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JACOBSTHAL FAMILY MODULO m

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Abstract. In this study, we investigate sets of remainder of the Jacobsthal and Jacobsthal-Lucas numbers modulo m for some positive integers m. Also some properties related to these sets and a new method to calculate the length of period modulo m is given.

Keywords: Jacobsthal numbers, Jacobsthal-Lucas numbers, modulo, period.

AMS Subject Classification: 11B39, 11B50

1. Introduction

Jacobsthal and Jacobsthal Lucas numbers have been used by scientists for their fundamental theory and their applications. For interesting applications of these numbers in computer science and nature, one can see the citations in [1,3,6,7]. For instance, in computer science, they have been used conditional instructions to change the flow of execution of a program by using Jacobsthal numbers. The properties of these numbers are first summarized by Horadam. As a reminder for the rest of this paper, the well-known Jacobsthal and Jacobsthal Lucas numbers $\{J_n\}_{n=0}^{\infty}$ and $\{j_n\}_{n=0}^{\infty}$ are defined by recurrence relations

$$J_{n+2} = J_{n+1} + 2J_n, \qquad (J_0 = 0,\ J_1 = 1) \tag{1}$$

$$j_{n+2} = j_{n+1} + 2j_n, \qquad (j_0 = 2,\ j_1 = 1) \tag{2}$$

respectively. The equations in (1) and (2) are the second order linear difference equation and their characteristic equation follows

$$r^2 = r + 2. \tag{3}$$

The equation in (3) has two real roots as $r_1 = 2$ and $r_2 = -1$. It means that the following relations hold for the numbers r_1, r_2:

$$r_1 + r_2 = 1,\ r_1 - r_2 = 3,\ r_1 r_2 = -2 \tag{4}$$

The Binet formulas of these numbers are given as follows

$$J_n = \frac{1}{3}\left(2^n - (-1)^n\right),\quad j_n = 2^n + (-1)^n. \tag{5}$$

There are many of study on modulo in literature. Authors developed some methods to find the length of period related to Fibonacci numbers modulo m, although there is no known formula for length of period [4,8,9]. Guo and Koch, in [5], studied the Fibonacci

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sequence mod n for some positive integer n. Also, some properties of the length of period for k-Fibonacci sequences modulo m, are presented in [2].

In this paper, we mainly focus on the lengths of period of Jacobsthal $\{J_n\}_{n=0}^{\infty}$ and Jacobsthal Lucas $\{j_n\}_{n=0}^{\infty}$ sequences modulo m and obtain a formula to find the lengths of period taking modulo m for some positive integer m. We also investigate the relations among lengths of period for these numbers.

2. Main Results

Let $L(J,m)$ and $l(j,m)$ denote the length of the period of $J_n \pmod{m}$ and $j_n \pmod{m}$, respectively. Suppose that the periods of sequences $\{J_n\}_{n=0}^{\infty}$ and $\{j_n\}_{n=0}^{\infty}$ have one or more zeros. In this case, the set of elements between two consecutive zeros including the first zero, but not the second one is called a chain.

There isn't directly specific formula to compute the length of period of Jacobsthal and Jacobsthal-Lucas numbers. However, in this study, we obtain the identities which are find the length of period for some specific cases.

Now, let us first consider the following lemma which will be needed later in this section.

Lemma 2.1. *The following properties are hold:*

a): *Assume that m is odd prime number and $i \in \mathbb{Z}^+$. Then* $L\left(J,m^i\right) = L(J,m)\, m^{i-1}$,

b): *For* $\gcd(a,b)=1,\ L(J,a.b) = \operatorname{lcm}(L(J,a), L(J,b))$.

Similarly, properties of Lemma 2.1 are hold for Jacobsthal Lucas numbers.

Theorem 2.1. *For $m \geqslant 3$, the following identities are hold:*

a): $L(J,m) = L(J,2m)$,

b): $l(j,m) = l(j,2m)$.

Proof. a) We proceed by induction on m. Since $L(J,3) = L(J,6) = 6$, the result is true for $m=3$. Assume that the result is true for k, that is, $L(J,k) = L(J,2k)$. We show that $L(J,k+1) = L(J,2k+2)$. If k is even, then $k+1$ is odd and we have $L(J,2(k+1)) = \operatorname{lcm}(L(J,2), L(J,k+1)) = L(J,k+1)$ .If k is odd, then $k+1$ is even. Thus, $k+1 = 2^n \cdot t$, such that $n \geqslant 1,\ t$ is odd and, t and n are not 1 at the same time. We first observe that if $m>1$, then $L(J,2^m) = 2$ because $3J_i \equiv 2^i - (-1)^i \equiv (-1)^{i+1} \equiv (-1)^{i+3} \equiv 3J_{i+2} (\operatorname{mod} 2^m)$ for each $i \geqslant 2^m$. Hence if $t=1$, then $L(J,2(k+1)) = L\left(J,2^{n+1}\right) = L(J,2^n) = L(J,k+1)$. Now suppose that $t>1$. Then

$$\begin{aligned} L(J,2(k+1)) &= \operatorname{lcm} L\left(J,2^{n+1}t\right) \\ &= \operatorname{lcm}\left(L\left(J,2^{n+1}\right), L(J,t)\right) \\ &= \operatorname{lcm}\left(L(J,2^n), L(J,t)\right) \\ &= L(J,2^n t) = L(J,k+1) \end{aligned}$$

as required.

b) The proof is similar to part (a).

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By considering above theorem, we give the following results.

Corollary 2.1. *For the special value of m in Theorem 2.1, it is obvious that the following results hold:*

a): *For $i \geqslant 2$,* $L\left(J,r_1^i\right) = l\left(j,r_1^i\right) = 2$,

b): *For $p > 2$ and $i \in \mathbb{N}$, $L\left(J, p.r_1^i\right) = L(J,p)$, $l\left(j, p.r_1^i\right) = l(j,p)$, where r_1 is root of characteristic equation of (1).*

Theorem 2.2. *Let r_1 and r_2 be roots of characteristic equation of (1) and $L^n\left(J,(r_1- r_2)^i\right) = L\left(J, L\left(J, ..., L\left(J,(r_1- r_2)^i\right)\right)\right)$. For $i, n \in \mathbb{Z}^+$, we have*

$$L^n\left(J,(r_1- r_2)^i\right) = L\left(J,(r_1- r_2)^i\right).$$

Proof. By induction on n, for $n = 1$, it is obvious. For $n = 2$, from Lemma 2.1, $L\left(J,(r_1- r_2)^i\right) = L\left(J,(r_1- r_2).(r_1- r_2)^{i-1}\right) = L(J,3).3^{i-1} = 6.3^{i-1} = 2.(r_1- r_2)^i$. If it is taken the length of period for every sides, then $L\left(J, L\left(J,(r_1- r_2)^i\right)\right) = L\left(J, 2.(r_1- r_2)^i\right)$. By considering Theorem 2.1, it is obtained

$$L\left(J, L\left(J,(r_1- r_2)^i\right)\right) = L\left(J,(r_1- r_2)^i\right) \quad (6)$$

and the relation holds. Suppose that the relation is true for any $m \leqslant n$. That is,

$$L^m\left(J,(r_1- r_2)^i\right) = L\left(J,(r_1- r_2)^i\right). \quad (7)$$

Then, we have

$$L^{m+1}\left(J,(r_1- r_2)^i\right) = L\left(J, L^m(J,(r_1- r_2)^i)\right).$$

Taking into account (6) and (7), we obtain

$$L^{m+1}\left(J,(r_1- r_2)^i\right) = L\left(J,(r_1- r_2)^i\right),$$

as required. ∎

Theorem 2.3. *Assume that m is odd and $i \in \mathbb{Z}^+$. The following statements are hold:*

a) For $m > J_i$,

$$\begin{aligned} J_{L(J,m)-1} &\equiv \frac{m+1}{2} \bmod (m), \\ J_{L(J,m)} &\equiv 0 \bmod (m), \\ J_{L(J,m)+i} &\equiv J_i \bmod (m). \end{aligned}$$

b) For $m > j_i$,

$$\begin{aligned} j_{l(j,m)-1} &\equiv \frac{m-1}{2} \bmod (m) \\ j_{l(j,m)} &\equiv 2 \bmod (m) \\ j_{l(j,m)+i} &\equiv j_i \bmod (m) \end{aligned}$$

Proof. a) Firstly, by using iteration method, we can write

$$\begin{aligned} J_{L(J,3)-1} &= J_5 \equiv \frac{3+1}{2} \bmod (3) \\ J_{L(J,5)-1} &= J_3 \equiv \frac{5+1}{2} \bmod (5) \\ J_{L(J,7)-1} &= J_5 \equiv \frac{7+1}{2} \bmod (7). \end{aligned}$$

Hence, by iterating this procedure, we can write $J_{L(J,m)-1} \equiv \frac{m+1}{2} \bmod (m)$. Secondly, it can be seen by using iteration the proof of $J_{L(J,m)} \equiv 0 \bmod (m)$.

Finally, from well-known Convolution product for Jacobsthal numbers (i.e. $J_{n+m} = J_n J_{m+1} + 2J_{n-1}J_m$), we have

$$\begin{aligned} J_{L(J,m)+i} - J_i &= J_{L(J,m)}J_{i+1} + 2J_{L(J,m)-1}J_i - J_i \\ &= J_{L(J,m)}J_{i+1} + J_i\left(2J_{L(J,m)-1} - 1\right). \end{aligned}$$

From above two properties, we obtain

$$\begin{aligned} J_{L(J,m)+i} - J_i &\equiv mJ_i \bmod (m) \\ &\equiv 0 \bmod (m). \end{aligned}$$

b) The proof can be seen easily in a similar manner with (a).

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Let us consider the remainders of the first two chains of the sequences $\{J_m \mod J_r\}$ and $\{j_m \mod j_r\}$:

$$0, J_1, J_2, \ldots, J_{r-2}, J_{r-1}, 0, a_1, a_2, \ldots, a_{J_{r-1}}, \tag{8}$$

$$0, j_1, j_2, \ldots, j_{r-2}, j_{r-1}, 0, b_1, b_2, \ldots, b_{j_{r-1}}. \tag{9}$$

The following theorem give us relations between a_i $(i = 1, 2, ..., J_{r-1})$ in (8) and Jacobsthal and between b_i $(i = 1, 2, ..., j_{r-1})$ in (9) and Jacobsthal Lucas numbers.

Theorem 2.4. *The following identities are hold:*

a) $a_i = \begin{cases} J_i, & r \text{ is even} \\ J_r - J_i, & r \text{ is odd} \end{cases}, for\ i=1,2,...,J_{r-1},$

b) $b_i = \begin{cases} j_r - j_i + 2(-1)^i, & r \text{ is even} \\ j_i - 2(-1)^i, & r \text{ is odd} \end{cases}, for\ i=1,2,...,j_{r-1}.$

Proof. In here, we will prove (a) since (b) can be thought in the same manner with it.

a): We use the principle of finite induction on m to prove. There are two cases of subscript r.

r **is even case:** By considering (5), for $m = 1$, $a_1 = \text{remainder}\left(\frac{J_{r+1}}{J_r}\right) = \text{remainder}\left(\frac{J_r+2J_{r-1}}{J_r}\right) = \text{remainder}\left(\frac{2J_{r-1}}{J_r}\right) = 1 = J_1$. As the usual next step of inductions, let us assume that it is true for all positive integers m. That is, $a_m = \text{remainder}\left(\frac{J_{r+m}}{J_r}\right) = J_m$. Therefore, we have to show that is true for $m+1$. $a_{m+1} = \text{remainder}\left(\frac{J_{r+m+1}}{J_r}\right) = \text{remainder}\left(\frac{J_{r+m}+2J_{r+m-1}}{J_r}\right) = \text{remainder}\left(\frac{J_{r+m}}{J_r}\right) + 2.\text{remainder}\left(\frac{J_{r+m-1}}{J_r}\right) = J_m + 2J_{m-1} = J_{m+1}$ which ends up the induction.

r **is odd case:** This case can be seen easily in a similar manner. ∎

By considering (8), (9) and above theorem, we obtain the relations between indices of Jacobsthal (Jacobsthal Lucas) numbers and length of period of Jacobsthal (Jacobsthal Lucas) numbers.

Corollary 2.2. *For $m > 2$, the following identities are hold:*

a): $L(J, J_m) = \begin{cases} 2m, & m \text{ is odd} \\ m, & m \text{ is even} \end{cases},$

b): $l(j, j_m) = 2m.$

As a result of above theorems, we can give the relation between $L(J, m)$ and $l(j, m)$.

Corollary 2.3. *Assume that p is prime number such that $p = 6k + 1$ for $k \in \mathbb{Z}^+$. Then, we have*

$$L(J, m) = \begin{cases} l(j, m), & 3 \nmid m \\ l(j, m), & 3 \mid m \text{ and } p \mid m \\ 3l(j, m), & 3 \mid m \text{ and } p \nmid m \end{cases}.$$

3. Numerical Examples

Taking into account the derived expressions for m and n, one can see the effectiveness of the above results.

J_n / m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	...
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
3	1	0	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	3	
5	1	2	1	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	
11	1	2	3	1	5	4	3	2	1	0	11	11	11	11	11	11	11	11	
21	1	0	1	1	3	0	5	3	1	10	9	8	7	6	5	4	3	2	
43	1	1	3	3	1	1	3	7	3	10	7	4	1	13	11	9	7	5	
85	1	1	1	0	1	1	5	4	5	8	1	7	1	10	5	0	13	9	
171	1	0	3	1	3	3	3	0	1	6	3	2	3	6	11	1	9	0	
341	1	2	1	1	5	5	5	8	1	0	5	3	5	11	5	1	17	18	
683	1	2	3	3	5	4	3	8	3	1	11	7	11	8	11	3	17	18	
1365	1	0	1	0	3	0	5	6	5	1	9	0	7	0	5	5	15	16	
2731	1	1	3	1	1	1	3	4	1	3	7	1	1	1	11	11	13	14	
5461	1	1	1	1	1	1	5	7	1	5	1	1	1	1	5	4	7	8	
10923	1	0	3	3	3	3	3	6	3	0	3	3	3	3	11	9	15	17	
21845	1	2	1	0	5	5	5	2	5	10	5	5	5	5	5	0	11	14	
43691	1	2	3	1	5	4	3	5	1	10	11	11	11	11	11	1	5	10	
87381	1	0	1	1	3	0	5	0	1	8	9	8	7	6	5	1	9	0	
174763	1	1	3	3	1	1	3	1	3	6	7	4	1	13	11	3	1	1	
349525	1	1	1	0	1	1	5	1	5	0	1	7	1	10	5	5	1	1	
⋮																			

Figure 1. Sets of remainder of Jacobsthal numbers for modulo m

j_n / m	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	...
2	0	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	2	
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	
5	1	2	1	0	5	5	5	5	5	5	5	5	5	5	5	5	5	5	
7	1	1	3	2	1	0	7	7	7	7	7	7	7	7	7	7	7	7	
17	1	2	1	2	5	3	1	8	7	6	5	4	3	2	1	0	17	17	
31	1	1	3	1	1	3	7	4	1	9	7	5	3	1	15	14	13	12	
65	1	2	1	0	5	2	1	2	5	10	5	0	9	5	1	14	11	8	
127	1	1	3	2	1	1	7	1	7	6	7	10	1	7	15	8	1	13	
257	1	2	1	2	5	5	1	5	7	4	5	10	5	2	1	2	5	10	
511	1	1	3	1	1	0	7	7	1	5	7	4	7	1	15	1	7	17	
1025	1	2	1	0	5	3	1	8	5	2	5	11	3	5	1	5	17	18	
2047	1	1	3	2	1	3	7	4	7	1	7	6	3	7	15	7	13	14	
4097	1	2	1	2	5	2	1	2	7	5	5	2	9	2	1	0	11	12	
8191	1	1	3	1	1	1	7	1	1	7	7	1	1	1	15	14	1	2	
16385	1	2	1	0	5	5	1	5	5	6	5	5	5	5	1	14	5	7	
32767	1	1	3	2	1	0	7	7	7	9	7	7	7	7	15	8	7	11	
65537	1	2	1	2	5	3	1	8	7	10	5	4	3	2	1	2	17	6	
131071	1	1	3	1	1	3	7	4	1	6	7	5	3	1	15	1	13	9	
262145	1	2	1	0	5	2	1	2	5	4	5	0	9	5	1	5	11	2	
524287	1	1	3	2	1	1	7	1	7	5	7	10	1	7	15	7	1	1	
1048577	1	2	1	2	5	5	1	5	7	2	5	10	5	2	1	0	5	5	
⋮																			

Figure 2. Sets of remainder of Jacobsthal Lucas numbers for modulo m

In Figure 1 and Figure 2, sets of remainder of Jacobsthal and Jacobsthal Lucas numbers for modulo m can be seen, respectively. Also the length of period for these numbers is illustrated for some values of n.

For example, in Figure 1, for $n = 5$ and $m = 11$, we have the remainders for modulo 11 as $\{0, 1, 1, 3, 5, 0, 10, 10, 8, 6, \cdots\}$ and the length of period of J_5 as $L(J, J_5) = 10$. Also, remainders of chain are $a_1 = J_5 - J_1 = 10,\ a_2 = 10,\ a_3 = 8,\ a_4 = 6$ in Theorem 2.4.

In Figure 2, for $n = 4$ and $m = 17$, we have the remainders for modulo 17 as $\{2, 1, 5, 7, 0, 14, 14, 8, \cdots\}$ and the length of period of j_4 as $L(j, j_4) = 8$. Also, remainders of chain are $b_1 = j_4 - j_1 + 2(-1)^1 = 14,\ b_2 = 14,\ b_3 = 8,\ b_4 = 2,\ b_5 = 1,\ b_6 = 5,\ b_7 = 7$ in Theorem 2.4.

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