TWMS J. App. Eng. Math. V.5, N.1, 2015, pp. 74-79.

GRAPHS WITH EQUAL DOMINATION AND INDEPENDENT DOMINATION NUMBER

S. K. VAIDYA[1], R. M. PANDIT[2], §

Abstract. A set S of vertices of a graph G is an independent dominating set of G if S is an independent set and every vertex not in S is adjacent to a vertex in S. The independent domination number of G, denoted by $i(G)$, is the minimum cardinality of an independent dominating set of G. In this paper, some new classes of graphs with equal domination and independent domination numbers are presented and exact values of their domination and independent domination numbers are determined.

Keywords: Dominating set, domination number, independent dominating set, independent domination number.

AMS Subject Classification: 05C69, 05C76.

1. Introduction

The domination in graphs is one of the concepts in graph theory which has attracted many researchers to work on it. Many variants of domination models are available in existing literature - Independent domination, Total domination, Global domination, Edge domination, just to name a few. Independent sets play an important role in graph theory and other area like discrete optimization. They appear in matching theory, coloring of graphs and in theory of trees. The present paper is focused on independent domination in graphs.

We begin with simple, finite, connected and undirected graph G with vertex set $V(G)$ and edge set $E(G)$. The set $S \subseteq V(G)$ of vertices in a graph G is called a dominating set if every vertex $v \in V(G)$ is either an element of S or is adjacent to an element of S. A dominating set S is called a minimal dominating set(MDS) if no proper subset S' of S is a dominating set.

The minimum cardinality of a minimal dominating set in G is called the domination number of G, denoted by $\gamma(G)$ and the corresponding dominating set is called a γ-set of G.

An independent set in a graph G is a set of pairwise non-adjacent vertices of G. A set S of vertices in a graph G is called an independent dominating set if S is both an independent and a dominating set of G. The independent domination number $i(G)$ of a graph G is the minimum cardinality of an independent dominating set.

[1] Department of Mathematics, Saurashtra University, Rajkot - 360005, Gujarat, India.
e-mail: samirkvaidya@yahoo.co.in;

[2] Department of Mathematics, A. V. Parekh Technical Institute, Rajkot - 360001, Gujarat, India.
e-mail: pandit.rajesh@ymail.com;

§ Manuscript received: May 29, 2014.

The theory of independent domination was formalized by Berge [4] and Ore [14] in 1962. The independent domination number and the notation $i(G)$ were introduced by Cockayne and Hedetniemi in [5, 6]. Independent dominating sets and variations of independent dominating sets are now extensively studied in the literature; see for example [3, 11, 15]. Independent dominating sets in regular graphs and in cubic graphs in particular, are also well studied; see for example [10, 9, 13]. Favaron [7] initiated the quest of finding sharp upper bounds for independent domination number in general graphs, as a function of order n and minimum degree $\delta(G)$. This work was extended by Sun and Wang [16]. Allan and Laskar [2] have shown that $K_{1,3}$ - free graphs are graphs with equal domination and independent domination numbers. The work of Allan and Laskar [2] is generalized by Topp and Volkmann [17] as well as by Acharya and Gupta [1]. The present work is aimed to investigate some new graphs whose domination number equals their independent domination number.

Notation. When discussing any graph G, we let p denote the cardinality of $V(G)$. For a vertex $v \in V(G)$, the open neighborhood of v, denoted by $N(v)$, is $\{u \in V(G) : uv \in E(G)\}$. We denote the degree of a vertex v in graph G by $d(v)$. The maximum degree (respectively minimum degree) among the vertices of G is denoted by $\triangle(G)$ (respectively $\delta(G)$).

For any real number n, $\lceil n \rceil$ denotes the smallest integer not less than n and $\lfloor n \rfloor$ denotes the greatest integer not greater than n. We denote by P_n the path on n vertices, C_n the cycle on n vertices and $K_{r,s}$ the complete bipartite graph with partite sets of size r and s.

For notation and graph theoretic terminology not defined herein, we refer the reader to West [18] while the terms related to the concept of domination are used in the sense of Haynes *et al.* [12].

2. Main Results

Definition 2.1. *Duplication of a vertex v_k by a new edge in a graph G produces a new graph G' by adding an edge $e' = u'v'$ such that $N(v') = \{v_k, u'\}$ and $N(u') = \{v_k, v'\}$.*

Theorem 2.1. *If G_1 is the graph obtained by duplication of each vertex of any graph G by a new edge, then $\gamma(G_1) = i(G_1) = p$.*

Proof. Let G be any graph with $\mid V(G) \mid = p$ and $\mid E(G) \mid = q$ and let each vertex of the graph G be duplicated by a new edge. Then, the resultant graph G_1 will have $3p$ vertices and $3p + q$ edges. Also, the resultant graph G_1 will have p vertex disjoint cycles, each of length three. To dominate these p disjoint cycles, at least p distinct vertices of G_1, one from each cycle, are required. These p vertices also dominate G_1. Hence, $\gamma(G_1) = p$. Moreover, it is also possible to take p pairwise non-adjacent vertices which will dominate the graph G_1. Therefore, for any independent dominating set S of G_1, $\mid S \mid \geq p$ implies that $i(G_1) = p$. Thus, $\gamma(G_1) = i(G_1) = p$. □

Definition 2.2. *Duplication of an edge $e = uv$ by a new vertex in a graph G produces a new graph G' by adding a vertex v' such that $N(v') = \{u, v\}$.*

Proposition 2.1. [14] *A dominating set S is a minimal dominating set if and only if for each vertex $u \in S$, one of the following two conditions holds:*
(a) u is an isolate of S,
(b) there exists a vertex $v \in V - S$ for which $N(v) \cap S = \{u\}$.

Theorem 2.2. *If G is the graph obtained by duplication of each edge of path P_n by a new vertex, then $\gamma(G) = i(G) = \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of path P_n and let $e_1, e_2, \ldots, e_{n-1}$ be the edges of path P_n which are duplicated by new vertices $u_1, u_2, \ldots, u_{n-1}$ respectively. Then, the resultant graph G will have $(2n-1)$ vertices and $3(n-1)$ edges.
First, we construct a vertex set $S \subset V(G)$ as follows:

$$S = \{v_2, v_4, v_6, \ldots, v_{2i}\} \text{ where } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ with } \mid S \mid = \left\lfloor \frac{n}{2} \right\rfloor.$$

Since each vertex in G is either in S or is adjacent to a vertex in S, it follows that the above set S is a dominating set of G. The set S is also an independent set of G because no two vertices in S are adjacent. Therefore, the set S is an independent dominating set of G.

For $n = 2$, the set $S = \{v_2\}$ is obviously an independent dominating set of G with minimum cardinality. Hence, $i(G) = 1 = \left\lfloor \frac{n}{2} \right\rfloor$ for $n = 2$.

For $n \geq 3$, $\triangle(G) = 4$ and $\mid S \mid = \left\lfloor \frac{n}{2} \right\rfloor$. Suppose, if possible, a vertex set $S_1 \subseteq V(G)$, $S_1 \neq S$ is an independent dominating set of G with $\mid S_1 \mid = \left\lfloor \frac{n}{2} \right\rfloor - 1 < \mid S \mid$. Now, to attain the minimum cardinality of S_1, only one vertex from $\left\lfloor \frac{n}{2} \right\rfloor - 1$ vertices of S_1 will dominate four distinct vertices of G and the remaining vertices of S_1 will dominate at most three distinct vertices of G. Hence, the set S_1 can dominate at most $4 + 3(\left\lfloor \frac{n}{2} \right\rfloor - 2) = 3\left\lfloor \frac{n}{2} \right\rfloor - 2$ vertices of G. But $3\left\lfloor \frac{n}{2} \right\rfloor - 2 < 2n - 2 < 2n - 1 = p$. Therefore, S_1 is not a dominating set of G, which is a contradiction. Hence, the above set S is an independent dominating set of G with minimum cardinality $\left\lfloor \frac{n}{2} \right\rfloor$. Thus, $i(G) = \left\lfloor \frac{n}{2} \right\rfloor$.

Moreover, for each vertex $u \in S$, there exists a vertex
$v \in V(G) - S$ for which $N(v) \cap S = \{u\}$. Therefore, by Proposition 2.1., the set S is a minimal dominating set of G. Hence, the set S is an MDS with minimum cardinality. Therefore, $\gamma(G) = \left\lfloor \frac{n}{2} \right\rfloor$.
Thus, $\gamma(G) = i(G) = \left\lfloor \frac{n}{2} \right\rfloor$. □

Definition 2.3. *The middle graph of a connected graph G denoted by $M(G)$ is the graph whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if*
(i) they are adjacent edges of G or
(ii) one is a vertex of G and the other is an edge incident with it.

Lemma 2.1. $\gamma(M(C_n)) = \left\lfloor \frac{n+1}{2} \right\rfloor$.

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and let $u_1, u_2, \ldots, u_n$ be the added vertices corresponding to the edges $e_1, e_2, \ldots, e_n$ of C_n to obtain $M(C_n)$. Then,
$\mid V(M(C_n)) \mid = 2n$ and $\mid E(M(C_n)) \mid = 3n$.
Now, we construct a vertex set $S \subset V(M(C_n))$ as follows:

$$S = \{v_2, v_4, v_6, \ldots, v_{2i}\} \cup \{v_n\} \text{ where } 1 \leq i \leq \left\lfloor \frac{n-1}{2} \right\rfloor \text{ with } \mid S \mid = \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Since each vertex in $V(M(C_n))$ is either in S or is adjacent to a vertex in S, it follows that the above set S is a dominating set of $M(C_n)$. Moreover, for each vertex $u \in S$, there exists a vertex $v \in V(M(C_n)) - S$ for which $N(v) \cap S = \{u\}$. Therefore, by Proposition 2.1., the set S is a minimal dominating set of $M(C_n)$.

Now, any two consecutive added vertices in $M(C_n)$ are adjacent to a common vertex. Therefore, at least $\left\lfloor \frac{n+1}{2} \right\rfloor$ vertices are required to dominate the added vertices $u_1, u_2, \ldots, u_n$ of $M(C_n)$. Moreover, these vertices also dominate the remaining vertices of C_n. That is, at least $\left\lfloor \frac{n+1}{2} \right\rfloor$ vertices are required to dominate $M(C_n)$ which implies that the above set S is of minimum cardinality.

Thus, the above set S is a minimal dominating set of $M(C_n)$ with minimum cardinality $\lfloor \frac{n+1}{2} \rfloor$ implying that $\gamma(M(C_n)) = \lfloor \frac{n+1}{2} \rfloor$. □

Proposition 2.2. [2] *For any graph G, $\gamma(M(G)) = i(M(G))$ where $M(G)$ denotes the middle graph of G.*

Theorem 2.3. *If G is the graph obtained by duplication of each edge of C_n by a new vertex, then $\gamma(G) = i(G) = \lfloor \frac{n+1}{2} \rfloor$.*

Proof. Let $v_1, v_2, \ldots, v_n$ be the vertices of cycle C_n and let $e_1, e_2, \ldots, e_n$ be the edges of cycle C_n which are duplicated by new vertices $u_1, u_2, \ldots, u_n$ respectively. Then, for the resultant graph G, $\mid V(G) \mid = 2n$ and $\mid E(G) \mid = 3n$.

Now, by Lemma 2.1., $\gamma(M(C_n)) = \lfloor \frac{n+1}{2} \rfloor$ and according to Proposition 2.2., $i(M(G_1)) = \gamma(M(G_1))$ where $M(G_1)$ denotes the middle graph of any graph G_1. Since G is the middle graph of C_n, it follows that $i(G) = \gamma(M(C_n)) = \lfloor \frac{n+1}{2} \rfloor$.
Thus, $\gamma(G) = i(G) = \lfloor \frac{n+1}{2} \rfloor$. □

Proposition 2.3. [8] $i(K_{r,s}) = min\ (r, s)$ *where $K_{r,s}$ denotes the complete bipartite graph.*

Theorem 2.4. *If G is the graph obtained by duplication of each edge of the complete bipartite graph $K_{r,s}$ with $r \leq s$ by a new vertex, then $\gamma(G) = i(G) = i(K_{r,s}) = min\ (r, s)$. That is, the independent domination number remains invariant under the operation of duplication of each edge by a new vertex in $K_{r,s}$.*

Proof. Let $K_{r,s}$ be a complete bipartite graph with a bipartition into two sets namely, X and Y with $r \leq s$. Let $x_1, x_2, \ldots, x_r \in X$ and $y_1, y_2, \ldots, y_s \in Y$. Let G be the graph obtained by duplication of each edge of $K_{r,s}$ by a new vertex. Let $e_1, e_2, \ldots, e_{rs}$ be the edges of $K_{r,s}$ which are duplicated by the vertices $v_1, v_2, \ldots, v_{rs}$ respectively. Then, $\mid V(G) \mid = r + s + rs$ and $\mid E(G) \mid = 3rs$.
First, we construct a vertex set $S \subset V(G)$ as follows:

$$S = \{x_1, x_2, \ldots, x_r\} \text{ with } \mid S \mid = r.$$

Since for every vertex $v \in V(G) - S$, there exists a vertex $u \in S$ such that v is adjacent to u, it follows that S is a dominating set of G. Moreover, the above set S is an independent set of G because the vertices in S are pairwise non-adjacent vertices. Therefore, the above set S is an independent dominating set of G.

Now, each partite set of a bipartite graph is an independent set and each of them dominates the other. Hence, at least $min\ (r, s)$ pairwise non-adjacent vertices are required to dominate $K_{r,s}$. Moreover, these vertices also dominate the vertices $v_1, v_2, \ldots, v_{rs}$ of G. Therefore, for any independent dominating set S_1 of G, $\mid S_1 \mid \geq min\ (r, s)$. Thus, the above set S is an independent dominating set of G with minimum cardinality implying that $i(G) = min\ (r, s)$.

Moreover, for each vertex $u \in S$, there exists a vertex $v \in V(G) - S$ for which $N(v) \cap S = \{u\}$. Therefore, by Proposition 2.1., the set S is a minimal dominating set of G. Hence, the set S is an MDS with minimum cardinality. Therefore, $\gamma(G) = min\ (r, s)$.
Thus, $\gamma(G) = i(G) = min\ (r, s)$.

Furthermore, by Proposition 2.3., $i(K_{r,s}) = min\ (r, s)$. Hence, $i(K_{r,s}) = i(G) = min\ (r, s)$. That is, the independent domination number remains invariant under the operation of duplication of each edge by a new vertex in $K_{r,s}$. □

3. Concluding Remarks

Allan and Laskar [2] have proved that if G is a claw-free graph then $\gamma(G) = i(G)$. Therefore, it is interesting and challenging as well to find out the graphs G containing claw as an induced subgraph with $\gamma(G) = i(G)$. Such graphs are presented in Theorem 2.1 and Theorem 2.4. It remains an open problem: When does the independent domination number equal the domination number for a graph G?

References

[1] Acharya, B. D. and Gupta, P., (2003), On graphs whose domination numbers equal their independent domination numbers, Electronic Notes in Discrete Math., 15, pp. 2-4.
[2] Allan, R. B. and Laskar, R., (1978), On domination and independent domination numbers of a graph, Discrete Math., 23, pp. 73-76.
[3] Ao, S., Cockayne, E. J., MacGillivray, G. and Mynhardt, C. M., (1996), Domination critical graphs with higher independent domination numbers, J. Graph Theory, 22, pp. 9-14.
[4] Berge, C., (1962), Theory of Graphs and its Applications, Methuen, London.
[5] Cockayne, E. J. and Hedetniemi, S. T., (1974), Independent graphs, Congr. Numer., X, pp. 471-491.
[6] Cockayne, E. J. and Hedetniemi, S. T., (1977), Towards a theory of domination in graphs, Networks 7, pp. 247-261.
[7] Favaron, O., (1988), Two relations between the parameters of independence and irredundance, Discrete Math., 70, pp. 17-20.
[8] Goddard, W., Henning, M., Lyle, J. and Southey, J., (2012), On the independent domination number of regular graphs, Ann. Comb., 16, pp. 719-732.
[9] Goddard, W. and Henning, M., (2013), Independent domination in graphs: A survey and recent results, Discrete Math., 313, pp. 839-854.
[10] Haviland, J., (1995), Independent domination in regular graphs, Discrete Math., 143, pp. 275-280.
[11] Haxell, P., Seamone, B. and Verstraëte, J., (2007), Independent dominating sets and Hamiltonian cycles, J. Graph Theory, 54, pp. 233-244.
[12] Haynes, T. W., Hedetniemi, S. T. and Slater, P. J., (1998), Fundamentals of Domination in Graphs, Marcel Dekker Inc., New York.
[13] Kostochka, A. V., (1993), The independent domination number of a cubic 3-connected graph can be much larger than its domination number, Graphs Combin., 9, pp. 235-237.
[14] Ore, O., (1962), Theory of graphs, Amer. Math. Soc. Transl., 38, pp. 206-212.
[15] Shiu, W. C., Chen, X. and Chan, W. H., (2010), Triangle-free graphs with large independent domination number, Discrete Optim., 7, pp. 86-92.
[16] Sun, L. and Wang, J., (1999), An upper bound for the independent domination number, J. Combin. Theory. Ser., B76, pp. 240-246.
[17] Topp, J. and Volkmann, L., (1991), On graphs with equal domination and independent domination numbers, Discrete Math., 96, pp. 75-80.
[18] West, D. B., (2003), Introduction to Graph Theory, Prentice-Hall of India, New Delhi.

S. K. Vaidya was born in 1961. He got his B. Sc. Degree and M. Sc. Degree in mathematics in 1982 and 1984 respectively. He obtained his Ph. D. degree in 1996 from Saurashtra University, Rajkot-360 005, Gujarat, India. At present he is working as a professor of mathematics at the department of mathematics in the Saurashtra University, which is one of the best universities in western India. He is having 30 years of teaching experience and published several research papers in scholarly and peer reviewed journals. He finds interest in the area of graph labeling and domination in graphs. He has been also serving as a referee and editorial board member for many journals of international repute.

R. M. Pandit was born in 1980. He got his B. Sc. Degree and M. Sc. Degree in mathematics in 2001 and 2003 respectively. He obtained his M. Phil. Degree in 2007. At present he is working as a Lecturer in Mathematics at the department of mathematics in the A. V. Parekh Technical Institute at Rajkot in Gujarat. He has published five research papers in scholarly and peer reviewed journals.