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FREE SURFACE FLOW OVER A TRIANGULAR DEPRESSION

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Abstract. Two-dimensional steady free-surface flows over an obstacle is considered. The fluid is assumed to be inviscid, incompressible and the flow is irrotational. Both gravity and surface tension are included in the dynamic boundary conditions. Far upstream, the flow is assumed to be uniform. Triangular obstruction is located at the channel bottom. In this paper, the fully nonlinear problem is formulated by using a boundary integral equation technique. The resulting integro-differential equations are solved iteratively by using Newton's method. When surface tension and gravity are included, there are two additional parameters in the problem known as the Weber number and Froude number. Finally, solution diagrams for all flow regimes are presented.

Keywords: Free surface flow, Potential flow, Weber number, Surface tension, Froud number.

AMS Subject Classification: 76B10, 76C05,76M25.

1. Introduction

Flow over submerged obstacles is one of the classical problems in fluid mechanics. This problem has many related physical applications ranging from the flow of water over rocks to atmospheric, and oceanic stratified flows encountering topographic obstacles, or even a moving pressure distribution over a free surface. Free surface flows over an obstacle have been investigated for different bottom topography by many researchers. Forbes and Schwartz [3] used the boundary integral method to find fully nonlinear solutions of subcritical and supercritical flows over a semi-circular obstacle. Their results confirmed and extended Lamb's solutions. In 2002, Dias and Vanden-Broeck [2] found a new solution called the "generalized hydraulic fall". Such solutions are characterized by downstream supercritical flow and a train of waves on the upstream side. This type of solution can be obtained by removing the radiation condition on the far upstream of the obstacle. Forbes [3] calculated numerical solutions of gravity-capillary flows over a semicircular obstruction. The fluid was subject to the combined effects of gravity and surface tension. Three different branches of solution were presented and compared between linear and fully nonlinear problems. In this work we compute accurate numerical solutions for the fully nonlinear problem. The problem is first formulated as an integral equation for the unknown shapes of the free surface. This equation is then discretized and the resulting algebraic equations are solved by Newton's method. Later on, we found numerical solutions of

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free-surface flows over a triangular depression with the effects of surface tension and the gravity. The problem is formulated in section 2. The numerical procedure is described in section 3 and the results are discussed in section 4.

2. Mathematical Formulation

We consider the steady two-dimensional flow of an inviscid and incompressible fluid over a triangular depression. The flow is assumed to be irrotational. The fluid domain is bounded below by a horizontal rigid wall $x'ox$ and the triangle BCD with angle γ, where $\frac{-\pi}{2} < \gamma < 0$ and above by the free surfaces EGF. (see Figure 1). Let us introduce Cartesian coordinates with the $x-$axis along the bottom and the $y-$axis directed vertically upwards. As $x \longrightarrow \infty$, the flow is assumed to approach a uniform stream with constant velocity U and constant length H. It is convenient to define dimensionless variables by taking U as the unit velocity and H as the unit length. The dimensionless parameters in the problem are the Froude number $Fr = \frac{U}{\sqrt{gH}}$ and the inverse Weber number $\delta = \dfrac{T}{\rho U^2 H}$. Here T is a surface tension, g is the gravity and ρ is a fluid density. Let's introduce

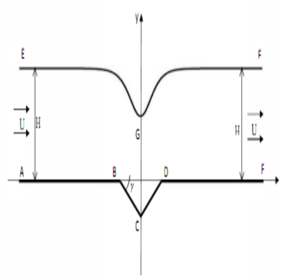


Figure 1. Sketch of the flow and the coordinate

the velocity potential $\phi(x, y)$ and the stream function $\psi(x, y)$, by defining the complex potential function f as :

$$f = \phi(x, y) + i\psi(x, y) \tag{1}$$

The complex velocity ω can be written as :

$$\omega = \frac{df}{dz} = u - iv \tag{2}$$

where u , v are the velocity components in the x and y directions, and $z = x + iy$.

We choose $\phi = 0$ at G and $\psi = 0$ along the streamline EGF. It then follows, from the choice of dimensionless variables, that $\psi = -1$ on the bottom $ABCDF$ (Figure 2) . The mathematical problem can be formulated in terms of the potential function ϕ satisfying the Laplace's equation

$$\Delta\phi = 0 \text{ in the fluid domain}$$

The effects of gravity and the surface tension are considered. Bernoulli's equation gives

$$\frac{1}{2}\left(u^{*2} + v^{*2}\right) + \frac{p^*}{\rho} + g^* y = B \tag{3}$$

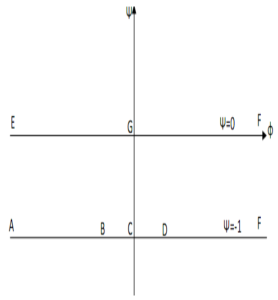


FIGURE 2. Flow configuration in the complex potential plane $f = \phi + i\psi$

where u^*, v^* are the dimensional horizontal and vertical components of velocity respectively, ρ is the density, p^* is the fluid pressure, g^* is the acceleration of the gravity and B is the dimensional Bernoulli constant.

The capillary Laplace's equation gives :

$$p^* - p_0 = \frac{T}{R} = K^* T \tag{4}$$

where $K^* = \dfrac{1}{R}$ is the curvature, p^* is the fluid pressure and p_0 is the atmospheric pressure. Substituting (4) into (3), used on the free surface and in terms of the dimensionless veriables, yields :

$$\frac{1}{2}\left(u^2 + v^2\right) + \delta K + \frac{1}{(Fr)^2}(y - 1) = \frac{1}{2} \tag{5}$$

The kinematic boundary conditions are :

$$\begin{cases} v = 0 \text{ on } \psi = -1 \text{ and } -\infty < \phi < \phi_B \text{ and } \phi_D < \phi < +\infty \\ v = u \tan\gamma \text{ on } \psi = -1 \text{ and } \phi_B < \phi < \phi_C \text{ and } \phi_C < \phi < \phi_D \end{cases} \tag{6}$$

We map the flow domain from the complex potential f-plane onto the lower half of complex ζ-plane by using the conformal mapping

$$\zeta = \alpha + i\beta = e^{-\pi f} = e^{-\pi\phi}\left(\cos\pi\psi - \sin\pi\psi\right) \tag{7}$$

as shown in (Figure 3).

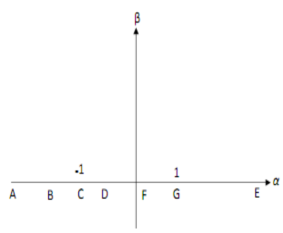


FIGURE 3. The flow in the complex ζ-plane $\zeta = \alpha + i\beta$

Next we introduce the new complex function $\tau - i\theta$ by :

$$\omega = u - iv = e^{\tau - i\theta} \tag{8}$$

The velocity terms, first, become :

$$u^2 + v^2 = e^{2\tau} \tag{9}$$

Secondly, the curvature K of a streamline, in terms of θ is :

$$K = -e^{\tau}\left|\frac{\partial\theta}{\partial\phi}\right| \tag{10}$$

Substituting (9) and (10) into (5), gives the final form of Bernoulli's equation that is needed for the numerical calculation. This is :

$$\frac{1}{2}e^{2\tau} - \delta e^{\tau}\left|\frac{\partial\theta}{\partial\phi}\right| + \frac{1}{(Fr)^2}(y-1) = \frac{1}{2} \quad \text{on} \quad EF \tag{11}$$

Using (8), the kinematic boundary conditions (6) in the $\zeta-$plane, become :

$$\begin{aligned}
\theta &= 0 \quad -\infty < \phi < \phi_B \quad \text{and } \phi_D < \phi < +\infty \text{ and } \psi = -1 \\
\theta &= \gamma \quad \phi_B < \phi < \phi_C \quad \text{and } \psi = -1 \\
\theta &= -\gamma \quad \phi_C < \phi < \phi_D \quad \text{and } \psi = -1 \\
\theta &= \text{unknown} \quad -\infty < \phi < +\infty \text{ and } \psi = 0
\end{aligned} \tag{12}$$

2.1. **Boundary integral techniques.** a differential equation was derived in terms of the new complex variables τ and θ on the free surface. Together with the differential equation it will define an integro-differential equation that will be solved numerically. The fluid region of the triangle problem, has an image region consisting of the upper half of the complex $\zeta-$ plane and applying Cauchy integral formula to it gives :

$$\tau(\zeta_0) - i\theta(\zeta_0) = \frac{1}{i\pi}\int_{\Gamma}\frac{\tau(\zeta) - i\theta(\zeta)}{\zeta - \zeta_0}d\zeta \tag{13}$$

The contour Γ is a simple, where ζ_0 is an image point of a point on the free surface, i.e. $\zeta_0 \in EF$. The path Γ consists of a large semi-circular arc of radius R, centred at the origin. Taking the limit of (13), as $R \longrightarrow \infty$, After taking the real part we obtain :

$$\tau(\alpha_0) = -\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{\theta(\alpha)}{\alpha - \alpha_0}d\alpha \tag{14}$$

Where $\tau(\alpha_0) = \tau(\alpha_0, 0)$ and $\theta(\alpha) = \theta(\alpha, 0)$ are used to simplify the notation. By using the boundary conditions (12) the equation (14) can be written as :

$$\tau(\alpha_0) = -\frac{\gamma}{\pi}\log\left|\frac{1+\alpha_0}{\alpha_B - \alpha_0}\right| + \frac{\gamma}{\pi}\log\left|\frac{\alpha_D + \alpha_0}{1+\alpha_0}\right| - \frac{1}{\pi}\int_0^{+\infty}\frac{\theta(\alpha)}{\alpha - \alpha_0}d\alpha \tag{15}$$

This equation holds along the free surface and so using (9), with $\psi = 0$ gives :

$$\alpha = e^{-\pi\phi} \tag{16}$$

Substituting (16) into (15), yields :

$$\begin{aligned}
\tau'(\phi_0) = &-\tfrac{\gamma}{\pi}\log\left|\tfrac{1+e^{-\pi\phi_0}}{e^{-\pi\phi_B}+e^{-\pi\phi_0}}\right| + \tfrac{\gamma}{\pi}\log\left|\tfrac{e^{-\pi\phi_D}+e^{-\pi\phi_0}}{1+e^{-\pi\phi_0}}\right| \\
&+ \int_{\infty}^{\phi}\tfrac{\theta'(\phi)e^{-\pi\phi}}{e^{-\pi\phi}-e^{-\pi\phi_0}}d\phi
\end{aligned} \tag{17}$$

for $-\infty < \phi < +\infty$

where $\tau'(\phi_0) = \tau\left(e^{-\pi\phi_0}\right)$ and $\theta'(\phi) = \theta\left(e^{-\pi\phi}\right)$. The equation (11) is now rewritten in terms of τ' and θ' as :

$$\frac{1}{2}e^{2\tau'} - \delta e^{\tau'}\frac{\partial\theta'}{\partial\phi} + \frac{1}{(Fr)^2}(y-1) = \frac{1}{2} \quad \text{on} \quad EF \tag{18}$$

Next we evaluate the values of y on the free surfaces by using (8) and integrating the identity

$$\frac{dz}{df} = \omega^{-1} \tag{19}$$

This gives

$$y(\alpha) = 1 - \frac{1}{\pi}\int_0^{\alpha} \frac{e^{-\tau(\alpha_0)}\sin\theta(\alpha_0)}{\alpha_0} d\alpha_0 \quad \text{for} \quad 0 < \alpha < +\infty \tag{20}$$

By using (16), we rewrite (20) as :

$$y'(\phi) = 1 + \int_{\infty}^{\phi} e^{-\tau'(\phi_0)}\sin\theta'(\phi_0)d\phi_0 \quad \text{for} \quad -\infty < \phi < +\infty \tag{21}$$

By substituting (17) and (21) into (18), an integro-differential equation is created and this is solved numerically in the following section.

3. Numerical procedure

The above system of nonlinear equations is solved numerically by using equally spaced points in the potential function ϕ. We introduce equally spaced mesh points in the potential function ϕ by :

$$\phi_I = \left[\frac{-(N-1)}{2} + (I-1)\right]\Delta, I = 1, ..., N, -\infty < \phi < +\infty \tag{22}$$

on the upstream and downstream free surface. Here $\Delta > 0$ is the mesh sizes on upstream and downstream free surface. The corresponding unknowns are :

$$\theta_I = \theta'(\phi_I)$$

There are only N unknowns θ_I. We evaluate the values τ_M of $\tau'(\phi)$ at the midpoints

$$\phi_M = \frac{\phi_{I+1} + \phi_I}{2}, I = 1, ..., N-1 \tag{23}$$

By applying the trapezoidal rule to the integrals in (17) with summations over the points ϕ_I. We evaluate $y_I = y'(\phi_I)$ by applying the trapezoidal rule to (21) and by using (19). This yields :

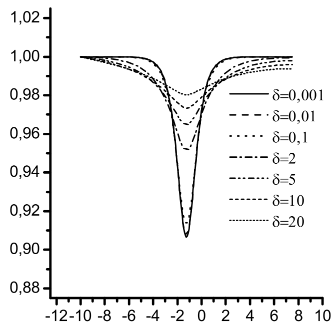
$$\begin{cases} y_1 = 1 \\ y_{I+1} = y_I + \Delta e^{-\tau_M}\sin\theta_M \end{cases} \text{and} \begin{cases} x_1 = x(\infty) \\ x_{I+1} = x_I + \Delta e^{-\tau_M}\sin\theta_M \end{cases} \quad I = 1, ..., N-1$$

Here $\theta_M = \dfrac{\theta_{I+1} + \theta_I}{2}$. We now satisfy (18) at the midpoints (23). This yields N nonlinear algebraic equations for the N unknowns $\theta_I, I = 1, ..., N$. The derivative $\frac{\partial\theta'}{\partial\phi}$ at the mesh points (22), is approximated by a finite difference, whereby

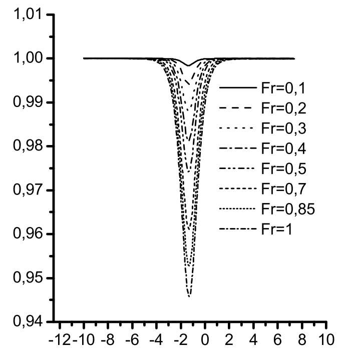
$$\frac{\partial\theta'}{\partial\phi} \approx= \frac{\theta'_{I+1} - \theta'_I}{\Delta}, I = 1, ..., N-1$$

4. Discussion of the results

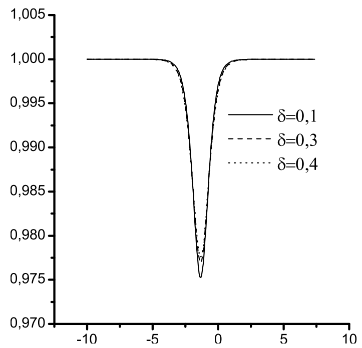
For a given value of δ and Fr, this system of N equations with N unknowns is solved by Newton's method. The results shown in Figures 4 are examples of free surface profiles calculated using the numerical scheme discussed in Section 3. Accuracy of numerical solutions depends on the grid spacing, Δ and the domain truncations. Most of the calculations are obtained with $N = 250$, $\Delta = 0.1$ and $\gamma = -\frac{\pi}{4}$. We computed solutions for various values of N and Δ until the numerical solutions were in agreement within graphical accuracy. Figure 4 shows the effect of surface tension and gravity on the upstream free surface profile.



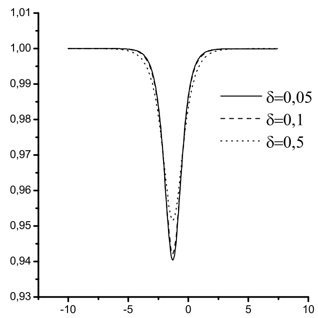
(a)



(b)



(c)



(d)

Figure 4. **(a)** Free streamline shapes for$F = 0$ and various inverse Weber number. **(b)**Free streamline shapes for $\delta = 0$ and various Froud number Fr. **(c)**Subcritical flows for Fr=0.5 and $\delta = 0.1$, 0.3 and 0.4 **(d)** Supercritical flows for Fr=1.2 and $\delta = 0.05$, 0.1 and 0.5.

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