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# ANALYSIS OF DISCRETE-TIME QUEUE WITH TWO HETEROGENEOUS SERVERS SUBJECT TO CATASTROPHES

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Abstract. This paper studies a discrete-time queueing system with two heterogeneous servers subject to catastrophes. We obtain explicit expressions for the steady-state probabilities at arbitrary epoch using displacement operator method. The waiting time distribution and outside observer's observation epoch probabilities are deduced. Various performance measures and numerical results have been investigated.

Keywords: Discrete-time, catastrophe, multi-server, heterogeneous servers, waiting, queue.

AMS Subject Classification: Primary 60K25; Secondary 90B22, 68M20.

## 1. Introduction

Queueing models with catastrophes gained considerable interest during the last few decades due to their applications in the analysis of computer- and communication systems where catastrophes induced by external effects have an important influence on costs and performance from an economic viewpoint. Whenever a catastrophe occurs at the system, all the customers present are forced to abandon the system immediately, the server gets inoperative instantaneously, and the server is ready for service when a new customer arrives. The modeling and analysis of queueing systems with catastrophes may be used to study the migration processes with catastrophes and computer networks with virus infections or a reset order.

Queueing systems with catastrophes have been investigated by many researchers (Chao [1], Chen and Renshaw [2], Di Crescenzo et al. [3] and Boudali and Economou [4]). The catastrophes occur as negative customers to the system and its characteristic is to annihilate all the customers in the system and the momentary deactivation of the service facilities till a new arrival of customers. The catastrophes might arrive either from outside the system or from some other service station. In a queueing system, whenever catastrophe occurs, it may be thought of as a clearing mechanism which causes all the jobs in the system to be lost. If a job infected with virus in computer systems, it carries virus to other processors deactivating files and perhaps the system itself. It has enormous applications in a broad areas especially in computer communication, industries, biosciences and population genetics. In real-world catastrophes appear in various situations in practice, for example, in the production sector, in the service sector, in the health care sector, in population genetics, in the transportation sector, in the telecommunication industry, etc. In most of the above cases, there is some sort of compensation for the jobs. Thus, the

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economic analysis of queueing systems with catastrophes who are forced to vacate the system pretend to be of concern from an applications point of view. Some articles on continuous-time queueing systems with catastrophes can be found in Kumar and Arivudainambi [5], Kumar and Madheswari [6], Kumar et al. [7]. The strategic behavior and social optimization in case of heterogeneous customers with Markovian vacation queues has been discussed in Guo and Hassin [8].

The study on multi-server queueing systems in general presumes the servers to be homogeneous. The heterogeneous service rates have many practical aspects in modeling real systems that permit customers to meet different qualities of service. For example, communications network supporting communication channels of various transmission rates, nodes in wireless systems serving different mobile users, nodes in telecommunications network with links of various capacities, servers formed with different processors as a consequence of system updates, multiprogramming computer system which spools its output for printing on a set of printers of different speeds, or scheduling jobs on functionally equivalent processors of a local computer network, manual assembly formed with different workers with the average task completion time differing from person to person, machines undergoing a process of rapid and constant technological renewal and depreciation, the transportation of goods with different abilities and capacities, etc. involve heterogeneous servers. The firms must give attention to the quality and service performance when designing and carrying out their operations as these are requirements in customer perceptions. In a heterogeneous environment, resources are autonomous, distributed, dense, and dynamic, hence they should be effectively scheduled so that maximum utilization of the resources is possible. As a result, heterogeneous multi-server queues can be used to obtain more insight into these systems and thus make them more manageable. But literature on this class is limited to the servers having homogeneous service rates as it simplifies the analysis. For more details on this topic, see Larsen and Agrawala [9] and Lin and Kumar [10].

The analysis of two heterogeneous servers queue subject to catastrophes in continuous-time has been carried out by Kumar et al. [11]. To evaluate system performance measures, discrete-time queueing models are better suited than their continuous-time counterparts for studying slotted digital computer communication systems, including mobile and broad integrated services digital networks (B-ISDN). It is more accurate and efficient than their continuous-time counterparts to analyze and design digital transmitting systems. Moreover, the modelling of discrete-time queues is more involved and rather different from the analysis applied for the corresponding continuous-time queueing models. The advantage of analyzing a discrete-time queue is that one can get the continuous-time results from it as a limiting case but the converse is not true. Comprehensive discussion of various kind of discrete-time queueing models can be found in Hunter [12], Gravey and Hébuterne [13], Bruneel and Kim [14], Takagi [15], Woodward [16]. The discrete-time $Geo/Geo/1$ queue with negative customers and disasters has been studied in Atencia and Moreno [17]. Multi-server discrete-time queueing systems $Geo/Geo/c$ have been reported in Goswami and Gupta [18] and Artalejo and Hernandez-Lerma [19]. Discrete-time two heterogeneous servers bulk-service infinite buffer queueing system has been discussed in Goswami and Samanta [20]. However, to the best of our knowledge, studies for the discrete-time two heterogeneous servers queueing system with catastrophes do not yet exist. It is the aim of the present paper to study an infinite buffer discrete-time two heterogeneous servers queueing system subject to catastrophes.

In the present paper, we investigate an infinite buffer discrete-time two heterogeneous servers queueing system with catastrophes, that is, two servers working with different service rates with possibility of catastrophe at the system. If both the servers are idle then an arriving customer always joins the server I. An arriving customer waits in a queue when both servers are busy. Whenever a catastrophe occurs at the system, all the customers present are destroyed instantly, both the servers become deactivated momentarily and the servers are ready for service to new arrivals. The inter-arrival times of customers and catastrophe times are assumed to be independent and geometrically distributed. The service times are also assumed to be independent and geometrically distributed with different mean service time at different servers.

This paper is organized as follows. The description of the queueing model and its analysis for the steady-state probabilities at arbitrary epoch is carried out in Section 2. Outside observer's distributions have been presented in Section 3. The waiting time analysis is carried out in Section 4. Some particular cases which are matched with existing results in the literature are demonstrated in Section 5. Numerical results to demonstrate the effect of the catastrophe on the behavior of the customers and on the various performance measures of the system are presented in Section 6. Section 7 concludes the paper. Finally, in the Appendix it has been shown that in the limiting case the results obtained in this paper tend to the continuous-time counterpart.

## 2. Model description and solution

We consider a discrete-time infinite waiting space queue with two heterogeneous servers under the early arrival system (EAS) and the late arrival system with delayed access (LAS-DA), which are also known as departure-first (DF) and arrival-first (AF) policies, respectively. Assume that the time axis is slotted into intervals of equal length with the length of a slot being unity, and it is marked as $0, 1, 2, \ldots, t, \ldots$. The detailed discussion about these concepts has been explained in the past at several places, see, e.g., Hunter [12] or Gravey and Hébuterne [13]. We assume that the inter-arrival times $A$ of jobs are independent and geometrically distributed with probability mass function (p.m.f.) $a_n = P(A = n) = (1 - \lambda)^{n-1}\lambda,\ n \geq 1$. The jobs are served by two heterogeneous servers with different mean service times. The service times of customers are independent and geometrically distributed with p.m.f. $P(S_i = n) = (1 - \mu_i)^{n-1}\mu_i,\ n \geq 1,\ i = 1, 2$. An arriving job waits in the queue when both the servers are busy. The catastrophes occur at the service-facility as a geometrically distributed with p.m.f. $P(B = n) = (1-\gamma)^{n-1}\gamma,\ n \geq 1$. Whenever a catastrophe occurs at the system, all the customers present are annihilated immediately, the server gets deactivated instantaneously, and the server is ready for service when a new job arrives. For any real number $x \in [0, 1]$, we denote $\overline{x} = 1 - x$.

If both the servers are idle, it is assumed that an arriving job always joins the server I. At every departure epoch, that is, before initiating service of the next job, we have any one of the following cases: $(i)$ both the servers are idle and there is no job waiting in the queue, $(ii)$ the server I is busy, the server II is idle, and no job is in the queue, $(iii)$ the server I is idle, the server II is busy, and no job is in the queue, and $(iv)$ both the servers are busy and there are $n \geq 0$ jobs waiting in the queue. To ensure the stability of the system, without loss of generality, we assume that $\rho = \lambda/(\mu_1 + \mu_2) < 1$.

### 2.1. The EAS system.

We first discuss the model for the early arrival system (EAS), that is, departure-first (DF) policy. A potential arrival occurs in the interval $(t, t+)$ and potential departures occur in the interval $(t-, t)$. The various time epochs at which events occur are depicted in Figure 1.

Let $Q_{0,0}(t)$ denote the probability that both the servers are idle at time $t$. Let $Q_{1,0}(t)$

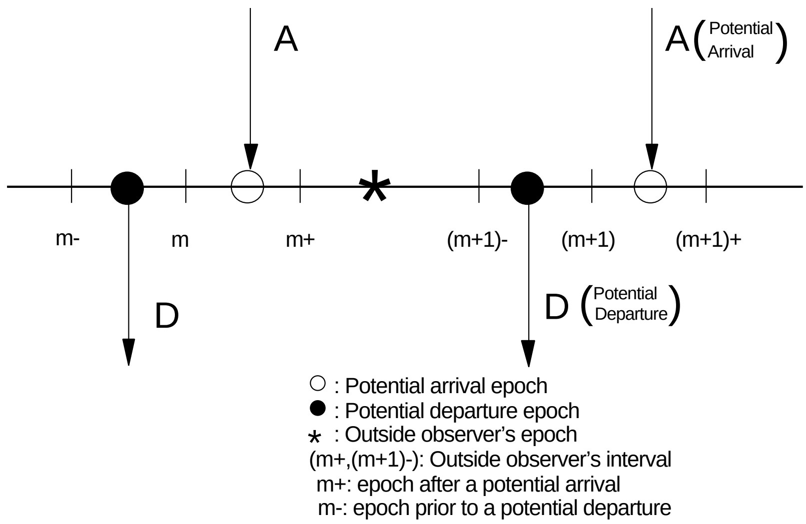


FIGURE 1. Various time epochs in EAS.

denote the probability that the server I is busy and the server II is idle, and $Q_{0,1}(t)$ denote the probability that the server I is idle and the server II is busy, when the queue is empty at time $t$. Further, let $Q_{n,2}(t)$ be the probability that both the servers are busy and $n \geq 0$ customers waiting in the queue at time $t$. In order to obtain the steady-state probabilities, we first construct the difference equations by relating the states of the system at two consecutive prior to potential arrival epochs $t$ and $(t+1)$. Using the probabilistic argument, we obtain

$$
\begin{aligned}
Q_{0,0}(t+1) &= \bar{\lambda} Q_{0,0}(t) + \bar{\gamma}\mu_1\left(\bar{\lambda}+\lambda\mu_2\right)Q_{1,0}(t) + \bar{\gamma}\mu_2\left(\bar{\lambda}+\lambda\mu_1\right)Q_{0,1}(t) \\
&\quad + \bar{\lambda}\mu_1\mu_2\bar{\gamma}Q_{0,2}(t) + \lambda\mu_1 Q_{0,0}(t) + \gamma(1-Q_{0,0}(t)), && (1)\\
Q_{1,0}(t+1) &= \bar{\gamma}\bar{\mu}_1\left(\bar{\lambda}+\lambda\mu_2\right)Q_{1,0}(t) + \lambda\bar{\mu}_1\mu_2\bar{\gamma}Q_{0,1}(t) + \lambda\bar{\mu}_1 Q_{0,0}(t) + \bar{\lambda}\mu_1\mu_2\bar{\gamma}Q_{1,2}(t) \\
&\quad + \bar{\gamma}\mu_2\left(\bar{\lambda}\bar{\mu}_1+\lambda\mu_1\right)Q_{0,2}(t), && (2)\\
Q_{0,1}(t+1) &= \bar{\gamma}\bar{\mu}_2\left(\bar{\lambda}+\lambda\mu_1\right)Q_{0,1}(t) + \lambda\mu_1\bar{\mu}_2\bar{\gamma}Q_{1,0}(t) + \bar{\lambda}\mu_1\bar{\mu}_2\bar{\gamma}Q_{0,2}(t), && (3)\\
Q_{0,2}(t+1) &= \bar{\gamma}\left(\bar{\lambda}\bar{\mu}_1\bar{\mu}_2+\lambda\mu_1\bar{\mu}_2+\lambda\bar{\mu}_1\mu_2\right)Q_{0,2}(t) + \lambda\bar{\mu}_1\bar{\mu}_2\bar{\gamma}\left(Q_{1,0}(t)+Q_{0,1}(t)\right) \\
&\quad + \bar{\gamma}\left(\bar{\lambda}\mu_1\bar{\mu}_2+\bar{\lambda}\bar{\mu}_1\mu_2+\lambda\mu_1\mu_2\right)Q_{1,2}(t) + \bar{\lambda}\mu_1\mu_2\bar{\gamma}Q_{2,2}(t), && (4)\\
Q_{n,2}(t+1) &= \bar{\gamma}\left(\bar{\lambda}\bar{\mu}_1\bar{\mu}_2+\lambda\mu_1\bar{\mu}_2+\lambda\bar{\mu}_1\mu_2\right)Q_{n,2}(t) + \bar{\lambda}\mu_1\mu_2\bar{\gamma}Q_{n+2,2}(t) \\
&\quad + \bar{\gamma}\left(\bar{\lambda}\mu_1\bar{\mu}_2+\bar{\lambda}\bar{\mu}_1\mu_2+\lambda\mu_1\mu_2\right)Q_{n+1,2}(t) + \lambda\bar{\mu}_1\bar{\mu}_2\bar{\gamma}Q_{n-1,2}(t),\ n\geq 1. && (5)
\end{aligned}
$$

Let us define in the steady-state as

$$Q_{0,0} = \lim_{t\to\infty} Q_{0,0}(t);\ Q_{1,0} = \lim_{t\to\infty} Q_{1,0}(t),\ Q_{0,1} = \lim_{t\to\infty} Q_{0,1}(t);\ Q_{n,2} = \lim_{t\to\infty} Q_{n,2}(t),\ n\geq 0.$$

In the steady-state, above equations (1) - (5) reduce to

$$
\begin{aligned}
0 &= -\lambda\bar{\mu}_1 Q_{0,0} + \bar{\gamma}\mu_1\left(\bar{\lambda}+\lambda\mu_2\right)Q_{1,0} + \bar{\gamma}\mu_2\left(\bar{\lambda}+\lambda\mu_1\right)Q_{0,1} + \bar{\lambda}\mu_1\mu_2\bar{\gamma}Q_{0,2} \\
&\quad + \lambda\mu_1 Q_{0,0} + \gamma(1-Q_{0,0}), && (6)
\end{aligned}
$$

$$\begin{aligned}
0 &= \bar{\gamma}\bar{\mu}_1\left(\bar{\lambda}+\lambda\mu_2-1\right)Q_{1,0}+\lambda\bar{\mu}_1\mu_2\bar{\gamma}Q_{0,1}+\lambda\bar{\mu}_1Q_{0,0}+\bar{\lambda}\mu_1\mu_2\bar{\gamma}Q_{1,2}\\
&\quad+\bar{\gamma}\mu_2\left(\bar{\lambda}\bar{\mu}_1+\lambda\mu_1\right)Q_{0,2}, \qquad (7)\\
0 &= \bar{\gamma}\bar{\mu}_2\left(\bar{\lambda}+\lambda\mu_1-1\right)Q_{0,1}+\lambda\mu_1\bar{\mu}_2\bar{\gamma}Q_{1,0}+\bar{\lambda}\mu_1\bar{\mu}_2\bar{\gamma}Q_{0,2}, \qquad (8)\\
0 &= \bar{\gamma}\left(\bar{\lambda}\bar{\mu}_1\bar{\mu}_2+\lambda\mu_1\bar{\mu}_2+\lambda\bar{\mu}_1\mu_2-1\right)Q_{0,2}+\bar{\gamma}\left(\bar{\lambda}\mu_1\bar{\mu}_2+\bar{\lambda}\bar{\mu}_1\mu_2+\lambda\mu_1\mu_2\right)Q_{1,2}\\
&\quad+\bar{\lambda}\mu_1\mu_2\bar{\gamma}Q_{2,2}, \qquad (9)\\
0 &= \bar{\gamma}\left(\bar{\lambda}\bar{\mu}_1\bar{\mu}_2+\lambda\mu_1\bar{\mu}_2+\lambda\bar{\mu}_1\mu_2\right)Q_{n,2}+\bar{\gamma}\left(\bar{\lambda}\mu_1\bar{\mu}_2+\bar{\lambda}\bar{\mu}_1\mu_2+\lambda\mu_1\mu_2\right)Q_{n+1,2}\\
&\quad+\bar{\lambda}\mu_1\mu_2\bar{\gamma}Q_{n+2,2}+\lambda\bar{\mu}_1\bar{\mu}_2\bar{\gamma}Q_{n-1,2}, \quad n\geq 1. \qquad (10)
\end{aligned}$$

The steady-state probabilities $Q_{0,0}$, $Q_{1,0}$, $Q_{0,1}$, and $Q_{n,2}$, $(n\geq 0)$ are computed by solving the system of equations (6) to (10). In order to obtain them, let us define the displacement operator $E$ as $E^jQ_{n,2}$=$Q_{n+j,2}$. We first solve the difference equation (10), and it can be simplified as

$$\left[\bar{\gamma}\Big(\bar{\lambda}+\lambda E^{-1}\Big)\Big(\bar{\mu}_1+\mu_1E\Big)\Big(\bar{\mu}_2+\mu_2E\Big)-1\right]Q_{n,2}=0. \qquad (11)$$

The characteristic equation associated with (11), after simplification, reduces to

$$h(z)\equiv\bar{\gamma}(\lambda+\bar{\lambda}z)(\bar{\mu}_1+\mu_1z)(\bar{\mu}_2+\mu_2z)-z=0.$$

Using Rouché's theorem it can be shown that only one zero of $h(z)$ falls inside the unit circle and, this root is real and unique if and only if $\rho<1$. We denote this root by $r$, $(0<r<1)$. Then $r$ satisfies the equation

$$\bar{\gamma}(\lambda+\bar{\lambda}r)(\bar{\mu}_1+\mu_1r)(\bar{\mu}_2+\mu_2r)-r=0. \qquad (12)$$

Now the solution of (10) can be written as

$$Q_{n,2}=Cr^n,\ n\geq 0,$$

where $C$ is constant. Setting $n=0$ yields $C=Q_{0,2}$, and hence

$$Q_{n,2}=r^nQ_{0,2},\ n\geq 1. \qquad (13)$$

Using (13) into (9), and simplifying, we obtain

$$Q_{1,0}+Q_{0,1}=\frac{Q_{0,2}}{r}. \qquad (14)$$

Substituting (14) into (8) yields

$$Q_{0,1}=\frac{\mu_1\bar{\mu}_2\bar{\gamma}(\lambda+\bar{\lambda}r)}{r(1-\bar{\lambda}\bar{\mu}_2\bar{\gamma})}Q_{0,2}. \qquad (15)$$

From (14) and (15), we obtain

$$Q_{1,0}=\frac{Q_{0,2}}{r}\left[1-\frac{\mu_1\bar{\mu}_2\bar{\gamma}(\lambda+\bar{\lambda}r)}{(1-\bar{\lambda}\bar{\mu}_2\bar{\gamma})}\right]. \qquad (16)$$

Again making use of (14) to (16) into (6), and after some algebraic manipulation, we obtain

$$Q_{0,0} = \frac{Q_{0,2}}{\lambda\bar{\mu}_1r}\left[(1-\bar{\lambda}\bar{\mu}_1\bar{\gamma})\left\{1-\frac{\mu_1\bar{\mu}_2\bar{\gamma}(\lambda+\bar{\lambda}r)}{(1-\bar{\lambda}\bar{\mu}_2\bar{\gamma})}\right\}-\bar{\gamma}\mu_2(\bar{\lambda}r+\lambda)(\bar{\mu}_1+\mu_1r)\right]. \qquad (17)$$

Finally, using the normalizing condition, we obtain

$$Q_{0,2} = \left[\frac{(1-\bar{\lambda}\bar{\mu}_1\bar{\gamma})}{\lambda\bar{\mu}_1r}\left\{1-\frac{\mu_1\bar{\mu}_2\bar{\gamma}(\lambda+\bar{\lambda}r)}{(1-\bar{\lambda}\bar{\mu}_2\bar{\gamma})}\right\}-\frac{\bar{\gamma}\mu_2(\bar{\lambda}r+\lambda)(\bar{\mu}_1+\mu_1r)}{\lambda\bar{\mu}_1r}+\frac{1}{r(1-r)}\right]^{-1} \qquad (18)$$

2.2. **The LAS-DA system.** We now discuss the model for the late arrival system with delayed access (LAS-DA), that is, arrival-first (AF) policy. Here, a potential arrival takes place in $(t-,\ t)$ and a potential departures occur in $(t,\ t+)$. The various time epochs at which events occur are depicted in Figure 2. Let $P_{0,0}(t-)$ denote the probability that both

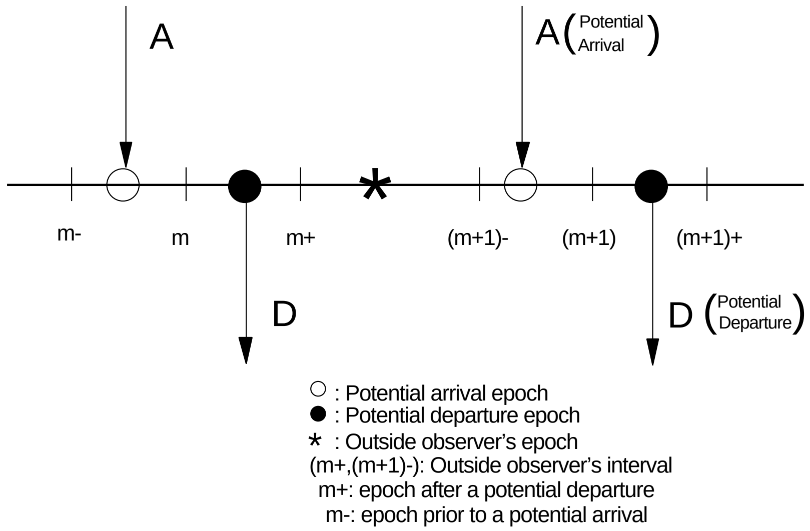


FIGURE 2. Various time epochs in LAS-DA.

the servers are idle at time $t-$. When the queue is empty at time $t-$, let $P_{1,0}(t-)$ denote the probability that the server I is busy and the server II is idle, and $P_{0,1}(t-)$ denote the probability that the server I is idle and the server II is busy. Further, let $P_{n,2}(t-)$ be the probability that both the servers are busy and there are $n \geq 0$ customers waiting in the queue at time $t-$. Relating the states of the system at two consecutive prior to potential arrival epochs $t-$ and $(t+1)-$, we obtain the following equations, where for the sake of simplicity we use $t$ instead of $t-$,

$$\begin{aligned}
P_{0,0}(t+1) &= \left(\bar{\lambda}-\gamma\right)P_{0,0}(t)+\bar{\lambda}\mu_1\bar{\gamma}P_{1,0}(t)+\bar{\lambda}\mu_2\bar{\gamma}P_{0,1}(t)+\bar{\lambda}\mu_1\mu_2\bar{\gamma}P_{0,2}(t)+\gamma, &(19)\\
P_{1,0}(t+1) &= \bar{\lambda}\bar{\mu}_1\bar{\gamma}P_{1,0}(t)+\bar{\gamma}\mu_2\left(\bar{\lambda}\bar{\mu}_1+\lambda\mu_1\right)P_{0,2}(t)+\lambda P_{0,0}(t)+\lambda\mu_2\bar{\gamma}P_{0,1}(t)\\
&\quad+\bar{\lambda}\mu_1\mu_2\bar{\gamma}P_{1,2}(t), &(20)\\
P_{0,1}(t+1) &= \bar{\lambda}\bar{\mu}_2\bar{\gamma}P_{0,1}(t)+\bar{\lambda}\mu_1\bar{\mu}_2\bar{\gamma}P_{0,2}(t)+\lambda\mu_1\bar{\gamma}P_{1,0}(t), &(21)\\
P_{0,2}(t+1) &= \bar{\gamma}\left(\bar{\lambda}\bar{\mu}_1\bar{\mu}_2+\lambda\mu_1\bar{\mu}_2+\lambda\bar{\mu}_1\mu_2\right)P_{0,2}(t)+\lambda\bar{\mu}_1\bar{\gamma}P_{1,0}(t)+\lambda\bar{\mu}_2\bar{\gamma}P_{0,1}(t)\\
&\quad+\bar{\gamma}\left(\bar{\lambda}\mu_1\bar{\mu}_2+\bar{\lambda}\bar{\mu}_1\mu_2+\lambda\mu_1\mu_2\right)P_{1,2}(t)+\bar{\lambda}\mu_1\mu_2\bar{\gamma}P_{2,2}(t), &(22)\\
P_{n,2}(t+1) &= \bar{\gamma}\left(\bar{\lambda}\bar{\mu}_1\bar{\mu}_2+\lambda\mu_1\bar{\mu}_2+\lambda\bar{\mu}_1\mu_2\right)P_{n,2}(t)+\bar{\lambda}\mu_1\mu_2\bar{\gamma}P_{n+2,2}(t)\\
&\quad+\bar{\gamma}\left(\bar{\lambda}\mu_1\bar{\mu}_2+\bar{\lambda}\bar{\mu}_1\mu_2+\lambda\mu_1\mu_2\right)P_{n+1,2}(t)+\lambda\bar{\mu}_1\bar{\mu}_2\bar{\gamma}P_{n-1,2}(t),\ n\geq 1 &(23)
\end{aligned}$$

In the steady-state, above equations (19) - (23) reduce to

$$\begin{aligned}
0 &= -\lambda P_{0,0}+\bar{\lambda}\mu_1\bar{\gamma}P_{1,0}+\bar{\lambda}\mu_2\bar{\gamma}P_{0,1}+\bar{\lambda}\mu_1\mu_2\bar{\gamma}P_{0,2}+\gamma(1-P_{0,0}), &(24)\\
0 &= \left(\bar{\lambda}\bar{\mu}_1\bar{\gamma}-1\right)P_{1,0}+\bar{\gamma}\mu_2\left(\bar{\lambda}\bar{\mu}_1+\lambda\mu_1\right)P_{0,2}+\lambda P_{0,0}+\lambda\mu_2\bar{\gamma}P_{0,1}\\
&\quad+\bar{\lambda}\mu_1\mu_2\bar{\gamma}P_{1,2}, &(25)
\end{aligned}$$

$$\begin{aligned}
0 &= \left(\bar{\lambda}\bar{\mu}_2\bar{\gamma}-1\right)P_{0,1}+\bar{\lambda}\mu_1\bar{\mu}_2\bar{\gamma}P_2+\lambda\mu_1\bar{\gamma}P_{1,0}, &(26)\\
0 &= \Big\{\bar{\gamma}\left(\bar{\lambda}\bar{\mu}_1\bar{\mu}_2+\lambda\mu_1\bar{\mu}_2+\lambda\bar{\mu}_1\mu_2\right)-1\Big\}P_{0,2}+\bar{\gamma}\left(\bar{\lambda}\mu_1\bar{\mu}_2+\bar{\lambda}\bar{\mu}_1\mu_2+\lambda\mu_1\mu_2\right)P_{1,2}\\
&\quad+\bar{\lambda}\mu_1\mu_2\bar{\gamma}P_{2,2}+\lambda\bar{\mu}_1\bar{\gamma}P_{1,0}+\lambda\bar{\mu}_2\bar{\gamma}P_{0,1}, &(27)\\
0 &= \Big\{\bar{\gamma}\left(\bar{\lambda}\bar{\mu}_1\bar{\mu}_2+\lambda\mu_1\bar{\mu}_2+\lambda\bar{\mu}_1\mu_2\right)-1\Big\}P_{n,2}+\bar{\gamma}\left(\bar{\lambda}\mu_1\bar{\mu}_2+\bar{\lambda}\bar{\mu}_1\mu_2+\lambda\mu_1\mu_2\right)P_{n+1,2}\\
&\quad+\bar{\lambda}\mu_1\mu_2\bar{\gamma}P_{n+2,2}+\lambda\bar{\mu}_1\bar{\mu}_2\bar{\gamma}P_{n-1,2},\ n\geq 1. &(28)
\end{aligned}$$

It is observed that equations (28) and (10) are identical but others are distinct. Therefore, the characteristic equation and hence the value of $r$ will be same in both (EAS and LAS-DA) cases. Applying the procedure discussed for EAS, we can yield

$$P_{n,2} = r^n P_{0,2},\ n\geq 1. \tag{29}$$

Substituting (29) into (27), and then using (26), we get

$$P_{0,1} = \frac{\mu_1\bar{\mu}_1\bar{\mu}_2\bar{\gamma}\left(\bar{\lambda}r+\lambda\right)}{r\left(\bar{\mu}_1+\bar{\mu}_2\bar{\gamma}(\lambda-\bar{\mu}_1)\right)}P_{0,2}, \tag{30}$$

$$P_{1,0} = \frac{\bar{\mu}_2P_{0,2}}{r}\left[1-\frac{\mu_1\bar{\mu}_2\bar{\gamma}\left(\bar{\lambda}r+\lambda\right)}{\bar{\mu}_1+\bar{\mu}_2\bar{\gamma}(\lambda-\bar{\mu}_1)}\right]P_{0,2}. \tag{31}$$

Using (29) - (31) into (24), and after simplification, we obtain

$$\begin{aligned}
P_{0,0} &= \frac{P_{0,2}}{\lambda}\left[\frac{\mu_1\bar{\mu}_2\bar{\gamma}\left(\bar{\lambda}r+\lambda\right)}{r\left(\bar{\mu}_1+\bar{\mu}_2\bar{\gamma}(\lambda-\bar{\mu}_1)\right)}\left\{(\bar{\lambda}\mu_2\bar{\gamma}+\gamma)\bar{\mu}_1-(\bar{\lambda}\mu_1\bar{\gamma}+\gamma)\bar{\mu}_2\right\}\right.\\
&\quad\left.+\frac{\bar{\lambda}\mu_1\bar{\gamma}}{r}(\bar{\mu}_2+\mu_2r)+\frac{\gamma\bar{\mu}_2}{r}+\frac{\gamma}{1-r}\right]. &(32)
\end{aligned}$$

Finally, using the normalizing condition, we obtain

$$\begin{aligned}
P_{0,2} &= \left[\frac{\mu_1\bar{\mu}_2\bar{\gamma}\left(\bar{\lambda}r+\lambda\right)}{\lambda r\left(\bar{\mu}_1+\bar{\mu}_2\bar{\gamma}(\lambda-\bar{\mu}_1)\right)}\left\{\lambda(\mu_2-\mu_1)+(\bar{\lambda}\mu_2\bar{\gamma}+\gamma)\bar{\mu}_1-(\bar{\lambda}\mu_1\bar{\gamma}+\gamma)\bar{\mu}_2\right\}\right.\\
&\quad\left.+\frac{\bar{\lambda}\mu_1\bar{\gamma}}{r\lambda}(\bar{\mu}_2+\mu_2r)+\frac{\bar{\mu}_2(\lambda+\gamma)}{r\lambda}+\frac{(\lambda+\gamma)}{\lambda(1-r)}\right]^{-1}. &(33)
\end{aligned}$$

## 3. Outside observer's distribution

In EAS, since an outside observer's observation epoch falls in a time interval after a potential arrival and before a potential departure, the probabilities $Q^o_{0,0}$, $Q^o_{1,0}$, $Q^o_{0,1}$ and $Q^o_{n,2}$ $(n\geq 0)$ that the outside observer sees both the servers idle, one server busy, and both the servers busy with $n$ customers in the queue, respectively, can be obtained by observing arbitrary epoch $(t)$ and outside observer's observation epoch $(*)$ in Figure 1. They are given by

$$\begin{aligned}
Q^o_{0,0} &= \bar{\lambda}Q_{0,0}, &(34)\\
Q^o_{1,0} &= \bar{\lambda}Q_{1,0}+\lambda Q_{0,0}, &(35)\\
Q^o_{0,1} &= \bar{\lambda}Q_{0,1}, &(36)\\
Q^o_{0,2} &= \bar{\lambda}Q_{0,2}+\lambda Q_{0,1}+\lambda Q_{1,0}, &(37)\\
Q^o_{n,2} &= \bar{\lambda}Q_{n,2}+\lambda Q_{n-1,2},\ n\geq 1. &(38)
\end{aligned}$$

Using (13) in (37) and (38), it follows that

$$Q^o_{n,2} = (\bar{\lambda} r + \lambda) r^{n-1} Q_{0,2} = \frac{r^n Q_{0,2}}{\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)},\ n \geq 0.$$

In LAS-DA, since an outside observer's observation epoch falls in a time interval after a potential departure and before a potential arrival, the probability $P^o_{0,0}$, $P^o_{1,0}$, $P^o_{0,1}$ and $P^o_{n,2}$ $(n \geq 0)$ that outside observer sees both the servers idle, one server busy, and both the servers busy with $n$ jobs in the queue are the same as $P_{0,0}$, $P_{1,0}$, $P_{0,1}$ and $P^o_{n,2}$ $(n \geq 0)$, respectively. Hence $P^o_{0,0} = P_{0,0}$, $P^o_{1,0} = P_{1,0}$, $P^o_{0,1} = P_{0,1}$ and $P^o_{n,2} = P_{n,2}$.

## 4. Performance measures

There are several system performance measures of the discussed queueing system, such as the expected number of jobs in the system, the expected number of jobs in the queue, the probability that an arriving job is expected to join the queue, the probability that the system has $n$ $(n = 1, 2)$ busy servers, the expected number of busy servers, the mean busy period of the system, etc. The expected number of jobs $(L_s)$ in the system is given by

$$L_s = Q_{1,0} + Q_{0,1} + \sum_{n=0}^{\infty} n Q_{n,2} = \left(\frac{1}{r} + \frac{r}{(1-r)^2}\right) Q_{0,2}.$$

The probability of an arriving jobs joining the queue is $\sum_{n=0}^{\infty} Q_{n,2} = \frac{Q_{0,2}}{1-r}$. Let $H$ denote the number of busy servers. The probability that the system has $n$ busy servers is given by

$$P\{H = n\} \quad = \quad \begin{cases} P\{H = 1\} = Q_{1,0} + Q_{0,1}, & \text{for } n = 1, \\ P\{H > 1\} = \sum_{n=0}^{\infty} Q_{n,2}, & \text{for } n = 2. \end{cases}$$

The mean number of busy servers is given by

$$E[H] = Q_{1,0} + Q_{0,1} + 2\sum_{n=0}^{\infty} Q_{n,2} = \left(\frac{1+r}{r(1-r)}\right) Q_{0,2}.$$

Similarly, we can evaluate performance measures for LAS-DA model.

### 4.1. **Expected lengths of the idle period, the busy period and the busy cycle.**

Let the expected length of the busy period, the idle period and the busy cycle be denoted by $E[B]$, $E[I]$ and $E[C]$, respectively. A cycle is the time that elapses between two consecutive arrivals finding an empty system. A cycle begins with a busy period during which the server is serving jobs, followed by an idle period during which the system is empty. The busy period starts from the instant when both the servers become busy and terminates when they go idle. The queue alternates between idle and busy periods and form an alternating renewal process. Due to the memoryless property of the geometric distribution, an idle period is geometrically distributed with mean $1/\lambda$. The long-run proportion of time that the server is idle equals

$$Q_{0,0} \quad = \quad \frac{E[I]}{E[I] + E[B]}.$$

Hence, the expected busy period is given by

$$E[B] = \frac{1 - Q_{0,0}}{\lambda Q_{0,0}},\ \textit{and}\ E[I] = \frac{1}{\lambda}.$$

A busy cycle is the time between two successive departures leaving an empty system or equivalently, the sum of a busy period and an adjacent idle period. The expected length of busy cycle (C) is $E[C] = 1/\lambda Q_{0,0}$.

4.2. **Waiting time distribution.** In this section, we obtain the actual waiting time distribution in the queue of an arrival job under the first-come, first-served (FCFS) queueing discipline. Let us define the random variable $T_q$ as the total amount of time measured in slots that an arrival spends in the queue and the corresponding p.m.f. $w_k = P(T_q = k), k \geq 0$. Further, let $W_q = \sum_{k=1}^{\infty} k w_k$ denote the average waiting time in the queue of an arrival job.

**Waiting time in EAS system:**

In EAS, an arriving job may observe the system in any one of the following two cases.

**Case 1.** $w_0 = P(T_q = 0)$.

If prior to an arrival, there are no jobs in the queue and at most one server is busy.

**Case 2.** $w_k = P(T_q = k),\ k \geq 1$.

Both the servers are busy and there may be $k \geq 0$ jobs waiting in the queue. Since there may occur at most two departures from the system in a slot so a new arriving job has to wait until the first departure or continue to wait for as many departures during $k$ slots as there were jobs waiting upon arrival. Therefore, the probability that an arriving job waits for greater than $k$ slots is

$$
\begin{aligned}
P(T_q > k) &= \Big(\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)\Big)^k \sum_{n=1}^{\infty} r^{n-1} Q_{0,2} \\
&= \frac{\Big(\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)\Big)^k Q_{0,2}}{1-r},\ k \geq 0.
\end{aligned}
$$

Then, consequently, we obtain

$$
\begin{aligned}
w_0 &= 1 - P(T_q > 0) = 1 - \frac{Q_{0,2}}{1-r}, && (39)\\
w_k &= P(T_q > k-1) - P(T_q > k) \\
&= \frac{Q_2}{1-r}\Big(\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)\Big)^{k-1}\Big(1 - \bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)\Big),\ k \geq 1. && (40)
\end{aligned}
$$

The average waiting time in the queue is given by

$$
W_q = \frac{Q_{0,2}}{(1-r)[1 - \bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)]}.
$$

**Remark 1.** The average queue length $(L_q^o)$ at outside observer's observation epoch is given by

$$
\begin{aligned}
L_q^o &= \sum_{n=1}^{\infty} n Q_{n,2}^o = \frac{r Q_{0,2}}{(1-r)^2 \bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)} \\
&= \frac{r[1 - \bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)]}{(1-r)\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)} \cdot \frac{Q_{0,2}}{(1-r)[1 - \bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)]} \\
&= \lambda W_q,
\end{aligned}
$$

where

$$
\lambda = \frac{r[1 - \bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)]}{(1-r)\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)} \qquad (41)
$$

is obtained from equation (12). Thus, the Little's formula $L_q^o = \lambda W_q$ is verified.

**Waiting time in LAS-DA system:**
In LAS-DA, an arriving job may observe the system in any one of the following two cases.
**Case 1.** $w_0 = P(T_q = 0)$.
This happens if prior to an arrival, there are (i) no jobs in the queue and at most one server is busy, or (ii) no jobs in the queue and first server becomes idle, or (iii) no jobs in the queue and second server becomes idle, or (iv) at most one job in the queue and both servers become idle.
**Case 2.** $w_k = P(T_q = k),\ k \geq 1$.
Similarly, this happens if prior to an arrival, both the servers are busy and there may or may not be jobs waiting in the queue. Therefore, the probability that an arriving job waits for greater than $k$ slots is

$$\begin{aligned} P(T_q > k) &= \Big(\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)\Big)^{k+1} \sum_{n=1}^{\infty} r^{n-1} P_{0,2}, \\ &= \frac{\Big(\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)\Big)^{k+1} P_{0,2}}{1-r},\ k \geq 0. \end{aligned} \tag{42}$$

Using (42), we obtain

$$\begin{aligned} w_0 &= 1 - P(T_q > 0) = 1 - \frac{P_{0,2}\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)}{1-r}, \\ w_k &= P(T_q > k-1) - P(T_q > k) \\ &= \frac{P_{0,2}}{1-r}\Big(\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)\Big)^k \Big(1 - \bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)\Big),\ k \geq 1. \end{aligned}$$

The average waiting time in the queue is given by

$$W_q = \frac{P_{0,2}\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)}{(1-r)[1 - \bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)]}.$$

**Remark 2.** The average queue length ($L_q^o$) at outside observer's observation epoch is given by

$$\begin{aligned} L_q^o &= \sum_{n=1}^{\infty} n P_{n,2}^o = \frac{r P_{0,2}}{(1-r)^2} \\ &= \frac{r[1 - \bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)]}{(1-r)\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)} \cdot \frac{P_{0,2}\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)}{(1-r)[1 - (\bar{\gamma}(\bar{\mu}_1 + \mu_1 r)(\bar{\mu}_2 + \mu_2 r)]} \\ &= \lambda W_q. \end{aligned}$$

## 5. Particular cases

In this section, some particular cases which are available in the literature are deduced from our model by taking specific values for the parameters $\mu_1,\ \mu_2$ and $\gamma$.
**Case 1**: $\mu_1 = \mu_2 = \mu$. The model reduces to *Geo*/*Geo*/2 queue with catastrophe, where jobs are served by two homogeneous servers.
The equation (12) is simplified as

$$r = \bar{\gamma}(\bar{\lambda} r + \lambda)(\bar{\mu} + \mu r)^2.$$

In the case of EAS, the steady state probabilities with the above value of $r$ are obtained from (13), (14), (17) and (18), and they are given by

$$\begin{aligned} Q_{n,2} &= r^n Q_{0,2}, \; n \geq 1, \\ \widetilde{Q}_{0,1} &= Q_{1,0} + Q_{0,1} = \frac{Q_{0,2}}{r}, \\ Q_{0,0} &= \frac{\bar{\gamma}(\lambda - \bar{\lambda}\mu r)Q_{0,2}}{\lambda r^2}, \\ Q_{0,2} &= \left[\frac{\bar{\gamma}(\lambda - \bar{\lambda}\mu r)Q_{0,2}}{\lambda r^2} + \frac{1}{r(1-r)}\right]^{-1}. \end{aligned}$$

The outside observer's observation epoch probabilities are obtained from (34) to (38) with the above value of $r$, and they are given by

$$\begin{aligned} Q^o_{n,2} &= \frac{r^n Q_{0,2}}{\bar{\gamma}(\bar{\mu} + \mu r)^2}, \; n \geq 0, \\ \tilde{Q}^o_{0,1} &= \bar{\lambda}\widetilde{Q}_{0,1} + \lambda Q_{0,0} = \frac{Q_{0,2}}{r}\Big(\bar{\lambda} + \frac{\bar{\gamma}(\lambda - \bar{\lambda}\mu r)}{r}\Big), \\ Q^o_{0,0} &= \frac{\bar{\lambda}\bar{\gamma}(\lambda - \bar{\lambda}\mu r)Q_{0,2}}{\lambda r^2}, \end{aligned}$$

where $\tilde{Q}^o_{0,1} = Q^o_{1,0} + Q^o_{0,1}$.
The average queue length and average waiting time in the queue are, respectively, given by

$$L^o_q = \frac{Q_{0,2} r}{(1-r)^2 \bar{\gamma}(\bar{\mu} + \mu r)^2}, \; W_q = \frac{Q_{0,2}}{(1-r)\left[1 - \bar{\gamma}(\bar{\mu} + \mu r)^2\right]}.$$

In the similar way, from equations (29)-(33) in the case of LAS-DA, we obtain

$$\begin{aligned} P_{n,2} &= r^n P_{0,2}, \; n \geq 1, \\ \widetilde{P}_{0,1} &= \frac{\bar{\mu} P_{0,2}}{r}, \\ P_{0,0} &= \frac{P_{0,2}}{\lambda}\Big(\frac{\bar{\lambda}\mu\bar{\gamma}}{r}(\bar{\mu} + \mu r) + \frac{\gamma\bar{\mu}}{r} + \frac{\gamma}{1-r}\Big), \\ P_{0,2} &= \left[\frac{\bar{\lambda}\mu\bar{\gamma}}{\lambda r}(\bar{\mu} + \mu r) + \frac{\bar{\mu}(\lambda + \gamma)}{\lambda r} + \frac{\lambda + \gamma}{\lambda(1-r)}\right]^{-1}, \end{aligned}$$

where $\widetilde{P}_{0,1} = P_{1,0} + P_{0,1}$.
The average queue length and average waiting time in the queue are, respectively, given by

$$L^o_q = \frac{P_{0,2} r}{(1-r)^2}, \; W_q = \frac{\bar{\gamma}(\bar{\mu} + \mu r)^2 P_{0,2}}{(1-r)[1 - \bar{\gamma}(\bar{\mu} + \mu r)^2]}.$$

**Case 2**: $\gamma = 0, \; \mu_1 = \mu_2 = \mu$. The model reduces to $Geo/Geo/2$ queue without catastrophes and two homogeneous servers. Let us define $P_0 = P_{0,0}$ and $P_1 = P_{0,1} + P_{1,0}$, where $P_n$=Pr$\{n$ jobs in the system$\}$, $n \geq 0$. The equation (12) is simplified as

$$r = (\bar{\lambda} r + \lambda)(\bar{\mu} + \mu r)^2.$$

Then from (29) - (33), we obtain

$$\begin{aligned} P_{n+2} &= r^n P_2, n \geq 1, \\ P_1 &= \frac{\bar{\mu} P_2}{r}, \\ P_0 &= \frac{\bar{\lambda}\mu(\bar{\mu}+\mu r)P_2}{\lambda r}, \\ P_2 &= \left[\frac{\bar{\lambda}\mu(\bar{\mu}+\mu r)}{\lambda r} + \frac{\bar{\mu}}{r} + \frac{1}{1-r}\right]^{-1}. \end{aligned}$$

Finally, the average queue length and average waiting time in the queue are, respectively, given by

$$L_q^o = \frac{P_2 r}{(1-r)^2}, \; W_q = \frac{P_2(\bar{\mu}+\mu r)^2}{(1-r)[1-(\bar{\mu}+\mu r)^2]},$$

which are matched with the results given in Artalejo and Hernández-Lerma [19] by taking $c = 2$.

## 6. Numerical results

In this section, we present numerical results in the form of table and graphs. The steady state probabilities at arbitrary, outside observer's observation epochs and waiting time distribution for both EAS and LAS-DA systems is given in Table 1. Various performance measures such as average queue length and average waiting time in the queue are given at the bottom of the tables. We have also obtained the average waiting time in the queue using Little's rule and found that it is the same as the one obtained using the p.m.f. of the actual waiting time in the queue.

We present the effect of the arrival rate $\lambda$ on the expected length of the busy period

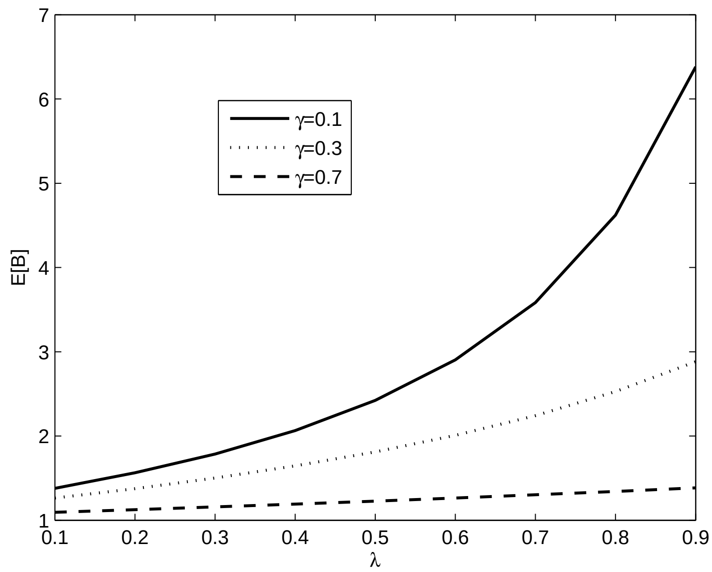


FIGURE 3. Effect of arrival rate on $E[B]$.

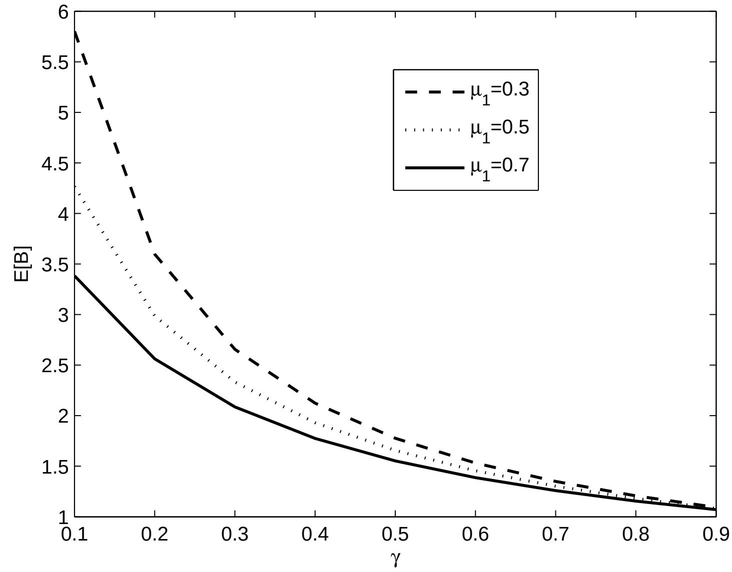


FIGURE 4. Effect of $\gamma$ on $E[B]$.

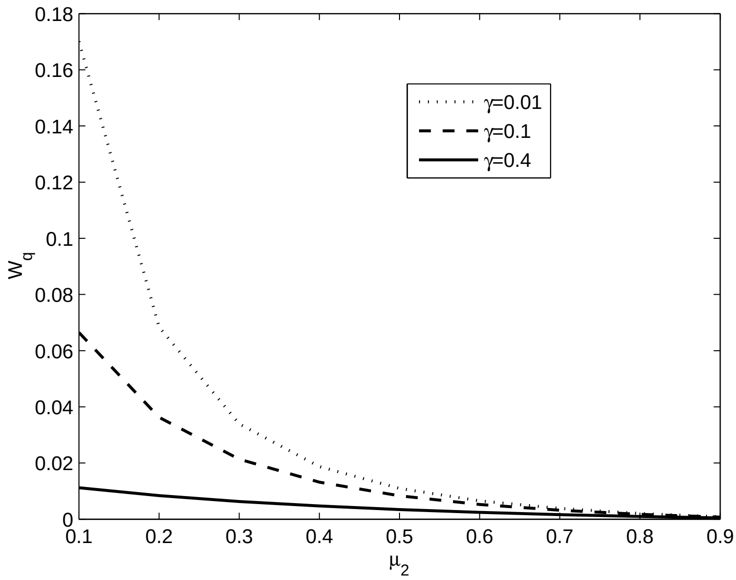


FIGURE 5. Effect of $\mu_2$ on $W_q$.

($E[B]$) in Figure 3 for different values of catastrophic rate $\gamma$ in case of LAS-DA system. The parameters for this graph are taken as $\mu_1 = 0.8$ and $\mu_2 = 0.6$. As expected, the expected length of the busy period increases with the increasing of the arrival rate $\lambda$, while it decreases with the increasing of the catastrophic rate $\gamma$. In Figure 4, we have plotted the effect of catastrophic rate $\gamma$ on the expected length of the busy period ($E[B]$) for different values of mean service rate $\mu_1$. The parameters for this graph are taken as

$\lambda = 0.5$, $\mu_1 = 0.1$ and $\mu_2 = 0.05$. One may observe that for fixed $\mu_1$, $E[B]$ decreases as catastrophic rate $\gamma$ increases and then it becomes almost static. However, the $E[B]$ reduces considerably when the mean service rate $\mu_1$ is increased. This is because when the mean service rate $\mu_1$ is large, more jobs can be served, which results in the decreasing of the expected length of the busy period.

Figure 5 presents the effect of service rate ($\mu_2$) on the average waiting time in the

TABLE 1. Queue length and waiting time distributions for $\lambda = 0.2, \mu_1 = 0.15, \mu_2 = 0.1$, $\gamma = 0.01, \rho = 0.8$.

| Queue length distributions | | | | Waiting time in slots | | |
|---|---|---|---|---|---|---|
| | | | | | EAS | LAS-DA |
| $(n,j)$ | $Q_{n,j}$ | $Q^o_{n,j}$ | $P_{n,j}$ | $k$ | $w_k$ | $w_k$ |
| $0,0$ | 0.222586 | 0.178069 | 0.183100 | 0 | 0.481382 | 0.484214 |
| $1,0$ | 0.170436 | 0.180866 | 0.162529 | 1 | 0.0470622 | 0.0468052 |
| $0,1$ | 0.088359 | 0.070687 | 0.087109 | 2 | 0.042792 | 0.042558 |
| $0,2$ | 0.172644 | 0.189875 | 0.188837 | 3 | 0.038908 | 0.038696 |
| $1,2$ | 0.115172 | 0.126667 | 0.125975 | 4 | 0.035378 | 0.035184 |
| $2,2$ | 0.076832 | 0.084500 | 0.084039 | 5 | 0.032167 | 0.031992 |
| $3,2$ | 0.051255 | 0.056371 | 0.056063 | 6 | 0.029248 | 0.029089 |
| $4,2$ | 0.034193 | 0.037605 | 0.037400 | 7 | 0.026594 | 0.026449 |
| $5,2$ | 0.022810 | 0.025087 | 0.024950 | 8 | 0.024181 | 0.024049 |
| $10,2$ | 0.003014 | 0.003315 | 0.003296 | 9 | 0.021987 | 0.021866 |
| $11,2$ | 0.002011 | 0.002211 | 0.002199 | 10 | 0.019991 | 0.019882 |
| $12,2$ | 0.001341 | 0.001475 | 0.001467 | 20 | 0.007721 | 0.007680 |
| $13,2$ | 0.000895 | 0.000984 | 0.000979 | 30 | 0.0029828 | 0.0029668 |
| $14,2$ | 0.000597 | 0.000656 | 0.000653 | 40 | 0.001152 | 0.001146 |
| $15,2$ | 0.000398 | 0.000438 | 0.000436 | 50 | 0.000445 | 0.000442 |
| $18,2$ | 0.000118 | 0.000130 | 0.000129 | 60 | 0.000172 | 0.000171 |
| $20,2$ | 0.000053 | 0.000058 | 0.000058 | 70 | 0.000066 | 0.000066 |
| $25,2$ | 0.000007 | 0.000008 | 0.000008 | 80 | 0.000026 | 0.000025 |
| $28,2$ | 0.000002 | 0.000002 | 0.000002 | 100 | 0.000004 | 0.000004 |
| $30,2$ | 0.000001 | 0.000001 | 0.000001 | 110 | 0.000001 | 0.000001 |
| $40,2$ | 0.000000 | 0.000000 | 0.000000 | $\geq 115$ | 0.000000 | 0.000000 |
| sum | 1.000000 | 1.000000 | 1.000000 | | 1.000000 | 1.000000 |
| $L^o_q$ | | 1.14302 | 1.13678 | | | |
| $W_q$ | | 5.71510 | 5.68388 | | 5.71510 | 5.68388 |
| $E[H]$ | | 1.39231 | 1.38416 | | | |
| $E[B]$ | | 23.0790 | 22.3076 | | | |
| $E[C]$ | | 28.0790 | 27.3076 | | | |

queue ($W_q$) for various $\gamma$ in case of EAS system. The parameters are taken as $\lambda = 0.8$ and $\mu_1 = 0.9$. It can be seen that the average waiting time in the queue decreases steadily with a increasing service rate $\mu_2$. The average waiting time in the queue will be very small when the service rate is large enough, which results in a decrease in the average waiting time in the queue. For fixed service rate $\mu_2$, the average waiting time in the queue decreases as $\gamma$ increases. The difference in EAS and LAS-DA system is highlighted in Figure 6 by considering the waiting time distribution. The parameters for these graphs are taken as $\lambda = 0.75$, $\mu_1 = 0.5$, $\mu_2 = 0.4$ and $\gamma = 0.03$. It can be seen that $W^o_q$ in case of

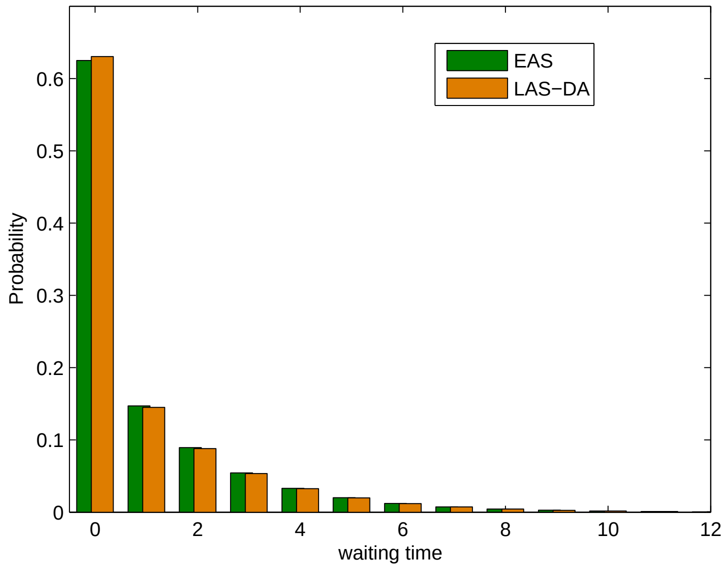


FIGURE 6. Distribution of the waiting time in the queue.

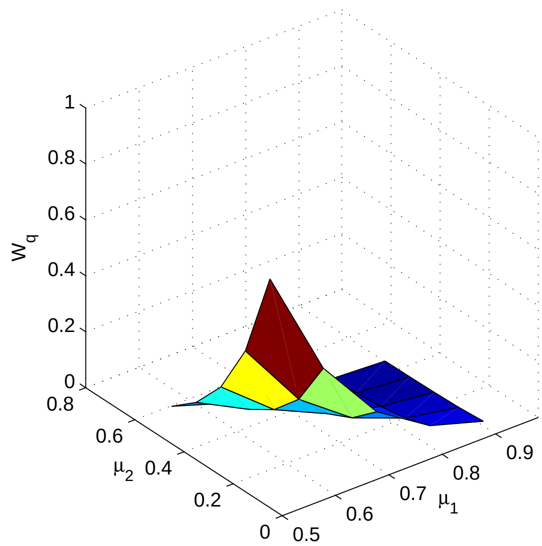


FIGURE 7. $W_q$ versus service rates $\mu_1$ and $\mu_2$.

EAS is slightly higher as compared to LAS-DA. This happens because in the EAS system, an outside observer's observation epoch falls in a time interval after a potential arrival and before a potential departure but in case of LAS-DA, it falls in a time interval after a potential departure and before a potential arrival. But the distribution of the waiting time in LAS-DA is larger than EAS when the queue is empty. This is because service to an arriving customer can starts immediately from the same slot in EAS if there is any free server, but in LAS-DA, it will enter into service in the next slot.

Figure 7 illustrates dependence of the mean waiting time in the queue ($W_q$) on service

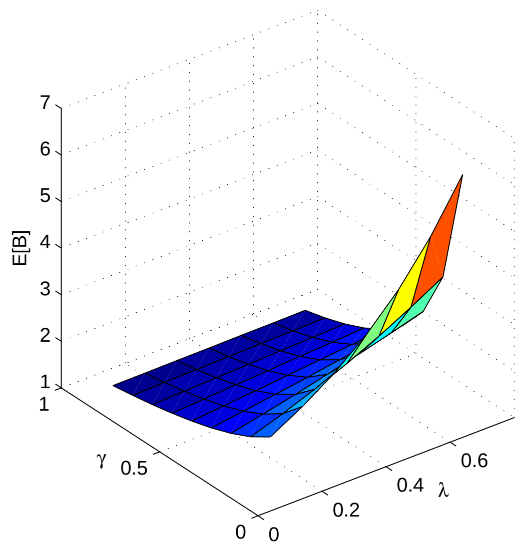


FIGURE 8. $E[B]$ versus arrival rate and catastrophic rate.

rates $\mu_1$ and $\mu_2$ in case of EAS system. It is observed that for fixed service rate $\mu_1$, the mean waiting time in the queue decreases when the service rate $\mu_2$ increases. Further, with fixed service rates $\mu_2$, the mean waiting time in the queue decreases when the service rate $\mu_1$ increases. To accomplish this, we can carefully setup the service rates $\mu_1$ and $\mu_2$ in the system in order to ensure the minimum mean waiting time in the queue. Figure 8 investigates dependence of the expected length of the busy period ($E[B]$) on arrival rate ($\lambda$) and catastrophic rate $\gamma$ in case of LAS-DA system. The parameters are taken as $\mu_1 = 0.5$ and $\mu_2 = 0.3$. It is observed that for fixed catastrophic rate the expected length of the busy period increases when the arrival rate $\lambda$ increases. Further, with fixed arrival rate $\lambda$ the expected length of the busy period decreases when the catastrophic rate $\gamma$ increases. To ensure the minimum expected length of the busy period, we can carefully setup the catastrophic rate $\gamma$ and the arrival rate in the system.

## 7. CONCLUSIONS

In this paper, we have carried out an analysis of discrete-time queueing system with two heterogeneous servers subject to catastrophes for the early arrival system and the late arrival system with delayed access that have applications in the analysis of computer and communication systems. We have developed an explicit expression of the steady-state probabilities at arbitrary epoch using displacement operator method. The waiting time distribution measured in slots and outside observer's observation epoch probabilities have been carried out. Some particular cases of the model have also been discussed. Various performance measures for the system under consideration are investigated using numerical illustrations. Finally, it is shown that in the limiting case the results presented in this paper tend to the continuous-time counterparts. The techniques used in this paper can be applied to analyze more complex models such as discrete-time multi-server queueing system of different service rates subject to catastrophes which is left for future investigation.

## Appendix

Here we study the relationship between our discrete-time queueing system with two heterogeneous servers subject to catastrophes and its continuous-time counterpart. We give below a succinct proof to get the continuous-time results from the corresponding discrete-time ones. Assume that the customers arrive according to a Poisson process with rate $\alpha$ and wait in the queue if both the servers are busy. The service times of the server I and server II are assumed to be exponentially distributed with mean service rates as $\beta_1$ and $\beta_2$, respectively. The catastrophes too take place according to a Poisson process with rate $\xi$. Let the time axis be slotted into intervals of equal length $\Delta > 0$, so that

$$\lambda = \alpha\Delta,\ \mu_1 = \beta_1\Delta,\ \mu_2 = \beta_2\Delta,\ \gamma = \xi\Delta \tag{43}$$

where $\Delta$ is sufficiently small. Using (43) into $\rho = \lambda/(\mu_1 + \mu_2) < 1$ and taking the limit as $\Delta \to 0$, we get, $\rho = \alpha/(\beta_1 + \beta_2) < 1$. Thus, the positive recurrence conditions for the discrete- and continuous-time systems are consistent.
Now, substituting (43) and taking the limit as $\Delta \to 0$ in (12), we obtain

$$\alpha - (\alpha + \beta_1 + \beta_2 + \xi)r + (\beta_1 + \beta_2)r^2 = 0. \tag{44}$$

Again, substituting (43) and taking the limit as $\Delta \to 0$ in (15) - (18), and from (13), we obtain the following results for the corresponding continuous-time model. They are

$$\begin{aligned}
Q_{0,1} &= \frac{\beta_1}{\alpha + \beta_2 + \xi} Q_{0,2}, \\
Q_{1,0} &= \left(\frac{\alpha + \beta_2 + \xi - \beta_1 r}{\alpha + \beta_2 + \xi}\right) Q_{0,2}, \\
Q_{0,0} &= \left((\alpha + \beta_1 + \xi)\left(\frac{\alpha + \beta_2 + \xi - \beta_1 r}{\alpha + \beta_2 + \xi}\right) - \beta_2\right) \frac{Q_{0,2}}{\alpha}, \\
Q_{n,2} &= r^n Q_{0,2},\ n \geq 1, \\
Q_{0,2} &= \left[\left(\frac{\alpha + \beta_1 + \xi}{\alpha}\right)\left(\frac{\alpha + \beta_2 + \xi - \beta_1 r}{\alpha + \beta_2 + \xi}\right) - \frac{\beta_2}{\alpha} + \frac{1}{r(1-r)}\right]^{-1}.
\end{aligned}$$

Similarly, in the limiting case one can obtain similar results from the outside observer's observation epoch probabilities as well as from the LAS-DA system. This leads to the conclusion that, in the continuous-time, results for both LAS-DA and EAS queues tend to same as it should be. One may note here that the results for the $M/M/2$ queue with heterogeneous servers subject to catastrophes presented in Kumar et al. [7] can be obtained by taking $p = 1$.

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