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ON RELATION OF TWO PROCESSES WITH INDEPENDENT INCREMENTS APPLIED IN QUEUEING SYSTEMS

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Abstract. In the paper, by using two processes ξ_t and $\eta(t), t \geq 0$ with independent increments, one of which is without negative overshoots and the second one is homogeneous in time, we study a homogeneous Markov process $\xi_t, t \geq 0$, and we find the Laplace transform of the generating function of transitional probabilities of the process $\xi_t, t \geq 0$.

Keywords: Process with independent increments, lattice distribution, Poison process.

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1. Introduction

Let $\eta(t)$, $t \geq 0$ be a non-decreasing process with independent and time homogeneous increments, and $\xi_t, t \geq 0$ be a process with independent increments without negative overshoots. Let's consider the Markov process $\xi_t, t \geq 0$ of the following form

$$d\xi_t = sign\xi_t \cdot d\xi_t + \eta^*\left(\frac{dt}{a\xi_t + b}\right), \quad a \geq 0, \quad b > 0 \tag{1}$$

where $\eta^*(x)$ is the process having the same distribution as $\eta(x) - \eta(0)$. From (1), it follows that as $\Delta \downarrow 0$ the process $\xi_t, t \geq 0$ has the following transitions

$$0 \overset{\Delta}{\to} \eta^*\left(\frac{\Delta}{b}\right),$$

$$x \overset{\Delta}{\to} x + \xi_\Delta + \eta^*\left(\frac{\Delta}{ax + b}\right), x > 0. \tag{2}$$

These transitions as $\Delta \downarrow 0$ determine infinitesimal operator of the process $\xi_t, t \geq 0$. Note that if the phase spaces $\eta(t)$ and $\xi(t)$ consists of integers $a = 0$, then ξ_t is the queueing length for queueing system M/M/1. And if $a > 0$, we have another Markov chain.

Introduce the denotation

$$P_t(A) = P\{\xi_t \in A\}, \quad A \subset (0, \infty),$$

$$P_t^0 = P\{\xi_t = 0\}.$$

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According to (2) we have

$$P_{t+\Delta} = P_t^0 \cdot P\left\{\eta^*\left(\frac{\Delta}{b}\right) \in A\right\} +$$

$$+\int_{0^+}^{\infty} P_t(dx)\, P\left\{x + \xi_\Delta + \eta^*\left(\frac{\Delta}{ax+b}\right) \in A\right\} + o(\Delta).$$

If

$$\varphi_t(\lambda) = P_t^0 + \int_{0^+}^{\infty} e^{-\lambda x} P_t(dx),$$

then

$$\varphi_{t+\Delta}(\lambda) = P_t^0\left(1 + \frac{h(\lambda)}{b}\Delta\right) +$$

$$+\int_{0^+}^{\infty} P_t(dx)\, e^{-\lambda x}\left\{1 + \left[\rho(\lambda) + \frac{h(\lambda)}{ax+b}\right]\Delta\right\} + o(\Delta), \tag{3}$$

where $h(\lambda)$ and $\rho(\lambda)$ are the cummulants for characteristic functions of the processes $\eta^*(t)$ and ξ_t, respectively.

From relation (3) we get

$$\frac{\partial \varphi_t(\lambda)}{\partial t} = P_t^0 \frac{h(\lambda)}{b} + \rho(\lambda)\left[\varphi_t(\lambda) - P_t^0\right] + h(\lambda)\int_{0^+}^{\infty} \frac{e^{-\lambda x}}{ax+b} P_t(dx). \tag{4}$$

Denote

$$\psi_t(\lambda) = \int_{0^+}^{\infty} \frac{e^{-\lambda x}}{ax+b} P_t(dx).$$

Then between $\varphi_t(\lambda)$ and $\psi_t(\lambda)$ we have such a relation

$$-a\frac{\partial \varphi_t(\lambda)}{\partial \lambda} = \varphi_t(\lambda) - b\psi_t(\lambda). \tag{5}$$

In Laplace transformations the relations (4) and (5) take the form

$$s\widetilde{\varphi}_s(\lambda) - e^{\lambda z} = \widetilde{P}_s^0 \frac{h(\lambda)}{b}\rho(\lambda)\left[\widetilde{\varphi}_s(\lambda) - \widetilde{P}_s^0(\lambda)\right] + h(\lambda)\widetilde{\psi}_s(\lambda)$$

$$-a\frac{\partial \widetilde{\psi}_s(\lambda)}{\partial \lambda} = \widetilde{\varphi}_s(\lambda) - b\widetilde{\psi}_s(\lambda).$$

With application of theory of semi-markov and markov processes to investigation of such characteristics as the queueing length, average service time, etc. of different queueing systems a lot of authors from this field have been engaged So, in the monograph of T.I. Nasirova [1], a full scheme of application of theory of semi-markov processes to investigation of the queueing system was studied. In [2], by studying the probability processes, in particular the processes with independent increments and by describing the queueing system, the so called updating method was used. In the paper [3], the distribution of the semi-markov walk process with positive drift and negative overshoots were studied.

Though the processes considered in these or other papers are directly connected with the problems of queueing theory, control of resources and so on, they have no property of sign variability. The peculiarity of the studied process $\xi_t, t \geq 0$ is that this process is characterized by its sign. More exactly, if for all $t \geq 0, \xi_t > 0$, then its investigation is reduced to studying the processes $\xi_t, t \geq 0$ with independent increments without negative overshoots and $\eta^* (x)$, $x \geq 0$ having the same distribution as $\eta(x) - \eta(0)$. If for all $t \geq 0, \xi_t \leq 0$, then we deal with one process $\eta^* (t), x \geq 0$.

In the lattice case, for $a = 0$ we find the Laplace transform of the generating function $P_k(t, \theta), |\theta| \leq 1$ of transitional probabilities $\left\{P_{kr}^{(t)}\right\}$, $t \geq 0$, $k, r \in \{0, 1, ...\}$ of the process $\xi_t, t \geq 0$. The obtained results are interpreted for the queueing system M/M/1 in the case of unordinary arrivals of the requirements flow. At the case of finiteness of intensity of arrivals $\lambda = \sum\limits_{k=1}^{\infty} \lambda_k < \infty$, the explicit form of the generating function of stationary distribution of the queue length of the considered queueing system is updated.

2. Basic results

Let us consider the case when $\xi_t, t \geq 0$ has a lattice distribution, and $\{q_{kj}\}$, $k, j \in \{0, 1, ...\}$ are its local characteristics. Then

$$q_{k+1,k} = \nu, \quad q_{kr} = \frac{\lambda r - k}{ak + b}, r > k,$$

and $\xi_t, t \geq 0$ is a Poisson process with parameters ν and downwards overshoots on 1. Let $\left\{P_{kr}^{(t)}\right\}$, $t \geq 0$, $k, r \in \{0, 1, ...\}$ be transitional probabilities of the chain $\xi_t, t \geq 0$.

Introduce the denotation

$$P_k(t, \theta) = \sum_{r=0}^{\infty} p_{kr}(t) \theta^r, \quad q_k(t, \theta) = \sum_{r=0}^{\infty} \frac{p_{kr}(t)}{ar + b} \theta^r, \quad |\theta| \leq 1$$

$$\widetilde{p}_k(s, \theta) = \int_0^{\infty} e^{-st} p_k(t, \theta)\, dt, \widetilde{q}_k(s, \theta) = \int_0^{\infty} e^{-st} q_k(t, \theta)\, dt,$$

$$\widetilde{p}_{k0}(s) = \int_0^{\infty} e^{-st} P_{k0}^{(t)} dt.$$

In these denotation we have the system

$$\begin{cases} s\widetilde{p}_k(s, \theta) - \theta^k = \nu\left(\frac{1}{\theta} - 1\right) + \lambda(\theta)\, \widetilde{q}_k(s, \theta) - \nu\left(\frac{1}{\theta} - 1\right) \widetilde{p}_{k0}(s) \\ a\theta \frac{\partial \widetilde{q}_k(s, \theta)}{\partial \theta} = \widetilde{p}_k(s, \theta) - b\widetilde{q}_k(s, \theta), \end{cases} \tag{6}$$

where

$$\lambda(\theta) = -\lambda + \sum_{k=1}^{\infty} \lambda_k \theta^k, \quad \lambda = \sum_{k=1}^{\infty} \lambda_k < \infty.$$

The system of equations (6) is equivalent to the equation

$$a(\nu + s)\theta(\theta - \theta_0) \frac{\partial \widetilde{q}_k(s, \theta)}{\partial \theta} = [a\lambda(\theta) - b(s + \nu)(\theta - \theta_0)]\, \widetilde{q}_k(s, \theta) +$$

$$+\theta^{k+1} - \nu(1 - \theta)\, \widetilde{p}_{k0}(s), \tag{7}$$

where $\theta_0 = \frac{\nu}{\nu+s}$.

Now solve equation (7). For $\theta = 0$ we have

$$\widetilde{q}_k(s, 0) = \frac{\nu}{b\nu - a\lambda} \cdot \widetilde{p}_{k0}(s),$$

where $\widetilde{p}_{k0}(s)$, $k \in \{0, 1, ...\}$ is to be determined.

Having divided both parts of (7) by $a(\nu+s)\cdot\theta(\theta-\theta_0)$, we get

$$\frac{\partial\widetilde{q}_k(s,\theta)}{\partial\theta}=\frac{a\lambda(\theta)-b(s+\nu)(\theta-\theta_0)}{a(\nu+s)\cdot\theta(\theta-\theta_0)}\widetilde{q}_k(s,\theta)+\frac{\theta^{k+1}-\nu(1-\theta)\widetilde{p}_{k0}(s)}{a(\nu+s)\cdot\theta(\theta-\theta_0)} \tag{8}$$

whence [4,5]

$$\widetilde{q}_k(s,\theta)=ce^{\int\limits_0^\theta\frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu}+$$

$$+\int\limits_0^\theta e^{\int\limits_\chi^\theta\frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu}\cdot\frac{\chi^{k+1}-\nu(1-\chi)\widetilde{p}_{k0}(s)}{a(\nu+s)\chi(\chi-\theta_0)}d\chi.$$

From (8) for $\theta=0$ we find

$$c=\widetilde{q}_k(s,0)=\frac{\nu}{b\nu-a\lambda}\widetilde{p}_{k0}(s).$$

Thus,

$$\widetilde{q}_k(s,\theta)=\frac{\nu}{b\nu-a\lambda}\widetilde{p}_{k0}(s)\,e^{\int\limits_0^\theta\frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu}+$$

$$+\int\limits_0^\theta e^{\int\limits_\chi^\theta\frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu}\cdot\frac{\chi^{k+1}-\nu(1-\chi)\widetilde{p}_{k0}(s)}{a(\nu+s)\chi(\chi-\theta_0)}d\chi$$

or

$$q_k(s,\theta)=\frac{\nu}{b\nu-a\lambda}\widetilde{p}_{k0}(s)\exp\int\limits_0^\theta\frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu+$$

$$+\int\limits_0^\theta\exp\int\limits_\chi^\theta\frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu\cdot\frac{\chi^{k+1}}{a(\nu+s)\chi(\chi-\theta_0)}d\chi-$$

$$-\widetilde{p}_{k0}(s)\int\limits_0^\theta\exp\int\limits_\chi^\theta\frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu\cdot\frac{\nu(1-\chi)}{a(\nu+s)\chi(\chi-\theta_0)}d\chi. \tag{9}$$

Further, writing the first equation of the system (6) in the form

$$\widetilde{p}_k(s,\theta)=\frac{1}{s}\left[\theta^k+\nu\left(\frac{1}{\theta}-1\right)+\lambda(\theta)\widetilde{q}_k(s,\theta)-\nu\left(\frac{1}{\theta}-1\right)\widetilde{p}_{k0}(s)\right],$$

and taking into account the found expression (9) for $\widetilde{q}_k(s,\theta)$, from the second equation of (6) we have

$$a\theta\left[\frac{\nu}{b\nu-a\lambda}\widetilde{p}_{k0}(s)\exp\int\limits_0^\theta\frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu\cdot\frac{a\lambda(\theta)-b(s+\nu)(\theta-\theta_0)}{a(\nu+s)\theta(\theta-\theta_0)}+\right.$$

$$+\frac{\theta^{k+1}}{a(\nu+s)\theta(\theta-\theta_0)}+\int\limits_0^\theta\exp\int\limits_\chi^\theta\frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu\times$$

$$\times\frac{a\lambda(\theta)-b(s+\nu)(\theta-\theta_0)}{a(\nu+s)\theta(\theta-\theta_0)}\frac{\chi^{k+1}}{a(\nu+s)\chi(\chi-\theta_0)}d\chi-$$

$$-\left(\widetilde{p}_k(s)\,\frac{\nu(1-\theta)}{a(\nu+s)\theta(\theta-\theta_0)}+\int\limits_0^\theta \exp\int\limits_\chi^\theta \frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu\times\right.$$
$$\left.\times\frac{a\lambda(\theta)-b(s+\nu)(\theta-\theta_0)}{a(\nu+s)\theta(\theta-\theta_0)}\frac{\nu(1-\chi)}{a(\nu+s)\chi(\chi-\theta_0)}d\chi\right)\Bigg]=$$
$$=\frac{1}{s}\left[\theta^k+\nu\left(\frac{1}{\theta}-1\right)+\lambda(\theta)\,\widetilde{q}_k(s,\theta)-\nu\left(\frac{1}{\theta}-1\right)\widetilde{p}_{k0}(s)\right]-b\widetilde{q}_k(s,\theta).$$

Introduce new denotation

$$A(s,\theta)=\frac{\nu}{b\nu-a\lambda}\int\limits_0^\theta \frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu\frac{a\lambda(\theta)-b(s+\nu)(\theta-\theta_0)}{a(\nu+s)\theta(\theta-\theta_0)},$$

$$B(s,\theta)=\frac{\nu(1-\theta)}{a(\nu+s)\theta(\theta-\theta_0)}+\int\limits_0^\theta \exp\int\limits_\chi^\theta \frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu\times$$
$$\times\frac{a\lambda(\theta)-b(s+\nu)(\theta-\theta_0)}{a(\nu+s)\theta(\theta-\theta_0)}\frac{\nu(1-\chi)}{a(\nu+s)\chi(\chi-\theta_0)}d\chi,$$

$$D(s,\theta)=\frac{\nu}{b\nu-a\lambda}\exp\int\limits_0^\theta \frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu,$$

$$E(s,\theta)=\int\limits_0^\theta \exp\int\limits_\chi^\theta \frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu,$$

$$F(s,\theta)=\int\limits_0^\theta \exp\int\limits_\chi^\theta \frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu,$$

$$M(s,\theta)=\frac{\theta^{k+1}}{a(\nu+s)\theta(\theta-\theta_0)}+\int\limits_0^\theta \exp\int\limits_\chi^\theta \frac{a\lambda(\mu)-b(s+\nu)(\mu-\theta_0)}{a(\nu+s)\mu(\mu-\theta_0)}d\mu\times$$
$$\times\frac{a\lambda(\theta)-b(s+\nu)(\theta-\theta_0)}{a(\nu+s)\theta(\theta-\theta_0)}\frac{\chi^{k+1}}{a(\nu+s)\chi(\chi-\theta_0)}d\chi.$$

Passing to these denotation in relations (9) and (10), we get:

$$a\theta A(s,\theta)\,\widetilde{p}_{k0}(s)-a\theta B(s,\theta)\,\widetilde{p}_{k0}(s)-$$
$$-\frac{1}{s}\lambda(\theta)\,\widetilde{p}_{k0}(s)\,D(s,\theta)+\frac{1}{s}\lambda(\theta)\,\widetilde{p}_{k0}(s)\,E(s,\theta)+$$
$$+\frac{1}{s}\nu\left(\frac{1}{\theta}-1\right)\widetilde{p}_{k0}(s)+b\widetilde{p}_{k0}(s)\,D(s,\theta)+b\widetilde{p}_{k0}(s)\,D(s,\theta)-b\widetilde{p}_{k0}(s)\,E(s,\theta)=$$
$$=\frac{1}{s}\left[\theta^k+\nu\left(\frac{1}{\theta}-1\right)\right]+\frac{1}{s}\lambda(\theta)\,F(s,\theta)-b(s,\theta)-a\theta M(s,\theta),$$

whence we find the Laplace transform $\widetilde{p}_{k0}(s)$ of transitional probability $p_{k0}(t)$, $t\geq 0, k\in\{0,1,...\}$ in the form

$$\widetilde{p}_{k0}(s)=\frac{\frac{1}{s}\left[\theta^k+\nu\left(\frac{1}{\theta}-1\right)\right]+\left[\frac{1}{s}\lambda(\theta)-b\right]F(s,\theta)-a\theta M(s,\theta)}{a\theta A(s,\theta)-a\theta B(s,\theta)+\left[2b-\frac{1}{s}\lambda(\theta)\right]D(s,\theta)+\left[\frac{1}{s}\lambda(\theta)-b\right]E(s,\theta)+\frac{1}{s}\nu\left(\frac{1}{\theta}-1\right)}. \quad (11)$$

In spite of the fact that the obtained expression for $\widetilde{p}_{k0}(s)$ do contain the parameter θ, the expression

$$\widetilde{p}_{k0}(s) = \int_0^\infty e^{-st} p_{k0}(t)\, dt$$

does not depend on the parameter θ, therefore if we differentiate the right hand side of (11) with respect to θ, we get 0. From the second equation of the system (6) we have

$$\widetilde{p}_k(s,\theta) = \frac{\theta^k}{s} + \frac{\nu}{s}\left(\frac{1}{\theta}-1\right) + \frac{1}{s}\lambda(\theta)\,\widetilde{q}_k(s,\theta) - \frac{\nu}{s}\left(\frac{1}{\theta}-1\right)\widetilde{p}_{k0}(s),$$

that contains the known variables $\widetilde{p}_{k0}(s)$ and $\widetilde{q}_k(s,\theta), k \in \{0,1,...\}$. Knowledge of $\widetilde{p}_k(s,\theta), k \in \{0,1,...\}$ enables to find the inverse Laplace transform $p_k(t,\theta)$, i.e.,

$$p_k(t,\theta) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{st}\widetilde{p}_k(s,\theta)\, dt, \quad k = 0,1,...$$

We can establish that if the chain $\xi_t, t \geq 0$ is ergodic, i.e., there exists the limit

$$\lim_{t\to\infty} p_{kr}(t) = p_r, k, r \in \{0,1,...\}$$

and

$$P(\theta) = \sum_{r=0}^{\infty} p_r\theta^r, \quad |\theta| \leq 1$$

then in the case of the system M/M/1 for $a > 0$ and $\sum\limits_{k=1}^{\infty} k\lambda_k < \infty$ for $P(\theta)$ it holds the following formula

$$P(\theta) = p_0\left[1 + \frac{\theta}{a\nu} + \frac{\lambda(\theta)}{\theta-1}\int_0^\theta \left(\frac{u}{\theta}\right)^{b/a} \exp\left\{\frac{1}{a\nu}\int_u^\theta \frac{\lambda(\omega)}{\omega-1}d\omega\right\}\frac{du}{u}\right]$$

where p_0 is found from the normalization condition $\sum\limits_{k=0}^{\infty} p_k = 1$.

3. Conclusion

In the paper, by the two processes ξ_t and $\eta(t), t \geq 0$ with independent increments; one of which is without negative overshoots and the second one is homogeneous in time, we study a homogeneous Markov process $\xi_t, t \geq 0$ given in the form

$$d\xi_t = sign\xi_t \cdot d\xi_t + \eta^*\left(\frac{dt}{a\xi_t + b}\right), \quad a \geq 0, \quad b > 0,$$

where $\eta^*(x)$ has the same distribution as $\eta(x) - \eta(0), \quad x \geq 0$.

In the lattice case, for $a = 0$ and integrity of phase spaces of the processes ξ_t and $\eta(t)$, $t \geq 0$ we find the Laplace transform of the generating function $P_k(t,\theta)$ of transitional probabilities $\{P_{kr}(t)\}, \quad t \geq 0, k, r \in \{0,1,...\}$ of the process $\xi_t, t \geq 0$.

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