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EXISTENCE AND NONEXISTENCE OF POSITIVE SOLUTIONS FOR A n-TH ORDER THREE-POINT BOUNDARY VALUE PROBLEM

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Abstract. The purpose of this paper is to establish some results on the existence and nonexistence of positive solutions for a type of nonlinear n-th order three-point boundary value problems. The main tool is a fixed point theorem of the cone expansion and compression of functional type due to Avery, Anderson, and O'Regan. Some examples are presented to illustrate the availability of the main results.

Keywords: positive solution, existence and nonexistence, boundary value problem, fixed point theorem.

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1. Introduction

We consider the existence and nonexistence of positive solution to the nonlinear n-th order three-point boundary value problem (BVP for short):

$$\begin{cases} u^{(n)}(t) + f(t, u(t)) = 0, \; t \in [a, b], \\ u^{(i)}(a) = 0, \; i = 0, 1, 2, ..., n-2, \\ u^{(p)}(b) = \xi u^{(p)}(\eta), \; (p \in \{1, 2, ..., n-2\} \text{ but fixed}) \end{cases} \tag{1}$$

where $n \geq 3$, $\xi \in (0, \infty)$, $\eta \in (a, b)$ are constants with $0 < \xi(\eta - a)^{n-p-1} < (b-a)^{n-p-1}$ and $f : [a, b] \times [0, \infty) \to [0, \infty)$ is continuous.

The theory of multi-point boundary value problems for ordinary differential equations arises in different areas of applied mathematics and physics. For example, the vibrations of a guy wire of uniform cross-section and composed of N parts of different densities can be set up as a multi-point boundary value problem; many problems in the theory of elastic stability can be handled as multi-point boundary value problems too. Recently, the existence and multiplicity of positive solutions for nonlinear ordinary differential equations have received a great deal of attention.

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Fixed point theorems have been applied to various boundary value problems to show the existence and multiplicity of positive solutions in the last two decades. Recently, Avery *et al.* [3] generalized the fixed point theorem of a cone expansion and compression of norm type by replacing the norms with two functions satisfying certain conditions to produce a fixed point theorem of the cone expansion and compression of functional type, and then they applied the fixed point theorem to verify the existence of a positive solution to a second order conjugate boundary value problem.

Motivated greatly by the above-mentioned work, by constructing a special cone and using the fixed point theorem of cone expansion and compression of functional type due to Avery et al., in this paper, we obtain some sufficient conditions for the existence of positive solutions for BVP (1). We also discuss the nonexistence of positive solutions. Here, as usual, by a positive solution to BVP (1), we mean a solution $u(t)$ such that $u(t) > 0$ on $(a, b]$.

We organize the rest of this paper as follows. In Section 2, we present some definitions and background results. We also state a fixed point theorem of cone expansion and compression of functional type due to Avery et al. The expression and properties of Green's function will be given in Section 3. The existence and nonexistence results are proved in Section 4. We end the paper with four examples of applications in Section 5.

2. Preliminaries

To make this paper self-contained, in this section, we recall some definitions and background results on cones and completely continuous operators. We also state a fixed point theorem of cone expansion and compression of functional type due to Avery, Henderson, and O'Regan.

Definition 2.1. *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone of E if it satisfies the following two conditions:*

(1) *$u \in P$, $\lambda > 0$ implies $\lambda u \in P$;*
(2) *$u \in P$, $-u \in P$ implies $u = 0$.*

Every cone $P \subset E$ induces an ordering in E given by $u \leq v$ if and only if $v - u \in P$.

Definition 2.2. *Let E be a real Banach space. An operator $T : E \to E$ is said to be completely continuous if it is continuous and maps bounded sets into precompact sets.*

Definition 2.3. *A map α is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\alpha : P \to [0, +\infty)$ is continuous and*

$$\alpha(\lambda u + (1 - \lambda)v) \geq \lambda\alpha(u) + (1 - \lambda)\alpha(v),\ u, v \in P,\ 0 \leq \lambda \leq 1.$$

Similarly we said the map β is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\beta : P \to [0, +\infty)$ is continuous and

$$\beta(\lambda u + (1 - \lambda)v) \leq \lambda\beta(u) + (1 - \lambda)\beta(v),\ u, v \in P,\ 0 \leq \lambda \leq 1.$$

We say the map γ is sublinear functional if

$$\gamma(\lambda u) \leq \lambda\gamma(u),\ u \in P,\ 0 \leq \lambda \leq 1.$$

All the concepts discussed above can be found in [7].

Property A1. Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property A1 if one of the following conditions holds:

(a) β is convex, $\beta(0) = 0$, $\beta(u) \neq 0$ if $u \neq 0$, and $\inf_{u \in P \cap \partial\Omega} \beta(u) > 0$;

(b) β is sublinaer, $\beta(0) = 0$, $\beta(u) \neq 0$ if $u \neq 0$, and $\inf_{u \in P \cap \partial\Omega} \beta(u) > 0$;
(c) β is concave and unbounded.

Property A2. Let P be a cone in a real Banach space E and Ω be a bounded open subset of E with $0 \in \Omega$. Then a continuous functional $\beta : P \to [0, \infty)$ is said to satisfy Property A2 if one of the following conditions holds:

(a) β is convex, $\beta(0) = 0$, $\beta(u) \neq 0$ if $u \neq 0$;
(b) β is sublinear, $\beta(0) = 0$, $\beta(u) \neq 0$ if $u \neq 0$;
(c) $\beta(u + v) \geq \beta(u) + \beta(v)$ for all $u, v \in P, \beta(0) = 0, \beta(u) \neq 0$ if $u \neq 0$.

The approach used in proving the existence results in this paper is the following fixed point theorem of cone expansion and compression of functional type due to Avery et al. [3].

Theorem 2.1. *Let Ω_1 and Ω_2 be two bounded open sets in a Banach space E such that $0 \in \Omega_1$ and $\overline{\Omega}_1 \subseteq \Omega_2$ and P is a cone in E. Suppose $T : P \cap (\overline{\Omega}_2 \backslash \Omega_1) \to P$ is a completely continuous operator, α and γ are nonnegative continuous functional on P, and one of the two conditions:*

(K1) *α satisfies Property A1 with $\alpha(Tu) \geq \alpha(u)$, for all $u \in P \cap \partial\Omega_1$, and γ satisfies Property A2 with $\gamma(Tu) \leq \gamma(u)$, for all $u \in P \cap \partial\Omega_2$, or*
(K2) *γ satisfies Property A2 with $\gamma(Tu) \leq \gamma(u)$, for all $u \in P \cap \partial\Omega_1$, and α satisfies Property A1 with $\alpha(Tu) \geq \alpha(u)$, for all $u \in P \cap \partial\Omega_2$,*

is satisfied, then T has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Expression and Properties of Green's Function

In this section we present the expression and properties of Green's function associated with BVP (1). In order to prove our main results, we need some preliminary results.

Lemma 3.1. *Let $h \in C[a, b]$, then the boundary value problem*

$$\begin{cases} u^{(n)}(t) + h(t) = 0, \ t \in [a, b], \\ u^{(i)}(a) = 0, \ i = 0, 1, 2, ..., n - 2, \\ u^{(p)}(b) = \xi u^{(p)}(\eta), \ (p \in \{1, 2, ..., n - 2\} \text{ but fixed}) \end{cases} \tag{2}$$

has a unique solution

$$u(t) = \int_a^b G(t, s)h(s)ds,$$

where

$$G(t, s) = G_1(t, s) + \frac{\xi(t - a)^{n-1}}{[(b - a)^{n-p-1} - \xi(\eta - a)^{n-p-1}]} G_2(\eta, s) \tag{3}$$

$$G_1(t, s) = \frac{1}{(n - 1)!} \begin{cases} \frac{(t-a)^{n-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}} - (t - s)^{n-1}, & a \leq s \leq t \leq b, \\ \frac{(t-a)^{n-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}}, & a \leq t \leq s \leq b, \end{cases} \tag{4}$$

$$G_2(\eta, s) = \frac{1}{(n - 1)!} \begin{cases} \frac{(\eta-a)^{n-p-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}} - (\eta - s)^{n-1}, & a \leq s \leq \eta, \\ \frac{(\eta-a)^{n-p-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}}, & \eta \leq s \leq b. \end{cases} \tag{5}$$

Proof. One can reduce equation $u^{(n)}(t) + h(t) = 0$ to an equivalent integral equation

$$u(t) = -\frac{1}{(n - 1)!} \int_a^t (t - s)^{n-1}h(s)ds + \sum_{i=0}^{n-1} A_i(t - a)^i \tag{6}$$

for some $A_i \in \mathbb{R}$ $(i = 0, 1, 2, ..., n-1)$. By $u^{(i)}(a) = 0$, $(i = 0, 1, 2, ..., n-2)$, it follows that $A_i = 0$ $(i = 0, 1, 2, ..., n-2)$. Thus,

$$u(t) = -\frac{1}{(n-1)!}\int_a^t (t-s)^{n-1}h(s)ds + A_{n-1}(t-a)^{n-1}. \tag{7}$$

Now we solve for A_{n-1} by $u^{(p)}(b) = \xi u^{(p)}(\eta)$. It follows from

$$u^{(p)}(t) = -\frac{1}{(n-p-1)!}\int_a^t (t-s)^{n-p-1}h(s)ds + \frac{(n-1)!}{(n-p-1)!}A_{n-1}(t-a)^{n-p-1} \tag{8}$$

that

$$\begin{aligned}
&-\frac{1}{(n-p-1)!}\int_a^b (b-s)^{n-p-1}h(s)ds + \frac{(n-1)!}{(n-p-1)!}A_{n-1}(b-a)^{n-p-1}\\
&= -\frac{\xi}{(n-p-1)!}\int_a^\eta (\eta-s)^{n-p-1}h(s)ds + \frac{(n-1)!}{(n-p-1)!}A_{n-1}\xi(\eta-a)^{n-p-1}
\end{aligned}$$

from which we get

$$\begin{aligned}
A_{n-1} &= \frac{1}{(n-1)![(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]}\\
&\quad\times\left[\int_a^b (b-s)^{n-p-1}h(s)ds - \xi\int_a^\eta (\eta-s)^{n-p-1}h(s)ds\right]\\
&= \frac{1}{(n-1)!(b-a)^{n-p-1}}\int_a^b (b-s)^{n-p-1}h(s)ds\\
&\quad+ \frac{\xi(\eta-a)^{n-p-1}}{(b-a)^{n-p-1}(n-1)![(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]}\int_a^b (b-s)^{n-p-1}h(s)ds\\
&\quad- \frac{\xi}{(n-1)![(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]}\int_a^\eta (\eta-s)^{n-p-1}h(s)ds\\
&= \frac{1}{(n-1)!(b-a)^{n-p-1}}\int_a^b (b-s)^{n-p-1}h(s)ds\\
&\quad+ \frac{\xi}{[(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]}\int_a^b G_2(\eta,s)h(s)ds
\end{aligned}$$

Therefore, substituting A_{n-1} into (7), one has

$$\begin{aligned}
u(t) &= -\frac{1}{n-1)!}\int_a^t (t-s)^{n-1}h(s)ds + \frac{(t-a)^{n-1}}{(n-1)!(b-a)^{n-p-1}}\int_a^b (b-s)^{n-p-1}h(s)ds\\
&\quad+ \frac{\xi(t-a)^{n-1}}{[(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]}\int_a^b G_2(\eta,s)h(s)ds\\
&= \int_a^b G_1(t,s)h(s)ds + \frac{\xi(t-a)^{n-1}}{[(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]}\int_a^b G_2(\eta,s)h(s)ds\\
&= \int_a^b G(t,s)h(s)ds
\end{aligned} \tag{9}$$

where $G(t,s)$ is defined by (3). This completes the proof. □

Lemma 3.2. *Suppose* $0 < \xi(\eta-a)^{n-p-1} < (b-a)^{n-p-1}$. *Then Green's function* $G(t,s)$ *defined by* (3) *has the following properties:*

(a) $\frac{\partial^j G(t,s)}{\partial t^j}$ *is continuous on* $[a,b]\times[a,b]$, $j = 0, 1, 2, ..., n-2$;

(b) $\frac{\partial G(t,s)}{\partial t} \geq 0$, *for all* $t, s \in [a,b]$;

(c) $\left(\frac{t-a}{b-a}\right)^{n-1} G(b,s) \leq G(t,s) \leq G(b,s)$, *for all* $t, s \in [a,b]$;

(d) $\int_a^b G(b,s)ds = \Big(p(b-a)^n[(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}] + n\xi(b-a)^{n-1}(b-\eta)(\eta - a)^{n-p-1}\Big) \times \Big(n!(n-p)[(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]\Big)^{-1} = A^{-1}$;

(e) $\int_\eta^b G(b,s)ds = \Big(n(b-a)^{n-1}(b-\eta)^{n-p} - (n-p)(b-\eta)^n[(b-a)^{n-p-1} - \xi(\eta - a)^{n-p-1}]\Big) \times \Big(n!(n-p)[(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]\Big)^{-1} = B^{-1}$;

(f) $\int_\eta^b G(\eta,s)ds = \Big((\eta-a)^{n-1}(b-\eta)^{n-p}\Big) \times \Big((n-1)!(n-p)[(b-a)^{n-p-1} - \xi(\eta - a)^{n-p-1}]\Big)^{-1} = C^{-1}$.

Proof. (a). This conclusion holds obviously, so we omit the proof here.
(b). First we prove

$$\frac{\partial G_1(t,s)}{\partial t} \geq 0, \ \ t, s \in [a,b].$$

In fact, if $t \leq s$, obviously, $\frac{\partial G_1(t,s)}{\partial t} \geq 0$. If $s \leq t$, from (4) we know

$$\begin{aligned} \frac{\partial G_1(t,s)}{\partial t} &= \frac{1}{(n-2)!}\left[\frac{(t-a)^{n-2}(b-s)^{n-p-1}}{(b-a)^{n-p-1}} - (t-s)^{n-2}\right] \\ &\geq \frac{1}{(n-2)!}\left[\frac{(t-a)^{n-2}(b-s)^{n-2} - (t-s)^{n-2}(b-a)^{n-2}}{(b-a)^{n-p-1}}\right] \\ &\geq 0. \end{aligned} \tag{10}$$

Now we prove

$$G_2(\eta, s) \geq 0, \ \ s \in [a,b]. \tag{11}$$

In fact, if $s \geq \eta$, obviously, (11) holds. If $s \leq \eta$, one has

$$\begin{aligned} &\frac{(\eta-a)^{n-p-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}} - (\eta-s)^{n-p-1} \\ &= \frac{(\eta-a)^{n-p-1}(b-s)^{n-p-1} - (\eta-s)^{n-p-1}(b-a)^{n-p-1}}{(b-a)^{n-p-1}} \\ &\geq 0, \end{aligned}$$

which implies that (11) is also true. Therefore, by (3), (10) and (11), we find

$$\frac{\partial G(t,s)}{\partial t} = \frac{\partial G_1(t,s)}{\partial t} + \frac{(n-1)\xi(t-a)^{n-2}}{[(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]} G_2(\eta, s) \geq 0.$$

The proof of (b) is completed.

(c). From (b), we know that $\frac{\partial G(t,s)}{\partial t} \geq 0$ for any $t, s \in [a,b]$. Thus $G(t,s)$ is increasing in t, so

$$G(t,s) \leq G(b,s), \ \text{ for } (t,s) \in [a,b] \times [a,b]. \tag{12}$$

On the other hand, if $s \le t$, then from (4), we have

$$\begin{aligned} G_1(t,s) &= \frac{1}{(n-1)!}\left[\frac{(t-a)^{n-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}} - (t-s)^{n-1}\right] \\ &= \frac{(t-a)^{n-1}}{(n-1)!(b-a)^{n-1}}\left[\frac{(b-a)^{n-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}} - (b-s)^{n-1}\right] \\ &\quad + \frac{1}{(n-1)!}\left[\frac{(t-a)^{n-1}(b-s)^{n-1}}{(b-a)^{n-1}} - (t-s)^{n-1}\right] \\ &\ge \frac{1}{(n-1)!}\left(\frac{t-a}{b-a}\right)^{n-1}\left[\frac{(b-a)^{n-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}} - (b-s)^{n-1}\right] \\ &= \left(\frac{t-a}{b-a}\right)^{n-1} G_1(b,s). \end{aligned} \tag{13}$$

If $t \le s$, then from (4), we have

$$\begin{aligned} G_1(t,s) &= \frac{1}{(n-1)!}\frac{(t-a)^{n-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}} \\ &= \frac{(t-a)^{n-1}}{(n-1)!(b-a)^{n-1}}\left[\frac{(b-a)^{n-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}} - (b-s)^{n-1}\right] \\ &\quad + \frac{1}{(n-1)!(b-a)^{n-1}}(t-a)^{n-1}(b-s)^{n-1} \\ &\ge \frac{1}{(n-1)!}\left(\frac{t-a}{b-a}\right)^{n-1}\left[\frac{(b-a)^{n-1}(b-s)^{n-p-1}}{(b-a)^{n-p-1}} - (b-s)^{n-1}\right] \\ &= \frac{1}{(n-1)!}\left(\frac{t-a}{b-a}\right)^{n-1} G_1(b,s). \end{aligned} \tag{14}$$

Thus form (13) and (14), we obtain that

$$G_1(t,s) \ge \left(\frac{t-a}{b-a}\right)^{n-1} G_1(b,s). \tag{15}$$

From (3) and (15), one has

$$\begin{aligned} G(t,s) &= G_1(t,s) + \frac{\xi(t-a)^{n-1}}{[(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]}G_2(\eta,s) \\ &\ge \left(\frac{t-a}{b-a}\right)^{n-1} G_1(b,s) + \frac{\xi(t-a)^{n-1}}{[(b-a)^{n-p-1} - \xi(\eta-a)^{n-p-1}]}G_2(\eta,s) \\ &= \left(\frac{t-a}{b-a}\right)^{n-1} G(b,s). \end{aligned}$$

The equations (d), (e), and (f) can be verified by direct calculations. Then the proof is completed. □

4. Main Results

In this section, we discuss the existence and nonexistence of positive non-decreasing solution of BVP (1). We shall consider the Banach space $E = C[a,b]$ equipped with norm $\|u\| = \max_{t\in[a,b]} |u(t)|$.

We define a cone $P \subset E$ by

$$P = \left\{ u \in C[a,b] : u(t) \geq 0,\ u(t) \text{ is increasing on } [a,b] \text{ and } u(t) \geq \left(\frac{t-a}{b-a}\right)^{n-1} \|u\|, t \in [a,b] \right\}. \tag{16}$$

Then P is a normal cone of E. It is easy to see that if $u \in P$, then $\|u\| = u(b)$. Define an integral operator $T : P \to E$ by

$$(Tu)(t) = \int_a^b G(t,s) f(s, u(s)) ds,\ t \in [a,b]. \tag{17}$$

Thus, to solve BVP (1), we only need to find a fixed point of the operator T in P.

Lemma 4.1. *The operator defined in* (17) *is completely continuous and satisfies* $T(P) \subseteq P$.

Proof. Applying the Arzela-Ascoli theorem and a standard arguments, we can prove that T is a completely continuous operator. We conclude that $T(P) \subseteq P$. In fact for any $u \in P$, it follows that

$$\begin{aligned}(Tu)(t) &= \int_a^b G(t,s) f(s, u(s)) ds \\ &\leq \int_a^b G(b,s) f(s, u(s)) ds\end{aligned}$$

which implies that

$$\|Tu\| \leq \int_a^b G(b,s) f(s, u(s)) ds.$$

On the other hand, by Lemma 3.2(c), we have

$$\begin{aligned}(Tu)(t) &= \int_a^b G(t,s) f(s, u(s)) ds \\ &\geq \left(\frac{t-a}{b-a}\right)^{n-1} \int_a^b G(b,s) f(s, u(s)) ds \\ &\geq \left(\frac{t-a}{b-a}\right)^{n-1} \|Tu\|, t \in [a,b].\end{aligned}$$

Therefore, $Tu \in P$. The proof is completed. □

Finally, let us define two continuous functionals α and γ on the cone P by

$$\begin{aligned}\alpha(u) &= \min_{t \in [\eta, b]} u(t) = u(\eta), \\ \gamma(u) &= \max_{t \in [a,b]} u(t) = u(b) = \|u\|.\end{aligned}$$

It is clear that $\alpha(u) \leq \gamma(u)$ for all $u \in P$.

Theorem 4.1. *Suppose that there exists positive numbers r, R with $r < \left(\frac{\eta-a}{b-a}\right)^{n-1} R$ such that the following conditions are satisfied:*

(C1) *$f(t,x) \geq Br$, for all $(t,x) \in [\eta, b] \times [r, R]$,*

(C2) *$f(t,x) \leq AR$, for all $(t,x) \in [a,b] \times [0, R]$.*

Then BVP (1) *has at least one positive and non-decreasing solution u^* satisfying*

$$r \leq \min_{t \in [\eta, b]} u^*(t) \text{ and } \max_{t \in [a,b]} u^*(t) \leq R.$$

Proof. Let

$$\Omega_1 = \{u : \alpha(u) < r\} \text{ and } \Omega_2 = \{u : \gamma(u) < R\},$$

it is easy to see that $0 \in \Omega_1, \Omega_2$, and Ω_2 being bounded open subsets of E. Let $u \in \overline{\Omega}_1$, then we have

$$r \geq \alpha(u) = \min_{t\in[\eta,b]} u(t) \geq \Big(\frac{\eta - a}{b-a}\Big)^{n-1} \|u\| = \Big(\frac{\eta - a}{b-a}\Big)^{n-1} \gamma(u).$$

Thus $R > r\big(\frac{b-a}{\eta-a}\big)^{n-1} > \gamma(u)$; i.e., $u \in \Omega_2$, so $\overline{\Omega}_1 \subseteq \Omega_2$.

Claim 1: If $u \in P \cap \partial\Omega_1$, then $\alpha(Tu) \geq \alpha(u)$.

To see this let $u \in P \cap \partial\Omega_1$, then $R = \gamma(u) \geq u(s) \geq \alpha(u) = r$, $s \in [\eta, b]$. Thus it follows from (C1), Lemma 3.2(f), and (17), that

$$\begin{aligned}
\alpha(Tu) = (Tu)(\eta) &= \int_a^b G(\eta, s) f(s, u(s)) ds \\
&\geq \int_\eta^b G(\eta, s) f(s, u(s)) ds \\
&\geq Br \int_\eta^b G(\eta, s) ds = r = \alpha(u).
\end{aligned}$$

Claim 2: If $u \in P \cap \partial\Omega_2$, then $\gamma(Tu) \leq \gamma(u)$.

To see this let $u \in P \cap \partial\Omega_2$, then $u(s) \leq \gamma(u) = R$, $s \in [a, b]$. Thus condition (C2) and Lemma 3.2(d) yield

$$\begin{aligned}
\gamma(Tu) = (Tu)(b) &= \int_a^b G(b, s) f(s, u(s)) ds \\
&\leq AR \int_\eta^b G(b, s) ds = R = \gamma(u).
\end{aligned}$$

Clearly α satisfies Property A1(c) and γ satisfies Property A2(a). Therefore the hypothesis (K1) of Theorem 2.1 is satisfied, and hence T has at least one fixed point $u^* \in P \cap (\overline{\Omega}_2 \backslash \Omega_1)$; i.e., BVP (1) has at least one positive solution $u^* \in P$ such that

$$r \leq \min_{t\in[\eta,b]} u^*(t) \text{ and } \max_{t\in[a,b]} u^*(t) \leq R.$$

This completes the proof. □

Theorem 4.2. *Suppose that there exist positive numbers r, R with $r < R$ such that the following conditions are satisfied:*

(C3) $f(t, x) \leq Ar$, *for all* $(t, x) \in [a, b] \times [0, r]$,

(C4) $f(t, x) \geq BR$, *for all* $(t, x) \in [\eta, b] \times [R, R\big(\frac{b-a}{\eta-a}\big)^{n-1}]$.

Then BVP (1) has at least one positive solution $u^* \in P$ satisfying

$$r \leq \max_{t\in[a,b]} u^*(t) \textit{ and } \min_{t\in[\eta,b]} u^*(t) \leq R.$$

Proof. For all $u \in P$, we have $\alpha(u) \leq \gamma(u)$. Thus if we let

$$\Omega_3 = \{u : \gamma(u) < r\} \text{ and } \Omega_4 = \{u : \alpha(u) < R\},$$

we have $0 \in \Omega_3$ and $\overline{\Omega}_3 \subseteq \Omega_4$, with Ω_3 and Ω_4 being bounded open subsets of E.

Claim 1: If $u \in P \cap \partial\Omega_3$, then $\gamma(Tu) \leq \gamma(u)$.

To see this let $u \in P \cap \partial\Omega_3$, then $u(s) \leq \gamma(u) = r$, $s \in [a,b]$. Thus condition (C3) and Lemma 3.2(d) yield

$$\begin{aligned}\gamma(Tu) = (Tu)(b) &= \int_a^b G(b,s)f(s,u(s))ds \\ &\leq Ar\int_\eta^b G(b,s)ds = r = \gamma(u).\end{aligned}$$

Claim 2: If $u \in P \cap \partial\Omega_4$, then $\alpha(Tu) \geq \alpha(u)$.

To see this let $u \in P \cap \partial\Omega_4$, then $R\big(\frac{b-a}{\eta-a}\big)^{n-1} = \alpha(u)\big(\frac{b-a}{\eta-a}\big)^{n-1} \geq \gamma(u) \geq u(s) \geq \alpha(u) = R$, $s \in [\eta, b]$. Thus it follows from (C4) and Lemma 3.2(f), that one has

$$\begin{aligned}\alpha(Tu) = (Tu)(\eta) &= \int_a^b G(\eta,s)f(s,u(s))ds \\ &\geq \int_\eta^b G(\eta,s)f(s,u(s))ds \\ &\geq BR\int_\eta^b G(\eta,s)ds = R = \alpha(u).\end{aligned}$$

Clearly α satisfies Property A1(c) and γ satisfies Property A2(a). Therefore the hypothesis (K2) of Theorem 2.1 is satisfied and hence T has at least one fixed point $u^* \in P\cap(\overline{\Omega}_4\backslash\Omega_3)$, i.e., BVP (1) has at least one positive solution $u^* \in P$ such that

$$r \leq \max_{t\in[a,b]} u^*(t) \text{ and } \min_{t\in[\eta,b]} u^*(t) \leq R.$$

This completes the proof. □

Now we discuss nonexistence of positive solutions of BVP (1).

Theorem 4.3. *Suppose that $f \in C([a,b]\times[0,\infty),[0,\infty))$, satisfies the condition*

$$\sup_{(t,x)\in[a,b]\times(0,\infty)} \frac{f(t,x)}{x} < A. \tag{18}$$

Then BVP (1) does not admit positive solutions.

Proof. Assume to the contrary that $u = u(t)$ is a positive solution of BVP (1), then from Lemma 3.2(d) and (18) we get

$$\begin{aligned}u(b) &= \int_a^b G(b,s)f(s,u(s))ds \\ &< A\int_a^b G(b,s)u(s)ds \\ &\leq A\|u\|\int_a^b G(b,s)ds = u(b).\end{aligned}$$

This is a contradiction and completes the proof. □

Theorem 4.4. *Suppose that $f \in C([a,b]\times[0,\infty),[0,\infty))$, satisfies the condition*

$$\inf_{(t,x)\in[\eta,b]\times(0,\infty)} \frac{f(t,x)}{x} > C\Big(\frac{b-a}{\eta-a}\Big)^{n-1}. \tag{19}$$

Then BVP (1) does not admit positive solutions.

Proof. Assume to the contrary that $u = u(t)$ is a positive solution of BVP (1), then by the definition of the cone P, we have $u(s) \geq \left(\frac{s-a}{b-a}\right)^{n-1}\|u\| \geq \left(\frac{\eta-a}{b-a}\right)^{n-1}u(b)$ for $s \in [\eta, b]$. Thus from Lemma 3.2(e) and (18) we get

$$\begin{aligned} u(b) &= \int_a^b G(b,s)f(s,u(s))ds \\ &\geq \int_\eta^b G(b,s)f(s,u(s))ds \\ &> C\Big(\frac{b-a}{\eta-a}\Big)^{n-1}\int_\eta^b G(b,s)u(s)ds \\ &\geq C\Big(\frac{b-a}{\eta-a}\Big)^{n-1}\int_\eta^b G(b,s)\Big(\frac{\eta-a}{b-a}\Big)^{n-1}u(b)ds = u(b). \end{aligned}$$

This is a contradiction and completes the proof. □

5. Examples

Now we provide several examples to demonstrate the applications of the theoretical results in the previous sections.

5.1. **Example.** Consider the boundary value problem

$$\begin{aligned} u^{(3)}(t) + \frac{t}{3}u^2(t) + u(t) + t^2 + t + 1 = 0,\ t \in (0,1) \\ u(0) = u'(0) = 0,\ u(1) = \xi u\Big(\frac{1}{2}\Big) \end{aligned} \tag{20}$$

In this problem, $f(t,x) = \frac{t}{3}x^2 + x + t^2 + t + 1$. It is easy to see that $f \in C([0,1] \times [0,\infty), [0,\infty))$. Let $\eta = \frac{1}{2}, \xi = \frac{1}{3}, r = \frac{1}{24}, R = 3$ then

$$f(t,x) \leq f(1,R) = 9 < \frac{261}{8} = AR,\ \text{for } (t,x) \in [0,1] \times [0,R],$$

and

$$f(t,x) \geq f(\frac{1}{2}, r) = \frac{6193}{3456} > \frac{259}{570} = Br,\ \text{for } (t,x) \in [\frac{1}{2}, 1] \times [r,R].$$

Hence, all the conditions of Theorem 4.1 are satisfied. Hence, Theorem 4.1 guarantees that BVP (20) has at least one positive solution $u^*(t)$ such that

$$\frac{1}{24} \leq \min_{t\in[\frac{1}{2},1]} u^*(t) \text{ and } \max_{t\in[0,1]} u^*(t) \leq 3.$$

5.2. **Example.** Consider the boundary value problem

$$\begin{aligned} u^{(4)}(t) + (2+t)u^4(t) + u^2(t) + tu(t) + t = 0,\ t \in (0,1) \\ u(0) = u'(0) = u''(0) = 0,\ u(1) = \xi u\Big(\frac{1}{2}\Big) \end{aligned} \tag{21}$$

In this problem, $f(t,x) = (2+t)x^4 + x^2 + tx + t$. Obviously $f \in C([0,1] \times [0,\infty), [0,\infty))$. Let $\eta = \frac{1}{2}, \xi = \frac{1}{3}, r = 1, R = 4$ then

$$f(t,x) \leq f(1,r) = 6 < \frac{121}{2} = Ar,\ \text{for } (t,x) \in [0,1] \times [0,r],$$

and

$$f(t,x) \geq f(\frac{1}{2}, R) = \frac{1317}{2} > \frac{21}{16} = BR,\ \text{for } (t,x) \in [\frac{1}{2}, 1] \times [4, 32].$$

Hence, all the conditions of Theorem 4.2 are satisfied. Hence, Theorem 4.2 guarantees that BVP (21) has at least one positive solution $u^*(t)$ such that

$$1 \leq \min_{t\in[\frac{1}{2},1]} u^*(t) \text{ and } \max_{t\in[0,1]} u^*(t) \leq 4.$$

5.3. **Example.** Consider the boundary value problem

$$\begin{aligned} &u^{(4)}(t) + \frac{3u^2(t)+tu(t)}{u(t)+1}(t+\sin t) = 0,\ t\in(0,1) \\ &u(0) = u'(0) = u''(0) = 0,\ u(1) = \xi u\Big(\frac{1}{2}\Big) \end{aligned} \tag{22}$$

In this problem, $f(t,x) = \frac{3x^2+tx}{x+1}(t+\sin t)$. It is easy to see that $f \in C([0,1]\times[0,\infty),[0,\infty))$. Let $\eta = \frac{1}{2}, \xi = \frac{1}{3}$ then

$$\frac{f(t,x)}{x} = \frac{3x+t}{x+1}(t+\sin t) < 6 < \frac{121}{2} = A, \text{ for } (t,x)\in[0,1]\times(0,\infty],$$

which implies (18) holds. Hence, by Theorem 4.3, BVP (22) does not admit positive solutions.

5.4. **Example.** Consider the boundary value problem

$$\begin{aligned} &u^{(3)}(t) + \frac{7u^2(t)+8u(t)}{u(t)+1}(2+t+\sin t) = 0,\ t\in(0,1) \\ &u(0) = u'(0) = 0,\ u(1) = \xi u\Big(\frac{1}{2}\Big) \end{aligned} \tag{23}$$

In this problem, $f(t,x) = \frac{7x^2+8x}{x+1}(2+t+\sin t)$. It is easy to see that $f \in C([0,1]\times[0,\infty),[0,\infty))$. Let $\eta = \frac{1}{2}, \xi = \frac{1}{3}$ then

$$\frac{f(t,x)}{x} = \frac{7x+8}{x+1}(2+t+\sin t) > 14 > \frac{5}{6} = 4C, \text{ for } (t,x)\in[0,1]\times(0,\infty],$$

which implies (19) holds. Hence, by Theorem 4.4, BVP (23) does not admit positive solutions.

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