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ON THE DISTRIBUTION OF THE MAXIMUM OF SUMS OF A SEQUENCE OF INDEPENDENT AND IDENTICALLY DISTRIBUTED RANDOM VARIABLES.

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ABSTRACT. The authors find in terms of generating functions the distribution of the maximum of sums of independent and identically distributed random variables with no negative jumps different from -1.
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1. Introduction

In the following paragraphs, we consider a sequence of independent and identically distributed random variables $\{\xi_1, \xi_2, \dots, \xi_n, \dots\}$ given on the probability space $\{\Omega, F, P(\cdot)\}$, with $\xi_i \in \{-1, 0, 1, 2, \dots\}$, $i \geq 1$ and $P\{\xi_i = k\} = P_k$, $i \geq 1$, $P_{-2} = P_{-3} = \dots = 0$.

In an integer phase it is natural to call such processes downwardly continuous [1].

From the sums $S_0 = 0$, $S_n = \xi_1 + \xi_2 + \dots + \xi_n$, $n \geq 1$, we obtain the sequence $\{S_n\}$, $n \geq 1$, which constitutes a random walk.

Let us denote

$$\mu_n = \max_{0 \leq i \leq n} S_i, \quad n = 0, 1, 2, \dots.$$

Our aim is to find the distribution of the variable μ_n, $n \geq 0$ for such a random walk. Problems of this kind arise while investigating queue systems, inventory management systems, etc.

Many authors studied a number of problems of theory of random walks in the plane. Recently, special emphasis is placed on the investigation of random walks in the subexponential case and two-boundary problems for them. For instance, D. A. Korshunov considers in his paper [2] the random walk $\{S_n\}$, $n \geq 1$ with the negative drift $M\xi_1 = -m < 0$ and an excess with heavy tails, when $Me^{\lambda \xi_1} = \infty$ for any $\lambda > 0$. The author investigated only the asymptotic behavior at infinity of the distribution density of the supremum $\sup_n S_n$. I. I. Yezhov, V. F. Kadankov, T. V. Kadankova [3] consider the sequence of independent and identically distributed random variables $\xi \in R = (-\infty, +\infty)$, $P\{\xi = 0\} < 1$ and ξ_n, $n \in N = \{1, 2, 3, \dots\}$.

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Considering the random walk $\xi_n \in R$, $n \in Z = \{0, 1, \ldots\}$ generated by the random variable $\xi = \xi(0) = 0$, $\xi_n = \xi_1 + \ldots + \xi_n$, $n \in N$ for all $x \geq 0$, they investigated the random variables

$$t^x = inf\{n : \xi(n) > x\}, \quad T^x = \xi(t^x) - x,$$
$$t_x = inf\{n : \xi(n) < -x\}, \quad T_x = -\xi(t_x) - x.$$

Using generating functions of the joint distributions $\{t^x, T^x\}$ and $\{t_x, T_x\}$ for the right half-plane $R(p) \geq 0$ (p is the parameter), in terms of generating functions and Laplace transform, the joint distribution of the first exit time from the interval and the value of the overshoot by the random walk was determined. However, practically, this solution method for two-boundary problems for random walks is unacceptable for special class problems, since it is based on the use of distribution of one-boundary functionals, whereas for determining the latter, combinatorial and factorization methods are used.

In the special class problems, using methods of the theory of generating functions and principles of random walks, we consider the problem of determining the distribution of the variable μ_n, $n \geq 0$, which, in turn, allows us to determine the distribution $\rho_n(k) = P\{S_n = k\}$ of the random walk $\{S_n\}$, $n \geq 0$. It is important that to do that we have to deal only with the distribution $P\{\mu_n = 0\}$, $n \geq 0$.

Let us proceed to determine the distribution of the random variable μ_n, $n \geq 0$. It stands to reason that the generating function $\varphi(z)$ of the random variable ξ_1, i.e.

$$\varphi(z) = Mz^{\xi_1} = \sum_{k=-1}^{\infty} P_k z^k, \quad |z| \leq 1$$

is not an analytic function at the point 0. And if we introduce the new variable

$$\xi_i^* = \xi_1 + 1, \quad i \geq 1,$$

its generating function

$$\varphi^*(z) = z\varphi(z) = \sum_{k=-1}^{\infty} \pi_k \, z^k, \quad \pi_k = P_{k-1}$$

is an analytic function in the circle $|z| \leq 1$.

For S_n, $n \geq 1$, we get

$$Mz^{S_n} = Mz^{(\xi_1^* + \ldots \xi_n^*)} z^n = \frac{[\varphi^*(z)]^n}{z^n}.$$

Seeing that

$$\varphi^*(z) = z\varphi(z) = \sum_{k=0}^{\infty} \pi_k \, z^k, \quad |z| \leq 1,$$

then

$$[\varphi^*(z)]^n = \sum_{k=0}^{\infty} \pi_k^{n^*} \, z^k, \quad |z| \leq 1,$$

where $\pi_k^{n^*}$ is n-fold convolution of distribution, π_k, $k \geq 1$.

According to the principles of the random walk theory [4], the following holds:

$$\begin{aligned} \pi_k^{2^*} &= \pi_0 \pi_k + \pi_1 \pi_{k-1} + \ldots + \pi_k \pi_0 \\ \pi_k^{3^*} &= \pi_0^{2^*} \pi_k + \pi_1^{2^*} \pi_{k-1} + \ldots + \pi_k^{2^*} \pi_0 \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{1}$$

Thus, the sequence $\left\{\pi_k^{n^*}\right\}$ is known for any $k,\, n \geq 1$.

2. Main results

With the known sequence (1), let us proceed to determine the explicit form of the generating function from the variables $x_n = P\{\mu_n = 0\}$. It should be noted that it follows from the formation of the sequence $\{\pi_k^{n^*}\}$ that $P\{S_n = k\} = \pi_{k+1}^{n^*} = \rho_n(k)$ is the probability of the event $\{S_n = k\}$.

Assume that we reach the trajectory peak of the walk $\{S_n\}$, $n \geq 1$, i.e. $\mu_i \geq k$. Two options are possible here: either the endpoint is below the level k, then $S_n \geq k$, or neither upper nor lower events cross on the level k - then $S_n < k$. In that case, $\mu_{n-1} \geq k$. Therefore, the following equality holds:

$$P\{\mu_n \geq k\} = P\{S_n \geq k\} + P\{\mu_{n-1} \geq k,\ S_n < k\}. \tag{2}$$

μ_i being the maximum among all peaks of our walk, at least one such i exists when $S_i \geq k$. If it turns out that $S_i = k$, there will be such instant j, when $S_j = k$. Since at some point we are above the level k, we cannot pass it getting below the level k, because we progress only by unity. If at that last instant $S_j = k$, $0 \leq j \leq n-1$, then we are below the level k at all subsequent instants.

Let us denote

$$C_j = \{S_j = k,\ S_{j+1} < k,\ \ldots,\ S_n < k\}.$$

Then the event $\{\mu_{n-1} \geq k,\ S_n < k\}$ is the union of events C_j, i.e.

$$\{\mu_{n-1} \geq k,\ S_n < k\} = \bigcup_{j=0}^{n-1} C_j. \tag{3}$$

By virtue of (2) and (3), we have the following:

$$P\{\mu_n \geq k\} = P\{S_n \geq k\} + \sum_{j=o}^{n-1} P\{S_j = k,\ S_{j+1} < k,\ \ldots,\ S_n < k\}.$$

In what follows

(4)
$$\begin{aligned} P\{S_j = k,\ S_{j+1} < k,\ \ldots,\ S_n < k\} &= P\{S_j = k\} P\{S_{j+1} < k,\ \ldots,\ S_n < k/S_j = k\} = \\ &= \rho_j(k) P\{\xi_1 + \ldots + \xi_{j+1} < k, \ldots, \xi_1 + \ldots + \xi_n < k/\xi_1 + \ldots + \xi_j = k\} = \\ &= \rho_j(k) P\{\xi_{j+1} < 0, \ldots, \xi_{j+1} + \ldots + \xi_n < 0/\xi_1 + \ldots + \xi_j = k\} = \\ &= \rho_j(k) P\{\xi_{j+1} < 0, \ldots, \xi_{j+1} + \ldots + \xi_n < 0\}. \end{aligned}$$

Since $\xi_1, \xi_2, \ldots$ are independent and identically distributed random variables,

(5)
$$\{S_1 < 0,\ S_2 < 0, \ldots,\ S_{n-j} < 0\} \equiv \{\xi_1 < 0,\ \xi_1 + \xi_2 < 0, \ldots \xi_1 + \xi_2 + \ldots + \xi_{n-j} < 0\},$$

i.e. these two events are identically equal. Considering that $\xi_1 = -1$, it follows from the right-hand side of (5) that $\xi_2 \leq 0$; and from the previous inequality, we get that $\xi_2 + \ldots + \xi_{n-j} \leq 0$.

By virtue of this reasoning from (4) we obtain

$$\begin{aligned} P\{S_j = k,\ S_{j+1} < k,\ \ldots,\ S_n < k\} &= \rho_j(k)\ P\{\xi_0 = -1,\ S_1 \leq 0, \ldots, S_{n-j-1} \leq 0\} = \\ &= \rho_j(k)\ \pi_0\ P\{S_1 \leq 0, \ldots,\ S_{n-j-1} \leq 0\} = \rho_j(k)\ \pi_0\ P\left\{\max_{1 \leq i \leq n-j-1} S_i \leq 0\right\}. \end{aligned} \tag{6}$$

By convention, $S_0 = 0$. Hence, it follows from (6) that

$$P\{S_j = k,\ S_{j+1} < k,\ \ldots,\ S_n < k\} = \rho_j(k)\,\pi_0\,P\left\{\max_{1\le i\le n-j-1} S_i \le 0\right\} =$$
$$= \rho_j(k)\,\pi_0\,P\left\{\max_{0\le i\le n-j-1} S_i = 0\right\}.$$

Thus,

$$P\{\mu_n \ge k\} = P\{S_n \ge k\} + \pi_0 \sum_{j=0}^{n-1} \rho_j(k)\,P\left\{\mu_{n-j-1} = 0\right\}.$$

Since

$$P\{S_n \ge k\} = \sum_{j\ge k} \rho_n(j),$$

then

$$P\{\mu_n \ge k\} = \sum_{j\ge k} \rho_n(j) + \pi_0 \sum_{j=0}^{n-1} \rho_j(k)\,P\left\{\mu_{n-j-1} = 0\right\}. \tag{7}$$

By virtue of (7)

$$P\{\mu_n = 0\} = P\{S_n = 0\} + \pi_0 \sum_{j=0}^{n-1} \left[\rho_j(1) - \rho_j(1)\right] P\{\mu_{n-j-1} = 0\}.$$

Let us denote

$$x_n = P\{\mu_n = 0\}.$$

We get

$$x_n = \rho_n(0) + \pi_0 \sum_{j=0}^{n-1} \left[\rho_j(0) - \rho_j(1)\right] x_{n-j-1}. \tag{8}$$

If

$$x(z) = \sum_{n=0}^{\infty} x_n\,z^n, \quad |z| \le 1$$

and

$$\tilde{\rho}_k(z) = \sum_{n=0}^{\infty} \rho_n(k)\,z^n, \quad |z| \le 1,$$

then the equation (8) in convolutions takes on the following form:

$$x(z) = \tilde{\rho}_0(z) + \pi_0(z)\left[\tilde{\rho}_0(z) - \tilde{\rho}_1(z)\right],$$

hence

$$x(z) = \frac{\tilde{\rho}_0(z)}{1 - \pi_0(z)\left[\tilde{\rho}_0(z) - \tilde{\rho}_1(z)\right]}.$$

Knowledge of the generating function $x(z)$ allows for determining the probabilities

$$\rho_n(k) = P\{S_n = k\} = \pi_{k+n}^{n^*}$$

of the event $\{S_n = k\}$.

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