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DOMINATION INTEGRITY OF TOTAL GRAPHS

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ABSTRACT. The domination integrity of a simple connected graph G is a measure of vulnerability of a graph. Here we determine the domination integrity of total graphs of path P_n, cycle C_n and star $K_{1,n}$.

Keywords: Integrity, Domination Integrity, total graph.

AMS Subject Classification: 05C38,05C69,05C76.

1. INTRODUCTION

We begin with simple, finite, connected and undirected graph G with vertex set $V(G)$ and edge set $E(G)$. For any undefined terminology and notation related to the concept of domination we refer to Haynes *et al.* [7] while for the fundamental concepts in graph theory we rely upon Harary [6]. In the remaining portion of this section we will give brief summary of definitions and information related to the present work.

The vulnerability of network have been studied in various contexts including road transportation system, information security, structural engineering and communication network. A graph structure is vulnerable if '*any small damage produces large consequences*'. In a communication network, the vulnerability measures the resistance of the network to disruption of operation after the failure of certain stations (junctions) or communication links (connections). In the theory of graphs, the vulnerability implies a lack of resistance(weakness) of graph network arising from deletion of vertices or edges or both. Communication networks must be so designed that they do not easily get disrupted under external attack and even if they get disturbed then they should be easily reconstructible. Many graph theoretic parameters have been introduced to describe the vulnerability of communication networks including binding number, rate of disruption, toughness, neighbor-connectivity, integrity, mean integrity, edge-connectivity and tenacity. In the analysis of the vulnerable communication network two quantities are playing vital role, namely (i) the number of elements that are not functioning (ii) the size of the largest remaining (survived) sub network within which mutual communication can still occur. In adverse relationship it is desirable that an opponent's network would be such that the above referred two quantities can be made simultaneously small. Here the first parameter provides an information about nodes which can be targeted for more disruption while the later gives the impact of damage after disruption. To estimate these quantities

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Barefoot *et al.* [2] have introduced the concept of integrity, which is defined as follows.

Definition 1.1 The integrity of a graph G is denoted by $I(G)$ and defined by $I(G) = min\{| S | + m(G-S) : S \subset V(G)\}$ where $m(G-S)$ is the order of a maximum component of $G-S$.

Many results are reported in a survey article on integrity by Bagga *et al.* [1]. Some general results on the interrelations between integrity and other graph parameters are investigated by Goddard and Swart [5] while Mamut and Vumar [8] have determined the integrity of middle graph of some graphs. It is also observed that bigger the integrity of network, more reliable functionality of the network after any disruption caused by non-functional devices (elements). The connectivity is useful to identify local weaknesses in some respect while integrity gives brief account of vulnerability of the graph network.

Definition 1.2 A subset S of $V(G)$ is called dominating set if for every $v \in V - S$, there exist a $u \in S$ such that v is adjacent to u.

Definition 1.3 The minimum cardinality of a minimal dominating set in G is called the domination number of G denoted as $\gamma(G)$ and the corresponding minimal dominating set is called a γ-set of G.

The theory of domination plays vital role in determining decision making bodies of minimum strength or weakness of a network when certain part of it is paralysed. In the case of disruption of a network, the damage will be more when vital node are under siege. This motivated the study of domination integrity when the sets of nodes disturbed are dominating sets. Sundareswaran and Swaminathan [9] have introduced the concept of domination integrity of a graph as a new measure of vulnerability which is defined as follows.

Definition 1.4 The domination integrity of a connected graph G denoted by $DI(G)$ and defined as $DI(G) = min\{| X | + m(G-X) : X \text{ is a dominating set}\}$ where $m(G-X)$ is the order of a maximum component of $G-X$.

Sundareswaran and Swaminathan [9] have investigated domination integrity of some standard graphs. In the same paper they have investigated domination integrity of Binomial trees and Complete k-ary trees while in [10] they have investigated domination integrity of middle graph of some standard graphs. Same authors in [11] have investigated the domination integrity of powers of cycles while in [12] they have discussed domination integrity of trees. Vaidya and Kothari [13, 14] have discussed domination integrity in the context of some graph operations and also investigated domination integrity of splitting graph of path P_n and cycle C_n while Vaidya and Shah [15] have investigated domination integrity of shadow graphs of P_n, C_n, $K_{m,n}$ and $B_{n,n}$.

Generally following types of problems are generally considered in the field of domination.
1. Introduce new type of domination parameters by combining domination with other graph theoretical property.
2. To find upper or lower bound of any particular dominating parameter with respect to graph parameters like $\delta(G), \triangle(G), \alpha_0(G), \beta_0(G), \kappa(G), \omega(G), diam(G)$ etc.

3. To obtain exact domination number for some graphs or graph families.
4. Characterize the graph or graph family which satisfies certain dominating parameter.
5. Study of algorithmic and complexity results for particular dominating parameter.
6. How a particular dominating parameter is affected under various graph operations.

The problems of first five types are largely discussed while the problems of sixth type are not so often but they are of great importance. The present work is aimed to discuss the problems of sixth kind in the context of domination integrity. We investigate domination integrity for total graphs of P_n, C_n and $K_{1,n}$.

The concept of total graph $T(G)$ of graph G was introduced by Behzad [3] which is defined as follows:

Definition 1.5 The total graph $T(G)$ of G is the graph whose vertex set is $V(G) \cup E(G)$ and two vertices are adjacent whenever they are either adjacent or incident in G.

It is obvious that $T(G)$ always contains both G and its line graph $L(G)$ as a induced subgraphs. Also $T(G)$ is the largest graph formed by adjacent and incidence relation between graph elements.

Dündar and Aytaç [4] discussed the integrity of total graphs via certain graph parameters while we discuss the domination integrity of total graphs.

Proposition 1.6 [4]

(i) $\gamma\left(T(P_n)\right) = \begin{cases} \dfrac{|V\left(T(P_n)\right)|}{5}; & if\ |V\left(T(P_n)\right)| \equiv 0 (\text{mod } 5) \\ \left\lfloor \dfrac{|V\left(T(P_n)\right)|}{5} \right\rfloor + 1; & otherwise \end{cases}$

(ii) $\gamma\left(T(C_n)\right) = \begin{cases} \dfrac{|V\left(T(C_n)\right)|}{5}; & if\ |V\left(T(C_n)\right)| \equiv 0 (\text{mod } 5) \\ \left\lfloor \dfrac{|V\left(T(C_n)\right)|}{5} \right\rfloor + 1; & otherwise \end{cases}$

2. Main Results

Proposition 2.1. [9]

(i) $DI\left(P_n\right) = \begin{cases} \left\lceil \dfrac{n}{2} \right\rceil + 1; & n = 2,3,4,5 \\ \left\lceil \dfrac{n}{3} \right\rceil + 2; & n \geq 6 \end{cases}$

(ii) $DI\left(C_n\right) = \begin{cases} 3; & n = 3,4 \\ \left\lceil \dfrac{n}{3} \right\rceil + 2; & n \geq 5 \end{cases}$

(iii) $DI\left(K_{m,n}\right) = \min\{m,n\} + 1$

Theorem 2.2. *$DI\left(T(P_n)\right) = n + 1$ for $n = 2$ to 7.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n and $u_1, u_2, \ldots, u_{n-1}$ be the added vertices corresponding to edges $e_1, e_2, \ldots, e_{n-1}$ to obtain $T(P_n)$. Let G be the graph

$T(P_n)$. Then $|V(G)| = 2n - 1$ and $|E(T(G)| = 4n - 5$.
To prove this result we consider following six cases.
Case 1: $n = 2$
Clearly $T(P_2)$ is C_3 so $DI(T(P_2)) = 3$ from proposition 2.1.
Case 2: $n = 3$
Consider $S = \{v_2, u_2\}$ then S is dominating set of $T(P_3)$ and $m(G-S) = 2$, so $|S|+m(G-S) = 4$. There does not exist any dominating set S_1 of $T(P_3)$ such that $|S_1|+m(G-S_1) < |S| + m(G - S)$. Hence, $DI(T(P_3)) = 4$.
Case 3: $n = 4$
Consider $S = \{v_2, u_2, u_4\}$ then S is dominating set of $T(P_4)$ and $m(G - S) = 2$, so $|S|+m(G-S) = 5$. If S_1 is any dominating set of $T(P_4)$ with $|S_1| \leq 2$ then $m(G-S_1) = 5$ so $|S_1| + m(G - S_1) > 5$. If we consider any dominating set S_2 of $T(P_4)$ such that $m(G - S_2) = 1$ then $|S_2| \geq 4$ hence, $|S_2| + m(G - S_2) \geq 5$. Therefore, $DI(T(P_4)) = 5$.
Case 4: $n = 5$
Consider $S = \{v_2, u_2, v_4, u_4\}$ then S is dominating set of $T(P_5)$ and $m(G - S) = 2$, so $|S|+m(G-S) = 6$. If S_1 is any dominating set of $T(P_5)$ with $|S_1| = 3$ then $m(G-S_1) \geq 4$ so $|S_1|+m(G-S_1) > 6$. If S_2 is any dominating set of $T(P_5)$ with $|S_2| = 2$ then $m(G-S_2) = 7$ so $|S_2| + m(G - S_2) = 9 > 6$. If we consider any dominating set S_3 of $T(P_5)$ such that $m(G - S_3) = 1$ then $|S_3| \geq 6$ hence, $|S_3| + m(G - S_3) \geq 7$. Therefore, $DI(T(P_5)) = 6$.
Case 5: $n = 6$
Consider $S = \{v_2, u_2, u_4, v_5\}$ then S is dominating set of $T(P_6)$ and $m(G - S) = 3$, so $|S| + m(G - S) = 7$. If S_1 is any dominating set of $T(P_6)$ with $m(G - S_1) \geq 4$ then $|S_1| + m(G - S_1) = 8 > 7$. If S_2 is any dominating set of $T(P_6)$ with $|S_2| = 2$ then clearly $|S_2| + m(G - S_2) > 7$. Therefore, $DI(T(P_6)) = 7$.
Case 6: $n = 7$
Consider $S = \{v_2, u_2, u_4, v_5, v_7\}$ then S is dominating set of $T(P_7)$ and $m(G - S) = 3$, so $|S| + m(G - S) = 8$. If S_1 is any dominating set of $T(P_7)$ with $m(G - S_1) \geq 4$ then $|S_1| + m(G - S_1) = 9 > 8$. If S_2 is any dominating set of $T(P_7)$ with $|S_2| = 2$ then clearly $|S_2| + m(G - S_2) > 8$. Therefore, $DI(T(P_7)) = 8$.
Hence $DI\,(T(P_n)) = n + 1$ for $n = 2$ to 7.

□

Theorem 2.3. *For $n \geq 8$,*

$$DI\,(T(P_n)) = \begin{cases} \frac{2n}{3} + 4; & if\ n \equiv 0 (\text{mod } 3) \\ \left\lceil \frac{2n}{3} \right\rceil + 4; & if\ n \equiv 1 (\text{mod } 3) \\ \left\lfloor \frac{2n}{3} \right\rfloor + 4; & if\ n \equiv 2 (\text{mod } 3) \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n and $u_1, u_2, \ldots, u_{n-1}$ be the added vertices corresponding to edges $e_1, e_2, \ldots, e_{n-1}$ to obtain $T(P_n)$. Let G be the graph $T(P_n)$.
Proposition 1.6 gives the value of $\gamma\,(T(P_n))$, here we provide $D(\gamma - set)$ for $T(P_n)$ for different possibilities of n as below:

- If $n \equiv 0 (\text{mod } 5)$ (i.e. $n = 5k$), consider $D = \{v_{2+5i}, u_{4+5i} | 0 \leq i < k\}$.
- If $n \equiv 1 (\text{mod } 5)$ (i.e. $n = 5k + 1$) or $n \equiv 2 (\text{mod } 5)$ (i.e. $n = 5k + 2$), consider $D = \{v_{2+5i}, u_{4+5i}, v_n | 0 \leq i < k\}$.
- If $n \equiv 3 (\text{mod } 5)$ (i.e. $n = 5k + 3$), consider $D = \{v_{2+5i}, u_{4+5j} | 0 \leq i \leq k, 0 \leq j < k\}$.

- If $n \equiv 4(\text{mod } 5)$ (i.e. $n = 5k+4$), consider $D = \{v_{2+5i}, u_{4+5j}, v_n | 0 \leq i \leq k, 0 \leq j < k\}$.

Hence, $\gamma\left(T(P_n)\right) = \begin{cases} \dfrac{2n-1}{5}; & if\ 2n-1 \equiv 0(\text{mod } 5) \\ \left\lfloor \dfrac{2n-1}{5} \right\rfloor + 1; & otherwise \end{cases}$

Clearly, $DI\left(T(P_n)\right) \leq |D| + m(G-D)$ .... (i)

Now we define another subset S of $V(T(P_n))$ as below:

- If $n \equiv 0(\text{ mod } 3)$ (i.e. $n = 3k$), consider $S = \{v_{2+3i}, u_{2+3i} | 0 \leq i < k\}$ and $|S| = 2k$.
- If $n \equiv 1(\text{mod } 3)$ (i.e. $n = 3k+1$) or $n \equiv 2(\text{mod } 3)$ (i.e. $n = 3k+2$), consider $S = \{v_{2+3i}, u_{2+3i}, v_n | 0 \leq i < k\}$ and $|S| = 2k+1$.

In all the above cases S is a dominating set for G as $u_{1+3t}, u_{3+3t} \in N(u_{2+3t})$ and $v_{1+3t}, v_{3+3t} \in N(v_{2+3t})$ for $t \in \mathbb{N} \cup \{0\}$ moreover $m(G-S) = 4$.
In order to compare the values of parameters $|D| + m(G-D)$ and $|S| + m(G-S)$ and to check the minimality of $|S| + m(G-S)$, we prepare the Table 1 for random values of n between 8 to 20.

Table 1

1	2	3	4	5	6	7	8
n	$2n-1$	$\lvert D\rvert$	$m(G-D)$	$\lvert D\rvert + m(G-D)$	$\lvert S\rvert$	$m(G-S)$	$\lvert S\rvert + m(G-S)$
8	15	3	12	15	5	4	9
9	17	4	13	17	6	4	10
10	19	4	15	19	7	4	11
11	21	5	16	21	7	4	11
16	31	7	24	31	11	4	15
20	39	8	31	39	13	4	17

From columns 5 and 8 of Table 1, we can observe that for D $(\gamma - set)$ and dominating set S,
$|S| + m(G-S) < |D| + m(G-D)$ .... (ii)
We have verified that the above relation (ii) is valid even for larger values of n.
From (i) and (ii), we have,
$DI\left(T(P_n)\right) \leq |S| + m(G-S) < |D| + m(G-D)$.
Hence, $DI\left(T(P_n)\right) \leq |S| + m(G-S)$ .... (iii)
We claim that $DI\left(T(P_n)\right) = |S| + m(G-S)$.
If we consider any dominating set S_1 of G such that, $|D| \leq |S_1| < |S|$ then due to construction of $T(P_n)$, it generates large value of $m(G-S_1)$ such that,
$|S| + m(G-S) < |S_1| + m(G-S_1)$.
Let S_2 be dominating set of G with minimal cardinality such that $m(G-S_2) = 3$ then,
$|S| + m(G-S) \leq |S_2| + m(G-S_2)$, for $8 \leq n \leq 13$ and
$|S| + m(G-S) < |S_2| + m(G-S_2)$, for $n \geq 14$.
Moreover if S_3 is any dominating set of G with $m(G-S_3) = 2$ or $m(G-S_3) = 1$ then clearly,
$|S| + m(G-S) < |S_3| + m(G-S_3)$

From above discussion we can say that among all dominating sets of G, S is such that $|S| + m(G - S)$ is minimum.
Therefore,
$|S| + m(G - S) = min\{| X | + m(G - X)|X$ is a dominating set$\}$
$= DI\,(T(P_n))$.
Hence, for $n \geq 8$

$$DI\,(T(P_n)) = \begin{cases} \frac{2n}{3} + 4; & if\ n \equiv 0 (\mathrm{mod}\ 3) \\ \\ \lceil \frac{2n}{3} \rceil + 4; & if\ n \equiv 1 (\mathrm{mod}\ 3) \\ \\ \lfloor \frac{2n}{3} \rfloor + 4; & if\ n \equiv 2 (\mathrm{mod}\ 3) \end{cases}$$

$\square$

Corollary 2.4. $$DI\,(T\,(P_n)) - DI\,(P_n) = \begin{cases} 1; & n = 2, 3 \\ 2; & n = 4, 5 \\ 3; & n = 6, 7 \\ \\ \frac{n}{3} + 2; & n \geq 8\ \&\ n \equiv 0 (\mathrm{mod}\ 3) \\ \\ \lfloor \frac{n}{3} \rfloor + 2; & n \geq 8\ \&\ n \equiv 1 (\mathrm{mod}\ 3)\ or\ n \equiv 2 (\mathrm{mod}\ 3) \end{cases}$$

Proof. In view of Proposition 2.1, Theorem 2.2 and Theorem 2.3 the result is obvious. $\square$

Theorem 2.5. $$DI\,(T(C_n)) = \begin{cases} 6; & n = 3, 4 \\ 7; & n = 5 \\ 8; & n = 6 \\ 9; & n = 7 \\ 10; & n = 8, 9 \\ 11; & n = 10 \end{cases}$$

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and $u_1, u_2, \ldots, u_n$ be the added vertices corresponding to edges $e_1, e_2, \ldots, e_n$ to obtain $T(C_n)$. Let G be the graph $T(C_n)$. Then $|V(G)| = 2n$ and $|E(T(G)| = 4n$.
To prove this result we consider following two cases.
Case 1: $n = 3, 4$

For $n = 3$, consider $S = \{v_2, v_3, u_3\}$ then $m(G - S) = 3$. There does not exist any dominating set S_1 of $T(C_3)$ such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $DI(T(C_3)) = 6$.
For $n = 4$, consider $S = \{v_2, v_4, u_2, u_4\}$ then $m(G - S) = 2$. There does not exist any dominating set S_1 of $T(C_4)$ such that $|S_1| + m(G - S_1) < |S| + m(G - S)$. Hence, $DI(T(C_4)) = 6$.
Case 2: $n = 5$ to 10

To explain this case we prepare the following Table 2.
The Table 2 gives dominating set S and corresponding values of $m(G - S)$ for $n = 5$ to 10. It can be observed that among all dominating sets of G, above given S gives the minimum value of $|S| + m(G - S)$.
Hence, for $n = 3$ to 10

TABLE 2

n	S	$\lvert S\rvert$	$m(G-S)$	$\lvert S\rvert+m(G-S)$
5	$\{v_2,u_2,u_4,v_5\}$	4	3	7
6	$\{v_2,u_2,v_5,u_5\}$	4	4	8
7	$\{v_2,u_2,u_4,v_5,u_7,v_7\}$	6	3	9
8	$\{v_2,u_2,u_4,v_5,u_7,v_7,v_8\}$	7	3	10
	$\{v_2,u_2,v_5,u_5,v_8,u_8\}$	6	4	10
9	$\{v_2,u_2,v_5,u_5,v_8,u_8\}$	6	4	10
10	$\{v_2,u_2,u_4,v_5,u_7,v_7,u_9,v_{10}\}$	8	3	11

$$DI\left(T(C_n)\right)=\begin{cases} 6; & n=3,4\\ 7; & n=5\\ 8; & n=6\\ 9; & n=7\\ 10; & n=8,9\\ 11; & n=10\end{cases}$$

□

Theorem 2.6. *For $n \geq 11$*

$$DI\left(T(C_n)\right)=\begin{cases} \frac{2n}{3}+4; & if\ n\geq 11\ \&\ n\equiv 0(\text{mod } 3)\\ \frac{2(n+2)}{3}+4; & if\ n\geq 11\ \&\ n\equiv 1(\text{mod } 3)\\ \frac{2(n+1)}{3}+4; & if\ n\geq 11\ \&\ n\equiv 2(\text{mod } 3)\end{cases}$$

Proof. Let $v_1, v_2, \dots, v_n$ be the vertices of cycle C_n and $u_1, u_2, \dots, u_n$ be the added vertices corresponding to edges $e_1, e_2, \dots, e_n$ to obtain $T(C_n)$. Let G be the graph $T(C_n)$. Then $|V(G)| = 2n$ and $|E(T(G)| = 4n$.
Proposition 1.6 gives the value of $\gamma\left(T(C_n)\right)$, here we provide $D(\gamma - set)$ for $T(C_n)$ for different possibilities of n as below:

- If $n \equiv 0(\text{mod } 5)$ (i.e. $n = 5k$), consider $D = \{v_{2+5i}, u_{4+5i} | 0 \leq i < k\}$.
- If $n \equiv 1(\text{mod } 5)$ (i.e. $n = 5k+1$) or $n \equiv 2(\text{mod } 5)$ (i.e. $n = 5k+2$), consider $D = \{v_{2+5i}, u_{4+5i}, v_n | 0 \leq i < k\}$.
- If $n \equiv 3(\text{mod } 5)$ (i.e. $n = 5k+3$), consider $D = \{v_{2+5i}, u_{4+5j} | 0 \leq i \leq k, 0 \leq j < k\}$.
- If $n \equiv 4(\text{mod } 5)$ (i.e. $n = 5k+4$), consider $D = \{v_{2+5i}, u_{4+5j}, v_n | 0 \leq i \leq k, 0 \leq j < k\}$.

Hence, $\gamma\left(T(C_n)\right)=\begin{cases} \dfrac{2n}{5}; & if\ 2n\equiv 0(\text{mod } 5)\\ \left\lfloor\dfrac{2n}{5}\right\rfloor+1; & otherwise\end{cases}$

Clearly, $DI\left(T(C_n)\right) \leq |D| + m(G-D) \dots\dots\dots (iv)$

Now we define another subset S of $V(T(C_n))$ as below:

- If $n \equiv 0(\text{mod } 3)$ (i.e. $n = 3k$) and $n \equiv 2(\text{mod } 3)$ (i.e. $n = 3k-1$), consider $S = \{v_{2+3i}, u_{2+3i} | 0 \leq i < k\}$ and $|S| = 2k$.

- If $n \equiv 1 (\mathrm{mod}\ 3)$ (i.e. $n = 3k+1$), consider $S = \{v_{2+3i}, u_{2+3i}, v_n | 0 \leq i < k\} \cup \{v_n, u_n\}$ and $|S| = 2(k+1) = \dfrac{2(n+2)}{3}$.

In all the above cases S is a dominating set for G as $u_{1+3t}, u_{3+3t} \in N(u_{2+3t})$ and $v_{1+3t}, v_{3+3t} \in N(v_{2+3t})$ for $t \in \mathbb{N} \cup \{0\}$ moreover $m(G-S) = 4$.
In order to compare the values of parameters $|D| + m(G-D)$ and $|S| + m(G-S)$ as well as to check the minimality of $|S| + m(G-S)$, we prepare the Table 3 for random values of n between 11 to 25.

TABLE 3

1	2	3	4	5	6	7	8
n	$2n$	$\|D\|$	$m(G-D)$	$\|D\| + m(G-D)$	$\|S\|$	$m(G-S)$	$\|S\| + m(G-S)$
11	22	5	17	22	8	4	12
12	24	5	19	24	8	4	12
13	26	6	20	26	10	4	14
14	28	6	22	28	10	4	14
16	32	7	25	32	12	4	16
25	50	10	40	50	18	4	22

From columns 5 and 8 of Table 3, we can observe that for D ($\gamma - set$) and dominating set S,
$|S| + m(G-S) < |D| + m(G-D)$ .... (v)
We have verified that the above relation (v) is valid even for larger values of n.
From (iv) and (v), we have,
$DI\,(T(C_n)) \leq |S| + m(G-S) < |D| + m(G-D)$.
Hence, $DI\,(T(C_n)) \leq |S| + m(G-S)$ .... (vi)
We claim that $DI\,(T(C_n)) = |S| + m(G-S)$.
If we consider any dominating set S_1 of G such that, $|D| \leq |S_1| < |S|$ then due to construction of $T(C_n)$, it generates large value of $m(G-S_1)$ such that,
$|S| + m(G-S) < |S_1| + m(G-S_1)$.
Let S_2 be dominating set of G with minimal cardinality such that $m(G-S_2) = 3$ then,
$|S| + m(G-S) \leq |S_2| + m(G-S_2)$, for $n = 13$ and
$|S| + m(G-S) < |S_2| + m(G-S_2)$, for $n = 11, 12$ and $n \geq 14$.
Moreover if S_3 is any dominating set of G with $m(G-S_3) = 2$ or $m(G-S_3) = 1$ then clearly,
$|S| + m(G-S) < |S_3| + m(G-S_3)$
From above discussion we can say that among all dominating sets of G, S is such that $|S| + m(G-S)$ is minimum.
Therefore,

$$\begin{aligned}|S| + m(G-S) &= min\{|\ X\ | + m(G-X) | X \text{ is a dominating set}\}\\ &= DI\,(T(C_n)).\end{aligned}$$

Hence, for $n \geq 11$

$$DI\left(T(C_n)\right)=\begin{cases}\frac{2n}{3}+4; & if\ n\geq 11\ \&\ n\equiv 0(\text{mod }3)\\ \\ \frac{2(n+2)}{3}+4; & if\ n\geq 11\ \&\ n\equiv 1(\text{mod }3)\\ \\ \frac{2(n+1)}{3}+4; & if\ n\geq 11\ \&\ n\equiv 2(\text{mod }3)\end{cases}$$

□

Corollary 2.7. $DI\left(T\left(C_n\right)\right)-DI\left(C_n\right)=\begin{cases}3; & n=3,4,5\\ 4; & n=6,7\\ 5; & n=8,9,10\\ \frac{n}{3}+2; & n\geq 11\ \&\ n\equiv 0(\text{mod }3)\\ \\ \lceil\frac{n}{3}\rceil+2; & n\geq 11\ \&\ n\equiv 1(\text{mod }3)\ or\ n\equiv 2(\text{mod }3)\end{cases}$

Proof. In view of Proposition 2.1, Theorem 2.5 and Theorem 2.6 the proof is obvious.

□

Theorem 2.8. $DI\left(T(K_{1,n})\right)=n+2$.

Proof. Let v be the apex vertex of $K_{1,n}$ and $v_1,v_2,\ldots,v_n$ be the pendant vertices of $K_{1,n}$ and $u_1,u_2,\ldots,u_n$ be the added vertices corresponding to edges $e_1,e_2,\ldots,e_n$ to obtain $T(K_{1,n})$. Let G be the graph $T(K_{1,n})$.
Consider $S=\{v,u_1,u_2,\ldots,u_n\}$ then $|S|=n+1$ and $m(G-S)=1$. Clearly S is a dominating set of G and $|S|+m(G-S)=n+2$.
For $S_1=\{v,u_1,u_2,\ldots,u_{n-1}\}$ then $|S_1|=n$ and $m(G-S_1)=2$ and $|S_1|+m(G-S_1)=n+2$.
For $S_2=\{v,u_1,u_2,\ldots,u_{n-2}\}$ then $|S_2|=n-1$ and $m(G-S_2)=4$ and $|S_2|+m(G-S_2)=n+3$.
Similarly for any other dominating set S_3 of G, $|S|+m(G-S)\leq|S_3|+m(G-S_3)$.
$|S|+m(G-S)=min\{|\ X\ |+m(G-X)|X \text{ is a dominating set}\}$
$\qquad\qquad\qquad = DI\left(T(K_{1,n})\right)$.
Hence, $DI\left(T(K_{1,n})\right)=n+2$. □

3. Concluding Remarks

The rapid growth of various modes of communication have emerged as a search for sustainable and secured network. The vulnerability of network is an important issue with special reference to defence objectives. We take up this problem in the context of expansion of graph network by means of total graph of a graph and investigate domination integrity of $T(P_n)$, $T(C_n)$ and $T(K_{1,n})$ and from Corollary 2.4 and Corollary 2.7 we conclude that the domination integrity increases in such circumstances. To investigate the domination integrity for line graph, shadow graph and the graph obtained by switching of a vertex in the context of P_n and C_n is an open area of research.

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