TWMS J. App. Eng. Math. V.5, N.2, 2015, pp. 276-285.

QUADRATIC SPLINE SOLUTION OF CALCULUS OF VARIATION PROBLEMS

REZA MOHAMMADI[1], AMENEH SADAT ALAVI[2] §

Abstract. In this paper, we developed numerical method of order $\mathcal{O}(h^2)$ and based on quadratic polynomial spline function for the numerical solution of class of two point boundary value problems arising in Calculus of Variation. The present approach gives better approximations over all the existing finite difference methods. Convergence analysis and a bound on the approximate solution are discussed. Numerical examples are also given to demonstrate the higher accuracy and efficiency of our method.

Keywords: Calculus of variation, Euler-Lagrange equation, quadratic polynomial spline, Convergence.

AMS Subject Classification: 65 L10, 65 L12

1. Introduction

The calculus of variations is concerned with finding extrema and, in this sense, it can be considered a branch of optimization. The problems and techniques in this branch, however, differ markedly from those involving the extrema of functions of several variables owing to the nature of the domain on the quantity to be optimized. A functional is a mapping from a set of functions to the real numbers. The calculus of variations deals with finding extrema for functionals as opposed to functions. The candidates in the competition for an extremum are thus functions as opposed to vectors in Rn, and this gives the subject a distinct character. The functionals are generally defined by definite integrals; the sets of functions are often defined by boundary conditions and smoothness requirements, which arise in the formulation of the problem/model.

Certainly there is an intimate relationship between mechanics and the calculus of variations, but this should not completely overshadow other fields where the calculus of variations also has applications. Aside from applications in traditional fields of continuum mechanics and electromagnetism, the calculus of variations has found applications in economics, urban planning, and a host of other nontraditional fields. Indeed, the theory of optimal control is centered largely around the calculus of variations. Finally it should be noted the calculus of variations does not exist in a mathematical vacuum or as a closed chapter of classical analysis. Historically, this field has always intersected with geometry and differential equations, and continues to do so.

[1] Department of Mathematics, University of Neyshabur, P. O. Box 91136-899, Neyshabur, Iran.
e-mail: rez.mohammadi@gmail.com, mohammadi@neyshabur.ac.ir;

[2] Department of Mathematics, Payame Noor University, P. O. Box 19395-3697, Tehran, Iran.
e-mail: author.second@net.com;

§ Manuscript received: November 24, 2014.

One might infer that the interest in this branch of Analysis is weakening and that the Calculus of Variations is a Chapter of Classical Analysis. In fact this inference would be quite wrong since new problems like those in control theory are closely related to the problems of the Calculus of Variations while classical theories, like that of boundary value problems for partial differential equations, have been deeply affected by the development of the Calculus of Variations. Moreover, the natural development of the Calculus of Variations has produced new branches of mathematics which have assumed different aspects and appear quite different from the Calculus of Variations [1].

Let us consider the following problem of the calculus of variations, which is the simplest form of a variational problem as

$$J[y(x)] = \int_a^b F(x, y(x), y'(x))dx, \tag{1}$$

where f is a function assumed to have at least second-order continuous partial derivatives with respect to x, y, and y'. Given two values $a, b \in R$, the fixed endpoint variational problem consists of determining the functions $y \in \mathcal{C}^2[a, b]$ such that $y(a) = \alpha$, $y(b) = \beta$, and J has a local extremum.

The necessary condition for $y(x)$ to extremize $J[y(x)]$ is that it should satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}(\frac{\partial F}{\partial y'}) = 0, \tag{2}$$

with boundary conditions as follows:

$$y(a) = \alpha, \quad y(b) = \beta. \tag{3}$$

The above boundary value problem, does not always have a solution and if the solution exists, it may not be unique. Note that if the solution of Euler's equation satisfies the boundary conditions, it is unique.

The general form of the variational problem in Eq. (1) is

$$J[y_1(x), y_2(x), ..., y_m(x)] = \int_a^b F(x, y_1(x), y_2(x), ..., y_m(x), y_1'(x), y_2'(x), ..., y_m'(x))dx, \tag{4}$$

with the given boundary conditions for all functions

$$y_1(a) = \alpha_1, \quad y_2(a) = \alpha_2, \quad ..., \quad y_m(a) = \alpha_m, \tag{5}$$

$$y_1(b) = \beta_1, \quad y_2(b) = \beta_2, \quad ..., \quad y_m(b) = \beta_m, \tag{6}$$

Here the necessary condition for the extremum of the functional in Eq. (4) is to satisfy the following system of second-order differential equations

$$\frac{\partial F}{\partial y_j} - \frac{d}{dx}(\frac{\partial F}{\partial y_j'}) = 0, \quad j = 1, 2, ..., m, \tag{7}$$

with boundary conditions given in Eqs. (5)-(6). However, the above system of differential equations can be solved easily only for simple cases. More historical comments about variational problems are found in [1, 2, 3].

Many efforts are going on to develop efficient and high accuracy methods for solving calculus of variation problems. Gelfand [2] and Elsgolts [3] investigated the Ritz and Galerkin direct methods for solving variational problems. The Walsh series method is introduced to variational problems by Chen and Hsiao [4]. Due to the nature of the Walsh functions, the solution obtained was piecewise constant. The authors in [5, 6, 7] applied some orthogonal polynomials on variational problems to find the continuous solutions for these problems.

Razzaghi and Marzban [8] introduced a new direct computational method via hybrid of Block-Pulse and Chebyshev functions to solve variational problems. Then, Razzaghi et al. [9, 10] presented direct methods for solving variational problems using Legendre wavelets. The rationalized Haar functions are applied to variational problems by Razzagi and Ordokhani [11, 12]. Dehghan and Tatari [13] aimed at producing approximate solutions of some variational problems, which are obtained in rapidly convergent series with elegantly computable components by the Adomian decomposition technique. Then, in their earlier research [14] the He's variational iteration method is employed for solving some problems in calculus of variations. Saadatmandi and Dehghan [15] used the Chebyshev finite difference method for finding the solution of the ordinary differential equations which arise from problems of calculus of variations. In [16] the homotopy-perturbation method has been intensively developed to obtain exact and approximate analytical solutions of variational problems by Abdulaziz and his co-authors. Dixit et al. [17] proposed a simple algorithm for solving variational problems via Bernstein orthonormal polynomials of degree six. In [18], the variational iteration method was implemented to give approximate solution of the Euler-Lagrange, Euler-Poisson and Euler-Ostrogradsky equations as ordinary (or partial) differential equations which arise from the variational problems. Maleki and Mashali-Firouzi [19] proposed a direct method using nonclassical parameterization and nonclassical orthogonal polynomials, for finding the extremal of variational problems. Nazemi et al. [20] employed the differential transform method (DTM) for solving some problems in calculus of variations.

The term spline in the spline function arises from the prefabricated wood or plastic curve board, which is called spline and is used by a draftman to plot smooth curves through connecting the known points. Spline functions can be integrated and differentiated due to being piecewise polynomials and since they have basis with small support, many of the integral that occur in the numerical methods are zero. Thus, spline functions are adapted to numerical methods to get the solution of the differential equations. Numerical methods with spline functions in getting the numerical solution of the differential equations lead to band matrices which are solvable easily with some algorithms in the market with low cost computation. During last four decades, there has been a growing interest in the theory of splines and their applications (see [21, 22]). For example, Rashidinia et al. [23] used cubic spline functions to develop a numerical method for the solution of second-order linear two-point boundary value problems.

The aim of this paper is to construct a new spline method based on a quadratic polynomial spline function for solving problems in calculus of variations. The main purpose is to analyze the efficiency of the exponential spline-difference method for such problems with sufficient accuracy. Application of our method is simple and in comparison with the existing well-known methods is accurate. The resulting spline difference scheme is analyzed for local truncation error and convergence. We have shown that by making use of the quadratic polynomial function, the resulting exponential spline difference scheme gives a tri-diagonal system which can be solved efficiently by using a tri-diagonal solver.

The outline of this paper is as follows: In Section 2, we derive our method. The method is formulated in a matrix form in this Section. Convergence analysis and a bound on the approximate solution are discussed in Section 3. Numerical results are presented to illustrate the applicability and accuracy in Section 4. Finally, in Section 5, we concluded the results of the proposed methods.

2. Derivation of the methods

2.1. **Formulation of the Quadratic spline approximations.** We introduce a finite set of grid points x_i by dividing the interval $[a,b]$ into n equal parts

$$x_i = a + ih, \quad i = 0, 1, ..., n,$$
$$x_0 = a, \quad x_n = b, \quad h = \frac{b-a}{n},$$

We also denote the function value $y(x_i)$ by y_i.

Let $\mathcal{S}_i(x)$ be the quadratic spline approximation of the function $y(x)$ at the grid point x_i and be given by

$$\mathcal{S}_i(x) = \sum_{j=0}^{2} a_{i,j}(x - x_i)^j, \tag{8}$$

for each $i = 0, ..., n$, where $a_{i,0}$, $a_{i,1}$ and $a_{i,2}$ are unknown coefficients. We first develop the explicit expressions for the three coefficients in (8) in terms of $y_{i+\frac{1}{2}}$, m_i and $M_{i+\frac{1}{2}}$, where:

$$\mathcal{S}_i(x_{i+\frac{1}{2}}) = y_{i+\frac{1}{2}}, \quad \mathcal{S}_i'(x_i) = m_i, \quad \mathcal{S}_i''(x_{i+\frac{1}{2}}) = M_{i+\frac{1}{2}}. \tag{9}$$

Now using (9), we can determine the three unknown coefficients in (8) as

$$a_{i,0} = y_{i+\frac{1}{2}} - \frac{h^2}{8}M_{i+\frac{1}{2}} - \frac{h}{2}m_i,$$
$$a_{i,1} = m_i,$$
$$a_{i,2} = M_{i+\frac{1}{2}}.$$

The continuity of the quadratic polynomial spline $\mathcal{S}_i(x)$ and it's first derivative $\mathcal{S}_i'(x)$ at the point (x_i, y_i) yield the following consistency relations for $i = 2, 3, ..., n-1$:

$$y_{i-\frac{3}{2}} - 2y_{i-\frac{1}{2}} + y_{i+\frac{1}{2}} = \frac{h^2}{8}\big[M_{i-\frac{3}{2}} + 6M_{i-\frac{1}{2}} + M_{i+\frac{1}{2}}\big]. \tag{10}$$

For the direct computation of $\mathcal{S}_{i\frac{1}{2}}$, $i = 1, 2, ..., n$, we need two more equations, one at each end of the range of integration. The two end conditions can be derived by Taylor series and the method of undetermined coefficients for each kind of boundary conditions [24]:

$$y_0 - 3y_{\frac{1}{2}} + y_{\frac{3}{2}} = \frac{h^2}{8}\big[5M_{\frac{1}{2}} + M_{\frac{3}{2}}\big], \tag{11}$$

and

$$y_{n-\frac{3}{2}} - 3y_{n-\frac{1}{2}} + 2y_n = \frac{h^2}{8}\big[M_{n-\frac{3}{2}} + 5M_{n-\frac{1}{2}}\big]. \tag{12}$$

Now, for analyzing the truncation error of the equation (10)-(12), we present the following lemma.

Lemma 2.1. *Suppose* $y(x) \in C^4[a,b]$. *Then* [24]

$$\mathcal{T}_i(h) = \begin{cases} -\dfrac{1}{64}h^4 y^{(4)}(\xi_1) + \mathcal{O}(h^5), \\ -\dfrac{1}{24}h^4 y^{(4)}(\xi_i) + \mathcal{O}(h^5), \quad i = 2, 3, ..., n-1, \\ -\dfrac{1}{64}h^4 y^{(4)}(\xi_n) + \mathcal{O}(h^5). \end{cases} \tag{13}$$

Proof. See [25]. □

2.2. **Numerical method.** In this subsection, we give the quadratic polynomial spline method for the following nonlinear boundary value problem which is general form of equations (2) as follow

$$y'' = f(x, y, y'), \quad a < x < b, \tag{14}$$

subjected to boundary conditions

$$y(a) = \alpha, \quad y(b) = \beta. \tag{15}$$

At the grid point (x_i), we may write differential equation (14) as

$$y_i'' = f(x_i, y_i, y_i'). \tag{16}$$

By using moment of spline in (16) we obtain

$$M_i = f(x_i, y_i, y_i'). \tag{17}$$

Now using Taylor series and the method of undetermined coefficients we introduce the following approximations for first derivative of y:

$$\begin{aligned} 3hy'_{\frac{1}{2}} &= -4y_0 + 3y_{\frac{1}{2}} + y_{\frac{3}{2}} + \mathcal{O}(h^4), \\ 3hy'_{\frac{3}{2}} &= 4y_0 - 9y_{\frac{1}{2}} + 5y_{\frac{3}{2}} + \mathcal{O}(h^4), \end{aligned} \tag{18}$$

$$\begin{aligned} 2hy'_{i-\frac{3}{2}} &= -3y_{i-\frac{3}{2}} + 4y_{i-\frac{1}{2}} - y_{i+\frac{1}{2}} + \mathcal{O}(h^4), \\ 2hy'_{i-\frac{1}{2}} &= -y_{i-\frac{3}{2}} + y_{i+\frac{1}{2}} + \mathcal{O}(h^4), \\ 2hy'_{i+\frac{1}{2}} &= y_{i-\frac{3}{2}} - 4y_{i-\frac{1}{2}} + 3y_{i+\frac{1}{2}} + \mathcal{O}(h^4), \end{aligned} \tag{19}$$

and

$$\begin{aligned} 3hy'_{n-\frac{3}{2}} &= -5y_{n-\frac{3}{2}} + 9y_{n-\frac{1}{2}} - 4y_n + \mathcal{O}(h^4), \\ 3hy'_{n-\frac{1}{2}} &= -y_{n-\frac{3}{2}} - 3y_{n-\frac{1}{2}} + 4y_n + \mathcal{O}(h^4). \end{aligned} \tag{20}$$

Now applying the difference formulas (10)-(12) to the nonlinear equations (14)-(15) and using (18)-(20), we obtain

$$\begin{gathered} y_0 - 3y_{\frac{1}{2}} + y_{\frac{3}{2}} = \tfrac{h^2}{8}[5f(x_{\frac{1}{2}}, y_{\frac{1}{2}}, \tfrac{-4y_0+3y_{\frac{1}{2}}+y_{\frac{3}{2}}}{3h}) + f(x_{\frac{3}{2}}, y_{\frac{3}{2}}, \tfrac{4y_0-9y_{\frac{1}{2}}+5y_{\frac{3}{2}}}{2h})], \\ y_{i-\frac{3}{2}} - 2y_{i-\frac{1}{2}} + y_{i+\frac{1}{2}} = \tfrac{h^2}{8}[f(x_{i-\frac{3}{2}}, y_{i-\frac{3}{2}}, \tfrac{-3y_{i-\frac{3}{2}}+4y_{i-\frac{1}{2}}-y_{i+\frac{1}{2}}}{2h}) + \\ 6f(x_{i-\frac{1}{2}}, y_{i-\frac{1}{2}}, \tfrac{-y_{i-\frac{3}{2}}+y_{i+\frac{1}{2}}}{2h}) + f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}, \tfrac{y_{i-\frac{3}{2}}-4y_{i-\frac{1}{2}}+3y_{i+\frac{1}{2}}}{2h})], \; i = 2, 3, ..., n-1, \\ y_{n-\frac{3}{2}} - 3y_{n-\frac{1}{2}} + 2y_n = \tfrac{h^2}{8}[f(x_{n-\frac{3}{2}}, y_{n-\frac{3}{2}}, \tfrac{-5y_{n-\frac{3}{2}}+9y_{n-\frac{1}{2}}-4y_n}{3h}) + \\ 5f(x_{n-\frac{1}{2}}, y_{n-\frac{1}{2}}, \tfrac{-y_{n-\frac{3}{2}}-3y_{n-\frac{1}{2}}+4y_n}{3h})]. \end{gathered} \tag{21}$$

The application of (21) at the points x_i, $i = 1, ..., n$ gives the $n \times n$ nonlinear system

$$\mathcal{A}\mathcal{Y} - \mathcal{F}(\mathcal{Y}) = \mathcal{T}(h), \tag{22}$$

where

$$\mathcal{A} = \begin{pmatrix} -3 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -3 \end{pmatrix}, \quad \mathcal{Y} = \begin{pmatrix} y_{\frac{1}{2}} \\ y_{\frac{3}{2}} \\ \vdots \\ y_{n-\frac{3}{2}} \\ y_{n-\frac{1}{2}} \end{pmatrix},$$

$$\mathcal{G}(\mathcal{Y}) = \begin{pmatrix} g_{\frac{1}{2}} \\ g_{\frac{3}{2}} \\ \vdots \\ g_{n-\frac{3}{2}} \\ g_{n-\frac{1}{2}} \end{pmatrix}, \quad \mathcal{T}(h) = \begin{pmatrix} t_1(h) \\ t_2(h) \\ \vdots \\ t_{n-1}(h) \\ t_n(h) \end{pmatrix}$$

and

$g_{\frac{1}{2}}(y_{\frac{1}{2}}, y_{\frac{3}{2}}) = \frac{h^2}{8}[5f(x_{\frac{1}{2}}, y_{\frac{1}{2}}, y'_{\frac{1}{2}}) + f(x_{\frac{3}{2}}, y_{\frac{3}{2}}, y'_{\frac{3}{2}})]$,

$g_{i-\frac{1}{2}}(y_{i-\frac{3}{2}}, y_{i-\frac{1}{2}}, y_{i+\frac{1}{2}}) = \frac{h^2}{8}[f(x_{i-\frac{3}{2}}, y_{i-\frac{3}{2}}, y'_{i-\frac{3}{2}}) + 6f(x_{i-\frac{1}{2}}, y_{i-\frac{1}{2}}, y'_{i-\frac{1}{2}}) + f(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}, y'_{i+\frac{1}{2}})]$,

$i = 2, 3, ..., n-1$,

$g_{n-\frac{1}{2}}(y_{n-\frac{3}{2}}, y_{n-\frac{1}{2}}) = \frac{h^2}{8}[f(x_{n-\frac{3}{2}}, y_{n-\frac{3}{2}}, y'_{n-\frac{3}{2}}) + 5f(x_{n-\frac{1}{2}}, y_{n-\frac{1}{2}}, y'_{n-\frac{1}{2}})]$.

3. Convergence Analysis

We next discuss the convergence of the method (21). In actual practice we use (13) and get

$$\mathcal{A}\breve{\mathcal{Y}} - \mathcal{F}(\breve{\mathcal{Y}}) = 0, \tag{23}$$

where $\breve{\mathcal{Y}}$ is an approximation of the solution vector $\mathcal{Y}$.

Subtracting (23) from (22), we get

$$\mathcal{A}\mathcal{E} - (\mathcal{F}(\mathcal{Y}) - \mathcal{F}(\breve{\mathcal{Y}})) = \mathcal{T}(h), \tag{24}$$

where $\mathcal{E} = \mathcal{Y} - \breve{\mathcal{Y}} = (e_1, e_2, ..., e_n)^T$. Using the mean-value theorem we write (24) as

$$(\mathcal{A} + \mathcal{B})\mathcal{E} = \mathcal{T}(h), \tag{25}$$

where $\mathcal{B}$ is the $n \times n$ tri-diagonal matrix. It is easy to see that for sufficiently small h, $(\mathcal{A}+\mathcal{B})$ is irreducible and monotone, and $\mathcal{B} \geq 0$. Therefore, $(\mathcal{A}+\mathcal{B})^{-1}$ exists, $(\mathcal{A}+\mathcal{B})^{-1} \geq 0$ and $(\mathcal{A}+\mathcal{B})^{-1} \leq \mathcal{A}^{-1}$. Suppose

$$\Omega_i = \sum_{j=1}^{n} \mathcal{A}^{-1}_{i,j}, \quad \text{and} \quad \Omega = (\Omega_1, \Omega_2, ..., \Omega_n)^T. \tag{26}$$

If $\|\mathcal{E}\| = max_i|e_i|$, then, for sufficiently small h, we have from (25),

$$\|\mathcal{E}\| \leq \|\Omega\|\|\mathcal{T}(h)\|. \tag{27}$$

From (13), $\|\mathcal{T}(h)\| = \mathcal{O}(h^4)$ and since $\|\Omega\| = \mathcal{O}(h^{-2})$, ([26]), from (27) it follows that for sufficiently small h,

$$\|\mathcal{E}\| = \mathcal{O}(h^2). \tag{28}$$

4. Illustrative computations

To illustrate the efficiency and applicability of our presented method computationally, we consider five examples of linear and nonlinear equations arising in calculus of variations, which their exact solutions are known. We solve these examples for various values of h and compare the results with some other methods. The Maximum absolute errors and classical convergence rates in the solutions are tabulated in tables. The numerical examples presented in this section confirm that our algorithm is numerically stable.

The Maximum absolute errors are measured by using following formula

$$L_\infty(h) = \max_{1\leq i\leq N-1}|Y_i - y_i|,$$

TABLE 1. The maximum absolute errors and classical convergence rates with various values of n for Example 4.1.

	$n=4$	$n=8$	$n=16$	$n=32$	$n=64$
$L_\infty(h)$	1.7835×10^{-3}	2.4162×10^{-4}	3.1299×10^{-5}	3.9784×10^{-6}	5.0133×10^{-7}
$R(h)$	—	2.8839	2.9485	2.9759	2.9883

where Y and y represent the exact and approximate solutions, respectively, we calculate the classical convergence rate

$$R(h)=\frac{\ln(L_\infty(h))-\ln(L_\infty(h/2))}{\ln 2}.$$

Example 4.1. *At first we consider the problem of finding the extremal of the functional* [15, 19, 20]

$$min \quad J[y(x)]=\int_0^1 \frac{1+(y(x))^2}{(y'(x))^2}dx, \tag{29}$$

subjected to given boundary conditions

$$y(0)=0, \quad y(1)=0.5. \tag{30}$$

The corresponding Euler-Lagrange equation is

$$y''(x)+y''(x)\big(y(x)\big)^2-y(x)\big(y'(x)\big)^2=0, \tag{31}$$

with boundary conditions (33). The exact solution of this problem is:

$$Y(x)=\sinh\big(0.4812118250x\big).$$

This example has been solved by using our scheme (21) with different values of $n=4,8,16,32,64$ *and the maximum absolute errors and classical convergence rates are given in Table 1. From Table 1, it can be seen that the numerical solutions are in excellent agreement with the exact solution. Also, the results for the scheme (21) clearly confirm the theoretical result stated in previous section.*

Example 4.2. *Consider the following problem* [15]

$$min \quad J[y(x)]=\int_0^1 (y(x)+y'(x)-4\exp(3x))^2dx, \tag{32}$$

subjected to given boundary conditions

$$y(0)=1, \quad y(1)=\exp(3). \tag{33}$$

The corresponding Euler-Lagrange equation is

$$y''(x)-y(x)-8\exp(3x)=0, \tag{34}$$

with boundary conditions (33). The exact solution of this problem is $Y(x)=\exp(3x)$.

The maximum absolute errors and the numerical rates of convergence are determined as in Example 4.1 and the results are shown in Table 2. As one clearly observes, the magnitude of the errors using the quadratic polynomial spline method becomes significantly smaller and remains uniform throughout the unit interval.

Example 4.3. *In this example, consider the following variational problem* [8, 9, 11, 19]*:*

$$min \quad J[y(x)]=\int_0^1 ((y'(x))^2+xy'(x)+(y(x))^2)dx, \tag{35}$$

TABLE 2. The maximum absolute errors and classical convergence rates with various values of n for Example 4.2.

	$n=4$	$n=8$	$n=16$	$n=32$	$n=64$
$L_\infty(h)$	1.16563×10^{-1}	3.25408×10^{-2}	8.44844×10^{-3}	2.13378×10^{-3}	5.34845×10^{-4}
$R(h)$	—	1.84079	1.94550	1.98527	1.99622

TABLE 3. The maximum absolute errors and classical convergence rates with various values of n for Example 4.3.

	$n=4$	$n=8$	$n=16$	$n=32$	$n=64$
$L_\infty(h)$	8.67532×10^{-5}	2.44226×10^{-5}	6.24253×10^{-6}	1.56640×10^{-6}	3.92283×10^{-7}
$R(h)$	—	1.82870	1.96801	1.99468	1.99749

with given boundary conditions

$$y(0)=0, \quad y(1)=\frac{1}{4}. \tag{36}$$

The exact solution of this problem is

$$Y(x)=\frac{1}{2}+\frac{2-e}{4(e^2-1)}e^x+\frac{e(1-2e)}{4(e^2-1)}e^{-x}.$$

The Euler-Lagrange equation of this problem is

$$y''(x)-y(x)+\frac{1}{2}=0, \tag{37}$$

with boundary conditions (36).

This example has been solved by using our scheme (21) with different values of $N=4,8,16,32,64$. *The maximum absolute errors in solution and the numerical convergence rates are tabulated in Tables 3. According to Table 3, we can see that the computational results are getting better as h become smaller. From this Table one can see that the presented method is more applicable and efficient for solving such calculus of variations equations.*

Example 4.4. *Consider the problem of finding the extremal of the functional* [3, 14, 15, 19]

$$J[y(x),z(x)]=\int_0^{\frac{\pi}{2}}((y'(x))^2+(z'(x))^2+2y(x)z(x))dx, \tag{38}$$

let the boundary conditions be

$$y(0)=0, \quad y(\frac{\pi}{2})=1, \quad z(0)=0, \quad z(\frac{\pi}{2})=-1, \tag{39}$$

which has the following analytical solution

$$Y(x)=\sin x, \quad and \quad Z(x)=-\sin x.$$

In this case the Euler-Lagrange equations are written in the following form:

$$y''(x)-z(x)=0, \quad and \quad z''(x)-y(x)=0, \tag{40}$$

with boundary conditions (39).

Tables 4 and 5 give the maximum absolute errors and the numerical convergence rates of the quadratic polynomial spline solution at all grid points for different n. *Table 3 shows that as we increase the grid points the numerical solutions approach to the exact solutions. Hence, the proposed scheme is stable and numerical solutions are convergent.*

Table 4. The maximum absolute errors and classical convergence rates for $y(x)$ with various values of n in Example 4.4.

	$n = 4$	$n = 8$	$n = 16$	$n = 32$	$n = 64$
$L_\infty(h)$	1.55679×10^{-3}	4.38554×10^{-4}	1.11821×10^{-4}	2.81217×10^{-5}	7.03727×10^{-6}
$R(h)$	—	1.82775	1.97155	1.99145	1.99860

Table 5. The maximum absolute errors and classical convergence rates for $z(x)$ with various values of n in Example 4.4.

	$n = 4$	$n = 8$	$n = 16$	$n = 32$	$n = 64$
$L_\infty(h)$	1.55679×10^{-3}	4.38554×10^{-4}	1.11821×10^{-4}	2.81217×10^{-5}	7.03727×10^{-6}
$R(h)$	—	1.82775	1.97155	1.99145	1.99860

5. Concluding Remarks

New method for solving calculus of variation problems is developed using quadratic polynomial spline method. This method is shown to be a second ordered convergent method. Convergence analysis for this method is discussed. The obtained numerical results show that the presented method maintain a remarkable high accuracy which make it encouraging for dealing with the solution of calculus of variation problems.

References

[1] Van Brunt, B., (2004), The Calculus of Variations, Springer-Verlag, New York.

[2] Gelfand, I. M., Fomin, S. V., (1963), Calculus of Variations, Prentice-Hall, NJ, (revised English edition translated and edited by R.A. Silverman).

[3] Elsgolts, L., (1977), Differential Equations and Calculus of Variations, Mir, Moscow, (translated from the Russian by G. Yankovsky).

[4] Chen, C. F. and Hsiao, C. H., (1975), A walsh series direct method for solving variational problems, J. Franklin Inst., 300, pp. 265-280.

[5] Chang, R. Y. and Wang, M. L., (1983), Shifted Legendre direct method for variational problems, J. Optim. Theory Appl., 39, pp. 299-306.

[6] Horng, I. R. and Chou, J. H., (1985), Shifted Chebyshev direct method for solving variational problems, Internat. J. Systems Sci., 16, pp. 855-861.

[7] Hwang, C. and Shih, Y. P., (1983), Laguerre series direct method for variational problems, J. Optim. Theory Appl., 39, 1, pp. 143-149.

[8] Razzaghi, M. and Marzban, H. R., (2000), Direct method for variational problems via of Block-Pulse and chebyshev functions, Mathematical Problems in Engineering, 6, pp. 85-97.

[9] Razzaghi, M. and Yousefi, S., (2000), Legendre wavelets direct method for variational problems, Math. Comput. Simulation, 53, pp. 185-192.

[10] Razzaghi, M. and Yousefi, S., (2001), Legendre Wavelets Method for the Solution of Nonlinear Problems in the Calculus of Variations, Mathematical and Computer Modellmg, 34, pp. 45-54.

[11] Razzaghi, M. and Ordokhani, Y., (2001), An application of rationalized Haar functions for variational problems, Appl. Math. Comput., 122, pp. 353-364.

[12] Razzaghi, M. and Ordokhani, Y., (2001), Solution for a classical problem in the calculus of variations via rationalized haar functions, Kybernetika, 37, 5, pp. 575-583.

[13] Dehghan, M. and Tatari, M., (2006), The use of Adomian decomposition method for solving problems in calculus of variations, Math. Probl. Eng., 2006, pp. 1-12.

[14] Tatari, M. and Dehghan, M., (2007), Solution of problems in calculus of variations via He's variational iteration method, Phys. Lett. A, 362, pp. 401-406.

[15] Saadatmandi, A. and Dehghan, M., (2008), The numerical solution of problems in calculus of variation using Chebyshev finite difference method, Phys. Lett. A, 372, pp. 4037-4040.

[16] Abdulaziz, O., Hashim, I. and Chowdhury, M. S. H., (2008), Solving variational problems by homotopy perturbation method, International Journal of Numerical Methods in Engngineering, 75, pp. 709-721.
[17] Dixit, S., Singh, V. K., Singh, A. K. and Singh, O. P., (2010), Bernstein Direct Method for Solving Variational Problems, International Mathematical Forum, 5, 48, pp. 2351-2370.
[18] Yousefi, S. A. and Dehghan, M., (2010), The use of He's variational iteration method for solving variational problems, Int. J. Comput. Math., 87, 6, pp. 1299-1314.
[19] Maleki, M. and Mashali-Firouzi, M., (2010), A numerical solution of problems in calculus of variation using direct method and nonclassical parameterization, Journal of Computational and Applied Mathematics, 234, 1364-1373.
[20] Nazemi, A. R., Hesam, S. and Haghbin, A., (2013), A fast numerical method for solving calculus of variation problems, AMO - Advanced Modeling and Optimization, 15, 2, pp. 133-149.
[21] Ahlberg, J. H., Nilson, J. H. and Walsh, E. N., (1967), The Theory of Splines and Their Applications. Academic Press, San Diego.
[22] De Boor, C., (1978), Practical Guide to Splines. Springer, Berlin.
[23] Rashidinia, J., Mohammadi, R. and Jalilian, R., (2008), Cubic spline method for two-point boundary value problems, IUST International Journal of Engineering Science, 19, 5-2, pp. 39-43.
[24] Ramadan, M. A., Lashien, I. F. and Zahra, W. K., (2007), Polynomial and nonpolynomial spline approaches to the numerical solution of second order boundary value problems, Appl. Math. Comput., 184, pp. 476-484.
[25] Zahra, W. K. and Elkholy, S. M., (2012), Quadratic spline solution for boundary value problem of fractional order, Numer. Algor., 59, pp. 373-391.
[26] Henrici, P., (1962), Discrete variable methods in ordinary differential equations, Wiley, New York.

Reza Mohammadi is an Assistant Professor of Applied Mathematics in School of Basic Sciences, University of Neyshabur, Neyshabur, Iran, since 2010. He received his Ph.D. degree in Applied Mathematics (Numerical analysis) from Iran University of Science and Technology in 2010 under supervision of Professor Jalil Rashidinia.
Research Interests: Numerical Analysis, Engineering Mathematics, Financial Mathematics, Numerical Methods for Ordinary Differential Equations, Numerical Methods for Partial Differential Equations, Spline Approximation, B-Spline Approximation, Numerical Linear Algebra and Optimal control and Calculus of variations.

Ameneh Sadat Alavi is a graduated student in Department of Mathematics, Payame Noor University, Khorasan Razavi branch, Neyshabur, Iran. She has M.Sc in Department of Mathematics, Payame Noor University, Khorasan Razavi branch, Neyshabur, Iran. Her research interests are Applied mathematics and computational sciences, Numerical solution of Ordinary Differential Equations and Spline approximations.