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EXISTENCE OF POSITIVE SOLUTIONS FOR A COUPLED SYSTEM OF HIGHER ORDER FRACTIONAL BOUNDARY VALUE PROBLEMS

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Abstract. The aim of this paper is to establish the existence of at least one positive solution for a coupled system of higher order two-point fractional order boundary value problems under suitable conditions. The approach is based on the Guo–Krasnosel'skii fixed point theorem.

Keywords: fractional derivative, coupled system, Green's function, boundary value problem, positive solution.

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1. Introduction

Fractional calculus is the field of mathematical analysis which unifies the theories of integration and differentiation of any arbitrary real order. In describing the properties of various real materials, the derivatives and integrals of non-integer order are very much suitable. The fractional order models are more general and adequate than integer order models. The study of fractional order differential equations has emerged as an important area of mathematics. It has wide range of applications in various fields of science and engineering such as physics, mechanics, control systems, flow in porous media and viscoelasticity.

Recently, much interest has been created in establishing positive solutions for boundary value problems associated with ordinary and fractional order differential equations. To mention the related papers along these lines, we refer to Erbe and Wang [6], Davis, Henderson, Prasad, and Yin [5] for ordinary differential equations, Henderson and Ntouyas [8, 9], Henderson, Ntouyas, and Purnaras [10] for systems of ordinary differential equations, Bai and Lü [3], Kauffman and Mboumi [11], Benchohra, Henderson, Ntoyuas, and Ouahab [4], Khan, Rehman, and Henderson [12], Prasad and Krushna [16, 17], Prasad, Krushna, and Sreedhar [18] for fractional order differential equations.

In this paper, we establish the existence of at least one positive solution by determining the values of λ for a coupled system of fractional order differential equations,

$$D_{0^+}^{q_1} u(t) + \lambda\, g_1(t) f_1\big(v(t)\big) = 0,\ \ t \in (0, b), \tag{1}$$

$$D_{0^+}^{q_2} v(t) + \lambda\, g_2(t) f_2\big(u(t)\big) = 0,\ \ t \in (0, b), \tag{2}$$

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satisfying two-point boundary conditions,

$$u^{(k)}(0) = 0,\ k = 1, 2, \cdots, n-1,\ u(b) - mD_{0^+}^{q_3} u(b) = 0, \tag{3}$$

$$v^{(k)}(0) = 0,\ k = 1, 2, \cdots, n-1,\ v(b) - mD_{0^+}^{q_3} v(b) = 0, \tag{4}$$

where $q_1, q_2 \in (n-1, n], n \geq 2, \lambda > 0, q_3 \in (0, 1]$ and $b > 0$. $f_i, g_i,\ i = 1, 2$ are given functions, m is a positive real number, $D_{0^+}^{q_j}, j = 1, 2, 3$ are the standard Riemann–Liouville fractional order derivatives.

We assume that the following conditions hold throughout the paper:

$(A1)$ $f_i : \mathbb{R}^+ \to \mathbb{R}^+, i = 1, 2$ are continuous,

$(A2)$ $g_i : [0, b] \to \mathbb{R}^+, i = 1, 2$ are continuous and does not vanish identically on any closed subinterval of $[0, b]$,

$(A3)$ $m\Gamma(q_1)b^{-q_3} > \Gamma(q_1 - q_3)$,

$(A4)$ each of

$$f_{i0} = \lim_{x \to 0^+} \frac{f_i(x)}{x} \text{ and } f_{i\infty} = \lim_{x \to \infty} \frac{f_i(x)}{x},\ i = 1, 2$$

exist as positive real numbers.

By a positive solution of the system of fractional order boundary value problem (1)-(4), we mean $\Big(u(t), v(t)\Big) \in \Big(C^n[0, b] \times C^n[0, b]\Big)$ satisfying (1)-(4) with $u(t) \geq 0$, $v(t) \geq 0$, for all $t \in [0, b]$ and $(u, v) \neq (0, 0)$.

The rest of the paper is organized as follows. In Section 2, we construct the Green functions for the associated linear fractional order boundary value problems and estimate the bounds for these Green functions. In Section 3, we develop criteria for the existence of at least one positive solution for a coupled system of fractional order boundary value problem (1)-(4), by applying the Guo–Krasnosel'skii fixed point theorem. In Section 4, as an application, we demonstrate our results with an example.

2. Green Functions and Bounds

In this section, we construct the Green functions for the associated boundary value problems and estimate the bounds for these Green functions, which are needed to establish the main results.

Lemma 2.1. *Let* $\Delta_1 = \big[m\Gamma(q_1)b^{-q_3} - \Gamma(q_1 - q_3)\big]b^{q_1-1}\Gamma(q_1) \neq 0$. *If* $h(t) \in C[0, b]$, *then the fractional order differential equations*

$$D_{0^+}^{q_1} u(t) + h(t) = 0,\ t \in [0, b], \tag{5}$$

satisfying the boundary conditions (3) *has a unique solution*

$$u(t) = \int_0^b G_1(t, s)h(s)ds,$$

where $G_1(t, s)$ *is the Green's function for the problem* (5), (3) *and is given by*

$$G_1(t, s) = \begin{cases} G_{11}(t, s), & 0 \leq t \leq s \leq b, \\ G_{12}(t, s), & 0 \leq s \leq t \leq b, \end{cases} \tag{6}$$

$$G_{11}(t, s) = \frac{1}{\Delta_1}\Big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1 - q_3)\Big]\Big[(b-s)t\Big]^{q_1-1},$$

$$G_{12}(t, s) = G_{11}(t, s) - \frac{(t-s)^{q_1-1}}{\Gamma(q_1)}.$$

Proof. Let $u \in C^n[0,b]$ be the solution of fractional order boundary value problem given by (5) and (3). An equivalent integral equation for (5) is given by

$$u(t) = \frac{-1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} h(s) ds + c_1 t^{q_1-1} + c_2 t^{q_1-2} + \cdots + c_n t^{q_1-n}.$$

From $u^{(k)}(0) = 0,\ 1 \le k \le n-1$, one can determine $c_n = c_{n-1} = c_{n-2} = \cdots = c_2 = 0$. Then

$$\left.\begin{aligned} u(t) &= \frac{-1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} h(s) ds + c_1 t^{q_1-1}, \\ D_{0^+}^{q_3} u(t) &= -\int_0^t \frac{(t-s)^{q_1-q_3-1}}{\Gamma(q_1-q_3)} h(s) ds + \frac{c_1 \Gamma(q_1)}{\Gamma(q_1-q_3)} t^{q_1-q_3-1}. \end{aligned}\right\} \tag{7}$$

By the condition $u(b) - mD_{a^+}^{q_3} u(b) = 0$ and (7), we obtain

$$c_1 = \frac{1}{\Delta_1} \int_0^b \Big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3) \Big] (b-s)^{q_1-1} h(s) ds.$$

Thus the unique solution of the fractional order boundary value problem given by (5) and (3) is

$$\begin{aligned} u(t) =& \frac{-1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} h(s) ds + \\ & \frac{t^{q_1-1}}{\Delta_1} \int_0^b \Big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3) \Big] (b-s)^{q_1-1} h(s) ds \\ =& \int_0^t \left[\frac{\big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3)\big]\big[t(b-s)\big]^{q_1-1}}{\Delta_1} - \frac{(t-s)^{q_1-1}}{\Gamma(q_1)} \right] h(s) ds \\ & + \frac{t^{q_1-1}}{\Delta_1} \int_t^b \Big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3) \Big] (b-s)^{q_1-1} h(s) ds. \end{aligned}$$

□

Lemma 2.2. *The Green's function $G_1(t,s)$ given by* (6) *is positive, for all $t, s \in (0,b)$.*

Proof. Consider the Green's function $G_1(t,s)$ given by (6). By the condition $(A3)$, we establish the positivity of the Green's function $G_1(t,s)$, for all $(t,s) \in (0,b) \times (0,b)$. □

Lemma 2.3. *The Green's function $G_1(t,s)$ is given in* (6) *satisfies the following inequalities*

$$G_1(t,s) \le G_1(b,s), \ \textit{for all}\ (t,s) \in [0,b] \times [0,b], \tag{8}$$

$$G_1(t,s) \ge \left(\frac{b}{4}\right)^{q_1-1} G_1(b,s), \ \textit{for all}\ (t,s) \in I \times [0,b], \tag{9}$$

where $I = \left[\frac{b}{4}, \frac{3b}{4}\right]$.

Proof. Consider the Green's function $G_1(t,s)$ given by (6). Then

$$G_1(b,s) = \frac{1}{\Delta_1} \Big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3) \Big] (b-s)^{q_1-1} - \frac{1}{\Gamma(q_1)} (b-s)^{q_1-1}.$$

Let $0 \le t \le s \le b$. Then

$$\frac{\partial G_1(t,s)}{\partial t} = \frac{(q_1-1)}{\Delta_1} \Big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3) \Big] (b-s)^{q_1-1} t^{q_1-2} \ge 0.$$

Let $0 \leq s \leq t \leq b$. Then

$$\begin{aligned}
&\frac{\partial G_1(t,s)}{\partial t}\\
&= \frac{(q_1-1)}{\Delta_1}\left[\Big(m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3)\Big)(b-s)^{q_1-1}t^{q_1-2} - \frac{(t-s)^{q_1-2}}{\Gamma(q_1)}\right]\\
&\geq \frac{(q_1-1)}{\Delta_1}\left[\Big(m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3)\Big)(b-s)^{q_1-1}t^{q_1-2} - \frac{(t-ts)^{q_1-2}}{\Gamma(q_1)}\right]\\
&= \frac{(q_1-1)t^{q_1-2}}{\Delta_1}\left[\Big(m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3)\Big)(b-s)^{q_1-1} - \frac{(1-s)^{q_1-2}}{\Gamma(q_1)}\right]\\
&\geq 0.
\end{aligned}$$

Therefore, $G_1(t,s)$ is increasing in t, which implies $G_1(t,s) \leq G_1(b,s)$.

Let $0 \leq t \leq s \leq b$ and $t \in I$. Then

$$\begin{aligned}
G_1(t,s) =&\frac{1}{\Delta_1}\Big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3)\Big]\Big[(b-s)t\Big]^{q_1-1}\\
\geq&\frac{1}{\Delta_1}\Big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3)\Big]\Big[(b-s)t\Big]^{q_1-1} - \frac{(t-ts)^{q_1-1}}{\Gamma(q_1)}\\
\geq&t^{q_1-1}\left[\Big(m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3)\Big)\frac{(b-s)^{q_1-1}}{\Delta_1} - \frac{(b-s)^{q_1-1}}{\Gamma(q_1)}\right]\\
=&t^{q_1-1}G_1(b,s) \geq \left(\frac{b}{4}\right)^{q_1-1}G_1(b,s).
\end{aligned}$$

Let $0 \leq s \leq t \leq b$ and $t \in I$. Then

$$\begin{aligned}
G_1(t,s) =&\frac{1}{\Delta_1}\Big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3)\Big]\Big[(b-s)t\Big]^{q_1-1} - \frac{(t-s)^{q_1-1}}{\Gamma(q_1)}\\
\geq&\frac{1}{\Delta_1}\Big[m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3)\Big]\Big[(b-s)t\Big]^{q_1-1} - \frac{(t-ts)^{q_1-1}}{\Gamma(q_1)}\\
\geq&t^{q_1-1}\left[\Big(m\Gamma(q_1)(b-s)^{-q_3} - \Gamma(q_1-q_3)\Big)\frac{(b-s)^{q_1-1}}{\Delta_1} - \frac{(b-s)^{q_1-1}}{\Gamma(q_1)}\right]\\
=&t^{q_1-1}G_1(b,s) \geq \left(\frac{b}{4}\right)^{q_1-1}G_1(b,s).
\end{aligned}$$

□

Lemma 2.4. *Let* $\Delta_2 = \big[m\Gamma(q_2)b^{-q_3} - \Gamma(q_2-q_3)\big]b^{q_2-1}\Gamma(q_2) \neq 0$. *If* $g(t) \in C[0,b]$, *then the fractional order differential equation*

$$D_{0^+}^{q_2}v(t) + g(t) = 0,\ \ t \in [0,b], \tag{10}$$

satisfying the boundary conditions (4) *has a unique solution*

$$v(t) = \int_0^b G_2(t,s)g(s)ds,$$

where $G_2(t,s)$ *is the Green's function for the problem* (10), (4) *and is given by*

$$G_2(t,s) = \begin{cases} G_{21}(t,s), & 0 \leq t \leq s \leq b,\\ G_{22}(t,s), & 0 \leq s \leq t \leq b,\end{cases} \tag{11}$$

$$G_{21}(t,s) = \frac{1}{\Delta_2}\Big[m\Gamma(q_2)(b-s)^{-q_3} - \Gamma(q_2-q_3)\Big]\Big[(b-s)t\Big]^{q_2-1},$$
$$G_{22}(t,s) = G_{21}(t,s) - \frac{(t-s)^{q_2-1}}{\Gamma(q_2)}.$$

Proof. Proof is similar to Lemma 2.1. □

Lemma 2.5. *The Green's function $G_2(t,s)$ given by* (11) *is positive, for all $t, s \in (0,b)$.*

Proof. Proof is similar to Lemma 2.2. □

Lemma 2.6. *The Green's function $G_2(t,s)$ is given in* (11) *satisfies the following inequalities*

$$G_2(t,s) \le G_2(b,s), \ \ for \ all \ (t,s) \in [0,b]\times[0,b], \tag{12}$$

$$G_2(t,s) \ge \left(\frac{b}{4}\right)^{q_2-1} G_2(b,s), \ \ for \ all \ (t,s) \in I\times[0,b], \tag{13}$$

where $I = \left[\frac{b}{4}, \frac{3b}{4}\right]$.

Proof. Proof is similar to Lemma 2.3. □

An order pair $\big(u(t), v(t)\big)$ is a solution of the fractional order boundary value problem (1)-(4) if and only if $u \in C^n[0,b]$ satisfies the following equation

$$u(t) = \lambda \int_0^b G_1(t,s)g_1(s)f_1\big(v(s)\big)ds, \tag{14}$$

where

$$v(t) = \lambda \int_0^b G_2(t,s)g_2(s)f_2\big(u(s)\big)ds.$$

To establish the existence of at least one positive solution, we will employ the following fixed point theorem due to Guo–Krasnosel'skii [7, 14].

Theorem 2.1. [7, 14] *Let X be a Banach Space, $P \subseteq X$ be a cone and suppose that Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Suppose further that $T : P\cap(\overline{\Omega}_2\backslash\Omega_1) \to P$ is completely continuous operator such that either*

(i) $\| Tu \| \le \| u \|,\ u \in P\cap\partial\Omega_1$ *and* $\| Tu \| \ge \| u \|,\ u \in P\cap\partial\Omega_2$, *or*
(ii) $\| Tu \| \ge \| u \|,\ u \in P\cap\partial\Omega_1$ *and* $\| Tu \| \le \| u \|,\ u \in P\cap\partial\Omega_2$ *holds.*

Then T has a fixed point in $P\cap(\overline{\Omega}_2\backslash\Omega_1)$.

3. Existence of at least one positive solution

In this section, we establish the existence of at least one positive solution for a coupled system of fractional order boundary value problem (1)-(4), by applying the Guo–Krasnosel'skii fixed point theorem.

Let $X = \Big\{u : u \in C[0,b]\Big\}$ be the Banach space equipped with the norm,

$$\|u\| = \max_{t\in[0,b]} |u(t)|.$$

Define a cone $P \subset X$ by

$$P = \Big\{u : u(t) \ge 0, t \in [0,b] \text{ and } \min_{t\in I} u(t) \ge \xi\|u\|\Big\},$$

where

$$\xi = \min\left\{\left(\frac{b}{4}\right)^{q_1-1}, \left(\frac{b}{4}\right)^{q_2-1}\right\}. \tag{15}$$

Now we define an integral operator $T : P \to X$, for $u \in P$, by

$$Tu(t) = \lambda \int_0^b G_1(t,s)g_1(s)f_1\bigg(\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr\bigg)ds \tag{16}$$

We notice from $(A1), (A2)$ and the Lemma 2.2 that, for $u \in P, Tu(t) \geq 0$ on $[0, b]$. Also for $u \in P$, we have from the Lemma 2.3, that

$$Tu(t) \leq \lambda \int_0^b G_1(b,s)g_1(s)f_1\bigg(\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr\bigg)ds$$

so that

$$\|Tu(t)\| \leq \lambda \int_0^b G_1(b,s)g_1(s)f_1\bigg(\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr\bigg)ds. \tag{17}$$

Next if $u \in P$, we have from the Lemma 2.3 and (17) that

$$\begin{aligned}\min_{t\in I} Tu(t) &= \min_{t\in I} \lambda \int_0^b G_1(t,s)g_1(s)f_1\bigg(\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr\bigg)ds \\ &\geq \xi\lambda \int_0^b G_1(b,s)g_1(s)f_1\bigg(\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr\bigg)ds \\ &\geq \xi\|Tu\|.\end{aligned}$$

Therefore $\min_{t\in I} Tu(t) \geq \xi\|Tu\|$. Hence $Tu \in P$ and so $T : P \to P$. Further the operator T is a completely continuous by an application of the Arzela–Ascoli's theorem.

For our results, we define positive constants ω and ρ, by

$$\omega = \max\left\{\int_0^b G_1(b,s)g_1(s)ds, \int_0^b G_2(b,s)g_2(s)ds\right\}$$

and

$$\rho = \min\left\{\int_{s\in I} \xi G_1(b,s)g_1(s)ds, \int_{s\in I} \xi G_2(b,s)g_2(s)ds\right\},$$

where ξ is given by (15).

Theorem 3.1. *Assume that the conditions* $(A1)$-$(A4)$ *hold and if*

$$\frac{1}{\rho\min\{f_{1\infty}, f_{2\infty}\}} < \lambda < \frac{1}{\omega\max\{f_{10}, f_{20}\}}. \tag{18}$$

Then the coupled system of fractional order boundary value problem given by (1)-(4) *has at least one positive solution.*

Proof. Let λ be given as in (19). Now $\epsilon > 0$ be chosen such that

$$\frac{1}{\rho\min\{f_{1\infty}-\epsilon, f_{2\infty}-\epsilon\}} \leq \lambda \leq \frac{1}{\omega\max\{f_{10}+\epsilon, f_{20}+\epsilon\}}. \tag{19}$$

By the condition $(A4)$ of f_{10} and f_{20}, there exists a constant $H_1 > 0$ such that

$$f_1(x) \leq (f_{10}+\epsilon)x, f_2(x) \leq (f_{20}+\epsilon)x, \text{ for } 0 \leq x \leq H_1. \tag{20}$$

Let $u \in P$ with $\|u\| = H_1$. First we have from (20) and the choice of ϵ,

$$\begin{aligned}\lambda \int_0^b G_2(s,r)g_2(r)f_2(u(r))dr \leq& \lambda \int_0^b G_2(b,r)g_2(r)\big(f_{20}+\epsilon\big)u(r)dr \\ \leq& \lambda \int_0^b G_2(b,r)g_2(r)dr\big(f_{20}+\epsilon\big)\|u\| \\ \leq& \|u\| = H_1.\end{aligned}$$

Consequently we next from (20) and the choice of ϵ, for $0 \leq t \leq b$,

$$\begin{aligned}Tu(t) =& \lambda \int_0^b G_1(t,s)g_1(s)f_1\Big(\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr\Big)ds \\ \leq& \lambda \int_0^b G_1(b,s)g_1(s)\big(f_{10}+\epsilon\big)\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)drds \\ \leq& \lambda\omega\big(f_{10}+\epsilon\big)H_1 \\ \leq& H_1 = \|u\|.\end{aligned}$$

Therefore $\|Tu\| \leq \|u\|$. If we set $\Omega_1 = \Big\{x \in B : \|x\| < H_1\Big\}$. Then

$$\|Tu\| \leq \|u\|, \text{ for } u \in P \cap \partial\Omega_1. \tag{21}$$

By the condition $(A4)$ of $f_{1\infty}, f_{2\infty}$, there exists a constant $\overline{H}_2 > 0$ such that

$$f_1(x) \geq (f_{1\infty} - \epsilon)x \text{ and } f_2(x) \geq (f_{2\infty} - \epsilon)x, \text{ for } x \geq \overline{H}_2. \tag{22}$$

Let $H_2 = \max\left\{2H_1, \dfrac{\overline{H}_2}{\xi}\right\}$. Then for $u \in P$ and $\|u\| = H_2$,

$$\min_{t\in I} u(t) \geq \xi\|u\| \geq \overline{H}_2.$$

Consequently from (22) and the choice of ϵ, we have

$$\begin{aligned}\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr \geq& \lambda \int_{r\in I} \xi G_2(b,r)g_2(r)\big(f_{2\infty}-\epsilon\big)u(r)dr \\ \geq& \lambda \int_{r\in I} \xi G_2(b,r)g_2(r)dr\big(f_{2\infty}-\epsilon\big)\|u\| \\ \geq& \|u\| = H_2.\end{aligned}$$

We have from (16) and the choice of ϵ,

$$\begin{aligned}Tu(t) =& \lambda \int_0^b G_1(t,s)g_1(s)f_1\Bigg(\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr\Bigg)ds \\ \geq& \lambda \int_{s\in I} \xi G_1(b,s)g_1(s)\big(f_{1\infty}-\epsilon\big)\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)drds \\ \geq& \lambda \int_{s\in I} \xi G_1(b,s)g_1(s)\big(f_{1\infty}-\epsilon\big)H_2 ds \\ \geq& \lambda\rho\big(f_{1\infty}-\epsilon\big)H_2 \\ \geq& H_2 = \|u\|.\end{aligned}$$

Therefore $\|Tu\| \geq \|u\|$. If we set $\Omega_2 = \Big\{x \in B : \|x\| < H_2\Big\}$, then

$$\|Tu\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_2. \tag{23}$$

From (21) and (23) we observe that the operator T satisfies the conditions stated in Theorem 2.1. Hence T has a fixed point. And this implies that the coupled system of fractional order boundary value problem given by (1)-(4) has at least one positive solution. $\square$

Theorem 3.2. *Assume that the conditions (A1)-(A4) hold and if*

$$\frac{1}{\rho\min\{f_{10}, f_{20}\}} < \lambda < \frac{1}{\omega\max\{f_{1\infty}, f_{2\infty}\}}. \tag{24}$$

Then the coupled system of fractional order boundary value problem given by (1)-(4) has at least one positive solution.

Proof. Let λ be given as in (24). Now $\epsilon > 0$ be chosen such that

$$\frac{1}{\rho\min\{f_{10}-\epsilon, f_{20}-\epsilon\}} \leq \lambda \leq \frac{1}{\omega\max\{f_{1\infty}+\epsilon, f_{2\infty}+\epsilon\}}.$$

By the condition $(A4)$ of f_{10} and f_{20}, there exists an $H_1 > 0$ such that

$$f_1(x) \leq (f_{10}-\epsilon)x, f_2(x) \leq (f_{20}-\epsilon)x, \text{ for } 0 < x < H_1. \tag{25}$$

Also from the condition $(A4)$ of f_{20} it follows that $f_{20}(0) = 0$ and there exists $0 < H_2 < H_1$ such that

$$\lambda f_2(x) \leq \frac{H_1}{\int_0^b G_2(b,s)g_2(s)ds}, \ 0 \leq x \leq H_2.$$

Choosing $u \in P$ and $\|u\| = H_2$, we have

$$\begin{aligned}\lambda\int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr \leq& \lambda\int_0^b G_2(b,r)g_2(r)f_2\big(u(r)\big)dr\\ \leq&\frac{\int_0^b G_2(b,r)g_2(r)H_1dr}{\int_0^b G_2(b,r)g_2(r)dr}\\ \leq& H_1.\end{aligned}$$

Then

$$\begin{aligned}Tu(t) =&\lambda\int_0^b G_1(t,s)g_1(s)f_1\Big(\lambda\int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr\Big)ds\\ \geq&\lambda\int_{s\in I}\xi G_1(b,s)g_1(s)\big(f_{10}-\epsilon\big)\lambda\int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)drds\\ \geq&\lambda\int_{s\in I}\xi G_1(b,s)g_1(s)\big(f_{10}-\epsilon\big)\lambda\int_{r\in I}\xi G_2(b,r)g_2(r)\big(f_{20}-\epsilon\big)u(r)drds\\ \geq&\lambda\int_{s\in I}\xi G_1(b,s)g_1(s)\big(f_{10}-\epsilon\big)\lambda\int_{r\in I}\xi G_2(b,r)g_2(r)\big(f_{20}-\epsilon\big)\xi\|u\|drds\\ \geq&\lambda\int_{s\in I}\xi G_1(b,s)g_1(s)\big(f_{10}-\epsilon\big)\lambda\rho\big(f_{20}-\epsilon\big)\xi\|u\|ds\\ \geq&\lambda\int_{s\in I}\xi G_1(b,s)g_1(s)(f_{10}-\epsilon)\xi\|u\|ds\\ \geq&\lambda\rho\big(f_{10}-\epsilon\big)\xi\|u\|\\ =&\xi\|u\| \geq \|u\|.\end{aligned}$$

Hence $\|Tu\| \geq \|u\|$. If we set $\Omega_1 = \Big\{x \in B : \|x\| < H_2\Big\}$, then

$$\|Tu\| \geq \|u\|, \text{ for } u \in P \cap \partial\Omega_1. \tag{26}$$

Now we establish in two cases.

Case (i)**:** f_2 is bounded. There exists a constant $\mathcal{N} > 0$ such that

$$f_2(x) \leq \mathcal{N}, \text{ for } x \in (0, \infty).$$

Then, for $0 \leq s \leq b$ and $u \in P$,

$$\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr \leq \mathcal{N}\lambda \int_0^b G_2(b,r)g_2(r)dr.$$

Let $\mathcal{M} = \max\left\{f(x) : 0 \leq x \leq \mathcal{N}\lambda \int_0^b G_2(b,r)g_2(r)dr\right\}$, and let

$$H_3 > \max\left\{2H_2, \mathcal{M}\lambda \int_0^b G_1(b,s)g_1(s)ds\right\}.$$

Then, for $u \in P$ with $\|u\| = H_3$,

$$\begin{aligned} Tu(t) &\leq \lambda \int_0^b G_1(b,s)g_1(s)\mathcal{M}ds \\ &\leq H_3 = \|u\|. \end{aligned}$$

Therefore $\|Tu\| \leq \|u\|$. If $\Omega_2 = \Big\{x \in B : \|x\| < H_3\Big\}$, then

$$\|Tu\| \leq \|u\|, \text{ for } u \in P \cap \partial\Omega_2. \tag{27}$$

Case (ii)**:** f_2 is unbounded. There exists $H_3 > \max\big\{2H_2, \overline{H}_1\big\}$ such that

$$f_2(x) \leq f_2(H_3), \text{ for } 0 < x \leq H_3.$$

Similarly, there exists $H_4 > \max\left\{H_3, \mathcal{M}\lambda\int_0^b G_2(b,r)g_2(r)f_2(H_3)dr\right\}$ such that

$$f_1(x) \leq f_1(H_4), \text{ for } 0 < x \leq H_4.$$

Choosing $u \in P$ with $\|u\| = H_4$, we have

$$\begin{aligned} Tu(t) =&\lambda \int_0^b G_1(t,s)g_1(s)f_1\left(\lambda \int_0^b G_2(s,r)g_2(r)f_2\big(u(r)\big)dr\right)ds \\ \leq&\lambda \int_0^b G_1(t,s)g_1(s)f_1\left(\lambda \int_0^b G_2(b,r)g_2(r)f_2\big(H_3\big)dr\right)ds \\ \leq&\lambda \int_0^b G_1(t,s)g_1(s)f_1\big(H_4\big)ds \\ \leq&\lambda \int_0^b G_1(b,s)g_1(s)ds\big(f_{1\infty} + \epsilon\big)H_4 \\ \leq&H_4 = \|u\|. \end{aligned}$$

Therefore, $\|Tu\| \leq \|u\|$. For this case, if we set $\Omega_2 = \big\{x \in B : \|x\| < H_4\big\}$, then

$$\|Tu\| \leq \|u\|, \text{ for } u \in P \cap \partial\Omega_2. \tag{28}$$

Applying Theorem 2.1 to (26), (27) and (28), we obtain that T has a fixed point $u \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$ together with $v(t) = \lambda \int_0^b G_2(t,s)g_2(s)f_2\big(u(s)\big)ds$ give us a positive solution (u,v) of the fractional order boundary value problem (1)-(4) with respect to cone P, for the chosen values λ. $\square$

4. An example

In this section, as an application, we demonstrate our results with an example. Consider the coupled system of fractional order boundary value problem,

$$\left.\begin{aligned} D_{0^+}^{3.5}u(t)+\lambda\frac{1}{t^2+1}\frac{(305v^2+15v)(3190-3129e^{-5v})}{50v+3}&=0,\\ t\in(0,1),&\end{aligned}\right\} \tag{29}$$

$$\left.\begin{aligned} D_{0^+}^{3.75}v(t)+\lambda\frac{1}{t^2+4}\frac{(786u^2+21u)(2176-2099e^{-2u})}{70u+5}&=0,\\ t\in(0,1),&\end{aligned}\right\} \tag{30}$$

$$u'(0)=u''(0)=u'''(0)=0, u(1)-D_{0^+}^{0.75}u(1)=0, \tag{31}$$

$$v'(0)=v''(0)=v'''(0)=0, v(1)-D_{0^+}^{0.75}v(1)=0. \tag{32}$$

By direct calculations, one can determine

$$f_{10}=305, f_{20}=323.4, f_{1\infty}=19459, f_{2\infty}=24433.37,$$

$$\xi=0.453334,\ \ \Omega=\max\{6.4327, 2.5281\} \text{ and } \rho=\min\{0.77459, 0.223291\}.$$

Applying Theorem 3.1, we get an eigenvalue interval $0.000230148<\lambda<0.0047366$, for which the fractional order boundary value problem (29)-(32) has at least one positive solution.

5. Conclusion

In this paper we have derived sufficient conditions for the existence of at least one positive solution for a coupled system of fractional higher order two-point boundary value problems on a suitable cone in a Banach space. We have determined the eigenvalue intervals of the parameter for which the two-point fractional order boundary value problem possess a positive solution.

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