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USING IMPLICIT RELATION TO PROVE COMMON COUPLED FIXED POINT THEOREMS FOR TWO HYBRID PAIRS OF MAPPINGS

BHAVANA DESHPANDE[1], AMRISH HANDA[2], DEEPMALA [3], §

Abstract. Using implicit relation we establish two common coupled fixed point theorems under the conditions of weakly commutativity and $w-$compatibility on a complete metric space, which is not partially ordered. We do not use the condition of continuity of any mapping for proving the existence of coupled coincidence and common coupled fixed point.

Keywords:Coupled fixed point, coupled coincidence point, implicit relation, $w-$compatibility, weakly commuting mappings.

AMS Subject Classification: 47H10, 54H25.

1. Introduction and Preliminaries

Let $(X,\ d)$ be a metric space and $CB(X)$ be the set of all nonempty closed bounded subsets of X. Let $D(x,\ A)$ denote the distance from x to $A \subset X$ and H denote the Hausdorff metric induced by d, that is,

$$
\begin{aligned}
D(x,\ A) &= \inf_{a\in A} d(x,\ a), \\
H(A,\ B) &= \max\left\{\sup_{a\in A} D(a,\ B),\ \sup_{b\in B} D(b,\ A)\right\},\ \text{for all } A,\ B \in CB(X).
\end{aligned}
$$

The study of fixed points for multivalued contractions and non-expansive mappings using the Hausdorff metric was studied by many authors under different conditions. This theory has found application in control theory, convex optimization, differential inclusions and economics. There exists considerable literature about fixed point properties for two hybrid pairs of mappings, including [2, 10, 11, 12, 20, 25, 26, 32].

Bhaskar and Lakshmikantham [7] introduced the concept of coupled fixed point for single-valued mappings and established some coupled fixed point results and found its application in the existence and uniqueness of solution for periodic boundary value problems. Lakshmikantham and Ciric [18] proved coupled coincidence and common coupled fixed point theorems for nonlinear contractive mappings in partially ordered complete metric spaces and extended the results established in [7]. Many authors focused on coupled fixed point theory for single-valued mappings and proved remarkable results including [5, 9, 13, 15, 16, 21, 33].

[1] Department of Mathematics, Govt. P. G. Arts & Science College, Ratlam (M. P.) India.
e-mail: bhavnadeshpande@yahoo.com;

[2] Department of Mathematics, Govt. P. G. Arts & Science College, Ratlam (M. P.) India.
e-mail: amrishhanda83@gmail.com;

[3] SQC & OR Unit, Indian Statistical Institute, 203 B. T. Road, Kolkata, 700 108, India.
e-mail: deepmaladm23@gmail.com;

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Recently Samet et al. [28] claimed that most of the coupled fixed point theorems for single-valued mappings on ordered metric spaces are consequences of well-known fixed point theorems.

The concepts related to coupled fixed point for single valued mappings have been extended by Abbas et al. [1] for multivalued mappings to obtained coupled coincidence point and common coupled fixed point theorems involving hybrid pair of mappings in complete metric spaces. At present, very few authors have been studied coupled fixed point theorems for hybrid pair of mappings including [1, 19].

In [1], Abbas, Ciric, Damjanovic and Khan introduced the following concept:

Definition 1.1. *Let X be a nonempty set, $F : X \times X \to 2^X$ (a collection of all nonempty subsets of X) and g be a self-mapping on X. An element $(x, y) \in X \times X$ is called*

(1) a coupled fixed point of F if $x \in F(x, y)$ and $y \in F(y, x)$.

(2) a coupled coincidence point of hybrid pair $\{F, g\}$ if $gx \in F(x, y)$ and $gy \in F(y, x)$.

(3) a common coupled fixed point of hybrid pair $\{F, g\}$ if $x = gx \in F(x, y)$ and $y = gy \in F(y, x)$.

We denote the set of coupled coincidence points of mappings F and g by $CF, g\}$. Note that if $(x, y) \in C(F, g)$, then (y, x) is also in $C(F, g)$.

Definition 1.2. *Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The mapping g is called $F-$weakly commuting at some point $(x, y) \in X \times X$ if $g^2x \in F(gx, gy)$ and $g^2y \in F(gy, gx)$.*

Definition 1.3. *Let $F : X \times X \to 2^X$ be a multivalued mapping and g be a self-mapping on X. The hybrid pair $\{F, g\}$ is called $w-$compatible if $gF(x, y) \subseteq F(gx, gy)$ whenever $(x, y) \in C(F, g)$.*

Lemma 1.1. [14] *Let (X, d) be a metric space. Then, for each $a \in X$ and $B \in CB(X)$, there is $b_0 \in B$ such that $D(a, B) = d(a, b_0)$, where $D(a, B) = \inf_{b \in B} d(a, b)$.*

On the other hand, several authors have been studied fixed point theorems satisfying an implicit relation for single-valued and multivalued mappings under different conditions including [3, 4, 6, 8, 17, 22, 23, 24, 27, 29, 30, 31, 34].

In this paper, we establish two common coupled fixed point theorems for two hybrid pairs of mappings satisfying an implicit relation under the conditions of weakly commutativity and $w-$compatibility on a complete metric space, which is not partially ordered. To prove our theorems we do not use condition of continuity of any mapping. We improve, extend and generalize the result of Sedghi et al. [29].

2. **Implicit relation**

Let R^+ be the set of all non-negative real numbers and let Ψ be the set of all continuous functions $\psi : (R^+)^{11} \to R$ satisfying the following conditions:

$\psi_1 : \psi(t_1, t_2, ..., t_{11})$ is non-decreasing in t_1 and non-increasing in $t_2, t_3, ..., t_{11}$.

ψ_2 : There exists $0 < k < 1$ such that for every $u, v, p, q \in R^+$ such that

$$\psi(u,\ v,\ v,\ u,\ u+v,\ 0,\ q,\ q,\ p,\ p+q,\ 0) \leq 0,$$

or

$$\psi(u,\ v,\ u,\ v,\ 0,\ u+v,\ q,\ p,\ q,\ 0,\ p+q) \leq 0,$$

then $\max\{u, p\} \leq k \max\{v, q\}$.

ψ_3 : For all $u, v > 0$,

$$\psi(u,\ u,\ 0,\ 0,\ u,\ u,\ v,\ 0,\ 0,\ v,\ v) > 0.$$

Example 2.1. *Let* $\psi(t_1, t_2, ..., t_{11}) = t_1 - h \max\{t_2, t_3, t_4, \frac{t_5+t_6}{2}, t_7, t_8, t_9, \frac{t_{10}+t_{11}}{2}\}$ *where* $0 < h < 1$.

(ψ_1) *Obvious.* (ψ_2) *Let* $\max\{u, p\} > 0$ *and* $\psi(u, v, v, u, u+v, 0, q, q, p, p+q, 0) = u - h \max\{u, v, p, q\} \leq 0$. *Thus* $u \leq h \max\{\max\{u, p\}, \max\{v, q\}\}$. *Similarly* $p \leq h \max\{\max\{u, p\}, \max\{v, q\}\}$. *Thus* $\max\{u, p\} \leq h \max\{\max\{u, p\}, \max\{v, q\}\}$. *Now, if* $\max\{u, p\} \geq \max\{v, q\}$, *then* $\max\{u, p\} \leq h \max\{u, p\} < \max\{u, p\}$, *which is a contradiction. Thus* $\max\{u, p\} < \max\{v, q\}$ *and* $\max\{u, p\} \leq h \max\{v, q\}$. *Similarly, let* $\max\{u, p\} > 0$ *and* $\psi(u, v, u, v, 0, u+v, q, p, q, 0, p+q) = u - h \max\{u, v, p, q\} \leq 0$, *then we have* $\max\{u, p\} \leq h \max\{v, q\}$. *Thus* (ψ_2) *is satisfying with* $k = h < 1$. *If* $\max\{u, p\} = 0$, *then* $\max\{u, p\} \leq k \max\{v, q\}$. (ψ_3) $\psi(u, u, 0, 0, u, u, v, 0, 0, v, v) = u - h \max\{u, v\} = \max\{u - hu, u - hv\} = \max\{u(1-h), u - hv\} > 0$. *Therefore* $\psi \in \Psi$.

Example 2.2. *Let* $\psi(t_1, t_2, ..., t_{11}) = t_1 - \alpha \max\{t_2, t_3, t_4, t_7, t_8, t_9\} - \beta \max\{t_5 + t_6, t_{10} + t_{11}\}$ *where* $\alpha, \beta \geq 0$ *and* $\alpha + 2\beta < 1$.

(ψ_1) *Obvious.* (ψ_2) *Let* $\max\{u, p\} > 0$ *and* $\psi(u, v, v, u, u+v, 0, q, q, p, p+q, 0) = u - \alpha \max\{v, u, q, p\} - \beta \max\{u+v, p+q\} \leq 0$, *then* $u \leq \alpha \max\{\max\{u, p\}, \max\{v, q\}\} + \beta[\max\{u, p\} + \max\{v, q\}]$, *it follows that* $u \leq \max\{(\alpha+\beta)\max\{u, p\} + \beta \max\{v, q\}, (\alpha+\beta)\max\{v, q\} + \beta \max\{u, p\}\}$. *Similarly* $p \leq \max\{(\alpha+\beta)\max\{u, p\} + \beta \max\{v, q\}, (\alpha+\beta)\max\{v, q\} + \beta \max\{u, p\}\}$. *Thus* $\max\{u, p\} \leq \max\{(\alpha+\beta)\max\{u, p\} + \beta \max\{v, q\}, (\alpha+\beta)\max\{v, q\} + \beta \max\{u, p\}\}$. *Now, if* $\max\{u, p\} \geq \max\{v, q\}$, *then* $\max\{u, p\} \leq (\alpha+2\beta)\max\{u, p\} < \max\{u, p\}$, *which is a contradiction. Thus* $\max\{u, p\} < \max\{v, q\}$ *and* $\max\{u, p\} \leq (\alpha+2\beta)\max\{v, q\}$. *Similarly, let* $\max\{u, p\} > 0$ *and* $\psi(u, v, u, v, 0, u+v, q, p, q, 0, p+q) = u - \alpha \max\{v, u, q, p\} - \beta \max\{u+v, p+q\} \leq 0$, *then we have* $\max\{u, p\} \leq (\alpha+2\beta)\max\{v, q\}$. *Thus* (ψ_2) *is satisfying with* $k = \alpha + 2\beta < 1$. *If* $\max\{u, p\} = 0$, *then* $\max\{u, p\} \leq k \max\{v, q\}$. (ψ_3) $\psi(u, u, 0, 0, u, u, v, 0, 0, v, v) = u - \alpha \max\{u, v\} - \beta \max\{2u, 2v\} = \max\{u - \alpha u - 2\beta u, u - \alpha v - 2\beta v\} = \max\{u(1-\alpha-2\beta), u - (\alpha+2\beta)v\} > 0$. *Therefore* $\psi \in \Psi$.

Example 2.3. *Let* $\psi(t_1, t_2, ..., t_{11}) = t_1 - a \max\{t_2, t_7\} - b \max\{t_3 + t_4, t_8 + t_9\} - c \max\{t_5 + t_6, t_{10} + t_{11}\}$, *where* $a, b, c \in [0, 1)$ *and* $a + 2b + 2c < 1$.

(ψ_1) *Obvious.* (ψ_2) *Let* $\max\{u, p\} > 0$ *and* $\psi(u, v, v, u, u+v, 0, q, q, p, p+q, 0) = u - a \max\{v, q\} - b \max\{u+v, p+q\} - c \max\{u+v, p+q\} \leq 0$, *then* $u \leq a \max\{v, q\} + b[\max\{u, p\} + \max\{v, q\}] + c[\max\{u, p\} + \max\{v, q\}]$. *Similarly* $p \leq a \max\{v, q\} + b[\max\{u, p\} + \max\{v, q\}] + c[\max\{u, p\} + \max\{v, q\}]$. *Thus* $\max\{u, p\} \leq a \max\{v, q\} + b[\max\{u, p\} + \max\{v, q\}] + c[\max\{u, p\} + \max\{v, q\}]$. *Now, if* $\max\{u, p\} \geq \max\{v, q\}$, *then* $\max\{u, p\} \leq (a+2b+2c)\max\{u, p\} < \max\{u, p\}$, *which is a contradiction. Thus* $\max\{u, p\} < \max\{v, q\}$ *and* $\max\{u, p\} \leq (a+2b+2c)\max\{v, q\}$. *Similarly, let* $\max\{u, p\} > 0$ *and* $\psi(u, v, u, v, 0, u+v, q, p, q, 0, p+q) = u - a \max\{v, q\} - b \max\{u+v, p+q\} - c \max\{u+v, p+q\} \leq 0$, *then we have* $\max\{u, p\} \leq (a+2b+2c)\max\{v, q\}$. *Thus* (ψ_2) *is satisfying with* $k = a + 2b + 2c < 1$. *If* $\max\{u, p\} = 0$, *then* $\max\{u, p\} \leq k \max\{v, q\}$. (ψ_3) $\psi(u, u, 0, 0, u, u, v, 0, 0, v, v) = u - a \max\{u, v\} - c \max\{2u, 2v\} = \max\{u - au - 2cu, u - av - 2cu\} = \max\{u(1-a-2c), u - (a+2c)v\} > 0$. *Therefore* $\psi \in \Psi$.

Example 2.4. *Let* $\psi(t_1, t_2, ..., t_{11}) = t_1 - h \max\{t_2, t_7\} - L \min\{t_3, t_4, t_5, t_6, t_8, t_9, t_{10}, t_{11}\}$, *where* $h \in [0, 1)$ *and* $L \geq 0$.

(ψ_1) *Obvious.* (ψ_2) *Let* $\max\{u, p\} > 0$ *and* $\psi(u, v, v, u, u+v, 0, q, q, p, p+q, 0) = u - h\max\{v, q\} \leq 0$, *then* $u \leq h\max\{v, q\}$. *Similarly* $p \leq h\max\{v, q\}$. *Thus* $\max\{u, p\} \leq h\max\{v, q\}$. *Similarly, let* $\max\{u, p\} > 0$ *and* $\psi(u, v, u, v, 0, u+v, q, p, q, 0, p+q) = u - h\max\{v, q\} \leq 0$, *then we have* $\max\{u, p\} \leq h\max\{v, q\}$. *Thus* ($\psi_2$) *is satisfying with* $k = h < 1$. *If* $\max\{u, p\} = 0$, *then* $\max\{u, p\} \leq k\max\{v, q\}$. ($\psi_3$) $\psi(u, u, 0, 0, u, u, v, 0, 0, v, v) = u - h\max\{u, v\} = \max\{u - hu, u - hv\} = \max\{u(1-h), u - hv\} > 0$. *Therefore* $\psi \in \Psi$.

Example 2.5. $\psi(t_1, t_2, ..., t_{11}) = t_1 - h\max\{t_2, t_3, t_4, \frac{t_5+t_6}{2}, t_7, t_8, t_9, \frac{t_{10}+t_{11}}{2}\} - L\min\{t_3, t_4, t_5, t_6, t_8, t_9, t_{10}, t_{11}\}$, *where* $h \in [0, 1)$ *and* $L \geq 0$.

(ψ_1) *Obvious.* (ψ_2) *Let* $\max\{u, p\} > 0$ *and* $\psi(u, v, v, u, u+v, 0, q, q, p, p+q, 0) = u - h\max\{u, v, p, q\} \leq 0$, *then* $u \leq h\max\{\max\{u, p\}, \max\{v, q\}\}$. *Similarly* $p \leq h\max\{\max\{u, p\}, \max\{v, q\}\}$. *Thus* $\max\{u, p\} \leq h\max\{\max\{u, p\}, \max\{v, q\}$. *Now, if* $\max\{u, p\} \geq \max\{v, q\}$, *then* $\max\{u, p\} \leq h\max\{u, p\} < \max\{u, p\}$, *which is a contradiction. Thus* $\max\{u, p\} < \max\{v, q\}$ *and* $\max\{u, p\} \leq h\max\{v, q\}$. *Similarly, let* $\max\{u, p\} > 0$ *and* $\psi(u, v, u, v, 0, u+v, q, p, q, 0, p+q) = u - h\max\{u, v, p, q\} \leq 0$, *then we have* $\max\{u, p\} \leq h\max\{v, q\}$. *Thus* ($\psi_2$) *is satisfying with* $k = h < 1$. *If* $\max\{u, p\} = 0$, *then* $\max\{u, p\} \leq k\max\{v, q\}$. ($\psi_3$) $\psi(u, u, 0, 0, u, u, v, 0, 0, v, v) = u - h\max\{u, v\} = \max\{u - hu, u - hv\} = \max\{u(1-h), u - hv\} > 0$. *Therefore* $\psi \in \Psi$.

Example 2.6. *Let* $\psi(t_1, t_2, ..., t_{11}) = t_1 - h\max\{t_2, t_7\}$, *where* $h \in [0, 1)$.

(ψ_1) *Obvious.* (ψ_2) *Let* $\max\{u, p\} > 0$ *and* $\psi(u, v, v, u, u+v, 0, q, q, p, p+q, 0\} = u - h\max\{v, q\} \leq 0$. *Thus* $u \leq h\max\{v, q\}$. *Similarly* $p \leq h\max\{v, q\}$. *Thus* $\max\{u, p\} \leq h\max\{v, q\}$. *Similarly, let* $\max\{u, p\} > 0$ *and* $\psi(u, v, u, v, 0, u+v, q, p, q, 0, p+q) = u - h\max\{v, q\} \leq 0$, *then we have* $\max\{u, p\} \leq h\max\{v, q\}$. *Thus* ($\psi_2$) *is satisfying with* $k = h < 1$. *If* $\max\{u, p\} = 0$, *then* $\max\{u, p\} \leq k\max\{v, q\}$. ($\psi_3$) $\psi(u, u, 0, 0, u, u, v, 0, 0, v, v) = u - h\max\{u, v\} = \max\{u - hu, u - hv\} = \max\{u(1-h), u - hv\} > 0$. *Therefore* $\psi \in \Psi$.

Example 2.7. *Let* $\psi(t_1, t_2, ..., t_{11}) = t_1 - b\max\{\frac{t_3+t_4}{2}, \frac{t_8+t_9}{2}\}$ *where* $b \in [0, \frac{1}{2})$.

(ψ_1) *Obvious.* (ψ_2) *Let* $\max\{u, p\} > 0$ *and* $\psi(u, v, v, u, u+v, 0, q, q, p, p+q, 0\} = u - b\max\{\frac{u+v}{2}, \frac{p+q}{2}\} \leq 0$. *Thus* $u \leq \frac{b}{2}[\max\{u, p\} + \max\{v, q\}]$. *Similarly* $p \leq \frac{b}{2}[\max\{u, p\} + \max\{v, q\}]$. *Thus* $\max\{u, p\} \leq \frac{b}{2}[\max\{u, p\} + \max\{v, q\}]$. *Now, if* $\max\{u, p\} \geq \max\{v, q\}$, *then* $\max\{u, p\} \leq b\max\{u, p\} < \max\{u, p\}$, *which is a contradiction. Thus* $\max\{u, p\} < \max\{v, q\}$ *and* $\max\{u, p\} \leq b\max\{v, q\}$. *Similarly, let* $\max\{u, p\} > 0$ *and* $\psi(u, v, u, v, 0, u+v, q, p, q, 0, p+q) = u - b\max\{\frac{u+v}{2}, \frac{p+q}{2}\} \leq 0$, *then we have* $\max\{u, p\} \leq b\max\{v, q\}$. *Thus* ($\psi_2$) *is satisfying with* $k = b < 1$. *If* $\max\{u, p\} = 0$, *then* $\max\{u, p\} \leq k\max\{v, q\}$. ($\psi_3$) $\psi(u, u, 0, 0, u, u, v, 0, 0, v, v) = u > 0$. *Therefore* $\psi \in \Psi$.

Example 2.8. *Let* $\psi(t_1, t_2, ..., t_{11}) = t_1 - c\max\{t_5 + t_6, t_{10} + t_{11}\}$ *where* $c \in [0, \frac{1}{2})$.

(ψ_1) *Obvious.* (ψ_2) *Let* $\max\{u, p\} > 0$ *and* $\psi(u, v, v, u, u+v, 0, q, q, p, p+q, 0\} = u - c\max\{u+v, p+q\} \leq 0$. *Thus* $u \leq c[\max\{u, p\} + \max\{v, q\}]$. *Similarly* $p \leq c[\max\{u, p\} + \max\{v, q\}]$. *Thus* $\max\{u, p\} \leq c[\max\{u, p\} + \max\{v, q\}]$. *Now, if* $\max\{u, p\} \geq \max\{v, q\}$, *then* $\max\{u, p\} \leq 2c\max\{u, p\} < \max\{u, p\}$, *which is a contradiction. Thus* $\max\{u, p\} < \max\{v, q\}$ *and* $\max\{u, p\} \leq 2c\max\{v, q\}$. *Similarly, let* $\max\{u, p\} > 0$ *and* $\psi(u, v, u, v, 0, u+v, q, p, q, 0, p+q) = u - c\max\{u+v, p+q\} \leq 0$, *then we have* $\max\{u, p\} \leq 2c\max\{v, q\}$. *Thus* ($\psi_2$) *is satisfying with* $k = 2c < 1$. *If* $\max\{u, p\} = 0$, *then* $\max\{u, p\} \leq k\max\{v, q\}$. ($\psi_3$) $\psi(u, u, 0, 0, u, u, v, 0, 0, v, v) = u - c\max\{2u, 2v\} = \max\{u - 2cu, u - 2cv\} = \max\{u(1-2c), u - 2cv\} > 0$. *Therefore* $\psi \in \Psi$.

Example 2.9. *Let* $\psi(t_1, t_2, ..., t_{11}) = t_1 - h\max\{t_2, \frac{t_3+t_4}{2}, \frac{t_5+t_6}{2}, t_7, \frac{t_8+t_9}{2}, \frac{t_{10}+t_{11}}{2}\}$ *where* $h \in [0, 1)$.

(ψ_1) *Obvious.* (ψ_2) *Let* $\max\{u, p\} > 0$ *and* $\psi(u, v, v, u, u+v, 0, q, q, p, p+q, 0\} = u - h\max\{v, q\} \leq 0$. *Thus* $u \leq h\max\{v, q\}$. *Similarly* $p \leq h\max\{v, q\}$. *Thus* $\max\{u, p\} \leq h\max\{v, q\}$. *Similarly, let* $\max\{u, p\} > 0$ *and* $\psi(u, v, u, v, 0, u+v, q, p, q, 0, p+q) = u - h\max\{v, q\} \leq 0$, *then we have* $\max\{u, p\} \leq h\max\{v, q\}$. *Thus* (ψ_2) *is satisfying with* $k = h < 1$. *If* $\max\{u, p\} = 0$, *then* $\max\{u, p\} \leq k\max\{v, q\}$. (ψ_3) $\psi(u, u, 0, 0, u, u, v, 0, 0, v, v) = u - h\max\{u, v\} = \max\{u - hu, u - hv\} = \max\{u(1-h), u - hv\} > 0$. *Therefore* $\psi \in \Psi$.

3. Main Results

Theorem 3.1. *Let* (X, d) *be a complete metric space. Assume* $F, G : X \times X \to CB(X)$ *and* $f, g : X \to X$ *be mappings satisfying*

(i) $F(X \times X) \subseteq g(X)$, $G(X \times X) \subseteq f(X)$,

(ii) *for all* $x, y, u, v \in X$, *where* $\psi \in \Psi$,

$$\psi\begin{pmatrix} H(F(x,\ y),\ G(u,\ v)), \\ d(fx,\ gu),\ D(fx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ D(fx,\ G(u,\ v)),\ D(gu,\ F(x,\ y)), \\ d(fy,\ gv),\ D(fy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ D(fy,\ G(v,\ u)),\ D(gv,\ F(y,\ x)) \end{pmatrix} \leq 0,$$

(iii) $f(X)$ *and* $g(X)$ *are closed subsets of* X, *then*

(a) F *and* f *have a coupled coincidence point,*

(b) G *and* g *have a coupled coincidence point,*

(c) F *and* f *have a common coupled fixed point, if* f *is* $F-$*weakly commuting at* (x, y) *and* $f^2x = fx$ *and* $f^2y = fy$ *for* $(x, y) \in C\{F, f\}$,

(d) G *and* g *have a common coupled fixed point, if* g *is* $G-$*weakly commuting at* $(\widetilde{x}, \widetilde{y})$ *and* $g^2\widetilde{x} = g\widetilde{x}$ *and* $g^2\widetilde{y} = g\widetilde{y}$ *for* $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$,

(e) F, G, f *and* g *have common coupled fixed point provided that both* (c) *and* (d) *are true.*

Proof. Let $x_0, y_0 \in X$ be arbitrary. Choose $u_1 = gx_1 \in F(x_0, y_0)$ and $v_1 = gy_1 \in F(y_0, x_0)$, as $F(X \times X) \subseteq g(X)$. Since $F, G : X \times X \to CB(X)$, therefore by Lemma 1.1, there exist $u_2 \in G(x_1, y_1)$ and $v_2 \in G(y_1, x_1)$ such that

$$\begin{aligned} d(u_1,\ u_2) &\leq H(F(x_0,\ y_0),\ G(x_1,\ y_1)), \\ d(v_1,\ v_2) &\leq H(F(y_0,\ x_0),\ G(y_1,\ x_1)). \end{aligned}$$

Since $G(X \times X) \subseteq f(X)$, there exist $x_2, y_2 \in X$ such that $u_2 = fx_2 \in G(x_1, y_1)$ and $v_2 = fy_2 \in G(y_1, x_1)$. Then we choose $u_3 \in F(x_2, y_2)$ and $v_3 \in F(y_2, x_2)$ such that

$$\begin{aligned} d(u_2,\ u_3) &\leq H(G(x_1,\ y_1),\ F(x_2,\ y_2)), \\ d(v_2,\ v_3) &\leq H(G(y_1,\ x_1),\ F(y_2,\ x_2)). \end{aligned}$$

Continuing this process, we obtain sequences $\{u_n\}$, $\{v_n\}$, $\{x_n\}$ and $\{y_n\}$ in X such that for all $n \geq 0$, we have

$$\begin{aligned} u_{2n} &= fx_{2n} \in G(x_{2n-1},\ y_{2n-1}),\ u_{2n+1} = gx_{2n+1} \in F(x_{2n},\ y_{2n}), \\ v_{2n} &= fy_{2n} \in G(y_{2n-1},\ x_{2n-1}),\ v_{2n+1} = gy_{2n+1} \in F(y_{2n},\ x_{2n}), \end{aligned}$$

and

$$\begin{aligned}
d(u_{2n-1},\ u_{2n}) &\leq H(F(x_{2n-2},\ y_{2n-2}),\ G(x_{2n-1},\ y_{2n-1})),\\
d(u_{2n},\ u_{2n+1}) &\leq H(G(x_{2n-1},\ y_{2n-1}),\ F(x_{2n},\ y_{2n})),\\
d(v_{2n-1},\ v_{2n}) &\leq H(F(y_{2n-2},\ x_{2n-2}),\ G(y_{2n-1},\ x_{2n-1})),\\
d(v_{2n},\ v_{2n+1}) &\leq H(G(y_{2n-1},\ x_{2n-1}),\ F(y_{2n},\ x_{2n})).
\end{aligned}$$

Then by condition (ii), we get

$$\psi\begin{pmatrix}
H(F(x_{2n},\ y_{2n}),\ G(x_{2n-1},\ y_{2n-1})),\\
d(fx_{2n},\ gx_{2n-1}),\\
D(fx_{2n},\ F(x_{2n},\ y_{2n})),\ D(gx_{2n-1},\ G(x_{2n-1},\ y_{2n-1})),\\
D(fx_{2n},\ G(x_{2n-1},\ y_{2n-1})),\ D(gx_{2n-1},\ F(x_{2n},\ y_{2n})),\\
d(fy_{2n},\ gy_{2n-1}),\\
D(fy_{2n},\ F(y_{2n},\ x_{2n})),\ D(gy_{2n-1},\ G(y_{2n-1},\ x_{2n-1})),\\
D(fy_{2n},\ G(y_{2n-1},\ x_{2n-1})),\ D(gy_{2n-1},\ F(y_{2n},\ x_{2n}))
\end{pmatrix}\leq 0.$$

Using (ψ_1), we get

$$\psi\begin{pmatrix}
d(u_{2n+1},\ u_{2n}),\\
d(u_{2n},\ u_{2n-1}),\ d(u_{2n},\ u_{2n+1}),\ d(u_{2n-1},\ u_{2n}),\\
0,\ d(u_{2n-1},\ u_{2n+1}),\\
d(v_{2n},\ v_{2n-1}),\ d(v_{2n},\ v_{2n+1}),\ d(v_{2n-1},\ v_{2n}),\\
0,\ d(v_{2n-1},\ v_{2n+1})
\end{pmatrix}\leq 0,$$

which implies that

$$\psi\begin{pmatrix}
d(u_{2n+1},\ u_{2n}),\\
d(u_{2n},\ u_{2n-1}),\ d(u_{2n},\ u_{2n+1}),\ d(u_{2n-1},\ u_{2n}),\\
0,\ d(u_{2n-1},\ u_{2n})+d(u_{2n},\ u_{2n+1}),\\
d(v_{2n},\ v_{2n-1}),\ d(v_{2n},\ v_{2n+1}),\ d(v_{2n-1},\ v_{2n}),\\
0,\ d(v_{2n-1},\ v_{2n})+d(v_{2n},\ v_{2n+1})
\end{pmatrix}\leq 0.$$

By (ψ_2), we get

$$\begin{aligned}
&\max\{d(u_{2n+1},\ u_{2n}),\ d(v_{2n+1},\ v_{2n})\}\\
\leq\ & k\max\{d(u_{2n},\ u_{2n-1}),\ d(v_{2n},\ v_{2n-1})\}.
\end{aligned}$$

Similarly, we can obtain

$$\begin{aligned}
&\max\{d(u_{2n},\ u_{2n-1}),\ d(v_{2n},\ v_{2n-1})\}\\
\leq\ & k\max\{d(u_{2n-1},\ u_{2n-2}),\ d(v_{2n-1},\ v_{2n-2})\}.
\end{aligned}$$

Thus, we have for all $n\in\mathbb{N}$,

$$\begin{aligned}
&\max\{d(u_n,\ u_{n+1}),\ d(v_n,\ v_{n+1})\}\\
\leq\ & k\max\{d(u_{n-1},\ u_n),\ d(v_{n-1},\ v_n)\}\\
\leq\ & k^n\max\{d(u_0,\ u_1),\ d(v_0,\ v_1)\}\\
\leq\ & k^n\delta.
\end{aligned}$$

Thus

$$\max\{d(u_n,\ u_{n+1}),\ d(v_n,\ v_{n+1})\}\leq k^n\delta, \tag{1}$$

where

$$\delta=\max\{d(u_0,\ u_1),\ d(v_0,\ v_1)\}.$$

Thus, for $m,\ n\in\mathbb{N}$ with $m>n$, by triangle inequality and (1), we get

$$\begin{aligned}
& \max\{d(u_n,\ u_{m+n}),\ d(v_n,\ v_{m+n})\} \\
\leq\ & \max\{d(u_n,\ u_{n+1}),\ d(v_n,\ v_{n+1})\} \\
& +\max\{d(u_{n+1},\ u_{n+2}),\ d(v_{n+1},\ v_{n+2})\} \\
& +...+\max\{d(u_{m+n-1},\ u_{m+n}),\ d(v_{m+n-1},\ v_{m+n})\} \\
\leq\ & k^n\delta+k^{n+1}\delta+...+k^{n+m-1}\delta \\
\leq\ & k^n(1+k+k^2+...+k^{m-1})\delta \\
\leq\ & \frac{k^n(1-k^m)}{1-k}\delta\to 0 \text{ as } n,\ m\to\infty,
\end{aligned}$$

which shows that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in X. Since X is complete, there exist $u,\ v\in X$ such that

$$\begin{aligned}
\lim_{n\to\infty} u_n &= \lim_{n\to\infty} fx_{2n}=\lim_{n\to\infty} gx_{2n+1}=u, \\
\lim_{n\to\infty} v_n &= \lim_{n\to\infty} fy_{2n}=\lim_{n\to\infty} gy_{2n+1}=v.
\end{aligned} \tag{2}$$

Since $f(X)$ and $g(X)$ are closed subsets of X, then there exist $x,\ y,\ \widetilde{x},\ \widetilde{y}\in X$, we have

$$u=fx=g\widetilde{x},\ v=fy=g\widetilde{y}. \tag{3}$$

Now, since $fx_{2n}\in G(x_{2n-1},\ y_{2n-1})$ and $fy_{2n}\in G(y_{2n-1},\ x_{2n-1})$, therefore by using condition (ii), we get

$$\psi\left(\begin{array}{c}
H(F(x,\ y),\ G(x_{2n-1},\ y_{2n-1})), \\
d(fx,\ gx_{2n-1}),\ D(fx,\ F(x,\ y)),\ D(gx_{2n-1},\ G(x_{2n-1},\ y_{2n-1})), \\
D(fx,\ G(x_{2n-1},\ y_{2n-1})),\ D(gx_{2n-1},\ F(x,\ y)), \\
d(fy,\ gy_{2n-1}),\ D(fy,\ F(y,\ x)),\ D(gy_{2n-1},\ G(y_{2n-1},\ x_{2n-1})), \\
D(fy,\ G(y_{2n-1},\ x_{2n-1})),\ D(gy_{2n-1},\ F(y,\ x))
\end{array}\right)\leq 0,$$

which implies, by (ψ_1), that

$$\psi\left(\begin{array}{c}
D(F(x,\ y),\ fx_{2n}), \\
d(fx,\ gx_{2n-1}),\ D(fx,\ F(x,\ y)),\ d(gx_{2n-1},\ fx_{2n}), \\
d(fx,\ fx_{2n}),\ D(gx_{2n-1},\ F(x,\ y)), \\
d(fy,\ gy_{2n-1}),\ D(fy,\ F(y,\ x)),\ d(gy_{2n-1},\ fy_{2n}), \\
d(fy,\ fy_{2n}),\ D(gy_{2n-1},\ F(y,\ x))
\end{array}\right)\leq 0.$$

Letting $n\to\infty$ in the above inequality, using the continuity of ψ, (2) and (3), we obtain

$$\psi\left(\begin{array}{c}
D(F(x,\ y),\ fx), \\
0,\ D(fx,\ F(x,\ y)),\ 0,\ 0,\ D(fx,\ F(x,\ y)), \\
0,\ D(fy,\ F(y,\ x)),\ 0,\ 0,\ D(fy,\ F(y,\ x))
\end{array}\right)\leq 0.$$

Thus, by (ψ_2), we obtain

$$D(fx,\ F(x,\ y))=0 \text{ and } D(fy,\ F(y,\ x))=0,$$

which implies that

$$fx\in F(x,\ y) \text{ and } fy\in F(y,\ x),$$

that is, $(x,\ y)$ is a coupled coincidence point of F and f. This proves (a). Again, since $gx_{2n+1} \in F(x_{2n},\ y_{2n})$ and $gy_{2n+1} \in F(y_{2n},\ x_{2n})$, therefore by using condition (ii), we get

$$\psi\begin{pmatrix} H(F(x_{2n},\ y_{2n}),\ G(\widetilde{x},\ \widetilde{y})), \\ d(fx_{2n},\ g\widetilde{x}),\ D(fx_{2n},\ F(x_{2n},\ y_{2n})),\ D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})), \\ D(fx_{2n},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{x},\ F(x_{2n},\ y_{2n})), \\ d(fy_{2n},\ g\widetilde{y}),\ D(fy_{2n},\ F(y_{2n},\ x_{2n})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x})), \\ D(fy_{2n},\ G(\widetilde{y},\ \widetilde{x})),\ D(g\widetilde{y},\ F(y_{2n},\ x_{2n})) \end{pmatrix} \leq 0,$$

which implies, by (ψ_1), that

$$\psi\begin{pmatrix} D(gx_{2n+1},\ G(\widetilde{x},\ \widetilde{y})), \\ d(fx_{2n},\ g\widetilde{x}),\ d(fx_{2n},\ gx_{2n+1}),\ D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})), \\ D(fx_{2n},\ G(\widetilde{x},\ \widetilde{y})),\ d(g\widetilde{x},\ gx_{2n+1}), \\ d(fy_{2n},\ g\widetilde{y}),\ d(fy_{2n},\ gy_{2n+1}),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x})), \\ D(fy_{2n},\ G(\widetilde{y},\ \widetilde{x})),\ d(g\widetilde{y},\ gy_{2n+1}) \end{pmatrix} \leq 0.$$

Letting $n \to \infty$ in the above inequality, using the continuity of ψ, (2) and (3), we obtain

$$\psi\begin{pmatrix} D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})), \\ 0,\ 0,\ D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})),\ 0, \\ 0,\ 0,\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x})),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x})),\ 0 \end{pmatrix} \leq 0.$$

Thus, by (ψ_2), we obtain

$$D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})) = 0 \text{ and } D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x})) = 0,$$

which implies that

$$g\widetilde{x} \in G(\widetilde{x},\ \widetilde{y}) \text{ and } g\widetilde{y} \in G(\widetilde{y},\ \widetilde{x}),$$

that is, $(\widetilde{x},\ \widetilde{y})$ is a coupled coincidence point of G and g. This proves (b).

Furthermore, from condition (c), we have f is $F-$weakly commuting at $(x,\ y)$, that is, $f^2x \in F(fx,\ fy)$, $f^2y \in F(fy,\ fx)$ and $f^2x = fx$, $f^2y = fy$. Thus $fx = f^2x \in F(fx,\ fy)$ and $fy = f^2y \in F(fy,\ fx)$, that is, $u = fu \in F(u,\ v)$ and $v = fv \in F(v,\ u)$. This proves (c). A similar argument proves (d). Then (e) holds immediately. □

Put $f = g$ in the Theorem 3.1, we get the following result:

Corollary 3.1. *Let $(X,\ d)$ be a complete metric space. Assume $F,\ G : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying*

(i) $F(X \times X) \subseteq g(X)$, $G(X \times X) \subseteq g(X)$,

(ii) for all $x,\ y,\ u,\ v \in X$ and $\psi \in \Psi$,

$$\psi\begin{pmatrix} H(F(x,\ y),\ G(u,\ v)), \\ d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ D(gx,\ G(u,\ v)),\ D(gu,\ F(x,\ y)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ D(gy,\ G(v,\ u)),\ D(gv,\ F(y,\ x)) \end{pmatrix} \leq 0,$$

(iii) $g(X)$ is a closed subset of X, then

(a) F and g have a coupled coincidence point,

(b) G and g have a coupled coincidence point,

(c) F and g have a common coupled fixed point, if g is $F-$weakly commuting at $(x,\ y)$ and $g^2x = gx$ and $g^2y = gy$ for $(x,\ y) \in C(F,\ g)$,

(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x},\ \widetilde{y})$ and $g^2\widetilde{x} = g\widetilde{x}$ and $g^2\widetilde{y} = g\widetilde{y}$ for $(\widetilde{x},\ \widetilde{y}) \in C\{G,\ g\}$,

(e) F, G and g have common coupled fixed point provided that both (c) and (d) are true.

Put $F = G$ and $f = g$ in the Theorem 3.1, we get the following result:

Corollary 3.2. *Let (X, d) be a complete metric space. Assume $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying*
(i) $F(X \times X) \subseteq g(X)$,
(ii) for all $x, y, u, v \in X$ and $\psi \in \Psi$,

$$\psi\begin{pmatrix} H(F(x,\ y),\ F(u,\ v)), \\ d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ D(gx,\ F(u,\ v)),\ D(gu,\ F(x,\ y)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ D(gy,\ F(v,\ u)),\ D(gv,\ F(y,\ x)) \end{pmatrix} \leq 0.$$

If (iii) of Corollary 3.1 holds, then
(a) F and g have a coupled coincidence point,
(b) F and g have a common coupled fixed point, if g is $F-$weakly commuting at (x, y) and $g^2x = gx$ and $g^2y = gy$ for $(x, y) \in C(F, g)$.

Examples 2.1-2.9 and Theorem 3.1 imply the following:

Corollary 3.3. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $f, g : X \to X$ be mappings satisfying (i) of Theorem 3.1 and*
(i) for all $x, y, u, v \in X$, where $0 < h < 1$,

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & h\max\left\{\begin{array}{c} d(fx,\ gu),\ D(fx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ d(fy,\ gv),\ D(fy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ \frac{D(fx,\ G(u,\ v))+D(gu,\ F(x,\ y))}{2},\ \frac{D(fy,\ G(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\}, \end{aligned}$$

or for all $x, y, u, v \in X$, where $\alpha, \beta \geq 0$ and $\alpha + 2\beta < 1$,

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & \alpha\max\left\{\begin{array}{c} d(fx,\ gu),\ D(fx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ d(fy,\ gv),\ D(fy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)) \end{array}\right\} \\ & +\beta\max\left\{\begin{array}{c} D(fx,\ G(u,\ v)) + D(gu,\ F(x,\ y)), \\ D(fy,\ G(v,\ u)) + D(gv,\ F(y,\ x)) \end{array}\right\}, \end{aligned}$$

or for all $x, y, u, v \in X$, where $a, b, c \in [0, 1)$ and $a + 2b + 2c < 1$,

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & a\max\{d(fx,\ gu),\ d(fy,\ gv)\} \\ & +b\max\left\{\begin{array}{c} D(fx,\ F(x,\ y)) + D(gu,\ G(u,\ v)), \\ D(fy,\ F(y,\ x)) + D(gv,\ G(v,\ u)) \end{array}\right\} \\ & +c\max\left\{\begin{array}{c} D(fx,\ G(u,\ v)) + D(gu,\ F(x,\ y)), \\ D(fy,\ G(v,\ u)) + D(gv,\ F(y,\ x)) \end{array}\right\}, \end{aligned}$$

or for all $x, y, u, v \in X$, where $h \in [0, 1)$ and $L \geq 0$,

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & h\max\{d(fx,\ gu),\ d(fy,\ gv)\} \\ & +L\max\left\{\begin{array}{c} D(fx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ D(fx,\ G(u,\ v)),\ D(gu,\ F(x,\ y)), \\ D(fy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ D(fy,\ G(v,\ u)),\ D(gv,\ F(y,\ x)) \end{array}\right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v \in X$, *where* $h \in [0,\ 1)$ *and* $L \geq 0$,

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & h \max \left\{ \begin{array}{c} d(fx,\ gu),\ D(fx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ d(fy,\ gv),\ D(fy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ \frac{D(fx,\ G(u,\ v))+D(gu,\ F(x,\ y))}{2},\ \frac{D(fy,\ G(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array} \right\} \\ & + L \max \left\{ \begin{array}{c} D(fx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ D(fx,\ G(u,\ v)),\ D(gu,\ F(x,\ y)), \\ D(fy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ D(fy,\ G(v,\ u)),\ D(gv,\ F(y,\ x)) \end{array} \right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v \in X$, *where* $h \in [0,\ 1)$,

$$H(F(x,\ y),\ G(u,\ v)) \leq h \max \{d(fx,\ gu),\ d(fy,\ gv)\},$$

or for all $x,\ y,\ u,\ v \in X$, *where* $b \in [0,\ \frac{1}{2})$,

$$H(F(x,\ y),\ G(u,\ v)) \leq b \max \left\{ \begin{array}{c} \frac{D(fx,\ F(x,\ y))+D(gu,\ G(u,\ v))}{2}, \\ \frac{D(fy,\ F(y,\ x))+D(gv,\ G(v,\ u))}{2} \end{array} \right\},$$

or for all $x,\ y,\ u,\ v \in X$, *where* $c \in [0,\ \frac{1}{2})$,

$$H(F(x,\ y),\ G(u,\ v)) \leq c \max \left\{ \begin{array}{c} D(fx,\ G(u,\ v)) + D(gu,\ F(x,\ y)), \\ D(fy,\ G(v,\ u)) + D(gv,\ F(y,\ x)) \end{array} \right\},$$

or for all $x,\ y,\ u,\ v \in X$, *where* $0 < h < 1$,

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & h \max \left\{ \begin{array}{c} d(fx,\ gu),\ d(fy,\ gv), \\ \frac{D(fx,\ F(x,\ y))+D(gu,\ G(u,\ v))}{2},\ \frac{D(fx,\ G(u,\ v))+D(gu,\ F(x,\ y))}{2}, \\ \frac{D(fy,\ F(y,\ x))+D(gv,\ G(v,\ u))}{2},\ \frac{D(fy,\ G(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array} \right\}. \end{aligned}$$

If (i) *of Theorem 3.1 holds, then*

(a) F *and* f *have a coupled coincidence point,*

(b) G *and* g *have a coupled coincidence point,*

(c) F *and* f *have a common coupled fixed point, if* f *is* $F-$*weakly commuting at* $(x,\ y)$ *and* $f^2x = fx$ *and* $f^2y = fy$ *for* $(x,\ y) \in C\{F,\ f\}$,

(d) G *and* g *have a common coupled fixed point, if* g *is* $G-$*weakly commuting at* $(\widetilde{x},\ \widetilde{y})$ *and* $g^2\widetilde{x} = g\widetilde{x}$ *and* $g^2\widetilde{y} = g\widetilde{y}$ *for* $(\widetilde{x},\ \widetilde{y}) \in C\{G,\ g\}$,

(e) $F,\ G,\ f$ *and* g *have common coupled fixed point provided that both* (c) *and* (d) *are true.*

Examples 2.1-2.9 and Corollary 3.1 imply the following:

Corollary 3.4. *Let* $(X,\ d)$ *be a complete metric space. Assume* $F,\ G : X \times X \to CB(X)$ *and* $g : X \to X$ *be mappings satisfying* (i) *of Corollary 3.1 and*

(i) *for all* $x,\ y,\ u,\ v \in X$, *where* $0 < h < 1$,

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & h \max \left\{ \begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ \frac{D(gx,\ G(u,\ v))+D(gu,\ F(x,\ y))}{2},\ \frac{D(gy,\ G(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array} \right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v \in X,$ *where* $\alpha,\ \beta \geq 0$ *and* $\alpha + 2\beta < 1,$

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & \alpha \max \left\{ \begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)) \end{array} \right\} \\ & + \beta \max \left\{ \begin{array}{c} D(gx,\ G(u,\ v)) + D(gu,\ F(x,\ y)), \\ D(gy,\ G(v,\ u)) + D(gv,\ F(y,\ x)) \end{array} \right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v \in X,$ *where* $a,\ b,\ c \in [0,\ 1)$ *and* $a + 2b + 2c < 1,$

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & a \max \{d(gx,\ gu),\ d(gy,\ gv)\} \\ & + b \max \left\{ \begin{array}{c} D(gx,\ F(x,\ y)) + D(gu,\ G(u,\ v)), \\ D(gy,\ F(y,\ x)) + D(gv,\ G(v,\ u)) \end{array} \right\} \\ & + c \max \left\{ \begin{array}{c} D(gx,\ G(u,\ v)) + D(gu,\ F(x,\ y)), \\ D(gy,\ G(v,\ u)) + D(gv,\ F(y,\ x)) \end{array} \right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v \in X,$ *where* $h \in [0,\ 1)$ *and* $L \geq 0,$

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & h \max \{d(gx,\ gu),\ d(gy,\ gv)\} \\ & + L \max \left\{ \begin{array}{c} D(gx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ D(gx,\ G(u,\ v)),\ D(gu,\ F(x,\ y)), \\ D(gy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ D(gy,\ G(v,\ u)),\ D(gv,\ F(y,\ x)) \end{array} \right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v \in X,$ *where* $h \in [0,\ 1)$ *and* $L \geq 0,$

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & h \max \left\{ \begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ \frac{D(gx,\ G(u,\ v)) + D(gu,\ F(x,\ y))}{2},\ \frac{D(gy,\ G(v,\ u)) + D(gv,\ F(y,\ x))}{2} \end{array} \right\} \\ & + L \max \left\{ \begin{array}{c} D(gx,\ F(x,\ y)),\ D(gu,\ G(u,\ v)), \\ D(gx,\ G(u,\ v)),\ D(gu,\ F(x,\ y)), \\ D(gy,\ F(y,\ x)),\ D(gv,\ G(v,\ u)), \\ D(gy,\ G(v,\ u)),\ D(gv,\ F(y,\ x)) \end{array} \right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v \in X,$ *where* $h \in [0,\ 1),$

$$H(F(x,\ y),\ G(u,\ v)) \leq h \max \{d(gx,\ gu),\ d(gy,\ gv)\},$$

or for all $x,\ y,\ u,\ v \in X,$ *where* $b \in [0,\ \frac{1}{2}),$

$$H(F(x,\ y),\ G(u,\ v)) \leq b \max \left\{ \begin{array}{c} \frac{D(gx,\ F(x,\ y)) + D(gu,\ G(u,\ v))}{2}, \\ \frac{D(gy,\ F(y,\ x)) + D(gv,\ G(v,\ u))}{2} \end{array} \right\},$$

or for all $x,\ y,\ u,\ v \in X,$ *where* $c \in [0,\ \frac{1}{2}),$

$$H(F(x,\ y),\ G(u,\ v)) \leq c \max \left\{ \begin{array}{c} D(gx,\ G(u,\ v)) + D(gu,\ F(x,\ y)), \\ D(gy,\ G(v,\ u)) + D(gv,\ F(y,\ x)) \end{array} \right\},$$

or for all $x,\ y,\ u,\ v \in X$, where $0 < h < 1$,

$$\begin{aligned} & H(F(x,\ y),\ G(u,\ v)) \\ \leq\ & h \max \left\{ \begin{array}{c} d(gx,\ gu),\ d(gy,\ gv), \\ \frac{D(gx,\ F(x,\ y))+D(gu,\ G(u,\ v))}{2},\ \frac{D(gx,\ G(u,\ v))+D(gu,\ F(x,\ y))}{2}, \\ \frac{D(gy,\ F(y,\ x))+D(gv,\ G(v,\ u))}{2},\ \frac{D(gy,\ G(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array} \right\}. \end{aligned}$$

If (iii) of Corollary 3.1 holds, then

(a) F and g have a coupled coincidence point,

(b) G and g have a coupled coincidence point,

(c) F and g have a common coupled fixed point, if g is $F-$weakly commuting at $(x,\ y)$ and $g^2x = gx$ and $g^2y = gy$ for $(x,\ y) \in C(F,\ g)$,

(d) G and g have a common coupled fixed point, if g is $G-$weakly commuting at $(\widetilde{x},\ \widetilde{y})$ and $g^2\widetilde{x} = g\widetilde{x}$ and $g^2\widetilde{y} = g\widetilde{y}$ for $(\widetilde{x},\ \widetilde{y}) \in C\{G,\ g\}$,

(e) F, G and g have common coupled fixed point provided that both (c) and (d) are true.

Examples 2.1-2.9 and Corollary 3.2 imply the following:

Corollary 3.5. *Let $(X,\ d)$ be a complete metric space. Assume $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying (i) of Corollary 3.2 and*

(i) for all $x,\ y,\ u,\ v \in X$, where $0 < h < 1$,

$$\begin{aligned} & H(F(x,\ y),\ F(u,\ v)) \\ \leq\ & h \max \left\{ \begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2},\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array} \right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v \in X$, where $\alpha,\ \beta \geq 0$ and $\alpha + 2\beta < 1$,

$$\begin{aligned} & H(F(x,\ y),\ F(u,\ v)) \\ \leq\ & \alpha \max \left\{ \begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)) \end{array} \right\} \\ & +\beta \max \left\{ \begin{array}{c} D(gx,\ F(u,\ v)) + D(gu,\ F(x,\ y)), \\ D(gy,\ F(v,\ u)) + D(gv,\ F(y,\ x)) \end{array} \right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v \in X$, where $a,\ b,\ c \in [0,\ 1)$ and $a + 2b + 2c < 1$,

$$\begin{aligned} & H(F(x,\ y),\ F(u,\ v)) \\ \leq\ & a \max \{d(gx,\ gu),\ d(gy,\ gv)\} \\ & +b \max \left\{ \begin{array}{c} D(gx,\ F(x,\ y)) + D(gu,\ F(u,\ v)), \\ D(gy,\ F(y,\ x)) + D(gv,\ F(v,\ u)) \end{array} \right\} \\ & +c \max \left\{ \begin{array}{c} D(gx,\ F(u,\ v)) + D(gu,\ F(x,\ y)), \\ D(gy,\ F(v,\ u)) + D(gv,\ F(y,\ x)) \end{array} \right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v \in X$, where $h \in [0,\ 1)$ and $L \geq 0$,

$$\begin{aligned} & H(F(x,\ y),\ F(u,\ v)) \\ \leq\ & h \max \{d(gx,\ gu),\ d(gy,\ gv)\} \\ & +L \max \left\{ \begin{array}{c} D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ D(gx,\ F(u,\ v)),\ D(gu,\ F(x,\ y)), \\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ D(gy,\ F(v,\ u)),\ D(gv,\ F(y,\ x)) \end{array} \right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v\in X$, *where* $h\in[0,\ 1)$ *and* $L\geq 0$,

$$\begin{aligned} & H(F(x,\ y),\ F(u,\ v)) \\ \leq\ & h\max\left\{\begin{array}{c} d(gx,\ gu),\ D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ d(gy,\ gv),\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2},\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\} \\ & +L\max\left\{\begin{array}{c} D(gx,\ F(x,\ y)),\ D(gu,\ F(u,\ v)), \\ D(gx,\ F(u,\ v)),\ D(gu,\ F(x,\ y)), \\ D(gy,\ F(y,\ x)),\ D(gv,\ F(v,\ u)), \\ D(gy,\ F(v,\ u)),\ D(gv,\ F(y,\ x)) \end{array}\right\}, \end{aligned}$$

or for all $x,\ y,\ u,\ v\in X$, *where* $h\in[0,\ 1)$,

$$H(F(x,\ y),\ F(u,\ v))\leq h\max\left\{d(gx,\ gu),\ d(gy,\ gv)\right\},$$

or for all $x,\ y,\ u,\ v\in X$, *where* $b\in[0,\ \frac{1}{2})$,

$$H(F(x,\ y),\ F(u,\ v))\leq b\max\left\{\begin{array}{c} \frac{D(gx,\ F(x,\ y))+D(gu,\ F(u,\ v))}{2}, \\ \frac{D(gy,\ F(y,\ x))+D(gv,\ F(v,\ u))}{2} \end{array}\right\},$$

or for all $x,\ y,\ u,\ v\in X$, *where* $c\in[0,\ \frac{1}{2})$,

$$H(F(x,\ y),\ F(u,\ v))\leq c\max\left\{\begin{array}{c} D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y)), \\ D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x)) \end{array}\right\},$$

or for all $x,\ y,\ u,\ v\in X$, *where* $0<h<1$,

$$\begin{aligned} & H(F(x,\ y),\ F(u,\ v)) \\ \leq\ & h\max\left\{\begin{array}{c} d(gx,\ gu),\ d(gy,\ gv), \\ \frac{D(gx,\ F(x,\ y))+D(gu,\ F(u,\ v))}{2},\ \frac{D(gx,\ F(u,\ v))+D(gu,\ F(x,\ y))}{2}, \\ \frac{D(gy,\ F(y,\ x))+D(gv,\ F(v,\ u))}{2},\ \frac{D(gy,\ F(v,\ u))+D(gv,\ F(y,\ x))}{2} \end{array}\right\}. \end{aligned}$$

If (iii) *of Corollary 3.1 holds, then*
(a) F *and* g *have a coupled coincidence point,*
(b) F *and* g *have a common coupled fixed point, if* g *is* $F-$*weakly commuting at* $(x,\ y)$ *and* $g^2x=gx$ *and* $g^2y=gy$ *for* $(x,\ y)\in C(F,\ g)$.

Theorem 3.2. *Let* $(X,\ d)$ *be a complete metric space. Assume* $F,\ G:X\times X\to CB(X)$ *and* $f,\ g:X\to X$ *be mappings satisfying* (i), (ii) *of Theorem 3.1 and*
(i) $\{F,\ f\}$ *and* $\{G,\ g\}$ *are* $w-$*compatible,*
(ii) $f(X)$ *or* $g(X)$ *is a closed subset of* X,
then $F,\ G,\ f$ *and* g *have a common coupled fixed point.*

Proof. We can prove like Theorem 3.1 that $\{u_n\}$ and $\{v_n\}$ are Cauchy sequences in X. Since X is complete, there exist $u,\ v\in X$ satisfying (2). Suppose that $f(X)$ is a closed subset of X, then there exist $x,\ y\in X$, we have

$$u=fx \text{ and } v=fy. \tag{4}$$

As in Theorem 3.1, we can prove that

$$fx\in F(x,\ y) \text{ and } fy\in F(y,\ x), \tag{5}$$

that is, $(x,\ y)$ is a coupled coincidence point of F and f. Hence $(x,\ y)\in C\{F,\ f\}$. From $w-$compatibility of $\{F,\ f\}$, we have $fF(x,\ y)\subseteq F(fx,\ fy)$, hence $f^2x\in F(fx,\ fy)$ and

$f^2y \in F(fy,\ fx)$, that is, $fu \in F(u,\ v)$ and $fv \in F(v,\ u)$. Now, by condition (ii) of Theorem 3.1, we get

$$\psi\begin{pmatrix} H(F(u,\ v),\ G(x_{2n-1},\ y_{2n-1})), \\ d(fu,\ gx_{2n-1}),\ D(fu,\ F(u,\ v)),\ D(gx_{2n-1},\ G(x_{2n-1},\ y_{2n-1})), \\ D(fu,\ G(x_{2n-1},\ y_{2n-1})),\ D(gx_{2n-1},\ F(u,\ v)), \\ d(fv,\ gy_{2n-1}),\ D(fv,\ F(v,\ u)),\ D(gy_{2n-1},\ G(y_{2n-1},\ x_{2n-1})), \\ D(fv,\ G(y_{2n-1},\ x_{2n-1})),\ D(gy_{2n-1},\ F(v,\ u)) \end{pmatrix} \leq 0.$$

From (ψ_1) and triangle inequality, we have

$$\psi\begin{pmatrix} d(fu,\ u_{2n}), \\ d(fu,\ u_{2n-1}),\ 0,\ d(u_{2n-1},\ u_{2n}),\ d(fu,\ u_{2n}),\ d(u_{2n-1},\ fu), \\ d(fv,\ v_{2n-1}),\ 0,\ d(v_{2n-1},\ v_{2n}),\ d(fv,\ v_{2n}),\ d(v_{2n-1},\ fv) \end{pmatrix} \leq 0.$$

Letting $n \to \infty$ in the above inequality, we get

$$\psi\begin{pmatrix} d(fu,\ u), \\ d(fu,\ u),\ 0,\ 0,\ d(fu,\ u),\ d(fu,\ u), \\ d(fv,\ v),\ 0,\ 0,\ d(fv,\ v),\ d(fv,\ v) \end{pmatrix} \leq 0.$$

Hence, by (ψ_3), we have $d(fu,\ u) = d(fv,\ v) = 0$. Thus

$$u = fu \in F(u,\ v) \text{ and } v = fv \in F(v,\ u).$$

Since $F(X \times X) \subseteq g(X)$, then there exist $\widetilde{x}, \widetilde{y} \in X$ such that $g\widetilde{x} = u = fu \in F(u, v)$ and $g\widetilde{y} = v = fv \in F(v, u)$. Now, by condition (ii) of Theorem 3.1, we get

$$\psi\begin{pmatrix} H(F(u,\ v),\ G(\widetilde{x},\ \widetilde{y})), \\ d(fu,\ g\widetilde{x}),\ D(fu,\ F(u,\ v)),\ D(g\widetilde{x},\ G(\widetilde{x},\ \widetilde{y})), \\ D(fu,\ G(\widetilde{x},\ \widetilde{y})),\ D(g\widetilde{x},\ F(u,\ v)), \\ d(fv,\ g\widetilde{y}),\ D(fv,\ F(v,\ u)),\ D(g\widetilde{y},\ G(\widetilde{y},\ \widetilde{x})), \\ D(fv,\ G(\widetilde{y},\ \widetilde{x})),\ D(g\widetilde{y},\ F(v,\ u)) \end{pmatrix} \leq 0,$$

and so we have

$$\psi\begin{pmatrix} D(u,\ G(\widetilde{x},\ \widetilde{y})), \\ 0,\ 0,\ D(u,\ G(\widetilde{x},\ \widetilde{y})),\ D(u,\ G(\widetilde{x},\ \widetilde{y})),\ 0 \\ 0,\ 0,\ D(v,\ G(\widetilde{y},\ \widetilde{x})),\ D(v,\ G(\widetilde{y},\ \widetilde{x})),\ 0 \end{pmatrix} \leq 0.$$

Hence, by (ψ_2), we have $D(u, G(\widetilde{x}, \widetilde{y})) = D(v, G(\widetilde{y}, \widetilde{x})) = 0$. Thus

$$u = g\widetilde{x} \in G(\widetilde{x},\ \widetilde{y}) \text{ and } v = g\widetilde{y} \in G(\widetilde{y},\ \widetilde{x}),$$

that is, $(\widetilde{x}, \widetilde{y})$ is a coupled coincidence point of G and g. Hence $(\widetilde{x}, \widetilde{y}) \in C\{G, g\}$. From $w-$compatibility of $\{G, g\}$, we have $gG(\widetilde{x}, \widetilde{y}) \subseteq G(g\widetilde{x}, g\widetilde{y})$, hence $g^2\widetilde{x} \in G(g\widetilde{x}, g\widetilde{y})$ and $g^2\widetilde{y} \in G(g\widetilde{y}, g\widetilde{x})$, that is, $gu \in G(u, v)$ and $gv \in G(v, u)$. Again, by condition (ii) of Theorem 3.1, we get

$$\psi\begin{pmatrix} H(F(u,\ v),\ G(u,\ v)), \\ d(fu,\ gu),\ D(fu,\ F(u,\ v)),\ D(gu,\ G(u,\ v)), \\ D(fu,\ G(u,\ v)),\ D(gu,\ F(u,\ v)), \\ d(fv,\ gv),\ D(fv,\ F(v,\ u)),\ D(gv,\ G(v,\ u)), \\ D(fv,\ G(v,\ u)),\ D(gv,\ F(v,\ u)) \end{pmatrix} \leq 0,$$

and so by triangle inequality, we have

$$\psi\begin{pmatrix} d(u,\ gu), \\ d(u,\ gu),\ 0,\ 0,\ d(u,\ gu),\ d(u,\ gu), \\ d(v,\ gv),\ 0,\ 0,\ d(v,\ gv),\ d(v,\ gv) \end{pmatrix} \leq 0.$$

Hence, by (ψ_3), we have $d(u, gu) = d(v, gv) = 0$. Thus

$$u = gu \in G(u,\ v) \text{ and } v = gv \in G(v,\ u).$$

Therefore (u, v) is a common coupled fixed point of F, G, f and g. The proof is similar when $g(X)$ is assumed to be a closed subset of X. □

Put $f = g$ in the Theorem 3.2, we get the following result:

Corollary 3.6. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying (i), (ii) of Corollary 3.1 and*
(i) $\{F, g\}$ and $\{G, g\}$ are $w-$compatible.
If (iii) of Corollary 3.1 holds, then F, G and g have a common coupled fixed point.

Put $F = G$ and $f = g$ in the Theorem 3.2, we get the following result:

Corollary 3.7. *Let (X, d) be a complete metric space. Assume $F : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying (i), (ii) of Corollary 3.2 and*
(i) $\{F, g\}$ is $w-$compatible.
If (iii) of Corollary 3.1 holds, then F and g have a common coupled fixed point.

Examples 2.1-2.9 and Theorem 3.2 imply the following:

Corollary 3.8. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $f, g : X \to X$ be mappings satisfying (i) of Theorem 3.1 , (i) of Corollary 3.3, (i) and (ii) of Theorem 3.2, then F, G, f and g have a common coupled fixed point.*

Examples 2.1-2.9 and Corollary 3.6 imply the following:

Corollary 3.9. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $f, g : X \to X$ be mappings satisfying (i), (iii) of Corollary 3.1, (i) of Corollary 3.4, and (i) of Corollary 3.6, then F, G and g have a common coupled fixed point.*

Examples 2.1-2.9 and Corollary 3.7 imply the following:

Corollary 3.10. *Let (X, d) be a complete metric space. Assume $F, G : X \times X \to CB(X)$ and $g : X \to X$ be mappings satisfying (iii) of Corollary 3.1, (i) of Corollary 3.2, (i) of Corollary 3.5 and (i) of Corollary 3.7, then F and g have a common coupled fixed point.*

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Bhavana Deshpande for the photography and short autobiography, see TWMS J. Appl. Eng. Math. V.5, No.1, 2015.

Amrish Handa received his M.Sc. in Mathematics degree from Vikram University of Ujjain, Madhya Pradesh, India in 2005. He secured first position in the whole university. Thenafter he cleared National Eligibility Test in Mathematical Sciences organized by Council of Scientific and Industrial Research, India in 2008. He worked as a Junior Research Fellow on Research Project entitled "A Study on Fixed Point Theory and its Applications." sanctioned by Madhya Pradesh Council of Science and Technology with Principal investigator Dr. Bhavana Deshpande from 2012 to 2014. His research area is Fixed point theory and its applications. His 34 research papers are published and 14 research papers are accepted for publication in reputed international journals

Deepmala is Visiting Scientist at SQC and OR Unit at Indian Statistical Institute, Kolkata, West Bengal, India. Her research interests are in the areas of pure and applied mathematics including Optimization, Mathematical Programming, Fixed Point Theory and Applications, Integral equations, Operator theory, Approximation Theory, Summability analysis etc. She is member of several scientific committees, advisory boards as well as member of editorial board of a number of scientific journals, like as Maejo International Journal of Science and Technology etc. Dr. Deepmala has more than 50 research papers to her credit published in several journals of repute as well as key note speaker in International conferences. Citations of her research contributions can be found in many books and monographs, PhD thesis, and scientific journal articles, much too numerous to be recorded here.