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NEW RESULTS ON CYCLIC NONLINEAR CONTRACTIONS IN PARTIAL METRIC SPACES

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Abstract. In this paper we study the concept of non-linear cyclic Kannan and Chatterjea contractions in partial metric spaces and we prove some new theorems on fixed point for these types of mappings extending some fixed point theorems in literature.
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1. Introduction and Preliminaries

In 1992, Matthews [1, 2] introduced the notion of partial metric space which is a generalization of the usual metric space in which the distance between two elements is no longer necessarily zero. After this remarkable contribution, many authors focused on partial metric spaces and its topological properties (see, e.g. [3]-[9]). The existence of fixed point for contraction type mappings on such spaces was considered by many authors [1]-[8]. In the sequel we recall the notion of a partial metric space and some of its properties which will be used later on.

Definition 1.1. *A partial metric is a function* $p : X \times X \to [0, \infty)$ *satisfying the following conditions*

(PM1) $p(x, y) = p(y, x)$ *(symmetry).*
(PM2) If $p(x, x) = p(x, y) = p(y, y)$, *then* $x = y$ *(equality).*
(PM3) $p(x, x) \le p(x, y)$ *(small self-distances).*
(PM4) $p(x, z) + p(y, y) \le p(x, y) + p(y, z)$ *(triangularity) for all* $x, y, z \in X$.

The pair (X, p) is then called a partial metric space (see, e.g. [1, 2]).
Notice that for a partial metric p on X, the function $d_p : X \times X \to [0, \infty)$ given by

$$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$$

is a usual metric on X. Observe that each partial metric p on X generates a T_0 topology τ_p on X which has the family of open $p-$balls $\{B_p(x, \varepsilon) : x \in X,\ \varepsilon > 0\}$, where $B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$ for all $x \in X$ and $\varepsilon > 0$ as a base. Matthews observed in ([2], p. 187) that a sequence (x_n) in a partial metric space (X, p) converges to some $x \in X$ with respect to p if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$. It is clear that, if $p(x, y) = 0$, then from (PM1), (PM2), and (PM3), $x = y$. But if $x = y$, $p(x, y)$ may not be 0.

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Example 1.1. *(See, [2]). Consider* $X = \mathbb{R}^+$ *with* $p(x,y) = \max\{x,y\}$. *Then* $(\mathbb{R}^+, p)$ *is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case* $d_p(x,y) = |x-y|$.

Definition 1.2. *([2],Definition 5.2). Let* (X,p) *be a partial metric space and* (x_n) *be a sequence in X. Then* (x_n) *is called a Cauchy sequence if* $\lim\limits_{n,m\to\infty} p(x_n, x_m)$ *is both exists and finite.*

Definition 1.3. *([2], Definition 5.3). A partial metric space* (X,p) *is said to be complete if every Cauchy sequence* (x_n) *in X converges, with respect to* τ_p *to a point* $x \in X$, *such that* $p(x,x) = \lim\limits_{n,m\to\infty} p(x_m, x_n)$.

Example 1.2. *(See, [2]). Let* $X := [0,1] \cup [2,3]$ *and define* $p : X \times X \to [0,\infty)$ *by*

$$p(x,y) = \begin{cases} \max\{x,y\}, & \{x,y\} \cap [2,3] \neq \phi \\ |x-y|, & \{x,y\} \subset [0,1] \end{cases}$$

Then, (X,p) *is a complete partial metric space.*

It is a well known fact (see, for instance [2], p.194) that a sequence in a partial metric space (X,p) is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,d_p), and that a partial metric space (X,p) is complete if and only if the metric space (X,d_p) is complete. Furthermore,

$$\lim_{n\to\infty} d_p(x,x_n) = 0 \text{ if and only if } p(x,x) = \lim_{n\to\infty} p(x,x_n) = \lim_{n,m\to\infty} p(x_n,x_m).$$

Generalizations of the Banach contraction mapping principle [10] have been proposed in various settings, see for example [11]-[14] and references therein. In [15], Kannan proved a fixed point theorem which extends the well-known Banach's contraction principle by considering the following definition.

Definition 1.4. *(See, [15]). A mapping* $T : X \to X$ *where* (X,d) *is a metric space is said to be a Kannan contraction if there exists* $\alpha \in \left[0, \frac{1}{2}\right)$ *such that for all* $x, y \in X$, *the following inequality*

$$d(Tx,Ty) \le \alpha\left[d(x,Tx) + d(y,Ty)\right],$$

holds.

Kannan [15] proved that if X is complete, then every Kannan contraction has a unique fixed point. Later on, a lot of papers were devoted to obtain fixed point theorems, following the Kannan's contraction, for various classes of contractive type conditions that do not require the continuity of T. One of them, which is a sort of dual to Kannan contraction, is presented by Chatterjea [17] as follows.

Definition 1.5. *(See, [17]). A mapping* $T : X \to X$, *where* (X,d) *is a metric space, is said to be a Chatterjea contraction if there exists* $\alpha \in \left[0, \frac{1}{2}\right)$ *such that for all* $x, y \in X$, *the following inequality*

$$d(Tx,Ty) \le \alpha\left[d(x,Ty) + d(y,Tx)\right],$$

holds.

Chatterjea [17] proved that if X is complete, then every Chatterjea contraction has a unique fixed point.

The cyclical extensions for these fixed point theorems were obtained at a later time, by considering non-empty closed subsets $\{A_i\}_{i=1}^m$ of a complete metric space X and a cyclical operator $T:\bigcup_{i=1}^{m}A_i\rightarrow\bigcup_{i=1}^{m}A_i$, i.e., satisfies $T(A_i)\subseteq A_{i+1}$ for all $i\in\{1,2,...,m\}$, where $A_{m+1}=A_1$. In [16], Rus presented the cyclical extension for the Kannan's theorem, and Petric in [18] presented cyclical extensions for Chatterjea theorem using fixed point structure arguments.

Redefining the concept of Chatterjea contraction was introduced by Choudhury in [19] as follows.

Definition 1.6. *(See, [19]). A mapping $T:X\rightarrow X$, where (X,d) is a metric space, is said to be a weak Chatterjea contraction if for all $x,y\in X$, the following inequality*

$$d(Tx,Ty)\leq\frac{1}{2}[d(x,Ty)+d(y,Tx)]-\psi(d(x,Ty),d(y,Tx)),$$

holds, where $\psi:[0,\infty)^2\rightarrow[0,\infty)$ is a continuous function such that $\psi(x,y)=0$ if and only if $x=y=0$.

Choudhury [19] proved the following theorem.

Theorem 1.1. *(See, [19]). If X is a complete metric space, then every weak Chatterjea contraction T has a unique fixed point.*

A new category of fixed point problems with the help of a control function in terms of altering distances was addressed by Khan *et. al.* [20]. Altering distances have been used in metric fixed point theory in many papers, see for example [21]-[23] and references therein.

We define in what follows, an altering distance function which will be used throughout the paper to get new fixed point theorems.

Definition 1.7. *The function $\phi:[0,\infty)\rightarrow[0,\infty)$ is called an altering distance function, if the following properties are satisfied.*
1. *ϕ is continuous and non-decreasing,*
2. *$\phi(t)=0$ if and only if $t=0$.*

The aim of this paper is to present a new general fixed point theorems of cyclic nonlinear contractions that extend some theorems in the literature, by the use of the continuous function ψ given in Definition 1.6 and the altering distance function ϕ given in Definition 1.7.

2. Main results

We begin this section by giving definitions of what we call a cyclic $(\phi-\psi)$-Kannan type contraction and a cyclic $(\phi-\psi)$-Chatterjea type contraction.

Definition 2.1. *Let $\{A_i\}_{i=1}^m$ be non-empty closed subsets of a partial metric space (X,p), and suppose $T:\bigcup_{i=1}^{m}A_i\rightarrow\bigcup_{i=1}^{m}A_i$ is a cyclical operator. Then T is said to be*

(1) *a cyclic $(\phi-\psi)$-Kannan type contraction if there exists nonnegative constants α,β with $0<\alpha+\beta\leq1$, $\alpha>0$ such that for any $x\in A_i, y\in A_{i+1}, i=1,2,...,m$, we have*

$$\phi(p(Tx,Ty))\leq\phi(\alpha p(x,Tx)+\beta p(y,Ty))-\psi(p(x,Tx),p(y,Ty)),$$

(2) *a cyclic* $(\phi-\psi)$*-Chatterjea type contraction if there exists constants* α,β *with* $0<\alpha\leq\beta$ *and* $0<\alpha+\beta<1$, *such that for any* $x\in A_i, y\in A_{i+1}, i=1,2,...,m$, *we have*

$$\phi\left(p\left(Tx,Ty\right)\right)\leq\phi\left(\alpha p\left(x,Ty\right)+\beta p\left(y,Tx\right)\right)-\psi\left(p\left(x,Ty\right),p\left(y,Tx\right)\right),$$

where $\phi:[0,\infty)\to[0,\infty)$ *is an altering distance function, and* $\psi:[0,\infty)^2\to[0,\infty)$ *is a continuous function with* $\psi(t,s)=0$ *if and only if* $t=s=0$.

Theorem 2.1. *Let* $\{A_i\}_{i=1}^m$ *be non-empty closed subsets of a complete partial metric space* (X,p) *and* $T:\bigcup\limits_{i=1}^m A_i\to\bigcup\limits_{i=1}^m A_i$ *be at least one of the following:*

1. *a cyclic* $(\phi-\psi)$*-Kannan type contraction,*
2. *a cyclic* $(\phi-\psi)$*-Chatterjea type contraction.*

Then T *has a unique fixed point* $z\in\bigcap\limits_{i=1}^m A_i$.

Proof. Take $x_0\in X$ and consider the sequence given by $x_{n+1}=Tx_n, n\geq 0$. If there exists $n_0\in N$ such that $x_{n_0+1}=x_{n_0}$, then the point of existence of the fixed point is proved. So, suppose that $x_{n+1}\neq x_n$ for any $n=0,1,....$ Then there exists $i_n\in\{1,...,m\}$ such that $x_{n-1}\in A_{i_n}$ and $x_n\in A_{i_{n+1}}$. Now, assume first that T is a cyclic $(\phi-\psi)$-Kannan type contraction. Then, we have

$$\begin{aligned}
\phi\left(p\left(x_n,x_{n+1}\right)\right) &= \phi\left(p\left(Tx_{n-1},Tx_n\right)\right)\\
&\leq \phi\left(\alpha p\left(x_{n-1},Tx_{n-1}\right)+\beta p\left(x_n,Tx_n\right)\right)\\
&\quad -\psi\left(p\left(x_{n-1},Tx_{n-1}\right),p\left(x_n,Tx_n\right)\right)\\
&= \phi\left(\alpha p\left(x_{n-1},x_n\right)+\beta p\left(x_n,x_{n+1}\right)\right)\\
&\quad -\psi\left(p\left(x_{n-1},x_n\right),p\left(x_n,x_{n+1}\right)\right)\\
&\leq \phi\left(\alpha p\left(x_{n-1},x_n\right)+\beta p\left(x_n,x_{n+1}\right)\right).
\end{aligned}$$

Since ϕ is a non-decreasing function, we get that

$$p\left(x_n,x_{n+1}\right)\leq\alpha p\left(x_{n-1},x_n\right)+\beta p\left(x_n,x_{n+1}\right),$$

which implies

$$p\left(x_n,x_{n+1}\right)\leq\left(\frac{\alpha}{1-\beta}\right)p\left(x_{n-1},x_n\right),\forall n.$$

Now if $\alpha+\beta<1$, by induction we get

$$p\left(x_n,x_{n+1}\right)\leq\left(\frac{\alpha}{1-\beta}\right)^n p\left(x_0,x_1\right)$$

and hence $\lim\limits_{n\to\infty}p\left(x_n,x_{n+1}\right)=0$. If $\alpha+\beta=1$ and since $\alpha>0$, we get

$$p\left(x_n,x_{n+1}\right)\leq p\left(x_{n-1},x_n\right).$$

Consequently $\{p\left(x_n,x_{n+1}\right)\}$ is a non-increasing sequence of nonnegative real numbers. Hence, there is $r\geq 0$ such that

$$\lim_{n\to\infty}p\left(x_n,x_{n+1}\right)=r.$$

Using the continuity of ϕ and ψ, we get

$$\phi(r)\quad\leq\quad\phi(r)-\psi(r,r),$$

which implies that $\psi(r,r)=0$, and hence, $r=0$.

Similarly, if T is a cyclic $(\phi-\psi)$-Chatterjea type contraction, then we have

$$\begin{aligned}
\phi\left(p\left(x_n,x_{n+1}\right)\right) &= \phi\left(p\left(Tx_{n-1},Tx_n\right)\right)\\
&\leq \phi\left(\alpha p\left(x_{n-1},Tx_n\right)+\beta p\left(x_n,Tx_{n-1}\right)\right)\\
&\quad -\psi\left(p\left(x_{n-1},Tx_n\right),p\left(x_n,Tx_{n-1}\right)\right)\\
&= \phi\left(\alpha p\left(x_{n-1},x_{n+1}\right)+\beta p\left(x_n,x_n\right)\right)\\
&\quad -\psi\left(p\left(x_{n-1},x_{n+1}\right),p\left(x_n,x_n\right)\right)\\
&\leq \phi\left(\alpha p\left(x_{n-1},x_{n+1}\right)+\beta p\left(x_n,x_n\right)\right).
\end{aligned}$$

Since ϕ is a non-decreasing function, we get

$$p\left(x_n,x_{n+1}\right)\leq \alpha p\left(x_{n-1},x_{n+1}\right)+\beta p\left(x_n,x_n\right),$$

and by triangular inequality, we have

$$\begin{aligned}
p\left(x_n,x_{n+1}\right) &\leq \alpha p\left(x_{n-1},x_{n+1}\right)+\beta p\left(x_n,x_n\right)\\
&\leq \alpha\left[p\left(x_{n-1},x_n\right)+p\left(x_n,x_{n+1}\right)-p\left(x_n,x_n\right)\right]+\beta p\left(x_n,x_n\right)\\
&= \alpha\left[p\left(x_{n-1},x_n\right)+p\left(x_n,x_{n+1}\right)\right]+\left(\beta-\alpha\right)p\left(x_n,x_n\right)\\
&\leq \alpha\left[p\left(x_{n-1},x_n\right)+p\left(x_n,x_{n+1}\right)\right]+\left(\beta-\alpha\right)p\left(x_n,x_{n-1}\right) \qquad \text{(by } PM3\text{)}\\
&= \beta p\left(x_{n-1},x_n\right)+\alpha p\left(x_n,x_{n+1}\right)
\end{aligned}$$

which implies

$$p\left(x_n,x_{n+1}\right)\leq\left(\frac{\beta}{1-\alpha}\right)p\left(x_{n-1},x_n\right).$$

Since $0<\alpha+\beta<1$, then $\frac{\beta}{1-\alpha}<1$, and by induction, we have

$$p\left(x_n,x_{n+1}\right)\leq\left(\frac{\beta}{1-\alpha}\right)^n p\left(x_0,x_1\right),$$

and hence, $\lim_{n\to\infty}p\left(x_n,x_{n+1}\right)=0$.

In the sequel, we show that (x_n) is a Cauchy sequence in X. To do so, we need to prove first, the claim that for every $\epsilon>0$, there exists $n\in\mathbb{N}$ such that if $p,q\geq n$ with $p-q\equiv 1\,(m)$, then $p\left(x_p,x_q\right)<\epsilon$. Suppose the contrary case, i.e., there exists $\epsilon>0$ such that for any $n\in\mathbb{N}$, we can find $p_n>q_n\geq n$ with $p_n-q_n\equiv 1\,(m)$ satisfying $p\left(x_{p_n},x_{q_n}\right)\geq\epsilon$. Now, we take $n>2m$. Then corresponding to $q_n\geq n$, we can choose p_n in such a way that it is the smallest integer with $p_n>q_n$ satisfying $p_n-q_n\equiv 1\,(m)$ and $p\left(x_{p_n},x_{q_n}\right)\geq\epsilon$. Therefore, $p\left(x_{q_n},x_{p_{n-m}}\right)<\epsilon$. Using the triangular inequality,

$$\begin{aligned}
\epsilon &\leq p\left(x_{p_n},x_{q_n}\right)\\
&\leq p\left(x_{q_n},x_{p_{n-m}}\right)+\sum_{i=1}^{m}p\left(x_{p_{n-i}},x_{p_{n-i+1}}\right)-\sum_{j=1}^{m}p\left(x_{p_{n-j}},x_{p_{n-j}}\right)\\
&\leq p\left(x_{q_n},x_{p_{n-m}}\right)+\sum_{i=1}^{m}p\left(x_{p_{n-i}},x_{p_{n-i+1}}\right)\\
&< \epsilon+\sum_{i=1}^{m}p\left(x_{p_{n-i}},x_{p_{n-i+1}}\right).
\end{aligned}$$

Letting $n \to \infty$ in the the last inequality, and taking into account that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we obtain $\lim_{n\to\infty} p(x_{p_n}, x_{q_n}) = \epsilon$. Again, by triangle inequality, we have

$$\begin{aligned}
\epsilon &\leq p(x_{q_n}, x_{p_n}) \\
&\leq p\left(x_{q_n}, x_{q_{n+1}}\right) + p\left(x_{q_{n+1}}, x_{p_{n+1}}\right) + p\left(x_{p_{n+1}}, x_{p_n}\right) \\
&\quad -p\left(x_{p_{n+1}}, x_{p_{n+1}}\right) - p\left(x_{q_{n+1}}, x_{q_{n+1}}\right) \\
&\leq p\left(x_{q_n}, x_{q_{n+1}}\right) + p\left(x_{q_{n+1}}, x_{q_n}\right) + p\left(x_{q_n}, x_{p_n}\right) \\
&\quad +p\left(x_{p_n}, x_{p_{n+1}}\right) + p\left(x_{p_{n+1}}, x_{p_n}\right) - p\left(x_{p_{n+1}}, x_{p_{n+1}}\right) \\
&\quad -p\left(x_{q_{n+1}}, x_{q_{n+1}}\right) - p\left(x_{q_n}, x_{q_n}\right) - p\left(x_{p_n}, x_{p_n}\right) \\
&\leq 2p\left(x_{q_n}, x_{q_{n+1}}\right) + p\left(x_{q_n}, x_{p_n}\right) + 2p\left(x_{p_n}, x_{p_{n+1}}\right).
\end{aligned}$$

Taking the limit as $n \to \infty$, and taking into account that $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we get $\lim_{n\to\infty} p\left(x_{q_{n+1}}, x_{p_{n+1}}\right) = \epsilon$. Since x_{p_n} and x_{q_n} lie in different adjacently labelled sets A_i and A_{i+1} for certain $1 \leq i \leq m$, assuming that T is a cyclic $(\phi - \psi)$-Kannan type contraction, we have

$$\begin{aligned}
\phi\left(p\left(x_{q_{n+1}}, x_{p_{n+1}}\right)\right) &= \phi\left(p\left(Tx_{q_n}, Tx_{p_n}\right)\right) \\
&\leq \phi\left(\alpha p\left(x_{q_n}, Tx_{q_n}\right) + \beta p\left(x_{p_n}, Tx_{p_n}\right)\right) \\
&\quad -\psi\left(p\left(x_{q_n}, Tx_{q_n}\right), p\left(x_{p_n}, Tx_{p_n}\right)\right) \\
&= \phi\left(\alpha p\left(x_{q_n}, x_{q_{n+1}}\right) + \beta p\left(x_{p_n}, x_{p_{n+1}}\right)\right) \\
&\quad -\psi\left(p\left(x_{q_n}, x_{q_{n+1}}\right), p\left(x_{p_n}, x_{p_{n+1}}\right)\right).
\end{aligned}$$

Letting $n \to \infty$ in the last equality, we obtain

$$\phi(\epsilon) \leq \phi(0) - \psi(0, 0) = 0.$$

Therefore, we get $\epsilon = 0$ which is a contradiction.

Similarly, assuming that T is a cyclic $(\phi - \psi)$-Chatterjea type contraction, we have

$$\begin{aligned}
\phi\left(p\left(x_{q_{n+1}}, x_{p_{n+1}}\right)\right) &= \phi\left(p\left(Tx_{q_n}, Tx_{p_n}\right)\right) \\
&\leq \phi\left(\alpha p\left(x_{q_n}, Tx_{p_n}\right) + \beta p\left(x_{p_n}, Tx_{q_n}\right)\right) \\
&\quad -\psi\left(p\left(x_{q_n}, Tx_{p_n}\right), p\left(x_{p_n}, Tx_{q_n}\right)\right) \\
&= \phi\left(\alpha p\left(x_{q_n}, x_{p_{n+1}}\right) + \beta p\left(x_{p_n}, x_{q_{n+1}}\right)\right) \\
&\quad -\psi\left(p\left(x_{q_n}, x_{p_{n+1}}\right), p\left(x_{p_n}, x_{q_{n+1}}\right)\right).
\end{aligned}$$

Letting $n \to \infty$ in the last equality, we obtain

$$\phi(\epsilon) \leq \phi((\alpha + \beta)\epsilon) - \psi(\epsilon, \epsilon).$$

Therefore, since $0 < \alpha + \beta < 1$, we get $\psi(\epsilon, \epsilon) = 0$, and hence, $\epsilon = 0$ which is a contradiction.

From the above proved claim for both cases, i.e., the case when T is a cyclic $(\phi - \psi)$-Kannan type contraction and the case when T is a cyclic $(\phi - \psi)$-Chatterjea type contraction, and for arbitrary $\epsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that if $p, q > n_0$ with $p - q \equiv 1\,(m)$, then $p(x_p, x_q) < \epsilon$. Since $\lim_{n\to\infty} p(x_n, x_{n+1}) = 0$, we can find $n_1 \in \mathbb{N}$ such that

$$p(x_n, x_{n+1}) \leq \frac{\epsilon}{m}, \text{ for } n > n_1.$$

Now, for $r, s > \max\{n_0, n_1\}$ and $s > r$, there exists $k \in \{1, 2, ..., m\}$ such that $s - r \equiv k\,(m)$. Therefore, $s - r + j \equiv 1\,(m)$ for $j = m - k + 1$. So, we have

$$\begin{aligned} p(x_r, x_s) &\leq p(x_r, x_{s+j}) + p(x_{s+j}, x_{s+j-1}) \\ &\quad + ... + p(x_{s+1}, x_s) - \sum_{i=1}^{j} p(x_{s+i}, x_{s+i}) \\ &\leq p(x_r, x_{s+j}) + p(x_{s+j}, x_{s+j-1}) + ... + p(x_{s+1}, x_s). \end{aligned}$$

This implies

$$p(x_r, x_s) \leq \epsilon + \frac{\epsilon}{m} \sum_{j=1}^{m} 1 = 2\epsilon.$$

Thus, (x_n) is a Cauchy sequence in $\bigcup_{i=1}^{m} A_i$. Consequently, (x_n) converges to some $z \in \bigcap_{i=1}^{m} A_i$. However in view of cyclical condition, the sequence (x_n) has an infinite number of terms in each A_i, for $i = 1, 2, ..., m$. Therefore $z \in \bigcap_{i=1}^{m} A_i$.

Now, we will prove that z is a fixed point of T. Suppose that $z \in A_i, Tz \in A_{i+1,}$ and we take a subsequence x_{n_k} of (x_n) with $x_{n_k} \in A_{i-1}$. Then, assuming that T is a cyclic $(\phi - \psi)$-Kannan type contraction, we have

$$\begin{aligned} \phi\left(p\left(x_{n_{k+1}}, Tz\right)\right) &= \phi\left(p\left(Tx_{n_k}, Tz\right)\right) \\ &\leq \phi\left(\alpha p\left(x_{n_k}, Tx_{n_k}\right) + \beta p(z, Tz)\right) \\ &\quad -\psi\left(p\left(x_{n_k}, Tx_{n_k}\right), p(z, Tz)\right) \\ &\leq \phi\left(\alpha p\left(x_{n_k}, Tx_{n_k}\right) + \beta p(z, Tz)\right). \end{aligned}$$

Letting $k \to \infty$, we have

$$\phi(p(z, Tz)) \leq \phi(\alpha p(z, Tz) + \beta p(z, Tz)),$$

since ϕ is a non-decreasing function, we get

$$p(z, Tz) \leq (\alpha + \beta)\, p(z, Tz).$$

Thus, since $0 < \alpha + \beta \leq 1$, we have $p(z, Tz) = 0$, and hence, $z = Tz$.

Similarly, assuming that T is a cyclic $(\phi - \psi)$-Chatterjea type contraction, we have

$$\begin{aligned} \phi\left(p\left(x_{n_{k+1}}, Tz\right)\right) &= \phi\left(p\left(Tx_{n_k}, Tz\right)\right) \\ &\leq \phi\left(\alpha p\left(x_{n_k}, Tz\right) + \beta p\left(z, Tx_{n_k}\right)\right) \\ &\quad -\psi\left(p\left(x_{n_k}, Tz\right), p\left(z, Tx_{n_k}\right)\right) \\ &\leq \phi\left(\alpha p\left(x_{n_k}, Tz\right) + \beta p\left(z, Tx_{n_k}\right)\right). \end{aligned}$$

Letting $k \to \infty$, we have

$$\phi(p(z, Tz)) \leq \phi(\alpha p(z, Tz) + \beta p(z, Tz)),$$

since ϕ is a non-decreasing function, we get

$$\begin{aligned} p(z, Tz) &\leq \alpha p(z, Tz) + \beta p(z, Tz) \\ &= (\alpha + \beta) p(z, Tz). \end{aligned}$$

Thus, since $0 < \alpha + \beta < 1$, we have $p(z, Tz) = 0$, and hence, $z = Tz$. □

Corollary 2.1. *Let X be a complete partial metric space, m positive integer, $A_1, A_2, ..., A_m$ non-empty closed subsets of X, and $X = \bigcup_{i=1}^{m} A_i$. Let $T : X \to X$ be an operator such that*

(i) $X = \bigcup_{i=1}^{m} A_i$ is a cyclic representation of X with respect to T,

(ii) for any $x \in A_i, y \in A_{i+1}, i = 1, 2, ..., m$, where $A_{m+1} = A_1$ and $\rho : [0, \infty) \to [0, \infty)$ is a Lebesgue integrable mapping satisfies $\int_0^t \rho(s)ds > 0$ for $t > 0$, we have one of the following:

$$\int_0^{p(Tx,Ty)} \rho(t)dt \leq \int_0^{\alpha p(x,Tx)+\beta p(y,Ty)} \rho(t)dt,$$

or

$$\int_0^{p(Tx,Ty)} \rho(t)dt \leq \int_0^{\alpha p(Tx,y)+\beta p(Ty,x)} \rho(t)dt.$$

Then T has a unique fixed point $z \in \bigcap_{i=1}^{m} A_i$.

Proof. Let $\phi : [0, \infty) \to [0, \infty)$ be defined as $\phi(t) = \int_0^t \rho(s)ds > 0$. Then ϕ is an altering distance function, and by taking $\psi = 0$, we get the result. □

Example 2.1. *Let $X \subset l^1, X = \{(x_n) \in l^1 : x_n \geq 0$ for all $n \in \mathbb{N}\}$. Define a partial metric p on X by*

$$p((x_n), (y_n)) = \sum_{n=1}^{\infty} \max\{x_n, y_n\}.$$

Let the set $\left\{ \delta_n = \left(\underbrace{0, 0, ..., 1}_{nth\ term}, 0, 0, ... \right), n \in \mathbb{N} \right\}$ be the standard basis for l^1. Let $\beta \in (0, 1)$ be fixed and consider the sets

$$\begin{aligned} A_1 &= \left\{ \sum_{k=l}^{\infty} \beta^{2k} \delta_{2k}, \quad l = 1, 2, 3.... \right\}, \\ B_2 &= \left\{ \sum_{k=l}^{\infty} \beta^{2k-1} \delta_{2k-1}, \quad l = 1, 2, 3.... \right\}. \end{aligned}$$

Let $A = A_1 \cup \{0\}$ and $B = B_1 \cup \{0\}$, and $Y = A \cup B$, where $0 = (0, 0, 0, 0...)$. Consider the map $T : Y \to Y$ given by

$$\begin{aligned} T\left(\sum_{k=l}^{\infty} \beta^{2k} \delta_{2k} \right) &= \sum_{k=l}^{\infty} \beta^{2k+1} \delta_{2k+1} \\ T\left(\sum_{k=l}^{\infty} \beta^{2k+1} \delta_{2k+1} \right) &= \sum_{k=l}^{\infty} \beta^{2k+2} \delta_{2k+2} \end{aligned}$$

It is easy to see that $T(A) \subset B$ and $T(B) \subset A$ and $Y = A \cup B$ is a cyclic representation of Y with respect to T. Now Let $A \ni x = \sum_{k=l}^{\infty} \beta^{2k} \delta_{2k}$ and $B \ni y = \sum_{k=m}^{\infty} \beta^{2k+1} \delta_{2k+1}$. Suppose *that $l \leq m$, (for the case $l > m$ is similar). Then*

$$T(x) = \sum_{k=l}^{\infty} \beta^{2k+1}\delta_{2k+1} \text{ and } T(y) = \sum_{k=m}^{\infty} \beta^{2k+2}\delta_{2k+2}$$

and

$$\begin{aligned} p(x,y) &= \sum_{k=l}^{m-1} \beta^{2k} + \frac{\beta^{2m}}{1-\beta} \\ p(T(x),T(y)) &= \sum_{k=l}^{m-1} \beta^{2k+1} + \frac{\beta^{2m+1}}{1-\beta} \\ p(x,T(x)) &= \frac{\beta^{2l}}{1-\beta},\ p(y,T(y)) = \frac{\beta^{2m+1}}{1-\beta}. \end{aligned}$$

Consequently for some positive real number $\alpha > 0$ we have,

$$\begin{aligned} p(T(x),T(y)) &= \sum_{k=l}^{m-1} \beta^{2k+1} + \frac{\beta^{2m+1}}{1-\beta} \\ &\leq \beta\frac{\beta^{2l}}{1-\beta} \leq \beta\frac{\beta^{2l}}{1-\beta} + \alpha\frac{\beta^{2m+1}}{1-\beta} \\ &= (\beta+1-1)\frac{\beta^{2l}}{1-\beta} + \alpha\frac{\beta^{2m+1}}{1-\beta} \\ &= \beta\frac{\beta^{2l}}{1-\beta} + \alpha\frac{\beta^{2m+1}}{1-\beta} - (1-\beta)\frac{\beta^{2l}}{1-\beta} \end{aligned}$$

Now if $\psi : [0,\infty)^2 \to [0,\infty)$ is taken such that $\psi(x,y) = (1-\beta)\max\{x,y\}$ we get

$$\begin{aligned} p(T(x),T(y)) &= \beta p(x,T(x)) + \alpha p(y,T(y)) - (1-\beta)\max\left\{\frac{\beta^{2l}}{1-\beta}, \frac{\beta^{2m+1}}{1-\beta}\right\} \\ &= \beta p(x,T(x)) + \alpha p(y,T(y)) - \psi(p(x,T(x)), p(y,T(y))) \end{aligned}$$

and so taking the altering distance ϕ to be $\phi(t) = t$,we get the result.

Example 2.2. *Let $X = [-1,1] \subseteq \mathbb{R}$ with*

$$p(x,y) = \begin{cases} \max\{x,y\} & x,y \in [0,1] \\ \max\{|x|,|y|\} & x,y \in [-1,0] \\ |x-y| & \text{otherwise} \end{cases}$$

.

It is not hard to see that p is a partial metric on $[-1,1]$.
Let $T : [-1,1] \to [-1,1]$ be given by

$$T(x) = \begin{cases} -\frac{1}{2}xe^{-\frac{1}{|x|}}, & x \in (0,1], \\ 0, & x = 0, \\ -\frac{1}{3}xe^{-\frac{1}{|x|}}, & x \in [-1,0). \end{cases}$$

By taking $\psi = 0, \phi(t) = t$ *and* $x \in [0,1], y \in [-1,0]$,

$$\begin{aligned} p(Tx,Ty) &= |Tx - Ty| = \left| -\frac{1}{2}xe^{-\frac{1}{|x|}} + \frac{1}{3}ye^{-\frac{1}{|y|}} \right| \\ &\leq \frac{1}{2}|x| + \frac{1}{3}|y| \\ &\leq \frac{1}{2}\left| x + \frac{1}{2}xe^{-\frac{1}{|x|}} \right| + \frac{1}{3}\left| y + \frac{1}{3}ye^{-\frac{1}{|y|}} \right| \\ &= \frac{1}{2}|Tx - x| + \frac{1}{3}|Ty - y| \\ &= \frac{1}{2}p(x,Tx) + \frac{1}{3}p(y,Ty) \end{aligned}$$

which implies that T *has a unique fixed point in* $[-1,0] \cap [0,1]$ *which is* $z = 0$.

References

[1] Mattews,S.G., (1992), Partial metric topology. research report 212. Department of Computer Science, University of warwick.

[2] Mattews,S.G., (1994), Partial metric topology. General Topology and its Applications. Proceedings of the 8th Summer Conference, Queen's College. Ann NY Acad Sci, 728, pp. 183-197.

[3] Oltra,S. and Valero,O., Banach's fixed point theorem for partial metric spaces. Rendiconti dell'Istituto di Mathematica dell'Universit di Trieste (2004), 36(1-2), pp. 17-26.

[4] Valero,O., On Banach's fixed point theorems for partial metric spaces. Appl Gen Topol. 62 (2005), pp. 229-240.

[5] Altun,I., Sola,F. and Simsek,H., Generalized contractions on partial metric spaces. Topol Appl 157, 18 (2010), pp. 2778-2785.

[6] Altun,I. and Erduran,A., Fixed point theorems for monotone mappings on partial metric spaces. Fixed Point Theory Appl (2011), pp. 10.

[7] Karapinar,E. and Inci,M., Fixed point theorems for operators on partial metric spaces. Appl Math Lett 24,11 (2011), pp. 1894-1899.

[8] Abdeljawad,T., Karapinar,E. and Tas,K., (2011), Existence and uniqueness of a common fixed point on partial metric spaces, Appl. Math. Lett. 24, pp. 1900-1904.

[9] Nashine,H.K. and Kadelburg,Z., Cyclic contractions and fixed point results via control functions on partial metric spaces, International Journal of Analysis Volume 2013, Article ID 726387, pp. 9, http://dx.doi.org/10.1155/2013/726387.

[10] Banach,S., (1922), Surles operations dans les ensembles et leur application aux equation sitegrales, Fund. Math., 3, pp. 133-181.

[11] Nieto,J. and Rodriguez-Lopez,R., (2007), Existence and uniqueness of fixed point in partially orderded sets and applications to ordinary differential equations, Acta Math. Sinica, 23,12, pp. 2205-2210.

[12] Agarwal,R., El-Gebeily,M.A. and O'regan,D.,(2008), Generalized contraction in partially ordered metric spaces, Appl, Anal. 87, 1, pp. 109-116.

[13] Harjani,J. and Sadarangani,K., (2008), Fixed point theorems for weakly contractive mappings in partially ordered sets, Nonlinear Anal., 71 (7-8), pp. 3403-3410.

[14] Harjani,J. and Sadarangani,K., (2010), Generalized contractions in partialy ordered metric spaces and applications to ordinary equations, Nonlinear Anal. 72 (3-4), pp. 1188-1197.

[15] Kannan,R., (1968), Some results on fixed points, Bull. Calcutta Math. Soc. 10, pp. 71-76.

[16] Rus,I., (2005), Cyclic representations and fixed points, Annals of the Tiberiu Popoviciu Seminar of Functional Equations, Approximation and Convexity, 3, pp. 171-178.

[17] Chatterjea,S., (1972), Fixed point theorems, C. R. Acad. Bulgare Sci. 25, pp. 727-730.

[18] Petric,M., (2010), On some cyclical contractive conditions, General Mathematics 18, No. 4, pp. 213-226.

[19] Choudhury,B.S., (2009), Unique fixed point theorem for weak C-contractive mappings, Kathmandu Univ. J. Sci. Eng. Tech. 5,1, pp. 6-13.

[20] Khan,M., Swaleh,M. and Sessa,S., (1984), Fixed point theorem by altering distances between points, bull. Austral. Math. Sos., 30(1), pp. 1-9.

[21] SastryK.R. and Babu,G.V., (1999), Some fixed point theorems by altering distances between the points, Indian Journal of Pure and Applied Mathematics, 30, 6, pp. 641-647.
[22] Sastry,K., Naidu,S., Babu,G. and Naidu,G.A., (2000), Generalization of common fixed point theorems for weakly commuting map by altering distances, Tamkang Journal of Mathematics, 31,3, pp. 243-250.
[23] Naidu,S., (2003), Some fixed point theorems in metric spaces by altering distances, Czechoslovak Mathematical Journal, 53,1, pp. 205–212.

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