TWMS J. App. Eng. Math. V.7, N.2, 2017, pp. 322-331

POSITION VECTOR OF A DEVELOPABLE h-SLANT RULED SURFACE

O. KAYA[1], M. ÖNDER[2], §

Abstract. In physics and geometry, the determination of position vector of a moving point is an important problem, since the trajectory of that point is a curve or a surface which are important in physics, geometry, and applied sciences. By considering this importance, in this paper, we give a new characterization for a special ruled surface called h-slant ruled surface in the Euclidean 3-space E^3. Later, using the obtained result, we study the position vector of a developable h-slant ruled surface in E^3. We obtain the natural representations for the striction curve and ruling of an h-slant ruled surface. Then, we give general parameterization of a developable h-slant ruled surface. Finally, we introduce some examples of obtained results.

Keywords: position vector, slant ruled surface, developable ruled surface.

AMS Subject Classification: 53A25, 14J26

1. Introduction

A general helix is a special curve whose tangent line makes a constant angle with a fixed straight line called the axis of the general helix. This definition allows us to think a general helix as a curve whose tangent indicatrix is a circle or an arc of a circle on the unit sphere. The well-known characterization of a general helix is that the function k_1/k_2 is constant where k_1 and k_2 are curvature and torsion of the curve, respectively [6]. Recently, Izumiya and Takeuchi have introduced another type of special curves. They have called this new curve as slant helix which is a curve whose principal normal lines make a constant angle with a fixed direction and they have given a characterization for slant helix in E^3 [8]. Moreover, slant helices have been studied by some mathematicians and new types of these curves have been introduced in higher dimensional spaces. Kula and Yaylı studied the spherical indicatrix of a slant helix and obtained that the spherical images of a slant helix are helices lying on unit sphere [13]. Later, Kula et al. obtained some new results characterizing slant helices in E^3 [14]. Moreover, Önder et al. defined a new type of slant helix called B_2-slant helix in Euclidean 4-space E^4 and introduced some characterizations for B_2-slant helix [17]. Furthermore, Önder, Zıplar, and Kaya introduced Eikonal slant helices and Eikonal Darboux helices in 3-dimensional Riemannian manifold and given the characterizations for these special curves [18].

[1] Manisa Celal Bayar University, Faculty of Arts and Sciences, Department of Mathematics, Muradiye Campus, 45140 Muradiye, Manisa, Turkey.
e-mail: onur.kaya@cbu.edu.tr, ORCID: http://orcid.org/0000-0002-4396-2483;
[2] Independent Researcher, Delibekirli Village, 31440, Kırıkhan, Hatay, Turkey.
e-mail: mehmetonder197999@gmail.com, ORCID: http://orcid.org/0000-0002-9354-5530;
§ Manuscript received: June 14, 2016; accepted: September 20, 2016.

Similar to the special curves, there exist some special surfaces in the surface theory. Ruled surface is a kind of such special surfaces which is generated with a continuous moving of a line along a curve. Önder generalized the theory of general helix and slant helix to ruled surfaces and called these ruled surfaces as slant ruled surfaces in E^3 [16]. He defined the slant ruled surfaces by the property that the vectors of the Frenet frame of a ruled surface make constant angles with fixed directions and obtained that the striction curves of developable slant ruled surfaces are helices or slant helices. Önder and Kaya defined Darboux slant ruled surfaces in E^3 such as the Darboux vector of the ruled surface makes a constant angle with a fixed direction and they gave characterizations for a ruled surface to be a Darboux slant ruled surface [15].

One of the most important problems of physics and differential geometry is to determine the position vector of a moving point. The trajectory of that point is a curve or a surface. The determination of parametric representation of the position vector of an arbitrary space curve or an arbitrary surface is still an open problem in the Euclidean space E^3 and in the Minkowski space E_1^3. It is not easy to solve this problem in general case. In ref. [1, 2, 3, 4, 5, 7, 9, 10] the authors tried to solve this problem in some special cases such as the curve lies on a special plane, or as the curve is a cylindrical helix, i.e., both the curvature k_1 and the torsion k_2 of the curve are non-vanishing constants or the curve is a general helix, i.e., the function k_1/k_2 is constant. All these studies are on curves. Of course, the determination of a parametric representation of a surface is more complicated and difficult since the surface has two parameters.

In this paper, we determine the parametric representation of an h-slant ruled surface in the Euclidean 3-space E^3. For this purpose, first we give a brief summary of ruled surfaces and slant ruled surfaces in Section 2. The study of position vector of an h-slant ruled surface in E^3 is given in Section 3. Finally, some examples of the obtained results are given in Section 4.

2. Ruled Surfaces in the Euclidean 3-space

This section contains a brief summary of the geometry of ruled surfaces and h-slant ruled surfaces in E^3.

A ruled surface S is a special surface generated by a continuous moving of a line along a curve $\vec{k}(u)$ and has the parameterization

$$\vec{r}(u,v) = \vec{k}(u) + v\,\vec{q}(u), \tag{1}$$

where the curve $\vec{k} = \vec{k}(u)$ is called base curve or generating curve of the surface and $\vec{q} = \vec{q}(u)$ is a unit direction vector of an oriented line in E^3 whose various positions are called rulings. The distribution parameter of S is the function $d = d(u)$defined by

$$d = \frac{\left|\dot{\vec{k}}, \vec{q}, \dot{\vec{q}}\right|}{\left\langle \dot{\vec{q}}, \dot{\vec{q}} \right\rangle}, \tag{2}$$

where $\dot{\vec{k}} = \frac{d\vec{k}}{du}$, $\dot{\vec{q}} = \frac{d\vec{q}}{du}$. If $\left|\dot{\vec{k}}, \vec{q}, \dot{\vec{q}}\right| = 0$, then the tangent planes are identical at all points of the same ruling. Such a ruling is called a torsal ruling. If $\left|\dot{\vec{k}}, \vec{q}, \dot{\vec{q}}\right| \neq 0$, then the tangent planes of the surface S are distinct at all points of same ruling. Such rulings are called nontorsal [11].

Definition 2.1. [11] *A ruled surface whose all rulings are torsal is called a developable ruled surface. The remaining ruled surfaces are called skew ruled surfaces.*

From (2), it is clear that a ruled surface is developable if and only if at all its points the distribution parameter is zero.

Let $\vec{m}$ be unit normal vector of the ruled surface S given in (1). Then we have

$$\vec{m} = \frac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|} = \frac{(\dot{\vec{k}} + v\dot{\vec{q}}) \times \vec{q}}{\sqrt{\left\langle \dot{\vec{k}} + v\dot{\vec{q}},\ \dot{\vec{k}} + v\dot{\vec{q}} \right\rangle - \left\langle \dot{\vec{k}}, \vec{q} \right\rangle^2}}, \tag{3}$$

If v infinitely decreases, then along a ruling $u = u_1$, the unit normal $\vec{m}$ approaches a limiting direction. This direction is called the asymptotic normal (or central tangent) direction and from (3) defined by

$$\vec{a} = \lim_{v \to \pm\infty} \vec{m}(u_1, v) = \frac{\vec{q} \times \dot{\vec{q}}}{\left\|\dot{\vec{q}}\right\|}.$$

The point at which the unit normal of S is perpendicular to $\vec{a}$ is called the striction point (or central point) C and the set of striction points of all rulings is called striction curve of the surface.

The vector $\vec{h}$ defined by $\vec{h} = \vec{a} \times \vec{q}$ is called central normal vector. Then the orthonormal system $\left\{C;\ \vec{q}, \vec{h}, \vec{a}\right\}$ is called Frenet frame of the ruled surface S where C is the central point and $\vec{q}$, $\vec{h}$, $\vec{a}$ are the unit vectors of ruling, the central normal vector, and the central tangent vector, respectively [11].

The set of all bounded vectors $\vec{q}(u)$ at the origin O constitutes a cone which is called *directing cone* of the ruled surface S [11]. The end points of unit vectors $\vec{q}(u)$ trace a spherical curve k_1 on the unit sphere S^2 and this curve is called *spherical image* of ruled surface S, whose arc length is denoted by s_1 [11]. A ruled surface and its directing cone have the same Frenet frame $\left\{\vec{q}, \vec{h}, \vec{a}\right\}$ and the derivative formulae of this frame with respect to the arc length s_1 are given as follows

$$\begin{bmatrix} d\vec{q}/ds_1 \\ d\vec{h}/ds_1 \\ d\vec{a}/ds_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & \kappa \\ 0 & -\kappa & 0 \end{bmatrix} \begin{bmatrix} \vec{q} \\ \vec{h} \\ \vec{a} \end{bmatrix}, \tag{4}$$

where $\kappa(s_1) = \|d\vec{a}/ds_1\|$ is called the conical curvature of the directing cone (For details [11]).

Let us now choose the base curve as a striction curve. Then the parameterization of ruled surface S is given by

$$\vec{r}\,(s, v) = \vec{c}(s) + v\vec{q}(s), \|\vec{q}(s)\| = 1, \tag{5}$$

where s is the arc length parameter of striction curve. If S is a developable ruled surface then the tangent vectors of striction curve coincide with the rulings, i.e., $\frac{d\vec{c}}{ds} = \vec{q}$. Then, for the tangent vector of the striction curve we have

$$\frac{d\vec{c}}{ds_1} = f(s_1)\vec{q}(s_1), \tag{6}$$

where $\frac{d\vec{c}}{ds} = \vec{q}$ and $f(s_1) = \frac{ds}{ds_1}$ [11].

Definition 2.2. [16] *Let S be a ruled surface in E^3 given by the parameterization*

$$\vec{r}\,(s, v) = \vec{c}(s) + v\vec{q}(s), \|\vec{q}(s)\| = 1,$$

where $\vec{c}(s)$ is striction curve of S and s is arc length parameter of $\vec{c}(s)$. Let the Frenet frame of S along $\vec{c}(s)$ be $\left\{\vec{q},\vec{h},\vec{a}\right\}$. Then S is called an h-slant ruled surface if the central normal vector $\vec{h}$ makes a constant angle θ with a fixed non-zero unit direction $\vec{u}$ in the space E^3, i.e.,

$$\left\langle \vec{h},\vec{u}\right\rangle = \cos\theta = \text{constant}, \quad (\theta \neq 0) \tag{7}$$

If S is both an h-slant ruled surface and developable, then it is called a developable h-slant ruled surface.

Theorem 2.1. [15] *A ruled surface Sin E^3 with conical curvature $\kappa \neq 0$ is an h-slant ruled surface if and only if the function*

$$\frac{\kappa'}{(1+\kappa^2)^{3/2}}$$

is constant.

3. Position vectors of developable h-slant ruled surfaces

In this section, first we give a characterization for h-slant ruled surfaces. Later, we study the position vectors of developable h-slant ruled surfaces. Unless mentioned otherwise, we will assume that S has the Frenet frame $\{\vec{q},\vec{h},\vec{a}\}$ and conical curvature $\kappa \neq 0$.

Theorem 3.1. *The central normal vector $\vec{h}$ satisfies the following differential equation of third order,*

$$\frac{1}{\kappa}\left[\frac{1}{\kappa'}\left(\vec{h}'' + \left(1+\kappa^2\right)\vec{h}\right)\right]' + \vec{h} = 0, \tag{8}$$

where quotation mark shows the derivative with respect to s_1.

Proof. From the Frenet formulae given in (4) we have

$$\vec{h}' = -\vec{q} + \kappa\vec{a}. \tag{9}$$

By differentiating (9) we obtain

$$\vec{a} = \frac{1}{\kappa'}\left(\vec{h}'' + \left(1+\kappa^2\right)\vec{h}\right).$$

If we take the derivative of the last equation and use the Frenet formulae, it follows

$$\frac{1}{\kappa}\left[\frac{1}{\kappa'}\left(\vec{h}'' + \left(1+\kappa^2\right)\vec{h}\right)\right]' + \vec{h} = 0,$$

which completes the proof. □

Theorem 3.2. *The ruled surface S is an h-slant ruled surface if and only if*

$$\kappa = \pm\frac{\cot\theta s_1}{\sqrt{1-\cot^2\theta s_1^2}}, \tag{10}$$

where θ is the constant angle between the fixed unit vector $\vec{u}$ and the central normal vector $\vec{h}$.

Proof. Since S is an h-slant ruled surface from (7), we have

$$\left\langle \vec{h}, \vec{u} \right\rangle = \cos\theta = constant. \tag{11}$$

Differentiating (11) and using (4) we have

$$\langle -\vec{q} + \kappa\vec{a}, \ \vec{u} \rangle = 0. \tag{12}$$

By taking $\langle \vec{a}, \vec{u} \rangle = x$, we write the unit vector $\vec{u}$ as follows,

$$\vec{u} = \kappa x\vec{q} + \cos\theta\vec{h} + x\vec{a}, \tag{13}$$

and since the vector $\vec{u}$ is a unit vector, we obtain

$$x = \pm\frac{\sin\theta}{\sqrt{1+\kappa^2}}.$$

Hence, (13) becomes

$$\vec{u} = \pm\frac{\kappa\sin\theta}{\sqrt{1+\kappa^2}}\vec{q} + \cos\theta\vec{h} \pm \frac{\sin\theta}{\sqrt{1+\kappa^2}}\vec{a}. \tag{14}$$

On the other hand, differentiating (12) gives us

$$\left\langle \kappa'\vec{a} - (1+\kappa^2)\vec{h}, \ \vec{u} \right\rangle = 0. \tag{15}$$

Now, substituting (14) in (15) we get

$$\frac{\kappa'}{(1+\kappa^2)^{3/2}} = \pm\cot\theta.$$

By integrating the last equation, it follows

$$\frac{\kappa}{\sqrt{1+\kappa^2}} = \pm\cot\theta\,(s_1+m)\,. \tag{16}$$

Thanks to a parameter change $s_1 \to s_1 - m$ which makes m disappear and (16) becomes

$$\frac{\kappa}{\sqrt{1+\kappa^2}} = \pm(\cot\theta)s_1.$$

Finally, from the last equation we obtain the desired result.

Conversely, let (10) holds. We define

$$\vec{u} = \cos\theta\left(s_1\vec{q} + \vec{h} \pm \frac{\sqrt{1-(\cot^2\theta)\,s_1^2}}{\cot\theta}\vec{a}\right). \tag{17}$$

From (17), it is clear that $\left\langle \vec{h}, \vec{u} \right\rangle = \cos\theta = constant$. By differentiating (17) and using (4) and (10) we get $\vec{u}' = 0$, which means that $\vec{u}$ is a constant vector. Therefore, the surface S is an h-slant ruled surface. □

Theorem 3.3. *If S is a developable h slant ruled surface, then the position vector of the striction curve $\vec{c} = (c_1, c_2, c_3)$ of $S$is given by:*

$$\begin{cases} c_1 = \sin\theta\int f\left[\int \cos\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right]\, ds_1\right] ds_1, \\ c_2 = \sin\theta\int f\left[\int \sin\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right]\, ds_1\right] ds_1, \\ c_3 = \int f\left[(\cos\theta)s_1 + n\right]\, ds_1, \end{cases}$$

or in the useful parametric form

$$\begin{cases} c_1 = -\sin^2\theta\int \gamma\left[\int \cos t\sin\left(t\cos\theta\right)\, dt\right] dt, \\ c_2 = -\sin^2\theta\int \gamma\left[\int \sin t\sin\left(t\cos\theta\right)\, dt\right] dt, \\ c_3 = \sin\theta\int \gamma\left[\cos\left(t\cos\theta\right) + n\right]\, dt, \end{cases}$$

where $f = ds/ds_1$, $\gamma = ds/dt$, n is a real constant and θ is the constant angle between the fixed unit vector $\vec{u}$ and the central normal vector $\vec{h}$.

Proof. Since S is an h-slant ruled surface, from Theorem 3.2 we have

$$\kappa = \pm \frac{(\cot\theta)s_1}{\sqrt{1-(\cot^2\theta)\,s_1^2}}. \tag{18}$$

If we substitute (18) in (8), it becomes

$$\left(1-\left(\cot^2\theta\right)s_1^2\right)\vec{h}''' - 3\left(\cot^2\theta\right)s_1\vec{h}'' + \vec{h}' = 0. \tag{19}$$

We write the central normal vector $\vec{h}$ as

$$\vec{h} = h_1\vec{e}_1 + h_2\vec{e}_2 + h_3\vec{e}_3, \tag{20}$$

where $h_1 = h_1(s_1),\ h_2 = h_2(s_1),\ h_3 = h_3(s_1)$ are smooth functions of s_1 and $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is the standard base of E^3. Now, let the surface S be an h-slant ruled surface. Without loss of generality we choose the fixed unit vector $\vec{u}$ as $\vec{e}_3$. Then, we get $h_3 = \cos\theta$ is constant. Since $\vec{h}$ is a unit vector it follows

$$h_1^2 + h_2^2 = 1 - \cos^2\theta = \sin^2\theta. \tag{21}$$

For the general solution of (21) we can write

$$h_1 = \sin\theta\cos t,\ \ h_2 = \sin\theta\sin t,$$

where $t = t(s_1)$ is a smooth function of s_1. Then, (20) becomes

$$\vec{h} = (\sin\theta\cos t,\ \sin\theta\sin t,\ \cos\theta)\,, \tag{22}$$

and since the vector $\vec{h}$ satisfies (19), we get the following differential equations

$$\begin{cases} \left(\left(\cot^2\theta\right)s_1\right)t' - \left(1-\left(\cot^2\theta\right)s_1^2\right)t'' = 0, \\ t' - \left(1-\left(\cot^2\theta\right)s_1^2\right)\left[(t')^3 - t'''\right] - \left(3\left(\cot^2\theta\right)s_1\right)t'' = 0. \end{cases} \tag{23}$$

The general solution of the first equation of (23) is

$$t(s_1) = n_1\arccos\left((\cot\theta)s_1\right) + n_2, \tag{24}$$

or

$$t(s_1) = n_1\arcsin\left((\cot\theta)s_1\right) + n_2, \tag{25}$$

where $n_1,\ n_2$ are real constants. The constant n_2 will disappear thanks to a parameter change $t \to t + n_2$. By substituting (24) or (25) in the second equation of (23) we obtain

$$n_1\cot\theta\left(1+\cot^2\theta\left(1-n_1^2\right)\right) = 0. \tag{26}$$

Since $\cot\theta \neq 0$ and $n_1 \neq 0$, from (26) we get $n_1 = \sec\theta$. Therefore, (24) and (25) becomes

$$t(s_1) = \sec\theta\arccos\left((\cot\theta)s_1\right), \tag{27}$$

and

$$t(s_1) = \sec\theta\arcsin\left((\cot\theta)s_1\right), \tag{28}$$

respectively. Thus, if we use (27), the components of the normal vector $\vec{h}$ become

$$\begin{cases} h_1 = \sin\theta\cos\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right], \\ h_2 = \sin\theta\sin\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right], \\ h_3 = \cos\theta, \end{cases} \tag{29}$$

and similarly from (28) they become

$$\begin{cases} h_1 = \sin\theta\cos\left[\sec\theta\arcsin\left((\cot\theta)s_1\right)\right], \\ h_2 = \sin\theta\sin\left[\sec\theta\arcsin\left((\cot\theta)s_1\right)\right], \\ h_3 = \cos\theta. \end{cases} \tag{30}$$

From the Frenet formulae we have $\vec{q}' = \vec{h}$. Therefore, if we integrate (29) we get

$$\begin{cases} q_1 = \sin\theta\int\cos\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right]\,ds_1, \\ q_2 = \sin\theta\int\sin\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right]\,ds_1, \\ q_3 = (\cos\theta)s_1 + n, \end{cases} \tag{31}$$

where n is an integration constant. Now, since the surface S is developable, we have $c' = f\vec{q}$. Then from (31), we have

$$\begin{cases} c_1 = \sin\theta\int f\left[\int\cos\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right]\,ds_1\right]ds_1, \\ c_2 = \sin\theta\int f\left[\int\sin\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right]\,ds_1\right]ds_1, \\ c_3 = \int f\left[(\cos\theta)s_1 + n\right]\,ds_1, \end{cases}$$

and if we take the parameter $t(s_1) = \sec\theta\arccos\left((\cot\theta)s_1\right)$, we achieve the parametric form that completes the proof. $\square$

From Theorem 3.3, we have the following corollaries:

Corollary 3.1. *If S is a developable h-slant ruled surface, then the parameterization of S with respect to the arc length parameter s_1 of the spherical image curve of the ruled surface is given by*

$$\vec{r}(s_1, v) = \vec{c}(s_1) + v\vec{q}(s_1), \tag{32}$$

where $\vec{c} = (c_1, c_2, c_3)$, $\vec{q} = (q_1, q_2, q_3)$ *and*

$$\begin{cases} c_1 = \sin\theta\int f\left[\int\cos\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right]\,ds_1\right]ds_1, \\ c_2 = \sin\theta\int f\left[\int\sin\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right]\,ds_1\right]ds_1, \\ c_3 = \int f\left[(\cos\theta)s_1 + n\right]\,ds_1, \end{cases}$$

$$\begin{cases} q_1 = \sin\theta\int\cos\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right]\,ds_1, \\ q_2 = \sin\theta\int\sin\left[\sec\theta\arccos\left((\cot\theta)s_1\right)\right]\,ds_1, \\ q_3 = (\cos\theta)s_1 + n. \end{cases}$$

Corollary 3.2. *If S is a developable h-slant ruled surface, then the parameterization of S with respect to the arbitrary parameter t is given by*

$$\vec{r}(t, v) = \vec{c}(t) + v\vec{q}(t), \tag{33}$$

where $\vec{c} = (c_1, c_2, c_3)$, $\vec{q} = (q_1, q_2, q_3)$ *and*

$$\begin{cases} c_1 = -\sin^2\theta\int\gamma\left[\int\cos t\sin\left(t\cos\theta\right)\,dt\right]dt, \\ c_2 = -\sin^2\theta\int\gamma\left[\int\sin t\sin\left(t\cos\theta\right)\,dt\right]dt, \\ c_3 = \sin\theta\int\gamma\left[\cos\left(t\cos\theta\right) + n\right]\,dt, \end{cases}$$

$$\begin{cases} q_1 = -\sin^2\theta\int\cos t\sin\left(t\cos\theta\right)\,dt, \\ q_2 = -\sin^2\theta\int\sin t\sin\left(t\cos\theta\right)\,dt, \\ q_3 = \sin\theta\cos\left(t\cos\theta\right) + n. \end{cases}$$

On the other hand, from equation (30) we give the following corollary:

Corollary 3.3. *If S is a developable h-slant ruled surface, then the position vector of the striction curve $c = (c_1, c_2, c_3)$ of $S$is given by:*

$$\begin{cases} c_1 = \sin\theta \int f \left[\int \cos\left[\sec\theta \arcsin\left((\cot\theta)s_1\right)\right] ds_1\right] ds_1, \\ c_2 = \sin\theta \int f \left[\int \sin\left[\sec\theta \arcsin\left((\cot\theta)s_1\right)\right] ds_1\right] ds_1, \\ c_3 = \int f\left[(\cos\theta)s_1 + n\right] ds_1, \end{cases}$$

or in the useful parametric form

$$\begin{cases} c_1 = \sin^2\theta \int \gamma \left[\int \cos t \cos\left(t\cos\theta\right) dt\right] dt, \\ c_2 = \sin^2\theta \int \gamma \left[\int \sin t \cos\left(t\cos\theta\right) dt\right] dt, \\ c_3 = \sin\theta \int \gamma \left[\sin\left(t\cos\theta\right) + n\right] dt, \end{cases}$$

where $f = ds/ds_1$, $\gamma = ds/dt$, s_1 is the arc length parameter of the spherical image curve of the ruled surface, t is an arbitrary parameter, n is a real constant, and θ is the constant angle between the fixed unit vector $\vec{u}$ and the central normal vector $\vec{h}$.

4. Examples

In this section, we take some special chosen of conical curvature κ and function γ and obtain some examples of developable h-slant ruled surfaces.

Example 4.1. *Let us consider the ruled surface S with conical curvature $\kappa = \frac{s_1}{\sqrt{1-s_1^2}}$ and function $\gamma = 1$. Then the parameterization of developable h-slant ruled surface S with axis $\vec{e}_3$ is obtained as follows*

$$\vec{r}(t, v) = (r_1, r_2, r_3),$$

where

$$r_1 = \frac{1}{4}\left(\frac{\sin\left(\left(1+\frac{1}{2}\sqrt{2}\right)t\right)}{\left(1+\frac{1}{2}\sqrt{2}\right)^2} + \frac{\sin\left(\left(-1+\frac{1}{2}\sqrt{2}\right)t\right)}{\left(-1+\frac{1}{2}\sqrt{2}\right)^2} + v\left(-\frac{\cos\left(\left(1+\frac{1}{2}\sqrt{2}\right)t\right)}{\left(1+\frac{1}{2}\sqrt{2}\right)} - \frac{\cos\left(\left(-1+\frac{1}{2}\sqrt{2}\right)t\right)}{\left(-1+\frac{1}{2}\sqrt{2}\right)}\right)\right),$$

$$r_2 = \frac{1}{4}\left(\frac{\cos\left(\left(-1+\frac{1}{2}\sqrt{2}\right)t\right)}{\left(-1+\frac{1}{2}\sqrt{2}\right)^2} - \frac{\cos\left(\left(1+\frac{1}{2}\sqrt{2}\right)t\right)}{\left(1+\frac{1}{2}\sqrt{2}\right)^2} + v\left(\frac{\sin\left(\left(-1+\frac{1}{2}\sqrt{2}\right)t\right)}{\left(-1+\frac{1}{2}\sqrt{2}\right)} - \frac{\sin\left(\left(1+\frac{1}{2}\sqrt{2}\right)t\right)}{\left(1+\frac{1}{2}\sqrt{2}\right)}\right)\right),$$

$$r_3 = \sin\left(\frac{1}{2}\sqrt{2}t\right) + v\sin(t)\cos\left(\frac{1}{2}\sqrt{2}t\right),$$

and $t(s_1) = \sec\theta \arccos(\cot\theta s_1)$ (See Fig. A).

Example 4.2. *If we take $\kappa = \frac{\sqrt{3}s_1}{\sqrt{9-3s_1^2}}$ and $\gamma = t$, the parameterization of developable h-slant ruled surface S with axis $\vec{e}_3$ is obtained as follows*

$$\vec{r}(t, v) = (r_1, r_2, r_3),$$

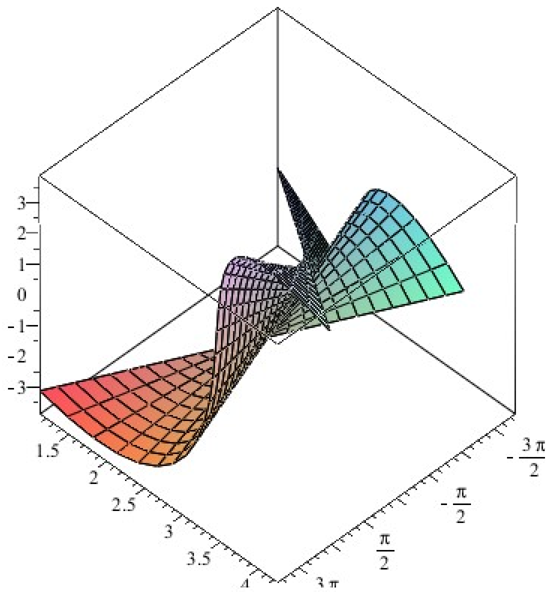
where

$$\begin{aligned} r_1 = {} & 2t\left(\tfrac{1}{3}\left(\cos\left(\tfrac{1}{2}t\right)\right)^2 \sin\left(\tfrac{1}{2}t\right) + \tfrac{2}{3}\sin\left(\tfrac{1}{2}t\right)\right) + \tfrac{4}{9}\left(\cos\left(\tfrac{1}{2}t\right)\right)^3 \\ & -\tfrac{10}{3}\cos\left(\tfrac{1}{2}t\right) - 3t\sin\left(\tfrac{1}{2}t\right) + v\left(-\tfrac{1}{4}\cos\left(\tfrac{3}{2}t\right) + \tfrac{3}{4}\cos\left(\tfrac{1}{2}t\right)\right) - t + 3 - 4v, \end{aligned}$$

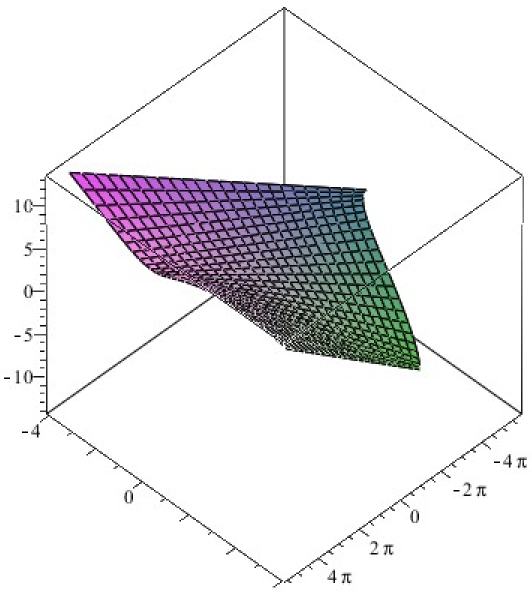
$$\begin{aligned} r_2 = {} & -2t\left(-\tfrac{1}{3}\left(\sin\left(\tfrac{1}{2}t\right)\right)^2 \cos\left(\tfrac{1}{2}t\right) - \tfrac{2}{3}\cos\left(\tfrac{1}{2}t\right)\right) - \tfrac{4}{9}\left(\sin\left(\tfrac{1}{2}t\right)\right)^3 \\ & -\tfrac{8}{3}\sin\left(\tfrac{1}{2}t\right) + v\left(\tfrac{3}{4}\sin\left(\tfrac{1}{2}t\right) - \tfrac{1}{4}\sin\left(\tfrac{3}{2}t\right)\right) - 2t + 1, \end{aligned}$$

$$r_3 = \frac{1}{2}\sqrt{3}\left(4\cos\left(\frac{1}{2}t\right) + 2t\sin\left(\frac{1}{2}t\right)\right) + v\sin(t)\cos\left(\frac{1}{2}t\right) + 2t - 2 - 2v,$$

and $t(s_1) = \sec\theta \arccos(\cot\theta s_1)$(See Fig. B).



(A) h-slant ruled surface



(B) h-slant ruled surface

Acknowledgement This research was supported by Manisa Celal Bayar University Scientific Research Projects Coordination Unit (Project No: 2014/07).

References

[1] Ali,A.T., (2010), Position vectors of spacelike general helices in Minkowski 3-space, Nonl. Anal. Theory Meth. Appl., 73, pp.1118-1126.

[2] Ali,A.T. and Turgut,M., (2013), Position vectors of a timelike general helices in Minkowski 3-space, Glob. J. Adv. Res. Class. Mod. Geom., 2(1), pp.1-10.

[3] Ali,A.T. and Turgut,M., (2010), Position vector of a timelike slant helix in Minkowski 3-space, J. Math. Anal. Appl., 365, pp.559-569.

[4] Ali,A.T., (2012), Position vectors of slant helices in Euclidean Space E^3, J of Egyptian Math Soc., 20 (1), pp.1-6.

[5] Ali,A.T., (2011), Position Vectors of General Helices in Euclidean 3-space, Bulletin of Math. Analysis and App., 3 (2), pp.198-205.

[6] Barros,M., (1997) General helices and a theorem of Lancret. Proc Amer Math Soc, 125(5), pp.1503-1509.

[7] Chen,B.Y., (2003), When does the position vector of a space curve always lie in its rectifying plane?, Amer. Math. Mounthly, 110 (2), pp.147-152.

[8] Izumiya,S and Takeuchi,N., (2004), New special curves and developable surfaces, Turk J Math., 28, pp.153-163.

[9] İlarslan,K. and Boyacıoğlu,O., (2007), Position vectors of a spacelike W-curve in Minkowski Space E_1^3, Bull. Korean Math. Soc., 44, pp.429-438.

[10] İlarslan,K. and Boyacıoğlu,O., (2008), Position vectors of a timelike and a null helix in Minkowski 3-space, Chaos, Solitons and Fractals, 38, pp.1383-1389.

[11] Karger,A. and Novak,J., (1978), Space Kinematics and Lie Groups. Prague, Czechoslovakia: STNL Publishers of Technical Lit.

[12] Kaya,O. and Önder,M., Characterizations of Slant Ruled Surfaces in the Euclidean 3-space, Caspian Journal of Mathematical Sciences, (in press).

[13] Kula,L. and Yaylı,Y., (2005), On slant helix and its spherical indicatrix. Appl. Math. Comp., 169, pp.600-607.

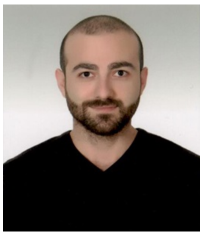
[14] Kula,L., Ekmekçi,N., Yaylı,Y., and İlarslan,K., (2009), Characterizations of Slant Helices in Euclidean 3-Space, Turk J Math., 33, pp.1-13.

[15] Önder,M. and Kaya,O., (2015), Darboux Slant Ruled Surfaces, Azerbaijan Journal of Mathematics, 5 (1), pp.64-72.

[16] Önder,M., (2013), Slant Ruled Surfaces in the Euclidean 3-space E^3, arXiv:1311.0627v1 [math.DG].

[17] Önder,M., Kazaz,M., Kocayiğit,H., and Kılıç, O.,(2008), B_2-slant helix in Euclidean 4-space E^4, Int J Contemp Math Sci., 3 (29-32), pp.1433-1440.

[18] Önder,M., Zıplar,E., and Kaya,O., (2014), Eikonal slant helices and Eikonal Darboux helices in 3-dimensional Riemannian Manifold, International Journal of Geometric Methods in Modern Physics, Vol. 11, 1450045 (10 pages).



Onur Kaya graduated from Ege University in 2010 and received his M.Sc. degree from Celal Bayar University in 2015. He is currently a Ph.D. student and a teaching assistant in the Department of Mathematics, Celal Bayar University. His research interests comprise differential geometry and Lorentzian geometry.

Mehmet Önder for the photograph and short biography, see TWMS J. Appl. and Eng. Math., V.5, No.2, 2015.