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AN ALGORITHM FOR SOLVING FUZZY RELATION PROGRAMMING WITH THE MAX-T COMPOSITION OPERATOR

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Abstract. This paper studies the problem of minimizing a linear objective function subject to max-T fuzzy relation equation constraints where T is a special class of pseudo-t-norms. Some sufficient conditions are presented for determination of its optimal solutions. Some procedures are also suggested to simplify the original problem. Some sufficient conditions are given for uniqueness of its optimal solution. Finally, an algorithm is proposed to find its optimal solution.

Keywords: Fuzzy relation equations; Max-T composition; Maximum solution; Minimal solution; Fuzzy optimization; Optimal solution.

AMS Subject Classification: 90C26, 90C70 .

1. Introduction

Fuzzy Relation Equations (FRE) and the problems related to them have been studied by many researchers since the resolution of fuzzy relation equations was proposed by Sanchez [12] in 1976 (see for instance, Refs. [1–5, 7–11, 15–18]). Their applications can be seen in many areas, for instance, fuzzy control, fuzzy decision making, fuzzy symptom diagnosis and especially fuzzy medical diagnosis [2, 8, 10, 11, 18]. An interested reader can find a comprehensive survey of done works about FRE and its applications in Ref. [18]. The problem of minimizing a linear objective function subject to fuzzy relation equations constraints with the max-min and the max-product composition has been widely investigated in the literature. Fang and Li [3] showed the problem with max-min composition can be converted into a 0-1 integer programming problem. This 0-1 integer programming problem is solved by the branch-and-bound method with jump-tracking technique. Wu et al. [15] enhanced Fang and Li's method by providing an efficient procedure that visits much fewer nodes in the solution tree than that of Fang and Li's procedure. Wu and Guu [16] proposed a necessary condition for an optimal solution to exist. Three rules for simplifying the work of computing an optimal solution are provided based on this necessary condition. Furthermore, the problem with the max-product composition was investigated by Loetamonphong and Fang [9] and they applied a similar method to Fang and Li's idea. Their method for solving the model was improved by Guu and Wu [4] by shrinking the region of search. Guu and Wu [4] identified a necessary condition for an optimal solution in terms of the maximum solution derived from fuzzy relational equations.

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This necessary condition provides that each component of an optimal solution is either 0 or the corresponding component in the vector of the maximum solution. Furthermore, this necessary condition was also extended for the problem with max-strict t-norm composition [17]. Guu and Wu [5] generalized the necessary condition mentioned in [4, 17] with max-T composition where T is a continuous Archimedean t-norm. In this paper, the condition is generalized to a more general class with respect to the max-strict t-norm and different to max-continuous Archimedean t-norm, i.e. max-T composition where T is a special class of pseudo-t-norms. Moreover, some sufficient conditions are presented for determination of the optimal solutions of the problem or some of their components. Under these conditions, optimal solutions of the problem or some of their components can be obtained easily. Some procedures are also proposed to simplify the original problem based on the conditions. Since computing of the maximum solution is easy, the conditions are very important to reduce the computation related to finding the optimal solutions. Some sufficient conditions are also given for uniqueness of the optimal solution of the problem. Finally, an algorithm is designed based on the conditions and the procedures. This paper is organized as follows. Section 2 contains some preliminary definitions. Also, the problem of fuzzy relation programming is briefly introduced and some required results are reviewed. In Section 3, some sufficient conditions are provided to determine the optimal solutions of the problem or some of their components. Some procedures are also given to reduce the original problem. In Section 4, an algorithm is proposed to solve the problem. Two numerical examples are presented to illustrate the procedures and the algorithm. Conclusions are expressed in Section 5.

2. Preliminaries and formulation of the fuzzy relation programming

We first express some assumptions. Throughout this paper, L denotes the real unit interval [0,1] and J always stands for any nonempty set of subscripts and $\underline{n} = \{1, ..., n\}$, for $\forall n \in N$. Also, notation $A \subset B$, for two sets A and B, is equivalent to $A \subseteq B$ and $A \neq B$. We now remind the definitions and results that they are needed in the next sections.

Definition 2.1. *[13] (A) A binary operation T on L is called a pseudo-t-norm if it satisfies the following conditions: (T1) $T(1, a) = a$ and $T(0, a) = 0$ for all $a \in L$, and (T2) $a, b, c \in L$ and $b \leq c \Rightarrow T(a, b) \leq T(a, c)$.*
(B) [13] A pseudo-t-norm T on L is said to be infinitely $\vee$-distributive if it satisfies the following condition: ($T_\vee$) $a, b_j \in L(j \in J) \Rightarrow T(a, \vee_{j\in J} b_j) = \vee_{j\in J} T(a, b_j)$.
(C) [14] A pseudo-t-norm T on L is said to be infinitely $\wedge$-distributive if it satisfies the following condition: ($T_\wedge$) $a, b_j \in L(j \in J) \Rightarrow T(a, \wedge_{j\in J} b_j) = \wedge_{j\in J} T(a, b_j)$.
(D) [14]A pseudo-t-norm T on L is said to be infinitely distributive if it is both infinitely $\vee$-distributive and infinitely $\wedge$-distributive.
(E) [7] Let T be an infinitely $\vee$-distributive pseudo-t-norm on L. Define $I(T), S(T) \in L^{L\times L}$ as: $I(T)(a, b) := \vee\{u \in L \mid T(a, u) \leq b\}$, and $S(T)(a, b) := \wedge\{u \in L \mid T(a, u) \geq b\}$, where $a, b \in L$. It is tacitly assumed that $\vee\emptyset = 0$ and $\wedge\emptyset = 1$.
(F) [6] A pseudo-t-norm T on L is said to be strong if it satisfies the condition: (T3) $T(a, 0) = 0$ for all $a \in L$.

Theorem 2.1. *[7] If T is an infinitely $\vee$-distributive pseudo-t-norm on L, then two conditions (1) $T(a, c) \leq b \Longleftrightarrow c \leq I(T)(a, b)$ for all $a, b, c \in L$ and (2) $T(a, 0) = 0$ for all $a \in L$ are equivalent each other.*

We are now ready to formulate the fuzzy relation programming as follows:

$$Min\ Z(x) = \sum_{i=1}^{m} c_i.x_i, \tag{2.1}$$

$$s.t.\ x \in X(A,b) := \{x \in [0,1]^m \mid A \bullet x = b\}. \tag{2.2}$$

Where $c_j \in R$ is the coefficient associated with the variable x_j; $A = [a_{ij}]$ is $m \times n$ fuzzy relation matrix with $0 \le a_{ij} \le 1$; and the operation "$\bullet$" represents the Max-T composition operator. The operator of T is an infinitely $\vee$-distributive strong pseudo-t-norm (see Ref. [7]). Furthermore, T satisfies the following condition:

$$\forall a,b,c \in (0,1],\ b < c \Longrightarrow T(a,b) < T(a,c). \tag{2.3}$$

To characterize $X(A,b)$, we define $X = \{x \in R^n \mid 0 \le x_j \le 1, \forall j \in \underline{n}\}$. For $x^1, x^2 \in X$, we say $x^1 \le x^2$ if and only if $x_j^1 \le x_j^2$, $\forall j \in \underline{n}$. In this way, "$\le$" forms a partial order relation on X and $(X, \le)$ becomes a lattice. Moreover, we call $\hat{x} \in X(A,b)$ a maximum solution if $x \le \hat{x}, \forall x \in X(A,b)$. Similarly, $\tilde{x} \in X(A,b)$ is called a minimal solution if $x \le \tilde{x}$ implies $x = \tilde{x}, \forall x \in X(A,b)$. According to [7], when $X(A,b) \ne \emptyset$, it can be completely determined by one maximum solution and a finite number of minimal solutions. The maximum solution can be obtained by assigning [7]:

$$\hat{x}_j = \wedge_{i \in \underline{m}} I(T)(a_{ij}, b_i),\ \forall j \in \underline{n}, \tag{2.4}$$

Moreover, if we denote the set of all minimal solutions by $\check{X}(A,b)$, then

$$X(A,b) = \bigcup_{\tilde{x} \in \check{X}(A,b)} \{x \in X \mid \tilde{x} \le x \le \hat{x}\}. \tag{2.5}$$

Definition 2.2. *(A) For a feasible solution $x \in X(A,b) \ne \emptyset$ in system (2.2), we call x_{j_0} a binding variable if $T(a_{ij_0}, x_{j_0}) = b_i$, for some $i \in \underline{m}$. (B) Let vector $\hat{x} = [\hat{x}_1, \hat{x}_2, ..., \hat{x}_n]^T$ be the maximum solution of $X(A,b)$. Then define $J_i = \{j \in \underline{n} \mid T(a_{ij}, \hat{x}_j) = b_i\}$, for all $i \in \underline{m}$, $I_j = \{i \in \underline{m} \mid T(a_{ij}, \hat{x}_j) = b_i$, for all $j \in \underline{n}$, and $\hat{J} = \prod_{i=1}^{m} J_i$.*

Remark 2.1. *[1] If x_j is a binding variable, then $T(a_{i_0 j}, x_j) = b_{i_0}$, for some $i_0 \in \underline{m}$, by Definition (2.2)(A). If $a_{i_0 j} = 0$, then b_{i_0} should be equal to zero. It is obvious that in this case we can arbitrarily select a value for x_j from $[0,1]$. Hence, this obvious case is taken out of our considerations.*

Lemma 2.1. *[1] (A) If $x \in X(A,b)$, then for each $i \in \underline{m}$ there exists $j_0 \in \underline{n}$ such that $T(a_{ij_0}, x_{j_0}) = b_i$ and $T(a_{ij}, x_j) \le b_i$, $\forall j \in \underline{n}$.*
(B) If $\hat{J} \ne \emptyset$, then $J_i \ne \emptyset$, for all $i \in \underline{m}$.

In order to study $X(A,b)$ in terms of the elements in $\hat{J}$, given $f \in \hat{J}$, we provide the following definition:

Definition 2.3. *[1] (1) Let $f = (f(1), ..., f(m)) \in \hat{J}$ such that for each $i \in \underline{m}$, $f(i) \in J_i$. Then we define vector $x^{(f)} = [x_1^{(f)}, ..., x_n^{(f)}]^T$ as follows:*

$$x_j^{(f)} = \begin{cases} \vee_{i \in I_j^{(f)}} S(T)(a_{ij}, b_i) & I_j^{(f)} \ne \emptyset, \\ 0 & I_j^{(f)} = \emptyset, \end{cases} \tag{2.6}$$

where $I_j^{(f)} = \{i \in \underline{m} \mid f(i) = j\}$.
(2) Define set F *as:* $F = \{x^{(f)} \mid f \in \hat{J}\}$.

We now study the relation between $X(A, b)$ and F in the following theorem.

Theorem 2.2. *[7] (A) The set of solutions* $X(A, b)$ *and the finite set* F *contain the same minimal elements, i.e.,* $\check{X}(A, b) = F_0$ *, where* F_0 *denotes the set of all the minimal solutions of* F *. (B)* $X(A, b) = \bigcup_{\tilde{x} \in F_0} [\tilde{x}, \hat{x}]$.

Now, we are ready to explain the process of solving problem (2.1)-(2.2), briefly. Similar to [3], to solve the problem (2.1)-(2.2), the problem is decomposed to two sub-problems as follows.

$$Min\ Z^1(x) = \sum_{i=1}^{m} c_i^1.x_i, \tag{2.7}$$

$$s.t.\ x \in X(A, b), \tag{2.8}$$

and

$$Min\ Z^2(x) = \sum_{i=1}^{m} c_i^2.x_i, \tag{2.9}$$

$$s.t.\ x \in X(A, b), \tag{2.10}$$

where $c_i^1 = \begin{cases} c_i & c_i < 0, \\ 0 & c_i \geq 0, \end{cases}$ and $c_i^2 = \begin{cases} 0 & c_i < 0, \\ c_i & c_i \geq 0, \end{cases}$ $\forall i \in \underline{m}$.

Obviously, $c_i = c_i^1 + c_i^2$, $\forall i \in \underline{m}$. Similar to [3], we can easily show that the maximum solution $\hat{x}$ is an optimal solution for problem (2.7)-(2.8) and one of minimal solutions $X(A, b)$, say $\tilde{x}^*$, is an optimal solution for problem (2.9)-(2.10). The vector $x^* = (x_i^*)_{i \in I}$ is now defined as follows.

$$x_i^* = \begin{cases} \hat{x}_i & if\ c_i < 0, \\ \tilde{x}_i^* & if\ c_i \geq 0, \end{cases} \quad \forall i \in \underline{m}. \tag{2.11}$$

Similar to [3], we can also show that x^* is an optimal solution of problem (2.1)-(2.2).

3. Some sufficient conditions and simplification of the problem (2.9)-(2.10)

In this section, some sufficient conditions are presented to determine the optimal solutions or some of their components. To do this, we briefly express the following lemma from [1]. The lemma shows the relation between the minimal solutions of $X(A, b)$ and its maximum solution.

Lemma 3.1. *[1] Suppose that* T *be an infinitely* $\vee$*-distributive strong pseudo-t-norm on* L*, and* T *satisfies condition* (2.3) *and* $\hat{x}$ *is the maximum solution. (A) For any feasible solution* x *of* $X(A, b)$ *with max-*T *composition, if* x_j *is a binding variable, then* $x_j = \hat{x}_j$*. (B) Let* $\tilde{x}$ *is a minimal solution of* $X(A, b)$ *with max-*T *composition, then* $\tilde{x}_j = 0$ *or* $\tilde{x}_j = \hat{x}_j$*, for each* $j \in \underline{n}$.

A direct result of Lemma (3.1) and the process of solving problem (2.1)-(2.2) in Section (2) are as follows.

Corollary 3.1. *Under the conditions of Lemma* (3.1)*, let* x^* *is an optimal solution of problem* (2.9)-(2.10) *with max-*T *composition, then* $x_j^* = 0$ *or* $x_j^* = \hat{x}_j$*, for each* $j \in \underline{n}$.

Proof. If x_j^* is not a binding variable, we can assign 0 to x_j^*. With attention to the objective of problem (2.9)-(2.10), the zero value is assigned to x_j^*. If x_j^* is a binding variable, then $x_j^* = \hat{x}_j$, by Lemma (3.1)(A). □

Corollary (3.1) reveals that the indices of an optimal solution x^* with $x_j^* = \hat{x}_j$ are contained in the index set of $\bigcup_{i \in \underline{m}} J_i$. Moreover, one can simply set
$x_j^* = 0$, for each $j \notin \bigcup_{i \in \underline{m}} J_i$.
Now, we can present lemmas to reduce the search domain of the optimal solutions of problem (2.9)-(2.10). The intuition behind those lemmas is to fix the components of optimal solutions as many as possible by assigning 0 or $\hat{x}_j$ to them, as stated in Corollary (3.1).
We now present some sufficient conditions to determine the optimal solutions of problem (2.9)-(2.10). Under the conditions, some the optimal solutions of problem (2.9)-(2.10) or some their components can be determined directly. In the following lemmas, assume $X(A, b) \neq \emptyset$.

Lemma 3.2. *If, for some $i_0 \in \underline{m}$, $J_{i_0} = \{j_0\}$, then for each optimal solution x^* of problem* (2.9)-(2.10)*, we have: $x_{j_0}^* = \hat{x}_{j_0}$.*

Proof. For $\forall x^* \in X(A, b)$, and for $i_0 \in \underline{m}$, $b_{i_0} = \vee_{j \in \underline{n}} T(a_{i_0 j}, x_j^*)$. Since $J_{i_0} = \{j_0\}$, therefore, $b_{i_0} = T(a_{i_0 j_0}, x_{j_0}^*)$. From Lemma (3.1)(A), we can conclude that $x_{j_0}^* = \hat{x}_{j_0}$. □

Under the conditions of Lemma (3.2), by setting $x_{j_0}^* = \hat{x}_{j_0}$, for each optimal solution x^*, we can remove row i_0 and column j_0 from matrix A and components j_0 and i_0 of vectors x and b, respectively.

Lemma 3.3. *If there exits $t \in \underline{n}$ such that $\bigcup_{j=1, j \neq t}^{n} I_j \subset I_t$, then problem* (2.9)-(2.10) *has only one optimal solution as $x^* = [x_1^*, ..., x_n^*]^T$, where $x_j^* = \begin{cases} \hat{x}_j & j = t, \\ 0 & j \neq t, \end{cases} \quad \forall j \in \underline{n}$.*

Proof. Since $X(A, b) \neq \emptyset$, it is concluded that $I_t = \underline{m}$. Otherwise, $\exists i_0 \in \underline{m}$ such that $J_{i_0} = \emptyset$. Hence, $X(A, b) = \emptyset$, which is a contradiction. Due to $I_j \subset I_t$, for each $j \in \underline{n} - \{t\}$, variable $x_t^* (:= \hat{x}_t)$ is binding in the constraints in which variable x_j^*, for each $j \in \underline{n} - \{t\}$, is binding as well. Regarding Corollary (3.1), we simply set $x_j^* = 0$ to be optimal vector x^*, for each $j \in \underline{n} - \{t\}$. Note that with $x_j^* = 0$, for each $j \in \underline{n} - \{t\}$, the feasibility of problem (2.9)-(2.10) can be maintained. Also, since $I_j \subset I_t$, $\forall j \in \underline{n} - \{t\}$, we conclude $\exists i_0 \in I_t$ such that $\forall j \in \underline{n} - \{t\}$, $i_0 \notin I_j$, i.e., $J_{i_0} = \{t\}$. From Lemma (3.2), we conclude that $\forall x^* \in X(A, b)$, $x_t^* = \hat{x}_t$. From the obtained results and $\bigcup_{j=1, j \neq t}^{n} I_j \subset I_t$, x^* is the only optimal solution of problem (2.9)-(2.10). □

Lemma 3.4. *If $\exists t_1, ..., t_r \in \underline{n}$ such that $\bigcup_{j=1, j \notin \{t_1, ..., t_r\}}^{n} I_j \subset I_{t_1}, ..., I_{t_r}$ and $I_{t_k} = \underline{m}$, for $\forall k \in \underline{r}$, then the optimal solutions of problem* (2.9)-(2.10) *are as x^* such that $c^{2^T} x^* = min_{k \in \{1, ..., r\}} \{c^{2^T} x^{(k)}\}$, where $x^{(k)} = [x_1^{(k)}, ..., x_n^{(k)}]^T$ and $x_j^{(k)} = \begin{cases} \hat{x}_j & j = t_k, \\ 0 & j \neq t_k, \end{cases} \quad \forall j \in \underline{n}$.*

Proof. We first show that for each $k \in \underline{r}$, the vector $\tilde{x}^{(k)}$ is a minimal solution of set $X(A, b)$. Then it is shown that $\tilde{x}^{(k)}$, for each $k \in \underline{r}$, are only minimal solutions of $X(A, b)$. Assume that $k \in \underline{r}$ be arbitrary. Due to $I_j \subset I_{t_k}$, for each $j \in \underline{n} - \{t_p | p \in \underline{r}\}$, and $I_{t_k} = \underline{m}$, variable $\tilde{x}_{t_k}^{(k)} (:= \hat{x}_{t_k})$ is binding in the constraints in which variable $\tilde{x}_j^{(k)}$, for each $j \in \underline{n} - \{t_k\}$, is binding as well. Regarding Corollary (3.1), we simply set $\tilde{x}_j^{(k)} = 0$ to be minimal vector $\tilde{x}^{(k)}$, for each $j \in \underline{n} - \{k\}$. Note that with $\tilde{x}_j^{(k)} = 0$, for each $j \in \underline{n} - \{k\}$, the feasibility of problem (2.9)-(2.10) can be maintained. Hence, vector $\tilde{x}^{(k)}$ is a minimal solution of $X(A, b)$. Since $\bigcup_{j=1, j \notin \{t_1,...,t_r\}}^{n} I_j \subset I_{t_1}, ..., I_{t_r}$, then $X(A, b)$ has only r minimal solutions. We can select the optimal vector(s) x^* such that $c^{2^T} x^* = min_{k \in \{1,...,r\}} \{c^{2^T} x^{(k)}\}$. Therefore, x^* is the optimal solution(s) of problem (2.9)-(2.10). □

Lemma 3.5. *If* $\exists t \in \underline{n}$ *such that* $\forall s \in \underline{n} - \{t\}$, $I_t \subset I_s$, *then for each optimal solution* x^* *of problem* (2.9)-(2.10), *we have* $x_t^* = 0$.

Proof. Assume x_t^* and x_s^*, for $\forall s \in \underline{n} - \{t\}$, be the associated optimal variables of I_t and I_s, for $\forall s \in \underline{n} - \{t\}$, respectively. Due to $I_t \subset I_s$, for $\forall s \in \underline{n} - \{t\}$, variables x_s^* are binding in the constraints in which variable x_t^* is binding as well. Therefore, for each optimal solution x^* of problem (2.9)-(2.10), we conclude that $x_t^* = 0$. □

Under the conditions of Lemma (3.5), by setting $x_t^* = 0$, for each optimal solution x^* of problem (2.9)-(2.10), we can remove column t of matrix A and component t of vector x.

Lemma 3.6. *If* $\exists t \in \underline{n}$ *such that* $I_t \neq \emptyset$ *and* $\forall s \in \underline{n} - \{t\}$, $I_t \bigcap I_s = \emptyset$, *then for each optimal solution* x^* *of problem* (2.9)-(2.10), *we have* $x_t^* = \hat{x}_t$.

Proof. Since $I_t \bigcap I_s = \emptyset$, for $\forall s \in \underline{n} - \{t\}$, and $I_t \neq \emptyset$, we have $J_i = \{t\}$, for each $i \in I_t$. Hence, from Lemma (3.2), we conclude that for each optimal solution x^* of problem (2.9)-(2.10), we have $x_t^* = \hat{x}_t$. □

Under the conditions of Lemma (3.6), by setting $x_t^* = \hat{x}_t$, we can remove rows $i \in I_t$ and column t from matrix A and components t and $i \in I_t$ from two vectors x and b, respectively.

Lemma 3.7. *If problem* (2.9)-(2.10) *satisfies two conditions: (1) For each* $t \in \underline{n}$, $I_t \neq \emptyset$ *and (2) For each* $t, s \in \underline{n}$ *such that* $t \neq s$ *and* $I_t \bigcap I_s = \emptyset$, *then the problem* (2.9)-(2.10) *will have only one optimal solution* x^* *such that* $x^* = \hat{x}$.

Proof. For $\forall t \in \underline{n}$, $I_t \neq \emptyset$, and $\forall s \in \underline{n} - \{t\}$, $I_t \bigcap I_s = \emptyset$, then regarding Lemma (3.6), for each optimal solution x^* of problem (2.9)-(2.10), we have $x_t^* = \hat{x}_t$. Since the current relation is true for each $t \in \underline{n}$, we conclude that $x^* = \hat{x}$. Therefore, problem (2.9)-(2.10) has only one optimal solution. □

The following corollary is a direct result of Lemma (3.7).

Corollary 3.2. *Under the conditions of Lemma* (3.7), *the feasible domain of problem* (2.1)-(2.2) *has only one feasible solution.*

Lemma 3.8. *If* $\exists t \in \underline{n}$ *such that* $I_t = \emptyset$, *then for each optimal solution* x^* *of problem* (2.9)-(2.10), *we have* $x_t^* = 0$.

Proof. Since $I_t = \emptyset$, we conclude that $I_t^{(f)} = \emptyset$ and $\forall x^{(f)} \in F$, $x_t^{(f)} = 0$. With regard to Theorem (2.2)(A), for each optimal solution x^* of problem (2.9)-(2.10), we have $x_t^* = 0$. □

Under the conditions of Lemma (3.8), by setting $x_t^* = 0$, we can remove column t from matrix A and component t of vector x.

4. An algorithm for solving problem (2.9)-(2.10)

We brief the obtained results in the previous section as an algorithm.

Algorithm 4.1. *Problem* (2.9)-(2.10) *has been given.*
Create the maximum solution $\hat{x}$ *of* $X(A,b)$. *If* $A \bullet \hat{x} = b$, *then* $X(A,b) \neq \emptyset$. *Go to Step 1. Otherwise, stop. The problem is infeasible.*
Step 1. Create sets J_i, *for* $\forall i \in \underline{m}$, *and* I_j, *for* $\forall j \in \underline{n}$.
Step 2. If $\exists t \in \underline{n}$ *such that* $\bigcup_{j=1, j\neq t}^{n} I_j \subset I_t$, *then problem* (2.9)-(2.10) *has only one optimal solution as* $x^* = [x_1^*, ..., x_n^*]^T$, *where* $x_j^* = \begin{cases} \hat{x}_j & j = t, \\ 0 & j \neq t, \end{cases}$ $\forall j \in \underline{n}$. *Stop.*
Step 3. If $\exists t_1, ..., t_r \in \underline{n}$ *such that* $\bigcup_{j=1, j\notin\{t_1,...,t_r\}}^{n} I_j \subset I_{t_1}, ..., I_{t_r}$ *and* $I_{t_k} = \underline{m}$, *for* $\forall k \in \underline{r}$, *then optimal solutions of problem* (2.9)-(2.10) *are as* x^* *such that*
$c^{2^T} x^* = min_{k\in\{1,...,r\}}\{c^{2^T} \tilde{x}^{(k)}\}$, *where* $\tilde{x}^{(k)} = [\tilde{x}_1^{(k)}, ..., \tilde{x}_n^{(k)}]^T$ *and*
$\tilde{x}_j^{(k)} = \begin{cases} \hat{x}_j & j = t_k, \\ 0 & j \neq t_k, \end{cases}$ $\forall j \in \underline{n}$. *Stop.*
Step 4. If problem (2.9)-(2.10) *satisfies two conditions: (1) For each* $t \in \underline{n}$, $I_t \neq \emptyset$ *and (2) For each* $t, s \in \underline{n}$, *such that* $t \neq s$ *and* $I_t \bigcap I_s = \emptyset$, *then problem* (2.9)-(2.10) *has only one optimal solution* x^* *with* $x^* = \hat{x}$. *Stop.*
Note: In Steps 5, 6, and 7, if matrixes A or b became empty, then assign zero to the remained variables. Stop.
Step 5. If $\exists i \in \underline{m}$ *such that* $|J_i| = 1$, *where* $J_i = \{j\}$, *then let* $x_j^* = \hat{x}_j$. *Remove rows i and column j from matrix A and components* j_0 *and* i_0 *of vectors x and b, respectively. Update* J_i, I_j, A, x *and* b. *Remove* J_i *and* I_j *which are empty.*
Step 6. If $\exists t \in \underline{n}$ *such that* $\forall s \in \underline{n} - \{t\}$, $I_t \subset I_s$, *then for each optimal solution* x^* *of problem* (2.9)-(2.10), $x_t^* = 0$. *Remove column t of matrix A and component t of vector x. Update* J_i, I_j, A, x *and* b. *Remove* J_i *and* I_j *which are empty.*
Step 7. If $\exists t \in \underline{n}$ *such that* $I_t \neq \emptyset$ *and* $\forall s \in \underline{n} - \{t\}$, $I_t \bigcap I_s = \emptyset$, *then for each optimal solution* x^* *of problem* (2.9)-(2.10), $x_t^* = \hat{x}_t$. *Remove rows* $i \in I_t$ *and column t from matrix A and components t and* $i \in I_t$ *from two vectors x and b, respectively. Update* J_i, I_j, A, x *and* b. *Remove* J_i *and* I_j *which are empty.*
Step 8. If matrixes A and b be nonempty and the conditions Steps 1, 2, 3, and 4 aren't satisfied for the remained problem, then solve the problem by branch-and-bound method. Produce the optimal solution. Stop.

Example 4.1. *Consider the following problem.*
$min\ z = x_1 + 2x_2 + x_3 + 3x_4,$

$$s.t. \quad \begin{pmatrix} \frac{4}{10} & 1 & \frac{2}{3} & \frac{8}{10} \\ \frac{2}{10} & \frac{3}{10} & \frac{8}{15} & \frac{4}{10} \\ \frac{8}{10} & \frac{45}{100} & \frac{8}{10} & \frac{26}{100} \\ \frac{4}{10} & \frac{6}{10} & \frac{4}{10} & \frac{2}{10} \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} \frac{25}{100} \\ \frac{2}{10} \\ \frac{3}{10} \\ \frac{15}{100} \end{pmatrix},$$

$x_i \in [0,1],\ i = 1,2,3,4.$

Where "$\bullet$*" is the operator of max-product composition. The maximum solution of the feasible domain of the problem is as:* $\hat{x} = \begin{pmatrix} \frac{375}{1000} & \frac{25}{100} & \frac{375}{1000} & \frac{3125}{10000} \end{pmatrix}^T$. *Since* $\hat{x}$ *satisfies the constraints of the problem, the problem is feasible.*

Step 1. $I_1 = \{3,4\}$, $I_2 = \{1,4\}$, $I_3 = \{1,2,3,4\}$, *and* $I_4 = \{1\}$. $J_1 = \{2,3,4\}$, $J_2 = \{3\}$, $J_3 = \{1,3\}$, *and* $J_4 = \{1,2,3\}$.

Step 2. $\bigcup_{j=1,j\neq 3}^{4} I_j \subset I_3$. *Therefore, the vector* $x^* = \begin{pmatrix} 0 & 0 & \frac{375}{1000} & 0 \end{pmatrix}^T$ *is the unique optimal solution of the problem with the objective function value* $z^* = 0.375$. *Stop.*

Example 4.2. *Consider the following problem.*
$min\ z = x_1 + 2x_2 + x_3 + 3x_4 + 1.5x_5 + 4x_6$,

$$s.t. \quad \begin{pmatrix} 0.5 & 0.4 & 0.35 & 0.97 & 0.97 & 0.42 \\ 0.92 & 0.88 & 0.34 & 1 & 1 & 0.44 \\ 0.71 & 0.19 & 0.6 & 0.47 & 0.63 & 0.22 \\ 0.54 & 0.25 & 0.35 & 0.54 & 0.78 & 0.342 \\ 0.87 & 0.82 & 0.51 & 0.95 & 0.95 & 0.41 \end{pmatrix} \bullet \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = \begin{pmatrix} 0.3 \\ 0.31 \\ 0.35 \\ 0.25 \\ 0.29 \end{pmatrix},$$

$x_i \in [0,1],\ i = 1, ..., 6.$

Where "$\bullet$" is the operator of max-T_Y composition that T_Y is defined as follows:

$$T_Y(a,b) = \begin{cases} b^{\frac{1}{a}}, & a.b > 0, \\ 0, & a.b = 0, \end{cases} \quad \text{where } a, b \in [0,1].$$

The maximum solution of the feasible domain of the problem is as:

$$\hat{x} = \begin{pmatrix} 0.34 & 0.36 & 0.53 & 0.31 & 0.31 & 0.6 \end{pmatrix}^T.$$

Since $\hat{x}$ satisfies the constraints of the problem, the problem is feasible.
Step 1. $I_1 = \{2,5\}$, $I_2 = \{2,5\}$, $I_3 = \{3,5\}$, $I_4 = \{1,2,5\}$, $I_5 = \{1,2,4,5\}$ *and* $I_6 = \{1,2,4,5\}$. $J_1 = \{3,4,5\}$, $J_2 = \{1,2,4,5,6\}$, $J_3 = \{3\}$, $J_4 = \{5,6\}$ *and* $J_5 = \{1,2,3,4,5,6\}$.
Step 2. The problem doesn't satisfy the conditions of this step.
Step 3. The problem doesn't satisfy the conditions of this step.
Step 4. The problem doesn't satisfy the conditions of this step.
Step 5. Since $J_3 = \{3\}$, *let* $x_3^* = \hat{x}_3 = 0.53$. *Put* $J_1' = \{4,5\}$, $J_5' = \{1,2,4,5,6\}$, $J_i' = J_i$, *for* $i = 2,4,6$, *and* $I_j' = I_j$, *for* $\forall j \neq 3$. *Remove* J_3 *and* I_3. *Remove row 3 and column 3 of matrix A and component 3 of vectors x and b.*
Step 6. The problem doesn't satisfy the conditions of this step.
Step 7. The problem doesn't satisfy the conditions of this step.
Step 8. The conditions of Step 2 are true for the reduced problem. Since $I_1' \bigcup I_2' \bigcup I_4' \bigcup I_5' \subset I_6'$, *according to step 2, the reduced problem has only one optimal solution as:* $x_6^* = \hat{x}_6 = 0.6$ *and* $x_j^* = 0$, *for* $j \neq 3,6$. *Therefore, the unique optimal solution of the original problem is as:*

$$x^* = \begin{pmatrix} 0 & 0 & 0.53 & 0 & 0 & 0.6 \end{pmatrix}^T,$$

with the objective function value: $z^* = 2.93$. *Stop.*

5. Conclusions

In this paper, the problem of linear objective function optimization was considered. Some sufficient conditions were proposed for determination of the optimal solutions of the problem in terms of the maximum solution of its feasible domain. Under these conditions, some optimal solutions of the problem were determined directly. Some procedures were also proposed to simplify the original problem based on the conditions. Moreover, some sufficient conditions were presented for uniqueness of the optimal solution of the original problem. Finally, an algorithm was suggested to find the optimal solution of the problem based on the procedures and the conditions.

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Ali Abbasi Molai for the photography and short autobiography, see TWMS J. App. Eng. Math., V.3, N.2.