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SOLUTIONS OF COMPLEX EQUATIONS WITH ADOMIAN DECOMPOSITION METHOD

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ABSTRACT. In this study, first order linear complex differential equations have been solved with adomian decomposition method.

Keywords: complex equation, adomian method.

AMS Subject Classification: 35F46, 39A45

1. Introduction

The Adomian Decomposition Method (ADM) is a method which is used in several areas of mathematics. Recently a great deal of interest has been focused on the application of Adomian's decomposition method to solve a wide variety of linear and nonlinear problems. This method has been introduced by Adomian[1] and it can be used in the linear and nonlinear differential equations, in the differential equations systems, in the integral equations, in the difference equations, in the differential-difference equations, and in the algebraic equations [2,3,4,5,6,15,16,17,18].

This method generates a solution in the form of a series whose terms are determined by a recursive relationship using the Adomian polynomials. Researchers who have used the ADM, have frequently enumerated on the many advantages that it offers. Since it was first presented in the 1980's, Adomian decomposition method has led to several modifications on the method made by various researchers in an attempt to improve the accuracy or expand the application of the original method[10,11,19]. Some of these modifications are Modified Adomian Method[12,19], Wazwaz modifications[10,13], Two step Adomian method[14], and restarted Adomian method[15,16]. Recently, the decomposition method has been used in fractional differential equations [7, 8, 9] . In this study, we solve the complex differential equations using ADM.

1.1. **Derivatives of Complex Functions .** Let $w = w(z, \overline{z})$ be a complex function. Here $z = x + iy$, $w(z, \overline{z}) = u(x, y) + i.v(x, y)$. A derivative according to z and $\overline{z}$ of $w(z, \overline{z})$ is defined as following

$$\frac{\partial w}{\partial z} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - i\frac{\partial w}{\partial y}\right)$$
$$\frac{\partial w}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + i\frac{\partial w}{\partial y}\right)$$

If we write $u + iv$ in place of w, we get that

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$$\frac{\partial w}{\partial z} = \frac{1}{2}\left[(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) + i(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})\right]$$
$$\frac{\partial w}{\partial \overline{z}} = \frac{1}{2}\left[(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}) + i(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})\right]$$

1.2. **Adomian Decomposion Method.** In this section, we mention from ADM. We consider $F(y(x)) = g(x)$, where F represents a general differential operator involving both the linear and nonlinear terms. The linear term is decomposed into $L+R$, where L is the highest order differential operator and R is the remainder of the linear operator. Thus the equation can be written

$$Ly + Ry + Ny = g(x),$$

where Ny represents the nonlinear terms. For solving Ly, we can write as follows

$$Ly = g(x) - Ry - Ny$$

Because L is invertible, an equivalent expression is as follows

$$L^{-1}Ly = L^{-1}g - L^{-1}Ry - L^{-1}Ny.$$

If L is first order, L^{-1} is a integral operator. If L is second order, L^{-1} is two fold integration operator. The nonlinear term Ny will be equated to $\sum_{n=0}^{\infty} A_n$, where A_n are the Adomian polynomials. Thus it can be written

$$\sum_{n=0}^{\infty} y_n = y_0 - L^{-1}R\left(\sum_{n=0}^{\infty} y_n\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right),$$

where y_0 is solution $Ly = g(x)$. Consequently we can write following equalities

$$y_1 = -L^{-1}Ry_0 - L^{-1}A_0$$

$$y_2 = -L^{-1}Ry_1 - L^{-1}A_1$$

$$y_3 = -L^{-1}Ry_2 - L^{-1}A_2$$

$$\vdots$$

$$y_{n+1} = -L^{-1}Ry_n - L^{-1}A_n,$$

where A_n polynomials are determined as follows

$$Ny = f(y)$$

$$A_0 = f(y_0)$$

$$A_1 = y_1\frac{df(y_0)}{dy_0}$$

$$A_2 = y_2\frac{df(y_0))}{dy_0} + \frac{y_1^2}{2}\frac{d^2f(y_0)}{d^2y_0}$$

$$A_3 = y_3\frac{df(y_0)}{dy_0} + y_1.y_2\frac{d^2f(y_0)}{d^2y_0} + \frac{y_1^3}{3!}\frac{d^3f(y_0)}{d^3y_0}$$

$$\vdots$$

2. Solution of Complex Equations with ADM.

Theorem 2.1. *Let A, B, C, F be functions of $z, \bar{z}$ and $w = u+iv$ a complex function. We consider following problem*

$$A(z,\overline{z})\frac{\partial w}{\partial z} + B(z,\overline{z})\frac{\partial w}{\partial \overline{z}} + C(z,\overline{z})w = F(z,\overline{z})$$
$$w(x,0) = f(x).$$

The solution of above mentioned problem is $w = u + iv$,where

$u = u_0 + \sum_{n=0}^{\infty} u_{n+1}$ and $v = v_0 + \sum_{n=0}^{\infty} v_{n+1}$. Therefore $u_0, v_0, u_{n+1}, v_{n+1}$ are as follows

$$u_0 = L_y^{-1}\left(\frac{2F_1}{A_2 - B_2}\right) + u(x,0), v_0 = L_y^{-1}\left(\frac{2F_2}{A_2 - B_2}\right) + v(x,0)$$

$$\begin{aligned} u_{n+1} &= -L_y^{-1}\left(\frac{2C_1}{A_2 - B_2}u_n\right) + L_y^{-1}\left(\frac{2C_2}{A_2 - B_2}v_n\right) + L_y^{-1}\left(\frac{A_2 + B_2}{A_2 - B_2}L_x v_n\right) \\ &\quad + L_y^{-1}\left(\frac{B_1 - A_1}{A_2 - B_2}L_y v_n\right) - L_y^{-1}\left(\frac{A_1 + B_1}{A_2 - B_2}L_x v_n\right) \end{aligned}$$

$$\begin{aligned} v_{n+1} &= -L_y^{-1}\left(\frac{2C_1}{A_2 - B_2}v_n\right) - L_y^{-1}\left(\frac{2C_2}{A_2 - B_2}u_n\right) + L_y^{-1}\left(\frac{A_1 - B_1}{A_2 - B_2}L_y u_n\right) \\ &\quad - L_y^{-1}\left(\frac{B_1 + A_1}{A_2 - B_2}L_x v_n\right) - L_y^{-1}\left(\frac{A_2 + B_2}{A_2 - B_2}L_x u_n\right) \end{aligned}$$

$$A_1 = ReA(z,\overline{z}), A_2 = ImA(z,\overline{z}), B_1 = ReB(z,\overline{z}), B_2 = ImB(z,\overline{z}),$$
$$C_1 = ReC(z,\overline{z}), C_2 = ImC(z,\overline{z}), F_1 = ReF(z,\overline{z}), F_2 = ImF(z,\overline{z})$$

Proof. Let's separate to real and imaginary parts that is given an equation using the definition of complex derivatives .

$$A(z,\overline{z})\frac{\partial w}{\partial z} + B(z,\overline{z})\frac{\partial w}{\partial \overline{z}} + C(z,\overline{z})w = F(z,\overline{z})$$

So we have following equality

$$\begin{aligned} &(A_1(x,y) + iA_2(x,y))\left[\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \\ &+ (B_1(x,y) + iB_2(x,y))\left[\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right] \\ &+2(C_1(x,y) + iC_2(x,y))(u + iv) \\ =\ & 2F_1(x,y) + 2iF_2(x,y) \end{aligned}$$

If the above equality is separated to real and imaginary parts, then we have following equalities:

$$\begin{aligned}
& A_1(x,y)\frac{\partial v}{\partial x} - A_1(x,y)\frac{\partial u}{\partial y} + A_2(x,y)\frac{\partial u}{\partial x} + A_2(x,y)\frac{\partial v}{\partial y} + B_1(x,y)\frac{\partial v}{\partial x} \\
& + B_1(x,y)\frac{\partial u}{\partial y} + B_2(x,y)\frac{\partial u}{\partial x} - B_2(x,y)\frac{\partial v}{\partial y} + 2C_1(x,y)v + 2C_2(x,y)u \\
= & \; 2F_2(x,y)
\end{aligned}$$

$$\begin{aligned}
& A_1(x,y)\frac{\partial u}{\partial x} + A_1(x,y)\frac{\partial v}{\partial y} + A_2(x,y)\frac{\partial u}{\partial y} - A_2(x,y)\frac{\partial v}{\partial x} + B_1(x,y)\frac{\partial u}{\partial x} \\
& - B_1(x,y)\frac{\partial v}{\partial y} - B_2(x,y)\frac{\partial v}{\partial x} - B_2(x,y)\frac{\partial u}{\partial y} + 2C_1(x,y)u - 2C_2(x,y)v \\
= & \; 2F_1(x,y)
\end{aligned}$$

If $A_2 - B_2 \neq 0$, then

$$L_y u = \frac{2F_1}{A_2 - B_2} - \frac{2C_1}{A_2 - B_2}u + \frac{2C_2}{A_2 - B_2}v + \frac{A_2 + B_2}{A_2 - B_2}L_x v + \frac{B_1 - A_1}{A_2 - B_2}L_y v - \frac{A_1 + B_1}{A_2 - B_2}L_x u$$

$$L_y v = \frac{2F_2}{A_2 - B_2} - \frac{2C_1}{A_2 - B_2}v - \frac{2C_2}{A_2 - B_2}u - \frac{A_1 + B_1}{A_2 - B_2}L_x v + \frac{A_1 - B_1}{A_2 - B_2}L_y u - \frac{A_2 + B_2}{A_2 - B_2}L_x u$$

$$\begin{aligned}
u &= \sum_{n=0}^{\infty} u_n, u_0 = L_y^{-1}\left(\frac{2F_1}{A_2 - B_2}\right) + u(x,0) \\
v &= \sum_{n=0}^{\infty} v_n, v_0 = L_y^{-1}\left(\frac{2F_2}{A_2 - B_2}\right) + v(x,0)
\end{aligned}$$

$$\begin{aligned}
u_{n+1} = & -L_y^{-1}\left(\frac{2C_1}{A_2 - B_2}u_n\right) + L_y^{-1}\left(\frac{2C_2}{A_2 - B_2}v_n\right) + L_y^{-1}\left(\frac{A_2 + B_2}{A_2 - B_2}L_x v_n\right) \\
& + L_y^{-1}\left(\frac{B_1 - A_1}{A_2 - B_2}L_y v_n\right) - L_y^{-1}\left(\frac{A_1 + B_1}{A_2 - B_2}L_x u_n\right)
\end{aligned}$$

$$\begin{aligned}
v_{n+1} = & -L_y^{-1}\left(\frac{2C_1}{A_2 - B_2}v_n\right) - L_y^{-1}\left(\frac{2C_2}{A_2 - B_2}u_n\right) - L_y^{-1}\left(\frac{A_1 + B_1}{A_2 - B_2}L_x v_n\right) \\
& + L_y^{-1}\left(\frac{A_1 - B_1}{A_2 - B_2}L_y u_n\right) - L_y^{-1}\left(\frac{A_2 + B_2}{A_2 - B_2}L_x u_n\right)
\end{aligned}$$

If $B_1 - A_1 \neq 0$, then

$$L_y u = \frac{2F_2}{B_1 - A_1} - \frac{2C_1}{B_1 - A_1}v - \frac{2C_2}{B_1 - A_1}u - \frac{A_1 + B_1}{B_1 - A_1}L_x v + \frac{B_2 - A_2}{B_1 - A_1}L_y v - \frac{A_2 + B_2}{B_1 - A_1}L_x u$$

$$L_y v = \frac{2F_1}{A_1 - B_1} - \frac{2C_1}{A_1 - B_1}u + \frac{2C_2}{A_1 - B_1}v + \frac{A_2 + B_2}{A_1 - B_1}L_x v + \frac{B_2 - A_2}{A_1 - B_1}L_y u - \frac{B_1 + A_1}{A_1 - B_1}L_x u$$

$$
\begin{aligned}
u &= \sum_{n=0}^{\infty} u_n, u_0 = L_y^{-1}\left(\frac{2F_2}{B_1 - A_1}\right) + u(x,0) \\
v &= \sum_{n=0}^{\infty} v_n, v_0 = L_y^{-1}\left(\frac{2F_1}{A_1 - B_1}\right) + v(x,0) \\
u_{n+1} &= -L_y^{-1}\left(\frac{2C_1}{B_1 - A_1} v_n\right) + L_y^{-1}\left(\frac{2C_2}{B_1 - A_1} u_n\right) - L_y^{-1}\left(\frac{A_1 + B_1}{B_1 - A_1} L_x v_n\right) \\
& +L_y^{-1}\left(\frac{B_2 - A_2}{B_1 - A_1} L_y v_n\right) - L_y^{-1}\left(\frac{A_2 + B_2}{B_1 - A_1} L_x u_n\right) \\
v_{n+1} &= -L_y^{-1}\left(\frac{2C_1}{A_1 - B_1} u_n\right) + L_y^{-1}\left(\frac{2C_2}{A_1 - B_1} v_n\right) + L_y^{-1}\left(\frac{A_2 + B_2}{A_1 - B_1} L_x v_n\right) \\
& +L_y^{-1}\left(\frac{B_2 - A_2}{A_1 - B_1} L_y u_n\right) - L_y^{-1}\left(\frac{B_1 + A_1}{A_1 - B_1} L_x u_n\right)
\end{aligned}
$$

□

Example 2.1. *Solve the following problem*

$$4w_z + w_{\overline{z}} = 0$$

with the condition

$$w(x,0) = -\frac{1}{3x}.$$

Solution 2.1. *Clearly the coefficients of equation which are as follows*

$$A = 4, B = 1, C = 0, F = 0$$

$$u_0 = u(x,0) = -\frac{1}{3x}, v_0 = v(x,0) = 0$$

$$u_{n+1} = -L_y^{-1}\left(\frac{5}{-3} L_x v_n\right), v_{n+1} = -L_y^{-1}\left(\frac{5}{-3} L_x u_n\right)$$

$$u_1 = 0, v_1 = \frac{5y}{9x^2}, u_2 = -\frac{25y^2}{27x^3}, v_2 = 0, u_3 = 0, v_3 = -\frac{125y^3}{81x^4}$$

$$u_4 = -\frac{625y^4}{243x^5}, v_4 = 0, ..., u_{2n+1} = v_{2n} = 0, u_{2n} = (-1)^{n-1}\frac{(5y)^{2n}}{(3x)^{2n+1}}, v_{2n-1} = (-1)^n \frac{(5y)^{2n-1}}{(3x)^{2n}}$$

Therefore,

$$
\begin{aligned}
u &= u_0 + u_1 + u_2 + u_3 + u_4 + \cdots = -\frac{1}{3x} + \frac{25y^2}{27x^3} - \frac{625y^4}{243x^5} + \cdots = -\frac{3x}{9x^2 + 25y^2} \\
v &= v_0 + v_1 + v_2 + v_3 + v_4 + \cdots = -\frac{5y}{9x^2} + \frac{125y^3}{81x^4} - \frac{3125y^5}{729x^6} + \cdots = -\frac{5y}{9x^2 + 25y^2}
\end{aligned}
$$

$$w = u + iv = \frac{-3x - 5iy}{9x^2 + 25y^2} = \frac{1}{z - 4\overline{z}}$$

Example 2.2. *Solve the following problem*

$$z.w_z - \overline{z}.w_{\overline{z}} = 2z^2 + 5\overline{z}$$

with the condition

$$w(x,0) = 2x^2 - 5x.$$

Solution 2.2. *The coefficients of equation are* $A = z, B = -\overline{z}, C = 0, F = 2z^2 + 5\overline{z}$.*If the coefficients separate real and imaginary parts we get that* $A_1 = x, A_2 = y, B_1 = -x, B_2 = y, C_1 = C_2 = 0, F_1 = 2x^2 - 2y^2 + 5x, F_2 = 4xy - 5y$

$$u_0 = L_y^{-1}\left(\frac{8xy - 10y}{-2x}\right) + 2x^2 - 5x = -2y^2 + \frac{5y^2}{2x} + 2x^2 - 5x$$

$$v_0 = L_y^{-1}\left(\frac{4x^2 - 4y^2 + 10x}{2x}\right) + v(x,0) = 2xy - \frac{2y^3}{3x} + 5y$$

$$\begin{aligned} u_{n+1} &= -L_y^{-1}\left(\frac{2y}{-2x}L_x u_n\right) \\ u_1 &= L_y^{-1}\left(\frac{y}{x}L_x u_0\right) = -\frac{5y^4}{8x^3} + 2y^2 - \frac{5y^2}{2x} \\ u_2 &= L_y^{-1}\left(\frac{y}{x}L_x u_1\right) = \frac{5y^6}{16x^5} + \frac{5y^4}{8x^3} \\ u_3 &= L_y^{-1}\left(\frac{y}{x}L_x u_2\right) = -\frac{25y^8}{128x^7} - \frac{5y^6}{16x^5} \\ &\vdots \end{aligned}$$

Similarly,

$$\begin{aligned} v_{n+1} &= L_y^{-1}\left(\frac{y}{x}L_x v_n\right) \\ v_1 &= L_y^{-1}\left(\frac{y}{x}L_x v_0\right) = \frac{2y^3}{3x} + \frac{2y^5}{15x^3} \\ v_2 &= L_y^{-1}\left(\frac{y}{x}L_x v_1\right) = -\frac{2y^5}{15x^3} - \frac{2y^7}{35x^5} \\ v_3 &= L_y^{-1}\left(\frac{y}{x}L_x v_2\right) = \frac{2y^7}{35x^5} + \frac{2y^9}{63x^7} \\ &\vdots \end{aligned}$$

Therefore

$$\begin{aligned} u &= \sum^{\infty} u_n = -2y^2 + \frac{5y^2}{2x} + 2x^2 - 5x - \frac{5y^4}{8x^3} + 2y^2 - \frac{5y^2}{2x} + \frac{5y^6}{16x^5} + \frac{5y^4}{8x^3} \\ &\quad -\frac{25y^8}{128x^7} - \frac{5y^6}{16x^5} + ... = 2x^2 - 5x \\ v &= \sum^{\infty} v_n = 2xy - \frac{2y^3}{3x} + 5y + \frac{2y^3}{3x} + \frac{2y^5}{15x^3} - \frac{2y^5}{15x^3} - \frac{2y^7}{35x^5} + \frac{2y^7}{35x^5} \\ &\quad +\frac{2y^9}{63x^7} - ... = 2xy \end{aligned}$$

$$w = u + iv = 2x^2 - 5x + i(2xy + 5y) = z^2 - 5\overline{z} + z.\overline{z}$$

3. CONCLUSION

In this study, we have studied solutions of first order complex partial differential equations by using ADM. Our next goal is a study to find solutions of first order nonlinear complex equations and more higher order linear complex equations with ADM.

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