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GRAPHS COSPECTRAL WITH MULTICONE GRAPHS $K_w \bigtriangledown L(P)$

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ABSTRACT. E. R. van Dam and W. H. Haemers [15] conjectured that almost all graphs are determined by their spectra. Nevertheless, the set of graphs which are known to be determined by their spectra is small. Hence, discovering infinite classes of graphs that are determined by their spectra can be an interesting problem. The aim of this paper is to characterize new classes of multicone graphs that are determined by their spectrum. A multicone graph is defined to be the join of a clique and a regular graph. It is proved that any graph cospectral with multicone graph $K_w \bigtriangledown L(P)$ is determined by its adjacency spectrum as well as its Laplacian spectrum, where K_w and $L(P)$ denote a complete graph on w vertices and the line graph of the Petersen graph, respectively. Finally, three problems for further researches are proposed.

Keywords: adjacency spectrum, Laplacian spectrum, DS graph, line graph of Petersen graph.

AMS Subject Classification: 05C50.

1. INTRODUCTION

All graphs considered here are simple and undirected. All terminology and notations on graphs not defined here, can be found in [3, 4, 5, 19]. Let G be a graph with the adjacency matrix A. We denote $det(\lambda I - A)$, the characteristic polynomial of G, by $P_G(\lambda)$. The multiset of eigenvalues of A is called the adjacency spectrum, or simply the spectrum of G. Since A is a symmetric matrix, the eigenvalues of G are real. Two non-isomorphic graphs with the same spectrum are called cospectral. We say that a graph is determined by the spectrum (DS for short) if there is no other non-isomorphic graph with the same spectrum. For two graphs G_1 and G_2 with disjoint vertex sets $V(G_1)$ and $V(G_2)$ and disjoint edge sets $E(G_1)$ and $E(G_2)$ the disjointunion of G_1 and G_2 is the graph $G = G_1 \cup G_2$ with the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. The join $G \bigtriangledown H$ of simple undirected graphs G and H is the graph with the vertex set $V(G \bigtriangledown H) = V(G) \cup V(H)$ and the edge set $E(G \bigtriangledown H) = E(G) \cup E(H) \cup \{uv| \, u \in V(G), \, v \in V(H)\}$. For a graph G, let $D(G) = diag(d_1, d_2, ..., d_n)$ be the diagonal matrix of vertex degrees, and let $A(G)$ be the adjacency matrix of G. The matrices $SL(G) = D(G) + A(G)$ and $Lp(G) = D(G) - A(G)$ are called the signless Laplacian matrix and Laplacian matrix of G, respectively. Line graph of a graph G denoted by $L(G)$, is a graph that its vertices are edges of G and two vertices of $L(G)$ are adjacent if their corresponding edges in G have a common vertex.
The background of the question " Which graphs are determined by their spectrum?"

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originates from Chemistry (in 1956, Günthadr and Primas [8] raised this question in the context of Hückel's theory). A remarkable fact is that there are many wonderful papers to studying cospectral graphs and introducing kinds of methods of constructing them (see [7], [9], [10], [11], [17] and [18], for example). In [15], it is conjectured that almost all graphs are DS. Nevertheless, the set of graphs which are known to be DS is small and therefore it would be interesting to find more examples of DS graphs. For a survey of the subject, see [15, 16]. Special classes of multicone graphs were characterizd in [17]. In [17], Authors investigated on the spectral characterization of multicone graphs and also they claimed that friendship graph F_n is DS with respect to its adjacency spectra. In [1], Abdian and Mirafzal characterized new classes of multicone graphs that were DS with respect to their spectra. In this paper, we characterize another new classes of multicone graphs that are DS with respect to their spectra.

2. Preliminaries

In this section, we give some results that will be used in the sequal.

Lemma 2.1. [2, 13] *Let G be a graph. For the adjacency matrix and Laplacian matrix, the following can be obtained from the spectrum:*
(i) The number of vertices,
(ii) The number of edges.
For the adjacency matrix, the following follows from the spectrum:
(iii) The number of closed walks of any length.
(iv) Being regular or not and the degree of regularity.
(v) Being bipartite or not.
For the Laplacian matrix, the following follows from the spectrum:
(vi) The number of components.

Theorem 2.1. [5] *If G_1 is r_1-regular with n_1 vertices, and G_2 is r_2-regular with n_2 vertices, then the characteristic polynomial of the join $G_1 \bigtriangledown G_2$ is given by:*

$$P_{G_1 \bigtriangledown G_2(x)} = \frac{P_{G_1}(x)P_{G_2}(x)}{(x-r_1)(x-r_2)}((x-r_1)(x-r_2)-n_1n_2).$$

Theorem 2.2. [1] *Let G be a simple graph with n vertices and m edges. Let $\delta = \delta(G)$ be the minimum degree of vertices of G and $\varrho(G)$ be the spectral radius of the adjacency matrix of G. Then*

$$\varrho(G) \leq \frac{\delta - 1}{2} + \sqrt{2m - n\delta + \frac{(\delta+1)^2}{4}}.$$

Equality holds if and only if G is either a regular graph or a bidegreed graph in which each vertex is of degree either δ or $n-1$.

Theorem 2.3. [12] *Let G and H be two graphs with Laplacian spectrum $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$ and $\mu_1 \geq \mu_2 \geq ... \geq \mu_m$, respectively. Then Laplacian spectra of $\overline{G}$ and $G \bigtriangledown H$ are $n-\lambda_1, n-\lambda_2, ..., n-\lambda_{n-1}, 0$ and $n+m, m+\lambda_1, ..., m+\lambda_{n-1}, n+\mu_1, ..., n+\mu_{m-1}, 0$, respectively.*

Lemma 2.2. [12] *Let G be a graph on n vertices. Then n is Laplacian eigenvalue of G if and only if G is the join of two graphs.*

Proposition 2.1. [1] *For a graph G, the following statements are equivalent:*
(i) G is d-regular.
(ii) $\varrho(G) = d_G$, the average vertex degree.
(iii) G has $v = (1, 1, ..., 1)^t$ as an eigenvector for $\varrho(G)$.

Theorem 2.4. [5, 14] *Let $G-j$ be the graph obtained from G by deleting the vertex j and all edges containing j. Then $P_{G-j}(x) = P_G(x)\sum_{i=1}^{m}\frac{\alpha_{ij}^2}{x-\mu_i}$, where m, α_{ij} and P_G are the number of distinct eigenvalues of graph G, main angle of G and characteristic polynomial of G, respectively.*

The rest of this paper is organized as follows. In Section 3, we prove that multicone graphs $K_w \bigtriangledown L(P)$ are determined by their adjacency spectrum. In Section 4, we show that these graphs are DS with respect to their Laplacian spectrum. In Section 5, we review what was said in the previous sections and finally we propose three conjectures for further researches.

Table 1. The line graph of the Petersen graph and some its graphical properties

The line graph of the Petersen graph	Some graphical properties
$\|V\|$	15
$\|E\|$	30
Girth	3
Graphical properties	Regular, Non-bipartite graph.
Adjacency spectrum of line graph of the Perersen graph	$\left\{[4]^1,\, [2]^5,\, [-1]^4,\, [-2]^5\right\}$

3. Main results

3.0.1. *Graphs cospectral with multicone graphs $K_w \bigtriangledown L(P)$.*

In this subsection, we show that any graph cospectral with a multicone graph$K_w \bigtriangledown L(P)$ is determined by its adjacency spectrum.

Proposition 3.1. *Let G be a graph. If $Spec(G) = Spec(K_w \bigtriangledown L(P))$. Then $Spec(G) = \left\{[-1]^{w-1},\, [2]^5,\, [-1]^4,\, [-2]^5,\, \left[\frac{\Omega+\sqrt{\Omega^2+4\Gamma}}{2}\right]^1,\, \left[\frac{\Omega-\sqrt{\Omega^2+4\Gamma}}{2}\right]^1\right\}$, where $\Omega = w+3$ and $\Gamma = 11w+4$.*

Proof. By Theorem 2.2 and $Spec(L(P)) = \left\{[4]^1,\, [2]^5,\, [-1]^4,\, [-2]^5\right\}$ the proof is completed. □

In Lemma 3.3 we show that any graph cospectral with a multicone graph $K_w \bigtriangledown L(P)$ must be bidegreed.

Lemma 3.1. *Let G be cospectral with a multicone graph $K_w \bigtriangledown L(P)$. Then $\delta(G) = w+4$.*

Proof. Assume $\delta(G) = w+4+x$, where x is an integer number. First, it is clear that in this case the equality in Theorem 2.3 happens, if and only if $x = 0$. We show that $x = 0$. By contrary, we suppose that $x \neq 0$. It follows from Theorem 2.3 and Proposition 3.1 that $\varrho(G) = \frac{w+3+\sqrt{8k-4l(w+4)+(w+5)^2}}{2} < \frac{w+3+x+\sqrt{8k-4l(w+4)+(w+5)^2+x^2+(2w+10-4l)x}}{2}$, where the integer numbers k and l denote the number of edges and the number of vertices of the graph G, respectively. A simple computation follows that

$$\sqrt{Q} - \sqrt{Q+g(x)} < x.$$

, where $Q = 8k - 4l(w+4) + (w+5)^2 \geq 0$ and $g(x) = x^2 + (2w + 10 - 4l)x$. In the following, we denote the absolute value of y by $|y|$, where y is an arbitrary integer number.

Now, we consider two cases:

Case 1. $x < 0$.

$|\sqrt{Q} - \sqrt{Q + g(x)}| > |x| \Longrightarrow 2Q + g(x) - 2\sqrt{Q(Q + g(x))} > x^2 \Longrightarrow Q + (w + 5 - 2l)x > \sqrt{Q(Q + g(x)} \Longrightarrow Q^2 + [(w + 5 - 2l)x]^2 + 2Q(w + 5 - 2l)x > Q^2 + Qg(x) \Longrightarrow [(w + 5 - 2l)x]^2 + 2Q(w + 5 - 2l)x > Qg(x) \Longrightarrow [(w + 5 - 2l)x]^2 + 2Q(w + 5 - 2l)x > Q(x^2+(2w+10-4l)x) \Longrightarrow [(w+5-2l)x]^2+Q(2w+10-4l)x > Q(x^2+(2w+10-4l)x) \Longrightarrow (w + 5 - 2l)^2 > Q \Longrightarrow (w + 5 - 2l)^2 > 8k - 4l(w + 4) + (w + 5)^2 \Longrightarrow k < \frac{l(l-1)}{2}$.

Therefore, if $x < 0$, then G cannot be a complete graph. In other words, if G is a complete graph, then $x > 0$. Or one can say that for a complete graph G cospectral with a multicone graph $K_w \bigtriangledown L(P)$, we must have $\delta(G) > w + 4$.

Case 2. $x > 0$.

In the same way of Case 1, one can conclude that for a complete graph G cospectral with a multicone graph $K_w \bigtriangledown L(P)$, we must have $\delta(G) < w + 4$.

But, Case 1 and Case 2 are contradict to each other. So, we must have $x = 0$. Therefore, the assertion holds. □

Lemma 3.2. *If G be a graph and $Spec(G) = Spec(K_w \bigtriangledown L(P))$, then G is bidegreed, in which any vertex of G is of degree $w + 4$ or $w + 14$.*

Proof. Lemma 3.2 together with Theorem 2.3 and Proposition 2.6 complete the proof. □

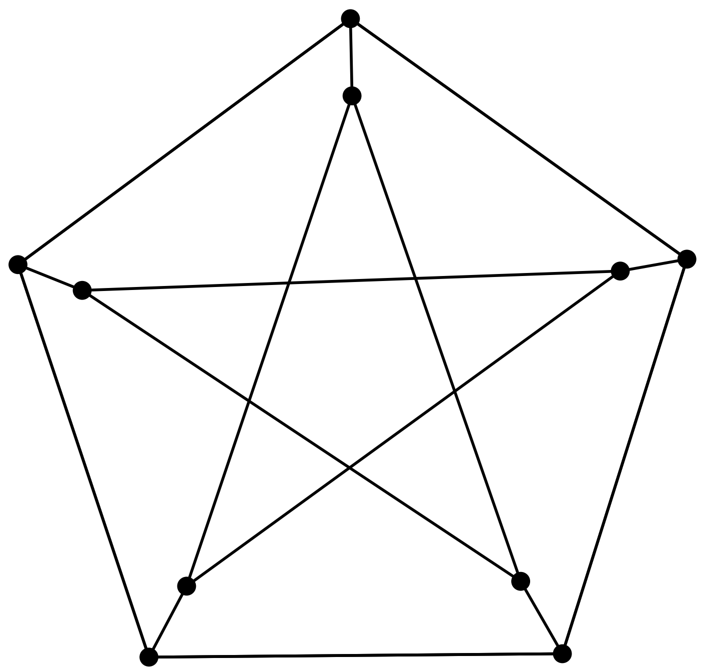

FIGURE 1. Petersen graph

Lemma 3.3. *Any graph cospectral with the multicone graph $K_1 \bigtriangledown L(P)$ is DS with respect to its adjacency spectrum.*

Proof. By Lemma 3.3, G has one vertex of degree 15, say j. On the other hand, Proposition 2.7, follows that $P_{G-j}(\lambda) = (\lambda-\mu_3)^4(\lambda-\mu_4)^3(\lambda-\mu_5)^4[\alpha_{1j}^2 A+\alpha_{2j}^2 B+\alpha_{3j}^2 C+\alpha_{4j}^2 D+\alpha_{5j}^2 E]$.

$A = (\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5)$,
$B = (\lambda - \mu_1)(\lambda - \mu_3)(\lambda - \mu_4)(\lambda - \mu_5)$,
$C = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_4)(\lambda - \mu_5)$,
$D = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_5)$,
$E = (\lambda - \mu_1)(\lambda - \mu_2)(\lambda - \mu_3)(\lambda - \mu_4)$,
where $\mu_1 = \dfrac{4+\sqrt{76}}{2}$, $\mu_2 = \dfrac{4-\sqrt{76}}{2}$, $\mu_3 = 2$, $\mu_4 = -1$ and $\mu_5 = -2$.
We know that $G - j$ has 15 eigenvalues. Also, if we remove the vertex j of graph G, then the number of edges and triangles that are removed of graph G is 15 and 30, respectively. Moreover, Lemma 3.3 follows that $G - j$ is regular and degree of its regularity is 4. By Lemma 2.1 (iii) for the closed walks of lengths 1, 2 and 3 we have:
$x + y + z + 4 = -(4\mu_3 + 3\mu_4 + 4\mu_5)$
$x^2 + y^2 + z^2 + 16 = 60 - (4\mu_3^2 + 3\mu_4^2 + 4\mu_5^2)$
$x^3 + y^3 + z^3 + 64 = 60 - (4\mu_3^3 + 3\mu_4^3 + 4\mu_5^3)$
, where x, y and z are eigenvalues of $P_{G-j}(\lambda)$. If we solve the above equations, then $x = 2$, $y = -1$ and $z = -2$. So, $Spec(G - j) = \left\{[4]^1, [2]^5, [-1]^4, [-2]^5\right\}$. It is well-known the line graph of the Petrsen graph is DS with respect to its adjacency spectrum. Therefore, $G - j \cong L(P)$ and so $G \cong K_1 \bigtriangledown L(P)$.

□

Up to now, we have shown the multicone graph $K_1 \bigtriangledown L(P)$ is DS. The natural questions is; what happen for multicone graphs $K_w \bigtriangledown L(P)$? we answer to this question in the next theorem.

Theorem 3.1. *Any graph cospectral with a multicone graphs $K_w \bigtriangledown L(P)$ is isomorphic to $K_w \bigtriangledown L(P)$.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing for proof. Let the problem be true for w; that is, if $Spec(G_1) = Spec(K_w \bigtriangledown L(P))$, then $G_1 \cong K_w \bigtriangledown L(P)$, where G_1 is an arbitrary graph cospectral with multicone graph $Spec(G) = Spec(K_w \bigtriangledown L(P))$. We show that $Spec(G) = Spec(K_{w+1} \bigtriangledown L(P))$ follows that $K_w \bigtriangledown L(P)$, where G is an arbitrary graph cospectral with multicone graph $Spec(G) = Spec(K_{w+1} \bigtriangledown L(P))$. First, it is obvious that G has one vertex and $w+15$ edges more than G_1. Now, by Lemma 3.3, it follows that G_1 has w vertices of degree $w + 14$ and 15 vertices of degree $w + 4$ and also G has $w + 1$ vertices of degree $w + 15$ and 15 vertices of degree $w + 5$. So, we must have $G \cong K_1 \bigtriangledown G_1$. Now, the induction hypothesis completes the proof. □

In the following, we present another proof of Theorem 3.5.

Proof. Let $Spec(G) = Spec(K_w \bigtriangledown L(P))$. By Lemma 3.3, G has graph Γ as its subgraph in which degree of any vertex of Γ is $w + 14$. On the other words, $G \cong K_w \bigtriangledown H$, where H is a subgraph of G. Now, we remove vertices of K_w and we consider 15 another vertices. Degree of graph consists of these vertices is 4, say H. By Theorem 2.2, $Spec(H) = \left\{[4]^1, [2]^5, [-1]^4, [-2]^5\right\} = Spec(L(P))$. This completes the proof. □

In the following, we show that multicone graphs $K_w \bigtriangledown L(P)$ are DS with respect to their adjacency spectrum.

4. Connected graphs cospectral with a multicone graph $K_w \bigtriangledown L(P)$ with respect to Laplacian spectrum

Theorem 4.1. *Multicone graphs $K_w \bigtriangledown L(P)$ are DS with respect to their Laplacian spectrum.*

Proof. We solve the problem by induction on w. If $w = 1$, there is nothing to prove. Let the problem be true for values of less than w; that is, $Spec(Lp(G_1)) = Spec(Lp(K_w \bigtriangledown L(P))$ implies that $G_1 \cong K_w \bigtriangledown L(P)$, where G_1 is a graph. We show that the problem is true for $w+1$; that is, we show that $Spec(Lp(G)) = Spec(Lp(K_{w+1} \bigtriangledown L(P))) = \Big\{[w+16]^{w+1}, [w+3]^4, [0]^1, [w+6]^4, [w+7]^5\Big\}$ follows that $G \cong K_{w+1} \bigtriangledown L(P)$, where G is a graph. But, $Spec(Lp(G_1)) = Spec(Lp(K_w \bigtriangledown L(P))) = \Big\{[w+15]^w, [w+2]^4, [0]^1, [w+5]^4, [w+6]^5\Big\}$ implies that $G_1 \cong K_w \bigtriangledown L(P)$. Theorem 2.5 implies that G_1 and G are the join of two graphs. On the other hand, $Spec(Lp(K_1 \bigtriangledown G_1)) = Spec(Lp(G)) = spec(Lp(K_{w+1} \bigtriangledown L(P)))$ and also G has one vertex and $w+15$ edges more than G_1. Therefore, we must have $G \cong K_1 \bigtriangledown G_1$. Now, the induction hypothesis completes the proof. □

5. Conclusion remark and Conjectures

Now, we review what were proved in the earlier sections and finally we pose three conjectures.

Corollary 5.1. *Let G be a graph. The following statements are eguivalent:*
(i) $G \cong K_w \bigtriangledown L(P)$,
(ii) $Spec(G) = Spec(K_w \bigtriangledown L(P))$,
(iii) $Spec(Lp(G)) = Spec(Lp(K_w \bigtriangledown L(P))$.

In the following, we pose three conjectures.

Conjecture 5.1. *Multicone graphs $K_w \bigtriangledown L(P)$ are DS with respect to their signless Laplacian spectrum.*

Conjecture 5.2. *The complement of multicone graphs $K_w \bigtriangledown L(P)$ are DS with respect to their adjacency spectrum.*

Conjecture 5.3. *The complement of multicone graphs $K_w \bigtriangledown L(P)$ are DS with respect to their signless Laplacian spectrum.*

References

[1] Abdian,A.Z. and Mirafzal,S.M., (2015), On new classes of multicone graphs determined by their spectrums, Alg. Struc. Appl., 2, pp.23-34.
[2] Abdollahi,A., Janbaz,S., and Oubodi,M., (2013), Graphs cospectral with a friendship graph or its complement, Trans. Comb., 2, pp.37-52.
[3] Bapat,R.B., (2010), Graphs and matrices, New York (NY), Springer.
[4] Biggs,N.L., (1933), Algebraic Graph Theory, Cambridge university press.
[5] Cvetković,D., Rowlinson,P., and Simić,S., (2010), An Introduction to the Theory of graph Spectra, London Mathematical Society Student Texts, 75, Cambridge University Press.
[6] Das,K.C., (2013), Proof of conjectures on adjacency eigenvalues of graphs, Disceret Math, 313 pp.19-25.
[7] Godsil,C.D. and McKay,B.D., (1982), Constructing cospectral graphs, Aequationes Math., 25, pp.252-268.
[8] Günthard,Hs.H. and Primas,H., (1956), Zusammenhang von Graphtheorie und Mo-Theorie von-Molekeln mit Systemen konjugierter Bindungen., Helv. Chim. Acta, 39, pp.1645-1653.
[9] Haemers,W.H. and Spence,E., (2004), Enumeration of cospectral graphs, Eur. J. Comb., 25 pp.199211.

[10] Harary,F., King,C., Mowshowitz,A., and Read,R., (1971), Cospectral graphs and digraphs, Bull. Lond. Math. Soc., 3, pp.321328.
[11] Johnson,C.R. and Newman,M., (1980), A note on cospectral graphs, J. Comb. Theory, Ser. B, 28, pp.96103.
[12] Mohammadian,A. and Tayfeh-Rezaie,B., (2011), Graphs with four distinct Laplacian eigenvalues, J. Algebraic Combin., 34, pp.671-682.
[13] Omidi,G.R., (2009), On graphs with largest Laplacian eignnvalues at most 4, Australas. J. Combin., 44, pp.163-170.
[14] Rowlinson,P., (2007), The main eigenvalues of a graph: a survey, Appl. Anal. Discrete Math., 1, pp.445-471.
[15] van Dam,E.R. and Haemers,W.H., (2003), Which graphs are determined by their spectrum?, Linear Algebra. Appl., 373, pp.241-272.
[16] van Dam,E.R. and Haemers,W.H., (2009), Developments on spectral characterizations of graphs, Discrete Math., 309, pp.576-586.
[17] Wang,J., Zhao,H., and Huang,Q., (2012), Spectral charactrization of multicone graphs, Czech. Math. J., 62, pp.117-126.
[18] Wang,J., Belardo,F., Huang,Q., and Borovićanin,B., (2010), On the two largest Q-eigenvalues of graphs, Discrete Math., 310, pp.2858-2866.
[19] West,D.B., (2001), Introduction to Graph Theory, Upper Saddle River, Prentice hall.

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