TWMS J. App. Eng. Math. V.5, N.1, 2015, pp. 118-123.

DISTANCE MAJORIZATION SETS IN GRAPHS

R.SUNDARESWARAN [1], V.SWAMINATHAN[2], §

Abstract. Let $G = (V, E)$ be a simple graph. A subset D of $V(G)$ is said to be a distance majorization set (or dm - set) if for every vertex $u \in V - D$, there exists a vertex $v \in D$ such that $d(u, v) \geq deg(u) + deg(v)$. The minimum cardinality of a dm - set is called the distance majorization number of G (or dm - number of G) and is denoted by $dm(G)$, Since the vertex set of G is a dm - set, the existence of a dm - set in any graph is guaranteed. In this paper, we find the dm - number of standard graphs like $K_n, K_{1,n}, K_{m,n}, C_n, P_n$, compute bounds on $dm-$ number and dm- number of self complementary graphs and mycielskian of graphs.

Keywords: Distance, Diameter, Degree

AMS Subject Classification: 05C07, 05C12, 05C35, 05C90.

1. Introduction

By a graph $G = (V, E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by n and m respectively. For basic definitions and terminologies we refer to [2]. The degree of a vertex v, denoted by $deg(v)$, is the cardinality of its adjacent vertices. Let $\delta(G)$ and $\Delta(G)$ be the minimum and maximum degree of a vertex of G. For vertices u and v in a connected graph G, the distance $d(u, v)$ is the length of a shortest $u - v$ path in G. A $u - v$ path of length $d(u, v)$ is called a $u - v$ geodesic. For a vertex v of G, the eccentricity $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, $rad(G)$ and the maximum eccentricity is its diameter, $diam(G)$ of G.

A vertex of a graph is said to dominate itself and all of its neighbors. A subset $D \subseteq V(G)$ is a dominating set of G if every vertex of $V(G) - D$ is dominated by at least one vertex of D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

A subset D of $V(G)$ is said to be a distance majorization set (or dm - set) if for every vertex $u \in V - D$, there exists a vertex $v \in D$ such that $d(u, v) \geq deg(u) + deg(v)$. The minimum cardinality of a dm - set is called the distance majorization number of G (or dm - number of G) and is denoted by $dm(G)$. A dominating set need not be a dm - set. For example, in $K_{1,n}$, the set consisting of the central vertex is a dominating set but it is not a dm - set if $n \geq 3$. A dm - set may not be a dominating set. For example, in P_5, the

[1] Department of Mathematics, SSN College of Engineering, Chennai.
e-mail: neyamsundar@yahoo.com;
[2] Department of Mathematics, S.N College, Madurai.
e-mail: sulanesri@yahoo.com;
§ Manuscript received: July 28, 2014.
TWMS Journal of Applied and Engineering Mathematics, Vol.5, No.1;

set containing the pendent vertices is a dm - set but it is not a dominating set. Thus, the concept of dm - sets is different from dominating sets.

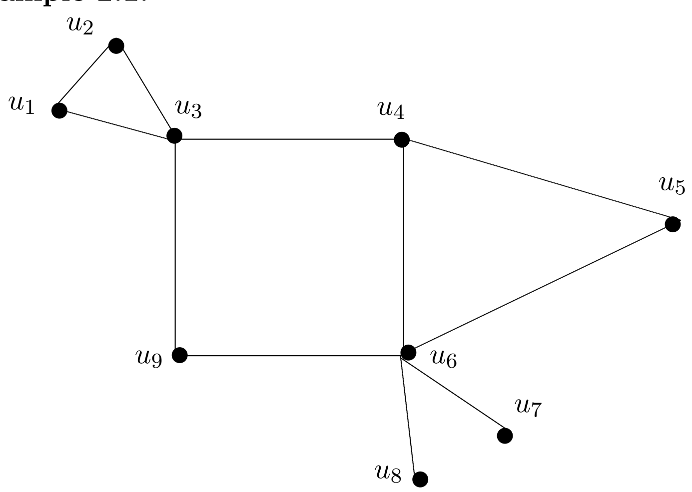
2. Main Results

Definition 2.1. *Let $G = (V, E)$ be a simple graph. A subset D of $V(G)$ is said to be distance majorization set (dm - set) if for every $u \in V - S$, there exists a vertex $v \in S$ such that $d(u, v) \geq d(u) + d(v)$. The minimum cardinality of a dm - set is called the distance majorization number (dm - number) and is denoted by $dm(G)$.*

Remark 2.1. *For any graph G, $V(G)$ is always a dm - set of G. Then the existence of a dm-set is guaranteed.*

Example 2.1.

G :



$S = \{u_3, u_4, u_5, u_6, u_7, u_9\}$ is a dm - set of G and hence it is easily seen that $dm(G) = 6$.

Remark 2.2. *Let $u, v \in V(G)$. Then u is dm - dominated by v if $d(u, v) \geq deg(u)+deg(v)$.*

Theorem 2.1. *$dm(G) = 1$ if and only if G has an isolate.*

Proof. If G has an isolate say u, then $\{u\}$is a dm - set of G and hence $dm(G) = 1$. Suppose $dm(G) = 1$. Let $\{u\}$ be a dm - set of G. Suppose u is not an isolate. Then there exists $v \in V(G)$ such that u and v are adjacent. Therefore, $d(u, v) = 1$ and $deg(u) + deg(v) \geq 2$, a contradiction. Hence u is an isolate of G. □

Theorem 2.2. *For a star graph $K_{1,n}$, $dm(K_{1,n}) = 2$.*

Proof. Let S be a dm - set of $K_{1,n}$. Let $V(K_{1,n}) = \{u, v_1, v_2, \cdots, v_n\}$. Let u be the central vertex of $K_{1,n}$. Thus $u \in S$. u can not dm-dominate any $v_i, 1 \leq i \leq n$, Since $d(v_i, v_j) = 2$, for all $i, j, 1 \leq i, j, \leq n$, $v_i \in S$ for some i. v_i dm-dominates v_j for all $j, j \neq i, i \neq j, 1 \leq j \leq n$, Therefore, $dm(K_{1,n}) = 2$.

□

Observation 2.1.

(1) Let $u \in V(G)$ be a full degree vertex of G. Then, clearly $d(u, v) = 1$, for all $v \in V(G)$. Thus any dm - set of G contain u.
(2) Every vertex of K_n is a full degree vertex. Therefore, $dm(K_n) = n$

(3) $dm(\overline{K_n}) = 1$, since each vertex is an isolate.
(4) For any connected graph G, $2 \leq dm(G) \leq n$.
(5) If for any vertex $u \in V(G)$ of degree greater than or equal to $diam(G)$, then u belongs to a dm - set of G.
(6) For any graph G, $1 \leq dm(G) \leq n$.

Theorem 2.3. *For a double star $D_{r,s}$, $dm(D_{r,s}) = 3$.*

Proof. Let S be a $dm-$ set of $D_{r,s}$. Let $V(D_{r,s}) = \{u, v, x_1, x_2, \cdots, x_r, y_1, y_2, \cdots, y_s\}$. Let u, v be the central vertices of $D_{r,s}$. Clearly, $d(u, x_i) = 1, d(u, y_j) = 2$ for all $i, j, 1 \leq i \leq r; 1 \leq j \leq s$ and $d(u, v) = 1$. Thus , $u, v \in S$. Since $d(x_i, y_j) = 3$, for all $i, j; 1 \leq i \leq r; 1 \leq j \leq s, x_1 \in S$. Hence , $dm(K_{1,n}) \geq 3$. $deg(x_i)+deg(x_j) < d(x_i, y_j), i, j, 1 \leq i \leq r; 1 \leq j \leq s$, $x_1 \in S$, and hence no other $x_i, y_j \in S$. Therefore, $dm(K_{1,n}) \leq 3$. Hence, $dm(K_{1,n}) = 3$. □

Theorem 2.4. *For a complete bipartite graph $K_{m,n}$, $dm(K_{m,n}) = m + n$.*

Proof. Let S be a dm - set of $K_{m,n}$. Let $V(K_{m,n}) = \{x_1, x_2, \cdots, x_m, y_1, y_2, \cdots, y_n\}$. Since $diam(K_{m,n}) = 2$ and $d(x_i) = n \geq 1, d(y_j) = m \geq 1$, for all $i, j; 1 \leq i \leq m; 1 \leq j \leq n$ Hence , $dm(K_{m,n}) \geq m + n$. $deg(x_i) + deg(x_j) < d(x_i, X_j)$, $i, j, 1 \leq i \leq m; 1 \leq j \leq n$, $dm(K_{m,n}) \leq m + n$. □

Theorem 2.5. *For a path P_n , $dm(P_n) = \begin{cases} 2 & n = 3 \text{ and } n \geq 7 \\ 3 & n = 4, 5, 6 \end{cases}$*

Proof. Let $V(P_n) = \{v_1, v_2, \cdots, v_n\}$. Let S be a dm - set of P_n. Let $n = 3$. Then $P_3 \cong K_{1,2}$, $dm(P_3) = 2$. Let $n = 4$. $diam(P_4) = 3$. Since $d(v_1, v_4) \geq deg(v_1) + deg(v_4)$ either v_1 or v_4 belongs to S. Thus $dm(P_4) = 3$. Let $n = 5$. Then,clearly S contains v_1, v_5, v_3 since v_2 and v_4 are $dm-$dominated by v_1 and v_5 respectively. Let $n = 6$. Then,clearly S contains v_1, v_6, v_3, since v_4, v_5 and v_2 are $dm-$dominated by v_1 and v_6 respectively.

Let $n \geq 7$. Then clearly, v_1 $dm-$ dominates the vertices $v_i, 4 \leq i \leq n - 1$ and v_n $dm-$ dominates the vertices $v_i, 2 \leq i \leq n - 3$. Therefore, $dm(P_n) = 2, n \geq 7$. □

Theorem 2.6. *For a cycle C_n, $dm(C_n) = \begin{cases} n & n \leq 7 \\ 4 & n = 8 \\ 3 & 9 \leq n \leq 13 \\ 2 & n \geq 14 \end{cases}$*

Proof. Let $V(C_n) = \{v_1, v_2, \cdots, v_n\}$. Let S be a dm - set of C_n. $diam(C_n) \leq 3, n \leq 7$. $deg(v) = 2$, for all $v \in V(C_n)$. Therefore, $d(v_i, v_j) < deg(v_i)+deg(v_j)$, for all $i, j, 1 \leq i, j \leq n$. Hence $dm(C_n) = n, n \leq 7$.

Let $n = 8$. Then,clearly S contains v_1, v_2, v_3, v_4. Since v_1 dm - dominates the vertices $v_i, 5 \leq i \leq n - 3$,v_2 dm - dominates the vertices $v_i, 6 \leq i \leq n - 2$, v_3 dm - dominates the vertices $v_i, 7 \leq i \leq n - 1$ and v_4 dm - dominates the vertices $v_i, 8 \leq i \leq n$. Therefore, $dm(C_n) = 4$.

Let $9 \leq n \leq 13$. Then $S = \{v_1, v_4, v_7\}$, since v_1 dm - dominates the vertices $v_i, 5 \leq i \leq n - 3$, v_4 dm - dominates the vertices $v_i, 8 \leq i \leq n$, v_7 dm - dominates the remaining vertices of C_n. Therefore, $dm(C_n) = 3$.

For $n \geq 14$. $diam(C_n) \geq \lceil \frac{n}{2} \rceil$, v_1 dm - dominates the vertices $v_i, 5 \leq i \leq n - 3$,$v_{\lceil \frac{n}{2} \rceil}$ dm - dominates the vertices $\{v_1, v_2, v_3, v_4\} \cup \{v_i, 12 \leq i \leq n\}$. Therefore, $dm(C_n) = 2$. □

Theorem 2.7. *If $d(u) + d(v) > diam(G)$ for every $u, v \in V(G)$, then $dm(G) = n$.*

Proof. Suppose $d(u) + d(v) > diam(G)$, for every $u, v \in V(G)$. Suppose $dm(G) < n$. Let S be a minimum dm - set of G. Let $u \in V - S$. Then , there exists $v \in S$ such that $d(u, v) \geq d(u) + d(v) > diam(G)$, a contradiction. Hence $dm(G) = n$.

□

Theorem 2.8. *If $2\delta(G) > d(u, v)$, for every $u, v \in V(G)$, then $dm(G) = n$.*

Proof. Suppose $2\delta(G) > d(u, v)$, for every $u, v \in V(G)$. Let S be a dm - set of G. By the definition of dm - set, each $u \in V - S$ there exists a vertex $v \in S$ such that $d(u, v) \geq d(u) + d(v) \geq \delta(G) + \delta(G) \geq 2\delta(G)$, a contradiction. Therefore, $dm(G) = n$. □

Theorem 2.9. *For any subgraph H of G , $dm(H) \leq dm(G)$.*

Proof. Let G be a graph. Let H be a subgraph of G. Let $u, v \in V(G)$. Suppose H contains an isolate. Then $dm(H) = 1 \leq dm(G)$. Suppose G does not contain an isolate. Let S be a dm - set of G. Clearly, $d_G(u, v) \geq d_H(u, v)$ and $deg_G(u) \geq deg_H(u), deg_G(v) \geq deg_H(v)$. Hence $dm(H) \leq dm(G)$.

□

Theorem 2.10. *For any spanning tree T of G , $dm(T) \leq dm(G)$.*

Proof. Let T be a spanning tree of G. Then clearly, $d_T(u, v) \geq d_G(u, v) \geq deg_G(u) + deg_G(v) \geq deg_T(u) + deg_T(v)$. Therefore,$dm(T) \leq dm(G)$.

□

Theorem 2.11. *For any tree T, $dm(T) \leq \lceil \frac{n}{2} \rceil$.*

Proof. Let T be a tree. Let S be a dm - set of T. Let u and v be diametrically opposite vertices of T. Let $u \in S$. Consider $T_1 = T - \{u, v\}$. Let $x, y \in T_1$ and x and y are diametrically opposite vertices of T_1. Then $S \cup \{x\}$. Consider $T_2 = T_1 - \{x, y\}$.Continuing this process until we get either K_1 or K_2, since any tree has exactly either one or two centers. Clearly, $dm(T) \leq \lceil \frac{n}{2} \rceil$.

□

Observation 2.2. *$dm(G) = n - 1$ if and only if $d(u) + d(v) > diam(G)$, for exactly one pair of vertices $u, v \in V(G)$.*

Theorem 2.12. *Let G be a graph. $diam(G) = 2$ and $dm(G) = 2$ if and only if G is a star.*

Proof. Let G be a graph. Let S be a dm - set of G. Suppose G is a star. Then clearly, $diam(G) = 2$ and $dm(G) = 2$.
Conversely, if $diam(G) = 2$ and $dm(G) = 2$. Since $diam(G) = 2$, G is non complete. Therefore, $deg(u) \leq n - 1, \forall u \in V(G)$. Let $u, v \in V(G)$. If $deg(x) \geq 2$, for every $x \in V - S, x \neq u, v$, then $x \in S$, a contradiction, $dm(G) = 2$. Since $diam(G) = 2$, G is connected. Therefore, u and v are adjacent with at least one vertex $x \in V - S$. As $deg(x) = 1, \forall x \in V - S$, either u or v adjacent with x. Without loss of generality, u is adjacent with x. Suppose v is not adjacent with u. Then v is an isolate, a contradiction. Therefore, u and v are adjacent. Hence G is a star. □

Lemma 2.1. *Let G be a self complementary graph. Then G contains exactly two pendent vertices.*

Proof. Suppose G contains a pendent vertex say u and its support v. If v is not adjacent with $n-2$ vertices of G, then $d(u, x) \geq 3, x \in V(G) - \{u, v\}$. Therefore, $deg_G(v) = n-1$. But $deg_{\overline{G}}(v) = 0$, a contradiction since $G \cong \overline{G}$. Therefore, G contains more than one pendent vertex. Suppose G contains more than 3 pendent vertices say u, v, w. Let u', v', w' be its support. Moreover, if x, y, z be three pendent vertices in $\overline{G}$, then $deg(x), deg(y), deg(z)$ is $n-2$ in G, a contradiction. Therefore, G contains exactly two pendent vertices. □

Theorem 2.13. *Let G be a self complementary graph. Then $dm(G) = n$ or $n-1$.*

Proof. Every nontrivial self-complementary graph G has diameter 2 or 3 [5].

By lemma 2.1, G has exactly two pendent vertices and degree of the remaining $n-2$ vertices is greater than or equal to 2. Therefore, $dm(G) = n$ or $n-1$. □

Definition 2.2. *Mycielski construction to create triangle-free graphs with large chromatic numbers. For a graph G, on n vertices $V(G) = \{v_1, v_2, \cdots, v_n\}$, let $\mu(G)$ be the graph on vertices $X \cup Y \cup \{z\} = \{x_1, x_2, \cdots, y_1, y_2, \cdots, y_n, z\}$ with edges zy_i for all i and edges x_ix_j, y_ix_j for all edges v_iv_j in G. For example, $\mu(K_2) = C_5$.*

Theorem 2.14. *For a graph G without isolated vertices, $dm(\mu(G)) = \max\{|V(\mu(G))|, 2n+2-l\}$,where l is the total number of pendent vertices in G.*

Proof. For a graph G without isolated vertices, $diam(\mu(G)) = min(max(2, diam(G)), 4)$ [6].

Clearly, $deg(z) = n, deg(x_i) = 2deg(v_i)$, $1 \leq i \leq n$ and $deg(y_j) = deg(v_j), 1 \leq j \leq n$.In $\mu(G)$, we have $d(z, x_i) = 2, d(x_i, y_i) = 2, d(y_i, y_j) = 2, d(x_i, y_j) \leq 3$ and $d(x_i, x_j) \leq 4$, for all $i \neq j$.

case(i): $diam(\mu(G)) = 2$.

Since $deg(v) \geq 2$, $v \in V(\mu(G))$, $dm(\mu(G)) = |V(\mu(G)|$.

case(ii): $diam(\mu(G)) = 4$.

Let S be a dm - set of G. z is not dm-dominated by x_i, y_j, x_i is not dm-dominated by y_i, y_i is not dm-dominated by y_j and x_i is not dm-dominated by y_j. $diam(G) \geq 4$.

Suppose $d(x_i, x_j) = 4$. In this case, $diam(\mu(G)) = 4$. $deg(x_i) = 2$ (Suppose $deg(x_i) = 3$. Then the degree of the vertex dm-dominates x_i is 1,a contradiction). $deg(x_i) = 2$ then $deg(v_i) = 1$. If x_i is dominated by x_j, then $deg(v_j) = 1$. Hence $dm(\mu(G)) = 2n+2-l$,where l is the total number of pendent vertices in G.

case(iii): $diam(\mu(G)) = diam(G)$.

In this case, if $diam(G) \geq 4$, $diam(\mu(G)) = min(diam(G), 4) = 4$. As the same lines in case(ii), we get the result. □

Lemma 2.2. *If G be a graph with $\alpha(G) = 1$, then $dm(L(G)) = n$, where $\alpha(G)$ is the vertex covering number of G.*

Proof. If $\alpha(G) = 1$, then G contains a spanning subgraph, that is star, then $L(G)$ is a complete graph. Hence $dm(L(G)) = n$. □

Lemma 2.3. *If $diam(L(G)) = 1$, then $dm(G) = 2$ or 3.*

Proof. $diam(L(G)) = 1$ if and only if G is either K_3 or $K_{1,n-1}$. Hence $dm(G) = 2$ or 3. □

3. Acknowledgement

I would like to express my sincere thanks to the Managment , SSN Institutions, Chennai.

References

[1] Chartrand, G. and Lesniak, L., (2004), Graphs and Digraphs (4th ed.), CRC Press, ISBN 978-1-58488-390-6.

[2] Buckley, F. and Harary, F.. (1990), Distance in Graphs, Addision-Wesley, Redwood City, CA.

[3] West, D. W., (2001), Introduction to Graph Theory - Second edition, Prentice Hall.
[4] Farrugia, A., (1999), Self-complementary graphs and generalisations: A comprehensive reference manual, University of Malta.
[5] Harary, F. and Robinson, R. W., (1985), The Diameter of a Graph and its Complement, The American Mathematical Monthly, Vol. 92, No. 3, pp. 211-212.
[6] Fisher, D. C., McKenna and Boyer, E. D., (1998), Hamiltonicity, diameter, domination, packing and biclique partitions of Mycielski's graphs, Discrete Applied Mathematics, 84, pp. 93-105.

R.Sundareswaran received his Masters degree in 1999, Ph. D degree in 2011 from Madurai Kamaraj University, Madurai, India. He did his Ph.D. degree in the major research project entitled Domination Integrity in graphs sponsored by Department of Science and Technology, New Delhi, India. His area of interest including vulnerability parameters of graphs, domination and colouring. He is currently working as Assistant Professor in the Department of Mathematics, SSN College of Engineering, Chennai, India. He has more than 10 years of teaching experience and 8 years of research experience. He published more than 10 research article in international referred journals. He is a reviewer of American Mathematical society.

V.Swaminathan received his Masters degree in 1968 from Bharathidasan University and Ph.D degree in 1982 from Andhra University. He is currently working as Coordinator in Ramanujan Research Center in Mathematics, S.N. College, Madurai, India. He has published more than 50 research articles . He has 45 years of teaching experience and 25 years of research experience. His research interest include domination, colouring, vulnerability parameters in graphs, Boolean like rings, metric dimension. He is also a reviewer of American Mathematical society and referee for many international jouranls.
