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SOLVING FRACTIONAL DIFFUSION AND FRACTIONAL DIFFUSION-WAVE EQUATIONS BY PETROV-GALERKIN FINITE ELEMENT METHOD

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Abstract. In the last few years, it has become highly evident that fractional calculus has been widely used in several areas of science. Because of this fact, their numerical solutions also have become urgently important. In this manuscript, numerical solutions of both the fractional diffusion and fractional diffusion-wave equations have been obtained by a Petrov-Galerkin finite element method using quadratic B-spline base functions as trial functions and linear B-spline base functions as the test functions. In those equations, fractional derivatives are used in terms of the Caputo sense. While the $L1$ discretizaton formula has been applied to fractional diffusion equation, the $L2$ discretizaton formula has been applied to the fractional diffusion-wave equation. Finally, the error norms L_2 and L_∞ have been calculated for testing the accuracy of the proposed scheme.

Keywords: Finite element method, Petrov-Galerkin method, Fractional diffusion equation, Fractional diffusion-wave equation, Quadratic B-Spline, Linear B-Spline.

AMS Subject Classification: 97N40, 65D07, 74S05, 26A33, 34A08, 65L60

1. Introduction

Recently, it has become increasingly evident that fractional derivatives are very useful in the analysis of a wide range of scientific areas such as engineering, physics, chemistry and some other branches. The problems in those areas can be described very successfully by models using mathematical tools from fractional calculus, i.e. the theory of derivatives and integrals of fractional (non-integer) order [18]. In reality, the concept of differentiation and integration to non-integer order is by no means new. Interest in this subject was evident almost as soon as the ideas of the classical calculus were known [8]. However, in the last several years numerous authors pointed out the fact that derivatives and integrals of non-integer order are more appropriate to describe various materials such as polymers. In reality, they model the physical problems more appropriately and accurately than non-integer order ones. The increasing number of fractional derivative applications in numerous areas of science and engineering clearly shows the significant demand for a better mathematical modelings of real objects, and it is clear that the fractional calculus provides one possible approach on the way to more adequate mathematical modelling of real objects and processes. For instance, the modelling of diffusion in a specific type of

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porous medium (in fractional media) can be given as one of the most significant applications of fractional-order derivatives [6], and the fractional diffusion-wave equations have been proposed to deal with viscoelastic problems such as propagation of stress waves in viscoelastic solids [9, 10]. Even though there have been numerous analytical methods [1, 19, 13] for dealing with those fractional equations, as also happens with ordinary (non-fractional) partial differential equations, in many situations the initial condition, and/or the external force are such that the only feasible choice is to apply to numerical techniques. However, even though we have witnessed a tremendous increase in the number of works on the topic in recent years [15, 21, 11, 20, 17, 5], this area of applied mathematics is not as well developed and understood as its non-fractional counterpart [16]. Even though the number of scientific and engineering problems including fractional calculus has already large and still growing, it can be said that there is still a long way to go in this field.

There are numerous numerical methods to solve fractional partial differential equations and those methods differ primarily in the style in which the normal and fractional derivatives are discretized [16]. In the present manuscript, the finite element method has been applied for solving fractional diffusion and fractional diffusion-wave equations. For the sake of simplicity, it can be summarized that the fundamental concept behind the finite element method is to divide the entire region of the solution domain into an approximately equivalent system of finite elements with associated nodes and to select the most suitable element type to model most closely the actual physical behavior . Doing so converts a huge problem into many solvable small problems. As a proposal, those elements must be made small enough to give usable results and yet large enough to reduce computational effort [3].

As parallel to the importance of fractional calculus, the number of recent studies about them have also increased. For instance, Sun et al. [4] have applied a semi-analytical finite element method for a class of time-fractional diffusion equations. Sweilam et al. [12] have solved time-fractional diffusion equation using Crank-Nicolson finite difference method. Monami and Odibat [18] implemented relatively new analytical techniques, the variational iteration method and the Adomian decomposition method, to solve linear fractional partial differential equations which are arising in fluid mechanics. Celik and Duman [2] utilized Crank-Nicolson method for the fractional diffusion equation with the Riesz fractional derivative and finally got numerical results by using fractional centered difference approach.

In the solution process, in place of the fractional diffusion and fractional diffusion-wave equations, the following general form is going to be used as a model

$$\frac{\partial^\gamma u}{\partial t^\gamma} = K\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where

$$\frac{\partial^\gamma}{\partial t^\gamma} f(t) = \frac{1}{\Gamma(n-\gamma)} \int_0^t (t-\tau)^{n-\gamma-1} \frac{\partial^n f(\tau)}{\partial t^n} d\tau \quad n-1 < \gamma < n \tag{2}$$

is the fractional derivative in the Caputo's sense [6, 1], K is generally known as the diffusion coefficient and n is an integer. For all calculations in the present paper, diffusion coefficient K is going to be taken as 1. Eq. (1) is called the fractional diffusion equation or sub-diffusion equation for $0 < \gamma \leq 1$, and it is called the fractional diffusion-wave equation for $1 < \gamma \leq 2$. In this manuscript, for diffusion equation, we will take the boundary conditions of the model problem (1) given in the interval $0 \leq x \leq \pi$ as

$$u(0,t) = 0,\, u(\pi,t) = 0 \tag{3}$$

and the initial condition as

$$u(x,0)=\sin x \tag{4}$$

and for diffusion-wave equation, together with the above boundary and the initial conditions, the following additional initial condition

$$\frac{\partial u(x,t)}{\partial t}\,|_{t=0}\;=0 \tag{5}$$

is going to be used. The exact solution of both the above problems which are obtained by Adomian Decomposotion Method is found as follows [19]

$$u(x,t)=E_\gamma(-t^\gamma)\sin x \tag{6}$$

where E_γ is the Mittag-Leffler function [6].

As has been used by Ref. [7] in his explicit finite difference method, in order to obtain a finite element scheme to solve the fractional diffusion equation ($0<\gamma\leq 1$), we will also discretize the Caputo derivative in terms of the so-called $L1$ formula [8]

$$\left.\frac{\partial^\gamma f}{\partial t^\gamma}\right|_{t_n}=\frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)}\sum_{k=0}^{n-1}b_k^\gamma\left[f(t_{n-k})-f(t_{n-1-k})\right]+O\left(\Delta t\right)$$

where

$$b_k^\gamma=(k+1)^{1-\gamma}-k^{1-\gamma}$$

and to solve the fractional diffusion-wave equation ($1<\gamma\leq 2$), we will discretize the Caputo derivative in terms of the so-called $L2$ formula [8]

$$\left.\frac{\partial^\gamma f}{\partial t^\gamma}\right|_{t_n}=\frac{(\Delta t)^{-\gamma}}{\Gamma(3-\gamma)}\sum_{k=0}^{n-1}b_k^\gamma\left[f\left(t_{n-k}\right)-2f\left(t_{n-1-k}\right)+f\left(t_{n-2-k}\right)\right]+O\left(\Delta t\right)$$

where

$$b_k^\gamma=(k+1)^{2-\gamma}-k^{2-\gamma}.$$

2. Quadratic B-spline Petrov-Galerkin Finite Element Solutions

To be able to continue solving Eq. (1) together with the given boundary conditions (3) and the initial conditions (4)-(5) using Petrov-Galerkin finite element method, firstly, we define quadratic B-spline base functions. Let's assume that the solution domain $[a,b]$ is divided into N finite elements of uniformly equal length by the nodes x_m, $m=0,1,2,...,N$ such that $a=x_0<x_1\cdots<x_N=b$ and $h=x_{m+1}-x_m$. The quadratic B-splines $Q_m(x)$, $(m=-1(1)N)$, at the nodes x_m are described over the solution domain $[a,b]$ by [14]

$$Q_m(x)=\frac{1}{h^2}\begin{cases}(x_{m+2}-x)^2-3(x_{m+1}-x)^2+3(x_m-x)^2 & x\in[x_{m-1},x_m],\\ (x_{m+2}-x)^2-3(x_{m+1}-x)^2 & x\in[x_m,x_{m+1}],\\ (x_{m+2}-x)^2 & x\in[x_{m+1},x_{m+2}],\\ 0 & \text{otherwise.}\end{cases} \tag{7}$$

Over the solution domain $[a,b]$, the set of splines $\{Q_{-1}(x),Q_0(x),\dots,Q_N(x)\}$ constitutes a basis for the functions described on this domain. Therefore, an approximation solution $U_N(x,t)$ over the domain can be given in terms of those quadratic B-spline trial functions as follows

$$U_N(x,t) = \sum_{m=-1}^{N} \delta_m(t) Q_m(x) \tag{8}$$

where $\delta_m(t)$'s are unknown, time dependent parameters to be determined from the boundary and weighted residual conditions. Due to the fact that each quadratic B-spline functions covers three consecutive elements, each element $[x_m, x_{m+1}]$ is covered by three quadratic B-spline functions. In this manuscript, the finite elements are identified with the interval $[x_m, x_{m+1}]$ and the elements nodes x_m, x_{m+1}. Using the nodal values U_m and U_m' given in terms of the parameter $\delta_m(t)$

$$\begin{aligned} U_N(x_m) &= U_m = \delta_{m-1} + \delta_m, \\ U_N'(x_m) &= U_m' = (2/h)(-\delta_{m-1} + \delta_m) \end{aligned} \tag{9}$$

the variation of $U_N(x,t)$ over the typical element $[x_m, x_{m+1}]$ is given by

$$U_N = \sum_{j=m-1}^{m+1} \delta_j Q_j. \tag{10}$$

The weight function Ψ is taken a linear B-spline L_m. Linear B-spline L_m at the knots x_m are defined over the interval $[a,b]$ by

$$L_m(x) = \frac{1}{h} \begin{cases} (x_{m+1} - x) - 2(x_m - x) & [x_{m-1}, x_m], \\ (x_{m+1} - x) & [x_m, x_{m+1}], \\ 0 & otherwise. \end{cases} \tag{11}$$

Using the local coordinate transformation for the finite element $[x_m, x_{m+1}]$, linear B-spline shape functions can be defined as

$$\begin{aligned} L_m &= 1 - \frac{\xi}{h} \\ L_{m+1} &= \frac{\xi}{h} \end{aligned} \tag{12}$$

Before proceeding with the application of Petrov-Galerkin method to the Eq. (1) with the appropriate boundary conditions, we initially need to construct the weak form of the Eq. (1). To do this, all terms in Eq. (1) are taken to one side of the equation and then multiplied by the weight function $\Psi(x)$. Then, by taking the integral of the resulting equation over the region $[0, \pi]$ and setting it to zero, we get

$$\int_0^\pi \left(\frac{\partial^\gamma U}{\partial t^\gamma} - \frac{\partial^2 U}{\partial x^2} \right) \Psi dx = 0 \tag{13}$$

where $\Psi(x)$ is the weighted function taken as linear B-spline function L_m. If we use integration by parts for distributing the degree of dependent variable between itself and test function, we obtain

$$\int_0^\pi \left(\frac{\partial^\gamma U}{\partial t^\gamma} \Psi + \frac{\partial U}{\partial x} \frac{\partial \Psi}{\partial x} \right) dx = \Psi \frac{\partial U}{\partial x} \bigg|_0^\pi . \tag{14}$$

If we consider the fact that the weak form (14) is valid for the whole region of the problem, particularly, it is also valid over the typical element $[x_m, x_{m+1}]$, therefore Eq. (14) can be particularly be written as follows

$$\int_{x_m}^{x_{m+1}} \left(\frac{\partial^\gamma U}{\partial t^\gamma}\Psi + \frac{\partial U}{\partial x}\frac{\partial \Psi}{\partial x} \right) dx = \Psi \frac{\partial U}{\partial x}\bigg|_{x_m}^{x_{m+1}}. \tag{15}$$

By using the transformation $\xi = x - x_m$, the weak form (15) transforms into the form

$$\int_0^h \left(\frac{\partial^\gamma U}{\partial t^\gamma}\Psi + \frac{\partial U}{\partial \xi}\frac{\partial \Psi}{\partial \xi} \right) d\xi = \Psi \frac{\partial U}{\partial \xi}\bigg|_0^h. \tag{16}$$

The newly obtained Eq. (16) is the element equation for a typical element "e". Now, Eqs. (7) can be rewritten as follows

$$\begin{matrix} Q_{m-1} \\ Q_m \\ Q_{m+1} \end{matrix} = \frac{1}{h^2} \left\{ \begin{matrix} (h-\xi)^2 \\ h2 + 2h\xi - 2\xi^2 \\ \xi^2 \end{matrix} \right. \tag{17}$$

Inserting Eqs. (7) into Eq. (16), we have

$$A^e \dot{\delta}(t) + B^e \delta(t) = C^e \delta(t) \tag{18}$$

where dot denotes γ^{th} fractional derivative with respect to time and A^e_{ij}, B^e_{ij} and C^e_{ij} are element matrices given by the following statements:

$A^e_{ij} = \int_0^h L_i Q_j d\xi,$
$B^e_{ij} = \int_0^h L_i' Q_j' d\xi,$
$C^e_{ij} = L_i Q_j' \Big|_0^h,$

where $i, j = m-1, m, m+1$. The element matrices are calculated as follows

$A^e_{ij} = \int L_i Q_j d\xi = \frac{h}{12} \begin{bmatrix} 3 & 8 & 1 \\ 1 & 8 & 3 \end{bmatrix}$

$B^e_{ij} = \int L_i' Q_j' d\xi = \frac{1}{h} \begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$

$C^e_{ij} = L_i Q_j' \Big|_0^h = \frac{2}{h} \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$

Assembling all contributions coming from all the elements, Eq. (18) yields the system

$$A\dot{\delta}(t) + B\delta(t) = C\delta(t) \tag{19}$$

where $\delta(t)$'s are unknown parameters and A, B and C are $(N+2)\times(N+2)$ global matrices with generalized m^{th} row as follows, respectively

$A : \frac{h}{12}\ (1 \quad 11 \quad 11 \quad 1),$
$B : \frac{1}{h}\ (-1 \quad 1 \quad 1 \quad -1 \quad),$
$C : \ (0 \quad 0 \quad 0 \quad 0 \quad).$

For the fractional diffusion equation ($0 < \gamma \leq 1$), if time parameters $\delta(t)$'s and its fractional time derivatives $\dot{\delta}(t)$'s in Eq. (19) are discretized by the Crank-Nicolson formula and $L1$ formula, respectively:

$$\delta_m = \frac{1}{2}(\delta_m^n + \delta_m^{n+1}) \tag{20}$$

and

$$\dot{\delta}_m = \frac{d^\gamma \delta}{dt^\gamma} = \frac{(\Delta t)^{-\gamma}}{\Gamma(2-\gamma)} \sum_{k=0}^{n-1} \left[(k+1)^{1-\gamma} - k^{1-\gamma} \right] \left[\delta_m^{n-k} - \delta_m^{n-k-1} \right],$$

we obtain a recurrence relationship between successive time levels relating unknown parameters $\delta_m^{n+1}(t)$

$$
\begin{array}{l}
(2-\alpha)\,\delta_{m-2}^{n+1}+(22+\alpha)\,\delta_{m-1}^{n+1}+(22+\alpha)\,\delta_{m}^{n+1}+(2-\alpha)\,\delta_{m+1}^{n+1}\\
=(2+\alpha)\,\delta_{m-2}^{n}+(22-\alpha)\,\delta_{m-1}^{n}+(22-\alpha)\,\delta_{m}^{n}+(2+\alpha)\,\delta_{m+1}^{n}\\
-2\sum\limits_{k=1}^{n}\left[(k+1)^{1-\gamma}-k^{1-\gamma}\right]\Big[(\delta_{m-2}^{n-k+1}-\delta_{m-2}^{n-k})+11(\delta_{m-1}^{n-k+1}-\delta_{m-1}^{n-k})\\
+11(\delta_{m}^{n-k+1}-\delta_{m}^{n-k})+(\delta_{m+1}^{n-k+1}-\delta_{m+1}^{n-k})\Big]
\end{array}
\tag{21}
$$

where

$$\alpha=\frac{24(\Delta t)^{\gamma}\Gamma(2-\gamma)}{h^2}.$$

Next, for fractional diffusion-wave equation ($1<\gamma\le 2$), if time parameters $\delta_m(t)$'s and its fractional time derivatives $\dot{\delta}_m(t)$'s in Eq. (18) are discretized by the Crank-Nicolson formula and $L2$ formula, respectively

$$\delta_m=\frac{1}{2}(\delta_m^n+\delta_m^{n+1}) \tag{22}$$

and

$$\dot{\delta}=\frac{d^{\gamma}\delta}{dt^{\gamma}}=\frac{(\Delta t)^{-\gamma}}{\Gamma(3-\gamma)}\sum_{k=0}^{n-1}\left[(k+1)^{2-\gamma}-k^{2-\gamma}\right]\left[\delta^{n-k}-2\delta_m^{n-k-1}+\delta_m^{n-k-2}\right],$$

we obtain a recurrence relationship between successive time levels relating unknown parameters $\delta_m^{n+1}(t)$

$$
\begin{array}{l}
(2-\alpha)\,\delta_{m-2}^{n+1}+(22+\alpha)\,\delta_{m-1}^{n+1}+(22+\alpha)\,\delta_{m}^{n+1}+(2-\alpha)\,\delta_{m+1}^{n+1}\\
=(4+\alpha)\,\delta_{m-2}^{n}+(44-\alpha)\,\delta_{m-1}^{n}+(44-\alpha)\,\delta_{m}^{n}+(4+\alpha)\,\delta_{m+1}^{n}\\
-2(\delta_{m-2}^{n-1}+11\delta_{m-1}^{n-1}+11\delta_{m}^{n-1}+\delta_{m+1}^{n-1})-2\sum\limits_{k=1}^{n}\left[(k+1)^{2-\gamma}-k^{2-\gamma}\right]\\
\times\Big[(\delta_{m-2}^{n-k+1}-2\delta_{m-2}^{n-k}+\delta_{m-2}^{n-k+1})+11(\delta_{m-1}^{n-k+1}-2\delta_{m-1}^{n-k}+\delta_{m-1}^{n-k+1})+11(\delta_{m}^{n-k+1}-2\delta_{m}^{n-k}+\delta_{m}^{n-k+1})\\
+(\delta_{m+1}^{n-k+1}-2\delta_{m+1}^{n-k}+\delta_{m+1}^{n-k+1})\Big]
\end{array}
\tag{23}
$$

where

$$\alpha=\frac{24(\Delta t)^{\gamma}\Gamma(3-\gamma)}{h^2}.$$

It is seen that both of the systems (21) and (23) consist of $N+2$ linear equations involving $N+2$ unknown parameters $(\delta_{-1},\dots,\delta_N)^T$. If we apply the boundary conditions (3) to the system (19), we easily obtain an $N\times N$ penta-diagonal matrix system.

2.1. **Initial state.** The initial vector $\mathbf{d}^0=(\delta_{-1},\delta_0,\delta_1,\dots,\delta_{N-2},\delta_{N-1},\delta_N)^T$ is determined from the initial condition $U(x,0)$ by interpolating using quadratic splines. If we use the relations at the knots $U_N(x_i,0)=U(x_i,0)$, $(i=0,\dots,N)$ together with $U_N'(x_N,0)=U'(x_N,0)$, initial vector $\mathbf{d}^0$ can be obtained from the following matrix equation

$$\begin{bmatrix} 1 & 1 & & & & & \\ 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & & \ddots & & & \\ & & & & 1 & 1 & \\ & & & & & \frac{-2}{h} & \frac{2}{h} \end{bmatrix} \begin{bmatrix} \delta_{-1}^0 \\ \delta_0^0 \\ \delta_1^0 \\ \vdots \\ \delta_{N-1}^0 \\ \delta_N^0 \end{bmatrix} = \begin{bmatrix} U_0^0 \\ U_1^0 \\ U_2^0 \\ \vdots \\ U_N^0 \\ U_N' \end{bmatrix}.$$

3. Numerical examples and results

In this section, the approximate solutions of the diffusion and diffusion-wave problems are numerically obtained by Petrov-Galerkin finite element method using quadratic B-spline base functions as element shape functions and linear B-spline functions as the weighted functions. The accuracy of the method is measured by the error norm L_2

$$L_2 = \left\|U^{exact} - U_N\right\|_2 \simeq \sqrt{h \sum_{j=0}^{N} \left|U_j^{exact} - (U_N)_j\right|^2} \tag{24}$$

and the error norm L_∞

$$L_\infty = \left\|U^{exact} - U_N\right\|_\infty \simeq \max_j \left|U_j^{exact} - (U_N)_j\right|. \tag{25}$$

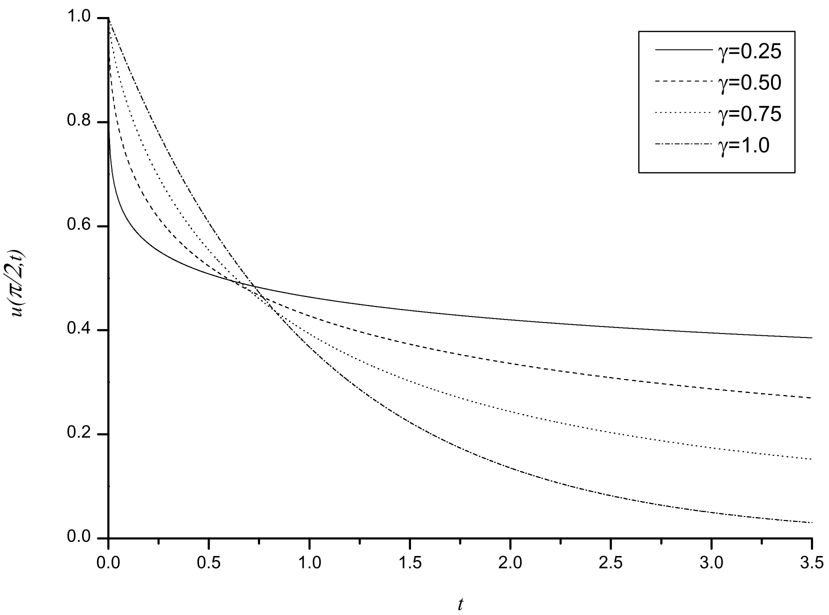

Figure 1. Numerical solutions of $u(\pi/2, t)$ for $\Delta t = 0.00007$ and $N = 40$ at different time levels.

In Fig. 1, numerical solutions of the fractional diffusion problem at the midpoint for various values of γ and $N = 40$ have been presented. The numerical and analytical solutions are so similar toeach other that they are indistinguishable. Table 1 compares the analytical solution and numerical solutions obtained for diffusion equation for values of $\gamma = 0.25$, $\gamma = 0.50$ and $\gamma = 0.75$. Table 1 clearly shows that the analytical and numerical solutions obtained by the present scheme are in good agreement with each other. As it is

Table 1. The comparison of the exact solutions with the numerical solutions of the diffusion problem with $N = 40$, $\Delta t = 0.00007$ and $t_f = 0.35$ for different values of γ and the error norms L_2 and L_∞.

	$\gamma = 0.25$		$\gamma = 0.50$		$\gamma = 0.75$	
x	Numerical	Exact	Numerical	Exact	Numerical	Exact
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.314159	0.164109	0.164109	0.176628	0.176627	0.194624	0.194621
0.628319	0.312153	0.312153	0.335966	0.335965	0.370197	0.370192
0.942478	0.429642	0.429642	0.462418	0.462416	0.509532	0.509525
1.256637	0.505075	0.505074	0.543605	0.543602	0.598991	0.598983
1.570796	0.531067	0.531066	0.571580	0.571577	0.629816	0.629808
1.884956	0.505075	0.505074	0.543605	0.543602	0.598991	0.598983
2.199115	0.429642	0.429642	0.462418	0.462416	0.509532	0.509525
2.513274	0.312153	0.312153	0.335966	0.335965	0.370197	0.370192
2.827433	0.164109	0.164109	0.176628	0.176627	0.194624	0.194621
3.141593	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.000614		0.003551		0.010828	
$L_\infty \times 10^3$	0.000490		0.002833		0.008639	

Table 2. The comparison of the exact solutions with the numerical solutions of the diffusion problem with $\gamma = 0.5$, $\Delta t = 0.00007$ and $t_f = 0.35$ for different values of N and the error norms L_2 and L_∞.

x	N=10	N=20	N=40	N=80	Exact
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.314159	0.176631	0.176628	0.176628	0.176628	0.176627
0.628319	0.335973	0.335967	0.335966	0.335966	0.335965
0.942478	0.462427	0.462419	0.462418	0.462418	0.462416
1.256637	0.543615	0.543606	0.543605	0.543605	0.543602
1.570796	0.571591	0.571581	0.571580	0.571580	0.571577
1.884956	0.543615	0.543606	0.543605	0.543605	0.543602
2.199115	0.462427	0.462419	0.462418	0.462418	0.462416
2.513274	0.335973	0.335967	0.335966	0.335966	0.335965
2.827433	0.176631	0.176628	0.176628	0.176628	0.176627
3.141593	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.017156	0.004349	0.003551	0.003501	
$L_\infty \times 10^3$	0.013689	0.003470	0.002833	0.002793	

seen from the table, as the values of γ increase, so the values of the error norms L_2 and L_∞ increase. Table 2 shows the numerical results for $\gamma = 0.5$, $\Delta t = 0.00007$ and $t_f = 0.35$ and for different values of N. It is obvious from Table 2 that as the number of division increases, the obtained numerical results become more accurate. The decreasing values of the error norms L_2 and L_∞ shows this fact clearly. In Fig. 2, the graphs of numerical solutions obtained for $\gamma = 0.50$ and $N = 40$ at different time levels are illusrated. Table 3 obviously indicates the fact that as the values of time steps become smaller, the level of agreement between the approximate solutions and analytical solutions also becomes better, resulting in the decrease of the values of the error norms L_2 and L_∞. Table 4 shows the error norms L_2 and L_∞ for $\gamma = 0.25$, $\gamma = 0.50$ and $\gamma = 0.75$ at several time levels. Moreover, it is clearly seen from the table that for each value of γ as the time increases, the values of error norms L_2 and L_∞ decease.

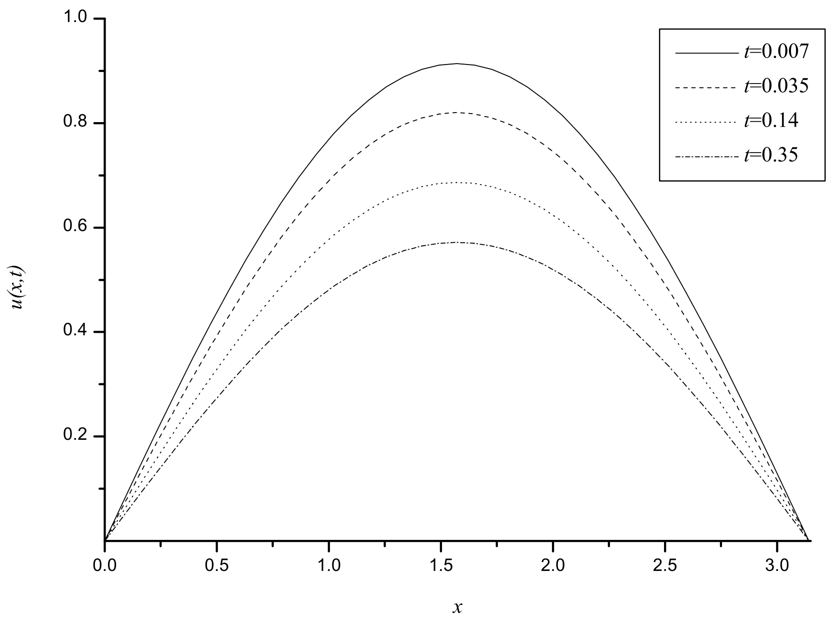

FIGURE 2. Numerical solutions of the diffusion problem for $\gamma = 0.50$ and $N = 40$ at different time levels.

TABLE 3. The comparison of the exact solutions with the numerical solutions of the diffusion problem with $\gamma = 0.5$, $N = 40$ and $t_f = 0.35$ for different values of Δt and the error norms L_2 and L_∞.

x	$\Delta t = 0.0035$	$\Delta t = 0.0007$	$\Delta t = 0.00035$	$\Delta t = 0.00007$	Exact
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.314159	0.176676	0.176636	0.176632	0.176628	0.176627
0.628319	0.336057	0.335982	0.335973	0.335966	0.335965
0.942478	0.462543	0.462439	0.462427	0.462418	0.462416
1.256637	0.543751	0.543630	0.543616	0.543605	0.543602
1.570796	0.571734	0.571607	0.571592	0.571580	0.571577
1.884956	0.543751	0.543630	0.543616	0.543605	0.543602
2.199115	0.462543	0.462439	0.462427	0.462418	0.462416
2.513274	0.336057	0.335982	0.335973	0.335966	0.335965
2.827433	0.176676	0.176636	0.176632	0.176628	0.176627
3.141593	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.196542	0.036649	0.018010	0.003551	
$L_\infty \times 10^3$	0.156818	0.029242	0.014370	0.002833	

Fig. 3 shows the numerical solutions of the fractional diffusion wave problem at the midpoint for various values of γ and $N = 40$. The graphs of the numerical and analytical

TABLE 4. The comparison of the exact solutions with the numerical solutions of the diffusion problem with $N = 40$, $\Delta t = 0.00007$ and $t_f = 0.35$ for different values of N and the error norms L_2 and L_∞.

t	$\gamma = 0.25$		$\gamma = 0.50$		$\gamma = 0.75$	
	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
0.07	0.010430	0.008322	0.035735	0.028512	0.043847	0.034984
0.14	0.003757	0.002998	0.016264	0.012977	0.028503	0.022742
0.21	0.001868	0.001491	0.009175	0.007320	0.020136	0.016067
0.28	0.001047	0.000836	0.005619	0.004483	0.014681	0.011714
0.35	0.000614	0.000490	0.003551	0.002833	0.010828	0.008639

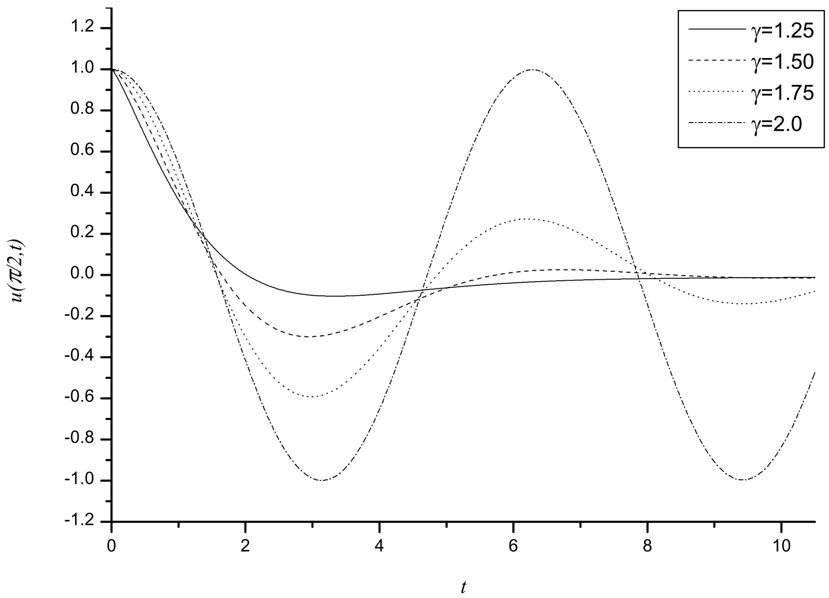

FIGURE 3. Numerical solutions of $u(\pi/2, t)$ for $\Delta t = 0.0015$ and $N = 40$ at different time levels.

solutions are indiscriminately similar to each other. Table 5 presents the comparison of the analytical solution and numerical solutions obtained by the scheme for diffusion-wave equation for values of $\gamma = 1.25$, $\gamma = 1.50$ and $\gamma = 1.75$. It is clearly demonstrated in the table that the obtained numerical results are satisfactorily in good agreement with the analytical ones. As the value of γ increases, so the values of the error norms L_2 and L_∞ increase. Table 6 gives the numerical results for $\gamma = 1.5$, $\Delta t = 0.00075$ and $t_f = 3.75$ for various values of N. As it is seen from the table, as the number of division of the problem domain increases, the values of error decrease. It is shown in Table 7 that as the values of time steps decrease, the agreement between the approximate solutions and analytical solutions becomes better, and the values of the error norms L_2 and L_∞ become smaller. Fig. 4 illusrates the numerical solutions for the values of $\gamma = 1.50$ and $N = 40$ at various time levels. Table 8 shows the error norms L_2 and L_∞ for $N = 40$, $\Delta t = 0.00075$ and $t_f = 3.75$ for various values of γ and t.

Table 5. The comparison of the exact solutions with the numerical solutions of the diffusion-wave problem with $N = 40$, $\Delta t = 0.00075$ and $t_f = 3.75$ for different values of γ and the error norms L_2 and L_∞.

	$\gamma = 1.25$		$\gamma = 1.50$		$\gamma = 1.75$	
x	Numerical	Exact	Numerical	Exact	Numerical	Exact
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.314159	-0.030452	-0.030452	-0.073478	-0.073478	-0.137937	-0.137943
0.628319	-0.057923	-0.057923	-0.139763	-0.139763	-0.262371	-0.262384
0.942478	-0.079724	-0.079724	-0.192367	-0.192367	-0.361123	-0.361140
1.256637	-0.093721	-0.093721	-0.226141	-0.226141	-0.424525	-0.424546
1.570796	-0.098545	-0.098545	-0.237779	-0.237779	-0.446372	-0.446394
1.884956	-0.093721	-0.093721	-0.226141	-0.226141	-0.424525	-0.424546
2.199115	-0.079724	-0.079724	-0.192367	-0.192367	-0.361123	-0.361140
2.513274	-0.057923	-0.057923	-0.139763	-0.139763	-0.262371	-0.262384
2.827433	-0.030452	-0.030452	-0.073478	-0.073478	-0.137937	-0.137943
3.141593	0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.000081		0.000632		0.027269	
$L_\infty \times 10^3$	0.000065		0.000504		0.021758	

Table 6. The comparison of the exact solutions with the numerical solutions of the diffusion-wave problem with $\gamma = 1.5$, $\Delta t = 0.00075$ and $t_f = 3.75$ for different values of N and the error norms L_2 and L_∞.

x	N=10	N=20	N=40	N=80	Exact
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.314159	-0.073482	-0.073478	-0.073478	-0.073478	-0.073478
0.628319	-0.139771	-0.139763	-0.139763	-0.139763	-0.139763
0.942478	-0.192378	-0.192368	-0.192367	-0.192367	-0.192367
1.256637	-0.226154	-0.226142	-0.226141	-0.226141	-0.226141
1.570796	-0.237792	-0.237780	-0.237779	-0.237779	-0.237779
1.884956	-0.226154	-0.226142	-0.226141	-0.226141	-0.226141
2.199115	-0.192378	-0.192368	-0.192367	-0.192367	-0.192367
2.513274	-0.139771	-0.139763	-0.139763	-0.139763	-0.139763
2.827433	-0.073482	-0.073478	-0.073478	-0.073478	-0.073478
3.141593	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.017421	0.001616	0.000632	0.000570	
$L_\infty \times 10^3$	0.013900	0.001290	0.000504	0.000455	

TABLE 7. The comparison of the exact solutions with the numerical solutions of the diffusion-wave problem with $\gamma = 1.5$, $N = 40$ and $t_f = 3.75$ for different values of Δt and the error norms L_2 and L_∞.

x	$\Delta t = 0.015$	$\Delta t = 0.0075$	$\Delta t = 0.0015$	$\Delta t = 0.00075$	Exact
0.000000	0.000000	0.000000	0.000000	0.000000	0.000000
0.314159	-0.073491	-0.073482	-0.073478	-0.073478	-0.073478
0.628319	-0.139788	-0.139771	-0.139764	-0.139763	-0.139763
0.942478	-0.192401	-0.192379	-0.192368	-0.192367	-0.192367
1.256637	-0.226181	-0.226155	-0.226142	-0.226141	-0.226141
1.570796	-0.237821	-0.237793	-0.237780	-0.237779	-0.237779
1.884956	-0.226181	-0.226155	-0.226142	-0.226141	-0.226141
2.199115	-0.192401	-0.192379	-0.192368	-0.192367	-0.192367
2.513274	-0.139788	-0.139771	-0.139764	-0.139763	-0.139763
2.827433	-0.073491	-0.073482	-0.073478	-0.073478	-0.073478
3.141593	0.000000	0.000000	0.000000	0.000000	0.000000
$L_2 \times 10^3$	0.053002	0.018505	0.001677	0.000632	
$L_\infty \times 10^3$	0.042289	0.014765	0.001338	0.000504	

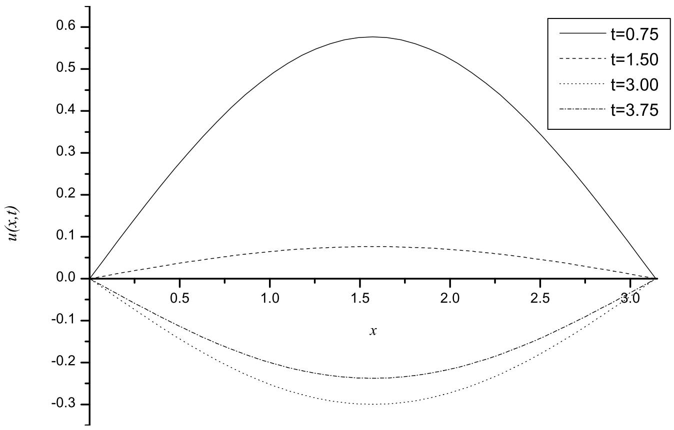

FIGURE 4. Numerical solutions of the diffusion-wave problem for $\gamma = 1.50$ and $N = 40$ at different time levels.

TABLE 8. he comparison of the exact solutions with the numerical solutions of the diffusion problem with $N = 40$, $\Delta t = 0.00075$ and $t_f = 3.75$ for different values of N and the error norms L_2 and L_∞.

	$\gamma = 1.25$		$\gamma = 1.50$		$\gamma = 1.75$	
t	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$	$L_2 \times 10^3$	$L_\infty \times 10^3$
1.5	0.000317	0.000253	0.002959	0.002361	0.001341	0.001070
3.0	0.000018	0.000015	0.001633	0.001303	0.053247	0.042485
4.5	0.000084	0.000067	0.001882	0.001502	0.016787	0.013394
6.0	0.000035	0.000028	0.001059	0.000845	0.053487	0.042677
7.5	0.000005	0.000004	0.000473	0.000378	0.009466	0.007553

4. Conclusion

In the present study, a Petrov-Galerkin finite element method has been successfully used to obtain the numerical solutions of diffusion and diffusion-wave equations. In these equations, the fractional derivative is considered of the Caputo form. The fractional derivative appearing in the fractional diffusion and diffusion-wave equations is approximated, respectively, by means of the so-called $L1$ and $L2$ formulae the same as used by Ref. [7] in the explicit finite difference method solution. One can easily conclude from the presented results that the applied method is a highly good one to obtain numerical solutions of this kind fractional partial differential equations.

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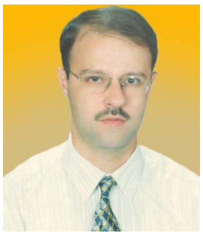

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