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SERIES SOLUTION OF EPIDEMIC MODEL

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Abstract. The present paper is concerned with the approximate analytic series solution of the epidemic model. In place of the traditional numerical, perturbation or asymtotic methods, Laplace-Adomian decomposition method (LADM) is employed.To demonstrate the effort of the LADM an epidemic model, which has been worked on recently, has been solved. The results are compared with the results obtained by Adomian decomposition method and homotopy perturbation method. Furthermore the results are compared with Fouth Order Runge Method and residual error. After examining the results, we see that LADM is a powerful method for obtaining aproximate solutions to epidemic model.

Keywords: Series solution, Epidemic model, Laplace-Adomian decomposition method, approximate solution, system of nonlinear differential equations.

AMS Subject Classification:

1. Introduction

Models of biological systems are represented by non-linear ordinary differential equations. In recent biological studies mathematical modelling and simulations of these models are playing very important roles. In scientific literature the epidemic model has taken attention in recent years. In [1] the problem of spreading of a non-fatal disease in a population which is assumed to have constant size over the period of the epidemic is considered. Adomian decomposition method [2] and homotopy perturbation method [3] have been used to solve this model. In this model the following system determines the progress of the disease:

$$\begin{aligned} \frac{dx}{dt} &= -\beta xy \\ \frac{dy}{dt} &= \beta xy - \gamma y \\ \frac{dz}{dt} &= \gamma y, \end{aligned} \tag{1.1}$$

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with the initial conditions

$$x(0) = N_1, y(0) = N_2, z(0) = N_3.$$

Where at time t

$x(t)$ susceptible population: those so far uninfected and therefore liable to infection;
$y(t)$ infective population: those who have the disease and are still at large;
$z(t)$ isolated population, or who have recovered and are therefore immune.

Purpose of this paper is to extend the application of LADM [4, 5, 6, 7, 8] to obtain an approximate solution of the epidemic model.

2. Solution Procedure

In this section, we will apply the LADM to epidemic model. We consider the (1.1) epidemic model. To solve this model by using the LADM, we recall that the Laplace transform of x_i' are defined by

$$L\{x'\} = s.L\{x\} - x(0) \quad ;\ i = 1, 2, \cdots, n.$$

Applying the Laplace transform to both side of (1.1) gives

$$\begin{aligned} s.L\{x(t)\} &= x(0) - L\{\beta x(t) y(t)\} \\ s.L\{y(t)\} &= y(0) + L\{\beta x(t) y(t)\} - L\{\gamma y(t)\} \\ s.L\{z(t)\} &= z(0) + L\{\gamma y(t)\} \end{aligned} \tag{2.1}$$

This can be reduced to

$$\begin{aligned} L\{x(t)\} &= \tfrac{x(0)}{s} - \tfrac{1}{s} L\{\beta x(t) y(t)\} \\ L\{y(t)\} &= \tfrac{y(0)}{s} + \tfrac{1}{s} L\{\beta x(t) y(t)\} - \tfrac{1}{s} L\{\gamma y(t)\} \\ L\{z(t)\} &= \tfrac{z(0)}{s} + \tfrac{1}{s} L\{\gamma y(t)\}. \end{aligned} \tag{2.2}$$

The Adomian decomposition method and the Adomian polynomials can be used to handle (2.2) and to address the nonlinear term $F = x(t).y(t)$ .In this method consists the next of representing the solutions as infinite series, namely

$$x = \sum_{k=0}^{\infty} x_k\ ,\ y = \sum_{k=0}^{\infty} y_k\ , z = \sum_{k=0}^{\infty} z_k \tag{2.3}$$

Where the components x_k are to be recursively computed. However, the nonlinear term $F = x(t).y(t)$ at the right side of (2.2) will be represented by an infinite series of Adomian polynomials

$$F(t, x, y) = \sum_{k=0}^{\infty} A_k \tag{2.4}$$

Where $A_k, k \geq 0$ are defined by

$$A_k = \frac{1}{k!} \frac{d^k}{d\lambda^k} \left[F\left(t, \sum_{j=0}^{k} \lambda^j x_j, \sum_{j=0}^{k} \lambda^j y_j \right) \right], k = 0, 1, 2, \cdots \tag{2.5}$$

and they so-called Adomian polynomials A_k can be evaluated for all forms of nonlinearity. Substituting (2.3) and (2.4) into (2.2) leads to

$$\begin{aligned} &L\left\{\sum_{k=0}^{\infty} x_k\right\} = \tfrac{x(0)}{s} - \tfrac{\beta}{s} L\left\{\sum_{k=0}^{\infty} A_k\right\} \\ &L\left\{\sum_{k=0}^{\infty} y_k\right\} = \tfrac{y(0)}{s} + \tfrac{\beta}{s} L\left\{\sum_{k=0}^{\infty} A_k\right\} - \tfrac{\gamma}{s} L\left\{\sum_{k=0}^{\infty} y_k\right\} \\ &L\left\{\sum_{k=0}^{\infty} z_k\right\} = \tfrac{z(0)}{s} + \tfrac{\gamma}{s} L\left\{\sum_{k=0}^{\infty} y_k\right\}. \end{aligned} \tag{2.6}$$

Matching both sides of (2.6) yields the following iterative algorithm.

$$\begin{aligned} &L\{x_0\} = \tfrac{x(0)}{s}, \quad L\{x_{k+1}\} = -\tfrac{\beta}{s} L\{A_k\} \\ &L\{y_0\} = \tfrac{y(0)}{s}, \quad L\{y_{k+1}\} = \tfrac{\beta}{s} L\{A_k\} - \tfrac{\gamma}{s} L\{y_k\} \\ &L\{z_0\} = \tfrac{z(0)}{s}, \quad L\{z_{k+1}\} = \tfrac{\gamma}{s} L\{y_k\} \end{aligned} \tag{2.7}$$

Applying the inverse Laplace transform to the first part of (2.7) gives x_0, y_0 and z_0 that will define A_0. By using A_0 will enable us to evaluate x_1, y_1 and z_1. The determination of x_1 and y_1 leads to the determination of A_1 that will allows us to determine x_2, y_2, z_2 and so on. This successively will lead to the complete determination of the components of x_k, y_k and z_k , $k \geq 0$ upon using the second part of (2.7). The series solution follows immediately after using the equation (2.3).

3. The Padé Approximate

Here we will give brief knowledge about Padé Approximate. The main advantage of the Padé approximation over the Taylor series approximation is that the Padé approximant often gives better approximation of the function than truncating its Taylor series, and it may still work where the Taylor series does not converge. Therefore Padé approximants are used extensively in computer calculations. The Padé approximation of a function is used to approximate functions by rational functions. The coefficients of the polynomial in both the numerator and the denominator are determined by using the coefficients in the Taylor series expansion of the function. The Padé approximation of a function, symbolized by $[m/n]$, is the quotient of two polynomials and of degrees m and n, respectively. The Padé approximation of a function deffined by

$$[m/n] = \frac{a_0 + a_1 x + a_2 x^2 + \cdots + a_m x^m}{1 + b_1 x + b_2 x^2 + \cdots + b_n x^n}$$

where we considered $b_0 = 1$, and the numerator and denominator have no common factors.

4. Application

In this section, we will apply the LADM to epidemic model. When we use the LADM to solve the epidemic model we consider the following parameter :

$N_1 = 20$	Initial population of $x(t)$, who are susceptible
$N_2 = 15$	Initial population of $y(t)$, who are infective
$N_3 = 10$	Initial population of y(t), who are immune
$\beta = 0.01$	Rate of change of susceptibles to infective population
$\gamma = 0.02$	Rate of change of infectives to immune population

Approximations having three terms, four terms and five terms for $x(t)$, $y(t)$ and $z(t)$ are calculated and presented below, approximations with one terms are not counted.

Approximations with three terms:

$$\begin{aligned} x(t) &= 20 - 3.t - 0.045t^2 + 0.02805t^3, \\ y(t) &= 15 + 2.7t + 0.018t^2 - 0.02817t^3, \\ z(t) &= 10 + 0.3t + 0.027t^2 + 0.00012t^3. \end{aligned}$$

Approximations with four terms:

$$\begin{aligned} x(t) &= 20 - 3.t - 0.045t^2 + 0.02805t^3 + 0.000795375t^4, \\ y(t) &= 15 + 2.7t + 0.018t^2 - 0.02817t^3 - 0.000654525t^4, \\ z(t) &= 10 + 0.3t + 0.027t^2 + 0.00012t^3 - 0.00014085t^4. \end{aligned}$$

Approximations with five terms:

$$\begin{aligned} x(t) &= 20 - 3.t - 0.045t^2 + 0.02805t^3 + 0.000795375t^4 - 0.00031655t^5, \\ y(t) &= 15 + 2.7t + 0.018t^2 - 0.02817t^3 - 0.000654525t^4 + 0.000319168t^5, \\ z(t) &= 10 + 0.3t + 0.027t^2 + 0.00012t^3 - 0.00014085t^4 - 2.6181E - 6\ t^5. \end{aligned}$$

A comparison between the results derived by this method with those which are presented in [2, 3], shows that we have derived exactly the same results with them.

We use Mathematica to calculate the [5/5] Padé approximate of the infinite series solution (1.1) for ten terms approximations, which gives the following rational fraction approximation to the solution:

$$px(t) = \frac{20 - 4.63203t + 0.505075t^2 - 0.0324804t^3 + 0.00123556t^4 - 0.0000223765t^5}{1 - 0.0816015t + 0.0152635t^2 - 0.000920593t^3 + 0.0000327091t^4 - 6.18204E - 7\ t^5}$$

$$py(t) = \frac{15 + 3.76451t + 0.407683t^2 + 0.025211t^3 + 0.000925777t^4 + 0.0000165658t^5}{1 + 0.0709677t + 0.0132047t^2 + 0.00109672t^3 + 0.0000253747t^4 + 1.83807E - 6\ t^5}$$

$$pz(t) = \frac{10 - 0.511677t + 0.156839t^2 - 0.00555202t^3 + 0.000496799t^4 - 6.58985E - 6\ t^5}{1 - 0.0811677t + 0.015419t^2 - 0.000810618t^3 + 0.0000474262t^4 - 9.59567E - 7\ t^5}$$

Table 1 gives the results from the solution of Epidemic Model and illustrate the absolute errors obtained by using Runge Kutta fourth order method and the [5/5] Padé approximate of the infinite series solution (1.1) for ten terms LADM approximations. We achieved a good approximation.

Table 1 Absolute errors obtained by using Runge Kutta fourth order method and Padé-LADM

t_i	$\lvert px(t_i) - RKM \rvert$	$\lvert py(t_i) - RKM \rvert$	$\lvert pz(t_i) - RKM \rvert$
0	0	0	0
1	4.728674838E-7	2.666530143E-7	2.062141355E-7
2	1.531118752E-7	5.559413907E-8	2.081567256E-7
3	6.102611660E-7	8.213127067E-7	1.761178154E-7
4	1.715516470E-6	1.536224918E-6	7.386365333E-7
5	1.574681681E-5	1.001382392E-5	9.721294774E-6
6	1.018548930E-4	5.372599448E-5	6.382178585E-5
7	4.664257195E-4	2.070983371E-4	2.922920797E-4
8	1.665940959E-3	6.263588701E-4	1.035143082E-3
9	4.933144396E-3	1.579403100E-3	3.014085391E-3
10	1.262249202E-2	3.454034028E-3	7.530199934E-3

These results are plotted in Figure 1 and Runge-Kutta Fourth order solution plotted in Figure 2.

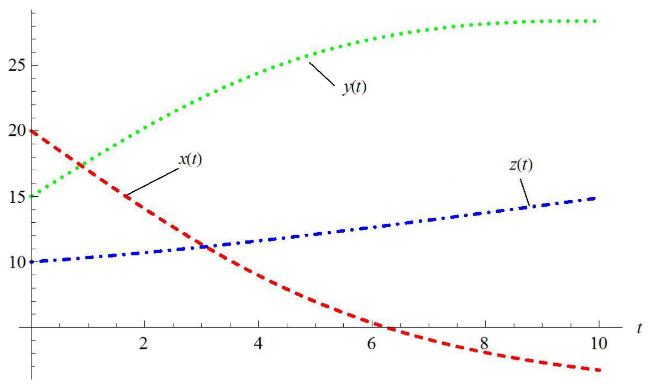

Figure 1. Plots of LADM solutions versus time

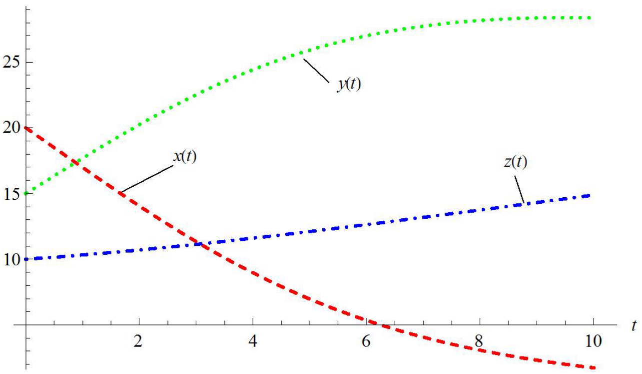

Figure 2. Plots of Fourth Order Runge-Kutta solutions versus time

To see the accuracy of the solution we define the residual error for the model as

$$\begin{aligned} resx &= \frac{d\widetilde{x}}{dt} - (-\beta\widetilde{x}\widetilde{y}) \\ resy &= \frac{d\widetilde{y}}{dt} - (\beta\widetilde{x}\widetilde{y} - \gamma\widetilde{y}) \\ resz &= \frac{d\widetilde{z}}{dt} - (\gamma\widetilde{y}) \end{aligned}$$

where $\widetilde{x}, \widetilde{y}$ and $\widetilde{z}$ are the Padé-LADM solutions for x, y and z, respectively. The residual errors are presented in Table 2.

Table 2

The residual errors obtained by Padé-LADM solutions

t_i	$resx$	$resy$	$resz$
0	4.44089E-16	1.55431E-15	4.996E-16
1	4.05932E-12	5.30842E-12	2.39868E-12
2	4.43151E-9	4.67863E-9	2.62688E-9
3	2.56259E-7	2.21589E-7	1.50603E-7
4	4.31442E-6	3.11613E-6	2.49225E-6
5	3.62864E-5	2.24366E-5	2.04283E-5
6	1.95021E-4	1.06069E-4	1.06061E-4
7	7.67129E-4	3.77125E-4	3.99392E-4
8	2.40181E-3	1.0946E-3	1.18641E-3
9	6.32905E-3	2.7337E-3	2.94113E-3
10	1.45964E-2	6.0851E-3	6.33213E-3

5.

6. Conclusion

The LADM which was used to solve the nonlinear system of differential equations, governing the problem of epidemic model, seems to be very easy to employ with reliable results. The results are just the same as those given in [2, 3], with the same degree of accuracy.. The obtained numerical results compared with the Fourth Order Runge Kutta Method and residual error. Results show that the present method provides remarkable accuracy . LADM is a powerful mathematical tool to solve epidemic model All the computations in the present work were carried out by using Mathematica 7.

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