MINIMAL DIMENSIONAL REPRESENTATIONS OF REDUCED ENVELOPING ALGEBRAS FOR $\mathfrak{gl}_n$

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ABSTRACT. Let $\mathfrak{g} = \mathfrak{gl}_N(\Bbbk)$, where $\Bbbk$ is an algebraically closed field of characteristic $p > 0$, and $N \in \mathbb{Z}_{\geq 1}$. Let $\chi \in \mathfrak{g}^*$ and denote by $U_\chi(\mathfrak{g})$ the corresponding reduced enveloping algebra. The Kac–Weisfeiler conjecture, which was proved by Premet, asserts that any finite dimensional $U_\chi(\mathfrak{g})$-module has dimension divisible by p^{d_χ}, where d_χ is half the dimension of the coadjoint orbit of χ. Our main theorem gives a classification of $U_\chi(\mathfrak{g})$-modules of dimension p^{d_χ}. As a consequence, we deduce that they are all parabolically induced from a 1-dimensional module for $U_0(\mathfrak{h})$ for a certain Levi subalgebra $\mathfrak{h}$ of $\mathfrak{g}$; we view this as a modular analogue of Mœglin's theorem on completely primitive ideals in $U(\mathfrak{gl}_N(\mathbb{C}))$. To obtain these results, we reduce to the case χ is nilpotent, and then classify the 1-dimensional modules for the corresponding restricted W-algebra.

1. Introduction

Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$, and $N \in \mathbb{Z}_{\geq 1}$. Let $G := \mathrm{GL}_N(\Bbbk)$ and $\mathfrak{g} := \mathfrak{gl}_N(\Bbbk) = \operatorname{Lie} G$. For $x \in \mathfrak{g}$, we write $x^{[p]}$ for the pth power of x as a matrix, and recall that $x \mapsto x^{[p]}$ is the p-power map for the restricted Lie algebra structure on $\mathfrak{g}$. Also we write x^p for the pth power of x in the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Then the elements $x^p - x^{[p]}$ are central elements in $U(\mathfrak{g})$, and the p-centre $Z_p(\mathfrak{g})$ of $U(\mathfrak{g})$ is defined to be the subalgebra generated by $\{x^p - x^{[p]} \mid x \in \mathfrak{g}\}$. It is well-known that $Z_p(\mathfrak{g})$ is G-isomorphic to the Frobenius twist $S(\mathfrak{g})^{(1)}$ of the symmetric algebra on $\mathfrak{g}$ and that $U(\mathfrak{g})$ is free of rank $p^{\dim \mathfrak{g}}$ over $Z_p(\mathfrak{g})$.

For an irreducible $U(\mathfrak{g})$-module M, the central elements $x^p - x^{[p]}$ act on M as $\chi(x)^p$ for some $\chi \in \mathfrak{g}^*$, thanks to Quillen's lemma. We define the ideal J_χ of $U(\mathfrak{g})$ to be generated by $\{x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}\}$, and the reduced enveloping algebra associated to χ to be $U_\chi(\mathfrak{g}) := U(\mathfrak{g})/J_\chi$. Then we have seen that any irreducible $U(\mathfrak{g})$-module factors through $U_\chi(\mathfrak{g})$ for some $\chi \in \mathfrak{g}^*$, and that $\dim U_\chi(\mathfrak{g}) = p^{\dim \mathfrak{g}}$.

Reduced enveloping algebras $U_\chi(\mathfrak{g})$, are defined more generally for the Lie algebra $\mathfrak{g}$ of a reductive algebraic group G over $\Bbbk$, and their representation theory attracted a great deal of research interest from leading mathematicians including Friedlander–Parshall, Humphreys, Jantzen, Kac and Premet in the late 20th century, we refer to the survey articles [Ja] and [Hu] for an overview. There has been continued interest and progress in the representation theory of reduced enveloping algebra, a notable advance being the proof by Bezrukavnikov–Mirkovic in [BM] of a conjecture of Lusztig regarding irreducible modules, for p sufficiently large. An important conjecture of Kac–Weisfeiler stated in [VK] asserts that, for G simple, the dimension of a $U_\chi(\mathfrak{g})$-module has dimension divisible by p^{d_χ}, where d_χ is half the dimension of the coadjoint orbit of χ, and was proved by Premet in [Pr1, Theorem I] (under some mild restrictions on G and p). The case $\mathfrak{g} = \mathfrak{gl}_N(\Bbbk)$ can be deduced directly if $p \nmid N$; also the

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case $p \mid N$ can now be obtained from an alternative proof by Premet in [Pr2, §2.6]. We also mention that Friedlander–Parshall previously proved the conjecture for $\mathfrak{g} = \mathfrak{sl}_N(\Bbbk)$ and $p \nmid N$ in [FP3, Theorem 5.1]. Consequently p^{d_χ} is smallest dimension of a $U_\chi(\mathfrak{g})$-module, so we refer to p^{d_χ}-dimensional modules for $U_\chi(\mathfrak{g})$ as *minimal dimensional modules.* We note that there is a relatively straightforward way, via parabolic induction, to construct minimal dimensional $U_\chi(\mathfrak{g})$-modules for $\mathfrak{g} = \mathfrak{gl}_N(\Bbbk)$, as was first observed by Friedlander–Parshall in [FP3, Corollary 5.2].

In this paper we classify the minimal dimensional $U_\chi(\mathfrak{g})$-modules (for $\mathfrak{g} = \mathfrak{gl}_N(\Bbbk)$), as stated in Theorem 1.1. As a consequence we show that they all can be obtained by parabolically inducing a 1-dimensional $U_0(\mathfrak{h})$-module for a certain Levi subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, as stated in Corollary 1.2. Both of these results are formulated for $\chi \in \mathfrak{g}^*$ nilpotent, but as explained in §2.1 there is a reduction to this case. We note that Corollary 1.2 can be viewed as a modular analogue of Mœglin's theorem, from [Mœ] on completely prime ideals of $U(\mathfrak{gl}_N(\mathbb{C}))$. Further, we remark some of our methods adapt those of Brundan in [Br], in which he gives an alternative proof of Mœglin's theorem.

We require some notation to state our main results. This is all set out in detail in Section 2, and here we only point out the necessary parts for the statements of Theorem 1.1 and Corollary 1.2.

The trace form on $\mathfrak{g} := \mathfrak{gl}_N(\Bbbk)$ is denoted $(\cdot\,,\cdot)$ and allows us to identify $\mathfrak{g} \cong \mathfrak{g}^*$. Consequently, we can talk about Jordan decomposition of elements of $\mathfrak{g}^*$ and nilpotent elements of $\mathfrak{g}^*$. We let $\mathfrak{b}$ be the Borel subalgebra of upper triangular matrices and $\mathfrak{t}$ the maximal toral subalgebra of diagonal matrices.

Let $\boldsymbol{p} = (p_1 \le p_2 \le \cdots \le p_n)$ be a partition of N, and let π be a pyramid associated to $\boldsymbol{p}$. This means that π is diagram with N boxes organised in rows, with row lengths given by $\boldsymbol{p}$ as defined in §2.2; further the boxes in π are labelled from 1 to N along rows starting from the top row. There is some choice of the pyramid π, and much of the notation below is dependant on this choice; as the results are all valid for any choice of π, we choose to work in this generality, and just note that the left justified pyramid is one choice that can be made.

From π we define the nilpotent element $e \in \mathfrak{g}$ as in (2.3), and let $\chi := (e, \cdot) \in \mathfrak{g}^*$. Then $\chi \in \mathfrak{g}^*$ is nilpotent and as we range over all partitions $\boldsymbol{p}$ of N, we get representatives of all coadjoint G-orbits of nilpotent elements of $\mathfrak{g}^*$. As explained in §2.2, we have that χ is in standard Levi form with respect to $\mathfrak{b}$. A good grading of $\mathfrak{g}$ for e is defined in (2.4), and from this we can define the parabolic subalgebra $\mathfrak{p}$ with Levi factor $\mathfrak{h}$ as in (2.5). We note that $\mathfrak{t}$ is contained in $\mathfrak{p}$ and $\mathfrak{h}$, but that $\mathfrak{b} \not\subseteq \mathfrak{p}$.

Let $\mathbb{F}_p \subseteq \Bbbk$ denote the field of p elements. We define $\mathrm{Tab}_\Bbbk(\pi)$ to be fillings of the boxes of π with elements from $\Bbbk$; and we define $\mathrm{Tab}_{\mathbb{F}_p}(\pi) \subseteq \mathrm{Tab}_\Bbbk(\pi)$ to be the filling with elements from $\mathbb{F}_p$. We refer to elements of $\mathrm{Tab}_\Bbbk(\pi)$ as π-tableau. Given $A \in \mathrm{Tab}_\Bbbk(\pi)$, we denote the entry in the box labelled i in π by a_i. Let $\varepsilon_1, \dots, \varepsilon_N$ be the standard basis of $\mathfrak{t}^*$ and define $\lambda_A = \sum_{i=1}^N a_i \varepsilon_i \in \mathfrak{t}^*$ for $A \in \mathrm{Tab}_\Bbbk(\pi)$. We say that $A \in \mathrm{Tab}_\Bbbk(\pi)$ is *column connected* if $a_i = a_j + 1$ whenever box i is directly above box j in π.

The weight $\rho \in \mathfrak{t}^*$ is defined in (2.6): it is a convenient renormalization of the half sum of positive roots corresponding to $\mathfrak{b}$. Also we define $\overline{\rho} \in \mathfrak{t}^*$ in (2.7), which is the half sum of positive roots for a Borel subalgebra of $\mathfrak{p}$, with a convenient renormalization.

We recap some established representation theory of $U_\chi(\mathfrak{g})$ and interpret it in our notation, see for example [Ja, Section 10], more detail is given in §2.5. Since $e \in \mathfrak{b}$, we have that

$\chi(\mathfrak{b}) = 0$. Given $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$ we define $\Bbbk_A$ to be the 1-dimensional $U_0(\mathfrak{b})$-module on which $\mathfrak{t}$ acts via $\lambda_A - \rho$, and the baby Verma module to be $Z_\chi(A) = U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} \Bbbk_A$. It is known that $Z_\chi(A)$ has a unique maximal submodule, and we denote the simple head of $Z_\chi(A)$ by $L_\chi(A)$. Further, any irreducible $U_\chi(\mathfrak{g})$-module is isomorphic to $L_\chi(A)$ for some $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$, and for $A, A' \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$, we have $L_\chi(A) \cong L_\chi(A')$ if and only if A is row equivalent to A'. We recall that we say that A is row equivalent to A' if we can obtain A' from A by reordering the entries in rows.

We are now in a position to state our main theorem giving a classification of minimal dimensional $U_\chi(\mathfrak{g})$-modules.

Theorem 1.1. *Let $\mathfrak{g} = \mathfrak{gl}_N(\Bbbk)$, let π be a pyramid corresponding to a partition $\boldsymbol{p}$ of N, and let χ be the nilpotent element of $\mathfrak{g}^*$ determined by π. For $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$, we have that $L_\chi(A)$ is a minimal dimensional $U_\chi(\mathfrak{g})$-module if and only if A is row equivalent to a column connected π-tableau.*

To state Corollary 1.2, we have to define certain 1-dimensional $U_0(\mathfrak{p})$-modules. As explained in §2.5, given $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$, we have that $\lambda_A - \overline{\rho}$ is the weight of a 1-dimensional $U_0(\mathfrak{h})$-module if and only A is column connected. For column connected $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$, we define $\overline{\Bbbk}_A$ to be the one dimensional $U_0(\mathfrak{p})$-module obtained by inflating the 1-dimensional $U_0(\mathfrak{h})$-module with weight $\lambda_A - \overline{\rho}$. In Theorem 2.2, we show that $L_\chi(A) \cong U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p})} \overline{\Bbbk}_A$ for column connected $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$. Combining this with Theorem 1.1, we immediately deduce.

Corollary 1.2. *Let $\mathfrak{g} = \mathfrak{gl}_N(\Bbbk)$, let π be a pyramid corresponding to a partition $\boldsymbol{p}$ of N, let χ be the nilpotent element of $\mathfrak{g}^*$ determined by π, and let $\mathfrak{p}$ be the parabolic subalgebra of $\mathfrak{g}$ determined by π. Let L be a minimal dimensional $U_\chi(\mathfrak{g})$-module. Then $L \cong U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p})} \overline{\Bbbk}_A$ for some column connected $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$.*

We give an outline of the main ideas in the proof of Theorem 1.1. The key step is to rephrase the problem in terms of W-algebras through Premet's equivalence. Let $U(\mathfrak{g}, e)$ be the finite W-algebra as in [GT, Definition 4.3]; in fact we use an equivalent definition in this paper as a subalgebra of $U(\mathfrak{p})$ as explained in §2.6. The restricted W-algebra $U_0(\mathfrak{g}, e)$ is as in [GT, Definition 8.5], though as explained in §2.6 our notation in this paper differs from that in [GT] and we view $U_0(\mathfrak{g}, e)$ as a subalgebra of $U_0(\mathfrak{p})$. The definitions of these W-algebras in [GT] are inspired by work of Premet, where $U(\mathfrak{g}, e)$ has appeared for p sufficiently large and is obtained from a characteristic 0 finite W-algebra via reduction modulo p, see for example [Pr3, §2.5].

We recall that Premet's equivalence, which is stated in Theorem 2.4, gives an equivalence of categories between $U_\chi(\mathfrak{g})$-mod and $U_0(\mathfrak{g}, e)$-mod. Moreover, through this equivalence a $U_0(\mathfrak{g}, e)$-module of dimension m corresponds to a $U_\chi(\mathfrak{g})$-module of dimension mp^{d_χ}. Therefore, in order to prove Theorem 1.1, we want to classify the 1-dimensional $U_0(\mathfrak{g}, e)$-modules.

In fact we classify all 1-dimensional $U(\mathfrak{g}, e)$-modules and determine which ones factor through the quotient map $U(\mathfrak{g}, e) \twoheadrightarrow U_0(\mathfrak{g}, e)$. We show that $U(\mathfrak{g}, e)$ is a modular truncated shifted Yangian, see Theorem 4.3. This is proved by following the methods of Brundan–Kleshchev in [BK1], but now using the PBW theorem for $U(\mathfrak{g}, e)$ given in [GT, Theorem 7.3] and reduction modulo p arguments. In particular, this allows us to determine the abelianisation $U(\mathfrak{g}, e)^{\mathrm{ab}}$ of $U(\mathfrak{g}, e)$, by observing that a calculation by Premet from [Pr3, Theorem 3.3] applies in characteristic p. As mentioned above we view $U(\mathfrak{g}, e)$ as a subalgebra of $U(\mathfrak{p})$. Thus we obtain 1-dimensional $U(\mathfrak{g},e)$-modules by restricting 1-dimensional $U(\mathfrak{p})$-modules. Rather than using the labelling of 1-dimensional $U(\mathfrak{p})$-modules as $\overline{\Bbbk}_A$ for column connected A in $\mathrm{Tab}_{\Bbbk}(\pi)$ above, we in fact consider $U(\mathfrak{p})$-modules $\widetilde{\Bbbk}_A$, where a different shift is used. Using the description of $U(\mathfrak{g},e)^{\mathrm{ab}}$, we are able to deduce that the restriction of the modules $\widetilde{\Bbbk}_A$ for column connected $A \in \mathrm{Tab}_{\Bbbk}(\pi)$ give all of the 1-dimensional $U(\mathfrak{g},e)$-modules. Moreover, for column connected $A, A' \in \mathrm{Tab}_{\Bbbk}(\pi)$ we deduce that the restrictions of $\widetilde{\Bbbk}_A$ and $\widetilde{\Bbbk}_{A'}$ are isomorphic if and only if A is row equivalent to A'. We denote $\widetilde{\Bbbk}_A$ restricted to $U(\mathfrak{g},e)$ by $\widetilde{\Bbbk}_{\overline{A}}$. Our methods for this classification of 1-dimensional $U(\mathfrak{g},e)$-modules are similar to those used by Brundan in [Br, Section 2].

Our next step is to show, for column connected $A \in \mathrm{Tab}_{\Bbbk}(\pi)$, that $\widetilde{\Bbbk}_{\overline{A}}$ factors to a module for $U_0(\mathfrak{g},e)$ if and only if $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$. This deduction is not immediate and is given in Theorem 6.1. From here we are in a position to apply Premet's equivalence to determine the minimal dimensional $U_\chi(\mathfrak{g})$-modules. A key step for this is given by Theorem 2.2, which says that $L_\chi(A) \cong U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p})} \overline{\Bbbk}_A$ for column connected $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$. This requires us to identify a vector in $U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p})} \overline{\Bbbk}_A$, which spans a 1-dimensional $U(\mathfrak{b})$-module with weight $\lambda_A - \rho$. From this we can deduce that $L_\chi(A)$ is minimal dimensional if A is column connected. By applying our classification of 1-dimensional $U_0(\mathfrak{g},e)$-modules and Premet's equivalence, we are thus able to conclude that the set $L_\chi(A)$ for $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$ column connected (up to row equivalence) gives all of the minimal dimensional $U_\chi(\mathfrak{g})$-modules, which proves Theorem 1.1. In fact it is possible to show that through Premet's equivalence $\widetilde{\Bbbk}_{\overline{A}}$ corresponds to $L_\chi(A)$; this is discussed in Remark 7.1.

We end the introduction with some remarks about minimal dimensional modules for reduced enveloping algebras $U_\chi(\mathfrak{g})$ for $\mathfrak{g}$ the Lie algebra of a reductive algebraic group over $\Bbbk$. The assertion that there is a $U_\chi(\mathfrak{g})$-module of dimension p^{d_χ} is now known as Humphreys' conjecture, see [Hu, §8], though we note that the question was asked earlier by Kac in his review of [Pr1] on the *Mathematical Reviews*. There has been lots of progress on this conjecture recently and thanks to the results of Premet in [Pr4] it is now known to be true for p sufficiently large; further Premet states that in forthcoming work he will give an explicit lower bound on p. The questions of whether the minimal dimensional modules can be classified, and whether they are parabolically induced are also of great interest. We plan to consider these in future work, and note that the characteristic 0 version of the latter is addressed in work of Premet and the second author in [PT].

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2. Preliminaries

2.1. **The general linear Lie algebra and reduced enveloping algebras.** Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$ and let $N \in \mathbb{Z}_{\geq 1}$. Throughout this paper $G := \mathrm{GL}_N(\Bbbk)$ and $\mathfrak{g} := \mathfrak{gl}_N(\Bbbk)$ is the Lie algebra of G, which is spanned by the matrix units

$\{e_{i,j} \mid 1 \le i, j \le N\}$. Let $(\cdot\,,\cdot) : \mathfrak{g} \times \mathfrak{g} \to \Bbbk$ denote the trace form associated to the natural representation of G, which we use to identify $\mathfrak{g} \cong \mathfrak{g}^*$ as G-modules. The universal enveloping algebra of $\mathfrak{g}$ is denoted $U(\mathfrak{g})$.

We occasionally need to call on some results from characteristic zero and so we fix some more notation. We let $\mathfrak{g}_{\mathbb{Z}}$ denote the general linear Lie $\mathbb{Z}$-algebra $\mathfrak{gl}_N(\mathbb{Z})$ and we write $\mathfrak{g}_{\mathbb{C}}$ for $\mathfrak{gl}_N(\mathbb{C})$. Throughout we use the identifications $\mathfrak{g} \cong \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ and $\mathfrak{g}_{\mathbb{C}} \cong \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, and by a slight abuse of notation we view the matrix units $e_{i,j}$ as elements of $\mathfrak{g}_{\mathbb{Z}}$ or $\mathfrak{g}_{\mathbb{C}}$ when it is convenient to do so. We often consider subalgebras of $\mathfrak{g}$, which are spanned by matrix units, so have analogues inside $\mathfrak{g}_{\mathbb{Z}}$ and $\mathfrak{g}_{\mathbb{C}}$ and we denote them by decorating with subscripts $\mathbb{Z}$ and $\mathbb{C}$. We mention that since $\mathfrak{g}_{\mathbb{Z}}$ is a free $\mathbb{Z}$-module the PBW theorem holds for $U(\mathfrak{g}_{\mathbb{Z}})$, so that $U(\mathfrak{g}_{\mathbb{Z}})$ is a free $\mathbb{Z}$-module with a basis consisting of ordered monomials in the matrix units with respect to any choice of total order.

Let $g \in G$, $x \in \mathfrak{g}$ and $\chi \in \mathfrak{g}^*$. We write $g \cdot x$ for the image of x under the adjoint action of g, so as matrices $g \cdot x = gxg^{-1}$; this action extends to an action on $U(\mathfrak{g})$ by algebra automorphisms. The centralizer of x in G is denoted $G^x := \{g \in G \mid g \cdot x = x\}$ and the centralizer of x in $\mathfrak{g}$ is denoted $\mathfrak{g}^x := \{y \in \mathfrak{g} \mid [y, x] = 0\}$; we note that we have $\mathfrak{g}^x = \mathrm{Lie}(G^x)$. We write $G \cdot \chi$ for the coadjoint orbit of χ. It is well-known that $\dim(G \cdot \chi)$ is even and we define $d_\chi := \frac{1}{2}\dim(G \cdot \chi)$.

Let $T \subseteq B \subseteq G$ be the maximal torus and Borel subgroup consisting of diagonal matrices and upper triangular matrices respectively, and let $\mathfrak{t} := \mathrm{Lie}(T)$, $\mathfrak{b} := \mathrm{Lie}(B)$. We use the notation $\mathrm{diag}(d_1, \dots, d_n)$ to denote the element of T with d_i in the ith entry of the diagonal. We write $X^*(T)$ for the group of characters, and let $\{\varepsilon_1, \dots, \varepsilon_N\}$ be the standard basis of $X^*(T)$ defined by $\varepsilon_i(\mathrm{diag}(d_1, \dots, d_n)) = d_i$. Let $\Phi \subseteq X^*(T)$ be the root system of G with respect to T, so $\Phi = \{\varepsilon_i - \varepsilon_j \mid 1 \le i, j \le n, i \ne j\}$. We write $e_{i,j}$ for the matrix unit that spans the root space corresponding to $\varepsilon_i - \varepsilon_j$. The root subgroup corresponding to $\varepsilon_i - \varepsilon_j$ is the image of $u_{i,j} : \Bbbk \to G$ defined by $u_{i,j}(s) := 1 + se_{i,j}$, and the adjoint action of $u_{i,j}(s)$ on $e_{k,l}$ is given by the formula

$$u_{i,j}(s) \cdot e_{k,l} = e_{k,l} + s\delta_{j,k}e_{i,l} - s\delta_{l,i}e_{k,j} - s^2\delta_{j,k}\delta_{l,i}e_{i,j}. \tag{2.1}$$

Where it is convenient we allow ourselves to view a character $\alpha \in X^*(T)$ as an element of $\mathfrak{t}^*$ by writing α for $d\alpha : \mathfrak{t} \to \Bbbk$; this is a slight abuse of notation, because $d\alpha = 0$ for any $\alpha \in pX^*(T)$.

There is a natural restricted structure on $\mathfrak{g}$, where the p-power map $x \mapsto x^{[p]}$ is given by taking the pth power of x as a matrix. In particular, we note that $e_{i,j}^{[p]} = \delta_{i,j}e_{i,j}$ for $1 \le i, j \le N$. The p-centre of $U(\mathfrak{g})$ is the subalgebra of the centre of $U(\mathfrak{g})$ generated by $\{e_{i,j}^p - e_{i,j}^{[p]} \mid 1 \le i, j \le N\}$. It follows from the PBW theorem that $U(\mathfrak{g})$ is a free $Z_p(\mathfrak{g})$-module of rank $p^{\dim \mathfrak{g}}$. Further, there is a natural identification $Z_p(\mathfrak{g}) \cong \Bbbk[(\mathfrak{g}^*)^{(1)}]$, where $(\mathfrak{g}^*)^{(1)}$ denotes the Frobenius twist of $\mathfrak{g}^*$. Given $\chi \in \mathfrak{g}^*$ we define J_χ to be the ideal of $U(\mathfrak{g})$ generated by $\{x^p - x^{[p]} - \chi(x)^p \mid x \in \mathfrak{g}\}$, and the *reduced enveloping algebra corresponding to* χ to be $U_\chi(\mathfrak{g}) := U(\mathfrak{g})/J_\chi$.

As stated in the introduction, the Kac–Weisfeiler conjecture, which is a theorem of Premet, states that p^{d_χ} is a factor of the dimension of any $U_\chi(\mathfrak{g})$-module. We refer to $U_\chi(\mathfrak{g})$-modules of dimension p^{d_χ} as *minimal dimensional modules*, and note that such modules are clearly irreducible.

Let $\chi \in \mathfrak{g}^*$. There is unique $x \in \mathfrak{g}$ such that $\chi = (x, \cdot)$. We have a Jordan decomposition $x = x_{\mathrm{s}} + x_{\mathrm{n}}$ of x, and thus a corresponding decomposition $\chi = \chi_{\mathrm{s}} + \chi_{\mathrm{n}}$. We say that χ is nilpotent if $\chi = \chi_{\mathrm{n}}$. Next we recall the "reduction" to the case χ nilpotent in the representation theory of $U_\chi(\mathfrak{g})$ from [FP1, Section 3]; as is noted in [FP1, Section 8], this reduction can also be deduced from [VK, Theorem 2]. Let $\mathfrak{l} = \mathfrak{g}^{x_{\mathrm{s}}}$, let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{l}$ and let $\mathfrak{u}$ denote the nilradical of $\mathfrak{q}$. We can parabolically induce a $U_\chi(\mathfrak{l})$-module M, to obtain the $U_\chi(\mathfrak{g})$-module $U_\chi(\mathfrak{g}) \otimes_{U_\chi(\mathfrak{q})} M$, where M is the $U_\chi(\mathfrak{q})$-module on which $\mathfrak{u}$ acts trivially. This gives a functor $U_\chi(\mathfrak{l})$-mod $\to U_\chi(\mathfrak{g})$-mod and it is proved in [FP1, Theorem 3.2] that this is an equivalence of categories; in turn there is an equivalence $U_\chi(\mathfrak{l})$-mod $\cong U_{\chi_{\mathrm{n}}}(\mathfrak{l})$-mod as follows from [FP1, Corollary 3.3]. Further, the theory of Jordan normal forms implies that $\dim(G \cdot \chi) = \dim(L \cdot \chi_{\mathrm{n}}) + 2 \dim \mathfrak{u}$. Therefore, through the equivalence of categories $U_{\chi_{\mathrm{n}}}(\mathfrak{l})$-mod $\cong U_\chi(\mathfrak{g})$-mod, minimal dimensional modules for $U_{\chi_{\mathrm{n}}}(\mathfrak{l})$ correspond to minimal dimensional modules for $U_\chi(\mathfrak{g})$-mod. This justifies our restriction to nilpotent χ in the statements of Theorem 1.1 and Corollary 1.2.

2.2. **Pyramids.** We require the combinatorics of pyramids to set up some notation. For more details on this we refer to [BK1, Section 7].

We fix a partition $\boldsymbol{p} = (p_1, \dots, p_n)$ on N with $p_1 \le \cdots \le p_n$. A pyramid π associated to $\boldsymbol{p}$ is a diagram with p_n boxes in the bottom row, p_{n-1} boxes in the row above it, and so forth, stacked in such a way that every box which is not in the bottom row lies directly above a box in the row beneath it, and boxes occur consecutively in each row. The boxes in the pyramid are numbered along rows from left to right and from top to bottom. For example, the pyramids associated to the partition $\boldsymbol{p} = (2, 5)$ are

$$\begin{array}{|c|c|ccc}\hline 1&2\\ \hline 3&4&\multicolumn{1}{c|}{5}&\multicolumn{1}{c|}{6}&\multicolumn{1}{c|}{7}\\ \hline\end{array}\,,\quad \begin{array}{c|c|c|cc}\cline{2-3} &1&2\\ \hline \multicolumn{1}{|c|}{3}&4&5&\multicolumn{1}{c|}{6}&\multicolumn{1}{c|}{7}\\ \hline\end{array}\,,\quad \begin{array}{cc|c|c|c}\cline{3-4} &&1&2\\ \hline \multicolumn{1}{|c|}{3}&\multicolumn{1}{c|}{4}&5&6&\multicolumn{1}{c|}{7}\\ \hline\end{array} \quad\text{and}\quad \begin{array}{ccc|c|c|}\cline{4-5} &&&1&2\\ \hline \multicolumn{1}{|c|}{3}&\multicolumn{1}{c|}{4}&\multicolumn{1}{c|}{5}&6&7\\ \hline\end{array}\,. \tag{2.2}$$

Let $l = p_n$. The columns of π are labelled $1, 2, ..., l$ from left to right and the rows are labelled $1, 2, ..., n$ from top to bottom. We denote the heights of the columns in π by $q_1, q_2, ..., q_l$. The box in π containing i is referred to as the ith box, and we write $\mathrm{row}(i)$ and $\mathrm{col}(i)$ for the row and column of the ith box respectively.

We fix a pyramid π corresponding to $\boldsymbol{p}$ for the rest of this paper. From π, we define the *shift matrix* $\sigma = (s_{i,j})$ as follows. For $1 \le i < j \le n$ we let $s_{j,i}$ be the left indentation of the ith row of π relative to the jth row, and we let $s_{i,j}$ be the right indentation of the ith row of π relative to the jth row; also we set $s_{i,i} = 0$. For example the shift matrices associated to the pyramids in (2.2) are

$$\begin{pmatrix} 0 & 3 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}.$$

2.3. **The nilpotent element and subalgebras.** We define the nilpotent element

$$e := \sum_{\substack{\mathrm{row}(i)=\mathrm{row}(j) \\ \mathrm{col}(i)=\mathrm{col}(j)-1}} e_{i,j} \in \mathfrak{g}. \tag{2.3}$$

For example for each of the pyramids in (2.2), we have $e = e_{1,2} + e_{3,4} + e_{4,5} + e_{5,6} + e_{6,7}$. Observe that e has Jordan blocks of size $p_1, p_2, ..., p_n$. We define $\chi := (e, \cdot) \in \mathfrak{g}^*$. We also note that χ is in standard Levi form (in the sense of [FP2, Definition 3.1]) with respect to the simple roots corresponding to the Borel subalgebra $\mathfrak{b}$.

The first part of the following lemma gives a basis of $\mathfrak{g}^e$, and can be verified by observing that the proof of [BK1, Lemma 7.3] is also valid in positive characteristic. The second part of the lemma is verified by direct calculation.

Lemma 2.1. *Let*

$$c_{i,j}^{(r)} := \sum_{\substack{1\le h,k,\le N \\ \mathrm{row}(h)=i,\mathrm{row}(k)=j \\ \mathrm{col}(k)-\mathrm{col}(h)+1=r}} e_{h,k}$$

for $0 \le i,j \le n$ *and* $r > s_{i,j}$.

(a) *The centralizer* $\mathfrak{g}^e$ *of* e *in* $\mathfrak{g}$ *has basis*

$$\{c_{i,j}^{(r)} \mid 0 \le i,j \le n, s_{i,j} < r \le s_{i,j} + p_{\min(i,j)}\},$$

(b) *We have*

$$[c_{i,j}^{(r)}, c_{k,l}^{(s)}] = \delta_{j,k} c_{i,l}^{(r+s-1)} - \delta_{i,l} c_{k,j}^{(r+s-1)}.$$

Consider the cocharacter $\mu : \Bbbk^\times \to T \subseteq G$ defined by $\mu(t) = \mathrm{diag}(t^{\mathrm{col}(1)}, \dots, t^{\mathrm{col}(n)})$. Using μ we define the $\mathbb{Z}$-grading

$$\mathfrak{g} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}(k) \quad \text{where} \quad \mathfrak{g}(k) := \{x \in \mathfrak{g} \mid \mu(t)x = t^k x \text{ for all } t \in \Bbbk^\times\}. \tag{2.4}$$

Since the adjoint action of $\mu(t)$ on a matrix unit is given by $\mu(t) \cdot e_{i,j} = t^{\mathrm{col}(j)-\mathrm{col}(i)} e_{i,j}$, we have $\mathfrak{g}(k) = \mathrm{span}\{e_{i,j} \mid \mathrm{col}(j) - \mathrm{col}(i) = k\}$. From the classification of good gradings in [EK, Section 4], we see that the grading in (2.4) is a good grading for e. In fact to get a good grading we should scale the grading by a factor of 2, as we have $e \in \mathfrak{g}(1)$. We refer also to [GT, Section 3] where good gradings are considered in positive characteristic, and it is shown that the "same classification" of good gradings holds. Since the grading in (2.4) is good we have that $\mathfrak{g}^e \subseteq \bigoplus_{k\ge0} \mathfrak{g}(k)$, which can also be seen directly from Lemma 2.1, Now it follows from [EK, Theorem 1.4] that $\dim \mathfrak{g}^e = \dim \mathfrak{g}(0)$; this can also be verified directly from the basis given in Lemma 2.1.

We define the following subalgebras of $\mathfrak{g}$

$$\mathfrak{p} := \bigoplus_{k\ge0} \mathfrak{g}(k), \quad \mathfrak{h} := \mathfrak{g}(0) \quad \text{and} \quad \mathfrak{m} := \bigoplus_{k<0} \mathfrak{g}(k). \tag{2.5}$$

Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$, and $\mathfrak{h}$ is the Levi factor of $\mathfrak{p}$ containing $\mathfrak{t}$. Further, $\mathfrak{m}$ is the nilradical of the opposite parabolic to $\mathfrak{p}$. We recall that the heights of the columns in π are $q_1, q_2, ..., q_l$, and we see that $\mathfrak{h}$ is isomorphic to $\mathfrak{gl}_{q_1}(\Bbbk) \oplus \mathfrak{gl}_{q_2}(\Bbbk) \oplus \cdots \oplus \mathfrak{gl}_{q_l}(\Bbbk)$. Also $\mathfrak{m}$ is the Lie algebra of the closed subgroup M of G generated by the root subgroups $u_{i,j}(\Bbbk)$ with $\mathrm{col}(j) < \mathrm{col}(i)$.

We recall that d_χ denotes half the dimension of the coadjoint G-orbit of χ. So we also have that d_χ is half the dimension of the adjoint G-orbit of e, and thus we see that $d_\chi := \dim \mathfrak{m}$, because $\dim \mathfrak{g}^e = \dim \mathfrak{g}(0)$.

2.4. **Tableaux and weights.** We require various weights in $\mathfrak{t}^*$, which are used as shifts and to label certain modules. These weights can be encoded by fillings of π as we explain below, then we move on to give the weights we need.

A π-*tableau* is a diagram obtained by filling the boxes of π with elements of $\Bbbk$. The set of all tableau of shape π is denoted $\mathrm{Tab}_\Bbbk(\pi)$, and we write $\mathrm{Tab}_{\mathbb{F}_p}(\pi) \subseteq \mathrm{Tab}_\Bbbk(\pi)$ for those tableaux with entries in $\mathbb{F}_p$. For $A \in \mathrm{Tab}_{\Bbbk}(\pi)$, we write a_i for the entry in the ith box of A. Two tableaux are called *row-equivalent* if one can be obtained from the other by permuting the entries in the rows. A tableau $A \in \mathrm{Tab}_{\Bbbk}(\pi)$ is *column-connected* if whenever the jth box of π is directly below the ith box we have $a_i = a_j + 1$.

For $A \in \mathrm{Tab}_{\Bbbk}(\pi)$ we define a weight $\lambda_A \in \mathfrak{t}^*$ by

$$\lambda_A := \sum_{i=1}^{N} a_i \varepsilon_i.$$

To understand the required weights it helps for us to give a decomposition of Φ. We define

$$\begin{aligned}\Phi_+ &:= \{\varepsilon_i - \varepsilon_j \in \Phi \mid \mathrm{row}(i) < \mathrm{row}(j)\},\\ \Phi_0 &:= \{\varepsilon_i - \varepsilon_j \in \Phi \mid \mathrm{row}(i) = \mathrm{row}(j)\} \text{ and}\\ \Phi_- &:= \{\varepsilon_i - \varepsilon_j \in \Phi \mid \mathrm{row}(i) > \mathrm{row}(j)\}.\end{aligned}$$

Also we define

$$\begin{aligned}\Phi(+) &:= \{\varepsilon_i - \varepsilon_j \in \Phi \mid \mathrm{col}(i) < \mathrm{col}(j)\},\\ \Phi(0) &:= \{\varepsilon_i - \varepsilon_j \in \Phi \mid \mathrm{col}(i) = \mathrm{col}(j)\} \text{ and}\\ \Phi(-) &:= \{\varepsilon_i - \varepsilon_j \in \Phi \mid \mathrm{col}(i) > \mathrm{col}(j)\}.\end{aligned}$$

Then for $\eta, \xi \in \{-, 0, +\}$, we define

$$\Phi(\eta)_\xi = \Phi(\eta) \cap \Phi_\xi.$$

We note that $\Phi(0)_0 = \varnothing$ and that $\Phi_+ \cup \Phi(+)_0$ is the system of positive roots corresponding to $\mathfrak{b}$. Further, $\Phi(+) \cup \Phi(0)_+$ is the system of positive roots corresponding to a Borel subalgebra contained in $\mathfrak{p}$, and $\Phi(-) \cup \Phi(0)_+$ is another system of positive roots.

Having set up this notation we are in a position to give the weights that we require. First we define

$$\rho := -\sum_{i=1}^{N} i\varepsilon_i \tag{2.6}$$

this is a shifted half sum of positive roots for $\mathfrak{b}$, and is given by

$$\rho = \frac{1}{2}\left(\sum_{\alpha \in \Phi_+ \cup \Phi(+)_0} \alpha\right) - \delta,$$

where

$$\delta = \frac{N+1}{2} \sum_{i=1}^{N} \varepsilon_i.$$

We note that we should be careful in the above formulas when $p = 2$, though as the final value of ρ only involves integer coefficients this is not a problem.

We also require a "choice of ρ" corresponding to the system of positive roots $\Phi(+) \cup \Phi(0)_+$, and we define

$$\overline{\rho} := \frac{1}{2}\left(\sum_{\alpha \in \Phi(+) \cup \Phi(0)_+} \alpha\right) - \delta \tag{2.7}$$

More explicitly, we have

$$\overline{\rho} = -\sum_{i=1}^{N}((q_1 + \cdots + q_{\mathrm{col}(i)-1}) + \mathrm{row}(i) - (n - q_{\mathrm{col}(i)}))\varepsilon_i.$$

The weight

$$\gamma := \sum_{\alpha \in \Phi(-)_+} \alpha,$$

is important for Theorem 2.2, because

$$\rho = \overline{\rho} + \gamma. \tag{2.8}$$

We define

$$\eta := \sum_{i=1}^{N}(n - q_{\mathrm{col}(i)} - \cdots - \cdots - q_l)\varepsilon_i, \tag{2.9}$$

and

$$\rho_{\mathfrak{h}} := -\sum_{i=1}^{N} \mathrm{row}(i)\varepsilon_i,$$

which is a shifted choice ρ for the Borel subalgebra $\mathfrak{b} \cap \mathfrak{h}$ of $\mathfrak{h}$. Further, we define

$$\beta := \sum_{i=1}^{N}(((q_1 + \cdots + q_{\mathrm{col}(i)-1}) - (q_{\mathrm{col}(i)+1} + \cdots + q_l))\varepsilon_i = \sum_{\alpha \in \Phi(-)} \alpha.$$

and

$$\widetilde{\rho} := \overline{\rho} + \beta.$$

We note that $\widetilde{\rho}$ is a shifted choice of ρ for the system of positive roots $\Phi(-) \cup \Phi(0)_+$. An important identity for us is

$$\widetilde{\rho} = \overline{\rho} + \beta = \eta + \rho_{\mathfrak{h}} = \sum_{\alpha \in \Phi(-) \cup \Phi(0)_+} \alpha. \tag{2.10}$$

2.5. **Some modules for $U_\chi(\mathfrak{g})$.** The weights introduced in the previous subsection are required to define some modules for $\mathfrak{h}$ and for $\mathfrak{g}$. In what follows it is helpful to note that $\chi|_{\mathfrak{p}} = 0$, so that we can view $U_0(\mathfrak{h}) \subseteq U_0(\mathfrak{p}) \subseteq U_\chi(\mathfrak{g})$.

We note that $\lambda \in \mathfrak{t}^*$ is the weight of 1-dimensional $U(\mathfrak{h})$-module if and only if $\lambda(e_{i,i}) = \lambda(e_{j,j})$ whenever $\mathrm{col}(i) = \mathrm{col}(j)$, and also that $\widetilde{\rho}(e_{i,i}) = \widetilde{\rho}(e_{j,j}) - 1$, when the ith box in π is directly above the jth box. Thus we deduce that, for $A \in \mathrm{Tab}_{\mathbb{k}}(\pi)$, we have $\lambda_A - \widetilde{\rho}$ is the weight of a 1-dimensional $U(\mathfrak{h})$-module if and only if A is column connected. For column connected A we denote this 1-dimensional $U(\mathfrak{h})$-module by $\widetilde{\mathbb{k}}_A$.

Similarly, given $A \in \mathrm{Tab}_{\mathbb{k}}(\pi)$, we have $\lambda_A - \overline{\rho}$ is the weight of a 1-dimensional $U(\mathfrak{h})$-module if and only if A is column connected. In this case we denote the 1-dimensional $U(\mathfrak{h})$-module by $\overline{\mathbb{k}}_A$, and note that it factors to a module for $U_0(\mathfrak{h})$ if and only if $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$. For $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$, we can inflate $\overline{\mathbb{k}}_A$ to a $U_0(\mathfrak{p})$-module and consider the induced module $N_\chi(A) := U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p})} \overline{\mathbb{k}}_A$. We have that $N_\chi(A) \cong U_\chi(\mathfrak{m})$ as a $U_\chi(\mathfrak{m})$-module, so that $\dim N_\chi(A) = p^{\dim \mathfrak{m}} = p^{d_\chi}$ and $N_\chi(A)$ is a minimal dimensional $U_\chi(\mathfrak{g})$-module.

Let $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$. We define $\mathbb{k}_A$ to be the 1-dimensional $U_0(\mathfrak{b})$-module where $\mathfrak{t}$ acts by $\lambda_A - \rho$, and the nilradical of $\mathfrak{b}$ acts trivially. The *baby Verma module* $Z_\chi(A)$ is defined to be $Z_\chi(A) := U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{b})} \Bbbk_A$. Since χ is in standard Levi form for the Levi subalgebra $\mathfrak{g}_0$ with basis $\{e_{i,j} \mid \mathrm{row}(i) = \mathrm{row}(j)\}$, $Z_\chi(A)$ has a simple head, which we denote by $L_\chi(A)$; this essentially follows from the results in [FP2, Section 3], see also [Ja, Proposition 10.7]. Moreover, we have that $L_\chi(A) \cong L_\chi(A')$ if and only if A is row equivalent to A', see [FP2, Corollary 3.5] or [Ja, Proposition 10.8]. To see this we note that the shift by ρ in our labelling of the simple modules, transforms the dot action of the W_0 on $\mathfrak{t}^*$ in [FP2] to the standard action, where W_0 is the Weyl group of $\mathfrak{g}_0$ with respect to T; and then this action corresponds to permutations of entries in rows of tableau. Given a $U_\chi(\mathfrak{g})$-module M we say $v \in M$ is a highest weight vector (for $\mathfrak{b}$) of weight $\lambda \in \mathfrak{t}^*$ if $\mathfrak{b}v \subseteq \Bbbk v$ and $tv = \lambda(t)v$ for all $t \in \mathfrak{t}$; so if $v \in M$ is a highest weight vector of weight $\lambda_A - \rho$, then there is a homomorphism $Z_\chi(A) \to M$ sending $1 \otimes 1_A$ to v, where 1_A denotes the generator of $\Bbbk_A$.

The following theorem is key to this paper and gives a compatibility between the modules $L_\chi(A)$ and $N_\chi(A)$.

Theorem 2.2. *For column connected $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$ we have $L_\chi(A) \cong N_\chi(A)$ and has dimension p^{d_χ}. In particular, for column connected $A, A' \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$, we have $N_\chi(A) \cong N_\chi(A')$ if and only if A is row equivalent to A'.*

Proof. From the discussion above, we know $N_\chi(A)$ has dimension p^{d_χ}, so it is a minimal module for $U_\chi(\mathfrak{g})$ and thus simple. It follows that if we can find a highest weight vector $v \in N_\chi(A)$ for $\mathfrak{b}$ of weight $\lambda_A - \rho$, then $L_\chi(A) \cong N_\chi(A)$ as required. This can be seen by noting that the homomorphism $Z_\chi(A) \to N_\chi(A)$ will factor to give this isomorphism. The claim regarding row equivalence was justified in the remarks preceding the statement of the theorem.

We observe that the root vectors corresponding to roots in $\Phi(-)_+$ span a p-nilpotent subalgebra $\mathfrak{a}$ of $\mathfrak{g}$. We let

$$I = \{(i,j) \mid \varepsilon_i - \varepsilon_j \in \Phi(-)_+\} = \{(i,j) \mid \mathrm{col}(i) > \mathrm{col}(j), \mathrm{row}(i) < \mathrm{row}(j)\} \subseteq \{1, ..., N\}^2,$$

so that $\mathfrak{a}$ has basis $\{e_{i,j} \mid (i,j) \in I\}$. Since all of the elements of this basis have nonzero $\mathfrak{t}^e$ weight, we see that the restriction of χ to $\mathfrak{a}$ is zero. Hence, the restricted enveloping algebra $U_0(\mathfrak{a})$ embeds in $U_\chi(\mathfrak{g})$, and consequently $e_{i,j}^p = 0$ in $U_0(\mathfrak{a}) \subseteq U_\chi(\mathfrak{g})$ for $(i,j) \in I$.

There is an action of T on $U_0(\mathfrak{a})$, and

$$u = \prod_{(i,j) \in I} e_{i,j}^{p-1}, \tag{2.11}$$

is in the unique weight space of maximal weight (with respect to the positive roots for $\mathfrak{b}$). Further, this weight space is 1-dimensional, which implies that the product in (2.11) can be taken in any order (up to rescaling).

Let $\overline{1}_A$ denote the generator of $\overline{\Bbbk}_A$. Observe that under the adjoint action $\mathfrak{t}$ acts on u with weight $(p-1)\gamma = -\gamma = \rho - \overline{\rho}$ by (2.8). Therefore, $v := u \otimes \overline{1}_A$ is a weight vector for $\mathfrak{t}$ with weight $\lambda_A - \overline{\rho} + (\overline{\rho} - \rho) = \lambda_A - \rho$. In order to complete the proof we must show v is a highest weight vector for the action of $\mathfrak{b}$, which requires us to show that $e_{i,i+1}v = 0$ for $i = 1, ..., N-1$.

We first deal with the case where $\mathrm{row}(i) = \mathrm{row}(i+1)$ and we let $r := \mathrm{row}(i)$. We begin by decomposing I into four subsets:

$$I_1 := \{(j,k) \in I \mid \mathrm{row}(j) = r\};$$
$$I_2 := \{(j,k) \in I \mid \mathrm{row}(k) = r\};$$
$$I_3 := \{(j,k) \in I \mid \mathrm{row}(j) < r, \mathrm{row}(k) > r\}; \text{ and}$$
$$I_4 := \{(j,k) \in I \mid (j,k) \notin I_1 \cup I_2 \cup I_3\}.$$

We record three facts about commuting elements which are straightforward to verify directly.
Fact (i). $e_{i,i+1}$ commutes with $e_{j,k}$ for $(j,k) \in I_3 \cup I_4$.
Fact (ii). The elements $\{e_{j,k} \mid (j,k) \in I_1 \cup I_3\}$ pairwise commute.
Fact (iii). The elements $\{e_{j,k} \mid (j,k) \in I_2 \cup I_3\}$ pairwise commute.

For $s = 1,2,3$, we see that $\{e_{j,k} \mid (j,k) \in I_s\}$ is the basis of an abelian subalgebra of $\mathfrak{a}$. Therefore, the element $u_s := \prod_{(j,k)\in I_s} e_{j,k}^{p-1}$ does not depend on the order of the product. We choose an arbitrary ordering of I_4 and let $u_4 := \prod_{(j,k)\in I_4} e_{j,k}^{p-1}$.

We proceed with three claims, which we use to show that $e_{i,i+1}v = 0$.
Claim 1. $(\mathrm{ad}(e_{i,i+1})u_1) \otimes \overline{1}_A = 0$.
Observe that $\mathrm{ad}(e_{i,i+1})u_1$ is a sum of expressions of the form

$$u_1^{(i+1,l)} := e_{i,l}(e_{i+1,l}^{p-2}) \prod e_{j,k}^{p-1}. \tag{2.12}$$

where $(i+1,l) \in I_1$, and the product is taken over all $(j,k) \neq (i+1,l) \in I_1$. Since all matrix units occurring in (2.12) are of the form $e_{a,b}$ with $\mathrm{row}(a) = r$ and $\mathrm{row}(b) > r$ all of these factors commute so can be reordered.

We consider two cases to complete the proof of Claim 1. The first case is when $(i,l) \in I_1$. Then $u_1^{(i+1,l)}$ contains a factor of $e_{i,l}^p$, so that $u_1^{(i+1,l)} = 0$. The second case is when $\mathrm{col}(i) = \mathrm{col}(l)$ and so $e_{i,l} \in [\mathfrak{h},\mathfrak{h}]$. In this case $e_{i,l}\overline{1}_A = 0$ and so $u_1^{(i+1,l)} \otimes \overline{1}_A = 0$.
Claim 2. $u_3 e_{j,k} u_1 \otimes \overline{1}_A = 0$ *whenever* $\mathrm{col}(j) = \mathrm{col}(k)$ *and* $\mathrm{row}(j) < \mathrm{row}(k) = r$.
We have $e_{j,k} \in [\mathfrak{h},\mathfrak{h}]$ so $e_{j,k}\overline{1}_A = 0$. Thus it suffices to show that $u_3(\mathrm{ad}(e_{j,k})u_1) = 0$. Observe that $\mathrm{ad}(e_{j,k})u_1$ is a sum of monomials of the form

$$e_{j,l}(e_{k,l}^{p-2}) \prod e_{k',l'}^{p-1} \tag{2.13}$$

where $(k,l) \in I_1$ and the product is taken over $(k',l') \neq (k,l) \in I_1$. Similar to the comments following (2.12) the matrix units occurring in (2.13) all commute and so can be reordered. Since $\mathrm{row}(j) < r$ and $\mathrm{row}(l) > \mathrm{row}(k) = r$ we have $e_{j,l} \in I_3$. Applying Fact (iii) above we see that $u_3 e_{j,l} e_{k,l}^{p-2} \prod e_{k',l'}^{p-1}$ contains a factor of $e_{j,l}^p$, hence is equal to 0. This proves Claim 2.
Claim 3. $u_3(\mathrm{ad}(e_{i,i+1})u_2)u_1 \otimes \overline{1}_A = 0$.
Observe that $\mathrm{ad}(e_{i,i+1})u_2$ is a sum of expressions of the form

$$u_2^{(l,i)} := -e_{l,i+1}(e_{l,i}^{p-2}) \prod e_{j',k}^{p-1}. \tag{2.14}$$

where $(l,i) \in I_2$ and the product is taken over all $(j,k) \in I_2$ with $(j,k) \neq (l,i)$. The matrix units occurring here all commute, so can be reordered. We consider two cases. The first case is when $(l,i+1) \in I_2$. Then $u_2^{(l,i)}$ contains a factor of $e_{l,i+1}^p$ and $u_2^{(l,i)} = 0$. The second case is when $\mathrm{col}(l) = \mathrm{col}(i+1)$. Then we can use Claim 2 along with Fact (iii) to show that $u_3 u_2^{(l,i)} u_1 \otimes \bar{1}_A = 0$. This completes the proof of Claim 3.

We now combine these claims to prove that $e_{i,i+1}v = 0$. Since $e_{i,i+1}$ lies in the nilradical of $\mathfrak{p}$ we have that $e_{i,i+1}\bar{1}_A = 0$. Thus it suffices to prove $\mathrm{ad}(e_{i,i+1})(u_4u_3u_2u_1) \otimes \bar{1}_A = 0$. Applying Fact (i) we only need to check $u_4u_3(\mathrm{ad}(e_{i,i+1})u_2)u_1 \otimes \bar{1}_A = 0$ and $u_4u_3u_2(\mathrm{ad}(e_{i,i+1})u_1) \otimes \bar{1}_A = 0$, which are given by Claim 3 and Claim 1 respectively.

We move on to deal with the case $\mathrm{row}(i) < \mathrm{row}(i+1)$, and show that $e_{i,i+1}v = 0$.

For this case first suppose that $\mathrm{col}(i) > \mathrm{col}(i+1)$. Then we have $e_{i,i+1} \in \mathfrak{a}$. By the remarks following (2.11) we can write $u = e_{i,i+1}^{p-1}u_0$ for some $u_0 \in U_0(\mathfrak{a})$ and so $e_{i,i+1}u = 0$, which implies that $e_{i,i+1}v = 0$.

The case where $\mathrm{col}(i) = \mathrm{col}(i+1)$, which only happens when $p_{\mathrm{row}(i)} = 1$ and $s_{i+1,i} = 0$. Then we have $e_{i,i+1} \in [\mathfrak{h}, \mathfrak{h}]$ and $e_{i,i+1}\bar{1}_A = 0$, so we are just required to show that $[e_{i,i+1}, u] = 0$. This is done with commutator arguments similar to those used above, so we omit the details. $\square$

2.6. **The W-algebra $U(\mathfrak{g}, e)$ and its p-centre.** Since $e \in \mathfrak{g}(1)$, we have that χ vanishes on $\mathfrak{g}(k)$ for $k \neq -1$. Therefore, χ restricts to a character of $\mathfrak{m}$. We define $\mathfrak{m}_\chi := \{x - \chi(x) \mid x \in \mathfrak{m}\} \subseteq U(\mathfrak{g})$, which is a Lie subalgebra of $U(\mathfrak{g})$. By the PBW theorem there is a direct sum decomposition

$$U(\mathfrak{g}) = U(\mathfrak{g})\mathfrak{m}_\chi \oplus U(\mathfrak{p})$$

We let $\mathrm{pr} : U(\mathfrak{g}) \to U(\mathfrak{p})$ be the projection onto the second factor. Also we abbreviate and write $I := U(\mathfrak{g})\mathfrak{m}_\chi$, and define $Q := U(\mathfrak{g})/I$.

As explained in [GT, §4.3] the adjoint action of M on $U(\mathfrak{g})$ gives an adjoint action of M on Q. In [GT, Definition 4.3] the W-algebra associated to e is defined to be

$$\{u + I \in Q \mid g \cdot u + I = u + I \text{ for all } g \in M\}.$$

In this paper, we prefer to work with an equivalent realization of $U(\mathfrak{g}, e)$ as a subalgebra of $U(\mathfrak{p})$. For this we require the *twisted adjoint action* of M on $U(\mathfrak{p})$, which is defined by

$$\mathrm{tw}(g) \cdot u := \mathrm{pr}(g \cdot u),$$

for $g \in M$ and $u \in U(\mathfrak{p})$. By using pr to identify $U(\mathfrak{g})/I$ with $U(\mathfrak{p})$, we can equivalently define the W-algebra associated to e to be the invariant subalgebra

$$U(\mathfrak{g}, e) := U(\mathfrak{p})^{\mathrm{tw}(M)} = \{u \in U(\mathfrak{p}) \mid \mathrm{tw}(g) \cdot u = u \text{ for all } g \in M\}.$$

We want to recast some of the material from [GT, Section 8] in our setting where $U(\mathfrak{g}, e) = U(\mathfrak{p})^{\mathrm{tw}(M)}$. We begin with the p-centre of $U(\mathfrak{g}, e)$, and to define this we note that the p-centre $Z_p(\mathfrak{p})$ of $U(\mathfrak{p})$ is stable under the twisted adjoint action of M of $U(\mathfrak{p})$. The *p-centre of* $U(\mathfrak{g}, e)$ is defined in [GT, Definition 8.1], and in our setting, it is given by

$$Z_p(\mathfrak{g}, e) := Z_p(\mathfrak{p})^{\mathrm{tw}(M)} \subseteq U(\mathfrak{g}, e).$$

Let $\psi \in \mathfrak{p}^* \subseteq \mathfrak{g}^*$. We write $J_\psi^{\mathfrak{p}}$ for the ideal of $U(\mathfrak{p})$ generated $\{x^p - x^{[p]} - \psi(x)^p \mid x \in \mathfrak{p}\}$, the *reduced W-algebra corresponding to ψ* as

$$U_\psi(\mathfrak{g}, e) := U(\mathfrak{g}, e)/(J_\psi^{\mathfrak{p}} \cap U(\mathfrak{g}, e)).$$

We note that our notation here differs from that used in [GT, Definition 8.5] by a shift of χ, i.e. $U_\psi(\mathfrak{g}, e)$ here would be denoted $U_{\chi+\psi}(\mathfrak{g}, e)$ there (to make sense of $\chi + \psi \in \mathfrak{g}^*$ we identify

$\mathfrak{p}^* = \mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{m}) \subseteq \mathfrak{g}^*$). This change in notation is partly justified by the fact that the kernel of the restriction of the projection $U(\mathfrak{p}) \twoheadrightarrow U_\psi(\mathfrak{p})$ to $U(\mathfrak{g}, e)$ is $J_\psi \cap U(\mathfrak{g}, e)$. Consequently, we can identify $U_\psi(\mathfrak{g}, e)$ with the image of $U(\mathfrak{g}, e)$ in $U_\psi(\mathfrak{p})$.

It turns out that for $\psi \neq \psi'$ we can have $J_\psi^{\mathfrak{p}} \cap U(\mathfrak{g}, e) = J_{\psi'}^{\mathfrak{p}} \cap U(\mathfrak{g}, e)$, so that $U_\psi(\mathfrak{g}, e) = U_{\psi'}(\mathfrak{g}, e)$. To explain precisely when this happens we need to translate some of the material from [GT, §8.2] to our setting. We write $\mathfrak{m}^\perp \subseteq \mathfrak{g}$ for the annihilator of $\mathfrak{m}$ with respect to $(\cdot\,,\cdot)$, and note that we can identify $\mathfrak{p}^* \cong e + \mathfrak{m}^\perp$ via $(\cdot\,,\cdot)$. There is an adjoint action of M on $e + \mathfrak{m}^\perp$, and this translates through the identification $\mathfrak{p}^* \cong e + \mathfrak{m}^\perp$ to an action of M on $\mathfrak{p}^*$, which we refer to as the *twisted action of M on $\mathfrak{p}^*$*. For $\phi \in \mathfrak{p}^*$, $g \in M$ and $x \in \mathfrak{p}$ this twisted adjoint action is given by $(\mathrm{tw}(g) \cdot \phi)(x) = \chi(g^{-1} \cdot x - x) + \phi(g^{-1} \cdot x)$.

Now we state the required part of [GT, Lemma 8.6] in our notation.

Lemma 2.3. *We have that $U_\psi(\mathfrak{g}, e) = U_{\psi'}(\mathfrak{g}, e)$ if and only if ψ and ψ' are conjugate under the twisted M-action on $\mathfrak{p}^*$.*

Thanks to Quillen's lemma, an irreducible $U(\mathfrak{g}, e)$-module L factors to a module for $U_\psi(\mathfrak{g}, e)$ for some $\psi \in \mathfrak{p}^*$. Further, it is clear from the definitions that, for $\psi, \psi' \in \mathfrak{p}^*$, the module L factors to a module for both $U_\psi(\mathfrak{g}, e)$ and for $U_{\psi'}(\mathfrak{g}, e)$ if and only if $J_\psi^{\mathfrak{p}} \cap U(\mathfrak{g}, e) = J_{\psi'}^{\mathfrak{p}} \cap U(\mathfrak{g}, e)$, which by the previous lemma occurs if only ψ and ψ' are conjugate under the twisted M-action.

We also recall Premet's equivalence in Theorem 2.4 below. This theorem is based on [Pr2, Theorem 2.4], and the statement here can be deduced from [Pr3, Lemma 2.2(c)] and [GT, Proposition 8.7 and Lemma 8.8], see also [GT, Remark 9.4]. For the statement, we view $\psi \in \mathfrak{p}^*$ as an element of $\mathfrak{g}^*$ via the identification $\mathfrak{p}^* = \mathrm{Ann}_{\mathfrak{g}^*}(\mathfrak{m}) \subseteq \mathfrak{g}^*$. Also we define $Q^\psi = Q/J_{\chi+\psi}Q$, and recall that as explained in [GT, §8.3] Q^ψ is a left $U_{\chi+\psi}(\mathfrak{g})$-module and a right $U_\psi(\mathfrak{g}, e)$-module

Theorem 2.4. *Let $\psi \in \mathfrak{p}^*$. We have*

(a) $U_{\chi+\psi}(\mathfrak{g}) \cong \mathrm{Mat}_{p^{d_\chi}} U_\psi(\mathfrak{g}, e)$;

(b) *the functor from* $U_\psi(\mathfrak{g}, e)$-mod *to* $U_{\chi+\psi}(\mathfrak{g})$-mod *given by*

$$M \mapsto Q^\psi \otimes_{U_\psi(\mathfrak{g},e)} M \tag{2.15}$$

is an equivalence of categories with quasi-inverse given by

$$V \mapsto V^{\mathfrak{m}_\chi} := \{v \in V \mid \mathfrak{m}_\chi v = 0\}. \tag{2.16}$$

(c) $\dim(Q^\psi \otimes_{U_\psi(\mathfrak{g},e)} M) = p^{d_\chi} \dim M$, *for a finite dimensional $U_\psi(\mathfrak{g}, e)$-module M.*

We also recall that $U(\mathfrak{g}, e)$ has a PBW basis, which is described in [GT, Theorem 7.3]. We summarize the properties that we require in Proposition 2.5 below and adapt the statement to the case $\mathfrak{g} = \mathfrak{gl}_N(\mathbb{k})$. For this we first have to give some notation. We fix a basis $x_1, ..., x_r$ of $\mathfrak{g}^e$, chosen so that $x_i \in \mathfrak{g}(n_i)$, where $n_i \in \mathbb{Z}_{\geq 0}$. Let $I_{\mathfrak{p}} = \{(i,j) \mid 1 \leq i, j, \leq N, e_{i,j} \in \mathfrak{p}\}$ and fix an order on $I_{\mathfrak{p}}$. For $\boldsymbol{a} = (a_{i,j}) \in \mathbb{Z}_{\geq 0}^{I_{\mathfrak{p}}}$ we write

$$\boldsymbol{e}^{\boldsymbol{a}} := \prod_{(i,j) \in I_{\mathfrak{p}}} e_{i,j}^{a_{i,j}} \in U(\mathfrak{p}), \tag{2.17}$$

and define $|\boldsymbol{a}| = \sum_{(i,j) \in I_{\mathfrak{p}}} a_{i,j}$ and $|\boldsymbol{a}|_e = \sum_{(i,j) \in I_{\mathfrak{p}}} (\mathrm{col}(j) - \mathrm{col}(i) + 1) a_{i,j}$.

We can now state our proposition about the PBW basis of $U(\mathfrak{g}, e)$; it is a consequence of [GT, Lemma 7.1 and Theorem 7.3]. We remind the reader that the graded degrees in this paper differ from those in *loc. cit.* by a factor of 2.

Proposition 2.5.

(a) *There are elements* $\Theta(x_1), \dots, \Theta(x_r)$ *of* $U(\mathfrak{g}, e)$ *of the form*

$$\Theta(x_i) = x_i + \sum_{|\boldsymbol{a}|_e \leq n_i+1} \lambda_{\boldsymbol{a},i} \boldsymbol{e}^{\boldsymbol{a}} \tag{2.18}$$

where $\lambda_{\boldsymbol{a},i} \in \Bbbk$ *satisfy* $\lambda_{\boldsymbol{a},i} = 0$ *whenever* $|\boldsymbol{a}|_e = n_i + 1$ *and* $|\boldsymbol{a}| = 1$.

(b) *Given any elements* $\Theta(x_1), \dots, \Theta(x_r) \in U(\mathfrak{g}, e)$ *of the form in* (2.18) *the ordered monomials in* $\Theta(x_1), \dots, \Theta(x_r)$ *form a basis of* $U(\mathfrak{g}, e)$.

Let $\Theta(x_i), \Theta(x_j)$ be elements of $U(\mathfrak{g}, e)$ of the form (2.18). Then a commutator calculation shows that

$$[\Theta(x_i), \Theta(x_j)] = [x_i, x_j] + \sum_{|\boldsymbol{a}|_e \leq n_i+n_j+1} \mu_{\boldsymbol{a}} \boldsymbol{e}^{\boldsymbol{a}}, \tag{2.19}$$

where $\mu_{\boldsymbol{a}} \in \Bbbk$ satisfy $\mu_{\boldsymbol{a}} = 0$ whenever $|\boldsymbol{a}|_e = n_i + n_j + 1$ and $|\boldsymbol{a}| = 1$. The key ingredient for this calculation is to observe that if we take the commutator $[\boldsymbol{e}^{\boldsymbol{a}}, \boldsymbol{e}^{\boldsymbol{b}}]$ for $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{Z}_{\geq 0}^{I_{\mathfrak{p}}}$, then we get a linear combinations of terms $\boldsymbol{e}^{\boldsymbol{c}}$ with $|\boldsymbol{c}|_e = |\boldsymbol{a}|_e + |\boldsymbol{b}|_e - 1$ and $|\boldsymbol{c}| = |\boldsymbol{a}| + |\boldsymbol{b}| - 1$, plus a linear combination of terms $\boldsymbol{e}^{\boldsymbol{d}}$ with $|\boldsymbol{d}|_e < |\boldsymbol{a}|_e + |\boldsymbol{b}|_e - 1$.

3. Modular truncated shifted Yangian

In this section we consider the modular shifted Yangian $Y_n(\sigma)$ and its truncation $Y_{n,l}(\sigma)$. The algebras $Y_n(\sigma)$ have been studied in recent work of Brundan and the second author, [BT]. Here we recall some of the results in *loc. cit.* and move on to verify that the truncation $Y_{n,l}(\sigma)$ has structure theory similar to that in characteristic 0. In the next section we exploit formulas from [BK1, Section 9] to show that the modular finite W-algebra $U(\mathfrak{g}, e)$ is isomorphic to the shifted truncated Yangian $Y_{n,l}(\sigma)$ of level $l = p_n$.

We recall that σ is the shift matrix for the pyramid π. The *modular shifted Yangian* $Y_n(\sigma)$ is the $\Bbbk$-algebra with generators

$$\begin{gathered} \{D_i^{(r)} \mid 1 \leq i \leq n, r > 0\} \cup \{E_i^{(r)} \mid 1 \leq i < n, r > s_{i,i+1}\} \\ \cup \{F_i^{(r)} \mid 1 \leq i < n, r > s_{i+1,i}\} \end{gathered} \tag{3.1}$$

and relations

$$\big[D_i^{(r)}, D_j^{(s)}\big] = 0, \tag{3.2}$$

$$\big[E_i^{(r)}, F_j^{(s)}\big] = -\delta_{i,j} \sum_{t=0}^{r+s-1} D_{i+1}^{(r+s-1-t)} \widetilde{D}_i^{(t)}, \tag{3.3}$$

$$\big[D_i^{(r)}, E_j^{(s)}\big] = (\delta_{i,j} - \delta_{i,j+1}) \sum_{t=0}^{r-1} D_i^{(t)} E_j^{(r+s-1-t)}, \tag{3.4}$$

$$\big[D_i^{(r)}, F_j^{(s)}\big] = (\delta_{i,j+1} - \delta_{i,j}) \sum_{t=0}^{r-1} F_j^{(r+s-1-t)} D_i^{(t)}, \tag{3.5}$$

$$\big[E_i^{(r)}, E_i^{(s)}\big] = \sum_{t=r}^{s-1} E_i^{(t)} E_i^{(r+s-1-t)} \qquad \text{if } r < s, \tag{3.6}$$

$$\big[F_i^{(r)}, F_i^{(s)}\big] = \sum_{t=s}^{r-1} F_i^{(r+s-1-t)} F_i^{(t)} \qquad \text{if } r > s, \tag{3.7}$$

$$\big[E_i^{(r+1)}, E_{i+1}^{(s)}\big] - \big[E_i^{(r)}, E_{i+1}^{(s+1)}\big] = E_i^{(r)} E_{i+1}^{(s)}, \tag{3.8}$$

$$\big[F_i^{(r)}, F_{i+1}^{(s+1)}\big] - \big[F_i^{(r+1)}, F_{i+1}^{(s)}\big] = F_{i+1}^{(s)} F_i^{(r)}, \tag{3.9}$$

$$\big[E_i^{(r)}, E_j^{(s)}\big] = 0 \qquad \text{if } |i-j| > 1, \tag{3.10}$$

$$\big[F_i^{(r)}, F_j^{(s)}\big] = 0 \qquad \text{if } |i-j| > 1, \tag{3.11}$$

$$\Big[E_i^{(r)}, \big[E_i^{(s)}, E_j^{(t)}\big]\Big] + \Big[E_i^{(s)}, \big[E_i^{(r)}, E_j^{(t)}\big]\Big] = 0 \quad \text{if } |i-j| = 1, r \neq s, \tag{3.12}$$

$$\Big[F_i^{(r)}, \big[F_i^{(s)}, F_j^{(t)}\big]\Big] + \Big[F_i^{(s)}, \big[F_i^{(r)}, F_j^{(t)}\big]\Big] = 0 \quad \text{if } |i-j| = 1, r \neq s \tag{3.13}$$

$$\Big[E_i^{(r)}, \big[E_i^{(r)}, E_j^{(t)}\big]\Big] = 0 \qquad \text{if } |i-j| = 1, \tag{3.14}$$

$$\Big[F_i^{(r)}, \big[F_i^{(r)}, F_j^{(t)}\big]\Big] = 0 \qquad \text{if } |i-j| = 1, \tag{3.15}$$

for all admissible i, j, r, s, t. In the relations, the shorthand $D_i^{(0)} = \widetilde{D}_i^{(0)} := 1$ is used, and the elements $\widetilde{D}_i^{(r)}$ for $r > 0$ are defined recursively by $\widetilde{D}_i^{(r)} := -\sum_{t=1}^{r} D_i^{(t)} \widetilde{D}_i^{(r-t)}$.

This presentation of $Y_n(\sigma)$ is given in [BT, Theorem 4.15] and is modelled on the Drinfeld presentation of the shifted Yangian defined over $\mathbb{C}$, as introduced in [BK1, Section 2]. It is proved in [BT, Theorem 4.14] that there is a PBW basis for $Y_n(\sigma)$, whose description does not depend on the characteristic p. Before stating this result it is necessary to introduce some additional elements. We define

$$E_{i,i+1}^{(r)} := E_i^{(r)},$$
$$F_{i,i+1}^{(r)} := F_i^{(r)}$$

for $i = 1, \dots, n-1$, and inductively define

$$E_{i,j}^{(r)} := [E_{i,j-1}^{(r-s_{j-1,j})}, E_{j-1}^{(s_{j-1,j}+1)}] \text{ for } 1 \le i < j \le n \text{ and } r > s_{i,j}, \tag{3.16}$$

$$F_{i,j}^{(r)} := [F_{j-1}^{(s_{j,j-1}+1)}, F_{i,j-1}^{(r-s_{j,j-1})}] \text{ for } 1 \le i < j \le n \text{ and } r > s_{j,i}. \tag{3.17}$$

Then [BT, Theorem 4.14] says that monomials in the elements

$$\begin{gathered}\{D_i^{(r)} \mid 1 \le i \le n, r > 0\} \cup \{E_{i,j}^{(r)} \mid 1 \le i < j \le n, r > s_{i,j}\} \\ \cup \{F_{i,j}^{(r)} \mid 1 \le i < j \le n, r > s_{j,i}\}\end{gathered} \tag{3.18}$$

in any fixed order give a basis of $Y_n(\sigma)$.

The shifted Yangian has the *canonical filtration* which we denote $Y_n(\sigma) = \bigcup_{r \ge 0} \mathcal{F}_r Y_n(\sigma)$ and is defined by declaring that $D_i^{(r)}, E_{i,j}^{(r)}, F_{i,j}^{(r)} \in \mathcal{F}_r Y_n(\sigma)$, i.e. that $\mathcal{F}_r Y_n(\sigma)$ is the spanned by the monomials in these elements of total degree $\le r$. It is immediate from the relations (3.2)–(3.14) that the associated graded algebra $\operatorname{gr} Y_n(\sigma)$ is commutative.

The *truncated shifted Yangian of level l* is denoted $Y_{n,l}(\sigma)$ and defined to be the quotient of $Y_n(\sigma)$ by the ideal generated by $\{D_1^{(r)} \mid r > p_1\}$; this definition is taken from [BK1, Section 6] where it is given for characteristic 0. We recall that $l = p_n$, so that $p_1 = l - s_{1,n} - s_{n,1}$. The truncated shifted Yangian inherits the canonical filtration from $Y_n(\sigma)$ and we write $Y_{n,l}(\sigma) = \bigcup_{i \ge 0} \mathcal{F}_i Y_{n,l}(\sigma)$. The associated graded algebra $\operatorname{gr} Y_{n,l}(\sigma)$ is certainly commutative, as it is a quotient of $\operatorname{gr} Y_n(\sigma)$. When working with $Y_{n,l}(\sigma)$ we often abuse notation by using the same symbols $D_i^{(r)}, E_{i,j}^{(r)}, F_{i,j}^{(r)}$ to refer to the elements of $Y_n(\sigma)$ and their images in $Y_{n,l}(\sigma)$.

The next lemma gives a spanning set for $Y_{n,l}(\sigma)$ and should be viewed as a modular version of [BK1, Lemma 6.1]; though we note that it is less general as we do not deal with parabolic presentations here. We recover the full PBW theorem for $Y_{n,l}(\sigma)$, i.e. that the spanning set given in the next lemma is actually a basis, once we have clarified the connection with $U(\mathfrak{g}, e)$ in Theorem 4.3.

Lemma 3.1. *The monomials in the elements*

$$\begin{gathered}\{D_i^{(r)} \mid 1 \le i \le n, 0 < r \le p_i\} \cup \{E_{i,j}^{(r)} \mid 1 \le i < j \le n, s_{i,j} < r \le s_{i,j} + p_i\} \\ \cup \{F_{i,j}^{(r)} \mid 1 \le i < j \le n, s_{j,i} < r \le s_{j,i} + p_i\}\end{gathered} \tag{3.19}$$

in any fixed order form a spanning set of $Y_{n,l}(\sigma)$.

Proof. Our proof uses the arguments in the proof of [BK1, Lemma 6.1]. As we are not using the more general parabolic presentations of the Yangian as in that proof, we outline the arguments required for the convenience of the reader.

During the proof we frequently refer to degree, by which we always mean filtered degree for the canonical filtration; on occasion we speak about the total degree of a monomial to make the intended meaning clearer. Until the final paragraph we use the word monomials to mean unordered monomials, as this simplifies the exposition. We frequently use that $\operatorname{gr} Y_n(\sigma)$ is commutative, so for $u \in Y_{n,l}(\sigma)$ of degree r and $v \in Y_{n,l}(\sigma)$ of degree s, the commutator $[u, v]$ has degree $\le r + s - 1$.

For $1 \le k \le n$ and $s \ge 1$, we let:

- Ω_k be the set of generators given in (3.19) with $i, j \le k$;
- $\Omega_{k,E}$ be the generators in Ω_k along with the generators $E_{i,k+1}^{(r)}$ with $1 \le i \le k$ and $s_{i,k+1} < r \le s_{i,k+1} + p_{k+1}$; and

- $\widehat{\Omega}_k$ be the set of generators from (3.18) with $i, j \leq k$.

A key observation for us is:

$(*)$ if $X \in \Omega_k$ with degree $r - s_{k,k+1}$, then $[X, E_k^{s_{k,k+1}+1}]$ can be written as a linear combination of monomials in $\Omega_{k,E}$ with total degree r.

This can be checked directly from the relations, and the definition of $E_{i,j}^{(r)}$ in (3.16). A similar statement holds with "F replacing E". Further, we have:

$(\dagger)$ if $X \in \Omega_{k,E}$ with degree $r - s_{k+1,k}$, then $[X, F_k^{s_{k+1,k}+1}]$ can be written as a linear combination of monomials in $\Omega_{k+1} \cup \{\widetilde{D}_k^{(s)} \mid s = p_k + 1, \dots, p_{k+1}\}$ with total degree r.

Again this is checked directly from the relations, and we note that $\widetilde{D}_k^{(s)}$ can be written in terms of $D_k^{(t)}$ for $t \leq s$.

We show by induction on k that any element in $\widehat{\Omega}_k$ of degree $r \geq 0$ can be written as a linear combination of monomials in the elements of Ω_k of total degree r.

To start the induction we note that the case $k = 1$ is trivial, because $D_1^{(r)} = 0$ for $r > p_1$ in $Y_{n,l}(\sigma)$. So suppose inductively we have proved the claim for $\widehat{\Omega}_k$ and we consider elements of $\widehat{\Omega}_{k+1}$.

First consider an element $E_{i,k+1}^{(r)}$ for $r > s_{i,k+1} + p_i$. For $i < k$, we use the definition of $E_{i,k+1}^{(r)}$ in (3.16) to write $E_{i,k+1}^{(r)} = [E_{i,k}^{(r-s_{k,k+1})}, E_k^{(s_{k,k+1}+1)}]$. Using the inductive hypothesis $E_{i,k}^{(r-s_{k,k+1})}$ can be written as a sum of monomials in Ω_k of total degree $r - s_{k,k+1}$. Now using $(*)$ we deduce that $E_{i,k+1}^{(r)}$ can be written as a sum of monomials in $\Omega_{k,E}$ with total degree r. For $i = k$, we have $E_{i,k+1}^{(r)} = E_k^{(r)}$ and we can use the relation (3.4) to write

$$E_k^{(r)} = \left[D_k^{(r-s_{k,k+1})}, E_k^{(s_{k,k+1}+1)}\right] - \sum_{t=1}^{r-s_{k,k+1}-1} D_k^{(t)} E_k^{(r-t)}.$$

The right hand side of the above is an expressions in elements of $\widehat{\Omega}_{k+1}$ of degree r. We use property $(*)$ to deduce that the first term above can be written as a sum of monomials in $\Omega_{k,E}$ with total degree r. To deal with the second term we do an induction on r.

We can deal with the elements $F_{i,k+1}^{(r)}$ similarly.

We are left to consider the elements $D_{k+1}^{(r)}$. Using (3.3), we write

$$D_{k+1}^{(r)} = -\left[E_k^{(r-s_{k+1,k})}, F_i^{(s_{k+1,k}+1)}\right] + \sum_{t=1}^{r} D_{i+1}^{(r-t)} \widetilde{D}_i^{(t)},$$

For the first term on the right hand side we use the above to write $E_k^{r-s_{k+1,k}}$ as a linear combination of monomials in $\Omega_{k,E}$ or total degree $r - s_{k+1,k}$. Using $(\dagger)$ we rewrite this in terms of monomials in $\Omega_{k+1} \cup \{\tilde{D}_k^{(s)} \mid s = p_k + 1, \dots, p_{k+1}\}$. Now we can use the inductive hypothesis to write this as linear combination of monomials in Ω_{k+1}. The second term can be dealt with by induction on r.

To finish the proof, we have to observe that for a fixed order on the elements given in (3.19), an unordered monomial can be written as a linear combination of ordered monomials. This is easily done using that $\operatorname{gr} Y_n(\sigma)$ is commutative, an induction on degree, and what has already been proved. $\square$

Let $Y_{n,l}(\sigma)^{\mathrm{ab}}$ denote the *maximal abelian quotient of* $Y_{n,l}(\sigma)$ obtained by factoring out the ideal generated by all commutators $\{[u,v] \mid u,v \in Y_{n,l}(\sigma)\}$. So the isomorphism classes of one dimensional representations of $Y_{n,l}(\sigma)$ are in one-to-one correspondence with maximal ideals of $Y_{n,l}(\sigma)^{\mathrm{ab}}$. A calculation due to Premet within the proof of [Pr3, Theorem 3.3] shows that $Y_{n,l}(\sigma)^{\mathrm{ab}}$ is generated by a particular subset of the elements (3.19) as stated in the following lemma. Although [Pr3, Theorem 3.3] is only stated in the characteristic 0 case, the required calculation works directly from the relations and we can observe that it is valid in characteristic p.

Lemma 3.2. *The algebra $Y_{n,l}(\sigma)^{\mathrm{ab}}$ is generated by the l elements*

$$\{\dot{D}_i^{(r)} \mid i = 1, ..., n, 0 < r \leq p_i - p_{i-1}\},$$

where $\dot{D}_i^{(r)}$ denotes the image of $D_i^{(r)}$ in $Y_{n,l}(\sigma)^{\mathrm{ab}}$.

4. $U(\mathfrak{g}, e)$ as modular truncated shifted Yangian

We proceed with the notation in Section 2 and recall that $U(\mathfrak{g}, e)$ is the invariant algebra $U(\mathfrak{p})^{\mathrm{tw}(M)}$ for the twisted adjoint action of M on $U(\mathfrak{p})$. The goal of the current section is to show that $U(\mathfrak{g}, e)$ is isomorphic to the truncated shifted Yangian $Y_{n,l}(\sigma)$ of level l.

First we recall some remarkable formulas from [BK1, §9] for elements of $U(\mathfrak{p})$, which are actually invariants for the twisted adjoint action of M as proved in Lemma 4.1. We refer also to [BK2, §3.3], as our notation is closer to the notation used there. The weight $\eta \in \mathfrak{t}^*$ from (2.9) is required to define these invariants, and we note that η extends to a character of $\mathfrak{p}$. For $e_{i,j} \in \mathfrak{p}$ we define

$$\tilde{e}_{i,j} := e_{i,j} + \eta(e_{i,j}).$$

Now for $1 \leq i, j \leq n$, $0 \leq x < n$ and $r \geq 1$, we let

$$T_{i,j;x}^{(r)} := \sum_{s=1}^{r} (-1)^{r-s} \sum_{\substack{i_1,\dots,i_s \\ j_1,\dots,j_s}} (-1)^{|\{t=1,\dots,s-1 | \mathrm{row}(j_t) \leq x\}|} \tilde{e}_{i_1,j_1} \cdots \tilde{e}_{i_s,j_s} \in U(\mathfrak{p}) \tag{4.1}$$

where the sum is taken over all $1 \leq i_1, \dots, i_s, j_1, \dots, j_s \leq N$ such that

(a) $\mathrm{col}(j_1) - \mathrm{col}(i_1) + \cdots + \mathrm{col}(j_s) - \mathrm{col}(i_s) + s = r$;
(b) $\mathrm{col}(i_t) \leq \mathrm{col}(j_t)$ for each $t = 1, \dots, s$;
(c) if $\mathrm{row}(j_t) > x$, then $\mathrm{col}(j_t) < \mathrm{col}(i_{t+1})$ for each $t = 1, \dots, s-1$;
(d) if $\mathrm{row}(j_t) \leq x$ then $\mathrm{col}(j_t) \geq \mathrm{col}(i_{t+1})$ for each $t = 1, \dots, s-1$;
(e) $\mathrm{row}(i_1) = i$, $\mathrm{row}(j_s) = j$;
(f) $\mathrm{row}(j_t) = \mathrm{row}(i_{t+1})$ for each $t = 1, \dots, s-1$.

Now define

$$D_i^{(r)} := T_{i,i;i-1}^{(r)} \text{ for } 1 \leq i \leq n, r > 0 \tag{4.2}$$

$$E_i^{(r)} := T_{i,i+1;i}^{(r)} \text{ for } 1 \leq i < j \leq n, r > s_{i,j} \tag{4.3}$$

$$F_i^{(r)} := T_{i+1,i;i}^{(r)} \text{ for } 1 \leq i < j \leq n, r > s_{j,i}. \tag{4.4}$$

These elements are denoted by the same symbols as the generators of the truncated shifted Yangian and this will be justified later. First we prove that they are invariants for the twisted adjoint action of M and thus are elements of $U(\mathfrak{g}, e)$.

Lemma 4.1. *The elements $D_i^{(r)}$, $E_i^{(r)}$ and $F_i^{(r)}$ of $U(\mathfrak{p})$ defined in* (4.2), (4.3) *and* (4.4) *are all invariant under the twisted adjoint action of M.*

Proof. Recall from §2.1 that $\mathfrak{g}_{\mathbb{C}} = \mathfrak{gl}_N(\mathbb{C})$ and $\mathfrak{g}_{\mathbb{Z}} = \mathfrak{gl}_N(\mathbb{Z})$; and we have the subalgebras $\mathfrak{p}_{\mathbb{C}}$, $\mathfrak{m}_{\mathbb{C}}$, $\mathfrak{p}_{\mathbb{Z}}$ and $\mathfrak{m}_{\mathbb{Z}}$.

Let $X_i^{(r)} \in U(\mathfrak{p})$ be one of the elements defined by (4.2), (4.3) or (4.4). Throughout this proof we abuse notation slightly by simultaneously viewing $X_i^{(r)}$ also as an element of $U(\mathfrak{p}_{\mathbb{Z}})$ and of $U(\mathfrak{p}_{\mathbb{C}})$.

According to [BK1, Lemma 10.12], we have that $X_i^{(r)} \in U(\mathfrak{p}_{\mathbb{C}})$ is invariant under the twisted adjoint action of $\mathfrak{m}_{\mathbb{C}}$. This twisted adjoint action is defined by

$$\mathrm{tw}(x)u := \mathrm{pr}(\mathrm{ad}(x)u)$$

for $x \in \mathfrak{m}_{\mathbb{C}}$ and $u \in U(\mathfrak{p}_{\mathbb{C}})$. We see that the twisted adjoint action of $\mathfrak{m}_{\mathbb{C}}$ exponentiates to give the twisted adjoint action of $M_{\mathbb{C}}$ as defined in (2.6). Thus we deduce that $X_i^{(r)}$ is an invariant for the twisted adjoint action of $M_{\mathbb{C}}$.

Now let $1 \leq i, j \leq N$ such that $e_{i,j} \in \mathfrak{m}_{\mathbb{C}}$. Let t be an indeterminate and consider the homomorphism $U(\mathfrak{g}_{\mathbb{C}})[t] \to U(\mathfrak{g}_{\mathbb{C}})[t]$ determined by $e_{k,l} \mapsto e_{k,l} + t\delta_{j,k}e_{i,l} - t\delta_{l,i}e_{k,j} - t^2\delta_{j,k}\delta_{l,i}e_{i,j}$, which "gives the action of $u_{i,j}(t)$ on $e_{k,l}$" as in (2.1). This preserves the integral form $U(\mathfrak{g}_{\mathbb{Z}})[t]$ of $U(\mathfrak{g}_{\mathbb{C}})[t]$ and, composing with the projection $U(\mathfrak{g}_{\mathbb{Z}})[t] \to U(\mathfrak{p}_{\mathbb{Z}})[t]$ along the direct sum decomposition $U(\mathfrak{g}_{\mathbb{Z}})[t] = U(\mathfrak{p}_{\mathbb{Z}})[t] \oplus U(\mathfrak{g}_{\mathbb{Z}})[t]\{x - \chi(x) \mid x \in \mathfrak{m}_{\mathbb{Z}}\}$, we obtain a $\mathbb{Z}$-module homomorphism $\psi_{i,j} : U(\mathfrak{p}_{\mathbb{Z}})[t] \to U(\mathfrak{p}_{\mathbb{Z}})[t]$. By the observations of the previous paragraph, $\psi_{i,j}(X_k^{(r)}) - X_k^{(r)} \in (t - s)U(\mathfrak{p}_{\mathbb{C}})[t]$ for every $s \in \mathbb{C}$. It follows that $\psi_{i,j}(X_k^{(r)}) - X_k^{(r)} = 0$ in $U(\mathfrak{p}_{\mathbb{C}})[t]$. Note that $\bigcap_{s\in\mathbb{C}}(t - s)U(\mathfrak{p}_{\mathbb{C}})[t] = 0$ follows from the fact that $U(\mathfrak{p}_{\mathbb{C}})[t]$ is a free $\mathbb{C}[t]$-module, and $\bigcap_{s\in\mathbb{C}}(t - s)\mathbb{C}[t] = 0$.

Now consider the equation

$$\psi_{i,j}(X_k^{(r)}) \otimes 1 - X_k^{(r)} \otimes 1 = 0,$$

valid in $U(\mathfrak{p}_{\mathbb{Z}})[t] \otimes_{\mathbb{Z}} \Bbbk \cong U(\mathfrak{p})[t]$. Examining the image in $U(\mathfrak{p}) \cong U(\mathfrak{p})[t]/(t - s)U(\mathfrak{p})[t]$ for all $s \in \Bbbk$, we deduce that $X_k^{(r)} \in U(\mathfrak{p})$ is invariant under the twisted adjoint action of the root subgroup $u_{i,j}(\Bbbk)$. Hence, $X_k^{(r)} \in U(\mathfrak{p})$ is invariant under the twisted adjoint action of M, and this completes the proof. □

We define elements $E_{i,j}^{(r)} \in U(\mathfrak{p})$ for $1 \leq i < j \leq n$ and $r > s_{i,j}$ from the expressions for $E_i^{(r)} \in U(\mathfrak{p})$ given in (4.3) and the recursive formula in (3.16); we define $F_{i,j}^{(r)} \in U(\mathfrak{p})$ similarly. From these definitions and Lemma 4.1, we have that these $E_{i,j}^{(r)}$ and $F_{i,j}^{(r)}$ are actually elements of $U(\mathfrak{g}, e)$. For the next lemma we recall the basis for $\mathfrak{g}^e$ from Lemma 2.1, and the notation for elements $\boldsymbol{e}^{\boldsymbol{a}}$ in $U(\mathfrak{p})$ given in (2.17).

Lemma 4.2.

(a) *For $1 \leq i \leq n$, and $1 \leq r \leq p_i$, we have $D_i^{(r)} = (-1)^{r-1}c_{i,i}^{(r)} + u$, where u is a linear combination of terms $\boldsymbol{e}^{\boldsymbol{a}}$ satisfying either $|\boldsymbol{a}|_e = r$ and $|\boldsymbol{a}| > 1$, or $|\boldsymbol{a}|_e < r$.*

(b) *For $1 \leq i < j \leq n$, and $s_{i,j} < r \leq p_i + s_{i,j}$, we have $E_{i,j}^{(r)} = (-1)^{r-1}c_{i,j}^{(r)} + u$, where u is a linear combination of terms $\boldsymbol{e}^{\boldsymbol{a}}$ satisfying either $|\boldsymbol{a}|_e = r$ and $|\boldsymbol{a}| > 1$, or $|\boldsymbol{a}|_e < r$.*

(c) *For $1 \leq i < j \leq n$, and $s_{j,i} < r \leq p_i + s_{j,i}$, we have $F_{i,j}^{(r)} = (-1)^{r-1}c_{j,i}^{(r)} + u$, where u is a linear combination of terms $\boldsymbol{e}^{\boldsymbol{a}}$ satisfying either $|\boldsymbol{a}|_e = r$ and $|\boldsymbol{a}| > 1$, or $|\boldsymbol{a}|_e < r$.*

(d) *The monomials in*

$$\{D_i^{(r)} \mid 1 \le i \le n, 1 \le r \le p_i\} \cup \{E_{i,j}^{(r)} \mid 1 \le i < j \le n, s_{i,j} < r \le p_i + s_{i,j}\}$$
$$\cup \{F_{i,j}^{(r)} \mid 1 \le i < j \le n, s_{j,i} < r \le p_i + s_{j,i}\}$$

taken in any fixed order form a basis of $U(\mathfrak{g}, e)$.

Proof. We begin by proving (a). Consider the expression given for $D_i^{(r)}$ given by (4.1) and (4.2). We can verify that the terms for $s > 1$ are a linear combination of terms $\boldsymbol{e}^{\boldsymbol{a}}$ satisfying $|\boldsymbol{a}|_e = r$ and $|\boldsymbol{a}| > 1$, or $|\boldsymbol{a}|_e < r$. So we are left to show that the $s = 1$ part is precisely $(-1)^{r-1}c_{i,i}^{(r)}$, and this follows directly from the definitions.

The cases of (b) and (c) for $j = i+1$ are proved similarly to (a). Then using the definitions in (3.16) and (3.17) along with Lemma 2.1(b), and (2.19), we deduce the statement for all i and j.

Part (d) is now an immediate consequence of Lemma 2.1(a) and Proposition 2.5. □

We are now in a position to prove the main result of the section, showing that $Y_{n,l}(\sigma)$ is isomorphic to $U(\mathfrak{g}, e)$, which is a modular analogue of [BK1, Theorem 10.1]

Theorem 4.3. *The map from* $Y_{n,l}(\sigma)$ *to* $U(\mathfrak{g}, e)$ *determined by sending each element of* $Y_{n,l}(\sigma)$ *in*

$$\{D_i^{(r)} \mid 1 \le i \le n, r \ge 1\} \cup \{E_i^{(r)} \mid 1 \le i < n, r > s_{i,i+1}\} \cup \{F_i^{(r)} \mid 1 \le i < n, r > s_{i+1,i}\}$$

to the element of $U(\mathfrak{g}, e)$ *denoted by the same symbol defines an isomorphism*

$$Y_{n,l}(\sigma) \xrightarrow{\sim} U(\mathfrak{g}, e).$$

Proof. Consider the elements $D_i^{(r)}, E_i^{(r)}, F_i^{(r)}$ defined in $U(\mathfrak{p}_{\mathbb{C}})$ by formulas (4.1), (4.2), (4.3), (4.4). It follows from [BK1, Theorem 10.1] that these elements satisfy relations (3.2)–(3.13). Also they satisfy (3.14) and (3.15) as these follow, over $\mathbb{C}$ from [BK1, (2.14), (2.15)]. Since all of these relations have integral coefficients they hold in $U(\mathfrak{p}_{\mathbb{Z}})$, and thus also in $U(\mathfrak{p}) \cong U(\mathfrak{p}_{\mathbb{Z}}) \otimes \Bbbk$. In addition it is clear from the definition of $D_1^{(r)} \in U(\mathfrak{g}, e)$ given in (4.1) and (4.2), that $D_1^{(r)} = 0$ for $r > p_1$.

Hence, the map described in the statement does give a homomorphism $Y_{n,l}(\sigma) \to U(\mathfrak{g}, e)$. By Lemma 4.2(d), we see that this homomorphism is surjective, and using Lemma 3.1, we deduce that it is injective. □

As mentioned before Lemma 3.1, we are now able to deduce a PBW theorem for $Y_{n,l}(\sigma)$, as given in [BK1, Corollary 6.3]. This says that the monomials in the elements given in (3.19) taken in any fixed order form a basis of $Y_{n,l}(\sigma)$, and follows immediately from Lemma 4.2 and Theorem 4.3.

5. 1-dimensional modules for $U(\mathfrak{g}, e)$

We follow similar methods to those in [Br, Section 2] to classify the 1-dimensional modules for $U(\mathfrak{g}, e)$. Before stating this in Theorem 5.1 we require some notation, and we also use the generators and relations for $U(\mathfrak{g}, e)$ given by Theorem 4.3 to make some initial deductions about 1-dimensional modules for $U(\mathfrak{g}, e)$.

It follows from (3.2) and Lemma 4.2 that $\{D_i^{(r)} \mid i = 1, \dots, n, 1 \leq r \leq p_i\}$ generates a subalgebra of $U(\mathfrak{g}, e)$ isomorphic to a polynomial algebra in N variables; we denote this subalgebra by $U(\mathfrak{g}, e)^0$.

Let $A \in \mathrm{Tab}_{\Bbbk}(\pi)$. For $i = 1, \dots, n$, we write $a_{i,1}, \dots, a_{i,p_i}$ for the entries in the ith row of A from left to right. We define the 1-dimensional $U(\mathfrak{g}, e)^0$-module $\Bbbk_{\overline{A}}$ by saying that $D_i^{(r)}$ acts on $\Bbbk_A$ by $e_r(a_{i,1} + i, \dots, a_{i,p_i} + i)$; here $e_r(x_1, \dots, x_{p_i})$ is the rth elementary symmetric polynomial in the indeterminates $x_1, \dots, x_{p_i}$. It is clear that $\Bbbk_{\overline{A}}$ depends only on the row equivalence class of A. We note that given $b_1, \dots, b_{p_i} \in \Bbbk$, finding $c_{i,1}, \dots, c_{i,p_i}$ such that $e_r(c_1, \dots, c_{p_i}) = b_r$ for each r is equivalent to finding solutions of the polynomial $t^{p_i} - b_1 t^{p_i - 1} + \dots + (-1)^{p_i - 1} b_{p_i - 1} t + (-1)^{p_i} b_{p_i}$. Therefore, since $\Bbbk$ is algebraically closed, we see that any 1-dimensional $U(\mathfrak{g}, e)^0$-module is isomorphic to $\Bbbk_{\overline{A}}$ for some $A \in \mathrm{Tab}_{\Bbbk}(\pi)$. Thus we see that the restriction of any 1-dimensional $U(\mathfrak{g}, e)$-module to $U(\mathfrak{g}, e)^0$ is isomorphic to $\Bbbk_{\overline{A}}$ for some $A \in \mathrm{Tab}_{\Bbbk}(\pi)$, and that such A is defined up to row equivalence. Using the relations (3.4) and (3.5) for $r = 1$, we see that the generators $E_i^{(s)}$ and $F_i^{(s)}$ act as 0 on any 1-dimensional $U(\mathfrak{g}, e)$-module, for all i and s. If the action of $U(\mathfrak{g}, e)^0$ on $\Bbbk_{\overline{A}}$ can be extended to a $U(\mathfrak{g}, e)$-module, on which $E_i^{(s)}$ and $F_i^{(s)}$ act as 0, then we denote this module by $\widetilde{\Bbbk}_{\overline{A}}$. Our goal is thus to determine when $\widetilde{\Bbbk}_{\overline{A}}$ exists, and this is achieved in the following theorem.

Theorem 5.1. *Let $A \in \mathrm{Tab}_{\Bbbk}(\pi)$. There is a 1-dimensional $U(\mathfrak{g}, e)$-module $\widetilde{\Bbbk}_{\overline{A}}$, which extends the action of $U(\mathfrak{g}, e)^0$ on $\Bbbk_{\overline{A}}$ if and only if A is row equivalent to a column connected tableau.*

Proof. First let $A \in \mathrm{Tab}_{\Bbbk}(\pi)$ be column connected, with entries in the ith row labelled $a_{i,1}, \dots, a_{i,p_i}$ for $i = 1, \dots, n$. Recall the 1-dimensional $U(\mathfrak{h})$-module $\widetilde{\Bbbk}_A$ defined in §2.5; this can be inflated to a $U(\mathfrak{p})$ module, which we also denote by $\widetilde{\Bbbk}_A$. Consider the action of the explicit elements $D_i^{(r)} \in U(\mathfrak{p})$ given in (4.1) and (4.2) on the module $\widetilde{\Bbbk}_A$. The only summands in the expression for $D_i^{(r)}$ that do not act as zero on $\widetilde{\Bbbk}_A$ are those which are products $\tilde{e}_{i_1,j_1} \cdots \tilde{e}_{i_s,j_s}$ such that $i_1 = j_1, \dots, i_s = j_s$: terms of this form only occur for $s = r$ and their sum is precisely $e_r(\tilde{e}_{i_1,j_1}, \dots, \tilde{e}_{ir,j_r})$. The $\mathfrak{t}$-weight of $\widetilde{\Bbbk}_A$ is $\lambda_A - \widetilde{\rho}$, and we have $\widetilde{\rho} = \eta + \rho_{\mathfrak{h}}$ by (2.10). Thus we see that each $\tilde{e}_{i,i}$ acts on $\widetilde{\Bbbk}_A$ by $(\lambda_A + \rho_{\mathfrak{h}})(e_{i,i})$, because of the shift of η in the definition of $\tilde{e}_{i,j}$. Combining all of these observations shows that the action of $D_i^{(r)}$ on $\widetilde{\Bbbk}_A$ is given by $e_r(a_{i,1} + i, \dots, a_{i,p_i} + i)$. This proves that $\widetilde{\Bbbk}_{\overline{A}}$ exists, under the assumption that A is column connected.

We move on to prove that $\widetilde{\Bbbk}_{\overline{A}}$ exists only if A is column connected. To do this first note that the action $U(\mathfrak{g}, e)$ on any 1-dimensional factors to an action of the abelianization $U(\mathfrak{g}, e)^{\mathrm{ab}}$ of $U(\mathfrak{g}, e)$. By Lemma 3.2 and Theorem 4.3 we know that $U(\mathfrak{g}, e)^{\mathrm{ab}}$ is generated by the images of the l elements

$$\{D_i^{(r)} \mid i = 1, \dots, n, 0 < r \leq p_i - p_{i-1}\}.$$

So any 1-dimensional $U(\mathfrak{g}, e)$-module is determined uniquely by the action of these elements. Thus to show that any one dimensional $U(\mathfrak{g}, e)$-module is of the form $\widetilde{\Bbbk}_{\overline{A}}$ for some column connected $A \in \mathrm{Tab}_{\Bbbk}(\pi)$, it suffices to show that for any set

$$\{a_i^{(r)} \mid i = 1, \dots, n, 0 < r \leq p_i - p_{i-1}\}.$$

where $a_i^{(r)} \in \Bbbk$, there is a column connected $A \in \mathrm{Tab}_{\Bbbk}(\pi)$ such that the action of $D_i^{(r)}$ on $\widetilde{\Bbbk}_{\overline{A}}$ is given by $a_i^{(r)}$ for $i = 1, \dots, n$ and $0 \leq r \leq p_i - p_{i-1}$. This is proved "over the complex numbers" at the end of [Br, Section 2], and depends crucially on [Br, Lemma 2.6]. It can be observed that the proof of this lemma is also valid over an algebraically closed field of characteristic p. From this we can deduce that all 1-dimensional $U(\mathfrak{g}, e)$-modules are of the form $\widetilde{\mathbb{k}}_{\overline{A}}$ for some column connected $A \in \mathrm{Tab}_{\mathbb{k}}(\pi)$. □

As mentioned after the statement of Lemma 3.2, the abelianization of $Y_{n,l}(\sigma)$ is actually a polynomial algebra on the generators given in that lemma. This can now be deduced immediately from the proof of Theorem 5.1, which shows that there are 1-dimensional modules for $Y_{n,l}(\sigma)$ on which these generators can act by arbitrary elements of $\mathbb{k}$.

6. 1-dimensional modules for $U_0(\mathfrak{g}, e)$

From Theorem 5.1 we have a classification of 1-dimensional $U(\mathfrak{g}, e)$-modules given by the modules $\widetilde{\mathbb{k}}_{\overline{A}}$ for $A \in \mathrm{Tab}_{\mathbb{k}}(\pi)$ ranging over a set of representatives of row equivalence classes of column connected tableaux. Our next theorem determines for which of these 1-dimensional modules the action of $U(\mathfrak{g}, e)$ factors through the quotient $U(\mathfrak{g}, e) \twoheadrightarrow U_0(\mathfrak{g}, e)$ to give a 1-dimensional $U_0(\mathfrak{g}, e)$-module. Therefore, we obtain a classification of the 1-dimensional $U_0(\mathfrak{g}, e)$-modules.

Theorem 6.1. *Let $A \in \mathrm{Tab}_{\mathbb{k}}(\pi)$ be column connected. Then $\widetilde{\mathbb{k}}_{\overline{A}}$ factors to a module for $U_0(\mathfrak{g}, e)$ if and only if $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$.*

Proof. Let $A \in \mathrm{Tab}_{\mathbb{k}}(\pi)$ be column connected with entries labelled $a_1, \dots, a_N$ as usual. We see that $\widetilde{\mathbb{k}}_A$ factors through $U_{\psi_A}(\mathfrak{p})$, where $\psi_A \in \mathfrak{p}^*$ is defined by $\psi_A(e_{i,j}) = 0$ for $i \neq j$, and $\psi_A(e_{i,i}) = a_i^p - a_i$. Therefore, $\widetilde{\mathbb{k}}_{\overline{A}}$ is a module for the reduced W-algebra $U_{\psi_A}(\mathfrak{g}, e)$ as defined in §2.6. From the discussion after Lemma 2.3, we see that $\widetilde{\mathbb{k}}_{\overline{A}}$ factors to a module for $U_0(\mathfrak{g}, e)$ if and only if 0 and ψ_A are conjugate under the twisted M-action.

We next show that ψ_A is conjugate to 0 under the twisted M-action only if $\psi_A = 0$. To do this note that under the identification $\mathfrak{p}^* \cong e + \mathfrak{m}^\perp$ we have that 0 corresponds to e and ψ_A corresponds to an element $e+\mathrm{diag}(a_1^p-a_1, \dots, a_N^p-a_N)$, where we recall that $\mathrm{diag}(d_1, \dots, d_N)$ denotes the diagonal matrix with ith entry d_i. We have $e + \mathrm{diag}(a_1^p - a_1, \dots, a_N^p - a_N) \in \mathfrak{t}$ is nilpotent only if $\mathrm{diag}(a_1^p - a_1, \dots, a_N^p - a_N) = 0$. Therefore, e is not in the same M-orbit as $e + \mathrm{diag}(a_1^p - a_1, \dots, a_N^p - a_N)$ unless $a_i^p - a_i = 0$ for all i.

Hence, we deduce that $\widetilde{\mathbb{k}}_{\overline{A}}$ factors to a module for $U_0(\mathfrak{g}, e)$ if only if $a_i^p - a_i = 0$ for all i. This is case if and only if $a_i \in \mathbb{F}_p$ for all i, so that $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$. □

7. Minimal dimensional modules for $U_\chi(\mathfrak{g})$

Armed with Premet's equivalence (Theorem 2.4), Theorem 2.2 and Theorem 6.1, we are ready to prove our main theorem.

Proof of Theorem 1.1. Let c_π the number of row equivalence classes in $\mathrm{Tab}_{\mathbb{F}_p}(\pi)$ containing a column connected tableau. By Theorem 6.1, we know that the number of isomorphism classes of 1-dimensional modules for $U_0(\mathfrak{g}, e)$-modules is c_π. Thus by Theorem 2.4 the number of minimal dimensional $U_\chi(\mathfrak{g})$-modules is c_π.

Given column connected $A \in \mathrm{Tab}_{\mathbb{F}_p}(\pi)$, we have that $L_\chi(A)$ is p^{d_χ}-dimensional by Theorem 2.2. Also up to isomorphism $L_\chi(A)$ depends only on the row equivalence class of A.

Therefore, the modules $L_\chi(A)$ for A ranging over a set of row equivalence classes of column connected tableaux in $\mathrm{Tab}_{\mathbb{F}_p}(\pi)$ give all c_π isomorphism classes of minimal dimensional modules. □

Now that Theorem 1.1 is proved, Corollary 1.2 follows, as explained in the introduction.

Remark 7.1. It is interesting to know the bijection given by Premet's equivalence between the sets of isomorphism classes of the 1-dimensional $U_0(\mathfrak{g},e)$-modules $\widetilde{\mathbb{k}}_{\overline{A}}$ and those of the minimal dimensional $U_\chi(\mathfrak{g})$-modules $L_\chi(A)$, as A ranges over a set of representatives of row equivalence classes of column connected tableau in $\mathrm{Tab}_{\mathbb{F}_p}(\pi)$. It turns out that this bijection sends $\widetilde{\mathbb{k}}_{\overline{A}}$ to $L_\chi(A)$ and we briefly outline some steps that can be used to verify this.

Use the fact that $L_\chi(A) \cong N_\chi(A) = U_\chi(\mathfrak{g}) \otimes_{U_0(\mathfrak{p})} \overline{\mathbb{k}}_A$ as is given in Theorem 2.2. Then show that $N_\chi(A)^{\mathfrak{m}_\chi} \cong \widetilde{\mathbb{k}}_{\overline{A}}$ using the following arguments; we recall here that $N_\chi(A)^{\mathfrak{m}_\chi}$ is defined in (2.16).

Consider the dual $N_\chi(A)^*$ viewed as a right module for $U_\chi(\mathfrak{g})$. We observe that $\lambda_A - \overline{\rho} - \beta = \lambda - \tilde{\rho}$ is the weight of a 1-dimensional right $U_0(\mathfrak{h})$-module, which we denote by $\mathbb{k}_{\lambda_A - \overline{\rho} - \beta}$. Using that any $U_\chi(\mathfrak{g})$-module is free as a $U_\chi(\mathfrak{m})$-module, it can be proved that $N_\chi(A)^* \cong \mathbb{k}_{\lambda_A - \overline{\rho} - \beta} \otimes_{U_0(\mathfrak{p})} U_\chi(\mathfrak{g})$. Note that it is more natural to consider weight $\lambda_A - \overline{\rho} + (p-1)\beta$ to prove this isomorphism, and use that this is the $\mathfrak{t}$-weight of $\prod_{\alpha \in \Phi(-)} e_\alpha^{p-1} \overline{1}_A \in N_\chi(A)$, where $\overline{1}_A$ is the generator of $\overline{\mathbb{k}}_A$.

Next consider the *Whittaker coinvariants* of $N_\chi(A)^*$. This is defined by $N_\chi(A)^*/N_\chi(A)^*\mathfrak{m}_\chi$, and is a right module for $U(\mathfrak{g},e)$. It is quite straightforward to show that the Whittaker coinvariants of $\mathbb{k}_{\lambda_A - \overline{\rho} - \beta} \otimes_{U_0(\mathfrak{p})} U_\chi(\mathfrak{g})$ is isomorphic to the restriction of right $U_0(\mathfrak{p})$-module $\mathbb{k}_{\lambda_A - \overline{\rho} - \beta}$ to $U(\mathfrak{g},e)$, so the same is true for the Whittaker coinvariants of $N_\chi(A)^*$. Standard arguments show that $N_\chi(A)^{\mathfrak{m}_\chi} \cong (N_\chi(A)^*/\mathfrak{m}_\chi N_\chi(A)^*)^*$. Then it can be deduced that $N_\chi(A)^{\mathfrak{m}_\chi}$ is isomorphic to the restriction of the left $U_0(\mathfrak{p})$-module $\mathbb{k}_{\lambda_A - \overline{\rho} - \beta}$ to $U_0(\mathfrak{g},e)$.

It just remains to just use the fact that $\overline{\rho} + \beta = \widetilde{\rho}$ to deduce that $N_\chi(A)^{\mathfrak{m}_\chi} \cong \widetilde{\mathbb{k}}_{\overline{A}}$.

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