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# COMMON COUPLED FIXED POINT THEOREM UNDER WEAK $\psi - \varphi$ CONTRACTION FOR HYBRID PAIR OF MAPPINGS WITH APPLICATION

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Abstract. We establish a common coupled fixed point theorem for hybrid pair of mappings under weak $\psi - \varphi$ contraction on a non-complete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. Moreover, an example and an application to integral equations are given here to illustrate the usability of the obtained results. We improve, extend, and generalize several known results.

Keywords: Coupled fixed point, coupled coincidence point, weak $\psi - \varphi$ contraction, $w-$compatibility, $F-$weakly commuting mappings.

AMS Subject Classification: 47H10, 54H25.

## 1. Introduction and Preliminaries

Let $(X, d)$ be a metric space. We denote by $2^X$ the class of all nonempty subsets of $X$, by $CL(X)$ the class of all nonempty closed subsets of $X$, by $CB(X)$ the class of all nonempty closed bounded subsets of $X$ and by $K(X)$ the class of all nonempty compact subsets of $X$. A functional $H : CL(X) \times CL(X) \to \mathbb{R}_+ \cup \{+\infty\}$ is said to be the Pompeiu-Hausdorff generalized metric induced by $d$ is given by

$$H(A,\ B) = \begin{cases} \max\left\{\sup_{a\in A} D(a,\ B),\ \sup_{b\in B} D(b,\ A)\right\}, & \text{if maximum exists,} \\ +\infty, & \text{otherwise,} \end{cases}$$

for all $A, B \in CL(X)$, where $D(x, A) = \inf_{a\in A} d(x, a)$ denote the distance from $x$ to $A \subset X$. For simplicity, if $x \in X$, we denote $g(x)$ by $gx$.

The existence of fixed points for various multivalued contractions and non-expansive mappings has been studied by many authors under different conditions which was initiated by Markin [22]. The theory of multivalued mappings has applications in control theory, convex optimization, differential inclusions and economics.

In 1987, Guo and Lakshmikantham [17] gave the notion of coupled fixed point. Following this paper, Gnana-Bhaskar and Lakshmikantham [4] introduced the concept of mixed monotone property for $F : X \times X \to X$ (where $X$ is an ordered metric space) and proved

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some results on the existence and uniqueness of coupled fixed points. Later on, Lakshmikantham and Ciric [18] generalized these results for nonlinear contraction mappings by introducing the notions of coupled coincidence point and mixed g-monotone property in 2009. These results are applied for proving the existence and uniqueness of the solution for periodic boundary value problems. Many authors focused on coupled fixed point theory including [3, 5, 10, 11, 20, 21, 25, 29].

Samet et al. [27] claimed that most of the coupled fixed point theorems in the setting of single-valued mappings on ordered metric spaces are consequences of well-known fixed point theorems.

The concepts related to coupled fixed point theory in the setting of multivalued mappings were extended by Abbas et al. [2] and obtained coupled coincidence point and common coupled fixed point theorems involving hybrid pair of mappings satisfying generalized contractive conditions in complete metric spaces. Very few papers were devoted to coupled fixed point problems for hybrid pair of mappings including [1, 2, 12, 13, 19, 28].

In [2], Abbas et al. introduced the following for multivalued mappings:

**Definition 1.1.** *Let $X$ be a non-empty set, $F : X \times X \to 2^X$ and $g$ be a self-mapping on $X$. An element $(x, y) \in X \times X$ is called*

(1) *a coupled fixed point of $F$ if $x \in F(x, y)$ and $y \in F(y, x)$.*

(2) *a coupled coincidence point of hybrid pair $\{F, g\}$ if $gx \in F(x, y)$ and $gy \in F(y, x)$.*

(3) *a common coupled fixed point of hybrid pair $\{F, g\}$ if $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$.*

*We denote the set of coupled coincidence points of mappings $F$ and $g$ by $C(F, g)$. Note that if $(x, y) \in C(F, g)$, then $(y, x)$ is also in $C(F, g)$.*

**Definition 1.2.** *Let $F : X \times X \to 2^X$ be a multivalued mapping and $g$ be a self-mapping on $X$. The hybrid pair $\{F, g\}$ is called $w-$compatible if $gF(x, y) \subseteq F(gx, gy)$ whenever $(x, y) \in C(F, g)$.*

**Definition 1.3.** *Let $F : X \times X \to 2^X$ be a multivalued mapping and $g$ be a self-mapping on $X$. The mapping $g$ is called $F-$weakly commuting at some point $(x, y) \in X \times X$ if $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$.*

**Lemma 1.1.** [26]. *Let $(X, d)$ be a metric space. Then, for each $a \in X$ and $B \in K(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b\in B} d(a, b)$.*

In [16], Gordji et al. established some fixed point theorems for $(\psi, \varphi)$-weak contractive mappings in a complete metric space endowed with a partial order. Our basic references are [5, 6, 7, 8, 9, 14, 15, 16, 23, 24, 26, 27].

In this paper, we establish a common coupled fixed point theorem for hybrid pair of mappings under weak $\psi - \varphi$ contraction on a non-complete metric space, which is not partially ordered. It is to be noted that to find coupled coincidence point, we do not employ the condition of continuity of any mapping involved therein. Moreover, an example and an application to integral equations are given here to illustrate the usability of the obtained results. We improve, extend and generalize the result of Gnana-Bhaskar and Lakshmikantham [4], Gordji et al. [16] and Lakshmikantham and Ciric [18].

## 2. Main results

Let $\Psi$ denote the set of all functions $\psi : [0, +\infty) \to [0, +\infty)$ satisfying
$(i_\psi)$ $\psi$ is continuous and non-decreasing,
$(ii_\psi)$ $\psi(t) = 0 \Leftrightarrow t = 0$,
$(iii_\psi)$ $\limsup_{s\to 0+} \frac{s}{\psi(s)} < \infty$.
Let $\Phi$ denote the set of all functions $\varphi : [0, +\infty) \to [0, +\infty)$ satisfying
$(i_\varphi)$ $\varphi$ is lower semi-continuous and non-decreasing,
$(ii_\varphi)$ $\varphi(t) = 0 \Leftrightarrow t = 0$,
$(iii_\varphi)$ for any sequence $\{t_n\}$ with $\lim_{n\to\infty} t_n = 0$, there exists $k \in (0, 1)$ and $n_0 \in \mathbb{N}$, such that $\varphi(t_n) \geq kt_n$ for each $n \geq n_0$.
Let $\Theta$ denote the set of all functions $\theta : [0, +\infty) \to [0, +\infty)$ satisfying
$(i_\theta)$ $\theta$ is continuous,
$(ii_\theta)$ $\theta(t) = 0 \Leftrightarrow t = 0$.

For simplicity, we define

$$\begin{aligned} & M(x,\ y,\ u,\ v) \\ = \ & \max\left\{ \begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2}, \\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array} \right\}, \end{aligned}$$

and

$$N(x,\ y,\ u,\ v) = \min\left\{ \begin{array}{c} D(gx,\ F(u,\ v)),\ D(gu,\ F(x,\ y)), \\ D(gy,\ F(v,\ u)),\ D(gv,\ F(y,\ x)) \end{array} \right\},$$

and

$$m(x,\ y,\ u,\ v) = \max\left\{ \begin{array}{c} d(x,\ u),\ d(x,\ F(x,\ y)),\ d(u,\ F(u,\ v)), \\ d(y,\ v),\ d(y,\ F(y,\ x)),\ d(v,\ F(v,\ u)), \\ \frac{d(x,\ F(u,\ v))+d(u,\ F(x,\ y))}{2}, \\ \frac{d(y,\ F(v,\ u))+d(v,\ F(y,\ x))}{2} \end{array} \right\},$$

and

$$n(x,\ y,\ u,\ v) = \min\left\{ \begin{array}{c} d(x,\ F(u,\ v)),\ d(u,\ F(x,\ y)), \\ d(y,\ F(v,\ u)),\ d(v,\ F(y,\ x)) \end{array} \right\}.$$

**Theorem 2.1.** *Let $(X, d)$ be a metric space, $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings. Suppose there exist some $\psi \in \Psi$, $\varphi \in \Phi$ and $\theta \in \Theta$ such that*

$$\begin{aligned} & \psi\left(H(F(x,\ y),\ F(u,\ v))\right) \\ \leq\ & \psi\left(M(x,\ y,\ u,\ v)\right) - \varphi\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right) + \theta\left(N(x,\ y,\ u,\ v)\right), \end{aligned} \tag{1}$$

*for all $x, y, u, v \in X$. Furthermore assume that $F(X\times X) \subseteq g(X)$ and $g(X)$ is a complete subset of $X$. Then $F$ and $g$ have a coupled coincidence point. Moreover, $F$ and $g$ have a common coupled fixed point, if one of the following conditions holds:*

*$(a)$ $F$ and $g$ are $w-$compatible. $\lim_{n\to\infty} g^n x = u$ and $\lim_{n\to\infty} g^n y = v$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$ and $g$ is continuous at $u$ and $v$.*

*$(b)$ $g$ is $F-$weakly commuting for some $(x, y) \in C(F, g)$ and $gx$ and $gy$ are fixed points of $g$, that is, $g^2x = gx$ and $g^2y = gy$.*

*$(c)$ $g$ is continuous at $x$ and $y$. $\lim_{n\to\infty} g^n u = x$ and $\lim_{n\to\infty} g^n v = y$ for some $(x, y) \in C(F, g)$ and for some $u, v \in X$.*

*$(d)$ $g(C(F, g))$ is a singleton subset of $C(F, g)$.*

*Proof.* Let $x_0, y_0 \in X$ be arbitrary. Then $F(x_0, y_0)$ and $F(y_0, x_0)$ are well defined. Choose $gx_1 \in F(x_0, y_0)$ and $gy_1 \in F(y_0, x_0)$, because $F(X\times X) \subseteq g(X)$. Since $F : X\times X \to K(X)$, therefore by Lemma 1.1, there exist $z_1 \in F(x_1,\ y_1)$ and $z_2 \in F(y_1,\ x_1)$ such that

$$\begin{aligned} d(gx_1,\ z_1) &\leq H(F(x_0,\ y_0),\ F(x_1,\ y_1)), \\ d(gy_1,\ z_2) &\leq H(F(y_0,\ x_0),\ F(y_1,\ x_1)). \end{aligned}$$

Since $F(X \times X) \subseteq g(X)$, there exist $x_2, y_2 \in X$ such that $z_1 = gx_2$ and $z_2 = gy_2$. Thus

$$\begin{aligned} d(gx_1,\ gx_2) &\leq H(F(x_0,\ y_0),\ F(x_1,\ y_1)), \\ d(gy_1,\ gy_2) &\leq H(F(y_0,\ x_0),\ F(y_1,\ x_1)). \end{aligned}$$

Continuing this process, we obtain sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that for all $n \in \mathbb{N}$, we have $gx_{n+1} \in F(x_n,\ y_n)$ and $gy_{n+1} \in F(y_n,\ x_n)$ such that

$$\begin{aligned} d(gx_{n+1},\ gx_{n+2}) &\leq H(F(x_n,\ y_n),\ F(x_{n+1},\ y_{n+1})), \\ d(gy_{n+1},\ gy_{n+2}) &\leq H(F(y_n,\ x_n),\ F(y_{n+1},\ x_{n+1})). \end{aligned}$$

Suppose first that $gx_{n_0} = gx_{n_0+1}$ and $gy_{n_0} = gy_{n_0+1}$ for some $n_0$. Then, the sequences $\{gx_n\}$ and $\{gy_n\}$ is constant for $n \geq n_0$. Indeed, let $n_0 = k$. Then $gx_k = gx_{k+1}$ and $gy_k = gy_{k+1}$. Now, by $(i_\psi)$ and (1), we obtain

$$\begin{aligned} & \psi\left(d(gx_{k+1},\ gx_{k+2})\right) \\ \leq\ & \psi\left(H(F(x_k,\ y_k),\ F(x_{k+1},\ y_{k+1}))\right) \\ \leq\ & \psi\left(M(x_k,\ y_k,\ x_{k+1},\ y_{k+1})\right) - \varphi\left(\psi\left(M(x_k,\ y_k,\ x_{k+1},\ y_{k+1})\right)\right) \\ & +\theta\left(N(x_k,\ y_k,\ x_{k+1},\ y_{k+1})\right), \end{aligned} \tag{2}$$

where

$$\begin{aligned} & M(x_k,\ y_k,\ x_{k+1},\ y_{k+1}) \\ =\ & \max\left\{ \begin{array}{c} d(gx_k, gx_{k+1}), D(gx_k, F(x_k, y_k)), D(gx_{k+1}, F(x_{k+1}, y_{k+1})), \\ d(gy_k, gy_{k+1}), D(gy_k, F(y_k, x_k)), D(gy_{k+1}, F(y_{k+1}, x_{k+1})), \\ \frac{D(gx_k, F(x_{k+1}, y_{k+1})) + D(gx_{k+1}, F(x_k, y_k))}{2}, \\ \frac{D(gy_k, F(y_{k+1}, x_{k+1})) + D(gy_{k+1}, F(y_k, x_k))}{2} \end{array} \right\} \\ \leq\ & \max\left\{ \begin{array}{c} d(gx_k,\ gx_{k+1}),\ d(gx_k,\ gx_{k+1}),\ d(gx_{k+1},\ gx_{k+2}), \\ d(gy_k,\ gy_{k+1}),\ d(gy_k,\ gy_{k+1}), d(gy_{k+1},\ gy_{k+2}), \\ \frac{d(gx_k,\ gx_{k+2}) + d(gx_{k+1},\ gx_{k+1})}{2}, \frac{d(gy_k,\ gy_{k+2}) + d(gy_{k+1},\ gy_{k+1})}{2} \end{array} \right\} \\ \leq\ & \max\left\{ \begin{array}{c} d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2}), \\ \frac{d(gx_k,\ gx_{k+2})}{2}, \frac{d(gy_k,\ gy_{k+2})}{2} \end{array} \right\} \\ \leq\ & \max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\}, \end{aligned}$$

and

$$\begin{aligned} & \theta\left(N(x_k,\ y_k,\ x_{k+1},\ y_{k+1})\right) \\ =\ & \min\left\{ \begin{array}{c} D(gx_k,\ F(x_{k+1},\ y_{k+1})),\ D(gx_{k+1},\ F(x_k,\ y_k)), \\ D(gy_k,\ F(y_{k+1},\ x_{k+1})),\ D(gy_{k+1},\ F(y_k,\ x_k)) \end{array} \right\} \\ =\ & 0. \end{aligned}$$

Thus, by (2) and $(ii_\theta)$, we get

$$\begin{aligned} & \psi\left(d(gx_{k+1},\ gx_{k+2})\right) \\ \leq\ & \psi\left(\max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\}\right) \\ & -\varphi\left(\psi\left(\max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\}\right)\right). \end{aligned}$$

Similarly

$$\begin{aligned}
& \psi\left(d(gy_{k+1},\ gy_{k+2})\right) \\
\leq\ & \psi\left(\max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\}\right) \\
& -\varphi\left(\psi\left(\max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\}\right)\right).
\end{aligned}$$

Combining them, we get

$$\begin{aligned}
& \max\left\{\psi\left(d(gx_{k+1},\ gx_{k+2})\right),\ \psi\left(d(gy_{k+1},\ gy_{k+2})\right)\right\} \\
\leq\ & \psi\left(\max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\}\right) \\
& -\varphi\left(\psi\left(\max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\}\right)\right).
\end{aligned}$$

Since $\psi$ is non-decreasing, it follows that

$$\begin{aligned}
& \psi\left(\max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\}\right) \\
\leq\ & \psi\left(\max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\}\right) \\
& -\varphi\left(\psi\left(\max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\}\right)\right),
\end{aligned}$$

which, by $(ii_\phi)$ and $(ii_\psi)$, implies that

$$\max\left\{d(gx_{k+1},\ gx_{k+2}),\ d(gy_{k+1},\ gy_{k+2})\right\} = 0.$$

It follows that

$$d(gx_{k+1},\ gx_{k+2}) = 0 \text{ and } d(gy_{k+1},\ gy_{k+2}) = 0,$$

and so $gx_{k+1} = gx_{k+2}$, $gy_{k+1} = gy_{k+2}$. Thus the sequences $\{gx_n\}$ and $\{gy_n\}$ are constants (starting from some $n_0$).

Suppose that $\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} > 0$ for each $n \in \mathbb{N}$. It is clear that $N(x_n,\ y_n,\ x_{n+1},\ y_{n+1}) = 0$ for all $n \in \mathbb{N}$. Now, by condition (1) and $(i_\psi)$, we have

$$\begin{aligned}
& \psi\left(d(gx_{n+1},\ gx_{n+2})\right) \\
\leq\ & \psi\left(H(F(x_n,\ y_n),\ F(x_{n+1},\ y_{n+1}))\right) \\
\leq\ & \psi\left(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\right) - \varphi\left(\psi\left(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\right)\right) \\
& +\theta\left(N(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\right),
\end{aligned}$$

which, by $(ii_\theta)$, implies

$$\begin{aligned}
& \psi\left(d(gx_{n+1},\ gx_{n+2})\right) \\
\leq\ & \psi\left(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\right) - \varphi\left(\psi\left(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\right)\right),
\end{aligned} \tag{3}$$

which by the fact that $\varphi \geq 0$ implies

$$\psi\left(d(gx_{n+1},\ gx_{n+2})\right) \leq \psi\left(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\right).$$

Since $\psi$ is non-decreasing, therefore we obtain

$$d(gx_{n+1},\ gx_{n+2}) \leq M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}).$$

Similarly, we can obtain that

$$d(gy_{n+1},\ gy_{n+2}) \leq M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}).$$

Combining them, we get

$$\max\left\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\right\} \leq M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}). \tag{4}$$

Hence

$$\begin{aligned}
&M(x_n, y_n, x_{n+1}, y_{n+1})\\
=\ & \max\left\{\begin{array}{c} d(gx_n, gx_{n+1}), D(gx_n, F(x_n, y_n)), D(gx_{n+1}, F(x_{n+1}, y_{n+1})),\\ d(gy_n, gy_{n+1}), D(gy_n, F(y_n, x_n)), D(gy_{n+1}, F(y_{n+1}, x_{n+1})),\\ \frac{D(gx_n, F(x_{n+1}, y_{n+1}))+D(gx_{n+1}, F(x_n, y_n))}{2},\\ \frac{D(gy_n, F(y_{n+1}, x_{n+1}))+D(gy_{n+1}, F(y_n, x_n))}{2} \end{array}\right\}\\
\leq\ & \max\left\{\begin{array}{c} d(gx_n, gx_{n+1}), d(gx_n, gx_{n+1}), d(gx_{n+1}, gx_{n+2}),\\ d(gy_n, gy_{n+1}), d(gy_n, gy_{n+1}), d(gy_{n+1}, gy_{n+2}),\\ \frac{d(gx_n, gx_{n+2})+d(gx_{n+1}, gx_{n+1})}{2}, \frac{d(gy_n, gy_{n+2})+d(gy_{n+1}, gy_{n+1})}{2} \end{array}\right\}\\
\leq\ & \max\left\{\begin{array}{c} \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\},\\ \max\{d(gx_{n+1}, gx_{n+2}), d(gy_{n+1}, gy_{n+2})\} \end{array}\right\}.
\end{aligned}$$

If $\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\} \geq \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}$. Then

$$\begin{aligned}
&M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\\
\leq\ & \max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\}.
\end{aligned} \tag{5}$$

From (4) and (5), we get

$$\begin{aligned}
&M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\\
=\ & \max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\}.
\end{aligned}$$

Thus, by (3), we have

$$\begin{aligned}
&\psi(d(gx_{n+1},\ gx_{n+2}))\\
\leq\ & \psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\})\\
&-\varphi(\psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\})).
\end{aligned}$$

Similarly, we can obtain that

$$\begin{aligned}
&\psi(d(gy_{n+1},\ gy_{n+2}))\\
\leq\ & \psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\})\\
&-\varphi(\psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\})).
\end{aligned}$$

Combining them, we get

$$\begin{aligned}
&\max\{\psi(d(gx_{n+1},\ gx_{n+2})),\ \psi(d(gy_{n+1},\ gy_{n+2}))\}\\
\leq\ & \psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\})\\
&-\varphi(\psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\})).
\end{aligned}$$

Since $\psi$ is non-decreasing, therefore

$$\begin{aligned}
&\psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\})\\
\leq\ & \psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\})\\
&-\varphi(\psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\})),
\end{aligned}$$

which is only possible when $\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\} = 0$, it is a contradiction. Hence, $\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\} \leq \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}$ for some $n \in \mathbb{N}$. Then

$$\begin{aligned}
&M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\\
\leq\ & \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}.
\end{aligned} \tag{6}$$

Notice that

$$\begin{aligned} & M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}) \\ \geq\ & \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}. \end{aligned} \tag{7}$$

From (6) and (7), we get

$$\begin{aligned} & M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}) \\ =\ & \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}. \end{aligned}$$

Thus, by (4), weget

$$\begin{aligned} & \max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\} \\ \leq\ & \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}. \end{aligned}$$

This shows that the sequence $\{\delta_n\}_{n=0}^{\infty}$ given by

$$\delta_n = \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\},\ \text{for each } n \in \mathbb{N},$$

is a non-increasing sequence. Thus there exists $\delta \geq 0$ such that

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} = \delta. \tag{8}$$

Then

$$\lim_{n\to\infty} M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}) = \delta.$$

We shall prove that $\delta = 0$. Assume to the contrary that $\delta > 0$. Then, by condition (1) and $(i_\psi)$, we have

$$\begin{aligned} & \psi(d(gx_{n+1},\ gx_{n+2})) \\ \leq\ & \psi(H(F(x_n,\ y_n),\ F(x_{n+1},\ y_{n+1}))) \\ \leq\ & \psi(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})) - \varphi(\psi(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}))) \\ & +\theta(N(x_n,\ y_n,\ x_{n+1},\ y_{n+1})), \end{aligned}$$

which, by $(ii_\theta)$, implies

$$\begin{aligned} & \psi(d(gx_{n+1},\ gx_{n+2})) \\ \leq\ & \psi(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})) - \varphi(\psi(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}))). \end{aligned}$$

Similarly

$$\begin{aligned} & \psi(d(gy_{n+1},\ gy_{n+2})) \\ \leq\ & \psi(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})) - \varphi(\psi(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}))). \end{aligned}$$

Combining them, we get

$$\begin{aligned} & \max\{\psi(d(gx_{n+1},\ gx_{n+2})),\ \psi(d(gy_{n+1},\ gy_{n+2}))\} \\ \leq\ & \psi(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})) - \varphi(\psi(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}))). \end{aligned}$$

Since $\psi$ is non-decreasing, therefore

$$\begin{aligned} & \psi(\max\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\}) \\ \leq\ & \psi(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})) - \varphi(\psi(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}))). \end{aligned}$$

Letting $n \to \infty$ in the above inequality, by using $(i_\psi)$ and $(i_\varphi)$, we get

$$\psi(\delta) \leq \psi(\delta) - \varphi(\psi(\delta)),$$

which, by $(ii_\varphi)$ and $(ii_\psi)$, implies that

$$\delta = \lim_{n\to\infty} \max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\} = 0. \tag{9}$$

Now we shall show that $\{\delta_n\}_{n=0}^{\infty}$ is a Cauchy sequence in $X$. Since

$$\lim_{n\to\infty} M(x_n,\ y_n,\ x_{n+1},\ y_{n+1}) = 0,$$

then, by $(iii_\varphi)$, there exist $k \in (0,\ 1)$ and $n_0 \in \mathbb{N}$ such that

$$\varphi\left(\psi\left(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\right)\right) \geq k\psi\left(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\right),\ \forall n \geq n_0.$$

For any natural number $n \geq n_0$, by (3), we have

$$\begin{aligned} & \psi\left(d(gx_{n+1},\ gx_{n+2})\right) \\ \leq\ & (1-k)\psi\left(M(x_n,\ y_n,\ x_{n+1},\ y_{n+1})\right) \\ \leq\ & (1-k)\psi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}\right\}\right). \end{aligned}$$

Thus

$$\begin{aligned} & \psi\left(d(gx_{n+1},\ gx_{n+2})\right) \\ \leq\ & (1-k)\psi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}\right\}\right),\ \forall n \geq n_0. \end{aligned}$$

Similarly

$$\begin{aligned} & \psi\left(d(gy_{n+1},\ gy_{n+2})\right) \\ \leq\ & (1-k)\psi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}\right\}\right),\ \forall n \geq n_0. \end{aligned}$$

Combining them, we get

$$\begin{aligned} & \max\left\{\psi\left(d(gx_{n+1},\ gx_{n+2})\right),\ \psi\left(d(gy_{n+1},\ gy_{n+2})\right)\right\} \\ \leq\ & (1-k)\psi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}\right\}\right),\ \forall n \geq n_0. \end{aligned}$$

Since $\psi$ is non-decreasing, therefore

$$\begin{aligned} & \psi\left(\max\left\{d(gx_{n+1},\ gx_{n+2}),\ d(gy_{n+1},\ gy_{n+2})\right\}\right) \\ \leq\ & (1-k)\psi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1}\right\}\right),\ \forall n \geq n_0. \end{aligned} \tag{10}$$

Denote

$$a_n = \psi\left(\max\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\}\right),\ \text{for all } n \geq 0.$$

From (10), we have

$$a_{n+1} \leq (1-k)a_n,\ \text{for all } n \geq n_0.$$

Then, we have

$$\sum_{n=0}^{\infty} a_n \leq \sum_{n=0}^{n_0} a_n + \sum_{n=n_0+1}^{\infty} (1-k)^{n-n_0} a_{n_0} < \infty. \tag{11}$$

On the other hand, by $(iii_\psi)$, we have

$$\limsup_{n\to\infty} \frac{\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\}}{\psi\left(\max\left\{d(gx_n,\ gx_{n+1}),\ d(gy_n,\ gy_{n+1})\right\}\right)} < \infty. \tag{12}$$

Thus, by (11) and (12), we have $\sum \max\{d(gx_n, gx_{n+1}), d(gy_n, gy_{n+1})\} < \infty$. It means that $\{gx_n\}_{n=0}^{\infty}$ and $\{gy_n\}_{n=0}^{\infty}$ are Cauchy sequences in $g(X)$. Since $g(X)$ is complete, therefore there exist $x,\ y \in X$ such that

$$\lim_{n\to\infty} gx_n = gx \text{ and } \lim_{n\to\infty} gy_n = gy. \tag{13}$$

Now, since $gx_{n+1} \in F(x_n,\ y_n)$ and $gy_{n+1} \in F(y_n,\ x_n)$, therefore by using condition (1) and $(i_\psi)$, we get

$$\begin{aligned} & \psi\left(D(gx_{n+1},\ F(x,\ y))\right) \\ \leq\ & \psi\left(H(F(x_n,\ y_n),\ F(x,\ y))\right) \\ \leq\ & \psi\left(M(x_n,\ y_n,\ x,\ y)\right) - \varphi\left(\psi\left(M(x_n,\ y_n,\ x,\ y)\right)\right) + \theta\left(N(x_n,\ y_n,\ x,\ y)\right), \end{aligned}$$

where

$$
\begin{aligned}
&M(x_n, y_n, x, y)\\
= &\max\left\{\begin{array}{c} d(gx_n, gx), D(gx_n, F(x_n, y_n)), D(gx, F(x, y)),\\ d(gy_n, gy), D(gy_n, F(y_n, x_n)), D(gy, F(y, x)),\\ \frac{D(gx_n, F(x,y)) + D(gx, F(x_n, y_n))}{2},\\ \frac{D(gy_n, F(y,x)) + D(gy, F(y_n, x_n))}{2} \end{array}\right\}\\
\le &\max\left\{\begin{array}{c} d(gx_n, gx), d(gx_n, gx_{n+1}), D(gx, F(x, y)),\\ d(gy_n, gy), d(gy_n, gy_{n+1}), D(gy, F(y, x)),\\ \frac{D(gx_n, F(x,y)) + d(gx, gx_{n+1})}{2},\\ \frac{D(gy_n, F(y,x)) + d(gy, gy_{n+1})}{2} \end{array}\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&N(x_n, y_n, x, y)\\
= &\min\left\{\begin{array}{c} D(gx_n, F(x, y)), D(gx, F(x_n, y_n)),\\ D(gy_n, F(y, x)), D(gy, F(y_n, x_n)) \end{array}\right\}.
\end{aligned}
$$

Letting $n \to \infty$ in the above inequality, by using $(i_\psi)$, $(i_\varphi)$, $(i_\theta)$, $(ii_\theta)$ and (13), we get

$$
\begin{aligned}
&\psi\left(D(gx,\ F(x,\ y))\right)\\
\le\ &\psi\left(\max\left\{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\right\}\right)\\
&-\varphi\left(\psi\left(\max\left\{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\right\}\right)\right).
\end{aligned}
$$

Similarly, we can obtain that

$$
\begin{aligned}
&\psi\left(D(gy,\ F(y,\ x))\right)\\
\le\ &\psi\left(\max\left\{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\right\}\right)\\
&-\varphi\left(\psi\left(\max\left\{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\right\}\right)\right).
\end{aligned}
$$

Combining them, we get

$$
\begin{aligned}
&\max\left\{\psi\left(D(gx,\ F(x,\ y))\right),\ \psi\left(D(gy,\ F(y,\ x))\right)\right\}\\
\le\ &\psi\left(\max\left\{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\right\}\right)\\
&-\varphi\left(\psi\left(\max\left\{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\right\}\right)\right).
\end{aligned}
$$

Since $\psi$ is non-decreasing, therefore

$$
\begin{aligned}
&\psi\left(\max\left\{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\right\}\right)\\
\le\ &\psi\left(\max\left\{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\right\}\right)\\
&-\varphi\left(\psi\left(\max\left\{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\right\}\right)\right),
\end{aligned}
$$

which, by $(ii_\varphi)$ and $(ii_\psi)$, implies that

$$\max\left\{D(gx,\ F(x,\ y)),\ D(gy,\ F(y,\ x))\right\} = 0,$$

it follows that

$$gx \in F(x,\ y) \text{ and } gy \in F(y,\ x),$$

that is, $(x,\ y)$ is a coupled coincidence point of $F$ and $g$. Hence $C(F,\ g)$ is non-empty.

Suppose now that $(a)$ holds. Assume that for some $(x,\ y) \in C(F,\ g)$,

$$\lim_{n\to\infty} g^n x = u \text{ and } \lim_{n\to\infty} g^n y = v, \tag{14}$$

where $u,\ v \in X$. Since $g$ is continuous at $u$ and $v$. We have, by (14), that $u$ and $v$ are fixed points of $g$, that is,

$$gu = u \text{ and } gv = v. \tag{15}$$

As $F$ and $g$ are $w-$compatible, so

$$(g^n x,\ g^n y) \in C(F,\ g), \text{ for all } n \geq 1,$$

that is,

$$g^n x \in F(g^{n-1}x,\ g^{n-1}y) \text{ and } g^n y \in F(g^{n-1}y,\ g^{n-1}x), \text{ for all } n \geq 1. \tag{16}$$

Now, by using (1), (16) and $(i_\psi)$, we obtain

$$\begin{aligned}
& \psi\left(D(g^n x,\ F(u,\ v))\right) \\
\leq\ & \psi\left(H(F(g^{n-1}x,\ g^{n-1}y),\ F(u,\ v))\right) \\
\leq\ & \psi\left(M(g^{n-1}x,\ g^{n-1}y,\ u,\ v)\right) - \varphi\left(\psi\left(M(g^{n-1}x,\ g^{n-1}y,\ u,\ v)\right)\right) \\
& +\theta\left(N(g^{n-1}x,\ g^{n-1}y,\ u,\ v)\right),
\end{aligned}$$

where

$$\begin{aligned}
& M(g^{n-1}x, g^{n-1}y, u, v) \\
=\ & \max\left\{\begin{array}{c} d(g^n x, gu), D(g^n x, F(g^{n-1}x, g^{n-1}y)), D(gu, F(u,v)), \\ d(g^n y, gv), D(g^n y, F(g^{n-1}y, g^{n-1}x)), D(gv, F(v,u)), \\ \frac{(g^n x, F(u,v)) + d(gu, F(g^{n-1}x, g^{n-1}y))}{2}, \\ \frac{D(g^n y, F(v,u)) + d(gv, F(g^{n-1}y, g^{n-1}x))}{2} \end{array}\right\} \\
\leq\ & \max\left\{\begin{array}{c} d(g^n x, gu), d(g^n x, g^n x), D(gu, F(u,v)), \\ d(g^n y, gv), D(g^n y, g^n y), D(gv, F(v,u)), \\ \frac{D(g^n x, F(u,v)) + D(gu, g^n x)}{2}, \\ \frac{D(g^n y, F(v,u)) + D(gv, g^n y)}{2} \end{array}\right\},
\end{aligned}$$

and

$$\begin{aligned}
& N(g^{n-1}x,\ g^{n-1}y,\ u,\ v) \\
=\ & \min\left\{\begin{array}{c} D(g^n x, F(u,v)), D(gu, F(g^{n-1}x, g^{n-1}y)), \\ D(g^n y, F(v,u)), D(gv, F(g^{n-1}y, g^{n-1}x)) \end{array}\right\}.
\end{aligned}$$

On taking limit as $n \to \infty$ in the above inequality, by using $(i_\psi)$, $(i_\varphi)$, $(i_\theta)$, $(ii_\theta)$, (14) and (15), we get

$$\begin{aligned}
& \psi\left(D(gu,\ F(u,\ v))\right) \\
\leq\ & \psi\left(\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right) \\
& -\varphi\left(\psi\left(\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right)\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
& \psi\left(D(gv,\ F(v,\ u))\right) \\
\leq\ & \psi\left(\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right) \\
& -\varphi\left(\psi\left(\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right)\right).
\end{aligned}$$

Combining them, we get

$$\begin{aligned}
& \max\left\{\psi\left(D(gu,\ F(u,\ v))\right),\ \psi\left(D(gv,\ F(v,\ u))\right)\right\} \\
\leq\ & \psi\left(\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right) \\
& -\varphi\left(\psi\left(\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right)\right).
\end{aligned}$$

Since $\psi$ is non-decreasing, therefore

$$\begin{aligned}&\psi\left(\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right)\\ \leq\ &\psi\left(\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right)\\ &-\varphi\left(\psi\left(\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}\right)\right),\end{aligned}$$

which, by $(ii_{\varphi})$ and $(ii_{\psi})$, implies that

$$\max\left\{D(gu,\ F(u,\ v)),\ D(gv,\ F(v,\ u))\right\}=0,$$

it follows that

$$gu\in F(u,\ v) \text{ and } gv\in F(v,\ u). \tag{17}$$

Now, from (15) and (17), we have

$$u=\ gu\in F(u,\ v) \text{ and } v=gv\in F(v,\ u),$$

that is, $(u,\ v)$ is a common coupled fixed point of $F$ and $g$.

Suppose now that $(b)$ holds. Assume that for some $(x,\ y)\in C(F,\ g)$, $g$ is $F-$weakly commuting, that is, $g^2x\in F(gx,\ gy)$ and $g^2y\in F(gy,\ gx)$ and $g^2x=gx$ and $g^2y=gy$. Thus $gx=g^2x\in F(gx,\ gy)$ and $gy=g^2y\in F(gy,\ gx)$, that is, $(gx,\ gy)$ is a common coupled fixed point of $F$ and $g$.

Suppose now that $(c)$ holds. Assume that for some $(x,\ y)\in C(F,\ g)$ and for some $u$, $v\in X$,

$$\lim_{n\to\infty}g^nu=x \text{ and } \lim_{n\to\infty}g^nv=y. \tag{18}$$

Since $g$ is continuous at $x$ and $y$. Therefore, by (18), we obtain that $x$ and $y$ are fixed points of $g$, that is,

$$gx=x \text{ and } gy=y. \tag{19}$$

Since $(x,\ y)\in C(F,\ g)$, therefore, by (19), we obtain

$$x=gx\in F(x,\ y) \text{ and } y=gy\in F(y,\ x),$$

that is, $(x,\ y)$ is a common coupled fixed point of $F$ and $g$.

Finally, suppose that $(d)$ holds. Let $g(C(F,\ g))=\{(x,\ x)\}$. Then, $\{x\}=\{gx\}=F(x,\ x)$. Hence $(x,\ x)$ is a common coupled fixed point of $F$ and $g$. $\square$

If we put $\theta(t)=0$ in the Theorem 2.1, we get the following result:

**Corollary 2.1.** *Let $(X,\ d)$ be a metric space. Assume $F:X\times X\to K(X)$ and $g:X\to X$ be two mappings. Suppose there exist some $\psi\in\Psi$ and $\varphi\in\Phi$ such that*

$$\psi\left(H(F(x,\ y),\ F(u,\ v))\right)\leq\psi\left(M(x,\ y,\ u,\ v)\right)-\varphi\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right),$$

*for all $x$, $y$, $u$, $v\in X$. Furthermore assume that $F(X\times X)\subseteq g(X)$ and $g(X)$ is a complete subset of $X$. Then, $F$ and $g$ have a coupled coincidence point. Moreover, if one of the conditions $(a)$ to $(d)$ of Theorem 2.1 holds. Then, $F$ and $g$ have a common coupled fixed point.*

If we put $\varphi(t)=t-t\widetilde{\varphi}(t)$ for all $t\geq 0$ in Corollary 2.1, then we get the following result:

**Corollary 2.2.** *Let $(X,\ d)$ be a metric space. Assume $F:X\times X\to K(X)$ and $g:X\to X$ be two mappings. Suppose there exist some $\psi\in\Psi$ and $\widetilde{\varphi}\in\Phi$ such that*

$$\psi\left(H(F(x,\ y),\ F(u,\ v))\right)\leq\widetilde{\varphi}\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right)\psi\left(M(x,\ y,\ u,\ v)\right),$$

*for all $x$, $y$, $u$, $v\in X$. Furthermore assume that $F(X\times X)\subseteq g(X)$ and $g(X)$ is a complete subset of $X$. Then, $F$ and $g$ have a coupled coincidence point. Moreover, if one of the* *conditions (a) to (d) of Theorem 2.1 holds. Then, $F$ and $g$ have a common coupled fixed point.*

If we put $\psi(t) = 2t$ for all $t \geq 0$ in Corollary 2.2, then we get the following result:

**Corollary 2.3.** *Let $(X, d)$ be a metric space. Assume $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings. Suppose there exists some $\widetilde{\varphi} \in \Phi$ such that*

$$H(F(x,\ y),\ F(u,\ v)) \leq \widetilde{\varphi}\,(2M(x,\ y,\ u,\ v))\,M(x,\ y,\ u,\ v),$$

*for all $x, y, u, v \in X$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of $X$. Then, $F$ and $g$ have a coupled coincidence point. Moreover, if one of the conditions (a) to (d) of Theorem 2.1 holds. Then, $F$ and $g$ have a common coupled fixed point.*

If we put $\widetilde{\varphi}(t) = k$ where $0 < k < 1$, for all $t \geq 0$ in Corollary 2.3, then, we get the following result:

**Corollary 2.4.** *Let $(X, d)$ be a metric space. Assume $F : X \times X \to K(X)$ and $g : X \to X$ be two mappings satisfying*

$$H(F(x,\ y),\ F(u,\ v)) \leq kM(x,\ y,\ u,\ v),$$

*for all $x, y, u, v \in X$, where $0 < k < 1$. Furthermore assume that $F(X \times X) \subseteq g(X)$ and $g(X)$ is a complete subset of $X$. Then, $F$ and $g$ have a coupled coincidence point. Moreover, if one of the conditions (a) to (d) of Theorem 2.1 holds. Then, $F$ and $g$ have a common coupled fixed point.*

If we take $F$ to be a singleton set and $g = I$ (the identity mapping) in Theorem 2.1, then, we get the following result:

**Corollary 2.5.** *Let $(X, d)$ be a complete metric space, $F : X \times X \to X$ be a mapping. Suppose there exist some $\psi \in \Psi$, $\varphi \in \Phi$ and $\theta \in \Theta$ such that*

$$\begin{aligned} & \psi\left(d(F(x,\ y),\ F(u,\ v))\right) \\ \leq\ & \psi\left(m(x,\ y,\ u,\ v)\right) - \varphi\left(\psi\left(m(x,\ y,\ u,\ v)\right)\right) + \theta\left(n(x,\ y,\ u,\ v)\right), \end{aligned} \tag{20}$$

*for all $x, y, u, v \in X$. Then, $F$ and $g$ have a coupled fixed point.*

**Example 2.1.** *Suppose that $X = [0, 1]$, equipped with the metric $d : X \times X \to [0, +\infty)$ defined as $d(x, y) = \max\{x, y\}$ and $d(x, x) = 0$ for all $x, y \in X$. Let $F : X \times X \to K(X)$ be defined as*

$$F(x,\ y) = \begin{cases} \{0\}, & \text{for } x,\ y = 1, \\ \left[0,\ \frac{x^2+y^2}{3}\right], & \text{for } x,\ y \in [0,\ 1), \end{cases}$$

*and $g : X \to X$ be defined as*

$$gx = x^2 \text{ for all } x \in X.$$

*Define $\psi : [0, +\infty) \to [0, +\infty)$ by*

$$\psi(t) = \frac{t}{2},\ \text{for all } t \geq 0,$$

*and $\varphi : [0, +\infty) \to [0, +\infty)$ by*

$$\varphi(t) = \frac{t}{3},\ \text{for all } t \geq 0,$$

*and* $\theta : [0, +\infty) \to [0, +\infty)$ *by*

$$\theta(t) = \frac{t}{4}, \text{ for all } t \geq 0.$$

*Now, for all* $x, y, u, v \in X$ *with* $x, y, u, v \in [0, 1)$, *we have*
*Case* $(a)$. *If* $x^2 + y^2 = u^2 + v^2$, *then*

$$\begin{aligned}
& \psi(H(F(x, y), F(u, v))) \\
= & \frac{1}{2}H(F(x, y), F(u, v)) \\
= & \frac{1}{6}(u^2 + v^2) \\
\leq & \frac{1}{6}\max\{x^2, u^2\} + \frac{1}{6}\max\{y^2, v^2\} \\
\leq & \frac{1}{6}d(gx, gu) + \frac{1}{6}d(gy, gv) \\
\leq & \frac{1}{3}M(x, y, u, v) \\
\leq & \psi(M(x, y, u, v)) - \varphi(\psi(M(x, y, u, v))) \\
\leq & \psi(M(x, y, u, v)) - \varphi(\psi(M(x, y, u, v))) + \theta(N(x, y, u, v)).
\end{aligned}$$

*Case* $(b)$. *If* $x^2 + y^2 \neq u^2 + v^2$ *with* $x^2 + y^2 < u^2 + v^2$, *then*

$$\begin{aligned}
& \psi(H(F(x, y), F(u, v))) \\
= & \frac{1}{2}H(F(x, y), F(u, v)) \\
= & \frac{1}{6}(u^2 + v^2) \\
\leq & \frac{1}{6}\max\{x^2, u^2\} + \frac{1}{6}\max\{y^2, v^2\} \\
\leq & \frac{1}{6}d(gx, gu) + \frac{1}{6}d(gy, gv) \\
\leq & \frac{1}{3}M(x, y, u, v) \\
\leq & \psi(M(x, y, u, v)) - \varphi(\psi(M(x, y, u, v))) \\
\leq & \psi(M(x, y, u, v)) - \varphi(\psi(M(x, y, u, v))) + \theta(N(x, y, u, v)).
\end{aligned}$$

*Similarly, we obtain the same result for* $u^2 + v^2 < x^2 + y^2$. *Thus, the contractive condition (1) is satisfied for all* $x, y, u, v \in X$ *with* $x, y, u, v \in [0, 1)$. *Again, for all* $x, y, u, v \in X$

*with $x,\ y \in [0,\ 1)$ and $u,\ v = 1$, we have*

$$\begin{aligned}
& \psi\left(H(F(x,\ y),\ F(u,\ v))\right) \\
= \;& \frac{1}{2}H(F(x,\ y),\ F(u,\ v)) \\
= \;& \frac{1}{6}(x^2+y^2) \\
\leq \;& \frac{1}{6}\max\left\{x^2,\ u^2\right\} + \frac{1}{6}\max\left\{y^2,\ v^2\right\} \\
\leq \;& \frac{1}{6}d(gx,\ gu) + \frac{1}{6}d(gy,\ gv) \\
\leq \;& \frac{1}{3}M(x,\ y,\ u,\ v) \\
\leq \;& \psi\left(M(x,\ y,\ u,\ v)\right) - \varphi\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right) \\
\leq \;& \psi\left(M(x,\ y,\ u,\ v)\right) - \varphi\left(\psi\left(M(x,\ y,\ u,\ v)\right)\right) + \theta\left(N(x,\ y,\ u,\ v)\right).
\end{aligned}$$

*Thus, the contractive condition (1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y \in [0,\ 1)$ and $u,\ v = 1$. Similarly, we can see that the contractive condition (1) is satisfied for all $x,\ y,\ u,\ v \in X$ with $x,\ y,\ u,\ v = 1$. Hence, the hybrid pair $\{F,\ g\}$ satisfies the contractive condition (1), for all $x,\ y,\ u,\ v \in X$. In addition, all the other conditions of Theorem 2.1 are satisfied and $z = (0,\ 0)$ is a common coupled fixed point of hybrid pair $\{F,\ g\}$. The function $F : X \times X \to K(X)$ involved in this example is not continuous at the point $(1,\ 1) \in X \times X$.*

## 3. Application to integral equations

As an application of the results established in section 2 of our paper, we study the existence of the solution to a Fredholm nonlinear integral equation. We shall consider the following integral equation

$$x(p) = \int_a^b \left(K_1(p,\ q) + K_2(p,\ q)\right)\left[f(q,\ x(q)) + g(q,\ x(q))\right] dq + h(p), \tag{21}$$

for all $p \in I = [a,\ b]$.

Let $\Upsilon$ denote the set of all functions $\gamma : [0,\ +\infty) \to [0,\ +\infty)$ satisfying
$(i_\gamma)$ $\gamma$ is non-decreasing,
$(ii_\gamma)$ $\gamma(p) \leq \frac{1}{3}p$.

**Theorem 3.1.** *Consider the integral equation (21) with $K_1,\ K_2 \in C(I \times I,\ \mathbb{R})$, $f,\ g \in C(I \times \mathbb{R},\ \mathbb{R})$ and $h \in C(I,\ \mathbb{R})$ satisfying the following conditions:*

$(i)$ $K_1(p,\ q) \geq 0$ and $K_2(p,\ q) \leq 0$ for all $p,\ q \in I$.
$(ii)$ There exist the positive numbers $\lambda$, $\mu$ and $\gamma \in \Upsilon$ such that for all $x,\ y \in \mathbb{R}$ with $x \geq y$, the following conditions hold:

$$0 \leq f(q,\ x) - f(q,\ y) \leq \lambda\gamma(x-y), \tag{22}$$

$$-\mu\gamma(x-y) \leq g(q,\ x) - g(q,\ y) \leq 0. \tag{23}$$

$(iii)$

$$\max\{\lambda,\ \mu\}\sup_{p\in I}\int_a^b [K_1(p,\ q)-K_2(p,\ q)]dq \leq 1. \tag{24}$$

Then, the integral equation (21) has a solution in $C(I,\ \mathbb{R})$.

*Proof.* Consider $X = C(I,\ \mathbb{R})$. It is well known that $X$ is a complete metric space with respect to the sup metric

$$d(x,\ y) = \sup_{p\in I}|x(p)-y(p)|.$$

Define $\psi : [0,\ +\infty) \to [0,\ +\infty)$ by

$$\psi(t) = \frac{t}{2},\ \text{for all } t \geq 0,$$

and $\varphi : [0,\ +\infty) \to [0,\ +\infty)$ by

$$\varphi(t) = \frac{t}{3},\ \text{for all } t \geq 0,$$

and $\theta : [0,\ +\infty) \to [0,\ +\infty)$ by

$$\theta(t) = \frac{t}{4},\ \text{for all } t \geq 0.$$

Define now the mapping $F : X \times X \to X$ by

$$\begin{aligned} F(x,\ y)(p) &= \int_a^b K_1(p,\ q)[f(q,\ x(q)) + g(q,\ y(q))]dq \\ &\quad + \int_a^b K_2(p,\ q)[f(q,\ y(q)) + g(q,\ x(q))]dq + h(p), \end{aligned}$$

for all $p \in I$. Now for all $x,\ y,\ u,\ v \in X$, we have

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ = & \int_a^b K_1(p,\ q)[(f(q,\ x(q)) - f(q,\ u(q))) - (g(q,\ v(q)) - g(q,\ y(q)))]dq \\ & - \int_a^b K_2(p,\ q)[(f(q,\ v(q)) - f(q,\ y(q))) - (g(q,\ x(q)) - g(q,\ u(q)))]dq. \end{aligned}$$

Thus, by using (22) and (23), we get

$$\begin{aligned} & F(x,\ y)(p) - F(u,\ v)(p) \\ \leq & \int_a^b K_1(p,\ q)\left[\lambda\gamma\left(x(q) - u(q)\right) + \mu\gamma\left(v(q) - y(q)\right)\right]dq \\ & - \int_a^b K_2(p,\ q)\left[\lambda\gamma\left(v(q) - y(q)\right) + \mu\gamma\left(x(q) - u(q)\right)\right]dq. \end{aligned} \tag{25}$$

Since the function $\gamma$ is non-decreasing and so we have

$$\begin{aligned} \gamma\left(x(q)-u(q)\right) &\leq \gamma\left(\sup_{q\in I}|x(q)-u(q)|\right)=\gamma(d(x,\ u)), \\ \gamma\left(v(q)-y(q)\right) &\leq \gamma\left(\sup_{q\in I}|v(q)-y(q)|\right)=\gamma(d(y,\ v)). \end{aligned}$$

Hence by (25), in view of the fact that $K_2(p,\ q)\leq 0$, we obtain

$$\begin{aligned} & |F(x,\ y)(p)-F(u,\ v)(p)| \\ \leq\ & \int_a^b K_1(p,\ q)\left[\lambda\gamma(d(x,\ u))+\mu\gamma(d(y,\ v))\right]dq \\ & -\int_a^b K_2(p,\ q)\left[\lambda\gamma(d(y,\ v))+\mu\gamma(d(x,\ u))\right]dq, \\ \leq\ & \int_a^b K_1(p,\ q)\left[\max\{\lambda,\ \mu\}\gamma(d(x,\ u))+\max\{\lambda,\ \mu\}\gamma(d(y,\ v))\right]dq \\ & -\int_a^b K_2(p,\ q)\left[\max\{\lambda,\ \mu\}\gamma(d(y,\ v))+\max\{\lambda,\ \mu\}\gamma(d(x,\ u))\right]dq, \end{aligned}$$

as all the quantities on the right hand side of (25) are non-negative. Now, taking the supremum with respect to $p$, by using (24), we get

$$\begin{aligned} & d(F(x,\ y),\ F(u,\ v)) \\ \leq\ & \max\{\lambda,\ \mu\}\sup_{p\in I}\int_a^b \left(K_1(p,\ q)-K_2(p,\ q)\right)dq.\left[\gamma(d(x,\ u))+\gamma(d(y,\ v))\right] \\ \leq\ & \gamma(d(x,\ u))+\gamma(d(y,\ v)). \end{aligned}$$

Thus,

$$\frac{1}{2}d(F(x,\ y),\ F(u,\ v))\leq\frac{\gamma(d(x,\ u))+\gamma(d(y,\ v))}{2}. \tag{26}$$

Now, since $\gamma$ is non-decreasing, we have

$$\begin{aligned} \gamma(d(x,\ u)) &\leq \gamma\left(m(x,\ y,\ u,\ v)\right), \\ \gamma(d(y,\ v)) &\leq \gamma\left(m(x,\ y,\ u,\ v)\right), \end{aligned}$$

which implies, by $(ii_\gamma)$, that

$$\begin{aligned} \frac{\gamma(d(x,\ u))+\gamma(d(y,\ v))}{2} &\leq \gamma\left(m(x,\ y,\ u,\ v)\right) \qquad (27) \\ &\leq \frac{1}{3}m(x,\ y,\ u,\ v). \end{aligned}$$

Thus, by (26) and (27), we have

$$\begin{aligned}
& \psi\left(d(F(x,\ y),\ F(u,\ v))\right) \\
= \ & \frac{1}{2}d(F(x,\ y),\ F(u,\ v)) \\
\leq \ & \frac{1}{3}m(x,\ y,\ u,\ v) \\
\leq \ & \psi\left(m(x,\ y,\ u,\ v)\right) - \varphi\left(\psi\left(m(x,\ y,\ u,\ v)\right)\right) \\
\leq \ & \psi\left(m(x,\ y,\ u,\ v)\right) - \varphi\left(\psi\left(m(x,\ y,\ u,\ v)\right)\right) + \theta\left(n(x,\ y,\ u,\ v)\right),
\end{aligned}$$

which is the contractive condition (20) in Corollary 2.5, which shows that all hypotheses of Corollary 2.5 are satisfied. This proves that $F$ has a coupled fixed point $(x,\ y) \in X \times X$ which is the solution in $X = C(I,\ \mathbb{R})$ of the integral equation (21). □

## References


[1] Abbas,M., Ali,B., and Amini-Harandi,A., (2012), Common fixed point theorem for hybrid pair of mappings in Hausdorff fuzzy metric spaces, Fixed Point Theory Appl., 225.

[2] Abbas,M., Ciric,L., Damjanovic,B., and Khan,M.A., Coupled coincidence point and common fixed point theorems for hybrid pair of mappings, Fixed Point Theory Appl.1687-1812-2012-4.

[3] Berinde,V., (2012), Coupled fixed point theorems for $\varphi-$contractive mixed monotone mappings in partially ordered metric spaces, Nonlinear Anal., 75, pp.3218-3228.

[4] Bhaskar,T.G. and Lakshmikantham,V., (2006), Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., 65(7), pp.1379-1393.

[5] Ciric,L., Damjanovic,B., Jleli,M., and Samet,B., (2012), Coupled fixed point theorems for generalized Mizoguchi-Takahashi contractions with applications, Fixed Point Theory Appl., 51.

[6] Deepmala, (2014), A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications, Ph.D. Thesis, Pt. Ravishankar Shukla University, Raipur (Chhatisgarh) India – 492 010.

[7] Deepmala and Pathak,H.K., (2013), Common fixed points for hybrid strict contractions in symmetric spaces under relaxed conditions, Antarct. J. Math., 10(6), pp.579-588.

[8] Deepmala and Pathak,H.K., (2014), Remarks on occasionally weakly compatible mappings versus occasionally weakly compatible mappings, Demonstr. Math., 47(3), pp.695-703.

[9] Deepmala and Pathak,H.K., (2013), Some common fixed point theorems for D-operator pair with applications to nonlinear integral equations, Nonlinear Funct. Anal. Appl., 18(2), pp.205-218.

[10] Deshpande,B. and Handa,A., (2015), Nonlinear mixed monotone-generalized contractions on partially ordered modified intuitionistic fuzzy metric spaces with application to integral equations, Afr. Mat., 26(3-4), pp.317-343.

[11] Deshpande,B. and Handa,A., (2014), Application of coupled fixed point technique in solving integral equations on modified intuitionistic fuzzy metric spaces, Adv. Fuzzy Syst., Volume 2014, Article ID 348069.

[12] Deshpande,B. and Handa,A., (2014), Common coupled fixed point theorems for two hybrid pairs of mappings under $\varphi - \psi$ contraction, ISRN, Volume 2014, Article ID 608725.

[13] Deshpande,B. and Handa,A., (2015), Common coupled fixed point for hybrid pair of mappings under generalized nonlinear contraction, East Asian Math.J., 31(1), pp.77-89.

[14] Debnath,P., (2014), Fixed points of contractive set valued mappings with set valued domains on a metric space with graph, TWMS J. App. Eng. Math., 4(2), pp.169-174.

[15] Gangopadhyay,M., Saha,M., and Baisnab,A.P., (2013), Some fixed point theorem in partial metric spaces, TWMS J. App. Eng. Math., 3(2), pp.206-213.

[16] Gordji,M.E., Baghani,H., and Kim,G.H., (2012), Common fixed point theorems for $(\psi,\ \varphi)-$weak nonlinear contraction in partially ordered sets, Fixed Point Theory Appl., 62.

[17] Guo,D. and Lakshmikantham,V., (1987), Coupled fixed points of nonlinear operators with applications, Nonlinear Anal., 11(5), pp.623-632.

[18] Lakshmikantham,V. and Ciric,L., (2009), Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal., 70(12), pp.4341-4349.

[19] Long,W., Shukla,S., and Radenovic,S., (2013), Some coupled coincidence and common fixed point results for hybrid pair of mappings in 0-complete partial metric spaces, Fixed Point Theory Appl., 145.

[20] Luong,N.V. and Thuan,N.X., (2011), Coupled fixed points in partially ordered metric spaces and application, Nonlinear Anal., 74, pp.983-992.
[21] Jain,M., Tas,K., Kumar,S., and Gupta,N., (2012), Coupled common fixed point results involving a $\varphi - \psi$ contractive condition for mixed g-monotone operators in partially ordered metric spaces, J. Inequal. Appl., 285.
[22] Markin,J.T., (1947), Continuous dependence of fixed point sets, Proc. Amer. Math. Soc., 38(1947), pp.545-547.
[23] Mishra,L.N., Tiwari,S.K., Mishra,V.N., and Khan, I.A., (2015), Unique Fixed Point Theorems for Generalized Contractive Mappings in Partial Metric Spaces, Journal of Function Spaces, Volume 2015, Article ID 960827, 8 pages.
[24] Mishra,L.N., Tiwari,S.K., and Mishra,V.N., (2015), Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces, J. Appl. Anal. Comput., 5(4), pp.600-612.
[25] Mursaleen,M., Mohiuddine,S.A., and Agarwal,R.P., (2012), Coupled fixed point theorems for alpha-psi contractive type mappings in partially ordered metric spaces, Fixed Point Theory Appl. 2012, 228.
[26] Rodriguez-Lopez,J. and Romaguera,S., (2004), The Hausdorff fuzzy metric on compact sets, Fuzzy Sets Syst., 147, pp.273-283.
[27] Samet,B., Karapinar,E., Aydi,H., and Rajic,V.C., (2013), Discussion on some coupled fixed point theorems, Fixed Point Theory Appl. 2013, 50.
[28] Singh,N. and Jain,R., (2012), Coupled coincidence and common fixed point theorems for set-valued and single-valued mappings in fuzzy metric space, Journal of Fuzzy Set Valued Analysis, Volume 2012, Article ID jfsva-00129.
[29] Sintunavarat,W., Kumam,P., and Cho,Y.J., (2012), Coupled fixed point theorems for nonlinear contractions without mixed monotone property, Fixed Point Theory Appl. 2012, 170.

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