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SHADOW OF OPERATORS ON FRAMES

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Abstract. Aldroubi introduced two methods for generating frames of a Hilbert space $\mathcal{H}$. In one of his method, one approach is to construct frames for $\mathcal{H}$ which are images of a given frame for $\mathcal{H}$ under $T \in \mathcal{B}(\mathcal{H}, \mathcal{H})$, a collection of all bounded linear operator on $\mathcal{H}$. The other method uses bounded linear operator on ℓ^2 to generate frames of $\mathcal{H}$. In this paper, we discuss construction of the retro Banach frames in Hilbert spaces which are images of given frames under bounded linear operators on Hilbert spaces. It is proved that the compact operators generated by a certain type of a retro Banach frame can construct a retro Banach frame for the underlying space. Finally, we discuss a linear block associated with a Schauder frame in Banach spaces.

Keywords: Hilbert frame, Banach frame, retro Banach frame, Schauder frame, compact operator.

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1. Introduction

Gabor [14] introduced a fundamental approach to signal decomposition in terms of elementary signals. Duffin and Schaeffer in [11], while addressing some difficult problems from the theory of nonharmonic Fourier series introduced *frames* (or *Hilbert frames*) for Hilbert spaces. Infact, Duffin and Schaeffer abstracted Gabor's method to define frames for Hilbert spaces. The theory of frames for Hilbert spaces was revived by Daubechies, Grossmann and Meyer in [10].

Let $\mathcal{H}$ be an infinite dimensional separable real (or complex) Hilbert space with inner product $\langle .,. \rangle$. A sequence $\{f_k\} \subset \mathcal{H}$ is called a *frame* (or *Hilbert frame*) for $\mathcal{H}$ if there exist constants $0 < \mathrm{A} \leq \mathrm{B} < \infty$ such that

$$\mathrm{A}\|f\|^2 \leq \|\{\langle f, f_k \rangle\}\|^2_{\ell^2} \leq \mathrm{B}\|f\|^2 \text{ for all } f \in \mathcal{H}. \quad (1)$$

The positive constants A and B are called *lower* and *upper bounds* of the frame $\{f_k\}$, respectively. They are not unique. The inequality (1) is called the *frame inequality* of the frame. The operator $\theta : \ell^2 \to \mathcal{H}$ given by

$$\theta(\{c_k\}) = \sum_{k=1}^{\infty} c_k f_k, \{c_k\} \in \ell^2$$

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is called the *synthesis operator* or *pre-frame operator.* The adjoint operator $\theta^* : \mathcal{H} \to \ell^2$ of θ is called the *analysis operator* of the frame and is given by

$$\theta^*(f) = \{\langle f, f_k \rangle\}, \ f \in \mathcal{H}.$$

Composing θ and θ^* we obtain the *frame operator* $S = \theta\theta^* : \mathcal{H} \to \mathcal{H}$ which is given by

$$S(f) = \sum_{k=1}^{\infty} \langle f, f_k \rangle f_k, \ f \in \mathcal{H}.$$

The frame operator S is a positive continuous invertible linear operator from $\mathcal{H}$ onto $\mathcal{H}$. Every vector $f \in \mathcal{H}$ can be written as:

$$f = SS^{-1}f = \sum_{k=1}^{\infty} \langle S^{-1}f, f_k \rangle f_k.$$

The series converges unconditionally and is called the *reconstruction formula* for the frame. The representation of f in the reconstruction formula need not be unique. Thus, frames are *redundant systems* in a Hilbert space which yield one natural representation for every vector in the concern Hilbert space, but which may have infinitely many different representations for a given vector.

Gröchenig in [15] generalized Hilbert frames to Banach spaces. Before the concept of Banach frames was formalized, it appeared in the foundational work of Feichtinger and Gröchenig [12, 13] related to the *atomic decompositions.* Feichtinger and Gröchenig studied the atomic decomposition via integrable group representation. Atomic decompositions appeared in the field of applied mathematics providing many applications [6, 7]. Casazza, Han and Larson [4] also carried out a study of atomic decompositions and Banach frames. An atomic decomposition allow a representation of every vector of the space via a series expansion in terms of a fixed sequence of vectors which we call *atoms.* On the other hand, a Banach frame for a Banach space ensure reconstruction via a bounded linear operator or synthesis operator. Recently, various generalization of frames in Banach spaces have been introduced and studied. Han and Larson [16] defined a Schauder frame for a Banach space $\mathcal{X}$ to be an inner direct summand (i.e. a compression) of a Schauder basis of $\mathcal{X}$. Retro Banach frames introduced and studied in [18]. For recent development in frames and retro Banach frames for Banach spaces one may refer to [9, 19, 22 - 28]. The reconstruction property in Banach spaces was introduced and studied by Casazza and Christensen in [5] and further studied in [19, 20, 21, 25]. The reconstruction property is an important tool in several areas of mathematics and engineering. As the perturbation result of Paley and Wiener preserves reconstruction property, it becomes more important from an application point of view. The reconstruction property is also used to study the geometry of Banach spaces. In fact, it is related to the bounded approximated property as observed in [3, 4]. An excellent approach towards the utility of frames in different direction is given in the book by Casazza and Kutyniok [2] (also see [8]) and in the paper by Heil and Walnut [17].

Motivated from a paper by Aldroubi [1], who introduced two methods for generating frames of a Hilbert space $\mathcal{H}$, in this paper we discuss the behaviour of bounded linear operators on retro Banach frames. Construction of retro Banach frames from bounded linear operator are given. It is proved that the image of a retro Banach frame under compact linear operator (generated by the given retro Banach frame) is a retro Banach frame for the underlying space. This is, not true, in general, for arbitrary compact linear operator. We present a sufficient condition (in terms of a certain closeness of a pair Schauder frames in a Banach space) for the existence of a compact operator on the underlying space. Finally, we discuss a linear block associated with Schauder frames in Banach spaces.

2. Preliminaries

Throughout this paper $\mathcal{X}$ will denotes an infinite dimensional real (or complex) separable Banach space, $\mathcal{X}^*$ the conjugate space (topological) of $\mathcal{X}$. For a sequence $\{f_k\} \subset \mathcal{X}$, $[f_k]$ denotes the closure of span$\{f_k\}$ in the norm topology of $\mathcal{X}$. The set of all positive integer is denoted by $\mathbb{N}$. The family of all bounded linear operator from a Banach space $\mathcal{X}$ into a Banach space $\mathcal{Y}$ is denoted by $\mathcal{B}(\mathcal{X}, \mathcal{Y})$. For $\Theta \in \mathcal{B}(\mathcal{X}, \mathcal{Y})$, Ker$\Theta$ and RanΘ denote the Kernel and range of the operator Θ, respectively. An operator $T : X \to Y$ from a normed space X into a normed space Y is said to be a *compact linear operator* if, T is linear and if for every bounded subset $W \subset X$, the image $T(W)$ is relatively compact. The collection of all compact linear operator on $\mathcal{X}$ is denoted by $\mathcal{K}(\mathcal{X}, \mathcal{X})$. A linear operator T on $\mathcal{H}$ is said to be a *coercive operator* if there is a constant $\delta > 0$ such that

$$\|T(f)\| \geq \delta \|f\| \text{ for all } f \in \mathcal{H}.$$

Definition 2.1. [18] *A system $\mathcal{F} \equiv (\{f_k\}, \Theta)$ $(\{f_k\} \subset \mathcal{X}, \Theta : \mathcal{Z}_d \to \mathcal{X}^*)$ is called a* retro Banach frame *for $\mathcal{X}^*$ with respect to an associated Banach space of scalar valued sequences $\mathcal{Z}_d$, if*

(1) $\{f^*(f_k)\} \in \mathcal{Z}_d$ *for all* $f^* \in \mathcal{X}^*$,

(2) *there exist positive constants* $0 < \mathrm{A}_0 \leq \mathrm{B}_0 < \infty$ *such that*

$$\mathrm{A}_0 \|f^*\| \leq \|\{f^*(f_k)\}\|_{\mathcal{Z}_d} \leq \mathrm{B}_0 \|f^*\| \text{ for all } f^* \in \mathcal{X}^*,$$

(3) Θ *is a bounded linear operator operator such that* $\Theta(\{f^*(f_k)\}) = f^*, \ f^* \in \mathcal{X}^*$.

The positive constant A_0, B_0 are called *lower* and *upper retro frame bounds* of the frame $\mathcal{F}$, respectively. The operator $\Theta : \mathcal{Z}_d \to \mathcal{X}^*$ is called the *retro pre-frame operator* (or simply *reconstruction operator*) associated with $\mathcal{F}$. If there exists no reconstruction operator Θ_m such that $(\{f_k\}_{k \neq m}, \Theta_m)$ ($m \in \mathbb{N}$ is arbitrary) is a retro Banach frame for $\mathcal{X}^*$, then $\mathcal{F}$ is called an *exact* retro Banach frame for $\mathcal{X}^*$.

Lemma 2.1. *Let $\mathcal{X}$ be a Banach space and $\{f_n^*\} \subset \mathcal{X}^*$ be a sequence such that $\{f \in \mathcal{X} : f_n^*(f) = 0,$ for all $n \in \mathbb{N}\} = \{0\}$. Then, $\mathcal{X}$ is linearly isometric to the Banach space $\mathcal{Z} = \{\{f_n^*(f)\} : f \in \mathcal{X}\}$, where the norm of $\mathcal{Z}$ is given by*

$$\|\{f_n^*(f)\}\|_{\mathcal{Z}} = \|f\|_{\mathcal{X}}, \ f \in \mathcal{X}.$$

Lemma 2.2. [18] *Let $\mathcal{F} \equiv (\{f_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$. Then, $\mathcal{F}$ is exact if and only if $f_n \notin [f_k]_{k \neq n}$, for all $n \in \mathbb{N}$.*

3. Operators on $\mathcal{H}$ For The Construction of Retro Banach Frames

Aldroubi in [1] gave fundamental methods for generating Hilbert frames for a Hilbert space $\mathcal{H}$. Let $\{f_n\} \subset \mathcal{H}$ be a frame for $\mathcal{H}$. In one of his method, Aldroubi considered the system $\Theta_n = T(f_n)$, where $T \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $n \in \mathbb{N}$. Aldroubi proved that the system $\{\Theta_n = T(f_n)\}$ is a frame for $\mathcal{H}$ if and only if the adjoint operator T^* is coercive.

Theorem 3.1. [1] *Let $\{f_n\}$ be a frame for $\mathcal{H}$ with frame bounds $0 < A \leq B$. If T is a bounded linear operator from $\mathcal{H}$ into $\mathcal{H}$, then $\{\Theta_n = T(f_n)\}$ is a frame for $\mathcal{H}$ if and only if there exists a positive constant γ such that the adjoint operator T^* satisfies*

$$\|T^*(f)\|^2 \geq \gamma \|f\|^2, \text{ for all } f \in \mathcal{H}.$$

Remark 3.1. *Let $\mathcal{F} \equiv (\{f_n\}, \Theta)$ be a retro Banach frame for $\mathcal{H}^*$ with respect to $\mathcal{Z}_d = \ell^2$ with frame bounds $0 < A, B < \infty$ and let $\mathcal{Q} \in \mathcal{B}(\mathcal{H}, \mathcal{H})$. Then, Theorem 3.1, provides necessary and sufficient conditions for the existence of the reconstruction operator $\widehat{\Theta}$ such that $\mathcal{G} \equiv (\{\mathcal{Q}(f_n)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to $\mathcal{Z}_d = \ell^2$. More precisely, from the idea given in the proof of sufficient part of Theorem 3.1, one of the choice for retro frame bounds of $\mathcal{G}$ are found to be $\sqrt{\gamma A}, \sqrt{B}\|\mathcal{Q}^*\|$. Aldroubi [1] gave* *construction of Hilbert frames for subspaces. By using certain ideas given in the proof of Theorem* 2 *in* [1, *p.* 1663], *we can construct a retro Banach frame for subspaces* $\mathcal{H}_1 \subset \mathcal{H}$ (*from a given retro Banach frame for* $\mathcal{H}$). *Other results proved by Aldroubi in* [1] *are also useful in the construction of retro Banach frames for* $\mathcal{H}^*$ *with respect to* $\mathcal{Z}_d = \ell^2$. *Aldroubi in* [1] *provides applications of the construction of frames from bounded linear operators in frames, for example, affine frames, multiresolution and wavelet theory.*

In this section, we discuss the construction of retro Banach frames for $\mathcal{H}^*$ as image of a given retro Banach frame under $\mathcal{Q} \in \mathcal{B}(\mathcal{H}, \mathcal{H})$. Let $\mathcal{F} \equiv (\{f_n\}, \Theta)$ be a retro Banach frame for $\mathcal{H}^*$ with respect to $\mathcal{Z}_d$ and let $\mathcal{Q} \in \mathcal{B}(\mathcal{H}, \mathcal{H})$. Then, in general, there exists no reconstruction operator $\widehat{\Theta}$ such that $(\{\mathcal{Q}(f_n)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$. In this direction, we have following example.

Example 3.1. *Consider the discrete signal space* $\mathcal{H} = L^2(\Omega, \mu)$, *where* $\Omega = \mathbb{N}$ *and* μ *is the counting measure. Let* $\{f_k\} \subset \mathcal{H}$ *be a sequence given by*

$$f_1 = \chi_1, \text{ and } f_k = \chi_{k-1}, \; k > 1 \; (k \in \mathbb{N}),$$

where $\chi_k = \{0, 0, 0, ..., \underbrace{1}_{k^{th}-place}, 0, 0, ...\}$ $(k \in \mathbb{N})$. *Choose* $\mathcal{Z}_d = \{\{f^*(f_k)\} : f^* \in \mathcal{H}^*\}$.
Then, by using Lemma 2.1, $\mathcal{Z}_d$ *is a Banach space of scalar valued sequences with the norm given by*

$$\|\{f^*(f_k)\}\|_{\mathcal{Z}_d} = \|f^*\|_{\mathcal{H}^*}, \; f^* \in \mathcal{H}^*.$$

Define $\Theta : \mathcal{Z}_d \to \mathcal{H}^*$ *by* $\Theta(\{f^*(f_k)\}) = f^*, f^* \in \mathcal{H}^*$. *Then,* Θ *is a bounded linear operator such that* $\mathcal{F} \equiv (\{f_k\}, \Theta)$ *is a retro Banach frame for* $\mathcal{H}^*$.

Let $\mathcal{Q} \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ *be given by* $\mathcal{Q}(f) = \mathcal{Q}(\{\eta_1, \eta_2, \eta_3, ...\}) = \{0, 0, \eta_3, ...\}$, $f = \{\eta_j\} \in \mathcal{H}$. *Then, there exists no reconstruction operator* $\widehat{\Theta}$ *such that* $\mathcal{G} \equiv (\{\mathcal{Q}(f_k)\}, \widehat{\Theta})$ *is a retro Banach frame for* $\mathcal{H}^*$. *Indeed, let* A^0, B^0 *be a choice of retro frame bounds for* $(\{\mathcal{Q}(f_k)\}, \widehat{\Theta})$.
Then

$$A^0\|f^*\| \leq \|\{f^*(\mathcal{Q}(f_k)\}\|_{\mathcal{Z}_d} \leq B^0\|f^*\| \text{ for all } f^* \in \mathcal{H}^*. \tag{2}$$

Choose $f^* = \chi_1$. *Then,* $f^*(\mathcal{Q}(f_k)) = 0$ *for all* $k \in \mathbb{N}$. *Therefore, by using retro frame inequality* (2), *we have* $f^* = 0$, *a contradiction.*

Remark 3.2. *Let* $\mathcal{Q}_0 \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ *be the operator given by*

$$\mathcal{Q}_0(f) = \mathcal{Q}_0(\{\eta_1, \eta_2, \eta_3, ...\}) = \left\{\frac{\eta_j}{j}\right\}, \; f = \{\eta_j\} \in \mathcal{H}.$$

Then, there exists a reconstruction operator $\widehat{\Theta}_0$ *such that* $\mathcal{G}_0 \equiv (\{\mathcal{Q}_0(f_k)\}, \widehat{\Theta}_0)$ *is a retro Banach frame for* $\mathcal{H}^*$ *with respect to* $\mathcal{Z}_\circ = \{\{f^*(\mathcal{Q}_0(f_k))\} : f^* \in \mathcal{H}^*\}$.

The first result of this section provides a necessary condition for the construction of retro Banach frames for $\mathcal{H}^*$ from bounded linear operators on $\mathcal{H}$. This can be extended to the construction of a Hilbert frame for Hilbert spaces.

Theorem 3.2. *Let* $\mathcal{F} \equiv (\{f_k\}, \Theta)$ *be a retro Banach frame for* $\mathcal{H}^*$ *with respect to* $\mathcal{Z}_d$ *and let* $\mathcal{Q} \in \mathcal{B}(\mathcal{H}, \mathcal{H})$. *If there exists a reconstruction operator* $\widehat{\Theta}$ *such that* $\mathcal{G} \equiv (\{\mathcal{Q}(f_k)\}, \widehat{\Theta})$ *is a retro Banach frame for* $\mathcal{H}^*$, *then*

$$\sup_{\xi \in (0,\infty)} \|\xi(\xi I + \mathcal{Q}\mathcal{Q}^*)^{-1}\| \leq 1, \tag{3}$$

where I *is the identity operator on* $\mathcal{H}$.

Proof. If $\mathcal{G}$ is a retro Banach frame for $\mathcal{H}^*$ with respect to $\mathcal{Z}_{\mathcal{Q}}$, then can find positive constants A_0 and B_0 such that

$$A_0\|f^*\| \leq \|\{f^*(\mathcal{Q}(f_k))\}\|_{\mathcal{Z}_{\mathcal{Q}}} \leq B_0\|f^*\| \text{ for each } f^* \in \mathcal{H}^*. \tag{4}$$

If $\langle \mathcal{Q}\mathcal{Q}^* f, f\rangle = 0$ for some $f \in \mathcal{H}$, then $\|\mathcal{Q}^* f\| = 0$. Therefore, by using retro frame inequality (4), we have $f = 0$. Indeed, by using the fact that $(KerQ^*)^{\perp} = [Ran\mathcal{Q}]$, lower inequality in (4) gives $Ker\mathcal{Q}^* = 0$, i.e. $\mathcal{Q}^*$ is injective. Therefore, $\|\mathcal{Q}^* f\| = 0$ implies that $f = 0$. Thus, $\langle \mathcal{Q}\mathcal{Q}^* f, f\rangle > 0$, whenever $f \neq 0$.
Therefore

$$\begin{aligned}\langle(\mathcal{Q}\mathcal{Q}^* + \xi I)f, f\rangle &= \langle \mathcal{Q}\mathcal{Q}^* f, f\rangle + \xi\langle f, f\rangle \\ &\geq \xi\|f\|^2, \text{ whenever } f \neq 0, \quad \xi \in (0, \infty).\end{aligned} \tag{5}$$

By using the Cauchy Schwartz inequality, (5) gives

$$\begin{aligned}\|(\mathcal{Q}\mathcal{Q}^* + \xi I)f\|\|f\| &\geq \langle(\mathcal{Q}\mathcal{Q}^* + \xi I)f, f\rangle \\ &\geq \xi\|f\|^2, \text{ whenever } f \neq 0, \quad \xi \in (0, \infty).\end{aligned}$$

Therefore

$$\|(\mathcal{Q}\mathcal{Q}^* + \xi I)f\| \geq \xi\|f\|, \text{ whenever } f \neq 0, \quad \xi \in (0, \infty).$$

This gives

$$\xi \ \|(\mathcal{Q}\mathcal{Q}^* + \xi I)^{-1} f\| \leq \|f\|, \text{ whenever } f \neq 0, \quad \xi \in (0, \infty).$$

Hence

$$\sup_{\xi\in(0,\infty)} \|\xi(\xi I + \mathcal{Q}\mathcal{Q}^*)^{-1}\| \leq 1.$$

The theorem is proved. □

Remark 3.3. *The condition (3) in Theorem* 3.2 *is not sufficient. Indeed, let $\mathcal{Q} = O$ be the zero operator on $\mathcal{H}$. Then, $\sup_{\xi\in(0,\infty)} \|\xi(\xi I + \mathcal{Q}\mathcal{Q}^*)^{-1}\| = 1$. But there exists no reconstruction operator $\widehat{\Theta}$ such that $(\{\mathcal{Q}(f_k)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to some associated Banach space of scalar valued sequences.*

Recall that if $\mathcal{F} \equiv (\{f_k\}, \Theta)$ is a retro Banach frame for $\mathcal{H}^*$. Then, in general, there exists no reconstruction operator $\widehat{\Theta}$ such that $(\{\mathcal{Q}(f_k)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to some associated Banach of scalar valued sequences (see Example 3.1). Regarding construction of Hilbert frames for Hilbert spaces from bounded linear operators, Aldroubi in one of his methods gave an explicit form of the operator under consideration in [1, p. 1663]. The following theorem gives sufficient condition for the existence of a bounded linear operator $\mathcal{Q}$ such that the pair $(\{\mathcal{Q}(f_k)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to some associated Banach space of scalar-valued sequences.

Theorem 3.3. *Let $\mathcal{F} \equiv (\{f_k\}, \Theta)$ $(\{f_k\} \subset \mathcal{H}, \Theta : \mathcal{Z}_d \to \mathcal{H}^*)$ be a retro Banach frame for $\mathcal{H}^*$ with respect to an associated sequence space $\mathcal{Z}_d$. Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ be such that*

$$\|\xi(\xi I + TT^*)^{-1}\| < 1, \ \textit{for some } \xi > 0, \tag{6}$$

where I is the identity operator on $\mathcal{H}$. Then, there exists a $\mathcal{Q} \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ and a reconstruction operator $\widehat{\Theta}$ such that $(\{\mathcal{Q}(f_k)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to some associated Banach of scalar valued sequences.

Proof. Assume that $\|\xi(\xi I + TT^*)\| < 1$, for some $\xi > 0$.
We compute

$$\begin{aligned} TT^*(\xi I + TT^*)^{-1} &= ((\xi I + TT^*) - \xi I)(\xi I + TT^*)^{-1} \\ &= I - \xi(\xi I + TT^*)^{-1}. \end{aligned} \tag{7}$$

By using (6), the operator $\xi(\xi I + TT^*)^{-1}$ is a bounded linear operator on $\mathcal{H}$ such that $\|\xi(\xi I + TT^*)^{-1}\| < 1$. Thus, $I - \xi(\xi I + TT^*)^{-1}$ is an invertible operator on $\mathcal{H}$. Therefore, by using (7), we conclude that $TT^*(\xi I + TT^*)^{-1}$ is invertible.
Choose $\mathcal{Q} = TT^*(\xi I + TT^*)^{-1}$. Then, $\mathcal{Q}$ is a bounded linear invertible operator on $\mathcal{H}$. By using Lemma 2.1, invertibility of $\mathcal{Q}$ and the fact that $\mathcal{F}$ is a retro Banach frame for $\mathcal{H}^*$, it is easy to verify that $\mathcal{Z}_\mathcal{Q} = \{\{f^*(\mathcal{Q}(f_k))\} : f^* \in \mathcal{H}^*\}$ is a Banach space with the norm given by

$$\|\{f^*(\mathcal{Q}(f_k))\}\| = \|f^*\|, \ f^* \in \mathcal{X}^*.$$

Define $\widehat{\Theta} : \mathcal{Z}_\mathcal{Q} \to \mathcal{H}^*$ by $\widehat{\Theta}(\{f^*(\mathcal{Q}(f_k))\}) = f^*$. Then, $\widehat{\Theta}$ is a bounded linear operator such that $(\{\mathcal{Q}(f_k)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to $\mathcal{Z}_\mathcal{Q}$. The theorem is proved. □

Remark 3.4. *The condition* (6) *in Theorem* 3.3 *can not be relaxed.*

In the definition of a retro Banach frame, three Banach spaces are involved. One of the Banach space is the associated Banach space of scalar valued sequences with respect to which a certain system admit retro Banach frame for the underlying space. Let $\mathcal{F}_o$ be a retro Banach frame for $\mathcal{H}^*$ with respect to an associated sequence space $\mathcal{Z}_d$. It would be interesting to know whether the image of $\mathcal{F}_o$ (under a bounded linear operator on $\mathcal{H}$) constitute a retro Banach frame for $\mathcal{H}^*$ with respect to $\mathcal{Z}_d$. The following theorem provides necessary and sufficient conditions for the construction of retro Banach frames from bounded linear operators on the underlying space with respect to the same associated Banach space of scalar valued sequences.

Theorem 3.4. *Let $\mathcal{F} \equiv (\{f_n\}, \Theta)$ $(\{f_n\} \subset \mathcal{H}, \Theta : \mathcal{Z}_d \to \mathcal{H}^*)$ be a retro Banach frame for $\mathcal{H}^*$ with respect to an associated sequence space $\mathcal{Z}_d$ and let $\mathcal{Q} \in \mathcal{B}(\mathcal{H}, \mathcal{H})$. Assume that $\mathcal{W} \in \mathcal{B}(\mathcal{Z}_d, \mathcal{Z}_d)$ is such that for all $f^* \in \mathcal{H}^*$, $\mathcal{W} : \{f^*(f_n)\} \to \{f^*(\mathcal{Q}(f_n))\}$. Then, there exists a bounded linear operator $\widehat{\Theta}$ such that $(\{\mathcal{Q}(f_n)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to $\mathcal{Z}_d$ if and only if*

$$\|\mathcal{W}(\{f^*(f_n)\})\|_{\mathcal{Z}_d} \geq c\|\mathcal{J}(\{f^*(\mathcal{Q}(f_n))\})\|_{\mathcal{Z}_d} \text{ for all } f^* \in \mathcal{H}^*, \tag{8}$$

where c is a positive constant and $\mathcal{J} \in \mathcal{B}(\mathcal{Z}_d, \mathcal{Z}_d)$ is an operator such that for all $f^* \in \mathcal{H}^*$

$$\mathcal{J} : \{f^*(\mathcal{Q}(f_n))\} \to \{f^*(f_n)\}.$$

Proof. Suppose first that $\mathcal{F}_\mathcal{Q} \equiv (\{\mathcal{Q}(f_n)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to $\mathcal{Z}_d$. Let $A_\mathcal{Q}$ and $B_\mathcal{Q}$ be a choice of retro frame bounds for $\mathcal{F}_\mathcal{Q}$ and let $\mathcal{P} : \mathcal{H}^* \to \mathcal{Z}_d$ be the analysis operator associated with $\mathcal{F}$ which is given by

$$\mathcal{P} : f^* \to \{f^*(f_n)\}, f^* \in \mathcal{H}^*.$$

Choose $\mathcal{J} = \mathcal{P}\widehat{\Theta}$ and $c = \frac{A_\mathcal{Q}}{\|\mathcal{P}\|} > 0$.
Then

$$\begin{aligned} \|\mathcal{W}(\{f^*(f_n)\})\|_{\mathcal{Z}_d} &= \|\{f^*(\mathcal{Q}(f_n))\}\|_{\mathcal{Z}_d} \\ &\geq A_\mathcal{Q}\|f^*\| \\ &\geq c\ \|\{f^*(f_n)\}\|_{\mathcal{Z}_d} \\ &= c\ \|\mathcal{J}(\{f^*(\mathcal{Q}(f_n))\})\|_{\mathcal{Z}_d} \text{ for all } f^* \in \mathcal{H}^*. \end{aligned}$$

The forward part is proved.

For the reverse part, suppose that (8) is satisfied.
We compute

$$\begin{aligned}\|\{f^*(\mathcal{Q}(f_n))\}\|_{\mathcal{Z}_d} &= \|\mathcal{W}(\{f^*(f_n)\})\|_{\mathcal{Z}_d} \\ &\le \|\mathcal{W}\|\|\{f^*(f_n)\}\|_{\mathcal{Z}_d} \\ &\le \|\mathcal{W}\|\|\mathcal{P}\|\|f^*\| \text{ for all } f^* \in \mathcal{H}^*. \end{aligned} \tag{9}$$

By using (8), we have

$$\begin{aligned} cA\|f^*\| &\le c\|\{f^*(f_n)\}\|_{\mathcal{Z}_d} \quad (\text{where } A \text{ is lower retro frame bound of } \mathcal{F}) \\ &= c\|\mathcal{J}(\{f^*(\mathcal{Q}(f_n))\})\|_{\mathcal{Z}_d} \\ &\le \|\mathcal{W}(\{f^*(f_n)\})\|_{\mathcal{Z}_d} \ (= \|\{f^*(\mathcal{Q}(f_n))\}\|_{\mathcal{Z}_d}). \end{aligned} \tag{10}$$

Set $a_0 = cA$ and $b_0 = \|\mathcal{W}\|\|\mathcal{P}\|$.
Then, by using (9) and (10), we have

$$a_0\|f^*\| \le \|\{f^*(\mathcal{Q}(f_n))\}\|_{\mathcal{Z}_d} \le b_0\|f^*\| \text{ for each } f^* \in \mathcal{H}^*.$$

Choose $\widehat{\Theta} = \Theta\mathcal{J}$. Then, $\widehat{\Theta} \in \mathcal{B}(\mathcal{Z}_d, \mathcal{H}^*)$ is such that $\widehat{\Theta}(\{f^*(\mathcal{Q}(f_n))\}) = f^*$ for all $f^* \in \mathcal{H}^*$. Hence $(\{\mathcal{Q}(f_n)\}, \widehat{\Theta})$ is a retro Banach frame for $\mathcal{H}^*$ with respect to $\mathcal{Z}_d$ and with one of the choice of retro frame bounds a_0, b_0. □

3.1. **Construction of Retro Banach Frames from Compact Operators.** Let us start this section with the promised counterexample.

" There exists a compact linear operator on a Banach space such that its image on a given retro Banach frame (even exact) need not a retro Banach frame for the underlying space."
This is given in the following example.

Example 3.2. *Let $\mathcal{F} \equiv (\{f_k\}, \Theta)$ be a retro Banach frame for $\mathcal{X}^*$ and let T be a compact linear operator on $\mathcal{X}$. Then, in general, there exists no reconstruction operator Ξ such that $(\{\mathrm{T}(f_k)\}, \Xi)$ is retro Banach frame for $\mathcal{X}^*$. Indeed, let $\mathcal{X} = L^2(\Omega, \mu)$ be the discrete signal space and let $\{\chi_k\} \subset \mathcal{X}$ be the standard orthornomal basis for $\mathcal{X}$. Then, there exists a reconstruction operator Θ such that $\mathcal{F} \equiv (\{\chi_k\}, \Theta)$ is an exact retro Banach frame for $\mathcal{X}^*$.*
Define $\mathrm{T}_o : \mathcal{X} \to \mathcal{X}$ by

$$\mathrm{T}_o(f = \{\xi_1, \xi_2, \xi_3, \xi_4, ...\}) = \left\{0, \xi_1, \frac{\xi_2}{2}, \frac{\xi_3}{3}, \frac{\xi_4}{4}, ...\right\}, \ f = \{\xi_j\} \in \mathcal{X}.$$

Then, T_o is a compact linear operator on $\mathcal{X}$.

Furthermore, there exists no reconstruction operator Ξ such that $(\{\mathrm{T_o}(\chi_k)\}, \Xi)$ is retro Banach frame for $\mathcal{X}^*$ with respect to any $\mathcal{Z}_d$. Indeed, let $(\gamma_0,\ \delta_0)$ be a choice of retro frame bounds for $(\{\mathrm{T_o}(\chi_k)\}, \Xi)$.
Then

$$\gamma_0\|f^*\| \le \|f^*(\mathrm{T_o}(\chi_k))\}\|_{\mathcal{Z}_d} \le \delta_0\|f^*\| \textit{ for all } f^* \in \mathcal{X}^*. \tag{11}$$

Choose $f_0^* = \chi_1 \in \mathcal{X}^*$. Then, we have $f_0^*(\mathrm{T_o}(\chi_k)) = 0$ for all $k \in \mathbb{N}$. Therefore, by using retro frame inequality (11), we obtain $f_0^* = 0$, a contradiction. Thus, the image of a retro Banach frame (even exact) under a compact linear operator on $\mathcal{X}$ need not be a retro Banach frame for the underlying space.

We now discuss construction of a compact linear operator from retro Banach frames, which can generate a retro Banach frame for the underlying space. Let $\mathcal{F} \equiv (\{f_k\}, \Theta)$ be an exact retro Banach frame for $\mathcal{X}^*$ (if $\mathcal{F}$ is not exact, then we can construct an exact retro Banach frame from $\{f_k\}$). Then, by Lemma 2.2, $f_j \notin [f_k]_{k\neq j}$ for all $j \in \mathbb{N}$. Therefore, by

the Hahn-Banach Theorem, we can find a sequence $\{f_k^*\} \subset \mathcal{X}^*$ such that $f_j^*(f_m) = \delta_{j,m}$ for all $j, m \in \mathbb{N}$. We call the sequence $\{f_k^*\} \subset \mathcal{X}^*$ an *admissible system* of $\mathcal{F}$.
Define $\mathrm{T} : \mathcal{X} \to \mathcal{X}$ by

$$\mathrm{T}(\bullet) = \sum_{i=1}^{\infty} \frac{f_i^*(\bullet) f_i}{i^p ||f_i||\ ||f_i^*||}. \quad (p > 1) \tag{12}$$

Then, one can verify that $\mathrm{T} \in \mathcal{K}(\mathcal{X}, \mathcal{X})$. This is summarized in the following lemma.

Lemma 3.1. *Let $\mathcal{F} \equiv (\{f_k\}, \Theta)$ be an exact retro Banach frame for $\mathcal{X}^*$ with admissible system $\{f_k^*\} \subset \mathcal{X}^*$. Then, $T : f \to \sum_{i=1}^{\infty} \frac{f_i^*(f) f_i}{i^p ||f_i||\ ||f_i^*||}$ $(p > 1)$ defines a compact linear operator on $\mathcal{X}$.*

Now a natural question arises: what is the importance of the compact linear operator T given in Lemma 3.1? The positive answer is that the compact linear operator given in Lemma 3.1 (and generalized for its prototype) always generates a retro Banach frame for the underlying space. This is given in the following theorem.

Theorem 3.5. *Let $\mathcal{F} \equiv (\{f_k\}, \Theta)$ be an exact retro Banach frame for a Banach space $\mathcal{X}^*$ with admissible system $\{f_k^*\} \subset \mathcal{X}^*$. Then, there exists a compact linear operator $\mathrm{T} : \mathcal{X} \to \mathcal{X}$ and a reconstruction operator $\Xi \in \mathcal{B}(\mathcal{Z}_\infty, \mathcal{X}^*)$ such that $(\{\mathrm{T}(f_k\}, \Xi)$ is a retro Banach frame for $\mathcal{X}^*$, where $\mathcal{Z}_\infty$ an associated Banach space of scalar valued sequences.*

Proof. The existence of a compact linear operator $\mathrm{T} \in \mathcal{B}(\mathcal{X}, \mathcal{X})$ associated with $\mathcal{F}$ is guaranteed by Lemma 3.1. More precisely, T is given by the equation (12).

For the second part, since $\mathcal{F}$ is a retro Banach frame for $\mathcal{X}^*$, we can find positive constants $\mathrm{A}_0, \mathrm{B}_0$ such that

$$\mathrm{A}_0 \|f^*\| \leq \|\{f^*(f_k)\}\|_{\mathcal{Z}_d} \leq \mathrm{B}_0 \|f^*\| \text{ for each } f^* \in \mathcal{X}^*. \tag{13}$$

Assume that there exists no reconstruction operator $\Theta^\times$ such that $(\{\mathrm{T}(f_k\}, \Xi)$ is a retro Banach frame for $\mathcal{X}^*$ with respect to any associated Banach space of scalar valued sequences. Then, by the Hahn-Banach Theorem there is a nonzero functional f_0^* such that $f_0^*(\mathrm{T} f_n) = 0$ for all $n \in \mathbb{N}$.
Therefore, by using (12), we have

$$\begin{aligned} 0 &= f_0^*(\mathrm{T} f_n) \text{ for all } n \in \mathbb{N} \\ &= f_0^* \left(\sum_{i=1}^{\infty} \frac{f_i^*(f_n) f_i}{i^p ||f_i||\ ||f_i^*||} \right) \text{ for all } n \in \mathbb{N} \\ &= f_0^* \left(\frac{f_n}{n^p ||f_n||\ ||f_n^*||} \right) \text{ for all } n \in \mathbb{N}. \end{aligned} \tag{14}$$

By using (14), we obtain $f_0^*(f_n) = 0$ for all $n \in \mathbb{N}$. Therefore, by lower retro frame inequality in (13), we have $f_0^* = 0$, a contradiction. Hence we can find a reconstruction operator $\Xi \in \mathcal{B}(\mathcal{Z}_\infty, \mathcal{X}^*)$ such that $(\{\mathrm{T}(f_k\}, \Xi)$ is retro Banach frame for $\mathcal{X}^*$, where $\mathcal{Z}_\infty$ an associated Banach space of scalar valued sequences. □

Remark 3.5. *Let $\mathcal{Y}$ be a closed subspace of $\mathcal{X}$ and let $\mathcal{F} \equiv (\{f_k\}, \Theta)$ be an exact retro Banach frame for the Banach space $\mathcal{X}^*$. Then, we can find a sequence $\{g_j\} \subset \mathcal{Y}$, a compact operator $\mathrm{T}_\mathcal{Y} \in \mathcal{B}(\mathcal{X}, \mathcal{X})$ and a reconstruction operator $\Xi_\mathcal{Y}$ such that $\mathcal{F}_\mathcal{Y} \equiv (\{\mathrm{T}_\mathcal{Y}(g_j)\}, \Xi_\mathcal{Y})$ is a retro Banach frame for $\mathcal{Y}^*$.*

4. Schauder Frames

Han and Larson [16] defined a Schauder frame for a Banach space $\mathcal{X}$ to be an inner direct summand (i.e. a compression) of a Schauder basis of $\mathcal{X}$.

Definition 4.1. [16] *Let $\{f_n\} \subset \mathcal{X}$ and $\{f_n^*\} \subset \mathcal{X}^*$. The pair $(\{f_n\}, \{f_n^*\})$ is called a* Schauder frame *for $\mathcal{X}$ if*

$$f = \sum_{n=1}^{\infty} f_n^*(f) f_n \text{ for all } f \in \mathcal{X}, \tag{15}$$

where the series in (15) *converges in the norm topology in $\mathcal{X}$.*
That is

$$\left\| f - \sum_{i=1}^{n} f_i^*(f) f_i \right\|_{\mathcal{X}} \to 0 \text{ as } n \to \infty.$$

The existence of a class of operators associated with redundant building blocks (for example, Schauder frames, Hilbert frames etc.) and vice-versa is one of the important topic in applied mathematics. The following proposition provides sufficient condition for the existence of a compact linear operator on $\mathcal{X}$ associated with Schauder frames for $\mathcal{X}$.

Proposition 4.1. *Let $\mathcal{G}_1 = (\{f_n\}, \{f_n^*\})$ and $\mathcal{G}_2 = (\{g_n\}, \{g_n^*\})$ be Schauder frames for $\mathcal{X}$. If*

$$\sum_{n=1}^{\infty} \|g_n^* - f_n^*\| \|f_n\| < \infty, \tag{16}$$

then $\widetilde{\Theta} : f \to \sum_{n=1}^{\infty} (g_n^* - f_n^*)(f) f_n$ defines a compact linear operator on $\mathcal{X}$.

Proof. By using (16), we have

$$\begin{aligned}
\left\| \sum_{k=p}^{p+n} (g_k^* - f_k^*)(f) f_k \right\| &\leq \left| \sum_{k=p}^{p+n} (g_k^* - f_k^*)(f) \right| \|f_k\| \\
&\leq \left(\sum_{k=p}^{p+n} \|g_k^* - f_k^*\| \|f_k\| \right) \|f\| \\
&\to 0 \text{ as } n, p \to \infty.
\end{aligned}$$

Therefore, $\widetilde{\Theta} : \mathcal{X} \to \mathcal{X}$ given by $\widetilde{\Theta}(f) = \sum_{k=1}^{\infty} (g_k^* - f_k^*)(f) f_k$ is a well defined linear operator. Moreover, $\widetilde{\Theta}$ is bounded.
Indeed

$$\|\widetilde{\Theta}\| \leq \sum_{k=1}^{\infty} \|g_k^* - f_k^*\| \|f_k\| < \infty.$$

Hence $\widetilde{\Theta}$ is bounded.
Choose $\Theta_n(\bullet) = \sum_{k=1}^{n} (g_k^* - f_k^*)(\bullet) f_k, \ n \in \mathbb{N}$. Then, each Θ_n is an operator of finite rank, and is compact.

We compute

$$\begin{aligned}\|\widetilde{\Theta} - \Theta_n\| &= \left\| \sum_{k=n+1}^{\infty} (g_k^* - f_k^*)(\bullet) f_k \right\| \\ &\leq \sum_{k=n+1}^{\infty} \|g_k^* - f_k^*\| \|f_k\| \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

Hence $\widetilde{\Theta} : f \to \sum_{k=1}^{\infty} (g_k^* - f_k^*)(f) f_k$ defines a compact linear operator on $\mathcal{X}$. □

To conclude the paper we discuss a linear block of the Schauder frame for $\mathcal{X}$ with respect to a given sequence in $\mathcal{X}$.

Definition 4.2. *Let $\mathcal{F} \equiv (\{f_k\}, \{f_k^*\})$ be a Schauder frame for $\mathcal{X}$ and let $\{g_k\} \subset \mathcal{X}$. A* linear block *of $\mathcal{F}$ with respect to $\{g_k\}$ is a sequence of the form*

$$\left\{ f_k + \sum_{i=k+1}^{\infty} f_i^*(g_k) f_i \right\}_{k=1}^{\infty}, \tag{17}$$

where the series converges in $\mathcal{X}$.

Recall that a Schauder frame $\mathcal{F} \equiv (\{f_k\}, \{f_k^*\})$ for $\mathcal{X}$ provides a series representation of each vector in $\mathcal{X}$ in terms of vectors f_k. If each f_k is perturbed by a nonzero vector in $\mathcal{X}$ (e.g. see (17)), then there is a question of the reconstruction of each vector in $\mathcal{X}$ as an infinite series in terms of the perturbed sequence $\{f_k + (\bullet)\}$. This is useful in applied mathematics. For example, in signal processing which depends on a series expansion of a signal. In this direction we have following result.

Theorem 4.1. *Assume that $\mathcal{F} \equiv (\{f_k\}, \{f_k^*\})$ is Schauder frame for a Banach space $\mathcal{X}$. Let $\{g_k\} \subset \mathcal{X}$ be a sequence of vectors and let*

$$\max \left\{ \limsup \sum_{i \geq k+1} \frac{\|f_i^*(g_k) f_i\|}{\|f_k\|}, \ \sum_{i=1}^{k} \|f_i^*(f) f_i\| \right\} < 1 \quad (k \in \mathbb{N}). \tag{18}$$

Then, there exists $\{g_k^*\} \subset \mathcal{X}^*$ such that $\left(\left\{ f_k + \sum_{i=k+1}^{\infty} f_i^*(g_k) f_i \right\}_{k=1}^{\infty}, \{g_k^*\} \right)$ is a Schauder frame for $\mathcal{X}$, where the series $\sum_{i=k+1}^{\infty} f_i^*(g_k) f_i$ converges in $\mathcal{X}$ $(k \in \mathbb{N})$.

Proof. Choose $\varphi_k = f_k + \sum_{i \geq k+1} f_i^*(g_k) f_i, \ k \in \mathbb{N}$.

By using (18) there exists a λ $(0 < \lambda < 1)$ such that

$$\|\varphi_k - f_k\| < \lambda \|f_k\| \text{ for all } k \in \mathbb{N}. \tag{19}$$

Then, by using (19), we have

$$\begin{aligned}\|f_i^*(f)(f_i - \varphi_i)\| &\leq \sum_{i=1}^{n} |f_i^*(f)| \|(f_i - \varphi_i)\| \\ &\leq \lambda \sum_{i=1}^{n} \|f_i^*(f) f_i)\|, \ f \in \mathcal{X}.\end{aligned} \tag{20}$$

Define $\Theta : \mathcal{X} \to \mathcal{X}$ by

$$\Theta \left(f = \sum_{i=1}^{\infty} f_i^*(f) f_i) \right) = \sum_{i=1}^{\infty} f_i^*(f) \varphi_i. \tag{21}$$

Then, Θ is a well defined bounded linear operator on $\mathcal{X}$.
By using (20), we compute

$$\begin{aligned}\|I-\Theta(f)\| &= \|f-\Theta(f)\| \\ &= \lim_{n\to\infty}\left\|\sum_{i=1}^{n} f_i^*(f)f_i - \sum_{i=1}^{n} f_i^*(f)\varphi_i\right\| \\ &\le \lim_{n\to\infty}\sum_{i=1}^{n}|f_i^*(f)|\|f_i-\varphi_i\| \\ &\le \lambda \lim_{n\to\infty}\sum_{i=1}^{n}\|f_i^*(f)f_i\| \text{ for all } f\in\mathcal{X}.\end{aligned}$$

This gives (by using (18)), $\|I-\Theta\|<1$, so Θ is an invertible operator on $\mathcal{X}$.

Choose $g_i^* = (\Theta^{-1})^* f_i^*, i\in\mathbb{N}$. Then, for all $f\in\mathcal{X}$, by using (21), we have

$$\begin{aligned}\sum_{i=1}^{\infty} g_i^*(f)\varphi_i &= \sum_{i=1}^{\infty}((\Theta^{-1})^* f_i^*)(f)\varphi_i \\ &= \sum_{i=1}^{\infty} f_i^*(\Theta^{-1}f)\varphi_i \\ &= \Theta\left(\Theta^{-1}f\right) \\ &= f.\end{aligned}$$

Hence $\left(\left\{f_k+\sum_{i=k+1}^{\infty} f_i^*(g_k)f_i\right\}_{k=1}^{\infty}, \{g_k^*\}\right)$ is a Schauder frame for $\mathcal{X}$. □

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