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G-(F,τ)-CONTRACTIONS IN PARTIAL RECTANGULAR METRIC SPACES ENDOWED WITH A GRAPH AND FIXED POINT THEOREMS

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Abstract. In this paper, the notion of G-(F,τ)-contractions in the context of partial rectangular metric spaces endowed with a graph is introduced. Some fixed point theorems for G-(F,τ)-contractions are also proved. The results of this paper generalize, extend, and unify some known results. Some examples are provided to illustrate the results proved herein.

Keywords: F-contraction, partial rectangular metric space, fixed point, graph.

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1. Introduction

The famous Banach contraction principle [9] has many applications in various branches of mathematics. Due to simplicity and usefulness, the Banach contraction principle is generalized by several authors (see, e.g., [4, 3, 12, 6, 5, 8, 2] and the references therein). In this sequel, Wardowski [2] introduced the notion of F-contractions (or (F,τ)-contractions) and generalized the Banach contraction principle in metric spaces. An example of [2] shows that this type of generalization is an actual generalization of Banach contraction principle. Jachymski [4] introduced G-contractions and initiate the fixed point theory in the metric spaces endowed with graphs. The results of Jachymski generalize and unify several known results in the literature.

In 1906, the famous french mathematician M. Fréchet [7] introduced the concept of a metric space. The concept of metric spaces is generalized by several authors. In this sequel, Branciari [1] introduced a class of generalized (rectangular) metric spaces by replacing triangular inequality by a similar one which involves four or more points instead of three and improved the Banach contraction principle. On the other hand, Matthews [10] introduced the notion of partial metric spaces as a part of the study of denotational semantics of dataflow network and proved the Banach contraction in this general setting.

Very recently, author [11] generalized both the concepts of rectangular metric spaces and partial metric spaces, by introducing the partial rectangular metric spaces and proved some extensions of Banach contraction principle.

In this paper, the notions of F-contractions (or (F,τ)-contractions) of Wardowski [2] and the notion of G-contractions of Jachymski [4] are combined to obtain some fixed point

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results in partial rectangular metric spaces endowed with a graph. The results of this paper generalize the results of Wardowski [2], Jachymski [4], and Branciari [1] in a general setting of partial rectangular metric spaces. Some examples which illustrate the results proved herein, are provided.

2. Preliminaries

First, we recall some definitions from partial metric, rectangular metric and partial rectangular metric spaces (see [1, 10, 11]).

Definition 2.1. [10] A partial metric on a nonempty set X is a mapping $p : X \times X \to \mathbb{R}$ such that, for all $x, y, z \in X$:

(P1) $p(x, y) \geq 0$;
(P2) $x = y$ if and only if $p(x, x) = p(x, y) = p(y, y)$;
(P3) $p(x, x) \leq p(x, y)$;
(P4) $p(x, y) = p(y, x)$;
(P5) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

A partial metric space is a pair (X, p) such that X is a nonempty set and p is a partial metric on X.

Definition 2.2. [1] Let X be a nonempty set and $d : X \times X \to \mathbb{R}$ be a mapping such that:

(R1) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
(R2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(R3) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$ for all $x, y \in X$ and for all distinct points $w, z \in X - \{x, y\}$ (rectangular property).

Then d is called a rectangular metric on X, and (X, d) is called a rectangular metric space. A sequence $\{x_n\}$ in X is called convergent and converges to $x \in X$, if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x) < \varepsilon$ for all $n > n_0$. Sequence $\{x_n\}$ is called a Cauchy sequence if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > n_0$. A rectangular metric space (X, d) is called complete if every Cauchy sequence in X converges in X.

Definition 2.3. [11] Let X be a nonempty set and ρ: $X \times X \to \mathbb{R}$ be a mapping such that:

(ρ1) $\rho(x, y) \geq 0$ for all $x, y \in X$;
(ρ2) $x = y$ if and only if $\rho(x, y) = \rho(x, x) = \rho(y, y)$ for all $x, y \in X$;
(ρ3) $\rho(x, x) \leq \rho(x, y)$ for all $x, y \in X$;
(ρ4) $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
(ρ5) $\rho(x, y) \leq \rho(x, w) + \rho(w, z) + \rho(z, y) - \rho(w, w) - \rho(z, z)$ for all $x, y \in X$ and for all distinct points $w, z \in X \setminus \{x, y\}$.

Then ρ is called a partial rectangular metric on X and the pair (X, ρ) is called a partial rectangular metric space.

Remark 2.1. [11] In a partial rectangular metric space (X, ρ) if $x, y \in X$ and $\rho(x, y) = 0$, then $x = y$ but the converse may not be true.

Remark 2.2. [11] It is clear that every rectangular metric space is a partial rectangular metric space with zero self-distance. However, the converse of this fact need not hold.

Example 2.1. Let $X = \{0, 1, 2, 3, 4\}$ and define a mapping $\rho\colon X \times X \to \mathbb{R}$ by

$$\rho(x, y) = \begin{cases} x, & \text{if } x = y; \\ 6 + \max\{x, y\}, & \text{if } x, y \in \{0, 1\}, x \neq y; \\ 2 + \max\{x, y\}, & \text{otherwise.} \end{cases}$$

Then (X, ρ) is a partial rectangular metric space. But it is neither rectangular metric space nor a partial metric space. Indeed, for any $x \in X$ with $x > 0$ we have $\rho(x, x) > 0$, so (X, ρ) is not a rectangular metric space. Also, $\rho(0, 1) = 7 > 6 = \rho(0, 2) + \rho(2, 1) - \rho(2, 2)$, so (X, ρ) is not a partial metric space.

Proposition 2.1. [11] For each partial rectangular metric space (X, ρ) the pair (X, ρ^r) is a rectangular metric space, where

$$\rho^r(x, y) = 2\rho(x, y) - \rho(x, x) - \rho(y, y) \text{ for all } x, y \in X.$$

Here (X, ρ^r) is called the induced rectangular metric space, and ρ^r is the induced rectangular metric. In further discussion until specified, (X, ρ^r) will represent the induced rectangular metric space. For more properties, examples and details of partial rectangular metric spaces the reader is referred to [11].

Now we define the convergence of a sequence and Cauchy sequence in partial rectangular metric spaces.

Definition 2.4. [11] Let (X, ρ) be a partial rectangular metric space, $\{x_n\}$ a sequence in X and $x \in X$. Then:

(i) The sequence $\{x_n\}$ is said to be convergent and converges to x, if $\lim\limits_{n\to\infty} \rho(x_n, x) = \rho(x, x)$.

(ii) The sequence $\{x_n\}$ is said to be Cauchy in (X, ρ) if $\lim\limits_{n,m\to\infty} \rho(x_n, x_m)$ exists and is finite.

(iii) (X, ρ) is said to be a complete partial rectangular metric space if for every Cauchy sequence $\{x_n\}$ in X there exists $x \in X$ such that

$$\lim_{n,m\to\infty} \rho(x_n, x_m) = \lim_{n\to\infty} \rho(x_n, x) = \rho(x, x).$$

Note that in a partial rectangular metric space the limit of convergent sequence may not be unique (see [11]).

Lemma 2.1. [11] *Let (X, ρ) be a partial rectangular metric space and $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ converges in (X, ρ^r) and converges to $x \in X$, that is, $\lim\limits_{n\to\infty} \rho^r(x_n, x) = 0$ if and only if $\lim\limits_{n\to\infty} \rho(x_n, x) = \lim\limits_{n\to\infty} \rho(x_n, x_n) = \rho(x, x)$.*

Lemma 2.2. [11] *Let (X, ρ) be a partial rectangular metric space and $\{x_n\}$ be a sequence in X. Then the sequence $\{x_n\}$ is a Cauchy sequence in (X, ρ) if and only if it is a Cauchy sequence in (X, ρ^r).*

Lemma 2.3. [11] *A partial rectangular metric space is complete if and only if its induced rectangular metric space is complete.*

Now we recall some basic notions from graph theory which we need subsequently (see also [4]).

Let X be a nonempty set and Δ denote the diagonal of the cartesian product $X \times X$. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops, that is, $E(G) \supseteq \Delta$. We assume G has no parallel edges, so we can identify G with the pair $(V(G), E(G))$. Moreover, we may treat

G as a weighted graph by assigning to each edge the rectangular distance between its vertices.

By G^{-1} we denote the conversion of a graph G, that is, the graph obtained from G by reversing the direction of edges. Thus we have

$$E(G^{-1}) = \{(x,y) \in X \times X \colon (y,x) \in E(G)\}.$$

The letter $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Actually, it will be more convenient for us to treat $\widetilde{G}$ as a directed graph for which the set of its edges is symmetric. Under this convention,

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}). \tag{1}$$

If x and y are vertices in a graph G, then a path in G from x to y of length l is a sequence $(x_i)_{i=0}^{l}$ of $l+1$ vertices such that $x_0 = x, x_l = y$ and $(x_{i-1}, x_i) \in E(G)$ for $i = 1, \dots, l$. A graph G is called connected if there is a path between any two vertices of G. G is weakly connected if $\widetilde{G}$ is connected.

Throughout this paper we assume that X is nonempty set, G is a directed graph such that $V(G) = X$ and $E(G) \supseteq \Delta$.

Definition 2.5. [2] Let $F \colon \mathbb{R}^+ \to \mathbb{R}$ (where $\mathbb{R}^+$ stands for the set of positive reals) be a mapping satisfying:

(F1) F is strictly increasing, that is, for $\alpha, \beta \in \mathbb{R}^+$ such that $\alpha < \beta$ implies $F(\alpha) < F(\beta)$;
(F2) for each sequence $\{\alpha_n\}$ of positive numbers, $\lim_{n\to\infty} \alpha_n = 0$ if and only if $\lim_{n\to\infty} F(\alpha_n) = -\infty$;
(F3) there exists $k \in (0,1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

For examples of function F, the reader is referred to [2]. We denote the set of all functions satisfying properties (F1)-(F3), by $\mathcal{F}$.

Motivated by [2] and [4] we give the following definitions in a partial rectangular metric spaces.

Definition 2.6. *Let (X,ρ) be a partial rectangular metric space. A mapping $T \colon X \to X$ is said to be an (F,τ)-contraction if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$Tx \neq Ty \Rightarrow \tau + F(\rho(Tx,Ty)) \leq F(\rho(x,y)) \quad \textit{for all } x,y \in X. \tag{2}$$

Remark 2.3. If $F(\alpha) = \ln(\alpha)$, $\alpha > 0$, then $F \in \mathcal{F}$ with any $k \in (0,1)$. Now for any mapping $T \colon X \to X$ the condition (2) reduces into the following form:

$$Tx \neq Ty \Rightarrow \rho(Tx,Ty) \leq e^{-\tau}\rho(x,y) \quad \text{for all } x,y \in X.$$

Thus, the Banach type contractions (see [11]) in partial rectangular metric spaces are a particular case of (F,τ)-contraction. Even the above inequality is a weaker form of Banach type contraction since it is satisfied only for those $x,y \in X$ for which $Tx \neq Ty$.

Definition 2.7. Let (X,ρ) be a partial rectangular metric space endowed with a graph G. A mapping $T \colon X \to X$ is said to be a G-(F,τ)-contraction if:

(GF1) T is edge preserving, that is, $(x,y) \in E(G)$ implies $(Tx,Ty) \in E(G)$ for all $x,y \in X$;
(GF2) there exist $\tau > 0$ and $F \in \mathcal{F}$ such that

$$Tx \neq Ty \ \Rightarrow \tau + F(\rho(Tx,Ty)) \leq F(\rho(x,y)) \tag{3}$$

for all $x,y \in X$ with $(x,y) \in E(G)$.

An obvious consequence of symmetry of $\rho(\cdot,\cdot)$ and (1) is the following remark.

Remark 2.4. If T is a G-(F,τ)-contraction then it is both a G^{-1}-(F,τ)-contraction and a $\widetilde{G}$-(F,τ)-contraction.

Example 2.2. Any (F,τ)-contraction on X is a G_0-(F,τ)-contraction, where $E(G_0) = X \times X$.

Example 2.3. Let (X,ρ) be a partial rectangular metric space, $\sqsubseteq$ a partial order on X and $T\colon X \to X$ be a nondecreasing, ordered (F,τ)-contraction, that is, $x \sqsubseteq y$ implies $Tx \sqsubseteq Ty$ and there exist $\tau > 0$ and $F \in \mathcal{F}$ such that

$$Tx \neq Ty \ \Rightarrow \tau + F(\rho(Tx,Ty)) \leq F(\rho(x,y))$$

for all $x, y \in X$ with $x \sqsubseteq y$. Then T is a G_1-(F,τ)-contraction, where $E(G_1) = \{(x,y) \in X \times X\colon x \sqsubseteq y\}$.

Remark 2.5. Let (X,ρ) be a partial rectangular metric space endowed with a graph G. Let the mapping $T\colon X \to X$ be a G-(F,τ)-contraction, then

$$\rho(Tx,Ty) < \rho(x,y) \text{ for all } x,y \in X \text{ with } Tx \neq Ty, (x,y) \in E(G).$$

Proof. Let $(x,y) \in E(G)$ and $Tx \neq Ty$, then since T is a G-(F,τ)-contraction, by (GF2) we have

$$\tau + F(\rho(Tx,Ty)) \leq F(\rho(x,y)),$$

that is,

$$F(\rho(Tx,Ty)) < F(\rho(x,y)).$$

Now the result follows from the property (F1). □

Definition 2.8. Let (X,ρ) be a partial rectangular metric space and $T\colon X \to X$ be a mapping. Then for $x_0 \in X$, a Picard sequence with initial value x_0 is defined by $\{x_n\}$, where $x_n = Tx_{n-1}$ for all $n \in \mathbb{N}$. The mapping T is called a Picard operator on X if T has a unique fixed point in X and for all $x_0 \in X$ the Picard sequence $\{x_n\}$ with initial value x_0 converges to the fixed point of T. The mapping T is called weakly Picard operator, if for any $x_0 \in X$, the limit of Picard sequence $\{x_n\}$ with initial value x_0, exists (it may depend on x_0) and it is a fixed point of T.

3. Main Results

Let (X,ρ) be a partial rectangular metric space endowed with a graph G and $T\colon X \to X$ be a mapping. We denote the set of all fixed points of T by $\mathrm{Fix}(T)$, that is, $\mathrm{Fix}(T) = \{x \in X\colon Tx = x\}$, also we use the notation $X_T = \{x \in X\colon (x,Tx),(x,T^2x) \in E(G)\}$. The set of all periodic points of T in X_T is denoted by $P_X(T)$, that is, $P_X(T) = \{x \in X_T\colon T^k x = x \text{ for some } k \in \mathbb{N}\}$. The space (X,ρ) is said to have the property (P) if:

$$\text{whenever a sequence } \{x_n\} \text{ in } X \text{ converges to } x \text{ with } (x_n,x_{n+1}) \in E(G) \text{ for all } n \in \mathbb{N},$$
$$\text{then there is a subsequence } \{x_{n_k}\} \text{ with } (x_{n_k},x) \in E(G) \text{ for all } k \in \mathbb{N}. \tag{P}$$

The following proposition will be useful in the further discussion.

Proposition 3.1. Let (X,ρ) be a partial rectangular metric space endowed with a graph G. Let $T\colon X \to X$ be a G-(F,τ)-contraction. If $x,y \in \mathrm{Fix}(T)$ are such that $(x,y) \in E(G)$, then $x = y$.

Proof. Let $x,y \in$Fix(T) be such that $(x,y) \in E(G)$. Suppose, $Tx \neq Ty$, then since T is a G-(F,τ)-contraction, by (GF2) we have

$$\tau + F(\rho(x,y)) = \tau + F(\rho(Tx,Ty)) \leq F(\rho(x,y)).$$

Since $\tau > 0$, we obtain from the above inequality that $F(\rho(x,y)) < F(\rho(x,y))$. This contradiction shows that $Tx = Ty$, that is, $x = y$. $\square$

Theorem 3.1. *Let (X,ρ) be a partial rectangular metric space endowed with a graph G. Let $T\colon X \to X$ be a G-(F,τ)-contraction and $X_T \neq \emptyset$, then:*

(I) *X_T is invariant under T.*

Suppose, $T^n x \neq T^{n+1}x$ for all $x \in X_T$ and $n \in \mathbb{N}$, then:

(II) *for any $x \in X_T$ and $n \in \mathbb{N}$ we have $F(\rho(T^n x, T^{n+1}x)) \leq F(\rho(x,Tx)) - n\tau$;*

(III) *T has no periodic point in X_T, that is, $P_X(T) = \emptyset$;*

(IV) *for $x_0 \in X_T$ all the terms of a Picard sequence $\{x_n\}$ with initial value x_0, are distinct.*

Proof. (I) Let $x \in X_T$, that is, $(x,Tx) \in E(G)$. Since T is a G-(F,τ)-contraction by (GF1) we have $(Tx,TTx) \in E(G)$, that is, $Tx \in X_T$.

(II) If $x \in X_T$ then by (I) we have $T^n x \in X_T$, that is, $(T^n x, T^{n+1}x) \in E(G)$ for all $n \in \mathbb{N}$, also, by assumption, $T^n x \neq T^{n+1}x$ for all $n \in \mathbb{N}$. Therefore by (GF2) we have

$$\tau + F(\rho(T^n x, T^{n+1}x)) = \tau + F(\rho(TT^{n-1}x, TT^n x)) \leq F(\rho(T^{n-1}x, T^n x)),$$

that is,

$$F(\rho(T^n x, T^{n+1}x)) \leq F(\rho(T^{n-1}x, T^n x)) - \tau.$$

Repetition of this process gives

$$F(\rho(T^n x, T^{n+1}x)) \leq F(\rho(x,Tx)) - n\tau.$$

(III) Suppose $P_X(T) \neq \emptyset$ and $x \in P_X(T)$. Then, there exists $k \in \mathbb{N}$ such that $T^k x = x$. By (II) we have

$$\begin{aligned} F(\rho(x,Tx)) &= F(\rho(T^k x, TT^k x)) \\ &= F(\rho(T^k x, T^{k+1}x)) \\ &\leq F(\rho(x,Tx)) - k\tau \\ &< F(\rho(x,Tx)). \end{aligned}$$

This contradiction shows that $P_X(T) = \emptyset$.

(IV) Let $x_0 \in X_T$, then by (I) X_T is invariant under T and $T^n x_0 = x_n \in X_T$ for all $n \in \mathbb{N}$. If $x_n = x_m$ for some $n,m \in \mathbb{N}$, $m > n$. Then we have

$$T^{m-n}T^n x_0 = T^m x_0 = T^n x_0.$$

Thus, $T^n x_0 \in P_X(T)$, which contradicts the result (III). Therefore, all the terms of the Picard sequence $\{x_n\}$ are distinct. $\square$

In the next theorem we show that the Picard sequence generated by a G-(F,τ)-contraction is a Cauchy sequence.

Theorem 3.2. *Let (X,ρ) be a partial rectangular metric space endowed with a graph G. Let $T\colon X \to X$ be a G-(F,τ)-contraction then for every $x_0 \in X_T$ the Picard sequence $\{x_n\}$ with initial value x_0, is a Cauchy sequence.*

Proof. Let $x_0 \in X_T$ and $\{x_n\}$ be the Picard sequence with initial value $x_0 \in X$. Then by (I) of Theorem 3.1, the set X_T is invariant under T therefore $T^n x_0 \in X_T$, that is, $(x_n, x_{n+1}) \in E(G)$ for all $n \geq 0$.

If $x_{n_0} = x_{n_0+1}$ for any $n_0 \in \mathbb{N}$, then by definition of Picard sequence we have $x_{n_0+r} = x_{n_0+r+1}$ for all $r \in \mathbb{N}$, and so, $x_{n_0} = x_{n_0+p}$ for all $p \in \mathbb{N}$. Therefore $\{x_n\}$ is a Cauchy sequence. Suppose, $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. For notational convenience, let $\rho_n = \rho(x_n, x_{n+1})$ for all $n \geq 0$. Since $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$, form (II) of Theorem 3.1 we have

$$F(\rho_n) \leq F(\rho_0) - n\tau \quad \text{for all} \quad n \in \mathbb{N}. \tag{4}$$

Letting $n \to \infty$ in the above inequality we obtain $\lim_{n\to\infty} F(\rho_n) = -\infty$, which together with (F2) yields

$$\lim_{n\to\infty} \rho_n = 0. \tag{5}$$

From (F3) there exists $k_1 \in (0,1)$ such that

$$\lim_{n\to\infty} \rho_n^{k_1} F(\rho_n) = 0. \tag{6}$$

By (4) we have

$$\begin{aligned} \rho_n^{k_1} F(\rho_n) - \rho_n^{k_1} F(\rho_0) &\leq \rho_n^{k_1}[F(\rho_0) - n\tau] - \rho_n^{k_1} F(\rho_0) \\ &= -\rho_n^{k_1} n\tau. \end{aligned}$$

Letting $n \to \infty$ in the above inequality and using (5) and (6) we obtain $\lim_{n\to\infty} n\rho_n^{k_1} = 0$. Therefore, there exists $n_1 \in \mathbb{N}$ such that $n\rho_n^{k_1} < 1$ for all $n > n_1$. Thus,

$$\rho_n < \frac{1}{n^{1/k_1}} \quad \text{for all} \quad n > n_1. \tag{7}$$

Due to (IV) of Theorem 3.1, we can assume that all the terms of the sequence $\{x_n\}$ are distinct. Since $x_0 \in X_T$ we have $(x_0, T^2x_0) = (x_0, x_2) \in E(G)$. By (GF1) we have $(Tx_0, Tx_2) = (x_1, x_3) \in E(G)$, which by induction gives $(x_n, x_{n+2}) \in E(G)$ for all $n \geq 0$. Therefore, by (GF2) we have

$$\begin{aligned} \tau + F(\rho(x_n, x_{n+2})) &= \tau + F(\rho(Tx_{n-1}, Tx_{n+1})) \\ &\leq F(\rho(x_{n-1}, x_{n+1})), \end{aligned}$$

that is, $F(\rho(x_n, x_{n+2})) \leq F(\rho(x_{n-1}, x_{n+1})) - \tau$. Repeating this process we obtain

$$F(\rho(x_n, x_{n+2})) \leq F(\rho(x_0, x_2)) - n\tau \quad \text{for all} \quad n \in \mathbb{N}. \tag{8}$$

Letting $n \to \infty$ in the above inequality we obtain $\lim_{n\to\infty} F(\rho(x_n, x_{n+2})) = -\infty$, which together with (F2) yields

$$\lim_{n\to\infty} \rho(x_n, x_{n+2}) = 0. \tag{9}$$

From (F3) there exists $k_2 \in (0,1)$ such that

$$\lim_{n\to\infty} [\rho(x_n, x_{n+2})]^{k_2} F(\rho(x_n, x_{n+2})) = 0. \tag{10}$$

By (8) we have

$$\begin{aligned} [\rho(x_n, x_{n+2})]^{k_2} F(\rho(x_n, x_{n+2})) &- [\rho(x_n, x_{n+2})]^{k_2} F(\rho(x_0, x_2)) \\ &\leq [\rho(x_n, x_{n+2})]^{k_2}[F(\rho(x_0, x_2)) - n\tau] \\ &\quad -[\rho(x_n, x_{n+2})]^{k_2} F(\rho(x_0, x_2)) \\ &= -[\rho(x_n, x_{n+2})]^{k_2} n\tau \end{aligned}$$

Letting $n \to \infty$ in the above inequality and using (9) and (10) we obtain $\lim\limits_{n\to\infty} n[\rho(x_n, x_{n+2})]^{k_2} = 0$. Therefore, there exists $n_2 \in \mathbb{N}$ such that $n[\rho(x_n, x_{n+2})]^{k_2} < 1$ for all $n > n_2$. Thus,

$$\rho(x_n, x_{n+2}) < \frac{1}{n^{1/k_2}} \quad \text{for all} \quad n > n_2. \tag{11}$$

Now we consider the quantity $\rho(x_n, x_{n+p})$ in two cases:
If p is even, say $2m$, then since all the terms of the sequence $\{x_n\}$ are distinct, using (ρ_5), (7) and (11) we have

$$\begin{aligned}\rho(x_n, x_{n+2m}) &\leq \rho_{n+2} + \rho_{n+3} + \cdots + \rho_{n+2m-1} + \rho(x_n, x_{n+2}) \\ &\quad -[\rho(x_{n+2}, x_{n+2}) + \rho(x_{n+3}, x_{n+3}) + \cdots + \rho(x_{n+2m-1}, x_{n+2m-1})] \\ &\leq \rho_{n+2} + \rho_{n+3} + \cdots + \rho_{n+2m-1} + \rho(x_n, x_{n+2}) \\ &\leq \sum_{i=n+2}^{n+2m-1} \frac{1}{i^{1/k_1}} + \frac{1}{n^{1/k_2}},\end{aligned}$$

that is,

$$\rho(x_n, x_{n+2m}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}, \tag{12}$$

where $k = \max\{k_1, k_2\}$.
If p is odd, say $2m+1$, then with similar reason we have

$$\begin{aligned}\rho(x_n, x_{n+2m+1}) &\leq \rho_n + \rho_{n+1} + \cdots + \rho_{n+2m} \\ &\quad -[\rho(x_{n+1}, x_{n+1}) + \rho(x_{n+2}, x_{n+2}) + \cdots + \rho(x_{n+2m}, x_{n+2m})] \\ &\leq \rho_n + \rho_{n+1} + \cdots + \rho_{n+2m} \\ &\leq \sum_{i=n}^{n+2m} \frac{1}{i^{1/k_1}},\end{aligned}$$

that is,

$$\rho(x_n, x_{n+2m+1}) \leq \sum_{i=n}^{\infty} \frac{1}{i^{1/k}}. \tag{13}$$

Since $k = \max\{k_1, k_2\} < 1$, by the convergence of series $\sum_{i=1}^{\infty}\left(\frac{1}{i^{1/k}}\right)$, (12) and (13) we have $\lim\limits_{n\to\infty} \rho(x_n, x_{n+p}) = 0$ for all $p > 0$, or equivalently $\lim\limits_{n,m\to\infty} \rho(x_n, x_m) = 0$. Therefore $\{x_n\}$ is a Cauchy sequence. □

In the next theorem, a sufficient condition for a G-(F,τ)-contraction to be a weakly Picard operator is provided.

Theorem 3.3. *Let (X, ρ) be a complete partial rectangular metric space endowed with a graph G and has the property (P). Let $T \colon X \to X$ be a G-(F,τ)-contraction such that $X_T \neq \emptyset$, then $T|_{X_T}$ is a weakly Picard operator.*

Proof. Suppose $X_T \neq \emptyset$. Let $x_0 \in X_T$, then by Theorem 3.2, the Picard sequence $\{x_n\}$ with initial value x_0, is a Cauchy sequence. By completeness of X and Theorem 3.2, there exists $x^* \in X$ such that

$$\lim_{n,m\to\infty} \rho(x_n, x_m) = \rho(x_n, x^*) = \rho(x^*, x^*) = 0. \tag{14}$$

We shall show that x^* is a fixed point of T. Without loss of generality we can assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$ and so by (IV) of Theorem 3.1, all the terms of sequence

$\{x_n\}$ are distinct and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$. Since the property (P) is satisfied, there exists a subsequence $\{x_{n_k}\}$ such that $(x_{n_k}, x^*) \in E(G)$ for all $k \in \mathbb{N}$. Without loss of generality, we can assume that there exists $j \in \mathbb{N}$ such that $x^* \neq x_{n_k}, Tx^* \neq x_{n_k}$ for all $k > j$. Note that such a j exists because the terms of the sequence $\{x_n\}$ are distinct. Thus, for all $k > j$, by Remark 2.5 we have

$$\begin{aligned}\rho(x^*, Tx^*) &\leq \rho(x^*, x_{n_k}) + \rho(x_{n_k}, x_{n_k+1}) + \rho(x_{n_k+1}, Tx^*)\\ &= \rho(x^*, x_{n_k}) + \rho_{n_k} + \rho(Tx_{n_k}, Tx^*)\\ &< \rho(x^*, x_{n_k}) + \rho_{n_k} + \rho(x_{n_k}, x^*).\end{aligned}$$

Using (14) in the above inequality we obtain $\rho(x^*, Tx^*) = 0$, that is, $Tx^* = x^*$. Thus, $x^* \in \mathrm{Fix}(T)$ and T is a weakly Picard operator. □

The following example shows that a G-(F,τ)-contraction in the above theorem may not be a Picard operator on X_T.

Example 3.1. Let $S_n = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, $n \in \mathbb{N}$, and $X = \{S_n \colon n \in \mathbb{N}\}$. Define $\rho \colon X \times X \to \mathbb{R}$ and a graph G by

$$\rho(x,y) = \begin{cases} 7, & \text{if } x = y;\\ 21 + |x-y|, & \text{if } x, y \in \{S_1, S_2\}, x \neq y;\\ 7 + |x-y|, & \text{otherwise}\end{cases}$$

and $V(G) = X$, $E(G) = \Delta \cup \{(S_n, S_m) \colon n > m, m \geq 3\} \cup \{(S_n, S_2) \colon n > 2\}$. Then (X, ρ) is a partial rectangular metric space endowed with the graph G. Note that (X, ρ) is not a rectangular metric space since $\rho(x, x) = 7 > 0$ for all $x \in X$. Also, it is not a partial metric space, because it lacks the property (P5). Indeed,

$$\rho(S_1, S_2) = 23 > \rho(S_1, S_3) + \rho(S_3, S_2) - \rho(S_3, S_3) = 12 + 10 - 7 = 15.$$

Define a mapping $T \colon X \to X$ by

$$TS_n = \begin{cases} S_n, & \text{if } n = 1, 2;\\ S_{n-1} & \text{if } n \geq 3.\end{cases}$$

Then $X_T = X \neq \emptyset$ and T is a G-(F,τ)-contraction with $\tau = 1$ and $F(\alpha) = \ln(\alpha) + \alpha$. Note that all the conditions of Theorem 3.3 are satisfied and $\mathrm{Fix}(T) = \{S_1, S_2\}$. Thus, the fixed point of T is not unique. Moreover, $T|_{X_T}$ is not a Picard operator.

In the next theorem a necessary and sufficient condition for the uniqueness of fixed point of a G-(F,τ)-contraction is provided.

Theorem 3.4. *Let (X, ρ) be a complete partial rectangular metric space endowed with a graph G and has the property (P). Let $T \colon X \to X$ be a G-(F,τ)-contraction such that $X_T \neq \emptyset$, then $T|_{X_T}$ is a weakly Picard operator. In addition, the subgraph $G_{Fix(T)}$ defined by $V(G_{Fix(T)}) = Fix(T)$ is weakly connected if and only if $T|_{X_T}$ is a Picard operator.*

Proof. By Theorem 3.3, we obtain that $T|_{X_T}$ is a Picard operator. If $V(G_{\mathrm{Fix}(T)}) = \mathrm{Fix}(T)$ is weakly connected then proof of uniqueness of fixed point follows from Proposition 3.1 and Remark 2.4. Thus $T|_{X_T}$ is a Picard operator. Conversely, if $T|_{X_T}$ is Picard operator then fixed point of T is unique, that is, $\mathrm{Fix}(T)$ is singleton and so $G_{\mathrm{Fix}(T)}$ it is weakly connected. □

Following corollary is a partial rectangular metric version of the results of Wardowski [2].

Corollary 3.1. *Let (X, ρ) be a complete partial rectangular metric space and $T \colon X \to X$ be an (F,τ)-contraction. Then T is a Picard operator.*

Proof. Define a graph G_0 by $V(G_0) = X$ and $E(G_0) = X \times X$, then (X, ρ) is a partial rectangular metric space endowed with the graph G_0 and T is a G_0-(F,τ)-contraction. Note that $E(G_0) = X \times X$, therefore $X_T = X$ and the condition (P) is satisfied trivially. Now the proof follows from Theorem 3.4. □

The following example illustrate the utility of the Theorem 3.4.

Example 3.2. Let $X = \{0, 1, 2, 3, 4\}$ and define $\rho\colon X \times X \to \mathbb{R}$ and a graph G by

$$\rho(x,y) = \begin{cases} x, & \text{if } x = y; \\ 6 + \max\{x,y\}, & \text{if } x, y \in \{0,1\}, x \neq y; \\ 2 + \max\{x,y\}, & \text{otherwise.} \end{cases}$$

$V(G) = X$ and $E(G) = \Delta \cup \{(0,2),(2,0),(2,3),(3,2)\}$. Then (X,ρ) is a partial rectangular metric space endowed with the graph G. Define a mapping $T\colon X \to X$ by

$$T0 = T1 = T2 = 0, T3 = 2, T4 = 1.$$

Then:

(i) T is not an (F,τ)-contraction in usual metric space $(X, |\cdot|)$;
(ii) T is not an (F,τ)-contraction in partial rectangular metric space (X,ρ);
(iii) T is not a G-(F,τ)-contraction in the induced rectangular metric space (X,ρ^r);
(iv) T is a G-(F,τ)-contraction in the partial rectangular metric space (X,ρ) and all the conditions of Theorem 3.4 are satisfied.

Proof. (i) To see this, take $x = 2, y = 3$, then $|Tx - Ty| = 2$ and $|x - y| = 1$. Therefore there exist no $F \in \mathcal{F}$ and $\tau > 0$ such that $\tau + F(|Tx - Ty|) \leq F(|x - y|)$ for all $x, y \in X$. Therefore, T is not an (F,τ)-contraction in usual metric space $(X, |\cdot|)$.

(ii) For this, take $x = 0, y = 4$, then $\rho(Tx,Ty) = \rho(0,1) = 6{+}1 = 7$ and $\rho(x,y) = 2{+}4 = 6$. Therefore there exist no $F \in \mathcal{F}$ and $\tau > 0$ such that $\tau + F(\rho(Tx,Ty)) \leq F(\rho(x,y))$ for all $x, y \in X$. Therefore, T is not an (F,τ)-contraction in partial rectangular metric space (X,ρ).

(iii) Note that, the rectangular metric induced by ρ is given by

$$\rho^r(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 12 + |x - y|, & \text{if } x, y \in \{0,1\}, x \neq y; \\ 4 + |x - y|, & \text{otherwise.} \end{cases}$$

Now take the point $x = 2, y = 3$, then $(2,3) \in E(G)$, $\rho^r(Tx,Ty) = \rho^r(0,2) = 6$ and $\rho^r(x,y) = \rho^r(2,3) = 5$. Therefore there exist no $F \in \mathcal{F}$ and $\tau > 0$ such that $\tau + F(\rho^r(Tx,Ty)) \leq F(\rho^r(x,y))$ for all $x, y \in X$ with $(x,y) \in E(G)$. Therefore, T is not a G-(F,τ)-contraction in the induced partial rectangular metric space (X,ρ^r).

(iv) For this, note that $(Tx,Ty) \in E(G)$ for all $(x,y) \in E(G)$, so (GF1) is satisfied. By easy calculations one can see that (GF2) is satisfied with $F(\alpha) = \ln(\alpha)$ and $\tau = \ln(5/4)$. Since the induced rectangular metric space (X,ρ^r) is complete therefore (X,ρ) is complete. Note that, the property (P) is satisfied trivially and $X_T = \{0,2,3\} \neq \emptyset$ therefore all the conditions of Theorem 3.4 are satisfied and $\mathrm{Fix}(T) = \{0\}$. □

Remark 3.1. (i) of the above example shows that the result from usual metric spaces (that is, the result of [2]) is not applicable. Also, (ii) of the above example shows that the non-graphical version of the Theorem 3.4, that is, the Corollary 3.1 of this paper is not applicable. As well as, by (iii) shows that the rectangular metric version of Theorem 3.4 is not applicable. But the Theorem 3.4 is applicable.

The following corollary is the fixed point theorem for an ordered (F,τ)-contraction in partial rectangular metric spaces.

Corollary 3.2. *Let (X,ρ) be a complete partial rectangular metric space endowed with a partial order $\sqsubseteq$. Let $T\colon X\to X$ be an ordered (F,τ)-contraction such that the following conditions hold:*

(A) there exists $x_0\in X$ such that $x_0\sqsubseteq Tx_0$;
(B) T is nondecreasing with respect to $\sqsubseteq$;
(C) if $\{x_n\}$ is a nondecreasing sequence in X and converging to some $z\in X$ then $x_n\sqsubseteq z$.

Then T is a weakly Picard operator. In addition, Fix(T) is well ordered (that is, all the elements of Fix(T) are comparable) if and only if T is a Picard operator.

Proof. Define a graph G_1 by $V(G_1)=X$ and $E(G_1)=\{(x,y)\in X\times X\colon x\sqsubseteq y\}$. Then (X,ρ) is a partial rectangular metric space endowed with the graph G_1 and T is a G_1-(F,τ)-contraction. By (A) and (B), we have $X_T\neq\emptyset$ and by (C) the condition (P) is satisfied. Note that Fix(T) is well ordered implies that the subgraph $G_{\mathrm{Fix}(T)}$ is weakly connected. Now the proof follows from Theorem 3.2. □

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