\ell)$ and $\ell = m \sim \sqrt{n}$ but nevertheless the extinction probability is much larger than the proposed upper bound $1/(r^4(k-1)(1-1/r)^2)$ from (3). For example, suppose that $\ell = m = k^{3/2}$. In this case (see Lemma 4.2), the extinction probability is at least

$$\frac{k}{2r(m+k)} = \frac{1}{2r(k^{1/2}+1)},$$

which is larger than $1/(r^4(k-1)(1-1/r)^2)$ for all sufficiently large k. Nevertheless, the bounds proposed by Jamieson-Lane and Hauert (3) seem to be close to the truth when k is very small compared to ℓ and m.

Our Corollary 4.6 identifies a wide class of parameters for which the extinction probability is provably at least $1/(1470r^4k)$. This is weaker than the suggested bound of Jamieson-Lane and Hauert by a factor of 1470. This constant factor is explained by the fact that our rigorous proof needs to show concentration of all random variables. We use lots of Chernoff bounds and other bounds on probabilities. In writing the proof, we optimised readability rather than optimising our constants, so our constants can presumably be improved.

There is recent work on other related aspects of the Moran process. For example, [Mertzios et al. 2013; Mertzios and Spirakis 2013] give fixation probability bounds on connected *undirected* graphs. Adlam et al. [2015] study amplification with respect to adversarial or "temperature-based" placement of the initial mutation, in which the "temperature" of a vertex is proportional to the sum of all incoming edge weights. Also,

[2]The reader who consults [Lieberman et al. 2005] might wonder why "k" as written in Equation (2) of [Lieberman et al. 2005] has become $k + 2$ here. The reason is just that we use a slightly different parameterisation from that of [Lieberman et al. 2005]. To allow appropriate comparison, we describe all previous work using the parameterisation that we give in Section 1.1.

[3]A full discussion of the argument of Jamieson-Lane and Hauert (and of the obstacles to making it a rigorous proof) are discussed in Section 9. This section is not necessary for the rest of the paper.

Mertzios and Spirakis [2013] consider the extent to which the number of "good starts" for fixation can be bounded.

1.5. Outline of the paper

Section 2, starting on Page 11, defines some notation and states some well-known probabilistic bounds (Chernoff bounds and analysis of gambler's ruin) which will be used in the proof.

Section 3, starting on Page 15, defines several stochastic processes which we use to study the Moran process. This section is important. It is impossible to read any of the proofs without understanding these processes.

Section 4, starting on Page 18, gives an upper bound on the fixation probability of superstars. The main result of the section is Theorem 4.1, which immediately implies Theorem 1.8. This is the technically easiest of our main proofs, so we present it first.

Section 5, starting on Page 32, gives a stronger upper bound on the fixation probability of metafunnels (and hence of funnels). The main result of the section is Theorem 5.1, which immediately implies Theorems 1.9 and 1.10. The proof of Theorem 5.1 has high-level similarity to the proof of Theorem 4.1, but it is much more difficult. Dependencies cause complications, and we must analyse several mutant processes to deal with these.

Section 6, starting on Page 52, establishes the existence of an infinite family of megastars which is strongly amplifying. The main theorem is Theorem 6.1, which immediately implies Theorem 1.6 and hence Theorem 1.4. In order to deal with dependencies, we study a mutant process called a "megastar process". We show in the proof of Theorem 6.1 that this process is dominated by the Moran process. Thus, the main work of the section is to prove the key lemma, Lemma 6.3, which analyses the megastar process.

Section 7, starting on Page 85, gives an upper bound showing that the analysis in Section 6 is fairly tight. The main theorem, Theorem 7.3, is straightforward and it immediately implies Theorem 1.11.

Section 8, starting on Page 87, gives a simple example of an evolutionary graph that is not isothermal but has fixation probability $\rho_{\mathrm{reg}}(r, n)$ (Proposition 1.12), clearing up a misconception in the literature.

Section 9, starting on Page 88, discusses earlier heuristic analysis of superstars.

2. DEFINITIONS AND PRELIMINARIES

2.1. Notation

We use $N^-(v)$ to refer to the set of in-neighbours of a vertex v and $N^+(v)$ to refer to the set of out-neighbours of v. We use $d^-(v) = |N^-(v)|$ and $d^+(v) = |N^+(v)|$.

We refer to the Lebesgue measure of a (measurable) subset $S \subseteq \mathbb{R}$ as the *measure* of that set, and denote it by $\mathrm{len}(S)$.

We use base-e for logarithms unless the base is given explicitly.

We write $\mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$, $\mathbb{Z}_{\geq 1} = \{1, 2, \dots\}$, and $[n] = \{1, 2, \dots, n\}$.

If $b < a$, we consider the interval $[a, b]$ to be well-defined but empty. Likewise if $b \leq a$, we consider the intervals (a, b), $(a, b]$ and $[a, b)$ to be well-defined but empty. We define empty sums, products, unions etc. to be the identities of the corresponding operations. For example, $\prod_{i=1}^{0} i = 1$ and $\bigcup_{i=1}^{0} A_i = \emptyset$.

Throughout the paper, we use lower case t's to denote fixed times and upper case T's to denote stopping times.

2.2. Chernoff bounds

We often use the following simple bound which applies to any real number $x \in [0,1]$.

$$x/2 \le 1 - e^{-x} \le x. \tag{4}$$

We will require the following well-known Chernoff bounds. The first appears as Theorem 5.4 of [Mitzenmacher and Upfal 2005].

LEMMA 2.1. *Let Y be a Poisson random variable with parameter $\rho \ge 0$. If $y > \rho$ and $z < \rho$, then*

$$\mathbb{P}(Y \ge y) \le \frac{e^{-\rho}(e\rho)^y}{y^y} \qquad \textit{and} \qquad \mathbb{P}(Y \le z) \le \frac{e^{-\rho}(e\rho)^z}{z^z}\,.$$

COROLLARY 2.2. *Let Y be a Poisson random variable with parameter $\rho \ge 0$. Then $\mathbb{P}(Y \ge 2\rho) \le e^{-\rho/3}$ and $\mathbb{P}(Y \le 2\rho/3) \le e^{-\rho/16}$.*

PROOF. Lemma 2.1 applied with $y = 2\rho$ and $z = 2\rho/3$ implies that

$$\mathbb{P}(Y \ge 2\rho) \le \frac{e^{-\rho}(e\rho)^{2\rho}}{(2\rho)^{2\rho}} = e^{-\rho}\left(\frac{e^2}{4}\right)^{\rho} = e^{(1-\log 4)\rho} \le e^{-\rho/3}\,,$$

$$\mathbb{P}(Y \le 2\rho/3) \le \frac{e^{-\rho}(e\rho)^z}{z^z} = e^{-\rho}\left(\frac{3e}{2}\right)^{2\rho/3} = e^{-(1-2/3-2\log(3/2)/3)\rho} \le e^{-\rho/16}\,. \quad \square$$

COROLLARY 2.3. *Let Y be a Poisson random variable with parameter $\rho > 0$. If $y \ge 8\rho$, then*

$$\mathbb{P}(Y \ge y) \le e^{-y}.$$

PROOF. Note that $y > e^2\rho$. Thus by Lemma 2.1, we have

$$\mathbb{P}(Y \ge y) \le \frac{e^{-\rho}(e\rho)^y}{y^y} \le \left(\frac{e\rho}{y}\right)^y \le e^{-y}. \quad \square$$

COROLLARY 2.4. *Let s be a positive integer and let Y be the sum of s i.i.d. exponential random variables, each with parameter λ. Then, for any $j \ge 3s/(2\lambda)$, $\mathbb{P}(Y < j) \ge 1 - e^{-\lambda j/16}$.*

PROOF. First, note that $\mathbb{P}(Y < j) = \mathbb{P}(Y \le j)$ since $\mathbb{P}(Y = j) = 0$. Then $\mathbb{P}(Y \le j)$ is equal to the probability that a Poisson process with parameter λ triggers at least s times in the interval $[0, j]$. This is the same as the probability that a Poisson random variable with parameter λj is at least s. Since $s \le 2\lambda j/3$, we can now use Corollary 2.2. $\square$

The following is Corollary 2.4 of [Janson et al. 2000].

LEMMA 2.5. *Suppose that Y follows the binomial distribution with n Bernoulli trials, each with success probability $p \in (0,1)$ and that $c > 1$. Then, for all $y \ge cnp$, $\mathbb{P}(Y \ge y) \le e^{-\varphi(c)y}$, where $\varphi(c) = \log c - 1 + 1/c$. Note that $\varphi(2) > 1/6$ and $\varphi(7) > 1$.*

We define the *geometric distribution* as follows. Given a biased coin which comes up heads with probability $p > 0$, imagine tossing it until it comes up heads. Then the total number of tosses which came up tails follows the geometric distribution with parameter p.

LEMMA 2.6. *Let $Y_1, \ldots, Y_t$ be a sequence of i.i.d. geometric variables with parameter $p \ge 13/14$. Then*

$$\mathbb{P}\big(Y_1 + \cdots + Y_t \ge 14t(1-p)\big) \le e^{-14t(1-p)}\,.$$

PROOF. Consider a series of independent coin tosses, each with probability p of coming up heads. Then the probability that $Y_1+\cdots+Y_t \geq 14t(1-p)$ is exactly the probability that at least $\lceil 14t(1-p)\rceil$ of the first $t+\lceil 14t(1-p)\rceil - 1$ coin tosses come up tails. By Lemma 2.5, the probability that at least $\lceil 14t(1-p)\rceil$ of the first $2t$ coin tosses come up tails is at most $e^{-14t(1-p)}$, and $2t \geq t+\lceil 14t(1-p)\rceil - 1$, so the result follows. □

2.3. Gambler's ruin

The following analysis of the classical gambler's ruin problem is well-known. See, for example, [Feller 1968, Chapter XIV].

LEMMA 2.7 (CLASSICAL GAMBLER'S RUIN). *Consider a random walk on $\mathbb{Z}_{\geq 0}$ that absorbs at 0 and a (for some positive integer a), starts at $z \in \{0,\ldots,a\}$ and from each state in $\{1,\ldots,a-1\}$ has probability $p \neq 1/2$ of increasing (by 1) and probability $q = 1-p$ of decreasing (by 1).*

(i) The probability of reaching state a is

$$\frac{(q/p)^z - 1}{(q/p)^a - 1}.$$

(ii) The expected number of transitions until absorption is

$$\frac{z}{q-p} - \left(\frac{a}{q-p}\right)\left(\frac{1-(q/p)^z}{1-(q/p)^a}\right).$$

COROLLARY 2.8 (GAMBLER'S RUIN INEQUALITIES). *Consider a random walk on $\mathbb{Z}_{\geq 0}$ that absorbs at 0 and a (for some positive integer a), starts at $z \in [0,a]$ and from each state in $\{1,\ldots,a-1\}$ has probability $p \neq 1/2$ of increasing (by 1) and probability $q = 1-p$ of decreasing (by 1).*

(i) If $p > q$ then the probability of reaching state a is at least $1-(q/p)^z$.
(ii) If $q > p$ then the expected number of transitions until absorption is at most $z/(q-p)$.
(iii) If $p > q$ then the expected number of transitions until absorption is at most $(a-z)/(p-q)$.

PROOF. Items (i) and (ii) are immediate. To see (iii) for $p > q$, rewrite the expected number of transitions as

$$\left(\frac{a}{p-q}\right)\left(\frac{1-(q/p)^z}{1-(q/p)^a}\right) - \frac{z}{p-q} \leq \frac{a-z}{p-q}. \quad \square$$

We also consider a variant of the gambler's ruin in which the probability of upwards transitions depends on the current state.

LEMMA 2.9. *Let a, b, c and d be integers satisfying $a < b < c-1$ and $c+1 < d$. Consider $p_1 \in (1/2,1)$. Consider the discrete Markov chain on states $\{a,\ldots,d\}$ with the following transition matrix.*

$$\begin{aligned} p_{a,a} &= 1, \\ p_{i,i+1} &= \begin{cases} p_1 & \text{if } a+1 \leq i \leq c, \\ 1/3 & \text{if } c < i \leq d-1, \end{cases} \\ p_{i,i-1} &= 1 - p_{i,i+1} \text{ for all } i \in \{a+1,\ldots,d-1\}, \\ p_{d,d} &= 1. \end{aligned}$$

For integers x, y and z in $\{a,\ldots,d\}$ and subset S of $\{a,\ldots,d\}$, let $p_{x\to y;z}$ denote the probability that, starting from state x, the chain visits state y without passing through z

and let $H_{x;S}$ be the number of transitions that the chain takes to hit a state in S, starting from state x.

(i) $p_{c\to d;b} \geq 1 - \left(\frac{1-p_1}{p_1}\right)^{c-b} 2^{d-c}$.

(ii) $\mathbb{E}[H_{c;\{a,d\}}] \leq 2^{d-c+1}\left(\frac{3p_1-1}{2p_1-1}\right)$.

PROOF. We first prove (i). It is immediate that

$$\begin{aligned} p_{c\to d;b} &= (1-p_1)p_{c-1\to d;b} + p_1 p_{c+1\to d;b} \\ &= (1-p_1)p_{c-1\to c;b}p_{c\to d;b} + p_1 p_{c+1\to d;c} + p_1(1-p_{c+1\to d;c})p_{c\to d;b}. \end{aligned}$$

Rearranging,

$$p_{c\to d;b} = \frac{p_1 p_{c+1\to d;c}}{(1-p_1)(1-p_{c-1\to c;b}) + p_1 p_{c+1\to d;c}}. \tag{5}$$

Now from Lemma 2.7(i),

$$p_{c+1\to d;c} = \frac{2-1}{2^{d-c}-1} \geq 2^{-(d-c)}. \tag{6}$$

Also, from Corollary 2.8(i),

$$p_{c-1\to c;b} \geq 1 - \left(\frac{1-p_1}{p_1}\right)^{c-b-1},$$

so

$$1 - p_{c-1\to c;b} \leq \left(\frac{1-p_1}{p_1}\right)^{c-b-1}.$$

Plugging these bounds into (5), we get

$$p_{c\to d;b} \geq \frac{1}{\left(\frac{1-p_1}{p_1}\right)^{c-b} 2^{d-c} + 1} = 1 - \frac{\left(\frac{1-p_1}{p_1}\right)^{c-b} 2^{d-c}}{\left(\frac{1-p_1}{p_1}\right)^{c-b} 2^{d-c} + 1} \geq 1 - \left(\frac{1-p_1}{p_1}\right)^{c-b} 2^{d-c}.$$

We now prove (ii). Clearly,

$$\mathbb{E}[H_{c;\{a,d\}}] = 1 + p_1\mathbb{E}[H_{c+1;\{a,d\}}] + (1-p_1)\mathbb{E}[H_{c-1;\{a,d\}}].$$

But

$$\mathbb{E}[H_{c+1;\{a,d\}}] = \mathbb{E}[H_{c+1;\{c,d\}}] + p_{c+1\to c;d}\mathbb{E}[H_{c;\{a,d\}}]$$

and

$$\mathbb{E}[H_{c-1;\{a,d\}}] = \mathbb{E}[H_{c-1;\{a,c\}}] + p_{c-1\to c;a}\mathbb{E}[H_{c;\{a,d\}}].$$

So solving, we get

$$\mathbb{E}[H_{c;\{a,d\}}] = \frac{1 + p_1\mathbb{E}[H_{c+1;\{c,d\}}] + (1-p_1)\mathbb{E}[H_{c-1;\{a,c\}}]}{1 - p_1 p_{c+1\to c;d} - (1-p_1)p_{c-1\to c;a}}.$$

Now $p_{c-1\to c;a} \leq 1$ and by Equation (6), $p_{c+1\to c;d} \leq 1 - 2^{-(d-c)}$. By Corollary 2.8(ii), $\mathbb{E}[H_{c+1;\{c,d\}}] \leq 1/(\frac{2}{3} - \frac{1}{3}) = 3$. Finally, by Corollary 2.8(iii),

$$\mathbb{E}[H_{c-1;\{a,c\}}] \leq \frac{1}{p_1 - (1-p_1)} = \frac{1}{2p_1 - 1}.$$

Plugging all of these in,

$$\mathbb{E}[H_{c;\{a,d\}}] \leq \frac{1 + 3p_1 + \frac{1-p_1}{2p_1-1}}{1 - p_1(1-2^{-(d-c)}) - (1-p_1)} = 2^{d-c+1}\left(\frac{3p_1-1}{2p_1-1}\right). \quad \square$$

3. STOCHASTIC PROCESSES

We will be concerned with the discrete-time Moran process [Moran 1958], as adapted by Lieberman, Hauert and Nowak [2005] and described in Section 1. This is a discrete model of evolution on an underlying directed graph G where the reproduction rate of mutants is a parameter $r > 0$ called the "fitness".

In this paper, we consider the situation $r > 1$, which corresponds to the situation in which a mutation is advantageous. The fitness r is a parameter of all of our processes. Our results apply to any fixed $r > 1$. Since the value of r is fixed, we simplify the presentation by not including it in the explicit notation and terminology. Thus, from now on, we say "Moran process" to signify "Moran process with fitness r".

Following Díaz et al. [2016] we will simplify our proofs by studying a continuous-time version of the Moran process. The continuous-time version is also parameterised by G and r and it has the same fixation probability as the discrete-time version, so our results will carry over immediately to the discrete process.

In order to deal with conditioning in the proofs we will in fact define several general stochastic processes, all of which depend on G and r — one of these will be equivalent to the continuous-time Moran process and others will be useful for dominations.

All of the processes that we study evolve over time. For any process P, we use $\mathcal{F}(P)$ to denote the filtration of P so $\mathcal{F}_t(P)$ captures the history of the process P up to and including time t.

3.1. The clock process

For each edge $e = (u, v)$ of G we define two Poisson processes — a Poisson process M_e with parameter $r/d^+(u)$ and a Poisson process N_e with parameter $1/d^+(u)$. We refer to these processes as *clocks*, and when an event occurs in one of them, we say that the relevant clock *triggers*. We refer to M_e as a *mutant clock with source u and target v* and N_e as a *non-mutant clock with source u and target v*.

We use $\mathcal{C}(G)$ to denote the set of all clocks so $\mathcal{C}(G) = \bigcup_{e \in E(G)} \{M_e, N_e\}$. We use $P(G)$ to denote the Cartesian product of all process in $\mathcal{C}(G)$. $P(G)$ is the stochastic process in which all clocks in $\mathcal{C}(G)$ evolve simultaneously and independently, starting at time 0.

With probability 1, the clocks trigger a countably infinite number of times and these can be indexed by an increasing sequence $\tau_1, \tau_2, \ldots$. Also, no clocks trigger simultaneously and the clocks trigger for an infinitely long period — that is, for every clock and every t, the clock triggers at some $\tau_i > t$. For convenience, we take $\tau_0 = 0$. We will use the random variables $\tau_0, \tau_1, \ldots$ (which depend on the process $P(G)$) in our arguments.

3.2. Mutant processes

A *mutant process* μ has an underlying graph $G(\mu)$ and initial state μ_0. At every time t, the state μ_t is a subset of $V(G(\mu))$, which we sometimes refer to as the "set of mutants" at time t. Every mutant process satisfies the following two constraints.

(1) For all $t \geq 0$, μ_t is determined by $\mathcal{F}_t(P(G(\mu)))$.
(2) For all $t, t' \geq 0$, if there is a non-negative integer i so that t and t' are both in the range $[\tau_i, \tau_{i+1})$ then $\mu_t = \mu_{t'}$.

We define some terminology associated with the mutant process μ.

— If the clock $M_{(u,v)}$ triggers at time t and $u \in \mu_t$ we say that *u spawns a mutant onto v in μ at time t* and that *μ spawns a mutant* onto v at time τ_i.
— If the clock $N_{(u,v)}$ triggers at time t and $u \notin \mu_t$ we say that *u spawns a non-mutant onto v in μ at time t*. We say that *μ spawns a non-mutant* onto v at time τ_i.
— If $v \in \mu_{\tau_i}$ and $v \notin \mu_{\tau_{i-1}}$ we say that *v becomes a mutant in μ at time τ_i*.

— If $v \in \mu_{\tau_{i-1}}$ and $v \notin \mu_{\tau_i}$ we say that v *becomes a non-mutant in* μ *at time* τ_i or that v *dies in* μ *at time* τ_i.

When the mutant process is absolutely clear from the context, we sometimes drop the phrase "in μ". Note that v does not necessarily become a mutant at time τ_i when some u spawns a mutant onto v at time τ_i since v may already be a mutant at that time.

For convenience, we include the filtration $\mathcal{F}_t(P(G))$ in the filtration $\mathcal{F}_t(\mu)$ of the mutant process so the sequence of trigger-times $\tau_0, \tau_1, \ldots$ up to time t can be determined from $\mathcal{F}_t(\mu)$.

Remark 3.1. Sometimes we will consider a mutant process μ in which the initial state μ_0 is a randomly chosen subset of $V(G(\mu))$. When we do this, we assume that the choice of the initial state μ_0 is independent of the triggering of the clocks in $\mathcal{C}(G(\mu))$.

We will define several mutant processes in the course of our proofs, but the most fundamental is the Moran process itself, which is a particular mutant process.

Definition 3.2 (*the Moran process*). The (continuous-time) Moran process on graph G with initial mutant $x_0 \in V(G)$ is a mutant process X with $G(X) = G$ and $X_0 = \{x_0\}$ defined as follows. Recall that, for every positive integer i, a clock $C \in \mathcal{C}(G)$ triggers at τ_i. For $t \in (\tau_{i-1}, \tau_i)$, we set $X_t = X_{\tau_{i-1}}$. Then we define X_{τ_i} as follows.

(i) If $C = M_{(u,v)}$ for some $(u, v) \in E(G)$ and $u \in X_{\tau_{i-1}}$ then $X_{\tau_i} = X_{\tau_{i-1}} \cup \{v\}$.
(ii) If $C = N_{(u,v)}$ for some $(u, v) \in E(G)$ and $u \notin X_{\tau_{i-1}}$ then $X_{\tau_i} = X_{\tau_{i-1}} \setminus \{v\}$.
(iii) Otherwise, $X_{\tau_i} = X_{\tau_{i-1}}$.

Considering the positive integers i in order, this completes the definition of the Moran process X_t. □

Remark 3.3. It is clear from Definition 3.2 that the Moran process X_t is a mutant process. In Definition 3.2, say that τ_i is a "relevant trigger time" if (i) or (ii) occurs rather than (iii). The discrete-time Moran process [Moran 1958], as adapted by Lieberman et al. [2005] is the Markov chain $X_{\tau_0}, X_{\tau_{i_1}}, X_{\tau_{i_2}}, \ldots$, where $\tau_{i_1}, \tau_{i_2}, \ldots$ is the increasing sequence of relevant trigger times. Note that the fixation probability of the discrete-time Moran process is the same as the fixation probability of the continuous-time process X_t, so we will study the process X_t in this paper.

Definition 3.4. We say that a mutant process is *extinct by time* t if, for all $t' \geq t$, $\mu_{t'} = \emptyset$. We say that it *fixates by time* t if, for all $t' \geq t$, $\mu_{t'} = V(G(\mu))$. We say that it *absorbs by time* t if it is extinct by time t or it fixates by time t. The *fixation probability* is the probability that, for some t, it fixates by time t. The *extinction probability* is the probability that, for some t, it is extinct by time t. □

Remark 3.5. The Moran process X_t is extinct by time t if $X_t = \emptyset$ and fixates by time t if $X_t = V(G(X))$. If G is strongly connected then the fixation probability and the extinction probability sum to 1.

Definition 3.6. For any mutant process μ, any vertex $u \in V(G(\mu))$, and any $t \geq 0$, we define $i_{\mathsf{m}}(\mu, u, t)$ to be the measure of the set $\{t' \leq t \mid u \in \mu_{t'}\}$. Similarly, we define $i_{\mathsf{n}}(\mu, u, t)$ to be the measure of the set $\{t' \leq t \mid u \notin \mu_{t'}\}$. □

The subscript "m" stands for "mutant" since $i_{\mathsf{m}}(\mu, u, t)$ is the amount of time that u is a mutant in μ, up until time t. Similarly, the subscript "n" stands for non-mutant. The random variables $i_{\mathsf{m}}(\mu, u, t)$ and $i_{\mathsf{n}}(\mu, u, t)$ are determined by $\mathcal{F}_t(\mu)$. Also, $i_{\mathsf{m}}(\mu, u, t) + i_{\mathsf{n}}(\mu, u, t) = t$.

3.3. The star-clock process

Consider a mutant process μ. We wish to be able to discuss events such as the event that a vertex u does not spawn a mutant until it has been a mutant for time t. In order to express such events in a clean way, making all conditioning explicit, we define additional stochastic processes.

For each edge $e = (u, v)$ of G we define four further Poisson processes — Poisson processes M_e^* and $\overline{M}_e^*$ each with parameter $r/d^+(u)$ and Poisson processes N_e^* and $\overline{N}_e^*$ each with parameter $1/d^+(u)$. We refer to these processes as *star-clocks*. We identify sources and targets of star-clocks in the same way that we did for clocks. For example, the star-clock $M_{(u,v)}^*$ has source u and target v.

We use $\mathcal{C}_{\text{mut}}^*(G)$ to denote the set $\mathcal{C}_{\text{mut}}^*(G) = \bigcup_{e \in E(G)} \{M_e^*, \overline{N}_e^*\}$. We use $\mathcal{C}_{\text{nmut}}^*(G)$ to denote the set $\mathcal{C}_{\text{nmut}}^*(G) = \bigcup_{e \in E(G)} \{N_e^*, \overline{M}_e^*\}$.

The star-clock process $P^*(G)$ is the stochastic process where all star-clocks in $\mathcal{C}_{\text{mut}}^*(G) \cup \mathcal{C}_{\text{nmut}}^*(G)$ evolve simultaneously and independently, starting at time 0.

3.4. A coupled process

Given a mutant process μ let $G = G(\mu)$. We will now define a stochastic process $\Psi(\mu)$ which is a coupling of μ (which includes the clock process $P(G)$) with the newly-defined star-clock process $P^*(G)$. Intuitively, the idea of the coupling is that each clock $M_{(u,v)}$ in $P(G)$ will evolve following $M_{(u,v)}^*$ when u is a mutant and following $\overline{M}_{(u,v)}^*$ when u is a non-mutant. Similarly, $N_{(u,v)}$ will evolve following $N_{(u,v)}^*$ when u is a non-mutant and $\overline{N}_{(u,v)}^*$ when u is a mutant. In the coupling, we pause the star-clocks in $\mathcal{C}_{\text{mut}}^*(G) \cup \mathcal{C}_{\text{nmut}}^*(G)$ while they are not being used to drive clocks in $\mathcal{C}(G)$, so that, e.g., the "local time" of a clock $M_{(u,v)}^*$ at global time t is $i_{\mathsf{m}}(\mu, u, t)$.

We will be able to deduce both $\mathcal{F}_t(\mu)$ and $\mathcal{F}_t(P^*(G))$ from the filtration $\mathcal{F}_T(\Psi(\mu))$ of the coupled process at an appropriate stopping time T — the details are given below. The fact that the coupling is valid (which we will show below) will ensure that both of the marginal processes, μ and $P^*(G)$, evolve according to their correct distributions.

To construct the coupling we start with a copy of the star-clock process $P^*(G)$ and with the initial state μ_0 of the mutant process μ. We define $\tau_0 = 0$ (so we have implicitly defined $\mathcal{F}_{\tau_0}(\mu)$).

Suppose that, for some non-negative integer j, we have defined $\mathcal{F}_{\tau_j}(\mu)$. Given this and the evolution of the star-clock process $P^*(G)$, we will show how to define τ_{j+1} and $\mathcal{F}_{\tau_{j+1}}(P(G))$ which determine $\mathcal{F}_{\tau_{j+1}}(\mu)$. To do this, let t_j be the minimum $t > 0$ such that one of the following occurs.

— For some $u \in \mu_{\tau_j}$, a star-clock in $\mathcal{C}_{\text{mut}}^*(G)$ with source u triggers at time $i_{\mathsf{m}}(\mu, u, \tau_j) + t$, or
— for some $u \notin \mu_{\tau_j}$, a star-clock in $\mathcal{C}_{\text{nmut}}^*(G)$ with source u triggers at time $i_{\mathsf{n}}(\mu, u, \tau_j) + t$.

We define $\tau_{j+1} = \tau_j + t_j$. No clocks in $\mathcal{C}(G)$ trigger in the interval (τ_j, τ_{j+1}). We now determine which clock from $\mathcal{C}(G)$ triggers at time τ_{j+1} by reconsidering each case.

— If $u \in \mu_{\tau_j}$ and $M_{(u,v)}^*$ triggers at time $i_{\mathsf{m}}(\mu, u, \tau_j) + t_j$ then $M_{(u,v)}$ triggers at time τ_{j+1}.
— If $u \in \mu_{\tau_j}$ and $\overline{N}_{(u,v)}^*$ triggers at time $i_{\mathsf{m}}(\mu, u, \tau_j) + t_j$ then $N_{(u,v)}$ triggers at time τ_{j+1}.
— If $u \notin \mu_{\tau_j}$, and $N_{(u,v)}^*$ triggers at time $i_{\mathsf{n}}(\mu, u, \tau_j) + t_j$ then $N_{(u,v)}$ triggers at time τ_{j+1}.
— If $u \notin \mu_{\tau_j}$ and $\overline{M}_{(u,v)}^*$ triggers at time $i_{\mathsf{n}}(\mu, u, \tau_j) + t_j$ then $M_{(u,v)}$ triggers at time τ_{j+1}.

This fully defines $\mathcal{F}_{\tau_{j+1}}(P(G))$ and hence $\mathcal{F}_{\tau_{j+1}}(\mu)$. So we have fully defined the coupling and therefore the process $\Psi(\mu)$.

Before showing that the coupling is valid, it will be helpful to state exactly what information is contained in $\mathcal{F}_t(\Psi(\mu))$. Certainly this includes $\mathcal{F}_t(\mu)$ which itself includes $\mathcal{F}_t(P(G))$. Also, $\mathcal{F}_t(P(G))$ defines a non-negative integer j so that $t \in [\tau_j, \tau_{j+1})$. We will use j to state the information that $\mathcal{F}_t(\Psi(\mu))$ contains about the evolution of $P^*(G)$.

— For each star-clock $C \in \mathcal{C}^*_{\text{mut}}(G)$ with source $u \in \mu_{\tau_j}$, $\mathcal{F}_t(\Psi(\mu))$ includes a list of the times in $[0, i_{\mathsf{m}}(\mu, u, \tau_j) + t - \tau_j]$ when C triggers.
— For each star-clock $C \in \mathcal{C}^*_{\text{mut}}(G)$ with source $u \notin \mu_{\tau_j}$, $\mathcal{F}_t(\Psi(\mu))$ includes a list of the times in $[0, i_{\mathsf{m}}(\mu, u, \tau_j)]$ when C triggers.
— For each star-clock $C \in \mathcal{C}^*_{\text{nmut}}(G)$ with source $u \notin \mu_{\tau_j}$, $\mathcal{F}_t(\Psi(\mu))$ includes a list of the times in $[0, i_{\mathsf{n}}(\mu, u, \tau_j) + t - \tau_j]$ when C triggers.
— For each star-clock $C \in \mathcal{C}^*_{\text{nmut}}(G)$ with source $u \in \mu_{\tau_j}$, $\mathcal{F}_t(\Psi(\mu))$ includes a list of the times in $[0, i_{\mathsf{n}}(\mu, u, \tau_j)]$ when C triggers.

To show that the coupling is valid we must show that both of the marginal processes, μ and $P^*(G)$, evolve according to their correct distributions. The fact that $P^*(G)$ does so is by construction. To show that μ does so, it suffices to prove that for all $j \in \mathbb{Z}_{\geq 0}$ and all possible values f_j of $\mathcal{F}_{\tau_j}(\mu)$, the distribution of $\mathcal{F}_{\tau_{j+1}}(\mu)$ conditioned on $\mathcal{F}_{\tau_j}(\mu) = f_j$ is correct. Note that the only information contained in $\mathcal{F}_{\tau_{j+1}}(\mu)$ but not $\mathcal{F}_{\tau_j}(\mu)$ is the value of τ_{j+1} and the identity of the clock in $\mathcal{C}(G)$ that triggers at time τ_{j+1}.

Let f'_j be an arbitrary possible value of $\mathcal{F}_{\tau_j}(\Psi(\mu))$ consistent with the event $\mathcal{F}_{\tau_j}(\mu) = f_j$, in the sense that the intersection of the events $\mathcal{F}_{\tau_j}(\Psi(\mu)) = f'_j$ and $\mathcal{F}_{\tau_j}(\mu) = f_j$ is non-empty. Recall from the definition of $\Psi(\mu)$ that, conditioned on $\mathcal{F}_{\tau_j}(\Psi(\mu)) = f_j$, $\mathcal{F}_{\tau_{j+1}}(\mu)$ depends only on particular star-clocks in particular intervals, as follows.

— For each $u \in \mu_{\tau_j}$, it depends on the evolution of each star-clock in $\mathcal{C}^*_{\text{mut}}(G)$ with source u only during the interval $(i_{\mathsf{m}}(\mu, u, \tau_j), \infty)$. It does not depend on the evolution of star-clocks in $\mathcal{C}^*_{\text{nmut}}(G)$ with source u.

— For each $u \notin \mu_{\tau_j}$, it depends on the evolution of each star-clock in $\mathcal{C}^*_{\text{nmut}}(G)$ with source u only during the interval $(i_{\mathsf{n}}(\mu, u, \tau_j), \infty)$. It does not depend on the evolution of star-clocks in $\mathcal{C}^*_{\text{mut}}(G)$ with source u.

For each star-clock, these intervals are disjoint from the intervals exposed in f'_j, and the start of each interval is determined by f'_j. Moreover, in the interval $(\tau_j, \tau_{j+1}]$, each clock in $\mathcal{C}(G)$ is triggered by a unique clock in $\mathcal{C}^*_{\text{mut}}(G) \cup \mathcal{C}^*_{\text{nmut}}(G)$ with the same rate. Thus all clocks in $\mathcal{C}(G)$ trigger with the correct rates in this period and they are independent of each other (since all of the star-clocks in $P^*(G)$ evolve independently). We conclude that $\mathcal{F}_{\tau_{j+1}}(P(G))$, and hence $\mathcal{F}_{\tau_{j+1}}(\mu)$, has the appropriate distribution. The coupling is therefore valid.

By construction, we have the following observation.

Observation 3.7. Let μ be a mutant process and consider $\Psi(\mu)$. Let (u, v) be an edge of $G(\mu)$. Given $t > 0$, let j be the maximum integer such that $\tau_j < t$. Then the following are true.

— $M_{(u,v)}$ triggers at time t if and only if either $u \in \mu_{\tau_j}$ and $M^*_{(u,v)}$ triggers at time $i_{\mathsf{m}}(\mu, u, t)$ or $u \notin \mu_{\tau_j}$ and $\overline{M}^*_{(u,v)}$ triggers at time $i_{\mathsf{n}}(\mu, u, t)$.
— $N_{(u,v)}$ triggers at time t if and only if either $u \notin \mu_{\tau_j}$ and $N^*_{(u,v)}$ triggers at time $i_{\mathsf{n}}(\mu, u, t)$ or $u \in \mu_{\tau_j}$ and $\overline{N}^*_{(u,v)}$ triggers at time $i_{\mathsf{m}}(\mu, u, t)$.

4. AN UPPER BOUND ON THE FIXATION PROBABILITY OF SUPERSTARS

Recall the definition of a (k, ℓ, m)-superstar from Section 1.1.2. We use $n = \ell(k+m)+1$ to denote the number of vertices of a (k, ℓ, m)-superstar.

Given any $i \in [\ell]$, we say that $v_{i,1}v_{i,2}\dots v_{i,k}$ is the *path associated with* the reservoir R_i. We will often consider the case that the initial mutant x_0 is in a reservoir. When it is possible, we simplify the notation by dropping the index i of the reservoir. Thus, we write R for the reservoir containing x_0 and we write $v_1 \dots v_k$ for the path associated with R. So if $R = R_i$ then for each $j \in [k]$, we write v_j as a synonym for $v_{i,j}$. The main result of this section is the following upper bound on the fixation probability of the superstar.

THEOREM 4.1. *Let $r > 1$. Then there exists a constant $c_r > 0$ (depending on r) such that the following holds for all positive integers k, ℓ and m. Choose x_0 uniformly at random from $V(\mathcal{S}_{k,\ell,m})$. Let X be the Moran process (with fitness r) with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then the probability that X goes extinct is at least $1/(c_r(n\log n)^{1/3})$.*

4.1. Proof Sketch

In this Section, we give an informal sketch of the proof of Theorem 4.1. The presentation of the proof itself does not depend upon the sketch so the reader may prefer to skip directly to the proof. In all of our proof sketches, we use the word "likely" to mean "sufficiently likely". We leave the details of "how likely" to the actual proofs.

If m is small relative to k (in particular, if $m < k(n\log n)^{1/3}$) then the initial mutant x_0 is likely to be placed in a path, rather than in a reservoir. If this happens, then it is likely to go extinct. This easy case is dealt with in Lemma 4.2 and corresponds to Case 2 in the proof of Theorem 4.1. (Case 1 is the trivial case where $n < n_0$.)

Another easy case arises if ℓ is sufficiently small relative to n (in particular, if $\ell = O((n\log n)^{1/3})$). This case is dealt with in Lemma 4.3 and corresponds to Case 3 in the proof of Theorem 4.1. In this case, even when x_0 is placed in a reservoir R, it is still likely that x_0 dies before v_2 ever becomes a mutant. This is because it takes roughly $\Theta(m)$ time for the mutation to spread from v_1 to v_2 since a mutant at v_1 has only probability $\Theta(1/m)$ of spawning a mutant before it dies. On the other hand, since ℓ is small, x_0 is sufficiently likely to die in $\Theta(m)$ time. For details, see the proof of Lemma 4.3.

The remaining case, Case 4 in the proof of Theorem 4.1, is deemed the "difficult regime" and is dealt with in Section 4.4. In this case, it is easy to show that $\ell = \Omega(\kappa\log n)$ and $m = \Omega(\kappa)$ where $\kappa = \max\{3k, 70r^4\log n\}$.

It is likely that the initial mutant x_0 is placed in a reservoir R, and the key lemma, showing that it is sufficiently likely to go extinct, is Lemma 4.5.

At a very high level, the argument proceeds as follows. Suppose that v^* does not spawn a mutant before x_0 dies. Then it is very easy to see that, after x_0 dies, the path of reservoir R is likely to go extinct quickly.

Thus, the crux of the argument is to show that x_0 is likely to die before v^* spawns a mutant. Each time v^* becomes a mutant it has an $O(1/\ell)$ chance of spawning a mutant before dying, so roughly our goal is to show that x_0 is sufficiently likely to die before v^* becomes a mutant $\Omega(\ell)$ times.

Very roughly, our high-level approach is to partition time into intervals of length $\kappa = O(m)$. In each block of $O(m/\kappa)$ such intervals, v_2 is likely to become a mutant $O(1)$ times. Each time this happens, it is likely that R's path will again fill with non-mutants within $O(\kappa)$ time, so it is likely that v_k is a mutant for at most $O(\kappa)$ time during the block and it is likely that v^* becomes a mutant at most $O(\kappa)$ times during the block. Combining $O(\ell/\kappa)$ blocks, it is likely that v^* becomes a mutant at most $O(\ell)$ times by time $\ell m/\kappa$. Since $N_{(v^*,x_0)}$ has rate $1/(\ell m)$, it is also likely that x_0 dies by time $\ell m/\kappa$.

In more detail, the proof of Lemma 4.5 shows that x_0 dies before v^* spawns a mutant as long as certain events called $\mathcal{P}_1$–$\mathcal{P}_5$ occur. These events are defined in the statement

of Lemma 4.11. They formalise the high-level approach that we have just described. It is important that most of these events are defined in terms of clock-triggers so that we can get good upper bounds on the probability that they fail and thus prove (in Lemma 4.11) that they are likely to occur simultaneously.

The proof of Lemma 4.5 tracks a quantity $\sigma(t)$ which is the number of times that v_k (the end of the path of the reservoir containing x_0) spawns a mutant onto the centre vertex v^* by time t. The proof uses $\mathcal{P}_1$–$\mathcal{P}_5$ to show that $\sigma(t)$ stays $O(\ell)$ up to a fixed time $t_{x_0} = O(\ell m/\kappa)$. As we noted, the analysis divides the period up to time t_{x_0} into intervals of size κ. Event $\mathcal{P}_5$ ensures that during most such intervals, non-mutant clocks with target v_1 and mutant clocks with targets v_1 and v_2 behave appropriately so that, if x_0 is the only mutant in R during the interval, then v_2 does not become a mutant during the interval. The fact that x_0 is indeed the only mutant in R follows from event $\mathcal{P}_1$ which ensures that v^* does not spawn a mutant while $\sigma(t)$ is small. Then since v_2 does not become a mutant during the interval, event $\mathcal{P}_3$ ensures that the clocks along the path trigger in such a way that (unless v_1 or v^* spawn a mutant) the only mutants remaining at the end of the interval are in $\{x_0, v_1\}$. This ensures that $\sigma(t)$ stays small through another interval. Event $\mathcal{P}_5$ only ensures the above during "most such intervals" but event $\mathcal{P}_4$ ensures that the mutant clock with source v_k does not trigger too often, so the remaining intervals are not too problematic. Thus, events $\mathcal{P}_1$, $\mathcal{P}_3$, $\mathcal{P}_4$ and $\mathcal{P}_5$, taken together, ensure that $\sigma(t_{x_0})$ is $O(\ell)$.

Given that $\sigma(t_{x_0})$ is $O(\ell)$, it is easy to show that the initial mutant goes extinct during the next two intervals (beyond time t_{x_0}). Event $\mathcal{P}_1$ ensures that v^* doesn't spawn any mutants. Event $\mathcal{P}_2$ ensures that the initial mutant x_0 has already died by time t_{x_0}. Finally, event $\mathcal{P}_3$ ensures that any remaining mutants die in the path during the next two intervals.

The difficult part of the proof is defining events $\mathcal{P}_1$–$\mathcal{P}_5$ in such a way that we can show (in Lemma 4.11) that they are likely to occur simultaneously. It turns out (Lemma 4.15, Corollary 4.17 and Lemma 4.18) that events $\mathcal{P}_3$–$\mathcal{P}_5$ are so unlikely to fail that we bound this probability with a simple union bound, avoiding any complicating conditioning. (Of course, for this it was necessary to express these events in terms of clocks rather than in terms of the underlying Moran process.) In order to simplify the presentation, we deal with $\mathcal{P}_1$ and $\mathcal{P}_2$ together, in Lemma 4.14. Roughly, they correspond to the event that, as long as $\sigma(t) = O(\ell)$ then v^* does not spawn a mutant at time t and, for $t = t_{x_0}$, x_0 dies by time t. This event is implied by the conjunction of three further events.

— $\mathcal{E}_1$ is the event that no star-clock $M^*_{(v^*,v)}$ (for any v) triggers in $[0, 1/r]$.

— $\mathcal{E}_2$ is the event that the star-clock $N^*_{(v^*,x_0)}$ triggers in $[0, t_{x_0} - 1]$.

— $\mathcal{E}_3$ corresponds informally to the event that v^* is a mutant for a period of time shorter than $1/r$ during the first $O(\ell)$ times that it becomes a mutant (though the formal definition is expressed in terms of clocks, and is a little more complicated). Note the intention, though, which is to ensure that v^* is a mutant for a period of time shorter than $1/r$, which makes $\mathcal{E}_1$ relevant.

Lemma 4.13 shows that $\mathcal{E}_3$ is very likely to hold. In the proof of Lemma 4.14, it is observed that $\mathcal{E}_1$ and $\mathcal{E}_2$ are independent (by the definition of the star-clocks) and that $\mathbb{P}(\mathcal{E}_1) = 1/e$. The proof demonstrates that $\mathcal{E}_2$ is sufficiently likely, giving the desired bound.

4.2. Glossary

clears before spawning ..... Definition 4.8, Page 22

clears within I_i ..... Definition 4.10, Page 23
c_r ..... Theorem 4.1, Page 19
$I_i = (i\kappa, (i+1)\kappa]$ ..... Definition 4.7, Page 22
$i_{\mathsf{m}}(\mu, u, t)$ ..... Definition 3.6, Page 16
$i_{\mathsf{n}}(\mu, u, t)$ ..... Definition 3.6, Page 16
$K = 70$ ..... Definition 4.4, Page 22
$\kappa = \max\{3k, Kr^4 \log n\}$ ..... Definition 4.4, Page 22
$(\mathcal{P}_1), (\mathcal{P}_2), (\mathcal{P}_3), (\mathcal{P}_4), (\mathcal{P}_5)$ ..... Lemma 4.11, Page 23
protected ..... Definition 4.9, Page 23
$\Psi(X)$ ..... Section 3.4, Page 17
$\sigma(t)$ ..... Definition 4.7, Page 22
$\mathcal{S}_{k,\ell,m}$ ..... Section 1.1.2, Page 4
$t_{\mathsf{max}} = 2\ell m$ ..... Definition 4.7, Page 22
T^h_{m} ..... Definition 4.12, Page 23
T^h_{n} ..... Definition 4.12, Page 23
$t_{x_0} = \ell m/(Kr^4\kappa)$ ..... Definition 4.7, Page 22
$Y_h = T^h_{\mathsf{n}} - T^h_{\mathsf{m}}$ ..... Definition 4.12, Page 23

4.3. The easy regimes

LEMMA 4.2. *Choose x_0 uniformly at random from $V(\mathcal{S}_{k,\ell,m})$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. The extinction probability of X is at least $k/(2r(m+k))$.*

PROOF. We have

$$\mathbb{P}(x_0 \notin R_1 \cup \cdots \cup R_\ell) = 1 - \frac{\ell m}{\ell(m+k)+1} \geq 1 - \frac{m}{m+k} = \frac{k}{m+k}.$$

Moreover, if $x_0 \notin R_1 \cup \cdots \cup R_\ell$, then x_0 has an in-neighbour of out-degree 1 so, with probability at least $1/(1+r) \geq 1/(2r)$, x_0 dies before spawning a mutant. The result therefore follows. □

LEMMA 4.3. *Suppose $m \geq 12r$ and $x_0 \in R_1 \cup \cdots \cup R_\ell$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. The extinction probability of X is at least $1/(26r^2\ell)$.*

PROOF. Let R be the reservoir containing x_0, and let $v_1 \ldots v_k$ be the path associated with R. Let $\xi = \lfloor m/(2r) \rfloor$, $t^* = m/(4r^2)$ and $J = [0, t^*]$. For all $t \geq 0$, let $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{E}^3_t$ be events defined as follows.

$\mathcal{E}^1$: $N_{(v^*,x_0)}$ triggers in J.
$\mathcal{E}^2$: $M_{(x_0,v_1)}$ triggers at most ξ times in J.
$\mathcal{E}^3_t$: $\min\{t' > t \mid \text{for some } v \neq x_0, N_{(v,v_1)} \text{ triggers at } t'\}$
$\qquad < \min\{t' > t \mid M_{(v_1,v_2)} \text{ triggers at } t'\}$.

Finally, let $T^i_{v_1}$ be the i'th time at which the clock $M_{(x_0,v_1)}$ triggers and define $\mathcal{E}^3 = \bigcap_{i=1}^{\xi} \mathcal{E}^3_{T^i_{v_1}}$.

Suppose that events $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{E}^3$ occur. We will show that X goes extinct. Let ξ' be the number of times that v_1 becomes a mutant in J. By $\mathcal{E}^2$, $\xi' \leq \xi$. By $\mathcal{E}^3$, for each of the first ξ' times that v_1 becomes a mutant, it dies before spawning a mutant. Thus, for all $t \in J$, $X_t \subseteq \{x_0, v_1\}$. Also, by $\mathcal{E}^1$, x_0 dies in J. As soon as x_0 dies, and v_1 dies for the (ξ')'th time, X is extinct.

We bound $\mathbb{P}(\mathcal{E}^1 \cap \mathcal{E}^2 \cap \mathcal{E}^3)$ below. Since $N_{(v^*,x_0)}$ has rate $1/(\ell m)$, we have

$$\mathbb{P}(\mathcal{E}^1) = 1 - e^{-t^*/(\ell m)} = 1 - e^{-m/(4r^2\ell m)} \geq 1 - \left(1 - \frac{1}{8r^2\ell}\right) = \frac{1}{8r^2\ell}. \tag{7}$$

Here the inequality follows by (4). Moreover, since $M_{(x_0,v_1)}$ has rate r, by Corollary 2.2 we have

$$\mathbb{P}(\mathcal{E}^2) \geq 1 - e^{-m/(12r)} \geq 1 - e^{-1}. \tag{8}$$

For any $t \in J$, let f be a possible value of $\mathcal{F}_t(X)$. Let Φ be the random variable containing the list of times in J at which $N_{(v^*,x_0)}$ and $M_{(x_0,v_1)}$ trigger. Let φ be a possible value of Φ which is consistent with the events $\mathcal{F}_t(X) = f$ and $\mathcal{E}^1 \cap \mathcal{E}^2$. Note that φ determines $\mathcal{E}^1 \cap \mathcal{E}^2$. By memorylessness and independence of clocks in $\mathcal{C}(\mathcal{S}_{k,\ell,m})$, we have

$$\mathbb{P}\left(\mathcal{E}^3_t \mid \mathcal{F}_t(X) = f, \Phi = \varphi\right) = \frac{m-1}{m-1+r} = 1 - \frac{r}{m-1+r} \geq 1 - \frac{r}{m}.$$

Thus for all $i \in [\xi]$, $\mathbb{P}\big(\mathcal{E}^3_{T^i_{v_1}} \mid \mathcal{E}^1 \cap \mathcal{E}^2\big) \geq 1 - r/m$. It follows by a union bound that

$$\mathbb{P}\left(\mathcal{E}^3 \mid \mathcal{E}^1 \cap \mathcal{E}^2\right) \geq 1 - \xi\left(\frac{r}{m}\right) \geq \frac{1}{2}. \tag{9}$$

Since $\mathcal{E}^1$ and $\mathcal{E}^2$ depend entirely on distinct clocks in $\mathcal{C}(\mathcal{S}_{k,\ell,m})$ in fixed intervals, the two events are independent. Thus by (7)–(9), we have

$$\mathbb{P}\left(\mathcal{E}^1 \cap \mathcal{E}^2 \cap \mathcal{E}^3\right) = \mathbb{P}\left(\mathcal{E}^1\right)\,\mathbb{P}\left(\mathcal{E}^2\right)\,\mathbb{P}\left(\mathcal{E}^3 \mid \mathcal{E}^1 \cap \mathcal{E}^2\right) \geq \left(\frac{1}{8r^2\ell}\right)\left(1 - \frac{1}{e}\right)\frac{1}{2} \geq \frac{1}{26r^2\ell},$$

and the result follows. □

4.4. The difficult regime

Definition 4.4. Let $K = 70$ and $\kappa = \max\{3k, Kr^4 \log n\}$. □

LEMMA 4.5. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose that $\ell \geq Kr^4\kappa \log n$, $m \geq 6r^2\kappa$ and $n \geq n_0$. Fix $x_0 \in R_1 \cup \cdots \cup R_\ell$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then the extinction probability of X is at least $1/(7Kr^4\kappa)$.*

The following corollary, which applies to the regime in which $\kappa = 3k$, is immediate.

COROLLARY 4.6. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose that $k \geq (K/3)r^4 \log n$, $\ell \geq 3Kr^4 k \log n$, $m \geq 18r^2 k$ and $n \geq n_0$. Fix $x_0 \in R_1 \cup \cdots \cup R_\ell$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then the extinction probability of X is at least $1/(21Kr^4k)$.*

The crux of our proof of Lemma 4.5 is Lemma 4.11. In order to state this lemma, we require the following additional definitions.

Definition 4.7. Let $t_{x_0} = \ell m/(Kr^4\kappa)$, and $t_{\mathsf{max}} = 2\ell m$. For all $i \in \mathbb{Z}_{\geq 0}$, let $I_i = (i\kappa, (i+1)\kappa]$. For all $t \in [0, t_{\mathsf{max}}]$, let

$$\sigma(t) = \left|\{t' \leq t \mid v_k \text{ spawns a mutant onto } v^* \text{ at time } t'\}\right|. \quad \square$$

The reason that we give t_{x_0} its name is that we will be most concerned with the case in which x_0 dies in the interval $[0, t_{x_0}]$.

Definition 4.8. Let $I \subseteq [0, \infty)$ be an interval, let $R \in \{R_1, \ldots, R_\ell\}$, suppose $x_0 \in R$, and let $v_1 \ldots v_k$ be the path associated with R. We say that v_1 *clears before spawning a mutant within* I if at least one of the following statements holds:

(i) $M_{(v_1,v_2)}$ does not trigger in I, or
(ii) for some $v \neq x_0$, $N_{(v,v_1)}$ triggers in I before $M_{(v_1,v_2)}$ first triggers in I. □

Definition 4.9. Let $i \in \mathbb{Z}_{\geq 0}$, let $R \in \{R_1, \ldots, R_\ell\}$, suppose $x_0 \in R$, and let $v_1 \ldots v_k$ be the path associated with R. We say that v_2 *is protected in* I_i if both of the following properties hold.

(i) v_1 clears before spawning a mutant within I_i.
(ii) For all $t \in I_i$ such that $M_{(x_0,v_1)}$ triggers at time t, v_1 clears before spawning a mutant within $(t, (i+1)\kappa]$. □

In particular, suppose that v_2 is protected in I_i and that x_0 is the only mutant in R for the duration of I_i. Then as we will see in the proof of Lemma 4.5, v_2 does not become a mutant in I_i.

Definition 4.10. Let $i \in \mathbb{Z}_{\geq 0}$, let $R \in \{R_1, \ldots, R_\ell\}$, suppose $x_0 \in R$, and let $v_1 \ldots v_k$ be the path associated with R. We say that $v_1 \ldots v_k$ *clears within* I_i if there exist $v_0 \in R \setminus \{x_0\}$ and times $i\kappa < t_1 < \cdots < t_{k+1} \leq (i+1)\kappa$ satisfying both of the following properties.

(i) For all $j \in [k]$, $N_{(v_{j-1},v_j)}$ triggers at time t_j, and $N_{(v_k,v^*)}$ triggers at time t_{k+1}.
(ii) $M_{(x_0,v_1)}$ does not trigger in the interval $[t_1, t_2]$. □

The purpose of this definition is the following. Suppose that $X_{i\kappa} \subseteq \{x_0, v_1, \ldots, v_k, v^*\}$, that neither v_1 nor v^* spawns a mutant within I_i, and that $v_1 \ldots v_k$ clears within I_i. Then, as we will see in the proof of Lemma 4.5, we will have $X_{(i+1)\kappa} \subseteq \{x_0, v_1\}$.

Our main task will be to prove the following lemma.

LEMMA 4.11. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose $\ell \geq Kr^4\kappa \log n$, $m \geq 6r^2\kappa$ and $n \geq n_0$. Let $R \in \{R_1, \ldots, R_\ell\}$, and let $v_1 \ldots v_k$ be the path associated with R. Fix $x_0 \in R$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then, with probability at least $1/(7Kr^4\kappa)$, the following events occur simultaneously.*

$\mathcal{P}_1$: *$\forall t \leq t_{\mathsf{max}}$, $\sigma(t) \geq \lfloor \ell/(2r) \rfloor + 1$ or v^* does not spawn a mutant at time t.*
$\mathcal{P}_2$: *$\sigma(t_{x_0}) \geq \lfloor \ell/(2r) \rfloor + 1$ or $x_0 \notin X_{t_{x_0}}$.*
$\mathcal{P}_3$: *For all integers i with $0 \leq i \leq t_{x_0}$, $v_1 \ldots v_k$ clears within I_i.*
$\mathcal{P}_4$: *For all integers i with $0 \leq i \leq t_{x_0}$, the clock $M_{(v_k,v^*)}$ triggers at most $\lfloor 2r\kappa \rfloor$ times within I_i.*
$\mathcal{P}_5$: *For all but at most $8r^2 t_{x_0}/m$ integers i with $0 \leq i \leq t_{x_0}/\kappa$, v_2 is protected in I_i.*

Note that the definition of $\mathcal{P}_5$ considers i up to t_{x_0}/κ, because it corresponds to at most t_{x_0}/κ intervals of length κ. The definitions of $\mathcal{P}_3$ and $\mathcal{P}_4$ consider larger values of i. In fact, it is only necessary to take i up to $t_{x_0}/\kappa + 2$ in $\mathcal{P}_3$ and $\mathcal{P}_4$ but we state the lemma as we did to avoid clutter. As a first step towards proving Lemma 4.11, we prove Lemmas 4.13 and 4.14 which give a lower bound on the probability that $\mathcal{P}_1$ and $\mathcal{P}_2$ hold.

Definition 4.12. Let $R \in \{R_1, \ldots, R_\ell\}$, let $v_1 \ldots v_k$ be the path associated with R, and suppose $x_0 \in R$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. We give two mutual recurrences to define stopping times T^h_{n} for all $h \in \mathbb{Z}_{\geq 0}$ and T^h_{m} for all $h \in \mathbb{Z}_{\geq 1}$. The subscript "$\mathsf{n}$" stands for "non-mutant" and the subscript "m" stands for

"mutant".

$$T_{\mathsf{n}}^h = \begin{cases} 0, & \text{if } h = 0, \\ \min\left\{ t > T_{\mathsf{m}}^h \;\middle|\; \begin{array}{l} t = t_{\mathsf{max}} \text{ or some clock } N_{(v,v^*)} \\ \text{with } v \neq v_k \text{ triggers at } t \end{array} \right\}, & \text{otherwise.} \end{cases}$$

$$T_{\mathsf{m}}^h = \min\{t > T_{\mathsf{n}}^{h-1} \mid t = t_{\mathsf{max}} \text{ or } v_k \text{ spawns a mutant onto } v^* \text{ at } t\}.$$

Finally, for all $h \in \mathbb{Z}_{\geq 1}$, let $Y_h = T_{\mathsf{n}}^h - T_{\mathsf{m}}^h$. □

LEMMA 4.13. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose $\ell \geq Kr^4\kappa \log n$ and $n \geq n_0$. Let $R \in \{R_1, \ldots, R_\ell\}$ and let $v_1 \ldots v_k$ be the path associated with R. Let $x_0 \in R$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then*

$$\mathbb{P}\left(\sum_{i=1}^{\lfloor \ell/(2r) \rfloor} Y_i < \frac{1}{r} \right) \geq 1 - \frac{1}{n^2}.$$

PROOF. Let n_0 be an integer which is sufficiently large with respect to r. We claim that $Y_1, \ldots, Y_{\lfloor \ell/(2r) \rfloor}$ are stochastically dominated above by $\lfloor \ell/(2r) \rfloor$ independent exponential variables, each with parameter $\ell - 1$. To see this claim, fix $i \in \mathbb{Z}_{\geq 1}$, $t \geq 0$, and $y_1, \ldots, y_{i-1}, t_{\mathsf{m}}^i > 0$. Let f_i be a possible value of $\mathcal{F}_{t_{\mathsf{m}}^i}(X)$. Suppose that the events $Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}$, $T_{\mathsf{m}}^i = t_{\mathsf{m}}^i$ and $\mathcal{F}_{t_{\mathsf{m}}^i}(X) = f_i$ are consistent, and note that in this case $\mathcal{F}_{t_{\mathsf{m}}^i}(X) = f_i$ determines the other events.

If $t \geq 0$ satisfies $t_{\mathsf{m}}^i + t \geq t_{\mathsf{max}}$, it follows that $t_{\mathsf{m}}^i + t \geq T_{\mathsf{n}}^i$ and hence if $\mathcal{F}_{t_{\mathsf{m}}^i}(X) = f_i$ then $Y_i = T_{\mathsf{n}}^i - t_{\mathsf{m}}^i \leq t$. Hence, for all such t,

$$\mathbb{P}\big(Y_i \leq t \mid \mathcal{F}_{t_{\mathsf{m}}^i}(X) = f_i\big) = 1 \geq 1 - e^{-(\ell-1)t}. \tag{10}$$

Suppose instead that $t_{\mathsf{m}}^i + t < t_{\mathsf{max}}$. If $\mathcal{F}_{t_{\mathsf{m}}^i}(X) = f_i$ then $Y_i \leq t$ if and only if some clock $N_{(v,v^*)}$ with $v \neq v_k$ triggers in the interval $(t_{\mathsf{m}}^i, t_{\mathsf{m}}^i + t]$. These clocks have total rate $\ell - 1$, and so by memorylessness we have

$$\mathbb{P}\big(Y_i \leq t \mid \mathcal{F}_{t_{\mathsf{m}}^i}(X) = f_i\big) = 1 - e^{-(\ell-1)t}. \tag{11}$$

Since (10) and (11) apply to every value of f_i consistent with $Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}$ and $T_{\mathsf{m}}^i = t_{\mathsf{m}}^i$, it follows that

$$\mathbb{P}\big(Y_i \leq t \mid Y_1 = y_1, \ldots, Y_{i-1} = y_{i-1}\big) \geq 1 - e^{-(\ell-1)t}.$$

Thus $\sum_{i=1}^{\lfloor \ell/(2r) \rfloor} Y_i$ is stochastically dominated above by a sum S of $\lfloor \ell/(2r) \rfloor$ i.i.d. exponential variables with parameter $\ell - 1$. Corollary 2.4 applies since

$$\frac{1}{r} \geq \frac{3\lfloor \ell/(2r) \rfloor}{2(\ell - 1)},$$

so we have

$$\mathbb{P}\left(\sum_{i=1}^{\lfloor \ell/(2r) \rfloor} Y_i < \frac{1}{r} \right) \geq \mathbb{P}\left(S < \frac{1}{r} \right) \geq 1 - e^{-(\ell-1)/(16r)} \geq 1 - \frac{1}{n^2},$$

as required. □

We are now in a position to prove that $\mathcal{P}_1$ and $\mathcal{P}_2$ occur with reasonable probability.

LEMMA 4.14. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose $\ell \geq Kr^4\kappa \log n$, $m \geq 2$ and $n \geq n_0$. Let $R \in \{R_1, \ldots, R_\ell\}$ and let $v_1 \ldots v_k$ be the path associated with R. Let $x_0 \in R$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then $\mathbb{P}(\mathcal{P}_1 \cap \mathcal{P}_2) \geq 1/(6Kr^4\kappa)$.*

PROOF. Let n_0 be an integer which is sufficiently large with respect to r. Consider the process $\Psi(X)$. Define the following three events.

$\mathcal{E}_1$: no star-clock $M^*_{(v^*,v)}$ (for any v) triggers in $[0, 1/r]$.
$\mathcal{E}_2$: the star-clock $N^*_{(v^*,x_0)}$ triggers in $[0, t_{x_0} - 1]$.
$\mathcal{E}_3$: $\sum_{i=1}^{\lfloor \ell/(2r) \rfloor} Y_i < 1/r$.

We will prove that $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3) \geq 1/(6Kr^4\kappa)$, and that if $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ occurs then so does $\mathcal{P}_1$ and $\mathcal{P}_2$.

We first bound $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3)$ below. The sum of the parameters of the star-clocks in $\{M^*_{(v^*,v)}\}$ is r, so $\mathbb{P}(\mathcal{E}_1) = e^{-1}$. We have $t_{x_0} = \ell m/(Kr^4\kappa) \geq m \log n \geq 2 \log n_0$ by hypothesis so, by choice of n_0, we may assume $t_{x_0} \geq 25$. The parameter of the star-clock $N^*_{(v^*,x_0)}$ is $1/(\ell m)$, so using (4) we have

$$\mathbb{P}(\mathcal{E}_2) = 1 - e^{-(t_{x_0}-1)/(\ell m)} \geq 1 - e^{-24t_{x_0}/(25\ell m)} \geq \frac{12t_{x_0}}{25\ell m} = \frac{12}{25Kr^4\kappa}.$$

Note that $\mathcal{E}_1$ and $\mathcal{E}_2$ are independent of each other by the definition of the star-clock process $P^*(\mathcal{S}_{k,\ell,m})$ and the fact that the intervals in the definitions of $\mathcal{E}_1$ and $\mathcal{E}_2$ are fixed: $t_{x_0} = \ell m/(Kr^4\kappa)$ does not depend on the evolution of $\Psi(X)$. So we have $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) \geq 12/(25eKr^4\kappa)$. Finally, by Lemma 4.13 together with the fact that $\kappa \leq n$, it follows that

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3) \geq \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2) - \mathbb{P}(\overline{\mathcal{E}_3}) \geq \frac{12}{25eKr^4\kappa} - \frac{1}{n^2} \geq \frac{1}{6Kr^4\kappa}. \tag{12}$$

We next show that $\mathcal{E}_1$ and $\mathcal{E}_3$ together imply that $\mathcal{P}_1$ occurs. If v^* does not spawn a mutant before time t_{max} then $\mathcal{P}_1$ occurs, so suppose instead that v^* spawns a mutant for the first time at some time $t_{\mathsf{sp}} \leq t_{\mathsf{max}}$. (The "s" subscript in t_{sp} stands for "spawn".) We must show that $\sigma(t_{\mathsf{sp}}) \geq \lfloor \ell/(2r) \rfloor + 1$. This will ensure that $\mathcal{P}_1$ occurs since $\sigma(t)$ is monotonically increasing.

Since v^* spawns no mutants before time t_{sp}, we have $X_t \subseteq \{x_0, v_1, \ldots, v_k, v^*\}$ for all $t < t_{\mathsf{sp}}$, and so (recalling Definition 3.6)

$$i_{\mathsf{m}}(X, v^*, t_{\mathsf{sp}}) \leq \sum_{\substack{i \geq 1, \\ T^i_{\mathsf{m}} < t_{\mathsf{sp}}}} Y_i. \tag{13}$$

Since $\mathcal{E}_3$ occurs, we have

$$\sum_{\substack{1 \leq i \leq \lfloor \ell/(2r) \rfloor, \\ T^i_{\mathsf{m}} < t_{\mathsf{sp}}}} Y_i < \frac{1}{r}. \tag{14}$$

However, since $\mathcal{E}_1$ occurs and v^* spawns a mutant at t_{sp}, we have $i_{\mathsf{m}}(X, v^*, t_{\mathsf{sp}}) \geq 1/r$. Therefore by (13) and (14),

$$\sum_{\substack{i \geq 1, \\ T^i_{\mathsf{m}} < t_{\mathsf{sp}}}} Y_i > \sum_{\substack{1 \leq i \leq \lfloor \ell/(2r) \rfloor, \\ T^i_{\mathsf{m}} < t_{\mathsf{sp}}}} Y_i.$$

Hence $T^{\lfloor \ell/(2r) \rfloor + 1}_{\mathsf{m}} < t_{\mathsf{sp}}$. Thus, $\sigma(t_{\mathsf{sp}}) \geq \lfloor \ell/(2r) \rfloor + 1$, and $\mathcal{P}_1$ occurs.

Finally, we show that $\mathcal{P}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ together imply that $\mathcal{P}_2$ occurs. Suppose $\sigma(t_{x_0}) \leq \lfloor \ell/(2r) \rfloor$. We have $t_{x_0} \leq \ell m \leq t_{\mathsf{max}}$, so by $\mathcal{P}_1$, v^* spawns no mutants in $[0, t_{x_0}]$. Hence as in (13), we have

$$i_{\mathsf{m}}(X, v^*, t_{x_0}) \;\leq\; \sum_{\substack{i \geq 1, \\ T^i_{\mathsf{m}} < t_{x_0}}} Y_i = \sum_{\substack{i \geq 1, \\ T^i_{\mathsf{m}} \leq t_{x_0}}} Y_i.$$

Since $\sigma(t_{x_0}) \leq \lfloor \ell/(2r) \rfloor$, it follows by $\mathcal{E}_3$ that $i_{\mathsf{m}}(X, v^*, t_{x_0}) < 1/r$ and therefore $i_{\mathsf{n}}(X, v^*, t_{x_0}) \geq t_{x_0} - 1/r > t_{x_0} - 1$. Since $\mathcal{E}_2$ occurs, it follows that v^* spawns a non-mutant onto x_0 at some time $t \leq t_{x_0}$. Since v^* spawns no mutants in $[0, t_{x_0}]$, x_0 cannot become a mutant in $(t, t_{x_0}]$, so $x_0 \notin X_{t_{x_0}}$ and $\mathcal{P}_2$ occurs.

Thus $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3$ implies that $\mathcal{P}_1$ and $\mathcal{P}_2$ occur, and so the result follows from (12). □

Lower bounds for the probabilities that properties $\mathcal{P}_3$–$\mathcal{P}_5$ hold follow from Chernoff bounds without too much difficulty.

LEMMA 4.15. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose $\ell \geq Kr^4\kappa \log n$, $m \geq 2$ and $n \geq n_0$. Let $R \in \{R_1, \ldots, R_\ell\}$ and let $v_1 \ldots v_k$ be the path associated with R. Let $x_0 \in R$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then $\mathbb{P}(\mathcal{P}_3) \geq 1 - 1/n$.*

PROOF. Let n_0 be an integer which is sufficiently large with respect to r. Fix $i \in \mathbb{Z}_{\geq 0}$ — we will bound the probability that $v_1 \ldots v_k$ clears within I_i (as in Definition 4.10). Let $v_0 \in R \setminus \{x_0\}$ be arbitrary. For all $h \in \mathbb{Z}_{\geq 0}$, let

$$T_{1,h} = \min\{t \geq i\kappa + h \mid t = i\kappa + h + 1/2 \text{ or } N_{(v_0,v_1)} \text{ triggers at } t\}$$
$$T_{2,h} = \min\left\{ t > T_{1,h} \;\middle|\; t = i\kappa + h + 1 \text{ or } \begin{pmatrix} M_{(x_0,v_1)} \text{ does not trigger in } [T_{1,h}, t) \\ \text{and } N_{(v_1,v_2)} \text{ triggers at } t \end{pmatrix} \right\}.$$

Let $\mathcal{E}_h$ be the event that $T_{1,h} < i\kappa + h + 1/2$ and $T_{2,h} < i\kappa + h + 1$.

The probability that the clock $N_{(v_0,v_1)}$ triggers in $[i\kappa + h, i\kappa + h + 1/2)$ is $1 - e^{-1/2}$. For any $t_1 \in [i\kappa + h, i\kappa + h + 1/2)$ the probability that there is a $t_2 \in (t_1, i\kappa + h + 1)$ such that $N_{(v_1,v_2)}$ triggers at t_2, and $M_{(x_0,v_1)}$ does not trigger in $[t_1, t_2]$ is $(1 - e^{-(r+1)(i\kappa+h+1-t_1)})/(r+1)$. To see this, note that the $1 - e^{-(r+1)(i\kappa+h+1-t_1)}$ factor corresponds to the probability that either $N_{(v_1,v_2)}$ or $M_{(x_0,v_1)}$ triggers in the relevant interval (together, they correspond to a Poisson process with rate $r+1$). The $1/(r+1)$ factor corresponds to the probability that it is actually $N_{(v_1,v_2)}$ rather than $M_{(x_0,v_1)}$ that triggers first. Since the relevant interval has length at least $1/2$, the product of these two factors is at least $(1 - e^{-(r+1)/2})/(r+1)$. So

$$\mathbb{P}(\mathcal{E}_h) \geq \left(1 - e^{-1/2}\right)\left(1 - e^{-(r+1)/2}\right)/\left(r+1\right) \geq (1 - e^{-1/2})(1 - e^{-1})/(r+1) \geq \frac{1}{5(r+1)} \geq \frac{1}{10r}.$$

Moreover, the events $\{\mathcal{E}_h \mid h \in \mathbb{Z}_{\geq 0}\}$ are mutually independent, as they depend only on the behaviour of clocks in $\mathcal{C}(\mathcal{S}_{k,\ell,m})$ in fixed disjoint intervals. Thus

$$\begin{aligned}
\mathbb{P}\big(\mathcal{E}_h \text{ holds for some } 0 \leq h \leq (Kr^4 \log n)/3\big) &\geq 1 - \left(1 - \frac{1}{10r}\right)^{\lfloor (Kr^4 \log n)/3 \rfloor + 1} \\
&\geq 1 - \left(1 - \frac{1}{10r}\right)^{(Kr^4 \log n)/3} \\
&\geq 1 - e^{-(Kr^3 \log n)/30}
\end{aligned}$$

$$\geq 1 - e^{-2\log n} = 1 - \frac{1}{n^2}\,. \tag{15}$$

Now, for all integers j with $3 \leq j \leq k+1$, define

$$T_j = \begin{cases} \min\{t > i\kappa + \kappa/2 \mid N_{(v_2,v_3)} \text{ triggers at } t\} & \text{if } j = 3, \\ \min\{t > T_{j-1} \mid N_{(v_{j-1},v_j)} \text{ triggers at } t\} & \text{if } 4 \leq j \leq k, \\ \min\{t > T_k \mid N_{(v_k,v^*)} \text{ triggers at } t\} & \text{if } j = k+1. \end{cases}$$

Note that if $T_{k+1} < (i+1)\kappa$, then $T_3, \ldots, T_{k+1} \in I_i$. By memorylessness and independence of distinct clocks in $\mathcal{C}(\mathcal{S}_{k,\ell,m})$, the random variables $T_3 - (i\kappa + \kappa/2), T_4 - T_3, \ldots, T_{k+1} - T_k$ are $k-1$ i.i.d. exponential variables with rate 1, and $T_{k+1} - (i\kappa + \kappa/2)$ is their sum. Corollary 2.4 applies because $\kappa/2 \geq 3(k-1)/2$, so

$$\begin{aligned} \mathbb{P}\big(T_{k+1} < (i+1)\kappa\big) &= \mathbb{P}\big(T_{k+1} - (i\kappa + \kappa/2) < \kappa/2\big) \\ &\geq 1 - e^{-\kappa/32} \geq 1 - e^{-(Kr^4 \log n)/32} \geq 1 - \frac{1}{n^2}. \end{aligned} \tag{16}$$

Suppose $\mathcal{E}_h$ holds for some $0 \leq h \leq (Kr^4 \log n)/3$. Note that $T_{2,h} \leq i\kappa + \kappa/2$. Suppose further that $T_{k+1} < (i+1)\kappa$. Then setting $t_1 = T_{1,h}$, $t_2 = T_{2,h}$ and $t_j = T_j$ for all $j \in \{3, \ldots, k+1\}$, we see that $t_1, \ldots, t_{k+1}$ satisfy the requirements of Definition 4.10 and so $v_1 \ldots v_k$ clears within I_i. It therefore follows by (15), (16), and a union bound that

$$\mathbb{P}(v_1 \ldots v_k \text{ clears within } I_i) \geq 1 - \frac{2}{n^2}.$$

Finally, since $t_{x_0} \leq n/4$, we can take a union bound over all integers i with $0 \leq i \leq t_{x_0}$ to deduce that

$$\mathbb{P}(\mathcal{P}_3) \geq 1 - 2t_{x_0} \cdot \frac{2}{n^2} \geq 1 - \frac{n}{2} \cdot \frac{2}{n^2} \geq 1 - \frac{1}{n}\,. \quad \square$$

LEMMA 4.16. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose $n \geq n_0$. Fix $x_0 \in R_1 \cup \cdots \cup R_\ell$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. With probability at least $1 - 1/n^2$, for all integers i with $0 \leq i \leq t_{x_0}$, every clock in $\mathcal{C}(\mathcal{S}_{k,\ell,m})$ triggers at most $\lfloor 2r\kappa \rfloor$ times in I_i.*

PROOF. Let n_0 be an integer which is sufficiently large with respect to r. Fix a given clock $C \in \mathcal{C}(\mathcal{S}_{k,\ell,m})$, fix $i \in \mathbb{Z}_{\geq 0}$ with $i \leq t_{x_0}$, and write $a \leq r$ for the rate of $\mathcal{C}(\mathcal{S}_{k,\ell,m})$. The number of times that C triggers in I_i follows the Poisson distribution with parameter $a\kappa$. Since the number of triggers is an integer and $2r\kappa \geq 2a\kappa$, by Corollary 2.2 we have

$$\begin{aligned} \mathbb{P}(C \text{ triggers at most } \lfloor 2r\kappa \rfloor \text{ times in } I_i) &= \mathbb{P}(C \text{ triggers at most } 2r\kappa \text{ times in } I_i) \\ &\geq \mathbb{P}(C \text{ triggers at most } 2a\kappa \text{ times in } I_i) \\ &\geq 1 - e^{-a\kappa/3} \geq 1 - e^{-(Kr^4 \log n)/3}. \end{aligned}$$

There are at most $2n^2$ clocks in $\mathcal{C}(\mathcal{S}_{k,\ell,m})$ and at most $t_{x_0} + 1 \leq 2n^2$ choices of i. Thus by a union bound, with probability at least $1 - 4n^4 e^{-Kr^4 \log n/3} \geq 1 - 1/n^2$, no single clock in $\mathcal{C}(\mathcal{S}_{k,\ell,m})$ triggers more than $\lfloor 2r\kappa \rfloor$ times in any interval I_i with $0 \leq i \leq t_{x_0}$. $\square$

The following corollary follows immediately from Lemma 4.16. Of course, the probability bound in the corollary can be strengthened to $1 - 1/n^2$, but we state what we will later use.

COROLLARY 4.17. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose $n \geq n_0$. Fix $x_0 \in R_1 \cup \cdots \cup R_\ell$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then $\mathbb{P}(\mathcal{P}_4) \geq 1 - 1/n$.* $\square$

The following lemma gives a lower bound on the probability that $\mathcal{P}_5$ occurs. In this lemma, we require that $m \geq 6r^2\kappa$, rather than $m \geq 2$, which we have so far been assuming.

LEMMA 4.18. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose $\ell \geq Kr^4\kappa \log n$, $m \geq 6r^2\kappa$ and $n \geq n_0$. Let $R \in \{R_1, \ldots, R_\ell\}$ and let $v_1 \ldots v_k$ be the path associated with R. Fix $x_0 \in R$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then $\mathbb{P}(\mathcal{P}_5) \geq 1 - 1/n$.*

PROOF. Let n_0 be an integer which is sufficiently large with respect to r. Fix $i \in \mathbb{Z}_{\geq 0}$. For all $t \in I_i$, define the following events $\mathcal{E}_t^1$ and $\mathcal{E}_i^2$.

$\mathcal{E}_t^1$: $\min\{t' > t \mid \text{for some } v \neq x_0,\ N_{(v,v_1)} \text{ triggers at } t'\}$
$\qquad < \min\{t' > t \mid M_{(v_1,v_2)} \text{ triggers at } t'\}$.
$\mathcal{E}_i^2$: $M_{(x_0,v_1)}$ triggers at most $\lfloor 2r\kappa \rfloor$ times in I_i.

Thus $\mathcal{E}_t^1$ occurs if and only if v_1 clears before spawning a mutant within (t, ∞). Let $T_{i,0} = i\kappa$, and let $T_{i,h}$ be the h'th time in I_i at which the clock $M_{(x_0,v_1)}$ triggers, or $(i+1)\kappa$ if no such time exists. Note that if $\bigcap_{h=0}^{\lfloor 2r\kappa \rfloor} \mathcal{E}_{T_{i,h}}^1 \cap \mathcal{E}_i^2$ occurs, then v_2 is protected in I_i.

Now consider any $t \in I_i$ and let f_t be a possible value of $\mathcal{F}_t(X)$. By memorylessness, we have

$$\mathbb{P}\big(\mathcal{E}_t^1 \mid \mathcal{F}_t(X) = f_t\big) = \frac{m-1}{m-1+r} = 1 - \frac{r}{m-1+r} \geq 1 - \frac{r}{m}.$$

In particular, since the event $T_{i,h} = t$ is determined by $\mathcal{F}_t(X)$, it follows by a union bound that

$$\mathbb{P}\left(\bigcap_{h=0}^{\lfloor 2r\kappa \rfloor} \mathcal{E}_{T_{i,h}}^1\right) \geq 1 - \frac{(\lfloor 2r\kappa \rfloor + 1)r}{m} \geq 1 - \frac{3r^2\kappa}{m}.$$

By Lemma 4.16 we have $\mathbb{P}(\mathcal{E}_i^2) \geq 1 - 1/n^2$, so it follows by a union bound that

$$\mathbb{P}(v_2 \text{ is protected in } I_i) \geq \mathbb{P}\left(\bigcap_{h=0}^{\lfloor 2r\kappa \rfloor} \mathcal{E}_{T_{i,h}}^1 \cap \mathcal{E}_i^2\right) \geq 1 - \frac{3r^2\kappa}{m} - \frac{1}{n^2}. \tag{17}$$

Since $I_0, I_1, \ldots$ are disjoint intervals, the events that v_2 is or is not protected in these intervals are independent by memorylessness. Thus the number of intervals I_i with $0 \leq i \leq t_{x_0}/\kappa$ in which v_2 is not protected is stochastically dominated above by a binomial distribution consisting of $\lfloor t_{x_0}/\kappa \rfloor + 1$ Bernoulli trials, each with success probability $3r^2\kappa/m + 1/n^2$. This distribution has expectation

$$\left(\left\lfloor \frac{t_{x_0}}{\kappa} \right\rfloor + 1\right)\left(\frac{3r^2\kappa}{m} + \frac{1}{n^2}\right) \leq \frac{4r^2 t_{x_0}}{m} = \frac{4r^2\ell}{Kr^4\kappa},$$

so by Lemma 2.5 we have

$$\mathbb{P}(\overline{\mathcal{P}_5}) \leq e^{-(1/6)8r^2 t_{x_0}/m} = e^{-(4/3)r^2\ell/(Kr^4\kappa)} \leq e^{-(4/3)r^2 \log n} \leq \frac{1}{n}.$$

Here the penultimate inequality follows since $\ell \geq Kr^4\kappa \log n$ by hypothesis. The result therefore follows. □

Now that we have proved lower bounds on the probability that each of $\mathcal{P}_1$–$\mathcal{P}_5$ occur, Lemma 4.11 follows easily.

LEMMA 4.11. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose $\ell \geq Kr^4\kappa \log n$, $m \geq 6r^2\kappa$ and $n \geq n_0$. Let $R \in \{R_1, \ldots, R_\ell\}$, and let $v_1 \ldots v_k$ be the path associated with R. Fix $x_0 \in R$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then, with probability at least $1/(7Kr^4\kappa)$, the following events occur simultaneously.*

$\mathcal{P}_1$: *$\forall t \leq t_{\mathsf{max}}$, $\sigma(t) \geq \lfloor \ell/(2r) \rfloor + 1$ or v^* does not spawn a mutant at time t.*
$\mathcal{P}_2$: *$\sigma(t_{x_0}) \geq \lfloor \ell/(2r) \rfloor + 1$ or $x_0 \notin X_{t_{x_0}}$.*
$\mathcal{P}_3$: *For all integers i with $0 \leq i \leq t_{x_0}$, $v_1 \ldots v_k$ clears within I_i.*
$\mathcal{P}_4$: *For all integers i with $0 \leq i \leq t_{x_0}$, the clock $M_{(v_k,v^*)}$ triggers at most $\lfloor 2r\kappa \rfloor$ times within I_i.*
$\mathcal{P}_5$: *For all but at most $8r^2 t_{x_0}/m$ integers i with $0 \leq i \leq t_{x_0}/\kappa$, v_2 is protected in I_i.*

PROOF. $\mathbb{P}(\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_5) \geq \mathbb{P}(\mathcal{P}_1 \cap \mathcal{P}_2) - \mathbb{P}(\overline{\mathcal{P}_3}) - \mathbb{P}(\overline{\mathcal{P}_4}) - \mathbb{P}(\overline{\mathcal{P}_5})$. Let n_0 be an integer which is sufficiently large with respect to r. Then we bound each term on the right-hand side by applying (in order) Lemma 4.14, Lemma 4.15, Corollary 4.17 and Lemma 4.18 to obtain

$$\mathbb{P}(\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_5) \geq \frac{1}{6Kr^4\kappa} - \frac{3}{n} \geq \frac{1}{7Kr^4\kappa},$$

as required. The final inequality follows since, by hypothesis, $\kappa \leq \ell/(Kr^4 \log n) \leq n/\log n$. □

We are now at last in a position to prove Lemma 4.5, which we will then use to prove Theorem 4.1.

LEMMA 4.5. *Consider any $r > 1$. There is an n_0, depending on r, such that the following holds. Suppose that $\ell \geq Kr^4\kappa \log n$, $m \geq 6r^2\kappa$ and $n \geq n_0$. Fix $x_0 \in R_1 \cup \cdots \cup R_\ell$. Let X be the Moran process with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then the extinction probability of X is at least $1/(7Kr^4\kappa)$.*

PROOF. Let n_0 be an integer which is sufficiently large with respect to r. Let R be a reservoir in $\{R_1, \ldots, R_\ell\}$ and let $v_1 \ldots v_k$ be the path associated with R. Suppose $x_0 \in R$. By Lemma 4.11, it suffices to assume that $\mathcal{P}_1 - \mathcal{P}_5$ occur and to prove that X goes extinct.

Recall the definition of $\sigma(t)$ from Definition 4.7. Note that $\sigma(0) = 0$ and $\sigma(t)$ is monotonically increasing in t. We will first bound $\sigma(t_{x_0})$ from above (assuming that $\mathcal{P}_1$–$\mathcal{P}_5$ occur). Consider an interval I_i with $0 \leq i \leq t_{x_0}$ (technically, we need only consider $0 \leq i \leq t_{x_0}/\kappa$, but the extra generality does no harm and we will later need to consider slightly larger i). Note that $I_i \subseteq [0, t_{\mathsf{max}}]$ since $(i+1)\kappa \leq 2t_{x_0}\kappa = t_{\mathsf{max}}/(Kr^4)$. Suppose that $\sigma(i\kappa) \leq \lfloor \ell/(2r) \rfloor - \lfloor 2r\kappa \rfloor$. We will derive an upper bound on $\sigma((i+1)\kappa)$ by splitting into cases.

Case 1: $i > 0$ and v_2 is protected in I_{i-1} and I_i. First note that since $\mathcal{P}_1$ occurs and $\sigma(i\kappa) \leq \lfloor \ell/(2r) \rfloor$, v^* does not spawn a mutant over the course of $[0, i\kappa]$ and so

$$X_t \subseteq \{x_0, v_1, \ldots, v_k, v^*\} \text{ for all } t \in [0, i\kappa].$$

Now, suppose for a contradiction that v_2 becomes a mutant at some time $\widehat{t}_2 \in I_{i-1}$. Then v_1 must have become a mutant beforehand. Let $\widehat{t}_1$ be the latest time in $[0, \widehat{t}_2]$ at which this occurs, and note that $M_{(x_0,v_1)}$ must have triggered at time $\widehat{t}_1$. Since v_2 is protected in I_{i-1}, if it were the case that $\widehat{t}_1 \in I_{i-1}$, then v_1 would clear before spawning within $(\widehat{t}_1, i\kappa]$ and so v_1 would die in $(\widehat{t}_1, \widehat{t}_2)$. This is impossible since v_1 spawns a mutant at time $\widehat{t}_2$, and v_1 does not become a mutant in $(\widehat{t}_1, \widehat{t}_2]$ by the definition of $\widehat{t}_1$. We therefore have $\widehat{t}_1 \notin I_{i-1}$, so $\widehat{t}_1 \leq (i-1)\kappa$. Since v_2 is protected in I_{i-1}, v_1 clears before

spawning a mutant within I_{i-1}, so v_1 dies in $((i-1)\kappa, \widehat{t_2})$ — again contradicting the fact that v_1 spawns a mutant at time $\widehat{t_2}$. Thus we can conclude that v_2 does not become a mutant in I_{i-1}.

Since $\mathcal{P}_3$ occurs, $v_1 \dots v_k$ clears within I_{i-1}. Let $v_0 \in R \setminus \{x_0\}$ and $t_1, \dots, t_{k+1} \in I_{i-1}$ be as in Definition 4.10. Since $N_{(v_0,v_1)}$ triggers at time t_1, it follows that $v_1 \notin X_{t_1}$. Since $M_{(x_0,v_1)}$ does not trigger in $[t_1, t_2]$, v_1 does not become a mutant in $[t_1, t_2]$ and so $v_1 \notin X_{t_2}$. Since $N_{(v_1,v_2)}$ triggers at time t_2, it follows that $v_2 \notin X_{t_2}$. We have already seen that v_2 does not become a mutant in I_{i-1}, so it follows that $v_2 \notin X_t$ for all $t \in [t_2, i\kappa]$.

Now, $v_3 \notin X_{t_3}$ since $N_{(v_2,v_3)}$ triggers at time $t_3 \in [t_2, i\kappa]$. Since v_2 is a non-mutant throughout $[t_2, i\kappa]$, it follows that $v_3 \notin X_t$ for all $t \in [t_3, i\kappa]$. Repeating the argument for $t_4, \dots, t_{k+1}$, we see that $v_2, \dots, v_k, v^* \notin X_{i\kappa}$ and hence $X_{i\kappa} \subseteq \{x_0, v_1\}$.

Since $X_{i\kappa} \subseteq \{x_0, v_1\}$, no mutants can be spawned in I_i until v_2 next becomes a mutant. However, by the same argument as above, the fact that v_2 is protected in I_i implies that v_2 does not become a mutant in I_i. Hence $X_t \subseteq \{x_0, v_1\}$ for all $t \in I_i$, and in particular v_k does not spawn a mutant onto v^* in I_i. Thus $\sigma((i+1)\kappa) = \sigma(i\kappa)$ This gives the desired upper bound on $\sigma((i+1)\kappa)$.

Case 2: Case 1 does not hold. Suppose for a contradiction that $\sigma((i+1)\kappa) \geq \sigma(i\kappa) + \lfloor 2r\kappa \rfloor + 1$. Then v_k spawns a mutant onto v^* at least $\lfloor 2r\kappa \rfloor + 1$ times in I_i, contradicting $\mathcal{P}_4$. Thus $\sigma((i+1)\kappa) \leq \sigma(i\kappa) + \lfloor 2r\kappa \rfloor$. Again, we have the desired upper bound on $\sigma((i+1)\kappa)$.

Combining Cases 1 and 2, we have proved that whenever $0 \leq i \leq t_{x_0}$ and $\sigma(i\kappa) \leq \lfloor \ell/(2r) \rfloor - \lfloor 2r\kappa \rfloor$,

$$\begin{aligned} &\sigma((i+1)\kappa) = \sigma(i\kappa), \text{ if } i > 0 \text{ and } v_2 \text{ is protected in } I_{i-1} \text{ and } I_i, \text{ and} \\ &\sigma((i+1)\kappa) \leq \sigma(i\kappa) + \lfloor 2r\kappa \rfloor, \text{ otherwise.} \end{aligned} \tag{18}$$

Since $\mathcal{P}_5$ occurs and $\ell \geq Kr^4\kappa \log n$, the number of intervals I_i such that $0 \leq i \leq \lfloor t_{x_0}/\kappa \rfloor$ and Case 1 does not hold is at most

$$\begin{aligned} 1 + 2\left|\left\{ i \in \mathbb{Z}_{\geq 0} \;\middle|\; i \leq \left\lfloor \frac{t_{x_0}}{\kappa} \right\rfloor,\ v_2 \text{ is not protected in } I_i \right\}\right| &\leq 1 + 2 \cdot \frac{8r^2 t_{x_0}}{m} \\ &= 1 + \frac{16r^2\ell}{Kr^4\kappa} \leq \frac{17r^2\ell}{Kr^4\kappa}. \end{aligned}$$

Moreover, again using the fact that $\ell \geq Kr^4\kappa \log n$,

$$\lfloor 2r\kappa \rfloor \cdot \frac{17r^2\ell}{Kr^4\kappa} \leq \frac{34r^3\ell}{Kr^4} = \frac{34\ell}{Kr} \leq \left\lfloor \frac{\ell}{2r} \right\rfloor - \lfloor 2r\kappa \rfloor.$$

Since $\sigma(0) = 0$, it therefore follows by repeated application of (18) that

$$\sigma(t_{x_0}) \leq \sigma\left(\left\lfloor \frac{t_{x_0}}{\kappa} + 1 \right\rfloor \kappa \right) \leq \left\lfloor \frac{\ell}{2r} \right\rfloor - \lfloor 2r\kappa \rfloor. \tag{19}$$

Now consider the behaviour of the process in the interval $(t_{x_0}, \lfloor t_{x_0}/\kappa + 2 \rfloor \kappa]$. From (19), we have that $\sigma(\lfloor t_{x_0}/\kappa + 1 \rfloor \kappa) \leq \lfloor \ell/(2r) \rfloor - \lfloor 2r\kappa \rfloor$, so by (18) it follows that $\sigma(\lfloor t_{x_0}/\kappa + 2 \rfloor \kappa) \leq \lfloor \ell/(2r) \rfloor$ and so, since $\mathcal{P}_1$ occurs, v^* does not spawn a mutant in the interval $[0, \lfloor t_{x_0}/\kappa + 2 \rfloor \kappa]$.

Since $\mathcal{P}_2$ occurs and (19) holds, we have $x_0 \notin X_{t_{x_0}}$, so for all $t \in (t_{x_0}, \lfloor t_{x_0}/\kappa + 2 \rfloor \kappa]$, we have $X_t \subseteq \{v_1, \dots, v_k, v^*\}$. Since $\mathcal{P}_3$ occurs, $v_1 \dots v_k$ clears within $I_{\lfloor t_{x_0}/\kappa + 1 \rfloor}$. Let $v_0 \in R \setminus \{x_0\}$ and the sequence of times $t_1, \dots, t_{k+1} \in I_{\lfloor t_{x_0}/\kappa + 1 \rfloor}$ be as in Definition 4.10. Then for all $i \in [k]$, $N_{(v_{i-1},v_i)}$ triggers at time t_i and so $v_i \notin X_t$ for all $t \in [t_i, \lfloor t_{x_0}/\kappa + 2 \rfloor] \kappa$.

Likewise, $v^* \notin X_t$ for all $t \in [t_{k+1}, \lfloor t_{x_0}/\kappa + 2 \rfloor]\kappa$. In particular, $X_{\lfloor t_{x_0}/\kappa+2\rfloor\kappa} = \emptyset$, so X goes extinct and the result holds. □

4.5. Proving the main theorem (Theorem 4.1)

We now have everything we need to prove Theorem 4.1, which follows relatively easily from Lemmas 4.2, 4.3 and 4.5.

THEOREM 4.1. *Let $r > 1$. Then there exists a constant $c_r > 0$ (depending on r) such that the following holds for all positive integers k, ℓ and m. Choose x_0 uniformly at random from $V(\mathcal{S}_{k,\ell,m})$. Let X be the Moran process (with fitness r) with $G(X) = \mathcal{S}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then the probability that X goes extinct is at least $1/(c_r(n \log n)^{1/3})$.*

PROOF. Fix $r > 1$ as in the statement of the theorem. Recall from Definitions 4.4 and 4.7 that $K = 70$ and $\kappa = \max\{3k, Kr^4 \log n\}$. Let n_0 be the smallest integer such that, for $n \geq n_0$, Lemma 4.5 and Lemma 4.14 applies and also

$$(n \log n)^{1/3} \geq n^{1/3} \geq \max\{18r^2, 6Kr^6 \log n, Kr^4(\log n)^2\}. \tag{20}$$

We split into cases depending on the values of k, ℓ, m and n. We show that in each case, the statement of the theorem holds, provided $c_r \geq \max\{2rn_0, 156r^6K\}$.

Case 1: $n < n_0$. We show that with probability at least $1/2rn_0$, x_0 dies before spawning a single mutant. Indeed, at the start of the process x_0 spawns a mutant with rate r, and every choice of $x_0 \in V(\mathcal{S}_{k,\ell,m})$ has an in-neighbour so x_0 dies with rate at least $1/n$. Thus X goes extinct with probability at least

$$\frac{\frac{1}{n}}{\frac{1}{n} + r} \geq \frac{1}{2rn} \geq \frac{1}{2rn_0},$$

so the result follows since $c_r \geq 2rn_0$.

Case 2: $n \geq n_0$, $m < k(n \log n)^{1/3}$. By Lemma 4.2, X goes extinct with probability at least

$$\frac{k}{2r(m+k)} \geq \frac{k}{2r(k(n \log n)^{1/3} + k)} \geq \frac{1}{4r(n \log n)^{1/3}},$$

where the final inequality holds since $(n \log n)^{1/3} \geq 1$. The result follows since $c_r \geq 4r$.

Case 3: $n \geq n_0$, $m \geq k(n \log n)^{1/3}$ and $\ell < 3Kr^4(n \log n)^{1/3}$. Note that

$$\mathbb{P}(x_0 \in R_1 \cup \cdots \cup R_\ell) = \frac{\ell m}{\ell(m+k)+1} \geq \frac{m}{m+k+1} \geq \frac{1}{2}, \tag{21}$$

where the final inequality is valid since $m \geq k(n \log n)^{1/3} \geq 2k$. We will therefore condition on $x_0 \in R_1 \cup \cdots \cup R_\ell$. Moreover, we have $m \geq k(n \log n)^{1/3} \geq 12r$. Thus by Lemma 4.3 and (21), X goes extinct with probability at least

$$\frac{1}{2} \cdot \frac{1}{26r^2\ell} \geq \frac{1}{156r^2Kr^4(n \log n)^{1/3}},$$

so the statement holds since $c_r \geq 156Kr^6$.

Case 4: $n \geq n_0$, $m \geq k(n \log n)^{1/3}$ and $\ell \geq 3Kr^4(n \log n)^{1/3}$. Note that

$$m \geq k(n \log n)^{1/3} \geq 6r^2 \max\{3k, Kr^4 \log n\} = 6r^2\kappa.$$

We will also show that $\ell \geq Kr^4\kappa \log n$, in order to apply Lemma 4.5. Since $\ell \geq 3Kr^4(n \log n)^{1/3}$ and $n = \ell(m+k)+1 \geq \ell m$, we have $m \leq n/\ell \leq (n^2/\log n)^{1/3}$. It is also immediate from (20) and the hypothesis on ℓ that

$$\ell \geq K^2r^8(\log n)^2. \tag{22}$$

Therefore,

$$3k \leq \frac{3m}{(n\log n)^{1/3}} \leq \frac{3n^{1/3}}{(\log n)^{2/3}} = \frac{3Kr^4(n\log n)^{1/3}}{Kr^4\log n} \leq \frac{\ell}{Kr^4\log n}. \tag{23}$$

It therefore follows from (22), (23) and the definition of κ (Definition 4.4) that $\ell \geq Kr^4\kappa\log n$, and so we may apply Lemma 4.5.

As in Case 3, (21) holds. Thus by (21) and Lemma 4.5, X goes extinct with probability at least $1/(14Kr^4\kappa)$. By (23) we have $3k \leq 3n^{1/3}$, and so

$$\frac{1}{14Kr^4\kappa} = \frac{1}{14Kr^4\max\{3k, Kr^4\log n\}} \geq \frac{1}{14Kr^4\max\{3n^{1/3}, Kr^4\log n\}} = \frac{1}{42Kr^4n^{1/3}},$$

and the result follows since $c_r \geq 156Kr^6$. □

5. AN UPPER BOUND ON THE FIXATION PROBABILITY OF METAFUNNELS

The (k,ℓ,m)-metafunnel is defined in Section 1.1.1. We use $n = 1+\ell\sum_{i=1}^{k} m^i$ to denote the number of vertices.

The main result of this section is the following upper bound on the fixation probability of the metafunnel.

THEOREM 5.1. *Let $r > 1$. Then there is a constant $c_r > 0$, depending on r, such that the following holds for all $k,\ell,m \in \mathbb{Z}_{\geq 1}$ such that the (k,ℓ,m)-metafunnel $\mathcal{G}_{k,\ell,m}$ has $n \geq 3$ vertices. Suppose that the initial state X_0 of the Moran process with fitness r is chosen uniformly at random from all singleton subsets of $V(\mathcal{G}_{k,\ell,m})$. The probability that the Moran process goes extinct is at least $e^{-\sqrt{\log r\cdot\log n}}(\log n)^{-c_r}$.*

5.1. Proof sketch

If $k = 1$ then $\mathcal{G}_{k,\ell,m}$ is a star and has extinction probability roughly $1/r^2$ so Theorem 5.1 follows easily. So for most of the proof (and the rest of this sketch) we assume $k \geq 2$. To prove the theorem, we divide the parameter space into two regimes.

In the first regime, $m < r^{\sqrt{\log_r n}}$. Since m is small, V_k is not too large compared to $V_0 \cup \cdots \cup V_{k-1}$. Thus, it is fairly likely that x_0 is born outside V_k, and dies before it can spawn a single mutant. This straightforward analysis is contained in the short Section 5.3.

Most of the proof (Section 5.4) focusses on the second regime, where $m \geq r^{\sqrt{\log_r n}}$ which, since $n \geq \ell m^k$, implies $k \leq \sqrt{\log_r n}$. In this regime it is likely that a uniformly-chosen initial mutant x_0 is born in V_k (Lemma 5.3) so we assume that this is the case in most of the proof (and the rest of this sketch). The key lemma is Lemma 5.28 which shows that, in this case, it is (sufficiently) likely that x_0 dies before v^* spawns a mutant.

In more detail, Definition 5.5 defines a stopping time T_{pa} which is the first time t that one of the following occurs.

(A1) $X_t = \emptyset$, or
(A2) $|X_t|$ exceeds a given threshold m^* which is a polynomial in $\log n$, or
(A3) By time t, v^* has already become a mutant in X more than b^* times, where b^* is about half as large as its number ℓm of in-neighbours, or
(A4) t exceeds some threshold t_{max} which is (very) exponentially large in n.

The subscript "pa" is for "pseudo-absorption time" because (A1) implies that the Moran process absorbs by going extinct and (A2) is a prerequisite for absorbing by fixating. The proof of Lemma 5.28 shows that, with sufficiently high probability, (A2)–(A4) do not hold, and so the Moran process X must go extinct by T_{pa}.

Conditioning makes it difficult to prove that (A2)–(A4) fail. To alleviate this, we divide the mutants into groups called "strains" which are easier to analyse. In particular, a strain contains all of the descendants of a particular mutant spawned by x_0. Formally, for each positive integer i, the i'th strain S^i is defined as a mutant process in Definition 5.6. Informally, S^i is "born" at the i'th time at which x_0 spawns a mutant in X. It "dies" when all of the descendants of this spawn have died. It is "dangerous" if one (or more) of these descendants spawns a mutant onto v^* before T_{pa}.

Lemma 5.7 defines eight events $\mathcal{P}_1$–$\mathcal{P}_8$. These are defined in such a way that we can show (in the proof of Lemma 5.28) that if $\mathcal{P}_1$–$\mathcal{P}_8$ simultaneously occur, then (A2)–(A4) do not hold. The definitions are engineered in such a way that we can also show that it is fairly likely that they do hold simultaneously — this takes up most of the proof. Informally, the events are defined as follows.

$\mathcal{P}_1$: No star-clock $M^*_{(v^*,v)}$ triggers in $[0,1]$.
$\mathcal{P}_2$: For some threshold $t_{x_0} < n$, the star-clock $N^*_{(v^*,x_0)}$ triggers in $[0, t_{x_0} - 2]$.
$\mathcal{P}_3$: v^* is a mutant for at most one unit of time up to time T_{pa}.
$\mathcal{P}_4$: The Moran process absorbs (either fixates or goes extinct) by time $t_{\mathsf{max}}/2$.
$\mathcal{P}_5$: Break $[0, t_{x_0}]$ into intervals of length $(\log n)^2$. During each interval, x_0 spawns at most $2r(\log n)^2$ mutants in X.
$\mathcal{P}_6$: Define s to be around $3rt_{x_0}$. Each of the strains $S^1, \ldots, S^s$ spawns at most $\log n$ mutants before T_{pa}.
$\mathcal{P}_7$: Each of the strains $S^1, \ldots, S^s$ dies within $(\log n)^2$ steps.
$\mathcal{P}_8$: At most $b^*/\log n$ of $S^1, \ldots, S^s$ are dangerous.

The rough sketch of Lemma 5.28 is as follows. $\mathcal{P}_1$ and $\mathcal{P}_3$ guarantee that v^* does not spawn a mutant in X until after T_{pa}. This together with $\mathcal{P}_2$ and $\mathcal{P}_3$ guarantees that the only mutants in the process before time T_{pa} are part of strains that are born before t_{x_0}. By $\mathcal{P}_5$, there are at most s such strains. By $\mathcal{P}_6$, each of these strains only has about $\log n$ mutants. Together with $\mathcal{P}_7$, this implies that (A2) does not hold at $t = T_{\mathsf{pa}}$. $\mathcal{P}_8$ and $\mathcal{P}_6$ imply that (A3) does not hold at $t = T_{\mathsf{pa}}$. Finally, $\mathcal{P}_4$ implies that (A4) does not hold at $t = T_{\mathsf{pa}}$.

The bulk of the proof involves showing (Lemma 5.7) that $\mathcal{P}_1$–$\mathcal{P}_8$ are sufficiently likely to simultaneously occur. Of these, $\mathcal{P}_3$–$\mathcal{P}_7$ are all so likely to occur that the probability that they do not occur can be subtracted off using a union bound (so conditioning on the other $\mathcal{P}_i$'s is not an issue). The majority of the failure probability comes from the probability that $\mathcal{P}_2$ does not occur. This is handled in the straightforward Lemma 5.8 which gives a lower bound on the probability that $\mathcal{P}_1$ and $\mathcal{P}_2$ both occur. The remaining event, $\mathcal{P}_8$, is sufficiently unlikely to occur that careful conditioning is required. This is (eventually) handled in Lemma 5.27, which shows that it is fairly likely to occur, conditioned on the fact that both $\mathcal{P}_1$ and $\mathcal{P}_2$ occur.

In order to get a good estimate on the probability that a strain is dangerous (in $\mathcal{P}_8$), we need to consider the number of mutants spawned from the "layer" of the strain closest to the centre vertex v^*. In order to do this, we define a new mutant process called the "head" of a strain. Strains and heads-of-strains share some common properties, and they are analysed together as "colonies" in Section 5.4.1. Informally (see Definition 5.12) a "colony" is a mutant process Z whose mutants are in $V_1 \cup \cdots \cup V_{k-1}$ (and not in V_0 or V_k). Once a colony becomes empty, it stays empty. Since a colony is a mutant process but not necessarily a Moran process, vertices may enter and/or leave whenever a clock triggers but we say that the colony is *hit* when a vertex leaves a colony specifically because a non-mutant is spawned onto it in the underlying Moran process. We define the "spawning chain" Y^Z of a colony and show that it increases whenever the colony spawns a mutant and that it only decreases when the colony is

hit. By analysing the jump chain of a spawning chain (see Definition 5.15) we are able to obtain the desired bounds on the probability that $\mathcal{P}_6$, $\mathcal{P}_7$ and $\mathcal{P}_8$ fail to occur.

5.2. Glossary

$\mathcal{A} = \{M_{(v^*,u)} \mid u \in V_k\} \cup \{N_{(v^*,x_0)}\}$	Definition 5.16, Page 41
$\mathcal{A}^* = \{M^*_{(v^*,u)} \mid u \in V_k\} \cup \{\overline{M}^*_{(v^*,u)} \mid u \in V_k\} \cup \{N^*_{(v^*,x_0)}, \overline{N}^*_{(v^*,x_0)}\}$	Definition 5.16, Page 41
(A1), (A2), (A3), (A4)	Definition 5.5, Page 35
$b^* = \lfloor \ell m/2 \rfloor$	Definition 5.4, Page 35
$C_r = \lceil 2 \log_r 20 \rceil$	Definition 5.4, Page 35
c_r	Theorem 5.1, Page 32
dangerous strain	Definition 5.6, Page 36
$\mathcal{G}_{k,\ell,m}$	Section 1.1.1, Page 4
hit	Definition 5.12, Page 40
$\mathcal{H}^i_t$ (head of a strain)	Definition 5.24, Page 47
$i_{\mathsf{m}}(\mu, u, t)$	Definition 3.6, Page 16
$i_{\mathsf{n}}(\mu, u, t)$	Definition 3.6, Page 16
$I_j = [(j-1)(\log n)^2, j(\log n)^2]$	Definition 5.4, Page 35
$m^* = \lceil 5r(\log n)^3 \rceil$	Definition 5.4, Page 35
$m' = m - m^*$	Definition 5.4, Page 35
$\mathcal{P}_1, \ldots, \mathcal{P}_8$	Lemma 5.7, Page 36
Φ	Definition 5.16, Page 41
$\Psi(X)$	Section 3.4, Page 17
$\Psi(X, Z)$	Definition 5.14, Page 41
$R_{a,b}$	Definition 5.13, Page 40
$s = \lceil 3rt_{x_0} \rceil$	Definition 5.4, Page 35
S^i_t (strain)	Definition 5.6, Page 36
T^i_{b}	Definition 5.6, Page 36
T^i_{d}	Definition 5.6, Page 36
$T_{\mathsf{e}}(Z)$	Definition 5.12, Page 40
$t_{\mathsf{max}} = n^{3n}$	Definition 5.4, Page 35
T_{pa}	Definition 5.5, Page 35
$T_{\mathsf{s}}(Z)$	Definition 5.12, Page 40
$t_{x_0} = \ell m^k/(r^k(\log n)^{C_r+5})$	Definition 5.4, Page 35
τ_i	Section 3.1, Page 15
Y^Z (spawning chain)	Definition 5.13, Page 40
$\widehat{Y}^Z$ (jump chain of Y^Z)	Definition 5.15, Page 41
Z (colony)	Definition 5.12, Page 40
(Z1), (Z2)	Page 41

5.3. The small m case

We first show that if m is small, then x_0 is likely to be born outside V_k and die before spawning a mutant. This is relatively easy.

LEMMA 5.2. *Suppose $k \geq 2$. Choose x_0 uniformly at random from $V(\mathcal{G}_{k,\ell,m})$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. The extinction probability of X is at least $1/(2(m+r))$.*

PROOF. Note that $\mathbb{P}(x_0 \in V_k) = \ell m^k/n$. First suppose $m = 1$ and $k \geq 2$. We have

$$\mathbb{P}(x_0 \in V_k) = \frac{\ell}{\ell k + 1} \leq \frac{\ell}{2\ell + 1} < \frac{1}{2}.$$

Moreover, if $x_0 \notin V_k$, then x_0 has an in-neighbour of out-degree 1. In this case, with probability at least $1/(1+r)$, x_0 dies before spawning a mutant. It follows that X goes extinct with probability at least $1/(2(1+r))$, as required.

Now suppose $m, k \geq 2$. We have

$$\mathbb{P}(x_0 \notin V_k) = \frac{1 + \ell \sum_{i=1}^{k-1} m^i}{1 + \ell \sum_{i=1}^{k} m^i} \geq \frac{\sum_{i=1}^{k-1} m^i}{\sum_{i=1}^{k} m^i} = \frac{m^k - m}{m(m^k - 1)} \geq \frac{1}{2m}.$$

If $x \in V_i$ for some $i \in \{0, \ldots, k-1\}$, then x has at least m^{i+1} in-neighbours with out-degree m^i, so with probability at least $m/(m+r)$, x_0 dies before spawning a mutant. Hence X has extinction probability at least

$$\frac{1}{2m} \cdot \frac{m}{m+r} = \frac{1}{2(m+r)}. \quad \square$$

5.4. The large m case

We now consider the case where m is large.

LEMMA 5.3. *Suppose $m \geq 2$. Choose x_0 uniformly at random from $V(\mathcal{G}_{k,\ell,m})$. Then $\mathbb{P}(x_0 \in V_k) \geq 1/2$.*

PROOF. We have

$$\mathbb{P}(x_0 \in V_k) = \frac{\ell m^k}{\ell \sum_{i=0}^{k} m^i - (\ell - 1)} \geq \frac{m^k}{\sum_{i=0}^{k} m^i} = \frac{m^k(m-1)}{m^{k+1} - 1} \geq 1 - \frac{1}{m} \geq \frac{1}{2}. \quad \square$$

For the remainder of Section 5, we will fix an arbitrary vertex $x_0 \in V_k$ and let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. We first define some constants. Then, we define a "pseudo-absorption time" T_{pa} which, by (A1) and (A2) in the definition below, is at most the absorption time of the Moran process X.

Definition 5.4 (*Constants*). We will use the following definitions for the rest of Section 5.

— $b^* = \lfloor \ell m/2 \rfloor$,
— $C_r = \lceil 2 \log_r 20 \rceil$,
— for each $j \in \mathbb{Z}_{\geq 1}$, $I_j = [(j-1)(\log n)^2, j(\log n)^2]$.
— $m^* = \lceil 5r(\log n)^3 \rceil$,
— $m' = m - m^*$,
— $t_{\mathsf{max}} = n^{3n}$,
— $t_{x_0} = \ell m^k / (r^k (\log n)^{C_r + 5})$, and
— $s = \lceil 3r t_{x_0} \rceil$. $\square$

Definition 5.5 (*The stopping time T_{pa}*). We define the stopping time T_{pa} to be the first time t that one of the following occurs:

(A1) $X_t = \emptyset$, or
(A2) $|X_t| \geq m^*$, or
(A3) v^* becomes a mutant in X at time t for the $(b^* + 1)$'st time, or
(A4) $t \geq t_{\mathsf{max}}$. $\square$

The definition of T_{pa} is motivated as follows. Certainly (A1) must hold when the process X goes extinct, and (A2) must hold before X fixates. If (A3) holds, we expect v^*

to spawn a mutant in X, which makes the process significantly harder to analyse (so we will stop the analysis before this). Actually, this is also why we stop at m^* mutants in (A2) — if the process contains too many mutants then it becomes harder to analyse. Finally, (A4) ensures that $T_{\mathsf{pa}} < \infty$.

We will prove that, with sufficiently high probability, (A2)–(A4) do not hold, and so the Moran process X must go extinct by T_{pa}. To do this, we will group the descendants of each mutant spawned by x_0 in X together and analyse each group as a separate mutant process.

Definition 5.6 (*Strains*). Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$ for some $x_0 \in V_k$. For each positive integer i, we define a mutant process S^i (called the *i'th strain*) with $G(S^i) = \mathcal{G}_{k,\ell,m}$. Let T^i_{b} be the i'th time at which x_0 spawns a mutant in X, or ∞ if x_0 spawns fewer than i mutants. The subscript "b" stands for the "birth" of the strain. Clearly, T^i_{b} is a function of the evolution of the process X. We let $S^i_t = \emptyset$ for all $t < T^i_{\mathsf{b}}$. If $T^i_{\mathsf{b}} < \infty$, then we have $T^i_{\mathsf{b}} = \tau_j$ for some j. Let u_i be the vertex onto which the mutant is spawned in X, and let $S^i_{\tau_j} = \{u_i\}$. The process S^i now evolves discretely as follows. Suppose we are given $S^i_{\tau_a}$ for some $a \geq j$. We define $S^i_t = S^i_{\tau_a}$ for all $t \in (\tau_a, \tau_{a+1})$. We then define $S^i_{\tau_{a+1}}$ by dividing into cases.

Case 1: Some vertex $u \in S^i_{\tau_a}$ spawns a mutant onto some vertex v in X at time τ_{a+1}. If $v \notin V_0 \cup V_k$, then we set $S^i_{\tau_{a+1}} = S^i_{\tau_a} \cup \{v\}$. Otherwise, we set $S^i_{\tau_{a+1}} = S^i_{\tau_a}$.

Case 2: Some vertex $v \in S^i_{\tau_a}$ dies in X at time τ_{a+1}. We set $S^i_{\tau_{a+1}} = S^i_{\tau_a} \setminus \{v\}$.

Case 3: Neither Case 1 nor Case 2 holds. We set $S^i_{\tau_{a+1}} = S^i_{\tau_a}$.

If $T^i_{\mathsf{b}} = \infty$ then we define $T^i_{\mathsf{d}} = \infty$. Otherwise, we define $T^i_{\mathsf{d}} = \sup\{t \mid S^i_t \neq \emptyset\}$. The subscript "d" stands for the "death" of the strain. Note that the definition maintains the invariant that $S^i_t \subseteq X_t$. Finally, we define the notion of a *dangerous* strain. The strain S^i is said to be *dangerous* if it spawns a mutant onto v^* during the interval $[0, T_{\mathsf{pa}}]$. □

Note that we allow S^i_t and $S^{i'}_t$ to intersect for $i \neq i'$. Intuitively, S^i_t is the set of all living descendants at time t (within $V_1 \cup \cdots \cup V_{k-1}$) of the i'th mutant spawned by x_0 in X.

We now set out a list of events $\mathcal{P}_1, \ldots, \mathcal{P}_8$ which, as we will see in the proof of Lemma 5.28, together imply extinction. We state these events and claim they hold with reasonable probability in Lemma 5.7.

LEMMA 5.7. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \geq n_0$, $m \geq r^{\sqrt{\log_r n}}$ and $2 \leq k \leq \sqrt{\log_r n}$. Suppose $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. With probability at least $r^{-k}/(\log n)^{C_r+7}$, all of the following events occur in $\Psi(X)$.*

$\mathcal{P}_1$: *no star-clock $M^*_{(v^*,v)}$ triggers in $[0,1]$.*
$\mathcal{P}_2$: *the star-clock $N^*_{(v^*,x_0)}$ triggers in $[0, t_{x_0} - 2]$.*
$\mathcal{P}_3$: $i_{\mathsf{m}}(X, v^*, T_{\mathsf{pa}}) \leq 1$.
$\mathcal{P}_4$: $X_{t_{\mathsf{max}}/2} \in \{\emptyset, V(\mathcal{G}_{k,\ell,m})\}$.
$\mathcal{P}_5$: *for all $j \leq \lceil t_{x_0}/(\log n)^2 \rceil$, x_0 spawns at most $2r(\log n)^2$ mutants in I_j in X.*
$\mathcal{P}_6$: *each of $S^1, \ldots, S^s$ spawns at most $\log n$ mutants in $(0, T_{\mathsf{pa}}]$.*
$\mathcal{P}_7$: *for all $i \in [s]$, $\min\{T^i_{\mathsf{d}}, T_{\mathsf{pa}}\} \leq T^i_{\mathsf{b}} + (\log n)^2$.*
$\mathcal{P}_8$: *at most $b^*/\log n$ of $S^1, \ldots, S^s$ are dangerous.*

The majority of the failure probability in Lemma 5.7 comes from $\mathcal{P}_2$. In addition, $\mathcal{P}_1$ and $\mathcal{P}_8$ may fail with reasonably high probability, so we will need to be careful with conditioning for these events. The remaining events each occur with high enough probability that we can apply a union bound.

We first show that $\mathcal{P}_1 \cap \mathcal{P}_2$ occurs with reasonable probability.

LEMMA 5.8. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \geq n_0$, $m \geq r^{\sqrt{\log_r n}}$ and $2 \leq k \leq \sqrt{\log_r n}$. Suppose $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then, in the process $\Psi(X)$,*

$$\mathbb{P}(\mathcal{P}_1 \cap \mathcal{P}_2) \geq \frac{1}{r^k (\log n)^{C_r+6}}.$$

PROOF. Let n_0 be a large integer relative to r. Note that $\mathcal{P}_1$ and $\mathcal{P}_2$ depend only on the star-clock process $P^*(G)$, and so they are independent by the definition of $P^*(G)$. The sum of the parameters of the star-clocks in $\{M^*_{(v^*,u)} \mid u \in V_k\}$ is r, so the definition of Poisson processes ensures that $\mathbb{P}(\mathcal{P}_1) = e^{-r}$.

The assumptions in the statement of the lemma guarantee that $t_{x_0} \geq 4$. The parameter of the star-clock $N^*_{(v^*,x_0)}$ is $1/(\ell m^k)$, so

$$\mathbb{P}(\mathcal{P}_2) = 1 - e^{-(t_{x_0}-2)/(\ell m^k)} \geq 1 - e^{-t_{x_0}/(2\ell m^k)}.$$

Using (4), we get

$$\mathbb{P}(\mathcal{P}_2) \geq 1 - e^{-t_{x_0}/(2\ell m^k)} \geq \frac{t_{x_0}}{4\ell m^k} = \frac{1}{4r^k(\log n)^{C_r+5}} \geq \frac{e^r}{r^k(\log n)^{C_r+6}}.$$

Since $\mathcal{P}_1$ and $\mathcal{P}_2$ are independent, the result follows. □

LEMMA 5.9. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \geq n_0$, $m \geq r^{\sqrt{\log_r n}}$ and $k \geq 2$. Suppose $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then $\mathbb{P}(\mathcal{P}_3) \geq 1 - 1/n$.*

PROOF. For every positive integer i, we define T^i_{m} and T^i_{n} as follows. If v^* becomes a mutant at least i times in $[0, T_{\mathsf{pa}}]$ then T^i_{m} is the i'th time that it does so. Otherwise, $T^i_{\mathsf{m}} = T_{\mathsf{pa}}$. If v^* becomes a non-mutant at least i times in $[0, T_{\mathsf{pa}}]$ then T^i_{n} is the i'th time that it does so. Otherwise, $T^i_{\mathsf{n}} = T_{\mathsf{pa}}$.

By item (A3) in the definition of T_{pa} (Definition 5.5), v^* may become a mutant at most b^* times in the interval $[0, T_{\mathsf{pa}})$, so

$$i_{\mathsf{m}}(X, v^*, T_{\mathsf{pa}}) = \sum_{i=1}^{b^*} (T^i_{\mathsf{n}} - T^i_{\mathsf{m}}).$$

Now consider any $i \in [b^*]$ and any $t_0 \geq 0$. Consider any possible value f for $\mathcal{F}_{t_0}(X)$ that is consistent with $T^i_{\mathsf{m}} = t_0$. We will show

$$\forall y \geq 0, \quad \mathbb{P}(T^i_{\mathsf{n}} - t_0 \leq y \mid \mathcal{F}_{t_0}(X) = f) \geq 1 - \exp(-\ell m' y). \tag{24}$$

The event $T^i_{\mathsf{m}} = t_0$ is determined by f. Since $T^i_{\mathsf{m}} \leq T_{\mathsf{pa}}$, we have $t_0 \leq T_{\mathsf{pa}}$. The value f also determines whether or not $t_0 = T_{\mathsf{pa}}$. If so, then (24) is trivial since $T^i_{\mathsf{n}} = T_{\mathsf{pa}} = t_0$, so $\mathbb{P}(T^i_{\mathsf{n}} - t_0 \leq y \mid \mathcal{F}_{t_0}(X) = f) = 1$.

From now on, we assume that f implies that $T_{\mathsf{pa}} > t_0$. Since $T^i_{\mathsf{m}} = t_0$, this implies that $v^* \in X_{t_0}$. Let $\mathcal{B}$ be the set of all non-mutant clocks with target v^* and let Ξ encapsulate the behaviour of every clock in $\mathcal{C}(\mathcal{G}_{k,\ell,m}) \setminus \mathcal{B}$ over the interval $(t_0, t_{\mathsf{max}}]$. (That is, let Ξ

contain a list, for each of these clocks, of all times that it triggered during this interval.) Consider any value ξ of Ξ which is consistent with $\mathcal{F}_{t_0}(X) = f$.

We now define a time t'_{pa} which depends only on the values f and ξ. To do so, consider the situation in which $\mathcal{F}_{t_0}(X) = f$, $\Xi = \xi$ and no clock in $\mathcal{B}$ triggers in $(t_0, t_{\mathsf{max}}]$, so that the evolution of X in this interval is entirely determined by f and ξ. Let t'_{pa} be the time at which T_{pa} would occur in this situation.

It is easy to see that $T^i_{\mathsf{n}} \leq t'_{\mathsf{pa}}$. If a non-mutant is spawned onto v^* in X at some time $t' \in (t_0, t'_{\mathsf{pa}}]$ then $T^i_{\mathsf{n}} \leq t' \leq t'_{\mathsf{pa}}$. Otherwise, the evolution of X in $(t_0, t'_{\mathsf{pa}}]$ is exactly the same as it would be if no clocks in $\mathcal{B}$ triggered, so $T_{\mathsf{pa}} = t'_{\mathsf{pa}} = T^i_{\mathsf{n}}$.

We will now prove (24). First, if $y \geq t'_{\mathsf{pa}} - t_0$, then since $T^i_{\mathsf{n}} \leq t'_{\mathsf{pa}}$, we have

$$\mathbb{P}(T^i_{\mathsf{n}} - t_0 \leq y \mid \mathcal{F}_{t_0}(X) = f, \Xi = \xi) \geq \mathbb{P}(t'_{\mathsf{pa}} - t_0 \leq y \mid \mathcal{F}_{t_0}(X) = f, \Xi = \xi) = 1. \tag{25}$$

So suppose $y < t'_{\mathsf{pa}} - t_0$. Let $t_1 < \cdots < t_z$ be the times in $(t_0, t_0 + y]$ at which clocks in $\mathcal{C}(\mathcal{G}_{k,\ell,m}) \setminus \mathcal{B}$ trigger and let $t_{z+1} = t_0 + y$. Thus $t_0 < \cdots < t_z \leq t_{z+1} < t'_{\mathsf{pa}}$. For all $h \in \{0, \ldots, z\}$, let $\chi(h)$ be the value that X_{t_h} would take in the situation where no clock in $\mathcal{B}$ triggers in $(t_0, t_h]$, $\mathcal{F}_{t_0}(X) = f$ and $\Xi = \xi$. Thus t'_{pa}, z, $t_0, \ldots, t_{z+1}$ and $\chi(0), \ldots, \chi(z)$ are all uniquely determined by f and ξ.

For each $h \in [z+1]$, let $\mathcal{E}_h$ be the event that a non-mutant is spawned onto v^* in the interval (t_{h-1}, t_h). Note that with probability 1, no non-mutant is spawned onto v^* at any time t_h. Thus

$$\begin{aligned}\mathbb{P}(T^i_{\mathsf{n}} - t_0 \leq y \mid \mathcal{F}_{t_0}(X) = f, \Xi = \xi) &= 1 - \mathbb{P}(\overline{\mathcal{E}_1} \cap \cdots \cap \overline{\mathcal{E}_{z+1}} \mid \mathcal{F}_{t_0}(X) = f, \Xi = \xi) \\ &= 1 - \prod_{h=1}^{z+1} \mathbb{P}(\overline{\mathcal{E}_h} \mid \mathcal{F}_{t_0}(X) = f, \Xi = \xi, \overline{\mathcal{E}_1} \cap \cdots \cap \overline{\mathcal{E}_{h-1}}).\end{aligned} \tag{26}$$

Now fix $h \in [z+1]$, and consider any possible value f_{h-1} of $\mathcal{F}_{t_{h-1}}(X)$ which implies that $\mathcal{F}_{t_0}(X) = f$ and $\overline{\mathcal{E}_1} \cap \cdots \cap \overline{\mathcal{E}_{h-1}}$ and is consistent with $\Xi = \xi$. Consider the evolution of X given $\mathcal{F}_{t_{h-1}}(X) = f_{h-1}$ and $\Xi = \xi$. Since $\overline{\mathcal{E}_1} \cap \cdots \cap \overline{\mathcal{E}_{h-1}}$ occurs, no non-mutant is spawned onto v^* in the interval $(t_0, t_{h-1}]$ and so $X_{t_{h-1}} = \chi(h-1)$. Moreover, X remains constant in $[t_{h-1}, t_h)$ unless a non-mutant is spawned onto v^*. Thus, given the condition that $\mathcal{F}_{t_{h-1}}(X) = f_{h-1}$ and $\Xi = \xi$, $\mathcal{E}_h$ occurs if and only if a non-mutant clock whose source is in $V_1 \setminus \chi(h-1)$ triggers in the interval (t_{h-1}, t_h). Since $t_{h-1} < t'_{\mathsf{pa}}$, property (A2) in the definition of T_{pa} (Definition 5.5) ensures that $|\chi(h-1)| < m^*$ so $|V_1 \setminus \chi(h-1)| \geq \ell m - m^* \geq \ell m'$. Thus,

$$\mathbb{P}(\overline{\mathcal{E}_h} \mid \mathcal{F}_{t_{h-1}}(X) = f_{h-1}, \Xi = \xi) \leq \exp(-\ell m'(t_h - t_{h-1})).$$

Combining this with (26) (by multiplying over all $h \in [z+1]$), we get

$$\mathbb{P}(T^i_{\mathsf{n}} - t_0 \leq y \mid \mathcal{F}_{t_0}(X) = f, \Xi = \xi) \geq 1 - \exp\left(-\ell m' y\right). \tag{27}$$

Equation (24) follows from (25) and (27). Equation (24) shows that $\sum_{i=1}^{b^*}(T^i_{\mathsf{n}} - T^i_{\mathsf{m}})$ is dominated from above by a sum S of b^* i.i.d. exponential random variables with rate $\ell m'$. It follows that

$$\mathbb{P}(\mathcal{P}_3) = \mathbb{P}(i_{\mathsf{m}}(X, v^*, T_{\mathsf{pa}}) \leq 1) = \mathbb{P}\left(\sum_{i=1}^{b^*}(T^i_{\mathsf{n}} - T^i_{\mathsf{m}}) \leq 1\right) \geq \mathbb{P}(S \leq 1).$$

The hypothesis of the lemma guarantees that $m \geq 4m^*$ so $b^* = \lfloor \ell m/2 \rfloor \leq 2\ell(m - m^*)/3 = 2\ell m'/3$. Therefore, by Corollary 2.4, $\mathbb{P}(S < 1) \geq 1 - e^{-\ell m'/16} \geq 1 - 1/n$, where the last inequality holds since $\log n \leq \ell(m - m^*)/16$ by the hypothesis of the lemma. □

LEMMA 5.10. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \geq n_0$ and $k \geq 2$. Suppose $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then $\mathbb{P}(\mathcal{P}_4) \geq 1 - 1/n$.*

PROOF. Let n_0 be a large constant. For all $i \in \mathbb{Z}_{\geq 1}$, let

$$J_i^- = (i-1)n^2, \quad J_i^+ = in^2,$$

and J_i be the interval $(J_i^-, J_i^+]$. Let $v_1, \ldots, v_n$ be a fixed ordering of $V(\mathcal{G}_{k,\ell,m})$. Define an event $\mathcal{E}_i$ as follows. If $X_{J_i^-} = \emptyset$, then $\mathcal{E}_i$ holds. Otherwise, let $j(i) = \min\{j \mid v_j \in X_{J_i^-}\}$. Let $e(i,1), \ldots, e(i,n-1)$ be the sequence of edges returned by a breadth-first search of $\mathcal{G}_{k,\ell,m}$ starting from $v_{j(i)}$. Then $\mathcal{E}_i$ holds if and only if clocks in $\mathcal{C}(\mathcal{G}_{k,\ell,m})$ trigger at least $n-1$ times in J_i, and the first $n-1$ such trigger events correspond to $M_{e(i,1)}, \ldots, M_{e(i,n-1)}$, in that order. Note that if $\mathcal{E}_i$ holds for some i then the Moran process reaches absorption no later than J_i^+.

Now let f_i be any possible value for the filtration $\mathcal{F}_{J_i^-}(\Psi(X))$. The event $\mathcal{F}_{J_i^-}(\Psi(X)) = f_i$ contains all information about $\mathcal{E}_1, \ldots, \mathcal{E}_{i-1}$ and $j(i)$. We will show that

$$\mathbb{P}\big(\mathcal{E}_i \,\big|\, \mathcal{F}_{J_i^-}(\Psi(X)) = f_i\big) \geq \frac{1}{(2n^2)^n}. \tag{28}$$

First, if $\mathcal{F}_{J_i^-}(\Psi(X)) = f_i$ implies that $X_{J_i^-} = \emptyset$, then the probability in (28) is 1. Otherwise, f_i implies that $X_{J_i^-}$ is non-empty and the first $n-1$ triggers of clocks in $\mathcal{C}(\mathcal{G}_{k,\ell,m})$ in the interval (J_i^-, ∞) are as in $\mathcal{E}_i$ with probability at least

$$\left(\frac{(r/n)}{(1+r)n}\right)^{n-1} \geq \frac{1}{(2n^2)^{n-1}}.$$

By Corollary 2.2, the probability that clocks in $\mathcal{C}(\mathcal{G}_{k,\ell,m})$ trigger at least n times in J_i is at least $1 - e^{-n^2/16}$. Thus by a union bound, we have established (28).

Let $i' = \min\{i \mid \mathcal{E}_i \text{ holds}\}$. Then $i' - 1$ is dominated below by a geometric distribution with parameter $1/(2n^2)^n$, so

$$\mathbb{P}(\overline{\mathcal{P}_4}) \leq \mathbb{P}(i' > n(2n^2)^n) \leq \left(1 - \frac{1}{(2n^2)^n}\right)^{n(2n^2)^n} \leq e^{-n} \leq \frac{1}{n}. \quad \square$$

LEMMA 5.11. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \geq n_0$ and $k \geq 2$. Suppose $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then $\mathbb{P}(\mathcal{P}_5) \geq 1 - 1/n$.*

PROOF. Let n_0 be large relative to r. For each $j \in \mathbb{Z}_{\geq 1}$, the number of times in I_j that x_0 spawns a mutant in X is dominated from above by the number of times in I_j that mutant clocks with source x_0 trigger, which follows a Poisson distribution with parameter $r\mathrm{len}(I_j) = r(\log n)^2$. By Corollary 2.2, we have

$$\mathbb{P}(x_0 \text{ spawns at least } 2r(\log n)^2 \text{ mutants in } I_j \text{ in } X) \leq e^{-r(\log n)^2/3} \leq e^{-(\log n)^2/3}.$$

We have $\lceil t_{x_0}/(\log n)^2 \rceil \leq \ell m^k \leq n$, so the result follows by a union bound over the I_j's with $j \leq \lceil t_{x_0}/(\log n)^2 \rceil$. $\square$

5.4.1. Colonies and spawning chains. To deal with $\mathcal{P}_6$, $\mathcal{P}_7$ and $\mathcal{P}_8$, we will need to analyse mutant spawns and deaths in strains and in the "bottom layers" of strains. We will use similar ideas for both cases, so to avoid redundancy we introduce the following general definitions.

Definition 5.12 (*Colonies*). Fix $\mathcal{G}_{k,\ell,m}$ and fix $x_0 \in V_k$. Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. A *colony* is a mutant process Z with $G(Z) = \mathcal{G}_{k,\ell,m}$ satisfying the following conditions.

— For all $t \geq 0$, $Z_t \subseteq X_t \cap (V_1 \cup \cdots \cup V_{k-1})$.
— If for some $t < t'$, Z_t is non-empty and $Z_{t'} = \emptyset$ then for all $t'' \geq t'$, $Z_{t''} = \emptyset$.

We define the start and end times of any colony Z as follows. The subscript "s" stands for "start" and the subscript "e" stands for "end".

$$T_{\mathsf{s}}(Z) = \min\{t \geq 0 \mid Z_t \neq \emptyset \text{ or } t = T_{\mathsf{pa}}\}.$$
$$T_{\mathsf{e}}(Z) = \min\{t \geq T_{\mathsf{s}}(Z) \mid Z_t = \emptyset \text{ or } t = T_{\mathsf{pa}}\}.$$

Further, we say that a colony Z in a Moran process X is *hit* at time t if $t = \tau_j$ for some $j \geq 1$ and there is a vertex $v \in Z_{\tau_{j-1}}$ such that some vertex u spawns a non-mutant onto v in X at time τ_j. □

Since a colony Z is a mutant process but not necessarily a Moran process, vertices may enter and/or leave Z at any time τ_j, as long as the conditions of Definition 5.12 are respected. A colony being hit at a particular time means that a vertex left the colony at that time specifically because a non-mutant was spawned onto it in the underlying Moran process.

Note that a strain (Definition 5.6) is a colony. Also, $T_{\mathsf{s}}(S^i) = \min\{T^i_{\mathsf{b}}, T_{\mathsf{pa}}\}$ and $T_{\mathsf{e}}(S^i) = \min\{T^i_{\mathsf{d}}, T_{\mathsf{pa}}\}$.

Definition 5.13 (*Spawning chains*). Fix $\mathcal{G}_{k,\ell,m}$ with $m > m^*$ and fix $x_0 \in V_k$. Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. The *spawning chain* Y^Z of the colony Z is a continuous-time stochastic process with states in $\mathbb{Z}$ which evolves as follows. First, for all $t \in [0, T_{\mathsf{s}}(Z)]$, we define $Y^Z_t = 1$. We next define Y^Z_t for all $t \in (T_{\mathsf{s}}(Z), T_{\mathsf{e}}(Z)]$. If $T_{\mathsf{s}}(Z) \geq T_{\mathsf{e}}(Z)$ there is nothing to define. Suppose instead that $T_{\mathsf{s}}(Z) < T_{\mathsf{e}}(Z)$, so that $T_{\mathsf{s}}(Z) = \tau_i$ for some i.

Now for any $j \geq i$ with $\tau_j < T_{\mathsf{e}}(Z)$, suppose that we are given $Y^Z_{\tau_j}$. If $\tau_{j+1} > T_{\mathsf{e}}(Z)$, then for all $t \in (\tau_j, T_{\mathsf{e}}(Z)]$, we set $Y^Z_t = Y^Z_{\tau_j}$. Otherwise, for all $t \in (\tau_j, \tau_{j+1})$, we set $Y^Z_t = Y^Z_{\tau_j}$ and we define $Y^Z_{\tau_{j+1}}$ according to the following cases.

Case 1: Z spawns a mutant at time τ_{j+1}. We set $Y^Z_{\tau_{j+1}} = Y^Z_{\tau_j} + 1$.

Case 2: Z is hit at time τ_{j+1}. With probability

$$\frac{m'|Z_{\tau_j}|}{\sum_{v \in Z_{\tau_j}} \sum_{u \in N^-(v) \setminus X_{\tau_j}} (1/d^+(u))} \tag{29}$$

(independently of all other events) we set $Y^Z_{\tau_{j+1}} = Y^Z_{\tau_j} - 1$; with the remaining probability, we set $Y^Z_{\tau_{j+1}} = Y^Z_{\tau_j}$. We will show below that the probability in (29) is well-defined.

Case 3: Neither Case 1 nor Case 2 holds. We set $Y^Z_{\tau_{j+1}} = Y^Z_{\tau_j}$.

We have now defined Y^Z_t for all $t \leq T_{\mathsf{e}}(Z)$. Finally, for $t > T_{\mathsf{e}}(Z)$ the spawning chain Y^Z_t evolves independently of $\Psi(X)$ as a continuous-time Markov chain on $\mathbb{Z}$ with start state $Y^Z_{T_{\mathsf{e}}(Z)}$ and the following transition rate matrix.

$$R_{a,b} = \begin{cases} r & \text{if } b = a+1, \\ m' & \text{if } b = a-1, \\ 0 & \text{otherwise.} \end{cases} \quad \square$$

Definition 5.14 (*The process* $\Psi(X,Z)$ *and its filtration*). Fix $\mathcal{G}_{k,\ell,m}$ with $m > m^*$ and fix $x_0 \in V_k$. Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Consider a colony Z. Let $\Psi(X,Z)$ be the stochastic process consisting of $\Psi(X)$ together with the spawning chain Y^Z whose evolution is coupled with that of $\Psi(X)$ in the manner described above. The filtration $\mathcal{F}_t(\Psi(X,Z))$ of $\Psi(X,Z)$ consists of $\mathcal{F}_t(\Psi(X))$ together with all information about the transitions of Y^Z up to time t. □

Definition 5.15 (*The jump chain* $\widehat{Y}^Z$). Fix $\mathcal{G}_{k,\ell,m}$ with $m > m^*$ and fix $x_0 \in V_k$. Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Consider a colony Z. Then the *jump chain* $\widehat{Y}^Z$ of Y^Z is defined straightforwardly. $\widehat{Y}^Z(0) = Y_0^Z = 1$, and if Y^Z makes its i'th transition at time t then $\widehat{Y}^Z(i) = Y_t^Z$. □

We now show that the probability in (29) is well-defined. First note that since $m > m^*$, the numerator of (29) is positive. Moreover, since Z is hit at time τ_{j+1}, there must be a vertex $v \in Z_{\tau_j}$ and a vertex $u \in N^-(v) \setminus X_{\tau_j}$ that spawned onto v at time τ_{j+1}. So, certainly, the denominator of (29) is nonzero. Now, let $v \in Z_{\tau_j}$ be arbitrary, and let i be the integer in $[k-1]$ such that $v \in V_i$. Then since $\tau_j < T_{\mathsf{e}}(Z) \le T_{\mathsf{pa}}$, by (A2) we have $|X_{\tau_j}| \le m^*$. It follows that

$$\sum_{u \in N^-(v) \setminus X_{\tau_j}} \frac{1}{d^+(u)} = \frac{|N^-(v) \setminus X_{\tau_j}|}{m^i} \ge \frac{m^{i+1} - |X_{\tau_j}|}{m^i} \ge \frac{m^{i+1} - m^*}{m^i} \ge m'.$$

It follows that

$$\sum_{v \in Z_{\tau_j}} \sum_{u \in N^-(v) \setminus X_{\tau_j}} \frac{1}{d^+(u)} \ge m' |Z_{\tau_j}|,$$

so (29) is, as claimed, a probability.

Note that either $T_{\mathsf{e}}(Z) = T_{\mathsf{pa}}$ or $Z_t = \emptyset$ for all $t \in [T_{\mathsf{e}}(Z), T_{\mathsf{pa}}]$. In either case, we see that spawning chains as defined above satisfy two important properties.

(Z1) $\{t \le T_{\mathsf{e}}(Z) \mid Y^Z \text{ increases at } t\} = \{t \le T_{\mathsf{pa}} \mid Z \text{ spawns a mutant at } t\}$.
(Z2) $\{t \le T_{\mathsf{e}}(Z) \mid Y^Z \text{ decreases at } t\} \subseteq \{t \le T_{\mathsf{pa}} \mid Z \text{ is hit at } t\}$.

Intuitively, we expect that for all $t \in (T_{\mathsf{s}}(Z), T_{\mathsf{e}}(Z)]$, the spawning chain Y_t^Z should behave similarly to a continuous-time Markov chain on $\mathbb{Z}$ which increments with rate $r|Z_t|$ and decrements with rate $m'|Z_t|$. For technical convenience we will not prove this. Instead, we will prove that the jump chain $\widehat{Y}^Z$ evolves as a random walk on $\mathbb{Z}$ with appropriate probabilities.

Definition 5.16. Let

$$\mathcal{A} = \{M_{(v^*,u)} \mid u \in V_k\} \cup \{N_{(v^*,x_0)}\}, \text{ and}$$
$$\mathcal{A}^* = \{M^*_{(v^*,u)} \mid u \in V_k\} \cup \{\overline{M}^*_{(v^*,u)} \mid u \in V_k\} \cup \{N^*_{(v^*,x_0)}, \overline{N}^*_{(v^*,x_0)}\}.$$

Let Φ contain, for each star-clock $C \in \mathcal{A}^*$, a list of the times at which C triggers in $[0, t_{\mathsf{max}}]$. □

The star-clocks in $\mathcal{A}^*$ are part of the star-clock process $P^*(\mathcal{G}_{k,\ell,m})$, so of course the times in Φ are "local" and don't necessarily correspond to the times that clocks in $\mathcal{A}$ trigger. However, these are related by Observation 3.7.

In order to prove Lemma 5.7, we will need to show that $\mathcal{P}_8$ is reasonably likely to occur, conditioned on $\mathcal{P}_1 \cap \mathcal{P}_2$. The following lemma (among others) will be used for this purpose, so we prove it conditioned on Φ.

LEMMA 5.17. *Let $k \geq 2$ and $m \geq 5m^*$. Fix $\mathcal{G}_{k,\ell,m}$ and fix $x_0 \in V_k$. Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Let Z be a colony. Let t_0 be a non-negative real number, and let f_0 be a possible value of the filtration $\mathcal{F}_{t_0}(\Psi(X,Z))$. Let φ be a possible value of Φ. If the three events $T_{\mathsf{s}}(Z) = t_0$, $\mathcal{F}_{t_0}(\Psi(X,Z)) = f_0$ and $\Phi = \varphi$ are consistent, then conditioned on these three events, the jump chain $\widehat{Y}^Z$ evolves as a random walk on $\mathbb{Z}$ with initial state 1 and the following transition matrix.*

$$\widehat{Y}^Z(a,b) = \begin{cases} r/(r+m') & \text{if } b = a+1, \\ m'/(r+m') & \text{if } b = a-1, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. The definition of the jump chain $\widehat{Y}^Z$ implies that $\widehat{Y}^Z(0) = 1$.

Let $T_0 = t_0$. For $i \in \mathbb{Z}_{\geq 1}$, let T_i be the random variable that is the time of Y^Z's i'th transition.

Now consider an $i \in \mathbb{Z}_{\geq 0}$ and a non-negative real number t_i. (If $i = 0$ then t_0 is already defined in the statement of the lemma. Otherwise, consider any $t_i \geq t_0$.) Suppose that f_i is a possible value for the filtration $\mathcal{F}_{t_i}(\Psi(X,Z))$ and that the events $T_{\mathsf{s}}(Z) = t_0$, $\mathcal{F}_{t_0}(\Psi(X,Z)) = f_0$, $\Phi = \varphi$, $\mathcal{F}_{t_i}(\Psi(X,Z)) = f_i$, and $T_i = t_i$ are consistent. Note that all of these events are determined by $\mathcal{F}_{t_i}(\Psi(X,Z)) = f_i$ and $\Phi = \varphi$, which also determine $\widehat{Y}^Z(0), \ldots, \widehat{Y}^Z(i)$. We therefore wish to show that

$$\begin{aligned} \mathbb{P}\big(\widehat{Y}^Z(i+1) = \widehat{Y}^Z(i) + 1 \,\big|\, \mathcal{F}_{t_i}(\Psi(X,Z)) = f_i, \Phi = \varphi\big) &= \frac{r}{r+m'}, \text{ and} \\ \mathbb{P}\big(\widehat{Y}^Z(i+1) = \widehat{Y}^Z(i) - 1 \,\big|\, \mathcal{F}_{t_i}(\Psi(X,Z)) = f_i, \Phi = \varphi\big) &= \frac{m'}{r+m'}. \end{aligned} \tag{30}$$

Let $t_{i+1} > t_i$ be arbitrary. Let Ξ contain, for each clock $C \in \mathcal{C}(\mathcal{G}_{k,\ell,m})$, a list of the times at which C triggers in (t_i, t_{i+1}). Suppose that ξ is a possible value of Ξ such that the event $\Xi = \xi$ is consistent with the events $\mathcal{F}_{t_i}(\Psi(X,Z)) = f_i$, $T_{i+1} = t_{i+1}$, and $\Phi = \varphi$. Let $\mathcal{F}$ be the intersection of these four events. Namely, $\mathcal{F}$ is the intersection of

— $\mathcal{F}_{t_i}(\Psi(X,Z)) = f_i$,
— $\Xi = \xi$,
— $T_{i+1} = t_{i+1}$, and
— $\Phi = \varphi$.

Let $\mathcal{E}_1$ be the event that $Y^Z_{t_{i+1}} = Y^Z_{t_i} + 1$ and let $\mathcal{E}_2$ be the event that $Y^Z_{t_{i+1}} = Y^Z_{t_i} - 1$. By integrating over all choices of t_{i+1} and ξ, Equation (30) will follow from

$$\begin{aligned} \mathbb{P}\left(\mathcal{E}_1 \mid \mathcal{F}\right) &= \frac{r}{r+m'}, \text{ and} \\ \mathbb{P}\left(\mathcal{E}_2 \mid \mathcal{F}\right) &= \frac{m'}{r+m'}. \end{aligned} \tag{31}$$

Since $\mathcal{F}$ implies $T_i = t_i$ and $T_{i+1} = t_{i+1}$, it implies $\mathcal{E}_1 \cup \mathcal{E}_2$. Also, $\mathcal{E}_1 \cap \mathcal{E}_2$ is empty. $\mathcal{F}$ determines the evolution of $\Psi(X,Z)$ throughout $[0, t_{i+1})$. In particular, it determines whether the event $T_{\mathsf{e}}(Z) < t_{i+1}$ occurs. We split into cases accordingly.

Case 1: $\mathcal{F}$ implies that $t_{i+1} > T_{\mathsf{e}}(Z)$. In this case, conditioned on $\mathcal{F}$, the behaviour of Y^Z_t at t_{i+1} is governed entirely by the transition rate matrix R and is therefore independent of Φ and $\Psi(X)$. The definition of the spawning chain Y^Z gives Equation (31).

Case 2: $\mathcal{F}$ implies that $t_{i+1} \leq T_{\mathsf{e}}(Z)$. Let $T^- = \max\{\tau_j \mid \tau_j < t_{i+1}\}$, and let t^- be the unique value of T^- consistent with $\mathcal{F}$. Let χ_{t^-} be the unique value of X_{t^-} consistent

with $\mathcal{F}$, and let ζ_{t^-} be the unique value of Z_{t^-} consistent with $\mathcal{F}$. Define

$$S = \sum_{v \in \zeta_{t^-}} \sum_{u \in N^-(v) \setminus \chi_{t^-}} (1/d^+(u)),$$
$$\mathcal{B}_1 = \{M_{(u,v)} \mid u \in \zeta_{t^-}\},$$
$$\mathcal{B}_2 = \{N_{(u,v)} \mid u \in V(\mathcal{G}_{k,\ell,m}) \setminus \chi_{t^-} \text{ and } v \in \zeta_{t^-}\}.$$

Consider the following events.

— $\widehat{\mathcal{E}}_1$ is the event that a clock in $\mathcal{B}_1$ triggers at t_{i+1}, and
— $\widehat{\mathcal{E}}_2$ is the event that
  — a clock in $\mathcal{B}_2$ triggers at t_{i+1}, and
  — an (independent) coin toss (part of the spawning chain), with probability $m'|\zeta_{t^-}|/S$ of coming up heads, comes up heads.

Note that, conditioned on $\mathcal{F}$, event $\widehat{\mathcal{E}}_1$ coincides with $\mathcal{E}_1$ and $\widehat{\mathcal{E}}_2$ coincides with $\mathcal{E}_2$. It is easy to see, using the definition of a Poisson process, that

$$\mathbb{P}\left(\widehat{\mathcal{E}}_1 \mid \widehat{\mathcal{E}}_1 \cup \widehat{\mathcal{E}}_2\right) = \frac{r|\zeta_{t^-}|}{r|\zeta_{t^-}| + S(m'|\zeta_{t^-}|/S)} = \frac{r}{r+m'}, \text{ and}$$
$$\mathbb{P}\left(\widehat{\mathcal{E}}_2 \mid \widehat{\mathcal{E}}_1 \cup \widehat{\mathcal{E}}_2\right) = \frac{m'}{r+m'}.$$

In order to establish (31), we would like to show that conditioning on $\mathcal{F}$ is equivalent to conditioning on $\widehat{\mathcal{E}}_1 \cup \widehat{\mathcal{E}}_2$. This is straightforward, apart from the event $\Phi = \varphi$, which is part of $\mathcal{F}$. Unfortunately, we need the result of the lemma to be conditioned on $\Phi = \varphi$ and not merely on the rest of $\mathcal{F}$, so the rest of this proof is merely technical, and is to deal with this.

To proceed, we consider the four events making up $\mathcal{F}$.

— First, consider the event $\mathcal{F}_{t_i}(\Psi(X,Z)) = f_i$. Let $\hat{f}_i$ be the induced value of $\mathcal{F}_{t_i}(\Psi(X))$. The value f_i consists of $\hat{f}_i$, together with the extra information about the transitions of Y^Z up to time t_i (giving the outcomes of the independent coin tosses that are part of the spawning chain Y^Z). The value $\hat{f}_i$ contains, for each clock $C \in \mathcal{C}(\mathcal{G}_{k,\ell,m})$, a list of the times at which C triggers in $[0, t_i]$. It also contains information about the times that the star-clocks trigger, according to the coupling in Section 3.4. Using Observation 3.7, we could translate $\hat{f}_i$ into a (unique) equivalent event which is a list of times at which certain star-clocks trigger. We will therefore write $\mathcal{F}^*_{t_i} = f^*_i$ to denote the event $\mathcal{F}_{t_i}(\Psi(X,Z)) = f_i$, expressed entirely in terms of star-clock triggers and outcomes of spawning-chain coin tosses.

— Similarly, given $\mathcal{F}^*_{t_i} = f^*_i$, we can uniquely express $\Xi = \xi$ as an event which is a list of times at which certain star-clocks trigger. We will denote this event as $\Xi^* = \xi^*$. The definitions of t^-, χ_{t^-} and ζ_{t^-} can be deduced from f^*_i and ξ^*.

— Now let $\mathcal{B}^*_1 = \{M^*_{(u,v)} \mid u \in \zeta_{t^-}\}$ and $\mathcal{B}^*_2 = \{N^*_{(u,v)} \mid u \in V(\mathcal{G}_{k,\ell,m}) \setminus \chi_{t^-} \text{ and } v \in \zeta_{t^-}\}$. Note that for every star-clock with source u in $\mathcal{B}^*_1 \cup \mathcal{B}^*_2$ the quantities $i_{\mathsf{m}}(u, X, t_{i+1})$ and $i_{\mathsf{n}}(u, X, t_{i+1})$ can be deduced from f^*_i and ξ^*. Let $\mathcal{E}^*_1$ be the event that a star-clock with source u in $\mathcal{B}^*_1$ triggers at $i_{\mathsf{m}}(u, X, t_{i+1})$. Note that by Observation 3.7, $\mathcal{F}^*_{t_i} = f^*_i$ and $\Xi^* = \xi^*$ implies that $\mathcal{E}^*_1$ is equivalent to $\widehat{\mathcal{E}}_1$. Similarly, let $\mathcal{E}^*_2$ be the event that a star-clock with source u in $\mathcal{B}^*_2$ triggers at $i_{\mathsf{n}}(u, X, t_{i+1})$ and the appropriate coin toss comes up heads, so that $\mathcal{F}^*_{t_i} = f^*_i$ and $\Xi^* = \xi^*$ implies that $\mathcal{E}^*_2$ is equivalent to $\widehat{\mathcal{E}}_2$.

We now have

$$\mathbb{P}\left(\mathcal{E}_1^* \mid \mathcal{F}_{t_i}^* = f_i^*, \Xi^* = \xi^*,\ \mathcal{E}_1^* \cup \mathcal{E}_2^*\right) = \frac{r}{r+m'}, \text{ and}$$
$$\mathbb{P}\left(\mathcal{E}_2^* \mid \mathcal{F}_{t_i}^* = f_i^*, \Xi^* = \xi^*, \mathcal{E}_1^* \cup \mathcal{E}_2^*\right) = \frac{m'}{r+m'}.$$

Finally, since $\mathcal{B}_1^* \cup \mathcal{B}_2^*$ is disjoint from $\mathcal{A}^*$, we conclude that, conditioned on $\mathcal{F}_{t_i}^* = f_i^*$, $\Xi^* = \xi^*$, and $\mathcal{E}_1^* \cup \mathcal{E}_2^*$, the event $\mathcal{E}_1^*$ is independent of Φ (by independence of star-clocks in the star-clock process). Thus, we obtain

$$\mathbb{P}\left(\mathcal{E}_1^* \mid \mathcal{F}_{t_i}^* = f_i^*, \Xi^* = \xi^*,\ \mathcal{E}_1^* \cup \mathcal{E}_2^*, \Phi = \varphi\right) = \frac{r}{r+m'}, \text{ and}$$
$$\mathbb{P}\left(\mathcal{E}_2^* \mid \mathcal{F}_{t_i}^* = f_i^*, \Xi^* = \xi^*, \mathcal{E}_1^* \cup \mathcal{E}_2^*, \Phi = \varphi\right) = \frac{m'}{r+m'},$$

which implies (31) by translating the events back to their original formulation. □

We next prove that, with high probability, Y^Z transitions many times shortly after $T_{\mathsf{s}}(Z)$.

LEMMA 5.18. *There exists $n_0 > 0$ such that the following holds for all $n \geq n_0$, $m \geq 5m^*$ and $k \geq 2$. Fix $\mathcal{G}_{k,\ell,m}$ with n vertices and fix $x_0 \in V_k$. Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Let Z be a colony. Let t_0 be a non-negative real number, and let f_0 be a possible value of the filtration $\mathcal{F}_{t_0}(\Psi(X,Z))$. If the events $T_{\mathsf{s}}(Z) = t_0$ and $\mathcal{F}_{t_0}(\Psi(X,Z)) = f_0$ are consistent then, conditioned on these events, with probability at least $1 - e^{-(\log n)^2/16}$, Y^Z increases at least $2\log n$ times in the interval $[t_0, t_0 + (\log n)^2]$.*

PROOF. Let P be a Poisson process with rate r. Conditioned on $T_{\mathsf{s}}(Z) = t_0$ and $\mathcal{F}_{t_0}(\Psi(X,Z)) = f_0$, we will couple the evolution of $\Psi(X,Z)$ from time t_0 with that of P. The coupling will have the property that every time P triggers, Y^Z increases.

Given the coupling, we can conclude that the probability that Y^Z increases at least $2\log n$ times in the interval $[t_0, t_0 + (\log n)^2]$ is at least the probability that P triggers at least $2\log n$ times in an interval of length $(\log n)^2$. This is the probability that a Poisson random variable W with parameter $\rho = r(\log n)^2$ has value at least $2\log n$. We have $2\log n < 2\rho/3$, so by Corollary 2.2, $\mathbb{P}(W \leq 2\log n) \leq \exp(-\rho/16) \leq \exp(-(\log n)^2/16)$.

It remains to give the details of the coupling. Roughly, the coupling will be constructed using the sequence $\tau_1, \tau_2, \ldots$. However, one technical detail arises, since $T_{\mathsf{e}}(Z)$ might not occur at one of the instants $\tau_1, \tau_2, \ldots$. So, for the purposes of the proof, let $\hat{\tau}_1, \hat{\tau}_2, \ldots$ be the increasing sequence containing $T_{\mathsf{e}}(Z)$ and all $\tau_1, \tau_2, \ldots$ from $\Psi(X)$ (and nothing else). This random sequence is a function of the evolution of $\Psi(X,Z)$.

Now, conditioned on $T_{\mathsf{s}}(Z) = t_0$ and $\mathcal{F}_{t_0}(\Psi(X,Z)) = f_0$ there is a non-negative integer j such that $\hat{\tau}_j = t_0$. We will define the coupling from each $\hat{\tau}_i$ for $i \geq j$.

So consider $i \geq j$ and suppose that for some time t_i and some filtration value f_i, we have $\hat{\tau}_i = t_i$ and $\mathcal{F}_{t_i}(\Psi(X,Z)) = f_i$. To continue the coupling in the open interval from t_i there are two cases. It is easy to determine which case applies, using f_i.

If $T_{\mathsf{e}}(Z) \leq t_i$, then the evolution of Y^Z after time t_i is a continuous-time Markov chain with transition matrix R, evolving independently of $\Psi(X)$. The rate of an upwards transition in R is r so use the triggering of P to dictate these upwards transitions.

If $T_{\mathsf{e}}(Z) > t_i$, then Z_{t_i} is non-empty, so choose some $u \in Z_{t_i}$. Now use the triggering of P to dictate the triggering of the mutant clocks with source u (which together have rate r). Thus, every time P triggers, a mutant clock in $\mathcal{C}(\mathcal{G}_{k,\ell,m})$ with source u is chosen uniformly at random to trigger. □

5.4.2. Continuing the proof of Lemma 5.7. We are now in a position to bound the probabilities with which events $\mathcal{P}_6$–$\mathcal{P}_8$ occur, using Lemmas 5.17 and 5.18. We start by showing that, up until $T_{\mathsf{e}}(S^i)$, the state of the spawning chain Y^{S^i} is at least as large as the size of the strain S^i (which is a colony).

LEMMA 5.19. *Let $k \geq 2$ and $m \geq 5m^*$. Fix $x_0 \in V_k$. Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Let $i \in \mathbb{Z}_{\geq 1}$. For all $t \leq T_{\mathsf{pa}}$, $|S_t^i| \leq \max\{Y_t^{S^i}, 0\}$.*

PROOF. For $t \leq T_{\mathsf{s}}(S^i)$ we have $|S_t^i| \leq 1 = Y_t^{S^i}$, and for $T_{\mathsf{e}}(S^i) \leq t \leq T_{\mathsf{pa}}$ we have $|S_t^i| = 0 \leq \max\{Y_t^{S^i}, 0\}$. Since S^i and Y^{S^i} remain constant except at $\tau_1, \tau_2, \ldots$, it suffices to prove that $|S_{\tau_j}^i| \leq Y_{\tau_j}^{S^i}$ for all $\tau_j \in [T_{\mathsf{s}}(S^i), T_{\mathsf{e}}(S^i))$. We will do so by induction on j.

If $T_{\mathsf{s}}(S^i) = T_{\mathsf{e}}(S^i) = T_{\mathsf{pa}}$ then there is nothing to prove, so suppose $T_{\mathsf{s}}(S^i) = \tau_x$ for some x. We have already seen that the claim holds for $j = x$. Suppose that the claim holds for some $j \geq x$, and that $\tau_{j+1} < T_{\mathsf{e}}(S^i)$. We will now prove the claim for $j+1$ by dividing into cases.

Case 1: $|S_{\tau_{j+1}}^i| > |S_{\tau_j}^i|$. This case may arise only if S^i spawns a mutant at time τ_{j+1}. Thus by (Z1), we have $Y_{\tau_{j+1}}^{S^i} = Y_{\tau_j}^{S^i} + 1 \geq |S_{\tau_j}^i| + 1 = |S_{\tau_{j+1}}^i|$.

Case 2: $Y_{\tau_{j+1}}^{S^i} < Y_{\tau_j}^{S^i}$. This case may arise only if the spawning chain Y^{S^i} is decremented at time τ_{j+1}, which by (Z2) may happen only if S^i is hit at time τ_{j+1}. Thus we have $Y_{\tau_{j+1}}^{S^i} = Y_{\tau_j}^{S^i} - 1 \geq |S_{\tau_j}^i| - 1 = |S_{\tau_{j+1}}^i|$.

Case 3: $Y_{\tau_{j+1}}^{S^i} \geq Y_{\tau_j}^{S^i}$ and $|S_{\tau_{j+1}}^i| \leq |S_{\tau_j}^i|$. In this case, we have $Y_{\tau_{j+1}}^{S^i} \geq Y_{\tau_j}^{S^i} \geq |S_{\tau_j}^i| \geq |S_{\tau_{j+1}}^i|$.

Thus the claim, and therefore the result, holds in all cases. □

COROLLARY 5.20. *Let $k \geq 2$ and $m \geq 5m^*$. Fix $x_0 \in V_k$. Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Let i be a positive integer. Suppose that $t \geq T_{\mathsf{s}}(S^i)$ and $Y_t^{S^i} = 0$. Then $t \geq T_{\mathsf{e}}(S^i)$.*

PROOF. If $t \geq T_{\mathsf{pa}}$, then $t \geq T_{\mathsf{e}}(S^i)$ by the definition of $T_{\mathsf{e}}(S^i)$. Otherwise $|S_t^i| = 0$ by Lemma 5.19, and so again $t \geq T_{\mathsf{e}}(S^i)$ by definition. □

It now follows by (Z1) and Corollary 5.20 that $\mathcal{P}_6$ occurs if, for each $i \in [s]$, Y^{S^i} increases at most $\log n$ times before reaching zero. We will then be able to apply Lemma 5.17 to reduce this to a simple question about random walks. We state the lemma below somewhat more generally since we will use it again to deal with $\mathcal{P}_8$.

LEMMA 5.21. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \geq n_0$, $m \geq 5m^*$ and $k \geq 2$. Fix $x_0 \in V_k$. Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Suppose that i and y are positive integers and that φ is a possible value for Φ. Then, conditioned on $\Phi = \varphi$, the probability in $\Psi(X)$ that S^i spawns at least y mutants in $(0, T_{\mathsf{pa}}]$ is at most $(20r/m)^y$.*

PROOF. By (Z1), the number of mutants S^i spawns in $(0, T_{\mathsf{pa}}]$ is equal to the number of times Y^{S^i} increases in $(T_{\mathsf{s}}(S^i), T_{\mathsf{e}}(S^i)]$. By Corollary 5.20, Y^{S^i} cannot reach zero until $T_{\mathsf{e}}(S^i)$, and so it suffices to prove that

$$\mathbb{P}\big(Y^{S^i} \text{ increases at least } y \text{ times before reaching } 0 \,\big|\, \Phi = \varphi\big) \leq (20r/m)^y. \tag{32}$$

Let t_0 be a non-negative real number and let f_0 be a possible value of $\mathcal{F}_{t_0}(\Psi(X, S^i))$ consistent with $T_{\mathsf{s}}(S^i) = t_0$ and $\Phi = \varphi$. Recall from Lemma 5.17 the transition probabilities of $\widehat{Y}^{S^i}$, conditioned on $T_{\mathsf{s}}(S^i) = t_0$, $\mathcal{F}_{t_0}(\Psi(X, S^i)) = f_0$ and $\Phi = \varphi$.

For all $j \in \mathbb{Z}_{\geq 1}$, let $\mathcal{E}_j$ be the event that Y^{S^i} takes exactly j forward steps from 1 before reaching 0. Let $\mathcal{E}_{\geq y} = \bigcup_{j=y}^{\infty} \mathcal{E}_j$. Since $m \geq 5m^* > r$, Y^{S^i} reaches 0 with probability 1, so we have

$$\mathbb{P}(\mathcal{E}_{\geq y}) = \sum_{j=y}^{\infty} \mathbb{P}(\mathcal{E}_j).$$

For Y^{S^i} to reach 0 after exactly j forward steps, Y^{S^i} must decrease exactly $j+1$ times for a total of $2j+1$ steps. Thus

$$\mathbb{P}(\mathcal{E}_{\geq y}) \;=\; \sum_{j=y}^{\infty} \binom{2j+1}{j} \left(\frac{r}{r+m'}\right)^j \left(\frac{m'}{r+m'}\right)^{j+1} \;\leq\; \sum_{j=y}^{\infty} 2^{2j+1} \left(\frac{r}{r+m'}\right)^j.$$

Since $4r/(r+m') \leq 1/2$, it follows that

$$\mathbb{P}(\mathcal{E}_{\geq y}) \leq 2 \sum_{j=y}^{\infty} \left(\frac{4r}{r+m'}\right)^j \leq 4 \left(\frac{4r}{m'}\right)^y \leq 4 \left(\frac{5r}{m}\right)^y \leq \left(\frac{20r}{m}\right)^y.$$

Here the penultimate inequality follows since $m \geq 5m^*$, so $m' \geq 4m/5$. Thus (32) holds, and the result follows. □

COROLLARY 5.22. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \geq n_0$, $m \geq 5m^*$ and $k \geq 2$. Fix $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then $\mathbb{P}(\mathcal{P}_6) \geq 1 - 1/n$.*

PROOF. Recall from Definition 5.4 that $s = \lceil 3rt_{x_0} \rceil$. Fix $i \in [s]$. Then by Lemma 5.21, the probability that S^i spawns at most $\lfloor \log n \rfloor$ mutants in $(0, T_{\mathsf{pa}}]$ is at least

$$1 - \left(\frac{20r}{m}\right)^{\lfloor \log n \rfloor + 1} \geq 1 - \left(\frac{1}{\log n}\right)^{\log n} = 1 - e^{-\log n \log\log n} \geq 1 - \frac{1}{n^2}.$$

By a union bound, the probability that each of $S^1, \ldots, S^s$ spawn at most $\log n$ mutants in $(0, T_{\mathsf{pa}}]$ is at least $1 - 1/n$ as required. □

Now, $\mathcal{P}_7$ is implied by $\mathcal{P}_6$ and Lemma 5.18.

LEMMA 5.23. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \geq n_0$, $k \geq 2$ and $m \geq 5m^*$. Fix $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then $\mathbb{P}(\mathcal{P}_7) \geq 1 - 2/n$.*

PROOF. By applying a union bound to Corollary 5.22 and Lemma 5.18, with probability at least $1 - 2/n$, $\mathcal{P}_6$ occurs and, for all $i \in [s]$, Y^{S^i} increases at least $2\log n$ times in $[T_{\mathsf{s}}(S^i), T_{\mathsf{s}}(S^i) + (\log n)^2]$. By (Z1) and the occurrence of $\mathcal{P}_6$, Y^{S^i} can increase at most $\log n$ times in $[T_{\mathsf{s}}(S^i), T_{\mathsf{e}}(S^i))$, so we must have $T_{\mathsf{e}}(S^i) \leq T_{\mathsf{s}}(S^i) + (\log n)^2$. Thus, $\mathcal{P}_7$ occurs by the definitions of $T_{\mathsf{s}}(S^i)$ and $T_{\mathsf{e}}(S^i)$. □

It remains only to bound the probability of $\mathcal{P}_8$. Note that while Lemma 5.21 bounds the number of mutants spawned by any vertex in a strain, to tightly bound the probability that the strain is dangerous we need to look at the number of mutants spawned from the "layer" of the strain closest to the centre vertex.

Definition 5.24 (*Head $\mathcal{H}^i_t$ of a strain*). Fix $\mathcal{G}_{k,\ell,m}$ and fix $x_0 \in V_k$. Consider the Moran process X with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. For each positive integer i, we define the *head* $\mathcal{H}^i$ of the strain S^i as follows. For $t \geq 0$, let

$$\mathcal{H}^i_t = \begin{cases} \emptyset & \text{if } S^i_t = \emptyset, \\ S^i_t \cap V_{\min\{j \mid S^i_t \cap V_j \neq \emptyset\}} & \text{otherwise.} \end{cases}$$

Note that, for all i, $\mathcal{H}^i$ is a colony with $T_{\mathsf{s}}(\mathcal{H}^i) = T_{\mathsf{s}}(S^i)$ and $T_{\mathsf{e}}(\mathcal{H}^i) = T_{\mathsf{e}}(S^i)$. □

The following lemma relates the behaviour of the spawning chain $Y^{\mathcal{H}^i}$ to the question of whether or not S^i is dangerous.

LEMMA 5.25. *Let $k \geq 2$ and $m \geq 5m^*$. Fix $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Let i be a positive integer. Suppose that S^i is dangerous, spawning a mutant onto v^* for the first time at some time $t_{\mathsf{sp}} \leq T_{\mathsf{pa}}$. Then $t_{\mathsf{sp}} \leq T_{\mathsf{e}}(S^i)$, and the following two statements hold in $\Psi(X, \mathcal{H}^i)$.*

(i) $|\{t \leq t_{\mathsf{sp}} \mid Y^{\mathcal{H}^i} \text{ increases at } t\}| \geq k - 1$.
(ii) $|\{t \leq t_{\mathsf{sp}} \mid S^i \text{ spawns a mutant at } t\}| \geq (k-1) + |\{t \leq t_{\mathsf{sp}} \mid Y^{\mathcal{H}^i} \text{ decreases at } t\}|$.

PROOF. First note that since $t_{\mathsf{sp}} \leq T_{\mathsf{pa}}$ and S^i is non-empty at t_{sp}, it is immediate that $T_{\mathsf{s}}(S^i) < t_{\mathsf{sp}} \leq T_{\mathsf{e}}(S^i)$. We now define some notation. Clearly t_{sp} is in the sequence $\tau_1, \tau_2, \ldots$, so write $t_{\mathsf{sp}} = \tau_j$. For $t \in [T_{\mathsf{s}}(S^i), T_{\mathsf{e}}(S^i))$, let $h^i(t) = \min\{y \mid S^i_t \cap V_y \neq \emptyset\}$, so that $\mathcal{H}^i_t = S^i_t \cap V_{h^i(t)}$. Let $a = |\{t \leq t_{\mathsf{sp}} \mid \mathcal{H}^i \text{ spawns a mutant at } t\}|$.

We first bound a below. Recall that $h^i(T_{\mathsf{s}}(S^i)) = k - 1$, and note that $h^i(\tau_{j-1}) = 1$ since S^i spawns a mutant onto v^* at time τ_j. Moreover, every time h^i decreases, it only decreases by 1 and $\mathcal{H}^i$ spawns a mutant. Also, h^i increases (by at least 1) at time τ_x whenever $\mathcal{H}^i$ is hit at τ_x and $|\mathcal{H}^i_{\tau_{x-1}}| = 1$. Finally, note that $\mathcal{H}^i$ spawns a mutant at time τ_j. Thus

$$\begin{aligned} a &\geq (k-2) + |\{x < j \mid \mathcal{H}^i \text{ is hit at } \tau_x \text{ and } |\mathcal{H}^i_{\tau_{x-1}}| = 1\}| + 1 \\ &= (k-1) + |\{x \leq j \mid \mathcal{H}^i \text{ is hit at } \tau_x \text{ and } |\mathcal{H}^i_{\tau_{x-1}}| = 1\}|, \end{aligned} \tag{33}$$

where the equality follows since no two clocks trigger at the same time. By (Z1) this implies part (i) of the result. (In fact it is substantially stronger, and we will use this extra strength later in the proof.)

We now bound a above. We say that a layer V_y is *empty* at time t if $S^i_t \cap V_y = \emptyset$, and *non-empty* otherwise. Since t_{sp} is the first time that $\mathcal{H}^i$ spawns a mutant onto v^*, every time $\mathcal{H}^i$ spawns a mutant in $[0, t_{\mathsf{sp}})$, a layer must become non-empty in S^i. Since $\mathcal{H}^i$ spawns no mutants in $[0, T_{\mathsf{s}}(S^i)]$, it follows that

$$\begin{aligned} a &= |\{t \in (T_{\mathsf{s}}(S^i), t_{\mathsf{sp}}] \mid \mathcal{H}^i \text{ spawns a mutant at } t\}| \\ &\leq |\{t \in (T_{\mathsf{s}}(S^i), t_{\mathsf{sp}}) \mid \text{for some } y \in [k-1],\ V_y \text{ becomes non-empty at } t\}| + 1. \end{aligned}$$

Since V_{k-1} becomes non-empty at $T_{\mathsf{s}}(S^i)$, it follows that

$$a \leq |\{t < t_{\mathsf{sp}} \mid \text{for some } y \in [k-1],\ V_y \text{ becomes non-empty at } t\}|.$$

Every time a layer becomes non-empty in $[0, t_{\mathsf{sp}}]$, either it subsequently becomes empty again in $[0, t_{\mathsf{sp}}]$ or it contains at least one mutant at time t_{sp}. Thus

$$a \leq |\{t \leq t_{\mathsf{sp}} \mid \text{for some } y \in [k-1],\ V_y \text{ becomes empty at } t\}| + |S^i_{t_{\mathsf{sp}}}|.$$

If a layer V_y becomes empty at time τ_x, then S^i is hit at time τ_x and $|S^i_{\tau_{x-1}} \cap V_y| = 1$. Thus

$$a \le |\{t \le t_{\mathsf{sp}} \mid S^i \setminus \mathcal{H}^i \text{ is hit at time } t\}| + |\{x \le j \mid \mathcal{H}^i \text{ is hit at } \tau_x \text{ and } |\mathcal{H}^i_{\tau_{x-1}}| = 1\}| + |S^i_{t_{\mathsf{sp}}}|.$$

Every time a vertex becomes a mutant in S^i during $[0, t_{\mathsf{sp}}]$, that mutant must either die in $[0, t_{\mathsf{sp}}]$ (at which point S^i is hit) or still be alive at t_{sp}. Thus

$$\begin{aligned} a \le\ & |\{t \le t_{\mathsf{sp}} \mid \text{a vertex becomes a mutant in } S^i \text{ at } t\}| \\ & - |\{x \le j \mid \mathcal{H}^i \text{ is hit at } \tau_x \text{ and } |\mathcal{H}^i_{\tau_{x-1}}| > 1\}|. \end{aligned}$$

Since the only time a vertex becomes a mutant in S^i without S^i spawning a mutant is $T_{\mathsf{s}}(S^i)$, and S^i spawns a mutant onto v^* at t_{sp}, it follows that

$$a \le |\{t \le t_{\mathsf{sp}} \mid S^i \text{ spawns a mutant at } t\}| - |\{x \le j \mid \mathcal{H}^i \text{ is hit at } \tau_x \text{ and } |\mathcal{H}^i_{\tau_{x-1}}| > 1\}|.$$

It now follows from (33) that

$$|\{t \le t_{\mathsf{sp}} \mid S^i \text{ spawns a mutant at } t\}| \ge (k-1) + |\{t \le t_{\mathsf{sp}} \mid \mathcal{H}^i \text{ is hit at time } t\}|.$$

Part (ii) of the result follows immediately from (Z2). □

We next prove that $Y^{\mathcal{H}^i}$ is unlikely to increase $k-1$ times before decreasing $C_r + 2$ times. This, combined with Lemma 5.25, will allow us to show that S^i is dangerous with probability at most roughly $(r/m)^{k-1}$.

LEMMA 5.26. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \ge n_0$, $m \ge m^*(\log n)^3$ and $2 \le k \le \sqrt{\log_r n}$. Fix $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Let φ be a possible value of Φ and let $i \in \mathbb{Z}_{\ge 1}$. Let $\mathcal{E}_i$ be the event in $\Psi(X, \mathcal{H}^i)$ that $Y^{\mathcal{H}^i}$ increases $k-1$ times before decreasing $C_r + 2$ times. Then*

$$\mathbb{P}(\mathcal{E}_i \mid \Phi = \varphi) \le (\log n)^{C_r+3} \left(\frac{r}{m}\right)^{k-1}.$$

PROOF. Let n_0 be a large integer relative to r. Consider any positive integer t_0 and filtration value f_0 such that the events $T_{\mathsf{s}}(\mathcal{H}^i) = t_0$, $\mathcal{F}_{t_0}(\Psi(X, \mathcal{H}^i)) = f_0$ and $\Phi = \varphi$ are consistent. Recall from Lemma 5.17 that, at each step and conditioned on these events, $\widehat{Y}^{\mathcal{H}^i}$ increases with probability $r/(r+m')$ and decreases with probability $m'/(r+m')$.

For $0 \le K \le C_r + 1$, let $\mathcal{E}_{i,K}$ be the event that $Y^{\mathcal{H}^i}$ increases precisely $k-1$ times within its first $k-1+K$ transitions. Thus $\mathcal{E}_i = \bigcup_{i=0}^{C_r+1} \mathcal{E}_{i,K}$.

The number of backward steps among the first $k-1+K$ transitions of $Y^{\mathcal{H}^i}$ follows the binomial distribution consisting of $k-1+K$ Bernoulli trials, each with success probability $m'/(r+m')$, and $\mathcal{E}_{i,K}$ holds if and only if this quantity is equal to K. Hence

$$\begin{aligned} &\mathbb{P}(\mathcal{E}_{i,K} \mid T_{\mathsf{s}}(\mathcal{H}^i) = t_0, \mathcal{F}_{t_0}(\Psi(X, \mathcal{H}^i)) = f_0, \Phi = \varphi) \\ &\quad = \binom{k-1+K}{K} \frac{(m')^K r^{k-1}}{(m'+r)^{k-1+K}} \le (k-1+K)^K \frac{m^K r^{k-1}}{(m')^{k-1+K}} \\ &\quad \le (\log n)^{C_r+1} \cdot \frac{m^K r^{k-1}}{(m')^{k-1+K}} = (\log n)^{C_r+1} \left(\frac{r}{m}\right)^{k-1} \left(\frac{m}{m'}\right)^{k-1+K}, \end{aligned}$$

where the final inequality holds since $k \le \sqrt{\log_r n}$ and $K \le C_r + 1$. Moreover,

$$\left(\frac{m}{m'}\right)^{k-1+K} = \left(1 + \frac{m^*}{m'}\right)^{k-1+K} \le \left(1 + \frac{2}{(\log n)^2}\right)^{k-1+K} \le e^{2(k+K)/(\log n)^2} \le e.$$

It therefore follows that

$$\mathbb{P}\big(\mathcal{E}_{i,K} \,\big|\, T_{\mathsf{s}}(\mathcal{H}^i) = t_0, \mathcal{F}_{t_0}(\Psi(X,\mathcal{H}^i)) = f_0, \Phi = \varphi\big) \le (\log n)^{C_r+2} \left(\frac{r}{m}\right)^{k-1}.$$

It now follows by a union bound over K that

$$\begin{aligned}\mathbb{P}\big(\mathcal{E}_{i} \,\big|\, T_{\mathsf{s}}(\mathcal{H}^i) = t_0, \mathcal{F}_{t_0}(\Psi(X,\mathcal{H}^i)) = f_0, \Phi = \varphi\big) &\le (C_r+2)(\log n)^{C_r+2} \left(\frac{r}{m}\right)^{k-1} \\ &\le (\log n)^{C_r+3} \left(\frac{r}{m}\right)^{k-1}. \quad \square\end{aligned}$$

We are now finally in a position to deal with $\mathcal{P}_8$.

LEMMA 5.27. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \ge n_0$, $m \ge r^{\sqrt{\log_r n}}$ and $2 \le k \le \sqrt{\log_r n}$. Fix $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then, in the process $\Psi(X)$, $\mathbb{P}(\mathcal{P}_8 \mid \mathcal{P}_1 \cap \mathcal{P}_2) \ge 1/2$.*

PROOF. Let n_0 be a large integer relative to r. Let φ be a possible value for Φ so that the event $\Phi = \varphi$ is consistent with $\mathcal{P}_1 \cap \mathcal{P}_2$. Note that $\mathcal{P}_1$ and $\mathcal{P}_2$ are determined by Φ.

For each $i \in [s]$, consider the process $\Psi(X, \mathcal{H}^i)$ on $\mathcal{G}_{k,\ell,m}$. Let $\mathcal{E}_i$ be the event that S^i spawns at most $k + C_r$ mutants in $(0, T_{\mathsf{pa}}]$ and let $\mathcal{E}_i'$ be the event that $Y^{\mathcal{H}^i}$ increases fewer than $k-1$ times before decreasing C_r+2 times. We first claim that $\mathcal{E}_i \cap \mathcal{E}_i'$ implies that S^i is not dangerous. Indeed, suppose $\mathcal{E}_i'$ holds and S^i is dangerous, spawning a mutant onto v^* for the first time at some time $t_{\mathsf{sp}} \le T_{\mathsf{pa}}$. By Lemma 5.25(i), $Y^{\mathcal{H}^i}$ must increase at least $k-1$ times in $[0, t_{\mathsf{sp}}]$, and so since $\mathcal{E}_i'$ holds $Y^{\mathcal{H}^i}$ must decrease at least $C_r + 2$ times in $[0, t_{\mathsf{sp}}]$. By Lemma 5.25(ii), it follows that S^i must spawn at least $k + C_r + 1$ mutants in $[0, t_{\mathsf{sp}}]$. Thus $\mathcal{E}_i$ cannot hold, and so $\mathcal{E}_i \cap \mathcal{E}_i'$ implies that S^i is not dangerous as claimed.

It therefore suffices to prove, conditioned on $\Phi = \varphi$, that with probability at least $1/2$, $\mathcal{E}_i \cap \mathcal{E}_i'$ holds for all but $b^*/\log n$ of the $i \in [s]$.

Fix $i \in [s]$. Then by Lemma 5.21,

$$\mathbb{P}(\overline{\mathcal{E}_i} \mid \Phi = \varphi) \le \left(\frac{20r}{m}\right)^{k+C_r+1} \le \left(\frac{20r}{m}\right)^{k+C_r} = \frac{20^{k+C_r} r^{C_r}}{m^{C_r}} \left(\frac{r}{m}\right)^k,$$

where

$$\frac{20^{k+C_r} r^{C_r}}{m^{C_r}} \le \frac{20^{\sqrt{\log_r n}+C_r} r^{C_r}}{r^{C_r\sqrt{\log_r n}}} = \frac{r^{(\sqrt{\log_r n}+C_r)(\log_r 20)+C_r}}{r^{C_r\sqrt{\log_r n}}} \le \frac{r^{2(\log_r 20)\sqrt{\log_r n}}}{r^{C_r\sqrt{\log_r n}}} \le 1.$$

(For the final inequality, we use the fact that $C_r = \lceil 2\log_r 20 \rceil$.) Moreover, by Lemma 5.26 we have

$$\mathbb{P}(\overline{\mathcal{E}_i'} \mid \Phi = \varphi) \le \left(\frac{r}{m}\right)^{k-1} (\log n)^{C_r+3}.$$

From a union bound and the fact that Φ determines $\mathcal{P}_1$ and $\mathcal{P}_2$, it follows that

$$\mathbb{P}(S^i \text{ is dangerous} \mid \mathcal{P}_1 \cap \mathcal{P}_2) \le \mathbb{P}\big(\overline{\mathcal{E}_i} \cup \overline{\mathcal{E}_i'} \,\big|\, \mathcal{P}_1 \cap \mathcal{P}_2\big) \le 2\left(\frac{r}{m}\right)^{k-1} (\log n)^{C_r+3}.$$

We now simply apply Markov's inequality. By linearity of expectation,

$$\mathbb{E}\big[|\{i \in [s] \mid S^i \text{ is dangerous}\}| \,\big|\, \mathcal{P}_1 \cap \mathcal{P}_2\big] \le 2s\left(\frac{r}{m}\right)^{k-1} (\log n)^{C_r+3}.$$

Hence by Markov's inequality, with probability at least $1/2$, at most $4s(r/m)^{k-1}(\log n)^{C_r+3}$ strains are dangerous. Since $t_{x_0} = \ell m^k/(r^k(\log n)^{C_r+5})$, $s = \lceil 3rt_{x_0} \rceil \leq 4rt_{x_0}$, and $b^* = \lfloor \ell m/2 \rfloor$, we have

$$4s\left(\frac{r}{m}\right)^{k-1}(\log n)^{C_r+3} \leq \frac{16r^k(\log n)^{C_r+3}}{m^{k-1}}t_{x_0} = \frac{16\ell m}{(\log n)^2} \leq \frac{b^*}{\log n},$$

so this implies that $\mathcal{P}_8$ occurs and the result follows. □

Lemma 5.7 now follows from everything we've done so far.

LEMMA 5.7. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \geq n_0$, $m \geq r^{\sqrt{\log_r n}}$ and $2 \leq k \leq \sqrt{\log_r n}$. Suppose $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. With probability at least $r^{-k}/(\log n)^{C_r+7}$, all of the following events occur in $\Psi(X)$.*

$\mathcal{P}_1$: *no star-clock $M^*_{(v^*,v)}$ triggers in $[0,1]$.*
$\mathcal{P}_2$: *the star-clock $N^*_{(v^*,x_0)}$ triggers in $[0, t_{x_0} - 2]$.*
$\mathcal{P}_3$: $i_{\mathsf{m}}(X, v^*, T_{\mathsf{pa}}) \leq 1$.
$\mathcal{P}_4$: $X_{t_{\mathsf{max}}/2} \in \{\emptyset, V(\mathcal{G}_{k,\ell,m})\}$.
$\mathcal{P}_5$: *for all $j \leq \lceil t_{x_0}/(\log n)^2 \rceil$, x_0 spawns at most $2r(\log n)^2$ mutants in I_j in X.*
$\mathcal{P}_6$: *each of $S^1, \ldots, S^s$ spawns at most $\log n$ mutants in $(0, T_{\mathsf{pa}}]$.*
$\mathcal{P}_7$: *for all $i \in [s]$, $\min\{T^i_{\mathsf{d}}, T_{\mathsf{pa}}\} \leq T^i_{\mathsf{b}} + (\log n)^2$.*
$\mathcal{P}_8$: *at most $b^*/\log n$ of $S^1, \ldots, S^s$ are dangerous.*

PROOF. Let $\mathcal{P} = \mathcal{P}_1 \cap \cdots \cap \mathcal{P}_8$.

$$\begin{aligned}\mathbb{P}(\mathcal{P}) &\geq \mathbb{P}(\mathcal{P}_1 \cap \mathcal{P}_2 \cap \mathcal{P}_8) - \mathbb{P}(\overline{\mathcal{P}_3}) - \mathbb{P}(\overline{\mathcal{P}_4}) - \mathbb{P}(\overline{\mathcal{P}_5}) - \mathbb{P}(\overline{\mathcal{P}_6}) - \mathbb{P}(\overline{\mathcal{P}_7}) \\ &= \mathbb{P}(\mathcal{P}_1 \cap \mathcal{P}_2)\,\mathbb{P}(\mathcal{P}_8 \mid \mathcal{P}_1 \cap \mathcal{P}_2) - \mathbb{P}(\overline{\mathcal{P}_3}) - \mathbb{P}(\overline{\mathcal{P}_4}) - \mathbb{P}(\overline{\mathcal{P}_5}) - \mathbb{P}(\overline{\mathcal{P}_6}) - \mathbb{P}(\overline{\mathcal{P}_7}).\end{aligned}$$

We bound each term on the right-hand side of the above by applying (in order) Lemmas 5.8, 5.27, 5.9, 5.10 and 5.11, Corollary 5.22, and Lemma 5.23. This yields

$$\mathbb{P}(\mathcal{P}) \geq \frac{1}{r^k(\log n)^{C_r+6}} \cdot \frac{1}{2} - \frac{6}{n} \geq \frac{1}{r^k(\log n)^{C_r+7}},$$

as required. The final inequality follows since $r^k \leq r^{\sqrt{\log_r n}} \leq \sqrt{n}$. □

5.4.3. Applying Lemma 5.7. We now prove that $\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_8$ implies extinction for the Moran process X, which together with Lemma 5.7 implies our lower bound on extinction probability.

LEMMA 5.28. *There exists $n_0 > 0$, depending on r, such that the following holds. Suppose $n \geq n_0$, $m \geq r^{\sqrt{\log_r n}}$ and $2 \leq k \leq \sqrt{\log_r n}$. Fix $x_0 \in V_k$. Let X be the Moran process with $G(X) = \mathcal{G}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then*

$$\mathbb{P}(X \text{ goes extinct}) \geq \frac{1}{r^k(\log n)^{C_r+7}}.$$

PROOF. Let n_0 be a large integer relative to r. By Lemma 5.7, it suffices to prove that $\mathcal{P}_1 \cap \cdots \cap \mathcal{P}_8$ implies extinction. We first show that we may restrict our attention to $S^1, \ldots, S^s$. Note that by Observation 3.7, $\mathcal{P}_1 \cap \mathcal{P}_3$ implies that v^* does not spawn a mutant in X in $[0, T_{\mathsf{pa}}]$. Thus the definition of the strains ensures that for all $t \leq T_{\mathsf{pa}}$,

$$X_t \setminus \{x_0, v^*\} \subseteq \bigcup_{i=1}^{\infty} S^i_t. \tag{34}$$

Let $s' = |\{i \mid T_{\mathsf{b}}^i \leq \min\{t_{x_0}, T_{\mathsf{pa}}\}\}|$. Again by Observation 3.7, $\mathcal{P}_2 \cap \mathcal{P}_3$ implies that, in the interval $[0, t_{x_0}]$, either x_0 dies in X or T_{pa} occurs or both. Thus no strains are born in $(t_{x_0}, T_{\mathsf{pa}}]$, so by (34), it follows that

$$X_t \setminus \{x_0, v^*\} \subseteq \bigcup_{i=1}^{s'} S_t^i \text{ for all } t \leq T_{\mathsf{pa}}. \tag{35}$$

Moreover, by $\mathcal{P}_5$ and the definition of each interval I_j in Definition 5.4, x_0 spawns at most $2r(\log n)^2 \cdot \lceil t_{x_0}/(\log n)^2 \rceil \leq 3rt_{x_0} \leq s$ mutants in $[0, t_{x_0}]$ in X and so $s' \leq s$.

We now show that $|X_{T_{\mathsf{pa}}}| < m^*$, and so (A2) does not hold with $t = T_{\mathsf{pa}}$. By (35), each mutant in $X_{T_{\mathsf{pa}}} \setminus \{x_0, v^*\}$ belongs to S_t^i for some $i \in [s']$. By $\mathcal{P}_6$, each such S^i contains at most $\log n + 1$ mutants in total. By $\mathcal{P}_7$ and the definition of s', each such S^i was born in the interval $[T_{\mathsf{pa}} - (\log n)^2, T_{\mathsf{pa}}] \cap [0, t_{x_0}]$. This interval spans at most two I_j's with $j \leq \lceil t_{x_0}/(\log n)^2 \rceil$, so by $\mathcal{P}_5$ there are at most $4r(\log n)^2$ such S^i's. Hence

$$|X_{T_{\mathsf{pa}}}| \leq (\log n + 1) \cdot 4r(\log n)^2 + 2 < m^*,$$

and so (A2) does not hold with $t = T_{\mathsf{pa}}$.

By $\mathcal{P}_8$, $\mathcal{P}_6$ and (35), v^* becomes a mutant in X at most $(b^*/\log n)\log n = b^*$ times in $[0, T_{\mathsf{pa}}]$. Thus (A3) does not hold with $t = T_{\mathsf{pa}}$.

Finally, since $\mathcal{P}_4$ occurs, $V_{t_{\mathsf{max}}/2} \in \{\emptyset, V(\mathcal{G}_{k,\ell,m})\}$. In either case, $T_{\mathsf{pa}} < t_{\mathsf{max}}$ by (A1) and (A2) and so (A4) does not hold with $t = T_{\mathsf{pa}}$. Since none of (A2)–(A4) holds at T_{pa}, (A1) must hold at T_{pa} by definition, and so the Moran process X goes extinct as required. □

5.5. Proof of the main theorem (Theorem 5.1)

Theorem 5.1 now follows easily from Lemmas 5.2 and 5.28.

THEOREM 5.1. *Let $r > 1$. Then there is a constant $c_r > 0$, depending on r, such that the following holds for all $k, \ell, m \in \mathbb{Z}_{\geq 1}$ such that the (k, ℓ, m)-metafunnel $\mathcal{G}_{k,\ell,m}$ has $n \geq 3$ vertices. Suppose that the initial state X_0 of the Moran process with fitness r is chosen uniformly at random from all singleton subsets of $V(\mathcal{G}_{k,\ell,m})$. The probability that the Moran process goes extinct is at least $e^{-\sqrt{\log r \cdot \log n}}(\log n)^{-c_r}$.*

PROOF. Let n_0 be the maximum of the value n_0 in Lemma 5.28 and $e^{2(r+1)}$. Recall that $C_r = \lceil 2\log_r 20 \rceil$ and take $c_r \geq C_r + 8$ large enough that the result holds whenever $n < n_0$ and whenever $k = 1$. (Note that if $k = 1$ then $\mathcal{G}_{k,\ell,m}$ is a star so as n tends to infinity, the extinction probability tends to $1/r^2$ [Lieberman et al. 2005; Broom and Rychtár 2008].)

We now consider the case where $n \geq n_0$ and $k \geq 2$. Consider the coupled process $\Psi(X)$, with x_0 taken uniformly at random from $V(\mathcal{G}_{k,\ell,m})$.

If $m \leq e^{\sqrt{\log r \cdot \log n}}$ then, by Lemma 5.2, we have

$$\mathbb{P}(X \text{ goes extinct}) \geq \frac{1}{2(m+r)} \geq \frac{1}{2(r+1)m} \geq \frac{1}{2(r+1)} e^{-\sqrt{\log r \cdot \log n}} \geq \frac{1}{\log n} e^{-\sqrt{\log r \cdot \log n}},$$

and the result follows. Suppose instead $m \geq e^{\sqrt{\log r \cdot \log n}} = r^{\sqrt{\log_r n}}$. Then we have $n \geq m^k$, so $k \leq \sqrt{\log_r n}$. By Lemma 5.3, we have $\mathbb{P}(x_0 \in V_k) \geq 1/2$. It follows from Lemma 5.28 that

$$\mathbb{P}(X \text{ goes extinct}) \geq \frac{1}{2} r^{-k} (\log n)^{-(C_r+7)} \geq e^{-\sqrt{\log r \cdot \log n}} (\log n)^{-c_r},$$

and again the result follows. □

6. A LOWER BOUND ON THE FIXATION PROBABILITY OF $\Upsilon_{\mathcal{M}}$

The definition of the (k, ℓ, m)-megastar $\mathcal{M}_{k,\ell,m}$ is given in Section 1.1.3. Note that each of the cliques $K_1, \ldots, K_\ell$ is the vertex set of a complete graph on k vertices (contrary to the fact that some authors use the notation K_i to denote a complete graph on i vertices). An infinite family of megastars is identified in Definition 1.5. Recall that

$$\Upsilon_{\mathcal{M}} = \{\mathcal{M}_{k(\ell),\ell,m(\ell)} \mid \ell \in \mathbb{Z}_{\geq 1}\},$$

where $m(\ell) = \ell$ and $k(\ell) = \lceil (\log \ell)^{23} \rceil$.

For convenience, we drop the argument ℓ in the functions $m(\ell)$ and $k(\ell)$ and simply write m and k. Also, we use $\mathcal{M}_\ell$ to denote the megastar $\mathcal{M}_{k(\ell),\ell,m(\ell)}$. We use $n = 1 + \ell(m+1+k)$ to denote the number of vertices of $\mathcal{M}_\ell$. Note that $\sqrt{n}/2 \leq \ell, m \leq \sqrt{n}$ when ℓ is sufficiently large. Our main theorem is the following.

THEOREM 6.1. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Consider the Moran process X with $G(X) = \mathcal{M}_\ell$ where the initial mutant x_0 is chosen uniformly at random from $V(\mathcal{M}_\ell)$. The fixation probability of X is at least $1 - (\log n)^{23}/n^{1/2}$.*

6.1. Glossary

$\alpha_i = \lfloor \max\{(2\log n)^2, m/(2\log n)^{2i}\} \rfloor$	Definition 6.26, Page 72
$\beta_i = \lfloor \max\{(2\log n)^2, \ell m/(2\log n)^{2i}\} \rfloor$	Definition 6.26, Page 72
$c_r = \lceil 2r/(r-1) \rceil$	Definition 6.9, Page 58
(D1), (D2), (D3), (D4)	Definition 6.28, Page 73
$\Delta = \lfloor c_r (\log n)^3 \rfloor$	Definition 6.11, Page 59
$\gamma_i = \beta_i (\log n)^{20}/k$	Definition 6.38, Page 80
good (filtration)	Definition 6.26, Page 72
$H_{x;S}$	Lemma 2.9, Page 13
$h_{\mathsf{in}}^{t_0,j} = \min\{h \in \mathbb{Z}_{\geq 0} \mid K_j \text{ is inactive at } T_{\mathsf{c}}^{t_0,j}(h)\}$	Definition 6.5, Page 57
$I_i = (I_i^-, I_i^+]$	Definition 6.30, Page 74
$I_i^- = n^8 + in(\log n)^3$	Definition 6.26, Page 72
$I_i^+ = n^8 + (i+1)n(\log n)^3$	Definition 6.26, Page 72
$i_{\mathsf{m}}(\mu, u, t)$	Definition 3.6, Page 16
$i_{\mathsf{n}}(\mu, u, t)$	Definition 3.6, Page 16
$J^{i,j}(h)$	Definition 6.30, Page 74
jump	Definition 6.7, Page 57
$k = k(\ell) = \lceil (\log \ell)^{23} \rceil$	Definition 1.5, Page 6
megastar process	Definition 6.2, Page 53
$\mathcal{M}_{k,\ell,m}$	Section 1.1.3, Page 4
$\mathcal{M}_\ell = \mathcal{M}_{k(\ell),\ell,m(\ell)}$	Section 6, Page 52
$m = m(\ell) = \ell$	Definition 1.5, Page 6
$\mathcal{P}_1(i), \mathcal{P}_2(i), \mathcal{P}_3(i), \mathcal{P}_4(i)$	Definition 6.26, Page 72
$p_{i,j}$	Definition 6.9, Page 58
$p'_{i,j}$	Lemma 6.8, Page 57
$p_{x \to y;z}$	Lemma 2.9, Page 13
$\Psi(X')$	Section 3.4, Page 17
$r' = (r+1)/2$	Definition 6.9, Page 58
$\mathcal{T}^i = \mathcal{T}^{i,1} \cup \cdots \cup \mathcal{T}^{i,\ell}$	Definition 6.40, Page 81
$\mathcal{T}^{i,j} = \left\{ t \in (t_i^-, t_i^- + \gamma_i] \;\middle\vert\; \begin{array}{l} \text{for some } v \in R_j, \\ N^*_{(v^*,v)} \text{ triggers at } t \end{array} \right\}$	Definition 6.40, Page 81
$T_{\mathsf{c}}^{t_0,j}(h)$	Definition 6.4, Page 57

T^i_{end} ..... Definition 6.28, Page 73
$t_i^- = i_{\mathsf{n}}(v^*, I_i^-)$ ..... Definition 6.40, Page 81
$t_i^+ = i_{\mathsf{m}}(v^*, I_i^-)$ ..... Definition 6.40, Page 81
$T^{t_0,j}_{\mathsf{in}} = T^{t_0,j}_{\mathsf{c}}(h^{t_0,j}_{\mathsf{in}})$ ..... Definition 6.6, Page 57
$T^{i,v}_{\mathsf{m}}(h)$ ..... Definition 6.29, Page 74
$T^{i,v}_{\mathsf{n}}(h)$ ..... Definition 6.29, Page 74
τ_i ..... Section 3.1, Page 15
$\mathcal{U}_i = \{v \in R_1 \cup \cdots \cup R_\ell \mid N^*_{(v^*,v)}$ triggers in $(t_i^-, t_i^- + \gamma_i]\}$. Definition 6.40, Page 81
$\Upsilon_{\mathcal{M}}$ ..... Definition 1.5, Page 6
$W^{i,j}_{\mathsf{a}}$ ..... Definition 6.33, Page 76
$W^{i,j}_{\mathsf{a}}(h), W^{i,j}_{\mathsf{in}}(h)$ ..... Definition 6.30, Page 74
X'_t ..... Definition 6.2, Page 53
$Y^{t_0,j}(h) = |X'_{T^{t_0,j}_{\mathsf{c}}(h)} \cap K_j|$ ..... Definition 6.7, Page 57
Z ..... Definition 6.9, Page 58

6.2. The megastar process

In working with the megastar, it will be helpful to isolate the evolution of the Moran process inside a clique K_j from the state of the process in the rest of the graph. To this end we define a new mutant process X', which has $G(X') = \mathcal{M}_\ell$. It is defined in the same way as the Moran process X, except that each feeder vertex is forced to be a non-mutant while its corresponding clique contains both mutants and non-mutants.

Definition 6.2. The *megastar process* on $\mathcal{M}_\ell$ with *initial mutant* $x_0 \in V(\mathcal{M}_\ell)$ is a mutant process X' with $G(X') = \mathcal{M}_\ell$ and $X'_0 = \{x_0\}$ defined as follows. Recall that $\tau_0 = 0$ and, for every $i \in \mathbb{Z}_{\geq 1}$, a clock $C \in \mathcal{C}(\mathcal{M}_\ell)$ triggers at τ_i. For $t \in (\tau_{i-1}, \tau_i)$, we set $X'_t = X'_{\tau_{i-1}}$. Then we define X'_{τ_i} as follows.

(i) If $C = M_{(u,v)}$ for some $(u,v) \in E(\mathcal{M}_\ell)$ such that $u \in X'_{\tau_{i-1}}$ and $u, v \notin \{a_1, \ldots, a_\ell\}$, then $X'_{\tau_i} = X'_{\tau_{i-1}} \cup \{v\}$.
(ii) If $C = M_{(a_j,v)}$ for some $j \in [\ell]$ and $v \in K_j$ such that $a_j \in X'_{\tau_{i-1}}$, and $K_j \cap X'_{\tau_{i-1}} = \emptyset$, then $X'_{\tau_i} = (X'_{\tau_{i-1}} \cup \{v\}) \setminus \{a_j\}$.
(iii) If $C = M_{(u,a_j)}$ for some $j \in [\ell]$ and $u \in R_j$ such that $u \in X'_{\tau_{i-1}}$, and $K_j \cap X'_{\tau_{i-1}} \in \{\emptyset, K_j\}$, then $X'_{\tau_i} = X'_{\tau_{i-1}} \cup \{a_j\}$.
(iv) If $C = N_{(u,v)}$ for some $(u,v) \in E(\mathcal{M}_\ell)$ such that $u \notin X'_{\tau_{i-1}}$, then $X'_{\tau_i} = X'_{\tau_{i-1}} \setminus \{v\}$.
(v) Otherwise, $X'_{\tau_i} = X'_{\tau_{i-1}}$.

For $j \in [\ell]$, we say K_j is *active at time* t if

$$\emptyset \subset X'_t \cap K_j \subset K_j,$$

and *inactive at time* t otherwise. □

X' is the only mutant process that will be considered in Section 6. Note that while a clique K_j is active, a_j is a non-mutant and the evolution of mutants in K_j depends only on clocks with sources in $K_j \cup \{a_j\}$ and not on the state of the rest of $\mathcal{M}_\ell$.

6.3. Proof of main theorem (Theorem 6.1) assuming key lemma (Lemma 6.3)

The key ingredient in the proof of our lower bound (Theorem 6.1) is the following lemma.

LEMMA 6.3. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Suppose that $x_0 \in R_1 \cup \cdots \cup R_\ell$. Then there exists a $t \geq 0$ such that $\mathbb{P}\big(X'_t = V(\mathcal{M}_\ell)\big) \geq 1 - 42(\log n)^2/n^{1/2}$.*

The precise value of t is not important, but in fact, our proof will work with $t = 2n^8$. Most of Section 6 will be devoted to the proof of Lemma 6.3. Before giving the proof, we show how to use Lemma 6.3 to prove Theorem 6.1, which we restate here for convenience.

THEOREM 6.1. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Consider the Moran process X with $G(X) = \mathcal{M}_\ell$ where the initial mutant x_0 is chosen uniformly at random from $V(\mathcal{M}_\ell)$. The fixation probability of X is at least $1 - (\log n)^{23}/n^{1/2}$.*

PROOF. Let X' be the megastar process on $\mathcal{M}_\ell$ with $X'_0 = X_0$. Recall that both X and X' are defined in terms of the same clock process $\mathcal{C}(\mathcal{M}_\ell)$. It is therefore immediate that for all $t \geq 0$, $X'_t \subseteq X_t$. Thus, if X' fixates at time t, then $X_t = X'_t = V(\mathcal{M}_\ell)$ and so X must also fixate at or before time t. Thus, $\mathbb{P}(X \text{ fixates}) \geq \mathbb{P}(X' \text{ fixates})$.

Let $\mathcal{R}$ be the event that the initial mutant x_0 is in a reservoir. Clearly,

$$\mathbb{P}(\mathcal{R}) = \frac{\ell m}{\ell(k+m+1)+1} > \frac{m}{k+m+2} > 1 - \frac{k+2}{m} \geq 1 - \frac{(\log \ell)^{23}+3}{m}\,.$$

By Lemma 6.3, we have

$$\mathbb{P}(X' \text{ fixates} \mid \mathcal{R}) \geq 1 - \frac{42(\log n)^2}{\sqrt{n}}\,.$$

Therefore,

$$\begin{aligned}\mathbb{P}(X' \text{ fixates}) &\geq \mathbb{P}(\mathcal{R})\,\mathbb{P}(X' \text{ fixates} \mid \mathcal{R})\\ &\geq 1 - \frac{(\log \ell)^{23}+3}{m} - \frac{42(\log n)^2}{\sqrt{n}}\\ &\geq 1 - \frac{2^{-23}(\log n)^{23}+3}{\sqrt{n}/2} - \frac{42(\log n)^2}{\sqrt{n}}\,,\end{aligned}$$

and the result follows. □

The rest of Section 6 is devoted to the proof of Lemma 6.3.

6.4. Sketch of the proof of the key lemma (Lemma 6.3)

In this Section, we give an informal sketch of the proof of Lemma 6.3. The presentation of the proof itself does not depend upon the sketch so the reader may prefer to skip directly to the proof. Throughout, we assume that n is "large" relative to r, leaving the details of how large to the actual proof.

At a very high level, the argument proceeds as follows. We set out some preliminary results concerning cliques in Section 6.5. With our choice of parameters, x_0 is very likely to spawn inside a reservoir, say R_1. Let $\Delta = \Theta((\log n)^3)$ (see the Glossary for the precise definition), and note that Δ is much smaller than k. In Section 6.6.1, we prove that K_1 is likely to fill with mutants before x_0 dies, and likely to contain at most Δ non-mutants at time n. In Section 6.6.2, we prove that $K_1, \ldots, K_\ell$ are all likely to contain at most Δ non-mutants at time n^8. Finally, in Section 6.7 we prove that the process is likely to fixate by time $2n^8$.

We now discuss each part of the argument in more detail. We say that a clique is *active* if it contains both mutants and non-mutants. The key idea of Section 6.5 is that

since we are working with the megastar process rather than the Moran process, the behaviour of an active clique is governed by a simple random walk on $\{0, \ldots, k\}$ (see Lemma 6.8). This walk is forward-biased for almost its entire length, so we dominate it below by two back-to-back gambler's ruins (see Definition 6.9 and Lemma 6.10). This, together with the fact that any clique containing both mutants and non-mutants changes state with rate at least r, allows us to prove several key properties of cliques in the megastar process which we state here in simplified form (see Corollary 6.18 and Lemma 6.19):

(C1) If a clique contains at least one mutant, then with at least constant probability it fills with mutants within time $k \log n$.
(C2) If a clique contains at most Δ non-mutants, then with very high probability it fills with mutants within time $(\log n)^7$.
(C3) Let I be an interval with $(\log n)^7 \leq \mathrm{len}(I) \leq e^{(\log n)^2}$. Then if a clique contains at most Δ non-mutants at the start of I, with very high probability it contains at most Δ non-mutants at the end of I and contains at most 2Δ non-mutants at any time in I.

Finally, we use (C2) and (C3) together with a careful domination to prove upper bounds on the likelihood of non-mutants being spawned onto v^* from an active clique (see Lemma 6.20).

We now discuss Section 6.6.1. Heuristically, the argument is quite simple. Consider the interval $J = [0, \sqrt{n}(\log n)^3]$. With probability at least $1 - O((\log n)^3/\sqrt{n})$, $N_{(v^*,x_0)}$ does not trigger in J and so v^* remains a mutant throughout. Conditioned on this event, by Chernoff bounds x_0 is very likely to spawn $\Omega(\sqrt{n}(\log n)^3)$ mutants onto a_1 in J. Each time a mutant is spawned onto a_1, either K_1 already contains a mutant or there is an $\Omega(1/m) = \Omega(1/\sqrt{n})$ chance of a_1 spawning a mutant into K_1 before dying. Whenever K_1 contains a mutant, by (C1) there is an $\Omega(1)$ chance that K_1 will fill with mutants, so in expectation K_1 will fill with mutants $\Omega((\log n)^3)$ times over the course of J. Finally, when K_1 has filled with mutants, by (C3) it is likely to contain at most Δ non-mutants at time n (see Lemma 6.22). Unfortunately, these events are not independent — for example, a mutant may be spawned onto a_j while K_j is already active from a previous spawn — so concentration is not guaranteed. To make the argument rigorous, we therefore divide J into sub-intervals and apply domination.

Section 6.6.2 is now relatively easy. (C3) tells us that K_1 is very likely to remain almost full of mutants for a superpolynomial length of time. While we could fill each subsequent clique with a similar argument to that used in Section 6.6.1, we have enough wiggle room that we can instead use a substantially simpler argument to prove that $K_1, \ldots, K_\ell$ each contain at most Δ non-mutants by time n^8 (see Lemma 6.25). A side effect of this is that our bound on t in the statement of Lemma 6.3 is very loose.

The meat of the proof is in Section 6.7. Suppose that $K_1, \ldots, K_\ell$ each contain at most Δ non-mutants. Since Δ is much smaller than k, it is tempting to simply dominate the number of mutants in reservoirs below by a random walk on $\{0, \ldots, \ell m\}$. We could argue that by (C3), for superpolynomial time most of v^*'s in-neighbours will be mutants, and so v^* will spawn far more mutants than non-mutants in this interval. While this is true, it will only take us so far — even if each clique only contained one non-mutant, we should still expect v^* to be a non-mutant for an $\Omega(1/k)$ proportion of the time, leaving us with $\Omega(m/k)$ non-mutants in each reservoir. However, all is not lost. Intuitively, when there are many mutants in a reservoir, the corresponding feeder vertex is more likely to be a mutant and so frequently its clique will contain no non-mutants at all. Developing this idea yields Lemma 6.27, the main result of the section.

For all $i \in \mathbb{Z}_{\geq 0}$, let

$$\begin{aligned} I_i^- &= n^8 + in(\log n)^3, \\ I_i^+ &= n^8 + (i+1)n(\log n)^3, \\ I_i &= (I_i^-, I_i^+], \\ \alpha_i &= \lfloor \max\{(2\log n)^2, m/(2\log n)^{2i}\} \rfloor, \\ \beta_i &= \lfloor \max\{(2\log n)^2, \ell m/(2\log n)^{2i}\} \rfloor. \end{aligned}$$

We say that the filtration at time I_i^- is *good* if the following events occur.

— $\mathcal{P}_1(i)$: $|(R_1 \cup \cdots \cup R_\ell) \setminus X'_{I_i^-}| \leq \beta_i$.
— $\mathcal{P}_2(i)$: For all $j \in [\ell]$, $|R_j \setminus X'_{I_i^-}| \leq \alpha_i$.
— $\mathcal{P}_3(i)$: For all $j \in [\ell]$, $|K_j \setminus X'_{I_i^-}| \leq \Delta$.
— $\mathcal{P}_4(i)$: For all but at most β_i choices of $j \in [\ell]$, $R_j \cup \{a_j\} \cup K_j \subseteq X'_{I_i^-}$.

Lemma 6.27 implies that if the filtration at I_i^- is good, then with very high probability so is the filtration at I_{i+1}^-. Thus the number of non-mutants in each reservoir drops by a factor of at least $(2\log n)^2$ to a minimum of $\lfloor (2\log n)^2 \rfloor$, as does the total number of non-mutants across all reservoirs. Moreover, if i is sufficiently large, then Lemma 6.27 also implies that the process fixates by time I_{i+1}^- with probability at least $1/2$.

Note that the filtration at I_0^- is good; indeed, $\alpha_0 = m$ and $\beta_0 = \ell m$, so $\mathcal{P}_1(0)$, $\mathcal{P}_2(0)$ and $\mathcal{P}_4(0)$ trivially occur, and $\mathcal{P}_3(0)$ is very likely to occur by Lemma 6.25. It is therefore relatively easy to prove Lemma 6.3 using Lemmas 6.25 and 6.27 (see Section 6.7.1).

The linchpin of the proof of Lemma 6.27 is a stopping time T^i_{end} defined to be the first time $t \geq I_i^-$ such that one of the following holds.

(D1) $t = I_i^+$.
(D2) v^* spawns β_{i+1} non-mutants in the interval $(I_i^-, t]$.
(D3) For some $j \in [\ell]$, v^* spawns α_{i+1} non-mutants onto vertices in R_j in the interval $(I_i^-, t]$.
(D4) For some $j \in [\ell]$, $|K_j \setminus X'_t| > 2\Delta$.

Note that the definition of T^i_{end} guarantees, without any need for conditioning, that throughout $(I_i^-, T^i_{\mathsf{end}}]$ our cliques remain almost full of mutants and not too many non-mutants are spawned onto reservoirs. We will therefore work in $(I_i^-, T^i_{\mathsf{end}}]$ for most of the proof to facilitate dominations, with the eventual goal of proving that $(I_i^-, T^i_{\mathsf{end}}] = I_i$.

In Section 6.7.2, we prove an upper bound on the number of times cliques are likely to become active over the interval $(I_i^-, T^i_{\mathsf{end}}]$ (see Lemma 6.34). In Section 6.7.3, we apply this together with Lemma 6.20 to prove an upper bound on the length of time for which v^* is likely to be a mutant over the interval $(I_i^-, T^i_{\mathsf{end}}]$ (see Lemma 6.37). Unfortunately, due to the use of T^i_{end}, these proofs require a fairly technical series of dominations. More details can be found in the relevant sections.

In Section 6.7.4, we put all of this together to prove Lemma 6.27 and hence Lemma 6.3. The key observation is that Lemma 6.37 combines with Chernoff bounds on star-clocks to give strong upper bounds on the number of non-mutants spawned by v^* in $(I_i^-, T^i_{\mathsf{end}}]$. These bounds, together with (C3), imply that none of (D2)–(D4) are likely to hold at T^i_{end} — in which case $(I_i^-, T^i_{\mathsf{end}}] = I_i$. Additional Chernoff bounds on star-clocks then imply that v^* is likely to spawn a mutant onto every reservoir vertex

over the first half of I_i, which implies that $\mathcal{P}_1(i+1)$ and $\mathcal{P}_2(i+1)$ are likely to occur. Then $\mathcal{P}_3(i+1)$ is likely to occur by (C3), and $\mathcal{P}_4(i+1)$ is likely to occur by (C2) combined with a relatively simple argument. This implies that the filtration at I_{i+1}^- is likely to be good, as required by Lemma 6.27. This part of the argument is mostly contained in Lemmas 6.41 and 6.42.

It remains only to prove that if i is sufficiently large, X fixates with probability at least $1/2$. In this case, we use similar arguments to the above to show that with probability at least $5/6$, v^* spawns no non-mutants at all over the course of I_i. In this case, similar arguments to those used to deal with $\mathcal{P}_4(i+1)$ work to show that X is likely to fixate as required.

6.5. The behaviour of mutants within cliques

In order to describe the behaviour of mutants in cliques in the megastar process, we require the following definitions.

Definition 6.4. Given a clique $j \in [\ell]$ and a time $t_0 \geq 0$, define the following stopping times $T_{\mathsf{c}}^{t_0,j}(h)$ for $h \in \mathbb{Z}_{\geq 0}$.

— If $h = 0$, let $T_{\mathsf{c}}^{t_0,j}(h) = t_0$.
— If $h > 0$ and K_j is active at $T_{\mathsf{c}}^{t_0,j}(h-1)$, then let

$$T_{\mathsf{c}}^{t_0,j}(h) = \min\left\{t > T_{\mathsf{c}}^{t_0,j}(h-1) \mid X'_t \cap K_j \neq X'_{T_{\mathsf{c}}^{t_0,j}(h-1)} \cap K_j\right\}.$$

— Otherwise, let $T_{\mathsf{c}}^{t_0,j}(h) = T_{\mathsf{c}}^{t_0,j}(h-1)$. □

The subscript "c" in $T_{\mathsf{c}}^{t_0,j}(h)$ stands for "change" because if K_j is active at t_0, then $T_{\mathsf{c}}^{t_0,j}(1), T_{\mathsf{c}}^{t_0,j}(2), \ldots$ are the times at which the number of mutants in K_j changes after t_0 until K_j next becomes inactive. We also use the following definition.

Definition 6.5. Let $h_{\mathsf{in}}^{t_0,j} = \min\{h \in \mathbb{Z}_{\geq 0} \mid K_j \text{ is inactive at } T_{\mathsf{c}}^{t_0,j}(h)\}$. □

The subscript "in" in $h_{\mathsf{in}}^{t_0,j}$ stands for "inactive". Note that $h_{\mathsf{in}}^{t_0,j}$ is finite with probability 1. Thus, with probability 1, the following is well-defined.

Definition 6.6. Let $T_{\mathsf{in}}^{t_0,j} = T_{\mathsf{c}}^{t_0,j}(h_{\mathsf{in}}^{t_0,j})$. □

Definition 6.7. For every $h \in \mathbb{Z}_{\geq 0}$, let $Y^{t_0,j}(h) = |X'_{T_{\mathsf{c}}^{t_0,j}(h)} \cap K_j|$. For $h \in \{1, \ldots, h_{\mathsf{in}}^{t_0,j}\}$, we say that $Y^{t_0,j}$ *jumps* at time $T_{\mathsf{c}}^{t_0,j}(h)$. □

We first show that $Y^{t_0,j}$ evolves as a Markov chain.

LEMMA 6.8. *Suppose $\ell \geq 3$, and consider any $j \in [\ell]$ and $t_0 \geq 0$. Let f be any possible value of $\mathcal{F}_{t_0}(X')$. Conditioned on the event $\mathcal{F}_{t_0}(X') = f$, $Y^{t_0,j}$ evolves as a discrete-time Markov chain on states $\{0, \ldots, k\}$ starting from state $Y^{t_0,j}(0)$ with the following transition matrix.*

$$\begin{aligned}
p'_{0,0} &= 1, \\
p'_{i,i+1} &= \frac{r(k-i)}{(r+1)(k-i)+1} \text{ for all } i \in [k-1], \\
p'_{i,i-1} &= 1 - p'_{i,i+1} \text{ for all } i \in [k-1], \\
p'_{k,k} &= 1, \\
p'_{i,j} &= 0, \text{ otherwise.}
\end{aligned}$$

PROOF. Consider any $t' \geq t_0$, any integer $h \geq 0$, any $y_0, \ldots, y_h \in \{0, \ldots, k\}$ and any possible value f' of $\mathcal{F}_{t'}(X')$ that implies that $\mathcal{F}_{t_0}(X') = f$, $Y^{t_0,j}(0) = y_0, \ldots, Y^{t_0,j}(h) = y_h$, and $T_{\mathsf{c}}^{t_0,j}(h) = t'$. We will show that, conditioned on $\mathcal{F}_{t'}(X') = f'$, the distribution of $Y^{t_0,j}(h+1)$ is as claimed in the lemma statement.

First, suppose that $y_h \in \{0, k\}$. In this case, Definition 6.4 guarantees that $T_{\mathsf{c}}^{t_0,j}(h+1) = T_{\mathsf{c}}^{t_0,j}(h)$ so $Y^{t_0,j}(h+1) = Y^{t_0,j}(h)$, which is consistent with $p'_{0,0} = p'_{k,k} = 1$.

Next, suppose that $y_h \in [k-1]$. Then the clique K_j is active throughout the interval $[t_0, T_{\mathsf{c}}^{t_0,j}(h+1))$. By the definition of the megastar process X', the feeder vertex a_j is a non-mutant throughout this interval.

The out-degree of a_j and every vertex in K_j is k. There are $y_h(k-y_h)$ mutant clocks whose sources are in $K_j \cap X'_{t'}$ and targets are in $K_j \setminus X'_{t'}$. Similarly, there are $(k-y_h+1)y_h$ non-mutant clocks whose sources are in $(K_j \cup \{a_j\}) \setminus X'_{t'}$ and targets are in $K_j \cap X'_{t'}$. Thus after t', the number of mutants in K_j increases with rate $u := ry_h(k-y_h)/k$ and decreases with rate $d := (k-y_h+1)y_h/k$. It follows that

$$\mathbb{P}(Y^{t_0,j}(h+1) = y_h + 1 \mid \mathcal{F}_{t'}(X') = f') = \frac{u}{u+d} = p'_{y_h, y_h+1},$$
$$\mathbb{P}(Y^{t_0,j}(h+1) = y_h - 1 \mid \mathcal{F}_{t'}(X') = f') = p'_{y_h, y_h-1},$$

as required. □

We will dominate the Markov chain $Y^{t_0,j}(h)$ in terms of a simpler Markov chain Z, which is defined as follows.

Definition 6.9. Let $c_r = \lceil 2r/(r-1) \rceil$ and let $r' = (r+1)/2$. Let Z be the discrete-time Markov chain on states $\{0, \ldots, k\}$ with the following transition matrix.

$$\begin{aligned}
p_{0,0} &= 1, \\
p_{i,i+1} &= \begin{cases} r'/(r'+1) & \text{if } 1 \leq i \leq k - c_r, \\ 1/3 & \text{if } k - c_r < i \leq k-1, \end{cases} \\
p_{i,i-1} &= 1 - p_{i,i+1} \text{ for all } i \in [k-1], \\
p_{k,k} &= 1, \\
p_{i,j} &= 0, \text{ otherwise.} \quad \square
\end{aligned}$$

Note that the value k used in Definition 6.9 is the same as $k = k(\ell)$ in our parameterisation of the megastar (Definition 1.5). We now show that $Y^{t_0,j}$ is dominated below by Z, starting from state $Y^{t_0,j}(0)$.

LEMMA 6.10. *Suppose $\ell \geq 3$, and consider any $j \in [\ell]$ and $t_0 \geq 0$. Let f be any possible value of $\mathcal{F}_{t_0}(X')$. Conditioned on the event $\mathcal{F}_{t_0}(X') = f$, $Y^{t_0,j}$ is dominated below by the Markov chain Z, starting from state $Y^{t_0,j}(0)$.*

PROOF. Given Lemma 6.8, the two things to show are

(i) for all $i \in [k-1]$, $p'_{i,i+1} \geq p_{i,i+1}$, and
(ii) for $0 < i < i+1 < k$, $p'_{i+1,i+2} \geq p_{i,i+1}$.

Since $p'_{i+1,i+2} \leq p'_{i,i+1}$, it suffices to prove the second of these, together with $p'_{k-1,k} \geq p_{k-1,k}$.

We start with the latter. We have

$$p'_{k-1,k} = \frac{r}{(r+1)+1} \geq \frac{1}{3},$$

which gives the desired bound since, from $c_r \geq 2$, we have $k - c_r < k-1$ and so $p_{k-1,k} = 1/3$.

So now we must consider i satisfying $0 < i < k-1$ and we must show $p'_{i+1,i+2} \geq p_{i,i+1}$.

Case 1: $i \leq k - c_r$. We have

$$p'_{i+1,i+2} - p_{i,i+1} = \frac{r(k-(i+1))}{(r+1)(k-(i+1))+1} - \frac{r'}{r'+1}.$$

Since $k > i+1$, the common denominator of these fractions is positive; the numerator of the difference is easily calculated to be $(k-i)(r-r') - r \geq c_r(r-1)/2 - r \geq 0$.

Case 2: $i > k - c_r$. Since i and k are integers, $i+1 < k$ implies that $1 \leq k-(i+1)$. Thus,

$$p'_{i+1,i+2} = \frac{r(k-(i+1))}{(r+1)(k-(i+1))+1} \geq \frac{r(k-(i+1))}{(r+2)(k-(i+1))} = \frac{r}{r+2} \geq \frac{1}{3}.$$

As $i > k - c_r$, $p_{i,i+1} = 1/3$, so this gives the desired bound. □

We now use our observations about the gambler's ruin problem in Section 2.3 to derive some simple bounds on the behaviour of the Markov chain Z.

Definition 6.11. Let $\Delta = \lfloor c_r(\log n)^3 \rfloor$. □

Note that $k \geq 2\Delta$ as long as ℓ is sufficiently large, and that having ℓ sufficiently large also implies a lower bound on n.

LEMMA 6.12. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Suppose that Z is started in state $k - c_r$. Then the following statements hold.*

(i) The probability of reaching state k without passing through state $k - \Delta$ is at least $1 - e^{-(\log n)^3}$.
(ii) The expected number of transitions that it takes to reach state k or state $k - 2\Delta$ is at most $\log n$.

PROOF. Recall the notation $H_{x;S}$ and $p_{x\to y;z}$ from Lemma 2.9. We first prove Item (i). Apply Lemma 2.9 with $p_1 = r'/(r'+1) = (r+1)/(r+3) > 1/2$, $a = k - 2\Delta$, $b = k - \Delta$, $c = k - c_r$ and $d = k$. By Item (i) of Lemma 2.9,

$$p_{c\to d;b} \geq 1 - \left(\frac{1-p_1}{p_1}\right)^{c-b} 2^{d-c}.$$

Note that $(1-p_1)/p_1 = 2/(r+1) = 1 - (r-1)/(r+1)$ and $\Delta \geq c_r(\log n)^3 - 1$, so, for all n sufficiently large with respect to r, it holds that

$$\begin{aligned}\left(\frac{1-p_1}{p_1}\right)^{\Delta - c_r} 2^{c_r} &\leq \left(1 - \frac{r-1}{r+1}\right)^{c_r(\log n)^3} (r+1)^{c_r+1} \\ &\leq \exp\left(-c_r\frac{r-1}{r+1}(\log n)^3 + (c_r+1)\log(r+1)\right) \leq e^{-(\log n)^3},\end{aligned}$$

where in the second inequality we used (4) and in the last inequality we used the fact that $c_r \frac{r-1}{r+1} > 1$. Item (i) thus follows.

We next prove Item (ii). Apply Lemma 2.9 with the same parameters as before. By Lemma 2.9(ii), $\mathbb{E}[H_{c,\{a,d\}}] \leq 2^{d-c+1}\left(\frac{3p_1-1}{2p_1-1}\right)$ so it suffices to show

$$2^{c_r+1}\left(\frac{3p_1-1}{2p_1-1}\right) \leq \log n\,,$$

which again holds as long as n is sufficiently large with respect to r. □

Next, we use the domination of $Y^{t_0,j}$ by Z (Lemma 6.10) and our observations about Z (Lemma 6.12) to derive some conclusions about mutants in cliques.

LEMMA 6.13. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let $j \in [\ell]$, $t_0 \geq 0$, and $y_0 \in \{0, \ldots, k\}$. Let f be a possible value of $\mathcal{F}_{t_0}(X')$ which implies that $Y^{t_0,j}(0) = y_0$. Then the following statements hold.*

(i) If $y_0 \geq k-1$, then

$$\mathbb{P}\Big(\forall t \in [t_0, T_{\mathsf{in}}^{t_0,j}], |K_j \setminus X'_t| \leq \Delta \,\Big|\, \mathcal{F}_{t_0}(X') = f\Big) \geq 1 - e^{-(\log n)^3}.$$

(ii) If $y_0 \geq k-\Delta$, then

$$\mathbb{P}\Big(\forall t \in [t_0, T_{\mathsf{in}}^{t_0,j}], |K_j \setminus X'_t| \leq 2\Delta \,\Big|\, \mathcal{F}_{t_0}(X') = f\Big) \geq 1 - 2e^{-(\log n)^3}.$$

PROOF. Recall the notation $p_{x\to y;z}$ from Lemma 2.9. The Markov chain Z is the same as the chain in Lemma 2.9 with $p_1 = r'/(r'+1)$, $a = 0$, $c = k - c_r$ and $d = k$. We start with simple lower bounds concerning Z.

— From any state $i \in \{k - c_r, \ldots, k\}$, Z must reach either state k or state $k - c_r$ before reaching state $k - \Delta$. By Lemma 6.12(i), $p_{k-c_r\to k;k-\Delta} \geq 1 - e^{-(\log n)^3}$, and we have

$$p_{i\to k;k-\Delta} \geq p_{k-c_r\to k;k-\Delta} \geq 1 - e^{-(\log n)^3}. \tag{36}$$

— Consider $i \in \{k-\Delta, \ldots, k-c_r-1\}$. By Corollary 2.8(i),

$$p_{i\to k-c_r;k-2\Delta} \geq 1 - (r')^{-\Delta} \geq 1 - e^{-\Delta(r'-1)/r'} \geq 1 - e^{-(\log n)^3},$$

where the last inequality holds for all n sufficiently large with respect to r, using the fact that $c_r \frac{r'-1}{r'} = c_r \frac{r-1}{r+1} > 1$. It follows from Lemma 6.12(i) that

$$p_{i\to k;k-2\Delta} \geq p_{i\to k-c_r;k-2\Delta} - p_{k-c_r\to k-2\Delta;k} \geq 1 - 2e^{-(\log n)^3}. \tag{37}$$

The result now follows easily. By Lemma 6.10, $Y^{t_0,j}$ is dominated below by Z with initial state y_0 conditioned on $\mathcal{F}_{t_0}(X') = f$. Thus Item (i) follows from (36) and Item (ii) follows from (36) and (37). □

Note that in the process of proving Lemma 6.13, we proved the following (see (36) and (37)).

COROLLARY 6.14. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let y_0 be an integer satisfying $k - \Delta \leq y_0 \leq k$. The probability that Z, when started from y_0, reaches state k without passing through state $k - 2\Delta$, is at least $1 - 2e^{-(\log n)^3}$.* □

We now use Lemma 6.12 and Corollary 6.14 to give lower bounds on the probability that Z reaches k in a relatively short time.

LEMMA 6.15. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Consider the Markov chain Z, starting from $y_0 \in [k-1]$.*

(i) If $k - \Delta \leq y_0 \leq k-1$, then $\mathbb{P}(Z_{\lceil 15c_r^2(\log n)^6\rceil} = k) \geq 1 - 3e^{-(\log n)^3}$.
(ii) If $1 \leq y_0 \leq k-\Delta$, then $\mathbb{P}(Z_{\lceil 5c_r^2 k\rceil} = k) \geq \frac{r-1}{5r}$.

PROOF. Recall the notation $H_{x;S}$ and $p_{x\to y;z}$ from Lemma 2.9. The Markov chain Z is the same as the Markov chain in Lemma 2.9 with $p_1 = r'/(r'+1)$, $a = 0$, $c = k - c_r$ and $d = k$. Let $\eta = \lceil 5ec_r\Delta\rceil \lceil (\log n)^3\rceil$ and $\eta' = \lceil 4c_r^2 k\rceil$.

We first establish Item (i). Note that $\lceil 15c_r^2(\log n)^6\rceil \geq \eta$, so it suffices to establish the stronger statement that for all y_0 satisfying $k-\Delta \leq y_0 \leq k-1$, we have $\mathbb{P}(Z_\eta = k) \geq 1 - 3e^{-(\log n)^3}$. So suppose $k-\Delta \leq y_0 \leq k-1$. Note that

$$\mathbb{P}(Z_\eta = k) \geq \mathbb{P}(H_{y_0;\{k-2\Delta,k\}} \leq \eta) - p_{y_0\to k-2\Delta;k}.$$

Corollary 6.14 shows that $p_{y_0\to k-2\Delta;k} \leq 2e^{-(\log n)^3}$. So, to show (i), we will show

$$\mathbb{P}(H_{y_0;\{k-2\Delta,k\}} \leq \eta) \geq 1 - e^{-(\log n)^3}. \tag{38}$$

To establish (38), we first show the following.

$$\text{For every integer } y \text{ satisfying } k-2\Delta < y < k,\ \mathbb{E}[H_{y;\{k-2\Delta,k\}}] \leq 5c_r\Delta. \tag{39}$$

To see (39), there are three cases to consider.

Case 1: $y = k - c_r$. By Lemma 6.12(ii), we have the (tighter) bound

$$\mathbb{E}[H_{y;\{k-2\Delta,k\}}] \leq \log n. \tag{40}$$

Case 2: $k - 2\Delta < y < k - c_r$. By Corollary 2.8(iii), we have

$$\mathbb{E}[H_{y;\{k-2\Delta,k-c_r\}}] \leq \frac{2\Delta(r'+1)}{r'-1} = \frac{2\Delta(r+3)}{r-1} \leq 4c_r\Delta.$$

Combining this and (40), we obtain

$$\mathbb{E}[H_{y;\{k-2\Delta,k\}}] \leq \mathbb{E}[H_{y;\{k-2\Delta,k-c_r\}}] + \mathbb{E}[H_{k-c_r;\{k-2\Delta,k\}}] \leq 5c_r\Delta.$$

Case 3: $k - c_r < y < k$. By Corollary 2.8(ii) we have $\mathbb{E}[H_{y;\{k-c_r,k\}}] \leq 3c_r$, and so

$$\mathbb{E}[H_{y;\{k-2\Delta,k\}}] \leq \mathbb{E}[H_{y;\{k-c_r,k\}}] + \mathbb{E}[H_{k-c_r;\{k-2\Delta,k\}}] \leq 3c_r + \log n \leq 2\log n.$$

These three cases establish (39). Applying Markov's inequality gives the following.

$$\text{For every integer } y \text{ satisfying } k-2\Delta < y < k,\quad \mathbb{P}(H_{y;\{k-2\Delta,k\}} \geq \lceil 5ec_r\Delta\rceil) \leq \frac{1}{e}. \tag{41}$$

Now (38) follows by subdividing the set $[\eta]$ (indexing η transitions from the initial state y_0) into $\lceil (\log n)^3\rceil$ disjoint sets of contiguous indices, each of size $\lceil 5ec_r\Delta\rceil$, then applying (41) to each subset.

We next establish item (ii), so suppose $1 \leq y_0 \leq k-\Delta$. By Corollary 2.8(iii), for all $y_0 \in \{1,\ldots,k-c_r\}$, we have

$$\mathbb{E}[H_{y_0;\{0,k-c_r\}}] \leq \frac{k(r'+1)}{r'-1} = \frac{k(r+3)}{r-1} \leq 2c_r k.$$

Applying Markov's inequality, we obtain

$$\mathbb{P}(H_{y_0;\{0,k-c_r\}} \leq \lceil 4c_r^2 k\rceil) \geq 1 - \frac{1}{2c_r} \geq 1 - \frac{r-1}{4r}.$$

Now by Corollary 2.8(i), $p_{y_0\to 0;k-c_r} \leq \frac{1}{r'}$. So

$$\mathbb{P}(H_{y_0;\{k-c_r\}} \leq \eta') \geq \mathbb{P}(H_{y_0;\{0,k-c_r\}} \leq \eta') - p_{y_0\to 0;k-c_r} \geq 1 - \frac{r-1}{4r} - \frac{2}{r+1}.$$

Now, using the fact that, starting from $Z_0 = k - c_r$, we have $\mathbb{P}(Z_\eta = k) \geq 1 - 3e^{-(\log n)^3}$, which we proved in the derivation of (i), we have, starting from $Z_0 = y_0$,

$$\mathbb{P}(Z_{\eta+\eta'} = k) \geq 1 - 3e^{-(\log n)^3} - \frac{r-1}{4r} - \frac{2}{r+1} = (r-1)\left(\frac{1}{r+1} - \frac{1}{4r}\right) - 3e^{-(\log n)^3} \geq \frac{r-1}{5r},$$

which completes the proof, since $\eta + \eta' \leq \lceil 5c_r^2 k \rceil$ (which we will now justify).

To see that $\eta + \eta' \leq \lceil 5c_r^2 k \rceil$, recall from the very start of Section 6 (just before the statement of Theorem 6.1) that $\sqrt{n}/2 \leq \ell \leq \sqrt{n}$ so $n \leq 4\ell^2$. Furthermore, $n \geq \ell^2$ so the condition $\ell \geq \ell_0$ in the statement of the lemma also guarantees that n is sufficiently large.

From the definition at the beginning of the proof, $\eta = \lceil 5ec_r\Delta \rceil \lceil (\log n)^3 \rceil$, so plugging in $\Delta = \lfloor c_r(\log n)^3 \rfloor$ (from the glossary in in Section 6.1) and using the fact that n is sufficiently large, we get $\eta \leq 6ec_r^2(\log n)^6$. Since c_r is a constant (depending on r, but not on n) and n is sufficiently large, this gives $\eta \leq (\log n)^7$. Now since $n \leq 4\ell^2$ and ℓ is sufficiently large, we have $\log n \leq 3\log \ell$ so $\eta \leq 3^7(\log \ell)^7 \leq (\log \ell)^8 = (\log \ell)^{23}/(\log \ell)^{15}$. Finally, plugging in $k = \lceil (\log \ell)^{23} \rceil$ (from the glossary in Section 6.1), $\eta \leq k/(\log \ell)^{15}$. Since ℓ is sufficiently large, this is (easily) smaller than $\lceil 5c_r^2 k \rceil - \eta' = \lceil 5c_r^2 k \rceil - \lceil 4c_r^2 k \rceil$. □

The following corollary follows immediately from Lemma 6.10 and Lemma 6.15.

COROLLARY 6.16. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let $j \in [\ell]$, let $t_0 \geq 0$, and let $y_0 \in [k-1]$. Let f be a possible value of $\mathcal{F}_{t_0}(X')$ which implies that $Y^{t_0,j}(0) = y_0$. Then the following statements hold.*

(i) If $k - \Delta \leq y_0 \leq k-1$, then

$$\mathbb{P}\Big(Y^{t_0,j}(\lceil 15c_r^2(\log n)^6 \rceil) = k \,\Big|\, \mathcal{F}_{t_0}(X') = f\Big) \geq 1 - 3e^{-(\log n)^3}.$$

(ii) If $1 \leq y_0 \leq k - \Delta$, then

$$\mathbb{P}\Big(Y^{t_0,j}(\lceil 5c_r^2 k \rceil) = k \,\Big|\, \mathcal{F}_{t_0}(X') = f\Big) \geq \frac{r-1}{5r}. \quad \square$$

To translate Corollary 6.16 into a bound on the time it takes K_j to fill with mutants, we will require the following lemma.

LEMMA 6.17. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let $j \in [\ell]$ and $t_0 > 0$. Let f be a possible value of $\mathcal{F}_{t_0}(X')$ which implies that K_j is active at t_0. Let $t^* \geq 16(\log n)^3$. Then, conditioned on the event $\mathcal{F}_{t_0}(X') = f$, with probability at least $1 - e^{-(\log n)^3}$, $T_{\mathsf{c}}^{t_0,j}(\lceil t^*/2 \rceil) < t_0 + t^*$.*

PROOF. We will first show that $T_{\mathsf{c}}^{t_0,j}(1) - T_{\mathsf{c}}^{t_0,j}(0), \ldots, T_{\mathsf{c}}^{t_0,j}(\lceil t^*/2 \rceil) - T_{\mathsf{c}}^{t_0,j}(\lceil t^*/2 \rceil - 1)$ are dominated above by i.i.d. exponential variables with rate 1.

Fix $0 \leq h \leq \lceil t^*/2 \rceil - 1$, let $x \geq 0$ and let $t_0, \ldots, t_h \geq 0$. Suppose f_h is a possible value of $\mathcal{F}_{t_h}(X')$ which implies that $T_{\mathsf{c}}^{t_0,j}(0) = t_0, \ldots, T_{\mathsf{c}}^{t_0,j}(h) = t_h$ and $\mathcal{F}_{t_0}(X') = f$. Fix $t \geq 0$. Note that f_h determines the event $T_{\mathsf{in}}^{t_0,j} \leq t_h$, as well as the value of $Y^{t_0,j}(h)$ — write $y_h = Y^{t_0,j}(h)$. If f_h is such that $T_{\mathsf{in}}^{t_0,j} \leq t_h$, then we have $T_{\mathsf{c}}^{t_0,j}(h+1) - t_h = 0$ and so

$$\mathbb{P}\big(T_{\mathsf{c}}^{t_0,j}(h+1) - t_h \leq x \mid \mathcal{F}_{t_h}(X') = f_h\big) = 1 \geq 1 - e^{-x}. \tag{42}$$

Suppose instead that f_h is such that $T_{\mathsf{in}}^{t_0,j} > t_h$, so that $1 \leq y_h \leq k-1$. Then $T_{\mathsf{c}}^{t_0,j}(h+1) - t_h$ is the amount of time it takes after t_h for a vertex in $K_j \cup \{a_j\}$ to spawn either a mutant onto a non-mutant in K_j or a non-mutant onto a mutant in K_j.

Thus, conditioned on $\mathcal{F}_{t_h}(X') = f_h$, (see the proof of Lemma 6.8) $T_{\mathsf{c}}^{t_0,j}(h+1) - t_h$ is an exponential variable with rate

$$\frac{ry_h(k-y_h)}{k} + \frac{y_h(k-y_h)}{k} + \frac{y_h}{k} \geq \frac{(r+1)y_h(k-y_h)}{k} \geq \frac{(r+1)(k-1)}{k} \geq \frac{2(k-1)}{k} \geq 1.$$

In particular, we have shown that

$$\mathbb{P}\big(T_{\mathsf{c}}^{t_0,j}(h+1) - t_h \leq x \mid \mathcal{F}_{t_h}(X') = f_h\big) \geq 1 - e^{-x}. \tag{43}$$

By (42) and (43) we have

$$\mathbb{P}\big(T_{\mathsf{c}}^{t_0,j}(h+1) - t_h \leq x \mid T_{\mathsf{c}}^{t_0,j}(0) = t_0, \ldots, T_{\mathsf{c}}^{t_0,j}(h) = t_h, \mathcal{F}_{t_0}(X') = f\big) \geq 1 - e^{-x},$$

and so $T_{\mathsf{c}}^{t_0,j}(1) - T_{\mathsf{c}}^{t_0,j}(0), \ldots, T_{\mathsf{c}}^{t_0,j}(\lceil t^*/2\rceil) - T_{\mathsf{c}}^{t_0,j}(\lceil t^*/2\rceil - 1)$ are dominated above by i.i.d. exponential variables with rate 1 as claimed.

It follows that $T_{\mathsf{c}}^{t_0,j}(\lceil t^*/2\rceil) - t_0$ is dominated above by a sum of $\lceil t^*/2\rceil$ i.i.d. exponential variables with rate 1. We have $t^* \geq 3\lceil t^*/2\rceil/2$, so by Corollary 2.4 we have

$$\mathbb{P}\big(T_{\mathsf{c}}^{t_0,j}(\lceil t^*/2\rceil) < t_0 + t^*\big) = \mathbb{P}\big(T_{\mathsf{c}}^{t_0,j}(\lceil t^*/2\rceil) - t_0 < t^*\big) \geq 1 - e^{-t^*/16} \geq 1 - e^{-(\log n)^3}. \quad \square$$

We now use Lemma 6.13, Corollary 6.16 and Lemma 6.17, to prove three results which contain all the properties of cliques that we will need to prove our key lemma, Lemma 6.3.

COROLLARY 6.18. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let $j \in [\ell]$, $t_0 \geq 0$, and $y_0 \in [k]$. Let f be a possible value of $\mathcal{F}_{t_0}(X')$ which implies that $Y^{t_0,j}(0) = y_0$. Then the following statements hold.*

(i) If $k - \Delta \leq y_0 \leq k$, then

$$\mathbb{P}\big(T_{\mathsf{in}}^{t_0,j} \leq t_0 + 30c_r^2(\log n)^6 \text{ and } K_j \subseteq X'_{T_{\mathsf{in}}^{t_0,j}} \,\big|\, \mathcal{F}_{t_0}(X') = f\big) \geq 1 - 4e^{-(\log n)^3}.$$

(ii) If $1 \leq y_0 \leq k - \Delta$, then

$$\mathbb{P}\big(T_{\mathsf{in}}^{t_0,j} \leq t_0 + 10c_r^2 k \text{ and } K_j \subseteq X'_{T_{\mathsf{in}}^{t_0,j}} \,\big|\, \mathcal{F}_{t_0}(X') = f\big) \geq \frac{r-1}{6r}.$$

PROOF. If $y_0 = k$, then $T_{\mathsf{in}}^{t_0,j} = t_0$ and the result follows immediately, so suppose instead that $y_0 \in [k-1]$. This implies $t_0 > 0$. Part (i) now follows on applying a union bound to Corollary 6.16(i) and Lemma 6.17, taking $t^* = 30c_r^2(\log n)^6$. Part (ii) likewise follows immediately by applying a union bound to Corollary 6.16(ii) and Lemma 6.17, taking $t^* = 10c_r^2 k$. $\square$

We combine Lemma 6.13 and Corollary 6.18 to prove the following.

LEMMA 6.19. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let $j \in [\ell]$. Let $I = (I^-, I^+]$ be a time interval with $30c_r^2(\log n)^6 < \mathrm{len}(I) \leq e^{(\log n)^2}$, and let f be a possible value of $\mathcal{F}_{I^-}(X')$ which implies that $|K_j \setminus X'_{I^-}| \leq \Delta$. Then, conditioned on $\mathcal{F}_{I^-}(X') = f$, with probability at least $1 - e^{-\frac{1}{2}(\log n)^3}$ the following statements all hold.*

(i) For all $t \in I$, $|K_j \setminus X'_t| \leq 2\Delta$.
(ii) $|K_j \setminus X'_{I^+}| \leq \Delta$.
(iii) For all $t_0 \in [I^-, I^+ - 30c_r^2(\log n)^6)$, there exists $t_1 \in [t_0, t_0 + 30c_r^2(\log n)^6]$ such that $K_j \subseteq X'_{t_1}$.

PROOF. We start with the following mutually recursive definitions. Let $T_{\mathsf{in}}(-1) = I^-$. Then, for $h \in \mathbb{Z}_{\geq 0}$,

$$T_{\mathsf{a}}(h) = \min\{t \geq T_{\mathsf{in}}(h-1) \mid K_j \text{ is active at } t \text{ or } t = I^+\},$$
$$T_{\mathsf{in}}(h) = \min\{t \geq T_{\mathsf{a}}(h) \mid K_j \text{ is inactive at } t \text{ or } t = I^+\}.$$

These definitions of $T_{\mathsf{in}}(h)$ and $T_{\mathsf{a}}(h)$ are local to this proof. The subscript "a" stands for "active" and the subscript "in" stands for "inactive". The notation must not be confused with the global variable $T_{\mathsf{in}}^{t_0,j}$. Let $\xi = \lceil 4r\mathrm{len}(I)\rceil$ and let $\kappa = 30c_r^2(\log n)^6$.

We define the following events.

— Let $\mathcal{E}_1$ be the event that $T_{\mathsf{a}}(\xi) = I^+$.
— For $h \in \mathbb{Z}_{\geq 0}$, let $\mathcal{E}_2(h)$ be the event that $T_{\mathsf{in}}(h) \leq T_{\mathsf{a}}(h) + \kappa$.
— The event $\mathcal{F}_{I^-}(X') = f$ determines whether K_j is active at time I^-.
  — If so, let $\mathcal{E}_3(0)$ be the event that, for all $t \in [T_{\mathsf{a}}(0), T_{\mathsf{in}}(0)]$, $|K_j \setminus X'_t| \leq 2\Delta$.
  — If not, let $\mathcal{E}_3(0)$ be the event that, for all $t \in [T_{\mathsf{a}}(0), T_{\mathsf{in}}(0)]$, $|K_j \setminus X'_t| \leq \Delta$.
— For $h \in \mathbb{Z}_{\geq 1}$, let $\mathcal{E}_3(h)$ be the event that, for all $t \in [T_{\mathsf{a}}(h), T_{\mathsf{in}}(h)]$, $|K_j \setminus X'_t| \leq \Delta$.
— For $h \in \mathbb{Z}_{\geq -1}$, let $\mathcal{E}_4(h)$ be the event that for all $t \in [T_{\mathsf{in}}(h), T_{\mathsf{a}}(h+1))$, $K_j \subseteq X'_t$.

Note that for all $h \geq 0$, $\mathcal{E}_3(h)$ implies $\mathcal{E}_4(h)$. This is easy to see as long as K_j is inactive at $T_{\mathsf{in}}(h)$. In this case, $\mathcal{E}_3(h)$ implies that $K_j \subseteq X'_{T_{\mathsf{in}}(h)}$ – since the number of non-mutants is at most 2Δ, but it is either 0 or k, it must be 0. On the other hand, if K_j is active at $T_{\mathsf{in}}(h)$ then $T_{\mathsf{in}}(h) = I^+$ so $T_{\mathsf{a}}(h+1) = I^+$ so the interval in $\mathcal{E}_4(h)$ is empty.

We next observe that $\mathcal{F}_{I^-}(X') = f$ implies that $\mathcal{E}_4(-1)$ occurs. From the statement of the lemma, $\mathcal{F}_{I^-}(X') = f$ implies that $|K_j \setminus X'_{I^-}| \leq \Delta$. If K_j is inactive at I^- then this implies that $K_j \subseteq X'_{I^-}$, which implies $\mathcal{E}_4(-1)$. If instead K_j is active at I^- then the interval in $\mathcal{E}_4(-1)$ is empty so $\mathcal{E}_4(-1)$ occurs vacuously.

For any integer q, let $\mathcal{E}_2^q = \bigcap_{h=0}^q \mathcal{E}_2(h)$, $\mathcal{E}_3^q = \bigcap_{h=0}^q \mathcal{E}_3(h)$, and $\mathcal{E}_4^q = \bigcap_{h=-1}^q \mathcal{E}_4(h)$. Let $\mathcal{E}_2 = \mathcal{E}_2^\xi$, $\mathcal{E}_3 = \mathcal{E}_3^\xi$ and $\mathcal{E}_4 = \mathcal{E}_4^\xi$.

We first show that if $\mathcal{F}_{I^-}(X') = f$ and $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ all occur, then statements (i), (ii) and (iii) hold. As we have just observed, $\mathcal{F}_{I^-}(X') = f$ and $\mathcal{E}_3$ imply that $\mathcal{E}_4$ also occurs. Then $\mathcal{E}_3$, $\mathcal{E}_4$ and $\mathcal{E}_1$ imply (i). They also imply (ii) except in the case where K_j is active at I^- and remains active for all of I. This case is ruled out by $\mathcal{E}_2(0)$ since $\mathrm{len}(I) > \kappa$. We now turn to statement (iii). Consider any $t_0 \in [I^-, I^+ - \kappa)$. Suppose first that K_j is inactive at time t_0. Since (i) holds, $K_j \subseteq X'_{t_0}$, so it suffices to take $t_1 = t_0$. Suppose instead that K_j is active at time t_0. Then, for some $h \geq 0$, $t_0 \in [T_{\mathsf{a}}(h), T_{\mathsf{in}}(h)]$. By $\mathcal{E}_1$, $h \leq \xi$. By $\mathcal{E}_2(h)$, we may assume $T_{\mathsf{in}}(h) \leq t_0 + \kappa$ so we can choose $t_1 = T_{\mathsf{in}}(h)$. Since $t_0 + \kappa < I^+$, $t_1 < I^+$ so $t_1 \in [T_{\mathsf{in}}(h), T_{\mathsf{a}}(h+1))$ and $\mathcal{E}_4(h)$ guarantees that $K_j \subseteq X'_{t_1}$.

During the remainder of the proof, we will show that

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \mid \mathcal{F}_{I^-}(X') = f) \geq 1 - e^{-\frac{1}{2}(\log n)^3}. \tag{44}$$

It is clear that $\mathcal{E}_1$ occurs if clocks with source a_j trigger (in total) fewer than ξ times in I. These clocks have total rate $1 + r \leq 2r$, so by Corollary 2.2 it follows that

$$\mathbb{P}(\mathcal{E}_1 \mid \mathcal{F}_{I^-}(X') = f) \geq 1 - e^{-(1+r)\mathrm{len}(I)/3} \geq 1 - e^{-(\log n)^3}. \tag{45}$$

Now consider any $h \in \{0, \ldots, \xi\}$ and any $t_h \in [I^-, I^+]$. If the events $T_{\mathsf{a}}(h) = t_h$, $\mathcal{F}_{I^-}(X') = f$, $\mathcal{E}_2^{h-1}$, $\mathcal{E}_3^{h-1}$ and $\mathcal{E}_4^{h-1}$ are consistent then let f_h be any value of $\mathcal{F}_{t_h}(X')$ such that $\mathcal{F}_{t_h}(X') = f_h$ implies all of these events. Suppose that $\mathcal{F}_{t_h}(X') = f_h$ and consider how many non-mutants K_j can have at time t_h.

Case 1. If $h = 0$ and K_j is active at I^- then $t_h = I^-$ and it follows from the assumption in the statement of the lemma that K_j has at most Δ non-mutants at time t_h.

Case 2. Otherwise, if $T_{\mathsf{in}}(h-1) < I^+$, then $[T_{\mathsf{in}}(h-1), T_{\mathsf{a}}(h))$ is non-empty, so it follows from $\mathcal{E}_4(h-1)$ that K_j has at most one non-mutant at time t_h.

Case 3. Otherwise, $T_{\mathsf{in}}(h-1) = t_h = I^+$ so it follows from $\mathcal{E}_3(h-1)$ that K_j has at most Δ non-mutants at $T_{\mathsf{in}}(h-1) = t_h$.

In any of these three cases, Corollary 6.18(i) implies that

$$\mathbb{P}(\mathcal{E}_2(h) \mid \mathcal{F}_{t_h}(X') = f_h) \geq 1 - 4e^{-(\log n)^3}. \tag{46}$$

In Case 1, Lemma 6.13(ii) implies that

$$\mathbb{P}(\mathcal{E}_3(h) \mid \mathcal{F}_{t_h}(X') = f_h) \geq 1 - 2e^{-(\log n)^3}. \tag{47}$$

In Case 2, Lemma 6.13(i) implies that

$$\mathbb{P}(\mathcal{E}_3(h) \mid \mathcal{F}_{t_h}(X') = f_h) \geq 1 - e^{-(\log n)^3}.$$

In Case 3, $T_{\mathsf{in}}(h-1) = T_{\mathsf{a}}(h) = T_{\mathsf{in}}(h)$ so $\mathcal{E}_3(h)$ is implied by $\mathcal{E}_3(h-1)$ hence

$$\mathbb{P}(\mathcal{E}_3(h) \mid \mathcal{F}_{t_h}(X') = f_h) = 1.$$

Equation (47) gives the worst bound of the three cases. Combining this with (46) using a union bound, we have

$$\mathbb{P}(\overline{\mathcal{E}_2(h)} \cup \overline{\mathcal{E}_3(h)} \mid \mathcal{F}_{I^-}(X') = f, \mathcal{E}_2^{h-1}, \mathcal{E}_3^{h-1}) \leq 6e^{-(\log n)^3},$$

so

$$\mathbb{P}(\mathcal{E}_2 \cap \mathcal{E}_3 \mid \mathcal{F}_{I^-}(X') = f) \geq 1 - (1+\xi)6e^{-(\log n)^3},$$

which, together with (45) gives (44). □

The final result of Section 6.5 shows that if an active clique is almost full of mutants, then with high probability it spawns no non-mutants at all onto v^* before becoming inactive, and with even higher probability it doesn't spawn too many non-mutants onto v^* before becoming inactive.

LEMMA 6.20. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let $j \in [\ell]$ and $t_0 \geq 0$. Let f be a possible value of $\mathcal{F}_{t_0}(X')$ which implies that $k - \Delta \leq |X'_{t_0} \cap K_j| \leq k-1$. Let S be the total number of non-mutants spawned onto v^* by vertices in K_j within $(t_0, T_{\mathsf{in}}^{t_0,j}]$. Then the following statements hold.*

(i) $\mathbb{P}(S = 0 \mid \mathcal{F}_{t_0}(X') = f) \geq 1 - (\log n)^{10}/k$.

(ii) $\mathbb{P}(S \leq (\log n)^3 \mid \mathcal{F}_{t_0}(X') = f) \geq 1 - 7e^{-(\log n)^3}$.

PROOF. Let $\mathcal{E}_1$ be the event that for all $t \in [t_0, T_{\mathsf{in}}^{t_0,j}]$, $|X'_t \cap K_j| \geq k - 2\Delta$, and let $\mathcal{E}_2$ be the event that $T_{\mathsf{in}}^{t_0,j} \leq t_0 + 30c_r^2(\log n)^6$. Let $\mathcal{A}$ be the set of all clocks which have both source and target in $K_j \cup \{a_j\}$. Let Φ contain, for each clock $C \in \mathcal{A}$, a list of the times at which C triggers in $(t_0, T_{\mathsf{in}}^{t_0,j}]$. Note that by the definition of the megastar process (Definition 6.2), for all $t \in [t_0, T_{\mathsf{in}}^{t_0,j}]$, $a_j \notin X'_t$. It follows that for all $t \in [t_0, T_{\mathsf{in}}^{t_0,j}]$, Φ and $\mathcal{F}_{t_0}(X')$ together uniquely determine $K_j \cap X'_t$. They therefore determine $T_{\mathsf{c}}^{t_0,j}(h)$ and $Y^{t_0,j}(h)$ for all $h \geq 0$ and hence they determine whether $\mathcal{E}_1$ and $\mathcal{E}_2$ occur.

Let $\chi_0 = K_j \cap X'_{t_0}$. Consider any integer $y \geq 0$, any sets $\chi_1, \ldots, \chi_y \subseteq K_j$ and any $t_1, \ldots, t_y > t_0$. Suppose that φ is a possible value of Φ such that f and φ together imply that $\mathcal{E}_1 \cap \mathcal{E}_2$ occurs, that $h_{\mathsf{in}}^{t_0,j} = y$, and that for all $h \in \{0, \ldots, y\}$, we have $T_{\mathsf{c}}^{t_0,j}(h) = t_h$

and $K_j \cap X'_{t_h} = \chi_h$. If $\mathcal{F}_{t_0}(X') = f$ and $\Phi = \varphi$, then $\mathcal{E}_1 \cap \mathcal{E}_2$ occurs and so

$$|K_j \setminus \chi_h| \leq 2\Delta \text{ for all } h \in \{0, \ldots, y\}, \tag{48}$$

$$t_y \leq t_0 + 30c_r^2(\log n)^6. \tag{49}$$

For all $h \in \{1, \ldots, y\}$, let

$$S_h = |\{t \in (t_{h-1}, t_h) \mid N_{(u,v^*)} \text{ triggers at time } t \text{ for some } u \in K_j \setminus \chi_{h-1}\}|.$$

We will show that, conditioning on $\Phi = \varphi$ and $\mathcal{F}_{t_0}(X') = f$, $S_1, \ldots, S_y$ are dominated above by independent Poisson variables $S'_1, \ldots, S'_y$, where S'_h has parameter $\lambda_h = 2\Delta(t_h - t_{h-1})/k$. Indeed, consider any $h \in [y]$. Consider any integers $s_1, \ldots, s_{h-1}$, and any possible value f_{h-1} of $\mathcal{F}_{t_{h-1}}(X')$ which is consistent with $\Phi = \varphi$ and which implies that $\mathcal{F}_{t_0}(X') = f$ and $S_1 = s_1, \ldots, S_{h-1} = s_{h-1}$.

S_h is independent of Φ by the definition of $\mathcal{C}(\mathcal{M}_\ell)$, since no clocks with target v^* are contained in $\mathcal{A}$, (t_{h-1}, t_h) is a fixed time interval, and χ_{h-1} is a fixed set. Moreover, S_h is independent of $\mathcal{F}_{t_{h-1}}(X')$ by memorylessness. Thus, conditioned on $\mathcal{F}_{t_{h-1}}(X') = f_{h-1}$ and $\Phi = \varphi$, S_h is simply a Poisson variable with parameter $(t_h - t_{h-1})(k - |\chi_{h-1}|)/k$, which is at most λ_h by (48). It therefore follows that for all $a \geq 0$,

$$\mathbb{P}(S_h \leq a \mid \mathcal{F}_{t_{h-1}}(X') = f_{h-1}, \Phi = \varphi) \geq \mathbb{P}(S'_h \leq a),$$

and hence

$$\mathbb{P}(S_h \leq a \mid \mathcal{F}_{t_0}(X') = f, \Phi = \varphi, S_1 = s_1, \ldots, S_{h-1} = s_{h-1}) \geq \mathbb{P}(S'_h \leq a).$$

Thus conditioned on $\mathcal{F}_{t_0}(X') = f$ and $\Phi = \varphi$, $S_1, \ldots, S_y$ are dominated above by $S'_1, \ldots, S'_y$ as claimed.

Note that with probability 1, no non-mutants are spawned onto v^* at times $t_1, \ldots, t_y$, and so $S = S_1 + \cdots + S_y$. It follows from (49) that conditioned on $\mathcal{F}_{t_0}(X') = f$ and $\Phi = \varphi$, S is dominated above by a Poisson variable S' with parameter

$$\lambda_1 + \cdots + \lambda_y = \frac{2\Delta}{k} \sum_{h=1}^{y} (t_h - t_{h-1}) = \frac{2\Delta}{k}(t_y - t_0) \leq \frac{60c_r^3(\log n)^9}{k}.$$

By a union bound applied to Lemma 6.13(ii) and Corollary 6.18(i), $\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \mid \mathcal{F}_{t_0}(X') = f) \geq 1 - 6e^{-(\log n)^3}$. Thus for all $a \geq 0$ we have

$$\begin{aligned}\mathbb{P}(S \leq a \mid \mathcal{F}_{t_0}(X') = f) &\geq \mathbb{P}(S \leq a \mid \mathcal{F}_{t_0}(X') = f, \mathcal{E}_1 \cap \mathcal{E}_2) - \mathbb{P}(\overline{\mathcal{E}_1 \cap \mathcal{E}_2} \mid \mathcal{F}_{t_0}(X') = f) \\ &\geq \mathbb{P}(S' \leq a) - 6e^{-(\log n)^3}. \end{aligned} \tag{50}$$

It is immediate from (50) using (4) that

$$\mathbb{P}(S = 0 \mid \mathcal{F}_{t_0}(X') = f) \geq e^{-60c_r^3(\log n)^9/k} - 6e^{-(\log n)^3} \geq 1 - \frac{(\log n)^{10}}{k},$$

and so part (i) of the result holds. Moreover, by (50) combined with Corollary 2.3, we have

$$\mathbb{P}(S \leq (\log n)^3 \mid \mathcal{F}_{t_0}(X') = f) \geq \mathbb{P}(S' \leq (\log n)^3) - 6e^{-(\log n)^3} \geq 1 - 7e^{-(\log n)^3},$$

and so part (ii) of the result holds. □

6.6. Filling cliques

Recall from Definition 6.2 that X'_0 is the set containing a single initial mutant, and write $X'_0 = \{x_0\}$. Because of the megastar's symmetry, without loss of generality we may assume that $x_0 \in R_1 \cup K_1 \cup \{a_1, v^*\}$. In Section 6.6, we will further restrict our attention to the case where x_0 is in a reservoir, i.e., $x_0 \in R_1$.

6.6.1. The first clique fills with mutants. In Section 6.6.1, we will show that if the initial mutant of X' lies in the reservoir R_1, then with high probability K_1 is almost full of mutants at time n (see Lemma 6.22). We first prove an ancillary lemma.

LEMMA 6.21. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Suppose $X'_0 \subseteq R_1$, and write $X'_0 = \{x_0\}$. Let $\mathcal{E}_1$ be the event that $N_{(v^*,x_0)}$ does not trigger in $[0, 17m(\log n)^2]$. Let $t_0 \in [0, 17m((\log n)^2 - 1)]$, and let f be a possible value of $\mathcal{F}_{t_0}(X')$ which is consistent with $\mathcal{E}_1$. Then*

$$\mathbb{P}(\text{there exists } t_1 \in [t_0, t_0 + 17m] \text{ such that } K_1 \subseteq X'_{t_1} \mid \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f) \geq \frac{r-1}{12r}.$$

PROOF. Let

$$\mathcal{C}_{a_1} = \{N_{(u,a_1)} \mid u \in R_1\} \cup \{M_{(a_1,v)} \mid v \in K_1\}.$$

For $h \in \{0, \ldots, 8m-1\}$, let

$$T_h = \min\{t > t_0 + 2h \mid M_{(x_0,a_1)} \text{ triggers at } t \text{ or } t = t_0 + 2h + 1\}.$$

Let $\mathcal{E}_2(h)$ be the event that $M_{(x_0,a_1)}$ triggers at time T_h. We have

$$\mathbb{P}(\mathcal{E}_2(h)) = 1 - e^{-r} > \tfrac{1}{2}. \tag{51}$$

Let

$$T'_h = \min\{t > T_h \mid \text{a clock in } \mathcal{C}_{a_1} \text{ triggers at } t \text{ or } t = T_h + 1\}.$$

Let $\mathcal{E}'_2(h)$ be the event that some mutant clock with source a_1 triggers at time T'_h. Note that $\mathcal{E}'_2(h)$ is independent of $\mathcal{E}_2(h)$. The probability that some clock in $\mathcal{C}_{a_1}$ triggers in $(T_h, T_h + 1]$ is $1 - e^{-r-m}$, and the probability that the first clock in $\mathcal{C}_{a_1}$ to trigger in (T_h, ∞) is a mutant clock with source a_1 is $r/(r+m)$. Hence by (51) and a union bound,

$$\mathbb{P}(\mathcal{E}_2(h) \cap \mathcal{E}'_2(h)) \geq \frac{1}{2}\left(\frac{r}{r+m} - e^{-r-m}\right) \geq \frac{1}{4(r+m)} \geq \frac{1}{8m}. \tag{52}$$

Note that for all h, the event $\mathcal{E}_2(h) \cap \mathcal{E}'_2(h)$ depends only on fixed clocks in $\mathcal{C}_{a_1} \cup \{M_{(x_0,a_1)}\}$ over the fixed interval $(t_0 + 2h, t_0 + 2h + 2]$. As such, the events $\mathcal{E}_2(h) \cap \mathcal{E}'_2(h)$ are independent from each other, from $\mathcal{E}_1$ and from $\mathcal{F}_{t_0}(X')$. It follows from (52) that

$$\mathbb{P}\left(\bigcup_{h=0}^{8m-1} (\mathcal{E}_2(h) \cap \mathcal{E}'_2(h)) \,\middle|\, \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f\right) \geq 1 - \left(1 - \frac{1}{8m}\right)^{8m} \geq 1 - e^{-1} \geq \frac{1}{2}. \tag{53}$$

Let $\mathcal{E}_2$ be the event that there exists $t \in (t_0, t_0 + 16m]$ such that $|K_1 \cap X'_t| \geq 1$. We now show that if $\mathcal{E}_1 \cap \mathcal{E}_2(h) \cap \mathcal{E}'_2(h)$ occurs for some $h \in \{0, \ldots, 8m-1\}$, then $\mathcal{E}_2$ occurs. Since $\mathcal{E}_1$ occurs and $T_h \leq 17m(\log n)^2$, we have $x_0 \in X'_{T_h}$. Since $\mathcal{E}_2(h)$ occurs, by the definition of X', either a_1 is a mutant at time T_h or K_1 is active at time $T_h \leq t_0 + 16m$ (and hence $\mathcal{E}_2$ occurs). If a_1 is a mutant at time T_h, then since $\mathcal{E}'_2(h)$ occurs, a_1 spawns a mutant at time $T'_h \leq t_0 + 16m$, and so $\mathcal{E}_2$ occurs. Thus in all cases, if $\mathcal{E}_1 \cap \mathcal{E}_2(h) \cap \mathcal{E}'_2(h)$ occurs then $\mathcal{E}_2$ occurs. Hence by (53),

$$\mathbb{P}(\mathcal{E}_2 \mid \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f) \geq \mathbb{P}\left(\bigcup_{h=0}^{8m-1} (\mathcal{E}_2(h) \cap \mathcal{E}'_2(h)) \,\middle|\, \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f\right) \geq \frac{1}{2}. \tag{54}$$

Let

$$T = \min\{t \geq t_0 \mid t = t_0 + 16m \text{ or } |K_1 \cap X'_t| \geq 1\}.$$

Let $\mathcal{E}_3$ be the event that $T_{\mathsf{in}}^{T,1} \leq T + m$ and K_1 is full of mutants at time $T_{\mathsf{in}}^{T,1}$. Let $\mathcal{A}$ be the set of all clocks which have both source and target in $K_1 \cup \{a_1\}$. For every $t \geq 0$, let Φ_t be a random variable which contains, for each clock $C \in \mathcal{A}$, a list of the times at which C triggers in $(t, t+m]$. Now consider any $t \geq t_0$. Let f' be any possible value of $\mathcal{F}_t(X')$ such that the following events are consistent — $\mathcal{F}_{t_0}(X') = f$, $\mathcal{F}_t(X') = f'$, $\mathcal{E}_2$, $T = t$, and $\mathcal{E}_1$. Note that the first four of these events are determined by $\mathcal{F}_t(X') = f'$. Conditioned on $\mathcal{F}_t(X') = f'$, $\mathcal{E}_1$ and Φ_t are independent. Also, conditioned on $\mathcal{F}_t(X') = f'$, $\mathcal{E}_3$ is determined by Φ_t (since the definition of the megastar process ensures that a_1 is a non-mutant throughout $[t, T_{\mathsf{in}}^{t,1}]$), so, conditioned on $\mathcal{F}_t(X') = f'$, $\mathcal{E}_3$ is independent of $\mathcal{E}_1$. Now, applying Corollary 6.18, we have

$$\mathbb{P}(\mathcal{E}_3 \mid \mathcal{E}_1, \mathcal{F}_t(X') = f') = \mathbb{P}(\mathcal{E}_3 \mid \mathcal{F}_t(X') = f') \geq \frac{r-1}{6r}.$$

Thus,

$$\mathbb{P}(\mathcal{E}_3 \mid \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f, \mathcal{E}_2) \geq \frac{r-1}{6r}. \tag{55}$$

It therefore follows from (54) and (55) that

$$\mathbb{P}(\mathcal{E}_3 \mid \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f) \geq \mathbb{P}(\mathcal{E}_3 \mid \mathcal{E}_1 \cap \mathcal{E}_2, \mathcal{F}_{t_0}(X') = f) \cdot \mathbb{P}(\mathcal{E}_2 \mid \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f) \geq \frac{r-1}{12r},$$

and so the result follows. □

We are now able to prove Lemma 6.22.

LEMMA 6.22. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Suppose $X'_0 \subseteq R_1$. Then with probability at least $1 - 19(\log n)^2/\ell$, K_1 contains at most Δ non-mutants at time n.*

PROOF. Let $\mathcal{E}_1$ be the event that $N_{(v^*, x_0)}$ does not trigger in $[0, 17m(\log n)^2]$. For all $i \in \mathbb{Z}_{\geq 0}$, let $\mathcal{E}_2(i)$ be the event that there exists $t \in [17im, 17(i+1)m]$ such that $K_1 \subseteq X'_t$. Let $\mathcal{E}_2 = \mathcal{E}_2(0) \cup \cdots \cup \mathcal{E}_2(\lfloor (\log n)^2 \rfloor - 1)$. By Lemma 6.21, for all integers i with $0 \leq i \leq \lfloor (\log n)^2 \rfloor - 1$ and all possible values f_i of $\mathcal{F}_{17im}(X')$ consistent with $\mathcal{E}_1$, we have

$$\mathbb{P}(\mathcal{E}_2(i) \mid \mathcal{E}_1, \mathcal{F}_{17im}(X') = f_i) \geq \frac{r-1}{12r}.$$

Since f_i determines $\mathcal{E}_2(0), \ldots, \mathcal{E}_2(i-1)$, it follows that

$$\mathbb{P}(\mathcal{E}_2 \mid \mathcal{E}_1) \geq 1 - \left(1 - \frac{r-1}{12r}\right)^{\lfloor (\log n)^2 \rfloor}. \tag{56}$$

Moreover, by (4),

$$\mathbb{P}(\mathcal{E}_1) = e^{-17(\log n)^2/\ell} \geq 1 - \frac{17(\log n)^2}{\ell}.$$

It therefore follows by (56) that

$$\mathbb{P}(\mathcal{E}_2) \geq \mathbb{P}(\mathcal{E}_2 \mid \mathcal{E}_1)\, \mathbb{P}(\mathcal{E}_1) \geq 1 - \frac{18(\log n)^2}{\ell}. \tag{57}$$

Now let $\mathcal{E}_3$ be the event that $|K_1 \setminus X'_n| \leq \Delta$. Let

$$T = \min\{t \geq 0 \mid t = 17m(\log n)^2 \text{ or } K_1 \subseteq X'_t\}.$$

Consider any $t \in [0, 17m(\log n)^2]$ and any possible value f' of $\mathcal{F}_t(X')$ which is consistent with $\mathcal{E}_2$ and $T = t$. Then by Lemma 6.19(ii) applied to the interval $(t, n]$, with probability at least $1 - e^{-\frac{1}{2}(\log n)^3}$ conditioned on $\mathcal{F}_t(X') = f'$, $\mathcal{E}_3$ occurs. Thus, using (57), we obtain

$$\mathbb{P}(\mathcal{E}_3) \geq \mathbb{P}(\mathcal{E}_3 \mid \mathcal{E}_2)\,\mathbb{P}(\mathcal{E}_2) \geq 1 - \frac{19(\log n)^2}{\ell},$$

as required. □

6.6.2. The other cliques become almost full. In Section 6.6.2, we will show that if the initial mutant of X' lies in a reservoir, without loss of generality R_1, then with high probability $K_1, \ldots, K_\ell$ are all almost full of mutants at time n^8 (see Lemma 6.25). The following lemma will be the linchpin of the proof.

LEMMA 6.23. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let $t_0 \geq 0$, and let $j \in [\ell - 1]$. Let f be a possible value of $\mathcal{F}_{t_0}(X')$ which implies that for all $j' \in [j]$, $K_{j'}$ contains at most Δ non-mutants at time t_0. Then*

$$\mathbb{P}(\text{for all } j' \in [j], |K_{j'} \setminus X'_{t_0+20c_r^2k}| \leq \Delta \mid \mathcal{F}_{t_0}(X') = f) \geq 1 - e^{-(\log n)^2}.$$

Moreover,

$$\mathbb{P}(|K_{j+1} \setminus X'_{t_0+20c_r^2k}| \leq \Delta \mid \mathcal{F}_{t_0}(X') = f) \geq 1/n^6.$$

PROOF. The first part of the result follows immediately from Lemma 6.19(ii) and a union bound over all $j \in [j']$ by taking $I = (t_0, t_0 + 20c_r^2k]$.

We now define some stopping times. Let

$$T_1 = \min\{t \geq t_0 \mid K_1 \subseteq X'_t \text{ or } t = t_0 + 30c_r^2(\log n)^6\}.$$

Let T_2 be the fourth time after T_1 at which a clock in $\mathcal{C}(\mathcal{M}_\ell)$ triggers. Let

$$T_3 = \min\{t \geq T_2 \mid K_{j+1} \subseteq X'_t \text{ or } t = T_2 + 10c_r^2k\}.$$

In addition, we define the following events.

— $\mathcal{E}_1$: $K_1 \subseteq X'_{T_1}$.
— $\mathcal{E}_2$: for some $v_1 \in K_1$, $v_2 \in R_{j+1}$ and $v_3 \in K_{j+1}$, the first four clocks in $\mathcal{C}(\mathcal{M}_\ell)$ to trigger in (T_1, ∞) are $M_{(v_1,v^*)}$, $M_{(v^*,v_2)}$, $M_{(v_2,a_{j+1})}$ and $M_{(a_{j+1},v_3)}$, in that order.
— $\mathcal{E}_2'$: $T_2 \leq T_1 + 1$.
— $\mathcal{E}_3$: $K_{j+1} \subseteq X'_{T_3}$.
— $\mathcal{E}_4$: $|K_{j+1} \setminus X'_{t_0+20c_r^2k}| \leq \Delta$.

Our goal is to prove $\mathbb{P}(\mathcal{E}_4 \mid \mathcal{F}_{t_0} = f) \geq 1/n^5$.

By Corollary 6.18(i), we have

$$\mathbb{P}(\mathcal{E}_1 \mid \mathcal{F}_{t_0}(X') = f) \geq 1 - 4e^{-(\log n)^3}. \tag{58}$$

It is immediate that

$$\mathbb{P}(\mathcal{E}_2 \mid \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f) = \frac{r}{n(1+r)} \cdot \frac{r}{\ell n(1+r)} \cdot \frac{r}{n(1+r)} \cdot \frac{r}{n(1+r)} \geq \frac{1}{n^5}. \tag{59}$$

For example, the first $r/(n(1+r))$ factor comes from the fact that the total rate of all clocks is $n(1+r)$ but the total rate of all mutant clocks with source in K_1 and target v^* is r since there are k such clocks, each with rate r/k.

Moreover, $\mathcal{E}_2'$ occurs if for all $h \in \{0,1,2,3\}$, at least one clock in $\mathcal{C}(\mathcal{M}_\ell)$ triggers in the interval $(T_2 + h/4, T_2 + (h+1)/4]$. Hence

$$\mathbb{P}(\mathcal{E}_2' \mid \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f) \geq 1 - 4e^{-(1+r)n/4}. \tag{60}$$

It follows from (58)–(60) and a union bound that

$$\begin{aligned}\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_2' \mid \mathcal{F}_{t_0}(X') = f) &\geq \mathbb{P}(\mathcal{E}_1 \mid \mathcal{F}_{t_0}(X') = f)\cdot \\ &\quad (\mathbb{P}(\mathcal{E}_2 \mid \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f) - \mathbb{P}(\overline{\mathcal{E}_2'} \mid \mathcal{E}_1, \mathcal{F}_{t_0}(X') = f)) \\ &\geq \left(1 - 4e^{-(\log n)^3}\right)(1/n^5 - 4e^{-(1+r)n/4}) \geq 1/(2n^5). \end{aligned} \tag{61}$$

Now consider any $t_2 > t_0$ and any possible value f_2 of $\mathcal{F}_{t_2}(X')$ which implies that $\mathcal{F}_{t_0}(X') = f$, $T_2 = t_2$, and that $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_2'$ occurs. Note that, if $\mathcal{F}_{t_2}(X') = f_2$ then, since $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_2'$ occurs, we must have $|K_{j+1} \cap X'_{t_2}| \geq 1$. It follows from Corollary 6.18 that

$$\mathbb{P}(\mathcal{E}_3 \mid \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_2', \mathcal{F}_{t_0}(X') = f) \geq \frac{r-1}{6r}.$$

It therefore follows from (61) that

$$\mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_2' \cap \mathcal{E}_3 \mid \mathcal{F}_{t_0}(X') = f) \geq \frac{r-1}{12rn^5}. \tag{62}$$

Finally, consider any $t_3 \geq t_0$ and any possible value f_3 of $\mathcal{F}_{t_3}(X')$ which implies that $\mathcal{F}_{t_0}(X') = f$, that $T_3 = t_3$, and that $\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_2' \cap \mathcal{E}_3$ occurs. Since $\mathcal{F}_{t_3}(X') = f_3$ implies that $\mathcal{E}_2'$ occurs, we have

$$t_3 \leq t_0 + 30c_r^2(\log n)^6 + 1 + 10c_r^2 k \leq t_0 + 20c_r^2 k - (\log n)^7.$$

If $\mathcal{F}_{t_3}(X') = f_3$ then $\mathcal{E}_3$ occurs so $K_{j+1} \subseteq X'_{t_3}$, which obviously implies $|K_{j+1} \setminus X'_{t_3}| \leq \Delta$. It therefore follows from Lemma 6.19(ii) applied to $(t_3, t_0 + 20c_r^2 k]$ that

$$\mathbb{P}(\mathcal{E}_4 \mid \mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_2' \cap \mathcal{E}_3, \mathcal{F}_{t_0}(X') = f) \geq 1 - e^{-\frac{1}{2}(\log n)^3},$$

and therefore by (62),

$$\mathbb{P}(\mathcal{E}_4 \mid \mathcal{F}_{t_0}(X') = f) \geq \mathbb{P}(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_2' \cap \mathcal{E}_3 \cap \mathcal{E}_4 \mid \mathcal{F}_{t_0}(X') = f) \geq 1/n^6.$$

The second part of the result therefore follows. □

We now apply Lemma 6.23 repeatedly to prove the following, which shows that, with high probability, as soon as, for some j, j of the cliques are almost full of mutants (having at most Δ non-mutants), after a little while this is true for at least $j+1$ of the cliques (so doing it again and again, we (nearly) fill all of the cliques).

LEMMA 6.24. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let $t_0 \geq 0$, and let $j \in [\ell - 1]$. Let f be a possible value of $\mathcal{F}_{t_0}(X')$ which implies that for all $j' \in [j]$, $K_{j'}$ contains at most Δ non-mutants at time t_0. Then*

$$\mathbb{P}\big(\text{for all } j' \in [j+1],\ |K_{j'} \setminus X'_{t_0 + 20c_r^2 n^7 k}| \leq \Delta \mid \mathcal{F}_{t_0}(X') = f\big) \geq 1 - n^8 e^{-(\log n)^2}.$$

PROOF. For all $i \in \mathbb{Z}_{\geq 1}$, let $t_i = t_0 + 20c_r^2 ki$. Let $\mathcal{E}_1(i)$ be the event that for all $j' \in [j]$, $|K_{j'} \setminus X'_{t_i}| \leq \Delta$. Let $\mathcal{E}_2(i)$ be the event that $|K_{j+1} \setminus X'_{t_i}| \leq \Delta$. For convenience, let $\mathcal{F}$ be the event that $\mathcal{F}_{t_0}(X') = f$. By a union bound, we have

$$\mathbb{P}\big(\cup_{i=1}^{n^7} \mathcal{E}_2(i) \mid \mathcal{F}\big) \geq 1 - \mathbb{P}\left(\cap_{i=1}^{n^7}(\overline{\mathcal{E}_2(i)} \cap \mathcal{E}_1(i)) \mid \mathcal{F}\right) - \mathbb{P}\left(\cup_{i=1}^{n^7}\overline{\mathcal{E}_1(i)} \mid \mathcal{F}\right). \tag{63}$$

Moreover,

$$\mathbb{P}\Big(\cap_{i=1}^{n^7} \left(\overline{\mathcal{E}_2(i)} \cap \mathcal{E}_1(i)\right) \,\big|\, \mathcal{F}\Big) = \prod_{i=1}^{n^7} \mathbb{P}\left(\overline{\mathcal{E}_2(i)} \cap \mathcal{E}_1(i) \,\big|\, \cap_{s=1}^{i-1} \left(\overline{\mathcal{E}_2(s)} \cap \mathcal{E}_1(s)\right) \cap \mathcal{F}\right)$$
$$\leq \prod_{i=1}^{n^7} \mathbb{P}\left(\overline{\mathcal{E}_2(i)} \,\big|\, \cap_{s=1}^{i-1} \left(\overline{\mathcal{E}_2(s)} \cap \mathcal{E}_1(s)\right) \cap \mathcal{F}\right).$$

Since for all i, the event $\bigcap_{s=1}^{i-1}(\overline{\mathcal{E}_2(s)} \cap \mathcal{E}_1(s)) \cap \mathcal{F}$ is determined by $\mathcal{F}_{t_{i-1}}(X')$, by Lemma 6.23 it follows that

$$\mathbb{P}\Big(\cap_{i=1}^{n^7} \left(\overline{\mathcal{E}_2(i)} \cap \mathcal{E}_1(i)\right) \,\big|\, \mathcal{F}\Big) \leq \left(1 - \frac{1}{n^6}\right)^{n^7} \leq e^{-n}. \tag{64}$$

We also have

$$\mathbb{P}\big(\cup_{i=1}^{n^7} \overline{\mathcal{E}_1(i)} \,\big|\, \mathcal{F}\big) \leq \sum_{i=1}^{n^7} \mathbb{P}\big(\overline{\mathcal{E}_1(i)} \,\big|\, \cap_{s=1}^{i-1} \mathcal{E}_1(s) \cap \mathcal{F}\big).$$

Since for all i, the event $\cap_{s=1}^{i-1} \mathcal{E}_1(s) \cap \mathcal{F}$ is determined by $\mathcal{F}_{t_{i-1}}(X')$, it follows from Lemma 6.23 that

$$\mathbb{P}\big(\cup_{i=1}^{n^7} \overline{\mathcal{E}_1(i)} \,\big|\, \mathcal{F}\big) \leq n^7 e^{-(\log n)^2}. \tag{65}$$

Hence by (63), (64) and (65), we have

$$\mathbb{P}\big(\cup_{i=1}^{n^7} \mathcal{E}_2(i) \,\big|\, \mathcal{F}\big) \geq 1 - 2n^7 e^{-(\log n)^2}. \tag{66}$$

Now, let $i' \in [n^7 - 1]$. Since $\mathcal{E}_2(i') \cap \mathcal{F}$ is determined by $\mathcal{F}_{t_{i'}}(X')$, it follows by Lemma 6.19(ii) applied to the interval $(t_{i'}, t_{n^7}]$ that

$$\mathbb{P}(\mathcal{E}_2(n^7) \mid \mathcal{E}_2(i') \cap \mathcal{F}) \geq 1 - e^{-\frac{1}{2}(\log n)^3}.$$

Hence by (66) we have $\mathbb{P}(\mathcal{E}_2(n^7) \mid \mathcal{F}) \geq 1 - 3n^7 e^{-(\log n)^2}$. In addition, by (65), we have $\mathbb{P}(\mathcal{E}_1(n^7) \mid \mathcal{F}) \geq 1 - n^7 e^{-(\log n)^2}$. By a union bound, it follows that

$$\mathbb{P}(\mathcal{E}_1(n^7) \cap \mathcal{E}_2(n^7) \mid \mathcal{F}) \geq 1 - n^8 e^{-(\log n)^2},$$

and so the result follows. □

The proof of Lemma 6.25, the goal of Section 6.6, now follows easily from Lemma 6.24.

LEMMA 6.25. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Suppose $X_0' \subseteq R_1$. Then with probability at least $1 - 20(\log n)^2/\ell$, for all $j \in [\ell]$, K_j contains at most Δ non-mutants at time n^8.*

PROOF. For each positive integer i, let $t_i = n + (i-1)20c_r^2 n^7 k$. Let $\mathcal{E}_i$ be the event that at time t_i, for all $j \in [i]$ we have $|X_{t_i}' \setminus K_j| \leq \Delta$. By Lemma 6.22, we have $\mathbb{P}(\mathcal{E}_1) \geq 1 - 19(\log n)^2/\ell$. For all $i \in \{2, \ldots, \ell\}$, by Lemma 6.24 (applied with $j = i-1$ starting at t_{i-1}), we have

$$\mathbb{P}(\mathcal{E}_i \mid \mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{i-1}) \geq 1 - n^8 e^{-(\log n)^2}.$$

It follows that

$$\mathbb{P}(\mathcal{E}_\ell) \geq 1 - \sum_{i=1}^{\ell} \mathbb{P}\big(\overline{\mathcal{E}_i} \mid \mathcal{E}_1 \cap \cdots \cap \mathcal{E}_{i-1}\big) \geq 1 - \frac{19(\log n)^2}{\ell} - \ell n^8 e^{-(\log n)^2}.$$

It therefore follows by Lemma 6.19(ii) applied to the interval $(t_\ell, n^8]$ combined with a union bound that with probability at least

$$1 - \frac{19(\log n)^2}{\ell} - \ell n^8 e^{-(\log n)^2} - \ell e^{-\frac{1}{2}(\log n)^3} \geq 1 - 20(\log n)^2/\ell,$$

for all $j \in [\ell]$, K_j contains at most Δ mutants at time n^8 as required. □

6.7. Filling reservoirs from cliques

6.7.1. Setting up an iteration scheme — Proof of Lemma 6.3. In this section, we outline an iterative argument which, together with Lemma 6.25, will allow us to prove our key lemma, Lemma 6.3.

Definition 6.26. For all $i \in \mathbb{Z}_{\geq 0}$, let

$$\begin{aligned} I_i^- &= n^8 + in(\log n)^3, \\ I_i^+ &= n^8 + (i+1)n(\log n)^3, \\ \alpha_i &= \lfloor \max\{(2\log n)^2, m/(2\log n)^{2i}\} \rfloor, \\ \beta_i &= \lfloor \max\{(2\log n)^2, \ell m/(2\log n)^{2i}\} \rfloor. \end{aligned}$$

Consider any $t \geq n^8$. Let i be the integer such that $t \in [I_i^-, I_i^+)$. Let f be a possible value of $\mathcal{F}_t(X')$. We say that f is *good* if the event $\mathcal{F}_t(X') = f$ implies that the following events occur.

— $\mathcal{P}_1(i)$: $|(R_1 \cup \cdots \cup R_\ell) \setminus X'_{I_i^-}| \leq \beta_i$.
— $\mathcal{P}_2(i)$: For all $j \in [\ell]$, $|R_j \setminus X'_{I_i^-}| \leq \alpha_i$.
— $\mathcal{P}_3(i)$: For all $j \in [\ell]$, $|K_j \setminus X'_{I_i^-}| \leq \Delta$.
— $\mathcal{P}_4(i)$: For all but at most β_i choices of $j \in [\ell]$, $R_j \cup \{a_j\} \cup K_j \subseteq X'_{I_i^-}$. □

Each interval $(I_i^-, I_i^+]$ corresponds to a phase of our iterative argument, which we state in Lemma 6.27. At the end of each interval, the number of non-mutants in each reservoir should drop by a factor of at least $(2\log n)^2$ to a minimum of $\lfloor (2\log n)^2 \rfloor$, as should the number of non-mutants in all reservoirs. In addition, every clique should remain almost full of mutants, and if there are fewer than ℓ non-mutants left in reservoirs then many branches should be completely full of mutants.

LEMMA 6.27. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let i be a non-negative integer, and let f be a good possible value of $\mathcal{F}_{I_i^-}(X')$. Then*

$$\mathbb{P}(\mathcal{F}_{I_i^+}(X') \textit{ is good} \mid \mathcal{F}_{I_i^-}(X') = f) \geq 1 - 15e^{-(\log n)^2}.$$

Moreover, if $\beta_i = \lfloor (2\log n)^2 \rfloor$, then

$$\mathbb{P}(X'_{I_i^+} = V(\mathcal{M}_\ell) \mid \mathcal{F}_{I_i^-}(X') = f) \geq 1/2.$$

Assuming Lemma 6.27 for the moment, we give the proof of our key lemma, Lemma 6.3, which we restate here for convenience.

LEMMA 6.3. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Suppose that $x_0 \in R_1 \cup \cdots \cup R_\ell$. Then there exists a $t \geq 0$ such that $\mathbb{P}\big(X'_t = V(\mathcal{M}_\ell)\big) \geq 1 - 42(\log n)^2/n^{1/2}$.*

PROOF. By the symmetry of the megastar, we may assume that $x_0 \in R_1$.

For a non-negative integer i, let $\mathcal{E}_i$ be the event that $\mathcal{F}_{I_i^-}(X')$ is good and let $\mathcal{E}_i'$ be the event that $X'_{I_i^+} = V(\mathcal{M}_\ell)$. To prove the lemma, it clearly suffices to show that

$$\mathbb{P}(\cup_{i=0}^{2n} \mathcal{E}_i') \geq 1 - \frac{42(\log n)^2}{\sqrt{n}}. \tag{67}$$

By a union bound, we have

$$\mathbb{P}(\cup_{i=0}^{2n} \mathcal{E}_i') \geq 1 - \mathbb{P}\Big(\cap_{i=0}^{2n} \left(\mathcal{E}_i \cap \overline{\mathcal{E}_i'}\right)\Big) - \mathbb{P}\big(\cup_{i=0}^{2n} \overline{\mathcal{E}_i}\big). \tag{68}$$

For all $i \in [2n]$, $\mathcal{F}_{I_i^-}(X')$ determines $\mathcal{E}_0 \cap \cdots \cap \mathcal{E}_{i-1}$ and thus, by Lemma 6.27, we have

$$\mathbb{P}(\mathcal{E}_i \mid \mathcal{E}_0 \cap \cdots \cap \mathcal{E}_{i-1}) \geq 1 - 15e^{-(\log n)^2}. \tag{69}$$

Also, since $x_0 \in R_1$, for $i = 0$, we have by Lemma 6.25 that

$$\mathbb{P}(\mathcal{E}_0) \geq 1 - \frac{20(\log n)^2}{\ell} \geq 1 - \frac{40(\log n)^2}{\sqrt{n}}, \tag{70}$$

since $\mathcal{P}_1(0)$, $\mathcal{P}_2(0)$ and $\mathcal{P}_4(0)$ hold trivially (from $\alpha_0 = m$ and $\beta_0 = \ell m$). Combining (69) and (70) we obtain that

$$\mathbb{P}\big(\cap_{i=0}^{2n} \mathcal{E}_i\big) \geq 1 - \frac{41(\log n)^2}{\sqrt{n}}, \text{ so } \mathbb{P}\big(\cup_{i=0}^{2n} \overline{\mathcal{E}_i}\big) \leq \frac{41(\log n)^2}{\sqrt{n}}. \tag{71}$$

For $i \geq n$, we have that $\beta_i = \lfloor (2\log n)^2 \rfloor$. Since $\mathcal{F}_{I_i^-}(X')$ determines $\bigcap_{h=0}^{i-1}(\mathcal{E}_i \cup \overline{\mathcal{E}_i'})$ for all $i \in \{n+1, \ldots, 2n\}$, it follows from Lemma 6.27 that

$$\mathbb{P}\Big(\left(\mathcal{E}_i \cap \overline{\mathcal{E}_i'}\right) \mid \left(\mathcal{E}_0 \cap \overline{\mathcal{E}_0'}\right) \cap \cdots \cap \left(\mathcal{E}_{i-1} \cap \overline{\mathcal{E}_{i-1}'}\right)\Big) \leq 1/2.$$

Hence, we obtain that

$$\mathbb{P}\Big(\cap_{i=0}^{2n} \left(\mathcal{E}_i \cap \overline{\mathcal{E}_i'}\right)\Big) \leq 1/2^n. \tag{72}$$

Plugging (71) and (72) in (68) yields (67), as wanted. □

Recall that we already proved Theorem 6.1 using Lemma 6.3 in Section 6.3. The remainder of Section 6 will therefore focus on the proof of Lemma 6.27. The following stopping time will be important in what follows.

Definition 6.28. Consider $i \in \mathbb{Z}_{\geq 0}$. We define T^i_{end} to be the first time $t \geq I_i^-$ such that one of the following holds.

(D1) $t = I_i^+$.
(D2) v^* spawns β_{i+1} non-mutants in the interval $(I_i^-, t]$.
(D3) For some $j \in [\ell]$, v^* spawns α_{i+1} non-mutants onto vertices in R_j in the interval $(I_i^-, t]$.
(D4) For some $j \in [\ell]$, $|K_j \setminus X'_t| > 2\Delta$. □

Note that if $\mathcal{F}_{I_i^-}(X')$ is good then $|K_j \setminus X'_{I_i^-}| \leq 2\Delta$ so $T^i_{\mathsf{end}} > I_i^-$.

The crux of our argument will be a proof that if $\mathcal{F}_{I_i^-}(X')$ is good, then $T^i_{\mathsf{end}} = I_i^+$ with high probability (see the proof of Lemma 6.42).

6.7.2. Bounding the number of times cliques become active. In Section 6.7.2, we bound the number of times that cliques become active in the interval $(I_i^-, T_{\mathsf{end}}^i]$ (see Lemma 6.34). We require the following definitions, which rely on Definition 6.26.

Definition 6.29. Let $v \in V(\mathcal{M}_\ell)$ and $i \in \mathbb{Z}_{\geq 0}$. Let $T_{\mathsf{m}}^{i,v}(-1) = I_i^-$. We recursively define times $T_{\mathsf{n}}^{i,v}(h)$ and $T_{\mathsf{m}}^{i,v}(h)$ for integers $h \geq 0$ as follows.

$$T_{\mathsf{n}}^{i,v}(h) = \min\{t \geq T_{\mathsf{m}}^{i,v}(h-1) \mid v \notin X_t' \text{ or } t = T_{\mathsf{end}}^i\},$$
$$T_{\mathsf{m}}^{i,v}(h) = \min\{t \geq T_{\mathsf{n}}^{i,v}(h) \mid v \in X_t' \text{ or } t = T_{\mathsf{end}}^i\}. \quad \square$$

The subscript "m" stands for "mutant" and the subscript "n" stands for "non-mutant".

Definition 6.30. For each $i \in \mathbb{Z}_{\geq 0}$, let $I_i = (I_i^-, I_i^+]$. Also, for $h \in \mathbb{Z}_{\geq 0}$ and $j \in [\ell]$, let $J^{i,j}(h)$ be the interval $(T_{\mathsf{n}}^{i,a_j}(h), T_{\mathsf{m}}^{i,a_j}(h))$, let $W_{\mathsf{a}}^{i,j}(h)$ be the number of times at which K_j becomes active in $J^{i,j}(h)$, and let $W_{\mathsf{in}}^{i,j}(h)$ be the number of times at which K_j becomes inactive in $J^{i,j}(h)$. $\square$

Note that the intervals $J^{i,j}(h)$ are disjoint from each other and they are all contained within $[I_i^-, I_i^+]$. Also, the interval $J^{i,j}(h)$ is empty if and only if $T_{\mathsf{n}}^{i,a_j}(h) = T_{\mathsf{end}}^i$. The subscript "$\mathsf{a}$" stands for "active" and the subscript "in" stands for "inactive". The following lemmas, Lemmas 6.31 and 6.32, are stated only for $i \in \mathbb{Z}_{\geq 1}$ — we will deal with $i = 0$ in the proof of Lemma 6.34.

LEMMA 6.31. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. For $i \in \mathbb{Z}_{\geq 1}$, let f be a good possible value of $\mathcal{F}_{I_i^-}(X')$. Let $j \in [\ell]$. Then,*

$$\mathbb{P}\big(W_{\mathsf{a}}^{i,j}(h) = 0 \textit{ for all } h \geq \lfloor 4\alpha_i \mathrm{len}(I_i) \rfloor + 1 \mid \mathcal{F}_{I_i^-}(X') = f\big) \geq 1 - e^{-n}.$$

PROOF. We will show that with high probability, the feeder vertex a_j becomes a non-mutant at most $\lfloor 4\alpha_i \mathrm{len}(I_i) \rfloor$ times within $(I_i^-, T_{\mathsf{end}}^i]$. Let $\mathcal{E}$ be the event that for all $v \in R_j$, $N_{(v,a_j)}$ triggers at most $2\mathrm{len}(I_i)$ times in I_i. Recall that $\mathrm{len}(I_i) = n(\log n)^3$, so by Corollary 2.2 combined with a union bound over all $v \in R_j$, we have

$$\mathbb{P}(\mathcal{E} \mid \mathcal{F}_{I_i^-}(X') = f) \geq 1 - e^{-n}.$$

Suppose that $\mathcal{E}$ occurs. Since $\mathcal{F}_{I_i^-}(X')$ is good, by $\mathcal{P}_2(i)$ we have $|R_j \setminus X'_{I_i^-}| \leq \alpha_i$. Moreover, by (D3), at most α_{i+1} non-mutants are spawned into R_j over the course of $(I_i^-, T_{\mathsf{end}}^i]$. Hence all but at most $\alpha_i + \alpha_{i+1} \leq 2\alpha_i$ vertices in R_j are mutants throughout $(I_i^-, T_{\mathsf{end}}^i]$, and therefore do not spawn any non-mutants onto a_j within $(I_i^-, T_{\mathsf{end}}^i]$. Since $\mathcal{E}$ occurs, the remaining vertices in R_j each spawn at most $2\mathrm{len}(I_i)$ non-mutants onto a_j within $(I_i^-, T_{\mathsf{end}}^i]$, and so at most $\lfloor 4\alpha_i \mathrm{len}(I_i) \rfloor$ non-mutants are spawned onto a_j in total over the course of $(I_i^-, T_{\mathsf{end}}^i]$.

Now, for all $h > 0$, $T_{\mathsf{n}}^{i,a_j}(h) = T_{\mathsf{end}}^i$ or a_j becomes a non-mutant at time $T_{\mathsf{n}}^{i,a_j}(h)$. It therefore follows that for all $h \geq \lfloor 4\alpha_i \mathrm{len}(I_i) \rfloor + 1$, $T_{\mathsf{n}}^{i,a_j}(h) = T_{\mathsf{m}}^{i,a_j}(h) = T_{\mathsf{end}}^i$ so the interval $J^{i,j}(h)$ is empty and hence $W_{\mathsf{a}}^{i,j}(h) = 0$ as required. $\square$

In Lemma 6.32 we will show that, with high probability, the sum $\sum_{h=0}^{\lfloor 4\alpha_i \mathrm{len}(I_i) \rfloor} W_{\mathsf{a}}^{i,j}(h)$ is small. We will use this in Lemma 6.34, to show that, with high probability, K_j doesn't become active too many times before a_j has become a mutant more than $\lfloor 4\alpha_i \mathrm{len}(I_i) \rfloor$ times.

LEMMA 6.32. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. For $i \in \mathbb{Z}_{\geq 1}$, let f be a good possible value of $\mathcal{F}_{I_i^-}(X')$. Let $j \in [\ell]$. Then*

$$\mathbb{P}\left(\sum_{h=0}^{\lfloor 4\alpha_i \mathrm{len}(I_i)\rfloor} W_{\mathsf{a}}^{i,j}(h) \leq \alpha_i\sqrt{n}(\log n)^4 - 1 \,\middle|\, \mathcal{F}_{I_i^-}(X') = f\right) \geq 1 - e^{-(\log n)^3}.$$

PROOF. We define the following random variables for $h \in \mathbb{Z}_{\geq 0}$. These variables are local to this proof. First, if K_j is inactive at $T_{\mathsf{n}}^{i,a_j}(h)$ (which is the left endpoint of $J^{i,j}(h)$) then $Q_h = W_{\mathsf{in}}^{i,j}(h)+1$. Otherwise, $Q_h = W_{\mathsf{in}}^{i,j}(h)$. It follows from the definition of $W_{\mathsf{in}}^{i,j}(h)$ and $W_{\mathsf{a}}^{i,j}(h)$ that $Q_h \geq W_{\mathsf{a}}^{i,j}(h)$. Next we define variables $T_h(0), \ldots, T_h(Q_h)$. First, we define $T_h(Q_h) = T_{\mathsf{m}}^{i,a_j}(h)$ (which is the right endpoint of $J^{i,j}(h)$). If $Q_h > 0$ then we define the remaining variables as follows.

Case 1. If K_j is inactive at the left endpoint of $J^{i,j}(h)$ then $T_h(0) = T_{\mathsf{n}}^{i,a_j}(h)$ and for $q \in [W_{\mathsf{in}}^{i,j}(h)]$, $T_{\mathsf{in}}^h(q)$ is the q'th time that K_j becomes inactive in $J^{i,j}(h)$.

Case 2. If K_j is active at the left endpoint of $J^{i,j}(h)$ then for $q \in [W_{\mathsf{in}}^{i,j}(h)]$, $T_h(q-1)$ is the q'th time that K_j becomes inactive in $J^{i,j}(h)$.

Now fix an integer $h \geq 0$. Consider any time $t_{\mathsf{n}}^{i,a_j}(h) \geq I_i^-$. Consider any integers $w_0, \ldots, w_{h-1}$ and $y \geq 0$, and any times $t_0, \ldots, t_y$ satisfying $t_{\mathsf{n}}^{i,a_j}(h) \leq t_0 \leq \cdots \leq t_y$. Suppose that f' is a value of $\mathcal{F}_{t_y}(X')$ which implies that

— $\mathcal{F}_{I_i^-}(X') = f$,
— $T_{\mathsf{n}}^{i,a_j}(h) = t_{\mathsf{n}}^{i,a_j}(h)$,
— $W_{\mathsf{a}}^{i,j}(0) = w_0, \ldots, W_{\mathsf{a}}^{i,j}(h-1) = w_{h-1}$,
— $W_{\mathsf{a}}^{i,j}(h) \geq y$, and
— $T_h(0) = t_0, \ldots, T_h(y) = t_y$.

The event $\mathcal{F}_{t_y}(X') = f'$ determines whether or not $t_y = T_{\mathsf{m}}^{i,a_j}(h)$. We split the analysis into two cases.

Case 1. If $\mathcal{F}_{t_y}(X') = f'$ implies that $t_y = T_{\mathsf{m}}^{i,a_j}(h)$, then since $y = Q_h \geq W_{\mathsf{a}}^{i,j}(h) \geq y$, we have $W_{\mathsf{a}}^{i,j}(h) = y$, so

$$\mathbb{P}(W_{\mathsf{a}}^{i,j}(h) = y \mid \mathcal{F}_{t_y}(X') = f') = 1. \tag{73}$$

Case 2. Suppose that $\mathcal{F}_{t_y}(X') = f'$ implies $t_y < T_{\mathsf{m}}^{i,a_j}(h)$. Let $\mathcal{E}$ be the event that in the interval (t_y, ∞), some mutant clock with source in $R_j \cap X'_{t_y}$ triggers before any non-mutant clock with source in $\{a_j, v^*\}$ triggers. If $\mathcal{F}_{t_y}(X') = f'$ then $t_y < T_{\mathsf{m}}^{i,a_j}(h) \leq T_{\mathsf{end}}^i$, so by (D3) and the fact that $i \geq 1$, it follows that R_j contains at least $m - \alpha_i - \alpha_{i+1} \geq m - 2\alpha_1 \geq m/2$ mutants at time t_y. Hence

$$\mathbb{P}(\mathcal{E} \mid \mathcal{F}_{t_y}(X') = f') \geq 1 - \frac{2}{2 + rm/2} \geq 1 - \frac{5}{m}. \tag{74}$$

We will now prove that if $\mathcal{F}_{t_y}(X') = f'$ and $\mathcal{E}$ occurs then $W_{\mathsf{a}}^{i,j}(h) = y$. Let T be the earliest time in the interval (t_y, ∞) at which some mutant clock with source in $R_j \cap X'_{t_y}$, say $M_{(v,a_j)}$, triggers.

If $t_y < T_{\mathsf{m}}^{i,a_j}(h)$ then $t_y \leq T_{\mathsf{end}}^i$ so (D4) implies that K_j has at most 2Δ non-mutants at t_y. On the other hand, $T_h(y) = t_y$ implies that K_j is inactive at t_y. Hence, it must be

full of mutants at t_y. This means that K_j remains inactive after t_y until a_j spawns a non-mutant onto a vertex in K_j. If $\mathcal{E}$ occurs, then no non-mutant clock with source a_j triggers in $(t_y, T]$, so K_j is inactive throughout $(t_y, T]$.
Recall that $v \in X'_{t_y}$. If $\mathcal{E}$ occurs, then no non-mutant clock with source v^* triggers in $(t_y, T]$, so $v \in X'_T$ also. Hence by definition, v spawns a mutant onto a_j at time T. Moreover, K_j is inactive at time T and so a_j becomes a mutant at time T. Thus $T_{\mathsf{m}}^{i,a_j}(h) \leq T$, and so K_j is inactive throughout $(t_y, T_{\mathsf{m}}^{i,a_j}(h)]$. Thus $W_{\mathsf{a}}^{i,j}(h) = y$ whenever $\mathcal{E}$ occurs, as claimed. By (74), it follows that if f' implies that $t_y < T_{\mathsf{m}}^{i,a_j}(h)$,

$$\mathbb{P}\big(W_{\mathsf{a}}^{i,j}(h) = y \mid \mathcal{F}_{t_y}(X') = f'\big) \geq \mathbb{P}\big(\mathcal{E} \mid \mathcal{F}_{t_y}(X') = f'\big) \geq 1 - 5/m. \tag{75}$$

Combining Cases 1 and 2 by considering all possible f', $t_{\mathsf{n}}^{i,a_j}(h)$, and $t_0, \ldots, t_y$ and combining Equations (73) and (75), it follows that

$$\mathbb{P}\left(W_{\mathsf{a}}^{i,j}(h) = y \;\middle|\; \begin{array}{l} W_{\mathsf{a}}^{i,j}(0) = w_0, \ldots, W_{\mathsf{a}}^{i,j}(h-1) = w_{h-1}, \\ W_{\mathsf{a}}^{i,j}(h) \geq y,\ \mathcal{F}_{I_i^-}(X') = f \end{array}\right) \geq 1 - \frac{5}{m}.$$

Let $W'_0, W'_1, \ldots$ be i.i.d. geometric variables with parameter $1 - 5/m$. It is immediate that $\mathbb{P}(W'_h = y \mid W'_h \geq y) = 1 - 5/m$. Moreover, $\mathbb{P}(W_{\mathsf{a}}^{i,j}(h) \geq 0) = \mathbb{P}(W'_h \geq 0) = 1$. It follows that conditioned on $\mathcal{F}_{I_i^-}(X') = f$, the random variables $W_{\mathsf{a}}^{i,j}(0), W_{\mathsf{a}}^{i,j}(1), \ldots$ are dominated above by $W'_0, W'_1, \ldots$.

Now, note that $(\log n)^3 \leq 14 \cdot 5(\lfloor 4\alpha_i \mathrm{len}(I_i) \rfloor + 1)/m \leq \alpha_i \sqrt{n}(\log n)^4 - 1$ and that $1 - 5/m \geq 13/14$. It therefore follows by Lemma 2.6 that

$$\mathbb{P}\left(\sum_{h=0}^{\lfloor 4\alpha_i \mathrm{len}(I_i) \rfloor} W'_h \geq \alpha_i \sqrt{n}(\log n)^4 - 1\right) \leq e^{-(\log n)^3}.$$

The result therefore follows. □

The following definition is related to Definition 6.30.

Definition 6.33. For each $i \in \mathbb{Z}_{\geq 0}$ and $j \in [\ell]$, let $W_{\mathsf{a}}^{i,j}$ be the number of times in $(I_i^-, T_{\mathsf{end}}^i]$ at which K_j becomes active. □

We now combine Lemma 6.31 and Lemma 6.32 to show that, with high probability, K_j does not become active too many times in $(I_i^-, T_{\mathsf{end}}^i]$.

LEMMA 6.34. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let i be a non-negative integer, and let f be a good possible value of $\mathcal{F}_{I_i^-}(X')$. Let $j \in [\ell]$. Then*

$$\mathbb{P}\big(W_{\mathsf{a}}^{i,j} \leq \alpha_i \sqrt{n}(\log n)^4 \mid \mathcal{F}_{I_i^-}(X') = f\big) \geq 1 - 2e^{-(\log n)^3}.$$

PROOF. First, note that K_j can become active either when K_j has no mutants and a mutant is spawned into it or when K_j is full of mutants and a_j spawns a non-mutant into it. By (D4), K_j has at most 2Δ non-mutants throughout the interval $(I_i^-, T_{\mathsf{end}}^i]$, so that

$$\text{for } t \in (I_i^-, T_{\mathsf{end}}^i],\ K_j \text{ can only become active at time } t \text{ if } t \in [T_{\mathsf{n}}^{i,a_j}(h), T_{\mathsf{m}}^{i,a_j}(h)] \text{ (with } h \in \mathbb{Z}_{\geq 0}) \text{ and a non-mutant clock with source } a_j \text{ triggers at time } t. \tag{76}$$

We next consider cases on the value of i, namely, whether $i = 0$ or not.

First suppose that $i = 0$, so $\alpha_i = m$. By Corollary 2.2 (see also the proof of Lemma 6.31), the probability that the non-mutant clocks with source a_j trigger more

than $2\text{len}(I_i)$ times in I_i is at most e^{-n}. It follows from (76) that with probability at least $1 - e^{-n}$,

$$W_{\mathsf{a}}^{i,j} \leq 2\text{len}(I_i) \leq 4m\sqrt{n}(\log n)^3 = 4\alpha_i\sqrt{n}(\log n)^3$$

as required.

Suppose instead that $i \geq 1$. We will show that with probability 1, it holds that

$$W_{\mathsf{a}}^{i,j} \leq 1 + \sum_{h=0}^{\infty} W_{\mathsf{a}}^{i,j}(h), \tag{77}$$

so that the lemma follows from Lemmas 6.31 and 6.32 (combined with a union bound).

It remains to justify that (77) holds with probability 1. By (76), in the interval $(I_i^-, T_{\mathsf{end}}^i]$, K_j can only become active within the intervals $[T_{\mathsf{n}}^{i,a_j}(h), T_{\mathsf{m}}^{i,a_j}(h)]$, $h \in \mathbb{Z}_{\geq 0}$. By the definition of the $W_{\mathsf{a}}^{i,j}(h)$'s, (77) will thus follow by showing that with probability 1, K_j becomes active at most once at times t with $t \in S$, where

$$S := \{T_{\mathsf{n}}^{i,a_j}(h) \mid h \in \mathbb{Z}_{\geq 0}\} \cup \{T_{\mathsf{m}}^{i,a_j}(h) \mid h \in \mathbb{Z}_{\geq 0}\}.$$

Note that a_j can become a non-mutant either because K_j was empty and a mutant was spawned into K_j or because a non-mutant was spawned onto a_j. Now let h be such that $T_{\mathsf{n}}^{i,a_j}(h) < T_{\mathsf{end}}^i$. By (D4), it must be the case that a non-mutant is spawned onto a_j at time $T_{\mathsf{n}}^{i,a_j}(h)$. Thus, for K_j to become active at time $T_{\mathsf{n}}^{i,a_j}(h)$, it must be the case from (76) that two clocks triggered at the same time, which happens with probability 0. Similarly, for all h such that $T_{\mathsf{m}}^{i,a_j}(h) < T_{\mathsf{end}}^i$, we have that a mutant is spawned onto a_j at $T_{\mathsf{m}}^{i,a_j}(h)$, and thus, with probability 1, K_j does not become active at $T_{\mathsf{m}}^{i,a_j}(h)$. Thus, K_j can only become active at a time $T_{\mathsf{n}}^{i,a_j}(h)$ (resp. $T_{\mathsf{m}}^{i,a_j}(h)$) if $T_{\mathsf{n}}^{i,a_j}(h) = T_{\mathsf{end}}^i$ (resp. $T_{\mathsf{m}}^{i,a_j}(h) = T_{\mathsf{end}}^i$). It thus follows that the number of times that K_j becomes active at times $t \in S$ is at most one (namely when $t = T_{\mathsf{end}}^i$), as desired. This proves (77) and concludes the proof of the lemma. □

6.7.3. The behaviour of the centre vertex. In Section 6.7.3, we will show that with high probability, v^* does not spend too much time as a non-mutant in the interval $(I_i^-, T_{\mathsf{end}}^i]$ (see Lemma 6.39). We first apply Lemma 6.20 and Lemma 6.34, to give an upper bound for the total number of non-mutants each clique spawns onto v^* within $(I_i^-, T_{\mathsf{end}}^i]$.

LEMMA 6.35. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let i be a non-negative integer, and let f be a good possible value of $\mathcal{F}_{I_i^-}(X')$. Let $j \in [\ell]$. Then with probability at least $1 - e^{-\frac{1}{2}(\log n)^3}$, conditioned on $\mathcal{F}_{I_i^-}(X') = f$, vertices in K_j spawn at most $10\alpha_i\sqrt{n}(\log n)^{17}/k$ non-mutants onto v^* within $(I_i^-, T_{\mathsf{end}}^i]$.*

PROOF. We start with the following mutually recursive definitions, which are the same as the ones in Lemma 6.19, except that the endpoint is T_{end}^i rather than I^+. Let $T_{\mathsf{in}}(-1) = I_i^-$. Then, for $h \in \mathbb{Z}_{\geq 0}$,

$$T_{\mathsf{a}}(h) = \min\{t \geq T_{\mathsf{in}}(h-1) \mid K_j \text{ is active at } t \text{ or } t = T_{\mathsf{end}}^i\},$$
$$T_{\mathsf{in}}(h) = \min\{t \geq T_{\mathsf{a}}(h) \mid K_j \text{ is inactive at } t \text{ or } t = T_{\mathsf{end}}^i\}.$$

For all $h \in \mathbb{Z}_{\geq 0}$, let S_h be the number of non-mutants spawned onto v^* in $(T_{\mathsf{a}}(h), T_{\mathsf{in}}(h))$ by vertices in K_j. By (D4), whenever K_j is inactive in the interval $(I_i^-, T_{\mathsf{end}}^i]$ it contains no non-mutants. Moreover, with probability 1 no non-mutants are spawned onto v^* at any time $T_{\mathsf{a}}(h)$ or $T_{\mathsf{in}}(h)$, except possibly I_i^-. (If $T_{\mathsf{a}}(h)$ or $T_{\mathsf{in}}(h)$ is in $(I_i^-, T_{\mathsf{end}}^i)$ this is

because a clock with source a_j triggers at the relevant time and the probability that two clocks trigger simultaneously is 0. Also, it is clear from the definition of T^i_{end} that the probability that a clock with target v^* triggers at this time is 0.) Thus precisely $\sum_{h=0}^{\infty} S_h$ non-mutants are spawned onto v^* in $(I_i^-, T^i_{\mathsf{end}}]$.

Let $\mathcal{E}_1$ be the event that $S_h = 0$ for all $h > \lfloor \alpha_i \sqrt{n} (\log n)^4 \rfloor$. Let $\mathcal{E}_2$ be the event that $S_h = 0$ for all but at most $10\alpha_i \sqrt{n} (\log n)^{14}/k$ values of $h \in \{0, \ldots, \lfloor \alpha_i \sqrt{n} (\log n)^4 \rfloor\}$. Let $\mathcal{E}_3$ be the event that for all $h \in \{0, \ldots, \lfloor \alpha_i \sqrt{n} (\log n)^4 \rfloor\}$, $S_h \leq (\log n)^3$. Note that to prove the result, it suffices to show that

$$\mathbb{P}\big(\mathcal{E}_1 \cap \mathcal{E}_2 \cap \mathcal{E}_3 \mid \mathcal{F}_{I_i^-}(X') = f\big) \geq 1 - e^{-\frac{1}{2}(\log n)^3}. \tag{78}$$

By Lemma 6.34, we have

$$\mathbb{P}\big(\mathcal{E}_1 \mid \mathcal{F}_{I_i^-}(X') = f\big) \geq 1 - 2e^{-(\log n)^3}.$$

Now, by $\mathcal{P}_3(i)$, every clique contains at most Δ non-mutants at time I_i^-. Moreover, by (D4), whenever K_j becomes active in $(I_i^-, T^i_{\mathsf{end}}]$ it contains only one non-mutant. It therefore follows that for all $h \in \mathbb{Z}_{\geq 0}$, either $T_{\mathsf{a}}(h) = T_{\mathsf{in}}(h) = T^i_{\mathsf{end}}$ or $|K_j \setminus X'_{T_{\mathsf{a}}(h)}| \in [\Delta]$.

Now, consider any integer $h \geq 0$ and any time $t_h \geq I_i^-$. Suppose that f_h is a possible value of $\mathcal{F}_{t_h}(X')$ which implies that $t_h = T_{\mathsf{a}}(h)$ and $\mathcal{F}_{I_i^-}(X') = f$. If f_h implies that $|K_j \setminus X'_{t_h}| \in [\Delta]$, then by Lemma 6.20(i) we have

$$\mathbb{P}(S_h > 0 \mid \mathcal{F}_{t_h}(X') = f_h) \leq (\log n)^{10}/k. \tag{79}$$

Otherwise, f_h must imply that $t_h = T^i_{\mathsf{end}}$ and hence $S_h = 0$, so (79) is valid for all choices of f_h. Moreover, by Lemma 6.20(ii) we have

$$\mathbb{P}(S_h \leq (\log n)^3 \mid \mathcal{F}_{t_h}(X') = f_h) \geq 1 - 7e^{-(\log n)^3}. \tag{80}$$

Now, recall that $\mathcal{F}_{t_h}(X')$ determines $S_0, \ldots, S_{h-1}$. It therefore follows from (79) that the number of integers $h \in \{0, \ldots, \lfloor \alpha_i \sqrt{n} (\log n)^4 \rfloor\}$ such that $S_h > 0$ is dominated above by a binomial variable with $\lfloor \alpha_i \sqrt{n} (\log n)^4 \rfloor + 1$ trials, each with success probability $(\log n)^{10}/k$. It follows by Lemma 2.5 that

$$\mathbb{P}(\mathcal{E}_2 \mid \mathcal{F}_{I_i^-}(X') = f) \geq 1 - e^{-10\alpha_i \sqrt{n} (\log n)^{14}/k} \geq 1 - e^{-(\log n)^3}.$$

Finally, by (80) combined with a union bound over all $h \in \{0, \ldots, \lfloor \alpha_i \sqrt{n} (\log n)^4 \rfloor\}$, we have

$$\mathbb{P}(\mathcal{E}_3 \mid \mathcal{F}_{I_i^-}(X') = f) \geq 1 - 7\big(\lfloor \alpha_i \sqrt{n} (\log n)^4 \rfloor + 1\big) e^{-(\log n)^3} \geq 1 - n^2 e^{-(\log n)^3}.$$

Thus (78) follows by a union bound, which implies the result. □

We are now in a position to prove that in total, not too many non-mutants are spawned onto v^* over the interval $(I_i^-, T^i_{\mathsf{end}}]$.

COROLLARY 6.36. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let i be a non-negative integer, and let f be a good possible value of $\mathcal{F}_{I_i^-}(X')$. Then with probability at least $1 - e^{-(\log n)^2}$, conditioned on $\mathcal{F}_{I_i^-}(X') = f$, at most $80\beta_i \sqrt{n} (\log n)^{19}/k$ non-mutants are spawned onto v^* within $(I_i^-, T^i_{\mathsf{end}}]$.*

PROOF. Let $\mathcal{E}$ be the event that for all $j \in [\ell]$, vertices in K_j spawn at most $10\alpha_i \sqrt{n} (\log n)^{17}/k$ non-mutants onto v^* over the course of $(I_i^-, T^i_{\mathsf{end}}]$.

Suppose that $\mathcal{E}$ occurs and that $\alpha_i > \lfloor (2\log n)^2 \rfloor$, so

$$\beta_i \geq \lfloor \ell m/(2\log n)^{2i} \rfloor \geq \ell \lfloor m/(2\log n)^{2i} \rfloor = \ell\alpha_i.$$

Then since $\mathcal{E}$ occurs, at most $10\ell\alpha_i\sqrt{n}(\log n)^{17}/k \leq 80\beta_i\sqrt{n}(\log n)^{19}/k$ non-mutants are spawned onto v^* over the course of $(I_i^-, T_{\mathsf{end}}^i]$.

Now suppose that $\mathcal{E}$ occurs and that $\alpha_i = \lfloor (2\log n)^2 \rfloor$. By (D2) and $\mathcal{P}_4(i)$, for all but at most $\beta_i + \beta_{i+1} \leq 2\beta_i$ values of $j \in [\ell]$, we have $K_j \subseteq X'_t$ for all $t \in (I_i^-, T_{\mathsf{end}}^i]$. Certainly these cliques cannot spawn non-mutants onto v^* in $(I_i^-, T_{\mathsf{end}}^i]$, so since $\mathcal{E}$ occurs, it follows that at most

$$20\beta_i\alpha_i\sqrt{n}(\log n)^{17}/k \leq 80\beta_i\sqrt{n}(\log n)^{19}/k$$

non-mutants are spawned onto v^* over the course of $(I_i^-, T_{\mathsf{end}}^i]$.

We have therefore shown that whenever $\mathcal{E}$ occurs, at most $80\beta_i\sqrt{n}(\log n)^{19}/k$ non-mutants are spawned onto v^* over the course of $(I_i^-, T_{\mathsf{end}}^i]$. By Lemma 6.35 combined with a union bound over all $j \in [\ell]$, we have

$$\mathbb{P}(\mathcal{E} \mid \mathcal{F}_{I_i^-}(X') = f) \geq 1 - e^{-(\log n)^2}.$$

The result therefore follows. □

We now use (D4) to show that when v^* is a non-mutant in $(I_i^-, T_{\mathsf{end}}^i]$, the time until either v^* becomes a mutant again or the interval ends is dominated above by an exponential variable with parameter $\ell/2$. The proof of this lemma is similar to the proof of Lemma 5.9.

LEMMA 6.37. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let $i \geq 0$ and $z \geq 0$ be integers, let $t_0 \in [I_i^-, I_i^+]$, and let f be a good possible value of $\mathcal{F}_{t_0}(X')$ which implies that $t_0 = T_{\mathsf{n}}^{i,v^*}(z)$. Let $S = T_{\mathsf{m}}^{i,v^*}(z) - t_0$. Then for all $t \geq 0$,*

$$\mathbb{P}(S \leq t \mid \mathcal{F}_{t_0}(X') = f) \geq 1 - e^{-\ell t/2}.$$

PROOF. Note that $t_0 \leq T_{\mathsf{end}}^i$, and f determines the event $t_0 = T_{\mathsf{end}}^i$. Moreover, if $t_0 = T_{\mathsf{end}}^i$, then $S = 0$ with probability 1 and the result follows. We may therefore assume that f implies that $t_0 < T_{\mathsf{end}}^i$, and in particular $v^* \notin X'_{t_0}$.

Let $\mathcal{A}$ be the set of all mutant clocks in $\mathcal{C}(\mathcal{M}_\ell)$ with target v^*, and let Φ encapsulate the behaviour of every clock in $\mathcal{C}(\mathcal{M}_\ell) \setminus \mathcal{A}$ over the interval I_i. In particular, by (D1), Φ determines the behaviour of these clocks in the interval $(I_i^-, T_{\mathsf{end}}^i]$. Consider any possible value φ of Φ which is consistent with $\mathcal{F}_{t_0}(X') = f$.

We now define a time t'_{end} which depends only on the values f and φ. To do so, consider the situation in which $\mathcal{F}_{t_0}(X') = f$, $\Phi = \varphi$ and no clock in $\mathcal{A}$ triggers in $(t_0, I_i^+]$, so that the evolution of X' in this interval is entirely determined by f and φ. Let t'_{end} be the time at which T_{end}^i would occur in this situation.

We claim that, if $\mathcal{F}_{t_0}(X') = f$ and $\Phi = \varphi$, then $T_{\mathsf{m}}^{i,v^*}(z) \leq t'_{\mathsf{end}}$. To see this, suppose that $\mathcal{F}_{t_0}(X') = f$ and $\Phi = \varphi$. If no mutant is spawned onto v^* in $(t_0, t'_{\mathsf{end}}]$, then X' evolves exactly as it would have done if no clocks in $\mathcal{A}$ triggered in $(t_0, t'_{\mathsf{end}}]$, and so $t'_{\mathsf{end}} = T_{\mathsf{end}}^i \geq T_{\mathsf{m}}^{i,v^*}(z)$. If a mutant is spawned onto v^* at some time $t' \in (t_0, t'_{\mathsf{end}}]$, then $T_{\mathsf{m}}^{i,v^*}(z) \leq t' \leq t'_{\mathsf{end}}$. So $T_{\mathsf{m}}^{i,v^*}(z) \leq t'_{\mathsf{end}}$ in all cases as claimed. Hence if $t \geq t'_{\mathsf{end}} - t_0$,

$$\mathbb{P}(S \leq t \mid \mathcal{F}_{t_0}(X') = f, \Phi = \varphi) = 1 \geq 1 - e^{-\ell t/2},$$

as required in the lemma statement. We now consider the case $t < t'_{\mathsf{end}} - t_0$.

Let $t_1 < \cdots < t_y$ be the times in $(t_0, t_0 + t]$ at which clocks in $\mathcal{C}(\mathcal{M}_\ell) \setminus \mathcal{A}$ trigger, and let $t_{y+1} = t_0 + t$. Thus $t_0 < \cdots < t_y \leq t_{y+1} < t'_{\mathsf{end}}$. For all $h \in \{0, \ldots, y\}$, let $\chi(h)$

be the value that X'_{t_h} would take in the situation where $\mathcal{F}_{t_0}(X') = f$, $\Phi = \varphi$ and no clock in $\mathcal{A}$ triggers in $(t_0, t_h]$. Thus t'_{end}, y, $t_0, \ldots, t_{y+1}$, and $\chi(0), \ldots, \chi(y)$ are all uniquely determined by f and φ.

For each $h \in [y+1]$, let $\mathcal{E}_h$ be the event that a mutant is spawned onto v^* in the interval (t_{h-1}, t_h). Note that with probability 1, no mutant is spawned onto v^* at any time t_h. Thus

$$\begin{aligned}\mathbb{P}(S \le t \mid \mathcal{F}_{t_0}(X') = f, \Phi = \varphi) &= 1 - \mathbb{P}(\overline{\mathcal{E}_1} \cap \cdots \cap \overline{\mathcal{E}_{y+1}} \mid \mathcal{F}_{t_0}(X') = f, \Phi = \varphi) \\ &= 1 - \prod_{h=1}^{y+1} \mathbb{P}(\overline{\mathcal{E}_h} \mid \mathcal{F}_{t_0}(X') = f, \Phi = \varphi, \overline{\mathcal{E}_1} \cap \cdots \cap \overline{\mathcal{E}_{h-1}}). \quad (81)\end{aligned}$$

Now fix $h \in [y+1]$, and consider any possible value f_{h-1} of $\mathcal{F}_{t_{h-1}}(X')$ which implies that $\mathcal{F}_{t_0}(X') = f$ and that $\overline{\mathcal{E}_1} \cap \cdots \cap \overline{\mathcal{E}_{h-1}}$ occurs, and is consistent with $\Phi = \varphi$. Consider the evolution of X' given $\mathcal{F}_{t_{h-1}}(X') = f_{h-1}$ and $\Phi = \varphi$. Since $\overline{\mathcal{E}_1} \cap \cdots \cap \overline{\mathcal{E}_{h-1}}$ occurs, no mutant is spawned onto v^* in the interval $(t_0, t_{h-1}]$ and so $X'_{t_{h-1}} = \chi(h-1)$. Moreover, X' remains constant in $[t_{h-1}, t_h)$ unless a mutant is spawned onto v^*. Thus, given the condition that $\mathcal{F}_{t_{h-1}}(X') = f_{h-1}$ and $\Phi = \varphi$, $\mathcal{E}_h$ occurs if and only if a mutant clock with source in $\chi(h-1)$ and target v^* triggers in the interval (t_{h-1}, t_h).

Now, since $\overline{\mathcal{E}_1} \cap \cdots \cap \overline{\mathcal{E}_{h-1}}$ occurs and $t'_{\mathsf{end}} > t_{h-1}$, by (D4) we have

$$|\chi(h-1) \cap (K_1 \cup \cdots \cup K_\ell)| \ge k\ell/2.$$

Hence

$$\mathbb{P}(\overline{\mathcal{E}_h} \mid \mathcal{F}_{t_{h-1}}(X') = f_{h-1}, \Phi = \varphi) = e^{-r(t_h - t_{h-1})|\chi(h-1) \cap (K_1 \cup \cdots \cup K_\ell)|/k} \le e^{-\ell(t_h - t_{h-1})/2}.$$

It therefore follows from (81) that

$$\mathbb{P}(S \le t \mid \mathcal{F}_{t_0}(X') = f, \Phi = \varphi) \ge 1 - \prod_{h=1}^{y+1} e^{-\ell(t_h - t_{h-1})/2} = 1 - e^{-\ell t/2}.$$

The result therefore follows. □

Definition 6.38. For all $i \in \mathbb{Z}_{\ge 0}$, let $\gamma_i = \beta_i (\log n)^{20}/k$. □

LEMMA 6.39. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \ge \ell_0$. Let i be a non-negative integer, and let f be a good possible value of $\mathcal{F}_{I_i^-}(X')$. Then*

$$\mathbb{P}\left(\mathrm{len}(\{t \in (I_i^-, T^i_{\mathsf{end}}] \mid v^* \notin X'_t\}) \le \gamma_i \;\middle|\; \mathcal{F}_{I_i^-}(X') = f\right) \ge 1 - 2e^{-(\log n)^2}.$$

PROOF. For all $h \in \mathbb{Z}_{\ge 0}$, write $S_h = T^{i,v^*}_{\mathsf{m}}(h) - T^{i,v^*}_{\mathsf{n}}(h)$. Consider any $s_0, \ldots s_{h-1} \ge 0$, $t_h \ge I_i^-$, and any possible value f_h of $\mathcal{F}_{t_h}(X')$ which implies that $\mathcal{F}_{I_i^-}(X') = f$, $S_0 = s_0, \ldots, S_{h-1} = s_{h-1}$ and $T^{i,v^*}_{\mathsf{n}}(h) = t_h$. Then by Lemma 6.37, for all $t \ge 0$ we have

$$\mathbb{P}(S_h \le t \mid \mathcal{F}_{t_h}(X') = f_h) \ge 1 - e^{-\ell t/2}.$$

It follows that $S_0, S_1, \ldots$ are dominated above by i.i.d. exponential variables $S'_0, S'_1, \ldots$ with parameter $\ell/2$.

Now, by the definition of $T^{i,v^*}_{\mathsf{n}}(h)$ and $T^{i,v^*}_{\mathsf{m}}(h)$, we have that

$$\mathrm{len}(\{t \in (I_i^-, T^i_{\mathsf{end}}] \mid v^* \notin X'_t\}) = \sum_{h=0}^{\infty} S_h. \quad (82)$$

By Corollary 6.36, we have

$$\mathbb{P}\big(S_h = 0 \text{ for all } h > 80\beta_i\sqrt{n}(\log n)^{19}/k \mid \mathcal{F}_{I_i^-}(X') = f\big) \geq 1 - e^{-(\log n)^2}. \tag{83}$$

Moreover, we have

$$\frac{3}{\ell}\left(\left\lfloor \frac{80\beta_i\sqrt{n}(\log n)^{19}}{k}\right\rfloor + 1\right) \leq \frac{241\beta_i\sqrt{n}(\log n)^{19}}{k\ell} \leq \frac{\beta_i(\log n)^{20}}{k} = \gamma_i,$$

and so by Corollary 2.4 we have

$$\mathbb{P}\left(\sum_{h=0}^{\left\lfloor \frac{80\beta_i(\log n)^{19}}{k}\right\rfloor} S_h < \gamma_i \,\middle|\, \mathcal{F}_{I_i^-}(X') = f\right) \geq \mathbb{P}\left(\sum_{h=0}^{\left\lfloor \frac{80\beta_i\sqrt{n}(\log n)^{19}}{k}\right\rfloor} S'_h < \gamma_i\right)$$
$$\geq 1 - e^{-\ell\gamma_i/32} \geq 1 - e^{-(\log n)^2}.$$

The result therefore follows by (82), (83) and a union bound. □

6.7.4. Proving Lemma 6.27. Recall that by Lemma 6.39, v^* is very unlikely to spend more than γ_i time as a non-mutant over the course of $(I_i^-, T^i_{\mathsf{end}}]$. This motivates the following definition.

Definition 6.40. Recall the definition of $\Psi(X')$ (Section 3.3). For all $i \in \mathbb{Z}_{\geq 0}$, let $t_i^- = i_{\mathsf{n}}(v^*, I_i^-)$ and let $t_i^+ = i_{\mathsf{m}}(v^*, I_i^-)$. For all $j \in [\ell]$, let

$$\mathcal{T}^{i,j} = \{t \in (t_i^-, t_i^- + \gamma_i] \mid \text{for some } v \in R_j,\ N^*_{(v^*,v)} \text{ triggers at } t\},$$
$$\mathcal{T}^i = \mathcal{T}^{i,1} \cup \cdots \cup \mathcal{T}^{i,\ell},$$
$$\mathcal{U}_i = \{v \in R_1 \cup \cdots \cup R_\ell \mid N^*_{(v^*,v)} \text{ triggers in } (t_i^-, t_i^- + \gamma_i]\}. \quad \square$$

In particular, if v^* spends at most γ_i time as a non-mutant in $(I_i^-, T^i_{\mathsf{end}}]$, then it only spawns non-mutants onto vertices in $\mathcal{U}_i$.

LEMMA 6.41. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let i be a non-negative integer, and let f^* be a possible value of $\mathcal{F}_{I_i^-}(\Psi(X'))$. Then with probability at least $1 - 3e^{-(\log n)^2}$ conditioned on $\mathcal{F}_{I_i^-}(\Psi(X')) = f^*$, the following statements all hold.*

(i) $|\mathcal{U}_i| \leq |\mathcal{T}^i| < \beta_{i+1}$.
(ii) For all $j \in [\ell]$, $|\mathcal{U}_i \cap R_j| \leq |\mathcal{T}^{i,j}| < \alpha_{i+1}$.
(iii) For all $v \in R_1 \cup \cdots \cup R_\ell$, $M^*_{(v^*,v)}$ *triggers in* $(t_i^+, t_i^+ + \mathrm{len}(I_i)/3]$.

PROOF. Note that t_i^- is determined by f^*, and that f^* does not determine the behaviour of star-clocks $N^*_{(v^*,v)}$ in (t_i^-, ∞). It follows that conditioned on $\mathcal{F}_{I_i^-}(\Psi(X')) = f^*$, $|\mathcal{T}^i|$ follows a Poisson distribution with parameter $\gamma_i = \beta_i(\log n)^{20}/k \leq \frac{1}{9}\frac{\beta_i}{(2\log n)^2}$.

If $\beta_i \leq (2\log n)^4$, then this parameter is at most $\frac{1}{9}(2\log n)^2$. We also have that $\beta_{i+1} \geq \lfloor (2\log n)^2 \rfloor$. Using also Corollary 2.3, we thus obtain that

$$\mathbb{P}(|\mathcal{T}^i| < \beta_{i+1} \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \geq \mathbb{P}(|\mathcal{T}^i| < \lfloor (2\log n)^2 \rfloor \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*)$$
$$\geq 1 - e^{-\lfloor (2\log n)^2 \rfloor} \geq 1 - e^{-(\log n)^2}.$$

If instead $\beta_i > (2\log n)^4$, then $\beta_{i+1} \geq \lfloor \frac{\beta_i}{(2\log n)^2} \rfloor \geq 8\gamma_i$ and so by Corollary 2.3,

$$\mathbb{P}(|\mathcal{T}^i| < \beta_{i+1} \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \geq 1 - e^{-\beta_{i+1}} \geq 1 - e^{-\lfloor (2\log n)^2 \rfloor} \geq 1 - e^{-(\log n)^2}.$$

Thus in all cases,

$$\mathbb{P}(|\mathcal{T}^i| < \beta_{i+1} \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \geq 1 - e^{-(\log n)^2}. \tag{84}$$

By a similar argument to the above, we see that for all $j \in [\ell]$, $|\mathcal{T}^{i,j}|$ follows a Poisson distribution with parameter γ_i/ℓ. Note that if $\beta_i = \lfloor (2\log n)^2 \rfloor$ then $\gamma_i/\ell \leq 1/\ell$, and if $\beta_i = \lfloor \ell m/(2\log n)^{2i} \rfloor$ then $\gamma_i/\ell \leq \frac{1}{10}\frac{m}{(2\log n)^{2i+2}}$. Thus $\gamma_i/\ell \leq \frac{1}{9}\frac{\alpha_i}{(2\log n)^2}$ in all cases. As in the above argument, it therefore follows from Corollary 2.3 that

$$\mathbb{P}(|\mathcal{T}^{i,j}| < \alpha_{i+1} \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \geq 1 - e^{-\lfloor (2\log n)^2 \rfloor}.$$

A union bound over $j \in [\ell]$ thus gives

$$\mathbb{P}(|\mathcal{T}^{i,j}| < \alpha_{i+1} \text{ for all } j \in [\ell] \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \geq 1 - \ell e^{-\lfloor (2\log n)^2 \rfloor} \geq 1 - e^{-(\log n)^2}. \tag{85}$$

Finally, let $\mathcal{E}$ be the event that $M^*_{(v^*,v)}$ triggers in $(t_i^+, t_i^+ + \mathrm{len}(I_i)/3]$ for all $v \in R_1 \cup \cdots \cup R_\ell$. Note that t_i^+ is determined by f^*, and that f^* does not determine the behaviour of star-clocks $M^*_{(v^*,v)}$ in (t_i^+, ∞). It follows that conditioned on $\mathcal{F}_{I_i^-}(\Psi(X')) = f^*$, for all $v \in R_1 \cup \cdots \cup R_\ell$, the number of times $M^*_{(v^*,v)}$ triggers in $(t_i^+, t_i^+ + \mathrm{len}(I_i)/3]$ is a Poisson variable with parameter $\mathrm{len}(I_i)/(3\ell m) \geq \frac{1}{3}(\log n)^3$. Hence

$$\mathbb{P}(M^*_{(v^*,v)} \text{ triggers in } (t_i^+, t_i^+ + \mathrm{len}(I_i)/3] \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \geq 1 - e^{-\frac{1}{3}(\log n)^3}.$$

Thus by a union bound over all $v \in R_1 \cup \cdots \cup R_\ell$, we have

$$\mathbb{P}(\mathcal{E} \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \geq 1 - \ell m e^{-\frac{1}{3}(\log n)^3} \geq 1 - e^{-(\log n)^2}. \tag{86}$$

The result therefore follows from a union bound over (84), (85) and (86). □

The following lemma will be the heart of the proof of Lemma 6.27.

LEMMA 6.42. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let i be a non-negative integer, and let f^* be a possible value of $\mathcal{F}_{I_i^-}(\Psi(X'))$ such that the induced value f of $\mathcal{F}_{I_i^-}(X')$ is good. Then with probability at least $1 - 10e^{-(\log n)^2}$ conditioned on $\mathcal{F}_{I_i^-}(\Psi(X')) = f^*$, the following statements all hold.*

(i) $(R_1 \cup \cdots \cup R_\ell) \setminus X'_{I_i^+} \subseteq \mathcal{U}_i$.

(ii) If $i \geq 6$, then for all $j \in [\ell]$ such that $R_j \cap \mathcal{U}_i = \emptyset$ and all $t \in [I_i^+ - \mathrm{len}(I_i)/4, I_i^+]$, we have $R_j \cup \{a_j\} \cup K_j \subseteq X'_t$.

PROOF. We first define events as follows. For all $h \in \mathbb{Z}_{\geq 0}$, let $J_h^- = I_i^- + \mathrm{len}(I_i)/2 + h(\log n)^7$, let $J_h^+ = J_h^- + (\log n)^7$, and let $J_h = (J_h^-, J_h^+]$.

— $\mathcal{E}_1$: $\mathrm{len}(\{t \in (I_i^-, T^i_{\mathsf{end}}] \mid v^* \notin X'_t\}) \leq \gamma_i$.
— $\mathcal{E}_2$: for all $t \in I_i$ and all $j \in [\ell]$, $|K_j \setminus X'_t| \leq 2\Delta$.
— $\mathcal{E}_3$: for all $j \in [\ell]$ and all $t \in [I_i^-, I_i^- + 2\mathrm{len}(I_i)/3]$, there exists $t' \in [t, t + (\log n)^7)$ such that $K_j \subseteq X'_{t'}$.
— $\mathcal{E}_4$: $|\mathcal{T}^i| < \beta_{i+1}$.

— $\mathcal{E}_5$: for all $j \in [\ell]$, $|\mathcal{T}^{i,j}| < \alpha_{i+1}$.
— $\mathcal{E}_6$: for all $v \in R_1 \cup \cdots \cup R_\ell$, $M^*_{(v^*,v)}$ triggers in $(t_i^+, t_i^+ + \mathrm{len}(I_i)/3]$.
— $\mathcal{E}_7$: for all $j \in [\ell]$, $W_{\mathsf{a}}^{i,j} \le \alpha_i \sqrt{n}(\log n)^4$
— $\mathcal{E}_8$: for all $j \in [\ell]$ and all $h \in \{0, \ldots, \lfloor 3\alpha_i \sqrt{n}(\log n)^4 \rfloor\}$, some clock $M_{(v,a_j)}$ triggers in the interval J_h.

Note that by Observation 3.7, $\mathcal{F}_{I_i^-}(\Psi(X'))$ is uniquely determined by $\mathcal{F}_{I_i^-}(X')$ and vice versa. Thus the value f from the statement of the lemma is the unique value of f such that $\mathcal{F}_{I_i^-}(X') = f$ if and only if $\mathcal{F}_{I_i^-}(\Psi(X')) = f^*$, allowing us to apply results like Lemma 6.39 to X' even though we are conditioning on $\mathcal{F}_{I_i^-}(\Psi(X'))$ rather than $\mathcal{F}_{I_i^-}(X')$.

Observe that by a union bound over all $j \in [\ell]$ and $h \in \{0, \ldots, \lfloor 3\alpha_i \sqrt{n}(\log n)^4 \rfloor\}$,

$$\mathbb{P}(\mathcal{E}_8 \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \ge 1 - \ell(\lfloor 3\alpha_i \sqrt{n}(\log n)^4 \rfloor + 1)e^{-rm(\log n)^7} \ge 1 - e^{-(\log n)^2}. \quad (87)$$

Moreover, by $\mathcal{P}_3(i)$, f satisfies the conditions of Lemma 6.19, which together with a union bound over $j \in [\ell]$ implies that

$$\mathbb{P}(\mathcal{E}_2 \cap \mathcal{E}_3 \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \ge 1 - \ell e^{-\frac{1}{2}(\log n)^3} \ge 1 - e^{-(\log n)^2}. \quad (88)$$

Thus by Lemma 6.39, (88), Lemma 6.41, Lemma 6.34 and (87) (applied in order),

$$\begin{aligned}
\mathbb{P}\left(\bigcap_{s=1}^{8} \mathcal{E}_s \,\middle|\, \mathcal{F}_{I_i^-}(\Psi(X')) = f^*\right) \ge 1 &- \mathbb{P}(\overline{\mathcal{E}_1} \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \\
&- \mathbb{P}(\overline{\mathcal{E}_2 \cap \mathcal{E}_3} \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \\
&- \mathbb{P}(\overline{\mathcal{E}_4 \cap \mathcal{E}_5 \cap \mathcal{E}_6} \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \\
&- \mathbb{P}(\overline{\mathcal{E}_7} \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \\
&- \mathbb{P}(\overline{\mathcal{E}_8} \mid \mathcal{F}_{I_i^-}(\Psi(X')) = f^*) \\
\ge 1 - 8e^{-(\log n)^2} &- 2\ell e^{-(\log n)^3} \ge 1 - 10e^{-(\log n)^2}. \quad (89)
\end{aligned}$$

It therefore suffices to show that when $\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_8$ occurs, (i) and (ii) both hold.

Suppose $\mathcal{E}_1 \cap \cdots \cap \mathcal{E}_8$ occurs. First note that since $\mathcal{E}_1$ occurs, by Observation 3.7, for all $t \in (I_i^-, T_{\mathsf{end}}^i]$,

$$v^* \text{ spawns a non-mutant at time } t \text{ only if } i_{\mathsf{n}}(v^*, t) \in \mathcal{T}^i.$$

Since $\mathcal{E}_4$ occurs, it follows that v^* spawns fewer than β_{i+1} non-mutants in the interval $(I_i^-, T_{\mathsf{end}}^i]$, and so (D2) does not hold at time T_{end}^i. Likewise, since $\mathcal{E}_1$ occurs, for all $t \in (I_i^-, T_{\mathsf{end}}^i]$ and all $j \in [\ell]$,

$$v^* \text{ spawns a non-mutant onto a vertex in } R_j \text{ at time } t \text{ only if } i_{\mathsf{n}}(v^*, t) \in \mathcal{T}^{i,j}.$$

and so since $\mathcal{E}_5$ occurs it follows that v^* spawns fewer than α_{i+1} non-mutants onto vertices in R_j in the interval $(I_i^-, T_{\mathsf{end}}^i]$. Hence (D3) does not hold at time T_{end}^i. Finally, since $\mathcal{E}_2$ occurs and $T_{\mathsf{end}}^i \in I_i$, (D4) does not hold at time T_{end}^i. Since none of (D2)–(D4) hold at T_{end}^i, (D1) must hold at T_{end}^i and so $T_{\mathsf{end}}^i = I_i^+$.

Since $\mathcal{E}_1$ occurs and $T_{\mathsf{end}}^i = I_i^+$, it follows that

$$\text{Throughout } I_i,\ v^* \text{ only spawns non-mutants onto vertices in } \mathcal{U}_i. \quad (90)$$

Also since $\mathcal{E}_1$ occurs and $T^i_{\mathsf{end}} = I_i^+$, we have

$$\mathrm{len}(\{t \in (I_i^-, I_i^- + \mathrm{len}(I_i)/2] \mid v^* \in X'_t\}) \geq \frac{\mathrm{len}(I_i)}{2} - \gamma_i > \frac{\mathrm{len}(I_i)}{3}.$$

Thus since $\mathcal{E}_6$ occurs, by Observation 3.7, v^* spawns a mutant onto every vertex in $(R_1 \cup \cdots \cup R_\ell) \setminus \mathcal{U}_i$ in the interval $(I_i^-, I_i^- + \mathrm{len}(I_i)/2]$. Thus by (90), every vertex in $(R_1 \cup \cdots \cup R_\ell) \setminus \mathcal{U}_i$ is a mutant throughout $[I_i^- + \mathrm{len}(I_i)/2, I_i^+]$, and in particular (i) holds.

To prove (ii), let $i \geq 6$ be an integer and let $j \in [\ell]$ be such that $R_j \cap \mathcal{U}_i = \emptyset$. By the above, we have $R_j \subseteq X'_t$ for all $t \in [I_i^- + \mathrm{len}(I_i)/2, I_i^+]$.

Let

$$\mathcal{J} = \{J_h \mid 0 \leq h \leq \lfloor 3\alpha_i \sqrt{n} (\log n)^4 \rfloor, K_j \text{ becomes active in } J_h\}$$
$$\mathcal{J}' = \{J_h \mid 0 \leq h \leq \lfloor 3\alpha_i \sqrt{n} (\log n)^4 \rfloor, \text{there exists } t \in J_h \text{ such that } K_j \text{ is active at } t.\}$$

From $i \geq 6$, we have $\alpha_i \leq \sqrt{n}/(\log n)^{12}$ and hence $J_h \subseteq (I_i^- + \mathrm{len}(I_i)/2, I_i^- + 2\mathrm{len}(I_i)/3]$ for all $h \leq \lfloor 3\alpha_i \sqrt{n} (\log n)^4 \rfloor$. Since $\mathcal{E}_3$ occurs, we have $|\mathcal{J}'| \leq 2|\mathcal{J}|$. Since $\mathcal{E}_7$ occurs, it therefore follows that $|\mathcal{J}'| \leq 2\alpha_i \sqrt{n} (\log n)^4$, and in particular there exists $h \in \{0, \ldots \lfloor 3\alpha_i \sqrt{n} (\log n)^4 \rfloor\}$ such that K_j is inactive throughout J_h. Moreover, since $\mathcal{E}_8$ occurs and $R_j \subseteq X_t$ for all $t \in J_h$, a_j must become a mutant in J_h. Thus $R_j \cup \{a_j\} \subseteq X'_t$ for all $t \in [I_i^- + 2\mathrm{len}(I_i)/3, I_i^+]$.

Finally, since $\mathcal{E}_3$ occurs, it follows that there exists $t \in [I_i^- + 2\mathrm{len}(I_i)/3, I_i^- - \mathrm{len}(I_i)/4]$ such that K_j is full of mutants at time t. It follows that $R_j \cup \{a_j\} \cup K_j \subseteq X'_t$ for all $t \in [I_i^+ - \mathrm{len}(I_i)/4, I_i^+]$. Thus (ii) holds, and so the result follows from (89). □

We are now at last in a position to prove Lemma 6.27, which completes the proof of Theorem 6.1.

LEMMA 6.27. *Consider any $r > 1$. There is an ℓ_0, depending on r, such that the following holds for any $\ell \geq \ell_0$. Let i be a non-negative integer, and let f be a good possible value of $\mathcal{F}_{I_i^-}(X')$. Then*

$$\mathbb{P}(\mathcal{F}_{I_i^+}(X') \textit{ is good} \mid \mathcal{F}_{I_i^-}(X') = f) \geq 1 - 15e^{-(\log n)^2}.$$

Moreover, if $\beta_i = \lfloor (2\log n)^2 \rfloor$, then

$$\mathbb{P}(X'_{I_i^+} = V(\mathcal{M}_\ell) \mid \mathcal{F}_{I_i^-}(X') = f) \geq 1/2.$$

PROOF. Consider the process $\Psi(X')$. We define the following events.

— $\mathcal{E}_1$: for all $j \in [\ell]$, $|K_j \setminus X'_{I_i^+}| \leq \Delta$.
— $\mathcal{E}_2$: $|\mathcal{U}_i| < \beta_{i+1}$.
— $\mathcal{E}_3$: for all $j \in [\ell]$, $|\mathcal{U}_i \cap R_j| < \alpha_{i+1}$.
— $\mathcal{E}_4$: $(R_1 \cup \cdots \cup R_\ell) \setminus X'_{I_i^+} \subseteq \mathcal{U}_i$.
— $\mathcal{E}_5$: either $i < 6$ or for all $j \in [\ell]$ such that $R_j \cap \mathcal{U}_i = \emptyset$ and all $t \in [I_i^+ - \mathrm{len}(I_i)/4, I_i^+]$, we have $R_j \cup \{a_j\} \cup K_j \subseteq X'_t$.
— $\mathcal{E}_6$: $\mathcal{U}_i = \emptyset$.
— $\mathcal{E}_7$: for some $v \in K_1 \cup \cdots \cup K_\ell$, $M_{(v,v^*)}$ triggers in $(I_i^+ - \mathrm{len}(I_i)/4, I_i^+]$.

Note that by Observation 3.7, there exists a unique value of f^* such that $\mathcal{F}_{I_i^-}(\Psi(X')) = f^*$ if and only if $\mathcal{F}_{I_i^-}(X') = f$. By $\mathcal{P}_3(i)$, f satisfies the conditions of Lemma 6.19, and so by Lemma 6.19(ii) and a union bound over $j \in [\ell]$ we have

$$\mathbb{P}(\overline{\mathcal{E}_1} \mid \mathcal{F}_{I_i^-}(X') = f) \geq 1 - \ell e^{-\frac{1}{2}(\log n)^3} \geq 1 - e^{-(\log n)^2}. \tag{91}$$

Thus, by (91), Lemma 6.41(i) and (ii), and Lemma 6.42 (applied in order) together with a union bound, we have

$$\begin{aligned}\mathbb{P}\left(\bigcap_{s=1}^{5}\mathcal{E}_s \,\middle|\, \mathcal{F}_{I_i^-}(X')=f\right) &\geq 1-\mathbb{P}(\overline{\mathcal{E}_1}\mid \mathcal{F}_{I_i^-}(X')=f)-\mathbb{P}(\overline{\mathcal{E}_2\cap\mathcal{E}_3}\mid \mathcal{F}_{I_i^-}(X')=f)\\ &\qquad -\mathbb{P}(\overline{\mathcal{E}_4\cap\mathcal{E}_5}\mid \mathcal{F}_{I_i^-}(X')=f)\\ &\geq 1-15e^{-(\log n)^2}. \end{aligned} \tag{92}$$

Suppose that $\mathcal{E}_1\cap\cdots\cap\mathcal{E}_5$ occurs. We claim that $\mathcal{F}_{I_{i+1}^-}(X')$ is good. Indeed, since $\mathcal{E}_2\cap\mathcal{E}_4$ occurs, $\mathcal{P}_1(i+1)$ is satisfied. Likewise, since $\mathcal{E}_3\cap\mathcal{E}_4$ occurs, $\mathcal{P}_2(i+1)$ is satisfied. Since $\mathcal{E}_1$ occurs, $\mathcal{P}_3(i+1)$ is satisfied. Finally, since $\mathcal{E}_2$ occurs, we have

$$|\{j\in[\ell]\mid R_j\cap\mathcal{U}_i\neq\emptyset\}|\leq|\mathcal{U}_i|\leq\beta_{i+1}.$$

If $i<6$, then $\beta_i>\ell$ and so $\mathcal{P}_4(i+1)$ holds vacuously. If instead $i\geq 6$, then $\mathcal{P}_4(i+1)$ holds since $\mathcal{E}_5$ occurs. Thus $\mathcal{F}_{I_{i+1}^-}(X')$ is good, and so the first part of the result follows from (92).

Now suppose $\beta_i=\lfloor(2\log n)^2\rfloor$, so $\gamma_i\leq 4(\log n)^{22}/k$ and $i\geq 6$. Note that t_i^- is determined by f, and that f does not determine the behaviour of clocks $N^*_{(v^*,v)}$ in (t_i^-,∞). It follows that conditioned on $\mathcal{F}_{I_i^-}(X')=f$, $|\mathcal{T}^i|$ follows a Poisson distribution with parameter γ_i. It therefore follows that

$$\mathbb{P}(\mathcal{E}_6\mid\mathcal{F}_{I_i^-}(X')=f)=e^{-\gamma_i}\geq 1-\gamma_i\geq 1-\frac{4(\log n)^{22}}{k}\geq\frac{5}{6}.$$

Moreover, it is immediate that

$$\mathbb{P}(\mathcal{E}_7\mid\mathcal{F}_{I_i^-}(X')=f)=1-e^{-r\ell\mathrm{len}(I_i)/4}\geq 5/6.$$

By Lemma 6.42(ii) and a union bound, it therefore follows that

$$\mathbb{P}(\mathcal{E}_5\cap\mathcal{E}_6\cap\mathcal{E}_7\mid\mathcal{F}_{I_i^-}(X')=f)\geq 1/2. \tag{93}$$

Suppose that $\mathcal{E}_5\cap\mathcal{E}_6\cap\mathcal{E}_7$ holds. Since $\mathcal{E}_5\cap\mathcal{E}_6$ holds and $i\geq 6$, for all $j\in[\ell]$ and all $t\in[I_i^+-\mathrm{len}(I_i)/4,I_i^+]$ we have $K_j\cup\{a_j\}\cup R_j\subseteq X'_t$. Since $\mathcal{E}_7$ holds, it therefore follows that $V(\mathcal{M}_\ell)\subseteq X'_{I_i^+}$. Thus the second part of the result follows from (93). □

7. AN UPPER BOUND ON FIXATION PROBABILITY OF MEGASTARS

The following lemma is similar to Lemma 4.2 for superstars.

LEMMA 7.1. *Let $r>1$, and let k, ℓ and m be arbitrary positive integers. Let $x_0\in\mathcal{M}_{k,\ell,m}$ be chosen uniformly at random. Let X be a Moran process with $G(X)=\mathcal{M}_{k,\ell,m}$ and $X_0=\{x_0\}$. Then X goes extinct with probability at least $k/(2r(m+k))$.*

PROOF. We have

$$\mathbb{P}(x_0\notin R_1\cup\cdots\cup R_\ell)=1-\frac{\ell m}{\ell(m+k+1)+1}\geq 1-\frac{m}{m+k}=\frac{k}{m+k}.$$

Moreover, let $\mathcal{E}$ be the event that x_0 dies before spawning a mutant. Then we have

$$\begin{aligned}\mathbb{P}(\mathcal{E}\mid x_0\in\{a_1,\ldots,a_\ell\})&=m/(m+r),\\ \mathbb{P}(\mathcal{E}\mid x_0\in K_1\cup\cdots\cup K_\ell)&=1/(1+r),\\ \mathbb{P}(\mathcal{E}\mid x_0=v^*)&=\ell/(\ell+r).\end{aligned}$$

Thus we have

$$\mathbb{P}(\mathcal{E}) \geq \mathbb{P}(\mathcal{E} \mid x_0 \notin R_1 \cup \cdots \cup R_\ell)\mathbb{P}(x_0 \notin R_1 \cup \cdots \cup R_\ell) \geq \frac{1}{1+r} \cdot \frac{k}{m+k} \geq \frac{k}{2r(m+k)}.$$

Since X goes extinct if $\mathcal{E}$ occurs, the result follows. □

The following lemma is similar to Lemma 4.3 for superstars.

LEMMA 7.2. *Let $r > 1$, and let k, ℓ and m be arbitrary positive integers with $m \geq 12r$. Let $x_0 \in R_1 \cup \cdots \cup R_\ell$ be arbitrary. Let X be the Moran process with $G(X) = \mathcal{M}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then X goes extinct with probability at least $1/(26r^2\ell)$.*

PROOF. Assume, without loss of generality, that $x_0 \in R_1$. Let $\xi = \lfloor m/(2r) \rfloor$, $t^* = m/(4r^2)$ and $J = [0, t^*]$. For all $t \geq 0$, let $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{E}^3_t$ be events defined as follows.

$\mathcal{E}^1$: $N_{(v^*,x_0)}$ triggers in J.
$\mathcal{E}^2$: $M_{(x_0,a_1)}$ triggers at most ξ times in J.
$\mathcal{E}^3_t$: $\min\{t' > t \mid \text{for some } u \neq x_0,\ N_{(u,a_1)} \text{ triggers at } t'\} <$
$\min\{t' > t \mid \text{some clock } M_{(a_1,v)} \text{ triggers at } t'\}$.

Finally, let $T^i_{v_1}$ be the i'th time at which the clock $M_{(x_0,a_1)}$ triggers and define $\mathcal{E}^3 = \bigcap_{i=1}^{\xi} \mathcal{E}^3_{T^i_{v_1}}$.

As in the proof of Lemma 4.3 X goes extinct if the events $\mathcal{E}^1$, $\mathcal{E}^2$ and $\mathcal{E}^3$ occur. Furthermore, these events have exactly the same probability as the corresponding events in the proof of Lemma 4.3, so the result follows. □

Our upper bound on fixation probability now follows easily from Lemmas 7.1 and 7.2.

THEOREM 7.3. *Let $r > 1$, and let k, ℓ and m be arbitrary positive integers. Let $x_0 \in V(\mathcal{M}_{k,\ell,m})$ be chosen uniformly at random. Let X be the Moran process with $G(X) = \mathcal{M}_{k,\ell,m}$ and $X_0 = \{x_0\}$. Then X fixates with probability at most $1 - 1/(52r^2\sqrt{n})$, where $n = |V(\mathcal{M}_{k,\ell,m})|$.*

PROOF. We prove the result by dividing into three cases.

Case 1: $n \leq 144r^2$. In this case, x_0 dies with rate at least $1/n$ and spawns a mutant with rate r, so x_0 dies before spawning a mutant with probability at least

$$\frac{\frac{1}{n}}{\frac{1}{n} + r} \geq \frac{1}{2rn} \geq \frac{1}{24r^2\sqrt{n}}. \tag{94}$$

Case 2: $n > 144r^2$ and $m \leq k\sqrt{n}$. In this case, by Lemma 7.1, X goes extinct with probability at least

$$\frac{k}{2r(m+k)} \geq \frac{1}{2r(\sqrt{n}+1)} \geq \frac{1}{3r\sqrt{n}}. \tag{95}$$

Case 3: $n > 144r^2$, $m > k\sqrt{n}$ and $\ell \leq \sqrt{n}$. In this case, we have

$$\mathbb{P}(x_0 \in R_1 \cup \cdots \cup R_\ell) = \frac{\ell m}{\ell(m+1+k)+1} \geq \frac{m}{m+3k} \geq \frac{\sqrt{n}}{\sqrt{n}+3} \geq \frac{1}{2}.$$

Moreover, we have $m \geq k\sqrt{n} \geq 12r$. Hence by Lemma 7.2, X goes extinct with probability at least

$$\frac{1}{2} \cdot \frac{1}{26r^2\ell} \geq \frac{1}{52r^2\sqrt{n}}. \tag{96}$$

Since $n \geq \ell m$, we have either $m \leq \sqrt{n} \leq k\sqrt{n}$ or $\ell \leq \sqrt{n}$, and so the above cases are exhaustive. The result therefore follows from (94)–(96). □

8. A CLARIFICATION OF THE ISOTHERMAL THEOREM

Lieberman et al. [2005] define the Moran process in the more general setting of weighted graphs. In this section, we consider this generalisation.

In the ordinary Moran process, the offspring produced by reproduction is equally likely to be placed at each neighbour of the reproducing vertex. Instead, given a graph G, we may assign to each edge (u, v) a weight w_{uv}, taking $w_{uv} = 0$ if there is no such edge. Without loss of generality, we require that, for each vertex u, $\sum_{v \in V(G)} w_{uv} = 1$, so the weights are probabilities. When a vertex u is chosen to reproduce in a weighted graph, its offspring goes to vertex v with probability w_{uv}; the process is, otherwise, identical to the process on unweighted graphs. Note that the unweighted process is recovered by assigning $w_{uv} = 1/d^+(u)$ for every edge (u, v).

A weighted graph G whose weights are probabilities as above is known as an *evolutionary graph* and is said to be *isothermal* [Lieberman et al. 2005; Shakarian et al. 2012] if, for all vertices u, $\sum_{v \in V(G)} w_{vu} = 1$. This corresponds to the condition that the weighted adjacency matrix of G is doubly-stochastic. Broom and Rychtár [2008] show that an undirected graph, when considered as a weighted graph with edge weights $w_{uv} = 1/d^+(u)$, is isothermal if and only if it is regular.

In the supplementary material to [Lieberman et al. 2005], Lieberman et al. state and prove the "isothermal theorem", which states that an evolutionary graph is "ρ-equivalent to the Moran process" if and only if it is isothermal. Being ρ-equivalent to the Moran process means that, for all sets $X \subseteq V(G)$, the probability of reaching fixation from the state in which the set of mutants is X is $(1 - 1/r^{|X|})/(1 - 1/r^n)$. In particular, this condition implies that the fixation probability given a single initial mutant placed uniformly at random is $\rho_{\text{reg}}(r, n) = (1 - 1/r)/(1 - 1/r^n)$.

The isothermal theorem has been incorrectly described in the literature. This may stem from the ambiguity of the informal statement of the theorem in the main text of [Lieberman et al. 2005, p. 313]. Shakarian, Roos and Johnson [2012] state the theorem in the following form, which is very similar to the informal statement by Lieberman et al.

PROPOSITION 8.1 ([SHAKARIAN ET AL. 2012, THEOREM 1]). *An evolutionary graph is isothermal if and only if the fixation probability of a randomly placed mutant is* $\rho_{\text{reg}}(r, n)$.

It is true that all n-vertex connected isothermal graphs do have fixation probability $\rho_{\text{reg}}(r, n)$. However, the converse direction of the proposition does not hold. We prove the following.

PROPOSITION 1.12. *There is an evolutionary graph that is not isothermal, but has fixation probability* $\rho_{\text{reg}}(r, n)$.

A counterexample to Proposition 8.1, proving Proposition 1.12, is the graph shown in Figure 4. This is an evolutionary graph: the total weight of outgoing edges from each vertex is 1. It is not isothermal, since, for example, $w_{10} + w_{20} \neq 1$. However, it can be shown that the fixation probability of a randomly placed mutant with fitness r is, nonetheless, $\rho_{\text{reg}}(r, n)$.

Towards calculating the fixation probability of the graph shown in Figure 4, let p_i be the probability of reaching fixation when the initial mutant is at vertex i, for $0 \leq i \leq 2$. For $0 \leq i < j \leq 2$, let p_{ij} be the probability of reaching fixation from the configuration with mutants at i and j and a non-mutant at the remaining vertex. Observe that, by

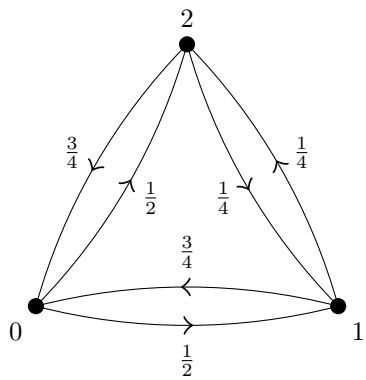

Fig. 4. A 3-vertex graph which is not isothermal but has fixation probability $\rho_{\mathrm{reg}}(r,3)$.

symmetry of the graph, $p_1 = p_2$ and $p_{01} = p_{02}$. The fixation probability of the graph is given by $\frac{1}{3}(p_0 + p_1 + p_2) = \frac{1}{3}(p_0 + 2p_1)$.

We obtain the following equations:

$$\begin{aligned}
p_0 &= \tfrac{1}{r+2}\left(r\left(\tfrac{1}{2}\,p_{01} + \tfrac{1}{2}\,p_{02}\right) + 2\cdot\tfrac{1}{4}\,p_0\right) = \tfrac{1}{r+2}\left(rp_{01} + \tfrac{1}{2}p_0\right),\\
p_1 &= \tfrac{1}{r+2}\left(r\left(\tfrac{3}{4}\,p_{01} + \tfrac{1}{4}\,p_{12}\right) + \left(\tfrac{1}{2} + \tfrac{3}{4}\right)p_1\right),\\
p_{01} &= \tfrac{1}{2r+1}\left(r\left(\tfrac{1}{2} + \tfrac{1}{4} + \left(\tfrac{1}{2} + \tfrac{3}{4}\right)p_{01}\right) + \tfrac{1}{4}\,p_0 + \tfrac{3}{4}\,p_1\right)\\
&= \tfrac{1}{2r+1}\left(\tfrac{3}{4}\,r + \tfrac{5}{4}\,rp_{01} + \tfrac{1}{4}\,p_0 + \tfrac{3}{4}\,p_1\right),\\
p_{12} &= \tfrac{1}{2r+1}\left(r\left(\tfrac{3}{4} + \tfrac{3}{4} + 2\cdot\tfrac{1}{4}\,p_{12}\right) + \tfrac{1}{2}p_1 + \tfrac{1}{2}p_2\right)\\
&= \tfrac{1}{2r+1}\left(r\left(\tfrac{3}{2} + \tfrac{1}{2}\,p_{12}\right) + p_1\right).
\end{aligned}$$

Rearranging these gives

$$\begin{aligned}
(2r+3)p_0 &= 2rp_{01} & (3r+4)p_{01} &= 3r + p_0 + 3p_1\\
(4r+3)p_1 &= 3rp_{01} + rp_{12} & (3r+2)p_{12} &= 3r + 2p_1\,.
\end{aligned}$$

Routine solution of this linear system gives

$$p_0 = \frac{r^2(2r+1)}{2(r+1)(r^2+r+1)} \qquad p_1 = \frac{r^2(4r+5)}{4(r+1)(r^2+r+1)}\,,$$

which yields the fixation probability

$$\frac{1}{3}(p_0 + 2p_1) = \frac{r^2(2r+1+4r+5)}{6(r+1)(r^2+r+1)} = \frac{r^2}{r^2+r+1} = \frac{r^2(r-1)}{r^3-1} = \rho_{\mathrm{reg}}(r,3)\,.$$

9. HEURISTIC ANALYSIS OF SUPERSTARS

As noted in Section 1.4, Jamieson-Lane and Hauert [2015] have already provided a heuristic analysis of the fixation probability of superstars. The heuristic analysis contains good intuition. The purpose of this final section is to explain some of the difficulties that arise when converting such a heuristic argument to a rigorous proof. This section is primarily for readers who are already familiar with the argument of [Jamieson-Lane and Hauert 2015]. This should not be regarded as a criticism of [Jamieson-Lane and Hauert 2015] — that paper provides an excellent heuristic analysis, so it does

what it intends. Rather, the purpose of this section is to illustrate the complicating factors that arise in rigorous proofs (these help to explain why our paper is so long!)

The evolution of the discrete-time Moran process on a superstar is extremely complicated. To avoid detailed analysis, Jamieson-Lane and Hauert [2015, Appendix E.2] use a simple random walk to stochastically dominate (from below) the number of mutants in reservoirs. They claim that there is always some forward bias in the actual process, in the sense that the number of mutants in reservoirs is always more likely to increase than to decrease. They say that the forward bias gets harder to quantify as the number of mutants increases, but that it is always positive. Thus, the dominating walk Q that they consider is forward-biased when there are relatively few mutants, and unbiased when there are more. More specifically, the dominating chain $Q(h)$ is a simple random walk on $\{0, \ldots, m\ell\}$. If h is below some threshold δ then Q increases by 1 during each (discrete) step with probability $\gamma/(1+\gamma)$ (for some $\gamma > 1$) and decreases by 1 with probability $1/(1+\gamma)$. If h is above δ then it increases or decreases (by 1) with probability $1/2$.

There are several problems with this domination. First, the domination is invalid because there are actually configurations in which the number of mutants is more likely to decrease than to increase. One such configuration is the configuration in which each reservoir contains $\frac{m}{2}$ mutants and $\frac{m}{2}$ non-mutants and the centre vertex v^* and all path vertices are non-mutants. It is easy to see that the number of mutants is more likely to decrease than to increase from this configuration, and even that the number of mutants is likely to decrease at least $k/(16r)$ times before it ever increases. Here is the idea. Before the mutant population of the reservoirs can possibly increase, v^* must become a mutant. This takes at least k reproductions: a mutant in a reservoir must reproduce and a chain of k reproductions must move the mutant down the corresponding path to the centre. This is very likely to take at least $nk/(2r)$ steps of the process. But during these $nk/(2r)$ steps, v^* is very likely to be chosen for reproduction at least $k/(4r)$ times. Since v^* is a non-mutant throughout this period, it must send a non-mutant into some reservoir each time it reproduces. Since half the reservoir vertices are mutants, it is very likely that these $k/(4r)$ reproductions of the centre will cause the mutant population of the reservoirs to decrease, not just once, but at least $k/(16r)$ times before it can even go up at all. Therefore, the assumption that the mutant population of the reservoirs is as likely to increase as to decrease does not hold for all configurations of mutants. A rigorous proof needs to cover all such possibilities.

Even in the early evolution of the process, when there are few mutants in reservoirs, there are still problems with making the domination rigorous. Jamieson-Lane and Hauert [2015, Section 3.5] say (translating their variable names to ours and adding a little notation for future reference)

> At any given time step, the probability of losing the initial mutant in the reservoir is $p_1 := 1/(F_t \ell m)$. Based on the dynamics in the path, we derive the per time step probability that a second mutant is generated in any reservoir as the product of the probability that a "train" is generated and the probability that the train succeeds in producing a second mutant, which yields approximately $p_2 := r^4 T/(F_t m \ell)$.

Here, F_t is taken to be the overall fitness (sum of individual fitnesses) in the configuration X_t and T is the expected length of a "train" which is a chain of mutants at the end of path. The dominating Markov chain Q is applied with $\gamma \sim p_2/p_1 \sim r^4 T$. Since Q is a Markov chain, the domination is only valid if it applies step-by-step to every configuration. It applies, if, from every fixed configuration, the probability that the number of mutants next goes down is proportional to p_1 and the probability that it next goes up is proportional to p_2. But this is not proved. First, note that the event

that the number of reservoir mutants increases does not occur with probability p_2 *at any particular step* (conditioned on the configuration prior to the step). Instead, the expression given in p_2 is a heuristic aggregated probability which may, roughly, apply at some step or block of steps in the future. In order to rigorously dominate the number of reservoir mutants using the Markov chain Q it is necessary to split the process into discrete pieces (whose length may be a random variable) so that the number of reservoir mutants decreases by at most one in each piece. It is important that, conditioned on any configuration at the start of any piece, the probability that the number of reservoir mutants goes up must be at least p_2/p_1 times the probability that it goes down. The paper does not provide such a domination. Nevertheless, we do believe that there is an infinite family of superstars that is strongly amplifying. Our Theorem 1.6 in Section 6 demonstrates strong amplification for megastars. A similar approach would presumably work for superstars, though of course it would not guarantee as strong amplification as Theorem 1.6, since this is impossible by Theorem 1.8.

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