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SOLVABILITY THE TELEGRAPH EQUATION WITH PURELY INTEGRAL CONDITIONS

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Abstract. In this paper a numerical technique is developed for the one-dimensional telegraph equation, we prove the existence, uniqueness, and continuous dependence upon the data of solution to a telegraph equation with purely integral conditions. The proofs are based on a priori estimates and Laplace transform method. Finally, we obtain the solution by using a simple and efficient algorithm for numerical solution.

Keywords: Telegraph equation, purely integral conditions, a priori estimates, Laplace transform method.

AMS Subject Classification: 35B45, 35A35, 44A10

1. Introduction

In the rectangular domain $D = \{(x,t) : 0 < x < 1,\ 0 < t \leq T\}$, we consider a second order telegraph equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + a\frac{\partial u}{\partial t} + bu = f(x,t), \qquad 0 < x < 1, \qquad 0 < t \leq T, \tag{1}$$

subject to the initial conditions

$$u(x,0) = \varphi(x), \qquad 0 < x < 1, \tag{2}$$

$$\frac{\partial u(x,0)}{\partial t} = \psi(x), \qquad 0 < x < 1, \tag{3}$$

and the purely integral conditions

$$\int_0^1 u(x,t)\,dx = 0, \qquad 0 < t \leq T, \tag{4}$$

$$\int_0^1 xu(x,t)\,dx = 0, \qquad 0 < t \leq T, \tag{5}$$

where f, φ, and $\psi$are known functions, c, a, b, and T are known positives constants.

The first investigation of this type of problems goes back to [4] in 1996, in which the author proved the existence, uniqueness, and continuous dependence of the solution upon the data of certain hyperbolic problems with only integral boundary conditions. Later, similar problems have been studied in [7, 10, 11, 13, 21] by using the energetic method and the Rothe time-discretization method. We refer the reader to [3, 4, 6, 8, 9, 10, 12, 15,

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20, 22, 23, 24] for hyperbolic equations with Neumann and integral condition. For other problems with nonlocal conditions, related to other equations, we refer to [2, 4, 10, 11, 12] and references therein.

In this paper a Laplace transform method is presented for the problem of obtaining numerical approximations.The main tool used in this paper is the Laplace transform and then used the numerical technique for the inverse Laplace transform to obtain the numerical solution. We use a numerical method for inverting the Laplace transform to get the solution.

The paper is organized as follows. In Section 2, we begin by introducing certain function spaces which are used in the next sections, and we reduce the posed problem to one with homogeneous integral conditions. In Section 3, we first establish the existence of the solution by the Laplace transform. In Section 4, we establish a priori estimates, which give the uniqueness and continuous dependence.

2. Preliminaries and Notations

Definition 2.1. *Denote by $L^2(0,T;H)$ the set of all measurable abstract functions $u(.,t)$ from $(0,T)$ into H equiped with the norm*

$$\|u\|_{L^2(0,T;H)} = \left(\int_0^T \|u(.,t)\|_H^2 \, dt\right)^{1/2} < \infty. \tag{6}$$

Definition 2.2. *We denote by $C_0(0,1)$ the vector space of continuous functions with compact support in $(0,1)$. Since such functions are Lebesgue integrable with respect to dx, we can define on $C_0(0,1)$ the bilinear form given by*

$$((u,w)) = \int_0^1 \Im_x^m u \cdot \Im_x^m w dx, \qquad m \geq 1, \tag{7}$$

where

$$\Im_x^m u = \int_0^x \frac{(x-\xi)^{m-1}}{(m-1)!} u(\xi,t) \, d\xi; \qquad for\ m \geq 1. \tag{8}$$

The bilinear form (2.2) is considered as a scalar product on $C_0(0,1)$ for which $C_0(0,1)$ is not complete.

Definition 2.3. *Denote by $B_2^m(0,1)$, the completion of $C_0(0,1)$ for the scalar product (2.2), which is denoted $(.,.)_{B_2^m(0,1)}$, introduced in [5]. By the norm of function u from $B_2^m(0,1)$, $m \in \mathbb{N}^*$, we understand the nonnegative number :*

$$\|u\|_{B_2^m(0,1)} = \left(\int_0^1 (\Im_x^m u)^2 \, dx\right)^{1/2} = \|\Im_x^m u\|; \qquad for\ m \geq 1. \tag{9}$$

Lemma 2.1. *For all $m \in \mathbb{N}^*$, the following inequality holds:*

$$\|u\|_{B_2^m(0,1)}^2 \leq \frac{1}{2} \|u\|_{B_2^{m-1}(0,1)}^2 . \tag{10}$$

Proof. See [5]. □

Corollary 2.1. *For all $m \in \mathbb{N}^*$, we have the elementary inequality*

$$\|u\|_{B_2^m(0,1)}^2 \leq \left(\frac{1}{2}\right)^m \|u\|_{L^2(0,1)}^2 . \tag{11}$$

Definition 2.4. *We denote by* $L^2(0,T;B_2^m(0,1))$ *the space of functions which are square integrable in the Bochner sense, with the scalar product*

$$(u,w)_{L^2(0,T;B_2^m(0,1))} = \int_0^T (u(.,t), w(.,t))_{B_2^m(0,1)}\, dt. \tag{12}$$

Since the space $B_2^m(0,1)$ *is a Hilbert space, it can be shown that* $L^2(0,T;B_2^m(0,1))$ *is a Hilbert space as well. The set of all continuous abstract functions in* $[0,T]$ *equipped with the norm*

$$\sup_{0\le t\le T} \|u(.,t)\|_{B_2^m(0,1)}$$

is denoted $C(0,T;B_2^m(0,1))$.

Corollary 2.2. *For every* $u \in L^2(0,1)$, *from which we deduce the continuity of the imbedding* $L^2(0,1) \longrightarrow B_2^m(0,1)$, *for* $m \ge 1$.

3. Existence of the Solution

In this section we shall apply the Laplace transform technique to find solutions of partial differential equations, we have the Laplace transform

$$U(x,s) = \mathcal{L}\{u(x,t); t \longrightarrow s\} = \int_0^\infty u(x,t)\exp(-st)\, dt, \tag{13}$$

where s is positive reel parameter. Taking the Laplace transforms on both sides of (1.1), we have

$$-c^2 \frac{d^2}{dx^2}[U(x,s)] + \left(s^2+as+b\right)U(x,s) = F(x,s) + (s+a)\varphi(x) + \psi(x) \ , \tag{14}$$

where $F(x,s) = \mathcal{L}\{f(x,t); t \longrightarrow s\}$. Similarly, we have

$$\int_0^1 U(x,s)\, dx = 0, \tag{15}$$

$$\int_0^1 xU(x,s)\, dx = 0, \tag{16}$$

Thus, considered equation is reduced in boundary value problem governed by second order inhomogeneous ordinary differential equation. We obtain a general solution of (3.2) as

$$U(x,s) = -\frac{c}{\sqrt{s^2+as+b}} \int_0^x [F(\tau,s) + (s+a)\varphi(\tau) + \psi(\tau)\ ] \sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right) d\tau$$

$$+C_1(s)\exp\left(-\frac{\sqrt{s^2+as+b}}{c}x\right) + C_2(s)\exp\left(\frac{\sqrt{s^2+as+b}}{c}x\right), \tag{17}$$

where $C_1$and C_2 are arbitrary functions of s. Substitution of (3.5) into $(3.3)-(3.4)$, we have

$$C_1(s)\int_0^1 \exp\left(-\frac{\sqrt{s^2+as+b}}{c}x\right) dx + C_2(s)\int_0^1 \exp\left(\frac{\sqrt{s^2+as+b}}{c}x\right) dx$$

$$= \frac{c}{\sqrt{s^2+as+b}} \int_0^1 \left[F(\tau,s) + (s+a)\varphi(\tau) + \psi(\tau) \int_\tau^1 \sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right) dx\right] d\tau,$$

$$\begin{aligned} & C_1(s)\int_0^1 x\exp\left(-\frac{\sqrt{s^2+as+b}}{c}x\right)dx + C_2(s)\int_0^1 x\exp\left(\frac{\sqrt{s^2+as+b}}{c}x\right)dx \\ = & \frac{c}{\sqrt{s^2+as+b}}\int_0^1\left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\int_\tau^1 x\sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right)dx\right]d\tau, \end{aligned}$$

where

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix}, \tag{18}$$

and

$$\begin{aligned} a_{11}(s) &= \int_0^1 \exp\left(-\frac{\sqrt{s^2+as+b}}{c}x\right)dx, \\ a_{12}(s) &= \int_0^1 \left(\frac{\sqrt{s^2+as+b}}{c}x\right)dx, \\ a_{21}(s) &= \int_0^1 x\left(-\frac{\sqrt{s^2+as+b}}{c}x\right)dx, \\ a_{22}(s) &= \int_0^1 x\left(\frac{\sqrt{s^2+as+b}}{c}x\right)dx, \\ b_1(s) &= \frac{c}{\sqrt{s^2+as+b}}\int_0^1\left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\int_\tau^1 \sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right)dx\right]d\tau, \end{aligned}$$

$$b_2(s) = \frac{c}{\sqrt{s^2+as+b}}\int_0^1\left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\int_\tau^1 x\sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right)dx\right]d\tau, \tag{19}$$

It is possible to evaluate the integrals in (3.5) and (3.7) exactly. In general, one may have to resort to numerical integration in order to compute them, however. For example, the Gauss's formula (25.4.30) given in Abramowitz and stegun [1] may be employed to calculate these integrals numerically, we have

$$\begin{aligned} & \int_0^1 \exp\left(\pm\frac{\sqrt{s^2+as+b}}{c}x\right)dx \\ \simeq & \frac{1}{2}\sum_{i=1}^N w_i \exp\left(\pm\frac{\sqrt{s^2+as+b}}{2c}[x_i+1]\right), \end{aligned}$$

$$\begin{aligned} & \int_0^1 x\exp\left(\pm\frac{\sqrt{s^2+as+b}}{c}x\right)dx \\ \simeq & \frac{1}{2}\sum_{i=1}^N w_i \left(\frac{1}{2}[x_i+1]\right)\exp\left(\pm\frac{\sqrt{s^2+as+b}}{2c}[x_i+1]\right), \end{aligned}$$

$$\int_0^x \left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\ \right]\sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right)d\tau$$

$$\simeq \frac{x}{2}\sum_{i=1}^{N} w_i\left[F\left(\frac{x}{2}[x_i+1];s\right)+(s+a)\varphi\left(\frac{x}{2}[x_i+1]\right)+\psi\left(\frac{x}{2}[x_i+1]\right)\right]\times$$

$$\times\sinh\left(\frac{\sqrt{s^2+as+b}}{c}\left[x-\frac{x}{2}[x_i+1]\right]\right),$$

$$\int_0^1\left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\int_\tau^1 \sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right)dx\right]d\tau$$

$$\simeq \frac{1}{4}\sum_{i=1}^{N} w_i\left[F\left(\frac{1}{2}[x_i+1];s\right)+(s+a)\varphi\left(\frac{1}{2}[x_i+1]\right)+\psi\left(\frac{1}{2}[x_i+1]\right)\right]\left(1-\frac{1}{2}[x_i+1]\right)\times$$

$$\times\sum_{i=1}^{N} w_j\sinh\left(\frac{\sqrt{s^2+as+b}}{c}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}[x_i+1]\right)x_j+\left(1+\frac{1}{2}[x_i+1]\right)\right]-\frac{1}{2}(x_i+1)\right]\right),$$

$$\int_0^1\left[F(\tau,s)+(s+a)\varphi(\tau)+\psi(\tau)\int_\tau^1 x\sinh\left(\frac{\sqrt{s^2+as+b}}{c}[x-\tau]\right)dx\right]d\tau$$

$$\simeq \frac{1}{4}\sum_{i=1}^{N} w_i\left[F\left(\frac{1}{2}[x_i+1];s\right)+(s+a)\varphi\left(\frac{1}{2}[x_i+1]\right)+\psi\left(\frac{1}{2}[x_i+1]\right)\right]\left(1-\frac{1}{2}[x_i+1]\right)\times$$

$$\times\left(\frac{1}{2}\left[\left(1-\frac{1}{2}[x_i+1]\right)x_j+\left(1+\frac{1}{2}[x_i+1]\right)\right]\right)$$

$$\sum_{i=1}^{N} w_j\sinh\left(\frac{\sqrt{s^2+as+b}}{c}\left[\frac{1}{2}\left[\left(1-\frac{1}{2}[x_i+1]\right)x_j+\left(1+\frac{1}{2}[x_i+1]\right)\right]-\frac{1}{2}(x_i+1)\right]\right) \tag{20}$$

where x_i and w_i are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n(x), \qquad \omega_i \ = 2/\left(1-x_i^2\right)\left[P_n'(x)\right]^2.$$

Their tabulated values can be found in [1] for different values of N.
Numerical inversion of Laplace transform. Sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; therefore a numerical inversion method must be used. A nice comparison of four frequently used numerical Laplace inversion algorithms is given by Hassan Hassanzadeh, Mehran Pooladi-Darvish [16]. In this work we use the Stehfest's algorithm [25] that is easy to implement. This numerical technique was first introduced by Graver [14] and its algorithm then offered by [25].Stehfest's algorithm approximates the time domain solution as

$$u(x,t)\approx\frac{\ln 2}{t}\sum_{n=1}^{2m}\beta_n U\left(x;\frac{n\ln 2}{t}\right), \tag{21}$$

where, m is the positive integer,

$$\beta_n=(-1)^{n+m}\sum_{k=\left[\frac{n+1}{2}\right]}^{\min(n,m)}\frac{k^m(2k)!}{(m-k)!k!(k-1)!(n-k)!(2k-n)!}, \tag{22}$$

and $[q]$ denotes the integer part of the real number q.The parameter m is a free parameter that should be optimized by trial and error. It was seen that with increasing m accuracy of result increases up to a point and then owing to the rounding errors it decreases [25]. Thus, for choosing optimum m, it is beneficial to apply an algorithm repeatedly for different values of m and study its effect on the solution. The other way to choose optimal value of m could be, to apply the Stehfest's algorithm for inverting the Laplace transform of some elementry functions which are known.

Remark 3.1. *1) Stehfest's method gives accurate results for many problems including diffusion problem, fractional functions in the Laplace domain. However, it fails to predict e^t type functions or those with oscillatory behavior such as sine and wave functions (see* [16]*). 2) Note that more than one numerical inversion algorithm can also be performed to check the accuracy of the result.*

4. Uniqueness and Continuous dependence of the Solution

We first establish an a priori estimate, the uniqueness and continuous dependence of the solution with respect to the data are immediate consequences.

Theorem 4.1. *If $u(x,t)$ is a solution of problem* $(1.1)-(1.5)$ *and* $f \in C\left(\overline{D}\right)$*, then we have*

$$\begin{aligned} &\|u(.,\tau)\|^2_{L^2(0,1)} \\ \leq\ & c_1\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\,dt + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)}\right), \end{aligned} \tag{23}$$

$$\begin{aligned} &\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)} \\ \leq\ & c_2\left(\int_0^\tau \|f(.,t)\|^2_{B_2^1(0,1)}\,dt + \|\varphi\|^2_{L^2(0,1)} + \|\psi\|^2_{B_2^1(0,1)}\right), \end{aligned} \tag{24}$$

where

$$c_1 = \frac{1}{(b+2c^2)}\max\left(1, \frac{1}{2a}, \frac{(b+2c^2)}{2}\right), c_2 = \max\left(1, \frac{1}{2a}, \frac{(b+2c^2)}{2}\right)$$

and $0 \leq \tau \leq T$.

Proof. Taking the scalar product in $B_2^1(0,1)$ of both sides of equation (1.1) with $\frac{\partial u}{\partial t}$, and integrating over $(0,\tau)$, we have

$$\begin{aligned} &\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial t^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt - c^2\int_0^\tau \left(\frac{\partial^2 u(.,t)}{\partial x^2}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt + \\ &a\int_0^\tau \left(\frac{\partial u(.,t)}{\partial t}, \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} + b\int_0^\tau \left(u(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} \\ &= \int_0^\tau \left(f(.,t), \frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)} dt. \end{aligned} \tag{25}$$

Integrating by parts on the left-hand side of (4.3), we obtain

$$\frac{1}{2}\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(0,1)}+\left(\frac{b}{2}+c^2\right)\|u(.,\tau)\|^2_{B_2^1(0,1)}+a\int_0^\tau\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)}dt\leq$$
$$\int_0^\tau\left(f(.,t),\frac{\partial u(.,t)}{\partial t}\right)_{B_2^1(0,1)}dt+\frac{1}{2}\|\psi\|^2_{B_2^1(0,1)}+\left(\frac{b+2c^2}{4}\right)\|\varphi\|^2_{L^2(0,1)} \tag{26}$$

By the $\varepsilon-$Cauchy inequality, the first term in the right-hand side of (4.4) is bounded by

$$\frac{\varepsilon}{2}\int_0^\tau\|f(.,t)\|^2_{B_2^1(0,1)}dt+\frac{1}{2\varepsilon}\int_0^\tau\left\|\frac{\partial u(.,t)}{\partial t}\right\|^2_{B_2^1(0,1)}dt. \tag{27}$$

We choose $\varepsilon=\frac{1}{2a}$ so that the second term will be simplified by the third term in the left-hand sid. Thus we have

$$\begin{aligned}&\left\|\frac{\partial u(.,\tau)}{\partial t}\right\|^2_{B_2^1(0,1)}+\left(b+2c^2\right)\|u(.,\tau)\|^2_{L^2(0,1)}\\ \leq\;&\frac{1}{2a}\int_0^\tau\|f(.,t)\|^2_{B_2^1(0,1)}dt+\|\psi\|^2_{B_2^1(0,1)}+\left(\frac{b+2c^2}{2}\right)\|\varphi\|^2_{L^2(0,1)}.\end{aligned} \tag{28}$$

From (4.6), we obtain estimates (4.1) and (4.2). □

Corollary 4.1. *If problem* $(1.1)-(1.5)$ *has a solution, then this solution is unique and depends continuously on* (f,φ,ψ).

5. Conclusion

In this work we study a Telegraph equation with purely integral conditions. The existence and uniqueness of the solution are proved. The proof is based on a priori estimates and Laplace transform method, sometimes, an analytical inversion of a Laplace domain solution is difficult to obtain; therefore a numerical inversion method must be used. We use the Stehfest's algorithm that is easy to implement to obtain approximate solution.

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