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NOTES ON CERTAIN HARMONIC STARLIKE MAPPINGS

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ABSTRACT. Complex-valued harmonic functions that are univalent and sense-preserving in the unit disk \mathbb{D} can be written in the form $f = h + \bar{g}$, where h and g are analytic in \mathbb{D} . We give some inequalities for normalized harmonic functions that are starlike.

Keywords: Harmonic mappings, starlike function, Carathéodory function.

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1. INTRODUCTION

A continuous function $f = u + iv$ is said to be a complex-valued harmonic function in a complex domain \mathcal{C} if both u and v are real harmonic in \mathcal{C} . In any simply connected domain $\mathcal{D} \subset \mathcal{C}$, we can write $f = h + \bar{g}$, where h and g are analytic in \mathbb{D} . We call h the analytic part and g the co-analytic part of f . A necessary and sufficient condition for f to be locally univalent and sense-preserving in \mathcal{D} is that $|h'(z)| > |g'(z)|$ ([1]).

Denote by $\mathcal{S}_{\mathcal{H}}$ the class of $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ for which $h(0) = f(0) = f_z(0) - 1 = 0$. Then for $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}$,

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad z \in \mathbb{D}.$$

It follows from the sense-preserving property if $f \in \mathcal{S}_{\mathcal{H}}$, then $|b_1| < 1$.

$f \in \mathcal{S}_{\mathcal{H}}$ reduces to the class of normalized analytic univalent functions if the co-analytic part of its members is zero. In 1984 Clunie and Sheil-Small [1] investigated the class $\mathcal{S}_{\mathcal{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. Many studies have been done on this class and its subclasses, and continued taking place. For more references see Duren [2].

A sense-preserving harmonic mapping $f \in \mathcal{S}_{\mathcal{H}}$ is said to be in the class $\mathcal{S}_{\mathcal{H}}^*$ if the range $f(\mathbb{D})$ is starlike with respect to the origin. A function $f \in \mathcal{S}_{\mathcal{H}}^*$ is called a harmonic starlike mapping in \mathbb{D} . A function $f = h + \bar{g}$ with such a property must satisfy the condition

$$\Re \left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + g(z)} \right) > 0$$

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for all $z \in \mathbb{D}$ [4].

Lemma 1.1. [2] *If $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^*$, then there exist angles α and β such that*

$$\Re \left\{ \left(e^{i\alpha} \frac{h(z)}{z} - e^{-i\alpha} \frac{g(z)}{z} \right) \left(e^{i\beta} - e^{-i\beta} z^2 \right) \right\} > 0 \quad (1)$$

for all $z \in \mathbb{D}$.

In view of the above lemma, we consider the class $\mathcal{S}_{\mathcal{H}}^{\otimes}(\alpha, \beta, \gamma)$ of $f = h + \bar{g}$ which satisfy

$$\Re \left\{ \left(e^{i\alpha} \frac{h(z)}{z} - e^{-i\alpha} \frac{g(z)}{z} \right) \left(e^{i\beta} - e^{-i\beta} z^2 \right) \right\} > \gamma, \quad (z \in \mathbb{D})$$

for some real numbers α, β, γ ($0 \leq \gamma < \Re \{ e^{i(\alpha+\beta)} (1 - e^{-i2\alpha} b_1) \}$)

2. MAIN RESULTS

For the class $f(z) \in \mathcal{S}_{\mathcal{H}}^{\otimes}(\alpha, \beta, \gamma)$, we have

Theorem 2.1. *If $f(z) \in \mathcal{S}_{\mathcal{H}}^{\otimes}(\alpha, \beta, \gamma)$, then*

$$\begin{aligned} & \frac{r}{1+r^2} \left\{ \frac{(1-r)(\Re p(0) - \gamma)}{1+r} - \sqrt{\gamma^2 + (\Im p(0))^2} \right\} \\ & \leq |h(z) - e^{-i2\alpha} g(z)| \leq \frac{r}{1-r^2} \left\{ \frac{(1+r)(\Re p(0) - \gamma)}{1-r} + \sqrt{\gamma^2 + (\Im p(0))^2} \right\} \end{aligned}$$

for $|z| = r < 1$, where $p(0) = e^{i(\alpha+\beta)} (1 - e^{-i2\alpha} b_1)$, $\Re p(0) = \cos(\alpha+\beta) - |b_1| \cos(\arg(b_1) + \beta - \alpha)$ and $\Im p(0) = \sin(\alpha+\beta) - |b_1| \sin(\arg(b_1) + \beta - \alpha)$.

Proof. Let

$$p(z) = \left(e^{i\alpha} \frac{h(z)}{z} - e^{-i\alpha} \frac{g(z)}{z} \right) \left(e^{i\beta} - e^{-i\beta} z^2 \right) \quad (z \in \mathbb{D})$$

for $f(z) \in \mathcal{S}_{\mathcal{H}}^{\otimes}(\alpha, \beta, \gamma)$. Then $p(z)$ is analytic in \mathbb{D} and

$$p(0) = e^{i(\alpha+\beta)} (1 - e^{-i2\alpha} b_1).$$

Further, let

$$\phi(z) = \frac{p(z) - \gamma - i\Im p(0)}{\Re p(0) - \gamma} \quad (z \in \mathbb{D}).$$

Then $\phi(z)$ is analytic in \mathbb{D} , $\phi(0) = 1$ and $\Re \phi(z) > 0$ for all z in \mathbb{D} . Since, $\phi(z)$ is the Carathéodory function, we can write that

$$\phi(z) \prec \frac{1+z}{1-z} \quad (z \in \mathbb{D}),$$

that is, that

$$\phi(z) = \frac{1+w(z)}{1-w(z)},$$

where $w(z)$ is analytic in \mathbb{D} , $w(0) = 0$, and $|w(z)| < 1$ for all $z \in \mathbb{D}$. Thus, the Schwarz lemma gives us that

$$|w(z)| = \left| \frac{\phi(z) - 1}{\phi(z) + 1} \right| \leq |z|$$

for $|z| = r < 1$. It follows that

$$\left| \phi(z) - \frac{1+r^2}{1-r^2} \right| \leq \frac{2r}{1-r^2},$$

that is, that

$$\frac{1-r}{1+r} \leq |\phi(z)| \leq \frac{1+r}{1-r}.$$

Replacing $\phi(z)$ by $p(z)$, we have that

$$\frac{1-r}{1+r} \leq \left| \frac{p(z) - \gamma - i\Im p(0)}{\Re p(0) - \gamma} \right| \leq \frac{1+r}{1-r}.$$

This shows us that

$$\begin{aligned} & \frac{(1-r)(\Re p(0) - \gamma)}{1+r} - \sqrt{\gamma^2 + (\Im p(0))^2} \\ & \leq |p(z)| \leq \frac{(1+r)(\Re p(0) - \gamma)}{1-r} + \sqrt{\gamma^2 + (\Im p(0))^2}. \end{aligned}$$

Noting that

$$p(z) = \frac{e^{i(\alpha+\beta)}}{z} (h(z) - e^{-i2\alpha}g(z)) (1 - e^{-i2\beta}z^2)$$

and

$$1 - r^2 \leq |1 - e^{-i2\beta}z^2| \leq 1 + r^2,$$

we have that

$$\begin{aligned} & \frac{r}{1+r^2} \left\{ \frac{(1-r)(\Re p(0) - \gamma)}{1+r} - \sqrt{\gamma^2 + (\Im p(0))^2} \right\} \\ & \leq |h(z) - e^{-i2\alpha}g(z)| \leq \frac{r}{1-r^2} \left\{ \frac{(1+r)(\Re p(0) - \gamma)}{1-r} + \sqrt{\gamma^2 + (\Im p(0))^2} \right\}. \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} p(0) &= e^{i(\alpha+\beta)} (1 - e^{-i2\alpha}b_1) \\ &= e^{i(\alpha+\beta)} (1 - |b_1|e^{i(\arg(b_1)-2\alpha)}) \\ &= e^{i(\alpha+\beta)} - |b_1|e^{i(\arg(b_1)+\beta-\alpha)} \\ &= \cos(\alpha + \beta) - |b_1| \cos(\arg(b_1) + \beta - \alpha) \\ &\quad + i(\sin(\alpha + \beta) - |b_1| \sin(\arg(b_1) + \beta - \alpha)), \end{aligned}$$

that is, that

$$\Re p(0) = \cos(\alpha + \beta) - |b_1| \cos(\arg(b_1) + \beta - \alpha)$$

and

$$\Im p(0) = \sin(\alpha + \beta) - |b_1| \sin(\arg(b_1) + \beta - \alpha).$$

This completes the proof of the theorem. □

Theorem 2.2. *If $f = h + \bar{g} \in \mathcal{S}_H^{\otimes}(\alpha, \beta, \gamma)$, then*

$$\begin{aligned} & \left| e^{i\alpha}(a_n - e^{-i2\beta}a_{n-2}) - e^{-i\alpha}(b_n - e^{-i2\beta}b_{n-2}) \right| \\ & \leq 2 \{ \cos(\alpha + \beta) - |b_1| \cos(\arg(b_1) + \beta - \alpha) - \gamma \}, \end{aligned}$$

where $a_0 = b_0 = 0$ and $a_1 = 1$.

Proof. In view of the fact that

$$\phi(z) = \frac{p(z) - \gamma - i\Im p(0)}{\Re p(0) - \gamma} \quad (z \in \mathbb{D})$$

is the Carathéodory function for $f(z) \in \mathcal{S}_{\mathcal{H}}^{\otimes}(\alpha, \beta, \gamma)$, where

$$p(z) = \left(e^{i\alpha} \frac{h(z)}{z} - e^{-i\alpha} \frac{g(z)}{z} \right) \left(e^{i\beta} - e^{-i\beta} z^2 \right),$$

$$\Re p(0) = \cos(\alpha + \beta) - |b_1| \cos(\arg(b_1) + \beta - \alpha)$$

and

$$\Im p(0) = \sin(\alpha + \beta) - |b_1| \sin(\arg(b_1) + \beta - \alpha),$$

if we write

$$\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots,$$

then we have that

$$|c_n| \leq 2$$

for $n = 1, 2, 3, \dots$ ([3]). It follows that

$$\left(e^{i\alpha} \frac{h(z)}{z} - e^{-i\alpha} \frac{g(z)}{z} \right) \left(e^{i\beta} - e^{-i\beta} z^2 \right) = (\Re p(0) - \gamma) \phi(z) + \gamma + i \Im p(0),$$

that is, that

$$\begin{aligned} & \left\{ e^{i\alpha} (1 + a_2 z + a_3 z^2 + a_4 z^3 + \dots) - e^{-i\alpha} (b_1 + b_2 z + b_3 z^2 + b_4 z^3 + \dots) \right\} \left(e^{i\beta} - e^{-i\beta} z^2 \right) \\ &= (\Re p(0) - \gamma) (1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots) + \gamma + i \Im p(0). \end{aligned}$$

Comparing coefficients of z^{n-1} in both side of the last equality, we obtain that

$$e^{i(\alpha+\beta)} a_n - e^{-i(\beta-\alpha)} a_{n-2} - e^{i(\beta-\alpha)} b_n + e^{-i(\alpha+\beta)} b_{n-2} = (\Re p(0) - \gamma) c_{n-1}.$$

Thus, we have that

$$\begin{aligned} & \left| e^{i\alpha} (a_n - e^{-i2\beta} a_{n-2}) - e^{-i\alpha} (b_n - e^{-i2\beta} b_{n-2}) \right| \\ & \leq 2 \{ \cos(\alpha + \beta) - |b_1| \cos(\arg(b_1) + \beta - \alpha) - \gamma \}, \end{aligned}$$

This completes the proof of the theorem. \square

Theorem 2.3. *Let*

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n$$

be analytic in \mathbb{D} . If $h(z)$ and $g(z)$ satisfy

$$\begin{aligned} \sum_{n=2}^{\infty} |a_n + e^{-i2\alpha} b_n| & \leq \frac{1}{4} \left\{ \left| 1 - \gamma + e^{i(\beta-\alpha)} b_1 + e^{i(\alpha+\beta)} \right| \right. \\ & \left. - \left| 1 + \gamma - e^{i(\beta-\alpha)} - e^{i(\alpha+\beta)} \right| \right\} \end{aligned}$$

for some real numbers α, β, γ with $0 \leq \gamma < \Re \{ e^{i(\alpha+\beta)} (1 + e^{-i2\alpha} b_1) \}$ then $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^{\otimes}(\alpha, \beta, \gamma)$ for all $z \in \mathbb{D}$.

Proof. Let us define the function $p(z)$ by

$$p(z) = e^{i(\alpha+\beta)} \left(\frac{h(z)}{z} + e^{-2i\alpha} \frac{g(z)}{z} \right) \left(1 + e^{-i2\beta} z^2 \right) \quad (z \in \mathbb{D}).$$

Then, if $p(z)$ satisfies

$$\Re p(z) > \gamma \quad (z \in \mathbb{D}),$$

which is equivalent to

$$\left| \frac{1 - (p(z) - \gamma)}{1 + (p(z) - \gamma)} \right| < 1 \quad (z \in \mathbb{D}),$$

then we say that $f = h + \bar{g} \in \mathcal{S}_H^*(\alpha, \beta, \gamma)$. It follows that

$$\begin{aligned} & |1 + (p(z) - \gamma)| - |1 - (p(z) - \gamma)| \\ &= \left| 1 - \gamma + e^{i(\alpha+\beta)} \left(1 + \sum_{n=2}^{\infty} a_n z^{n-1} + e^{-i2\alpha} b_1 + e^{-i2\alpha} \sum_{n=2}^{\infty} b_n z^{n-1} \right) (1 + e^{-i2\beta} z^2) \right| \\ &\quad - \left| 1 + \gamma - e^{i(\alpha+\beta)} \left(1 + \sum_{n=2}^{\infty} a_n z^{n-1} + e^{-i2\alpha} b_1 + e^{-i2\alpha} \sum_{n=2}^{\infty} b_n z^{n-1} \right) (1 + e^{-i2\beta} z^2) \right| \\ &= \left| 1 - \gamma + e^{i(\alpha+\beta)} \left((1 + e^{-i2\alpha} b_1) (1 + e^{-i2\beta} z^2) + \sum_{n=2}^{\infty} (a_n + e^{-i2\alpha} b_n) z^{n-1} \right. \right. \\ &\quad \left. \left. + e^{-i2\beta} \sum_{n=2}^{\infty} (a_n + e^{-i2\alpha} b_n) z^{n+1} \right) \right| \\ &\quad - \left| 1 + \gamma - e^{i(\alpha+\beta)} \left((1 + e^{-i2\alpha} b_1) (1 + e^{-i2\beta} z^2) + \sum_{n=2}^{\infty} (a_n + e^{-i2\alpha} b_n) z^{n-1} \right. \right. \\ &\quad \left. \left. + e^{-i2\beta} \sum_{n=2}^{\infty} (a_n + e^{-i2\alpha} b_n) z^{n+1} \right) \right| \\ &= \left| 1 - \gamma + e^{i(\alpha+\beta)} (1 + e^{-i2\alpha} b_1) + e^{i(\alpha+\beta)} (1 + e^{-i2\alpha} b_1) e^{-i2\beta} z^2 + e^{i(\alpha+\beta)} (a_2 + e^{-i2\alpha} b_2) z \right. \\ &\quad \left. + e^{i(\alpha+\beta)} (a_3 + e^{-i2\alpha} b_3) z^2 + e^{i(\alpha+\beta)} \sum_{n=2}^{\infty} \left((a_{n+2} + e^{-i2\alpha} b_{n+2}) + e^{-i2\beta} (a_n + e^{-i2\alpha} b_n) \right) z^{n+1} \right| \\ &\quad - \left| 1 + \gamma - e^{i(\alpha+\beta)} (1 + e^{-i2\alpha} b_1) - e^{i(\alpha+\beta)} (1 + e^{-i2\alpha} b_1) e^{-i2\beta} z^2 - e^{i(\alpha+\beta)} (a_2 + e^{-i2\alpha} b_2) z \right. \\ &\quad \left. - e^{i(\alpha+\beta)} (a_3 + e^{-i2\alpha} b_3) z^2 - e^{i(\alpha+\beta)} \sum_{n=2}^{\infty} \left((a_{n+2} + e^{-i2\alpha} b_{n+2}) + e^{-i2\beta} (a_n + e^{-i2\alpha} b_n) \right) z^{n+1} \right| \\ &> \left| 1 - \gamma + e^{i(\beta-\alpha)} b_1 + e^{i(\alpha+\beta)} \right| - |a_2 + e^{-i2\alpha} b_2| - |a_3 + e^{-i2\alpha} b_3| \\ &\quad - \sum_{n=2}^{\infty} (|a_n + e^{-i2\alpha} b_n| + |a_{n+2} + e^{-i2\alpha} b_{n+2}|) - |1 + \gamma - e^{i(\beta-\alpha)} b_1 - e^{i(\alpha+\beta)}| \\ &\quad - |a_2 + e^{-i2\alpha} b_2| - |a_3 + e^{-i2\alpha} b_3| - \sum_{n=2}^{\infty} (|a_n + e^{-i2\alpha} b_n| + |a_{n+2} + e^{-i2\alpha} b_{n+2}|) \\ &= \left| 1 - \gamma + e^{i(\beta-\alpha)} b_1 + e^{i(\alpha+\beta)} \right| - \left| 1 + \gamma - e^{i(\beta-\alpha)} - e^{i(\alpha+\beta)} \right| - 4 \sum_{n=2}^{\infty} |a_n + e^{-i2\alpha} b_n|. \end{aligned}$$

Therefore, if $h(z)$ and $g(z)$ satisfy

$$\sum_{n=2}^{\infty} |a_n + e^{-i2\alpha} b_n| \leq \left| 1 - \gamma + e^{i(\beta-\alpha)} b_1 + e^{i(\alpha+\beta)} \right| - \left| 1 + \gamma - e^{i(\beta-\alpha)} - e^{i(\alpha+\beta)} \right|,$$

then we see that

$$\left| \frac{1 - (p(z) - \gamma)}{1 + (p(z) - \gamma)} \right| < 1 \quad (z \in \mathbb{D}).$$

This completes the proof of the theorem. \square

Theorem 2.4. *If $f = h + \bar{g} \in \mathcal{S}_{\mathcal{H}}^{\otimes}(\alpha, \beta, \gamma)$, then*

$$|a_{2m} - e^{-i2\alpha}b_{2m}| \leq 2m(\Re p(0) - \gamma) \quad (m = 1, 2, 3, \dots)$$

and

$$|a_{2m+1} - e^{-i2\alpha}b_{2m+1}| \leq 2m(\Re p(0) - \gamma) + |1 - e^{-i2\alpha}b_1| \quad (m = 1, 2, 3, \dots)$$

where

$$\Re p(0) = \cos(\alpha + \beta) - |b_1| \cos(\arg(b_1) + \beta - \alpha).$$

Proof. Defining

$$p(z) = \left(e^{i\alpha} \frac{h(z)}{z} - e^{-i\alpha} \frac{g(z)}{z} \right) (e^{i\beta} - e^{-i\beta} z^2) \quad (z \in \mathbb{D})$$

and

$$\phi(z) = \frac{p(z) - \gamma - i\Im p(0)}{\Re p(0) - \gamma} \quad (z \in \mathbb{D}).$$

for $f(z) \in \mathcal{S}_{\mathcal{H}}^{\otimes}(\alpha, \beta, \gamma)$, we know that $\phi(z)$ is the Carathéodory function in \mathbb{D} . Therefore, if we write

$$\phi(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \dots,$$

then

$$|c_n| \leq 2$$

for $n = 1, 2, 3, \dots$ ([3]). It follows that

$$\begin{aligned} p(z) - \gamma - i\Im p(0) &= e^{i(\alpha+\beta)}(1 - e^{-i2\alpha}b_1) - i\Im p(0) - \gamma \\ &\quad + e^{i(\alpha+\beta)}(a_2 - e^{-i2\alpha}b_2)z \\ &\quad + \left(e^{i(\alpha+\beta)}(a_3 - e^{-i2\alpha}b_3) - e^{i(\alpha-\beta)}(1 - e^{-i2\alpha}b_1) \right) z^2 \\ &\quad + \left(e^{i(\alpha+\beta)}(a_4 - e^{-i2\alpha}b_4) - e^{i(\alpha-\beta)}(a_2 - e^{-i2\alpha}b_2) \right) z^3 \\ &\quad + \left(e^{i(\alpha+\beta)}(a_5 - e^{-i2\alpha}b_5) - e^{i(\alpha-\beta)}(a_3 - e^{-i2\alpha}b_3) \right) z^4 \\ &\quad + \left(e^{i(\alpha+\beta)}(a_6 - e^{-i2\alpha}b_6) - e^{i(\alpha-\beta)}(a_4 - e^{-i2\alpha}b_4) \right) z^5 + \dots \end{aligned}$$

and

$$(\Re p(0) - \gamma)\phi(z) = \Re p(0) - \gamma + (\Re p(0) - \gamma)c_1 z + (\Re p(0) - \gamma)c_2 z^2 + \dots$$

Comparing the coefficient of z^n , we have that

$$e^{i(\alpha+\beta)}(a_{n+1} - e^{-i2\alpha}b_{n+1}) - e^{i(\alpha-\beta)}(a_{n-1} - e^{-i2\alpha}b_{n-1}) = (\Re p(0) - \gamma)c_n,$$

for $n = 1, 2, 3, \dots$ where $a_0 = b_0 = 0$ and $a_1 = 1$.

If we take $n = 1$, then we have that

$$|a_2 - e^{-i2\alpha}b_2| \leq 2(\Re p(0) - \gamma).$$

We also have that for $n = 2, 3, 4$

$$|a_3 - e^{-i2\alpha}b_3| \leq 2(\Re p(0) - \gamma) + |1 - e^{-i2\alpha}b_1|,$$

$$\begin{aligned} |a_4 - e^{-i2\alpha}b_4| &\leq 2(\Re p(0) - \gamma) + |a_2 - e^{-i2\alpha}b_2| \\ &\leq 4(\Re p(0) - \gamma), \end{aligned}$$

and

$$\begin{aligned} |a_5 - e^{-i2\alpha}b_5| &\leq 2(\Re p(0) - \gamma) + |a_3 - e^{-i2\alpha}b_3| \\ &\leq 4(\Re p(0) - \gamma) + |1 - e^{-i2\alpha}b_1|, \end{aligned}$$

respectively. Thus applying the mathematical induction, we obtain that

$$|a_{2m} - e^{-i2\alpha}b_{2m}| \leq 2m(\Re p(0) - \gamma) \quad (m = 1, 2, 3, \dots)$$

and

$$|a_{2m+1} - e^{-i2\alpha}b_{2m+1}| \leq 2m(\Re p(0) - \gamma) + |1 - e^{-i2\alpha}b_1| \quad (m = 1, 2, 3, \dots).$$

This completes the proof of the theorem. \square

Since

$$|1 - e^{-i2\alpha}b_1| \leq 1 + |b_1| < 2$$

and

$$|a_n - e^{-i2\alpha}b_n| \geq ||a_n| - |b_n|| \quad (n = 2, 3, 4, \dots),$$

we have:

Corollary 2.1. *If $f = h + \bar{g} \in \mathcal{S}_H^*(\alpha, \beta, \gamma)$, then*

$$||a_{2m}| - |b_{2m}|| \leq 2m(\Re p(0) - \gamma) \quad (m = 1, 2, 3, \dots)$$

and

$$\begin{aligned} ||a_{2m+1}| - |b_{2m+1}|| &\leq 2m(\Re p(0) - \gamma) + 1 + |b_1| \\ &< 2m(\Re p(0) - \gamma) + 2 \quad (m = 1, 2, 3, \dots). \end{aligned}$$

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