TWMS J. App. Eng. Math. V.7, N.1, 2017, pp. 142-153

ADAPTIVE METHODS FOR SOLVING OPERATOR EQUATIONS BY USING FRAMES OF SUBSPACES

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Abstract. In this paper, using a frame of subspaces we transform an operator equation to an equivalent ℓ_2-problem. Then, we propose an adaptive algorithm to solve the problem and investigate the optimality and complexity properties of the algorithm.

Keywords: Hilbert space, dual space, frame of subspaces, best N-term approximation, adaptive algorithm.

AMS Subject Classification: 65F10, 65F08

1. Introduction and preliminaries

The aim of the paper is to study the application of frames of subspaces in designing adaptive iterative methods for solving operator equations. Usually these operators are defined on a bounded domain or a closed manifold where a wavelet basis with specific properties is needed to be constructed. Most importantly, during the approach some serious drawbacks such as stability may not be avoided. Therefore, it is suggested to use a slightly weaker concept, namely frame. In [7, 8, 5, 6] some adaptive numerical methods for elliptic operator equations have been developed by using wavelet bases and frames. One of the advantages of frame of subspaces is that they facilitate the construction of frames for special applications and meanwhile it is easier to construct or choose already known frames for smaller spaces.
The main focus of the paper is to find $u \in H$ such that

$$Lu = f, \tag{1}$$

where $L : H \to H$ is a bounded, invertible and self adjoint linear operator on a separable Hilbert space H. In general, it is impossible to find the exact solution of the problem (1), because the separable Hilbert space H is infinite dimensional. A natural approach to construct an approximate solution is to solve a finite dimensional counterpart of the problem (1). First, we briefly recall the definitions and basic properties of frames and frames of subspaces.
Assume that H is a separable Hilbert space, Λ is a countable set of indices and $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} \subset H$ is a frame for H. This means that there exist constants $0 < A_\Psi \le B_\Psi < \infty$ such that

$$A_\Psi \|f\|_H^2 \le \sum_{\lambda\in\Lambda} |\langle f, \psi_\lambda\rangle|^2 \le B_\Psi \|f\|_H^2, \quad \forall f \in H. \tag{2}$$

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§ Manuscript received: May 09, 2016; accepted: September 22, 2016.
TWMS Journal of Applied and Engineering Mathematics, Vol.7, No.1;

For the frame Ψ, the frame operator $S : H \to H$ is defined by $S(f) = \sum_{\lambda\in\Lambda}\langle f, \psi_\lambda\rangle\psi_\lambda$. It was shown that S is a positive definite and invertible operator satisfying $A_\Psi I_H \leq S \leq B_\Psi I_H$. Also, the sequence $\widetilde{\Psi} = (\widetilde{\psi}_\lambda)_{\lambda\in\Lambda} = (S^{-1}\psi_\lambda)_{\lambda\in\Lambda}$ is a frame (called the canonical dual frame) for H with bounds B_Ψ^{-1}, A_Ψ^{-1} and every $f \in H$ has the expansion

$$f = \sum_{\lambda\in\Lambda}\langle f, \psi_\lambda\rangle\widetilde{\psi}_\lambda = \sum_{\lambda\in\Lambda}\langle f, \widetilde{\psi}_\lambda\rangle\psi_\lambda. \tag{3}$$

For an index set $\widetilde{\Lambda} \subset \Lambda$, $(\psi_\lambda)_{\lambda\in\widetilde{\Lambda}}$ is called a frame sequence, if it is a frame for its closed span. For more details see [1, 3].

For an index set Λ and a family of weights $\{v_\lambda\}_{\lambda\in\Lambda}$, i.e., $v_\lambda > 0$ for all $\lambda \in \Lambda$, a family of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$ of a Hilbert space H is called a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H, if there exist constants $0 < A \leq B < \infty$ such that

$$A\|f\|^2 \leq \sum_{\lambda\in\Lambda} v_\lambda^2\|\pi_{H_\lambda}(f)\|^2 \leq B\|f\|^2 \quad \forall f \in H, \tag{4}$$

where π_{H_λ} denotes the orthogonal projection onto the subspace H_λ.
The constants A and B is called the frame bounds of the frame of subspaces. If $A = B$ then the frame $\{H_\lambda\}_{\lambda\in\Lambda}$ with respect to$\{v_\lambda\}_{\lambda\in\Lambda}$, is called a A -tight frame of subspaces. It is clear, the family $\{H_\lambda\}_{\lambda\in\Lambda}$ of the frame of subspaces is complete, in the sense that $\overline{span}_{\lambda\in\Lambda}\{H_\lambda\} = H$.
The following theorem [2], shows how we are able to string together frames for each of the subspaces H_λ to get a frame for H.

Theorem 1.1. *Let Λ be an index set, $v_\lambda > 0$ for each $\lambda \in \Lambda$, and $\{\psi_{\lambda_i}\}_{i\in I_\Lambda}$ be a frame sequence in H with frame bounds A_λ and B_λ. Define $H_\lambda = \overline{span}_{i\in I_\Lambda}\{\psi_{\lambda i}\}$ for all $\lambda \in \Lambda$, and suppose that $0 < A = \inf_{\lambda\in\Lambda} A_\lambda \leq B = \sup_{\lambda\in\Lambda} B_\lambda < \infty$. Then $\{v_\lambda\psi_{\lambda_i}\}_{\lambda\in\Lambda, i\in I_\Lambda}$ is a frame for H if and only if $\{H_\lambda\}_{\lambda\in\Lambda}$ is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H.*

For a frame of subspaces $\{H_\lambda\}_{\lambda\in\Lambda}$ with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ define

$$(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2} = \{\{\psi_\lambda\}_{\lambda\in\Lambda} | \psi_\lambda \in H_\lambda, \ \sum_{\lambda\in\Lambda}\|\psi_\lambda\|^2 < \infty\}$$

with inner product given by $\langle\{\psi_\lambda\}_{\lambda\in\Lambda}, \{\varphi_\lambda\}_{\lambda\in\Lambda}\rangle = \sum_{\lambda\in\Lambda}\langle\psi_\lambda, \varphi_\lambda\rangle$. Now the synthesis operator $T_{H,v} : (\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2} \to H$ for the frame of subspace $\{H_\lambda\}_{\lambda\in\Lambda}$ with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ is defined by

$$T_{H,v}(f) = \sum_{\lambda\in\Lambda} v_\lambda f_\lambda \quad \forall f = \{f_\lambda\}_{\lambda\in\Lambda} \in (\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}.$$

Also, the adjoint $T^*_{H,v}$ of the synthesis operator is called the analysis operator. In fact $T^*_{H,v} : H \to (\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}$ is given by $T^*_{H,v}(f) = \{v_\lambda\pi_{H_\lambda}(f)\}_{\lambda\in\Lambda}$. It is proved that the synthesis operator $T_{H,v}$ is bounded, linear and onto. [2]. Also, the analysis operator $T^*_{H,v}$ is an (possibly into) isomorphism. As in the well known frame situation, the frame operator $S_{H,v}$ for $\{H_\lambda\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ is defined by

$$S_{H,v}(f) = T_{H,v}T^*_{H,v}(f) = T_{H,v}(\{v_\lambda\pi_{H_\lambda}(f)\}_{\lambda\in\Lambda}) = \sum_{\lambda\in\Lambda} v_\lambda^2\pi_{H_\lambda}(f).$$

The frame operator $S_{H,v}$ for $\{H_\lambda\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$ is self-adjoint, invertible on H with $AI \leq S_{H,v} \leq BI$, where A and B are the bounds of the frame of subspaces. Furthermore, the following reconstruction formula satisfies:

$$f = \sum_{\lambda \in \Lambda} v_\lambda^2 S_{H,v}^{-1} \pi_{H_\lambda}(f) \quad \forall f \in H.$$

It is proved that $\{S_{H,v}^{-1} H_\lambda\}_{\lambda\in\Lambda}$ is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$. [2].

Proposition 1.1. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$, and let $L : H \to H$ be a bounded invertible operator on H. Then $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ is a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$.*

Proof. See [2]. □

2. Preconditioning by using frames of subspaces

The most straight forward approach to an iterative solution of a linear system is to rewrite the equation (1) as a linear fixed-point iteration. One way to do this is the Richardson iteration. The abstract method reads as follows:
write $Lu = f$ as $u = (I - L)u + f$. For given $u_0 \in H$, define for $n \geq 0$,

$$u_{n+1} = (I - L)u_n + f. \tag{5}$$

Since $Lu - f = 0$,

$$\begin{aligned} u_{n+1} - u &= (I - L)u_n + f - u - (f - Lu) = (I - L)u_n - u + Lu \\ &= (I - L)(u_n - u). \end{aligned}$$

Hence $\|u_{n+1} - u\|_H \leq \|I - L\|_{H\to H}\|u_n - u\|_H$, so that (5) converges if $\|I - L\|_{H\to H} < 1$.
It is sometimes possible to precondition (1) by multiplying both sides by a matrix B,

$$BLu = Bf,$$

so that convergence of iterative methods is improved. This is a very effective technique for solving differential equations, integral equations, and related problems [2, 3]. We shall apply this technique by using frames of subspaces.
Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for a separable Hilbert space H with the frame operator $S_{H,v}$. By Proposition 1.1, $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ also is a frame with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$. We denote the frame operator for $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ and $\{v_\lambda\}_{\lambda\in\Lambda}$, by $S'_{H,v}$ and we note that $S'_{H,v}f = \sum_{\lambda\in\Lambda} v_\lambda^2 L\pi_{H_\lambda} L^{-1} f = L\sum_{\lambda\in\Lambda} v_\lambda^2 \pi_{H_\lambda} L^{-1} f = LS_{H,v}L^{-1}f$, that means, $S'_{H,v} = LS_{H,v}L^{-1}$.
Also since L is bounded invertible then there exist two positive constants c_1 and c_2 such that

$$c_1\|u\|_H \leq \|Lu\|_H \leq c_2\|u\|_H, \quad \forall u \in H. \tag{6}$$

Now we design an algorithm in order to approximate the solution u of the equation (1). The convergence rate of the algorithm depends on the values of the bounds of the frames.

Theorem 2.1. *Let $\{H_\lambda\}_{\lambda\in\Lambda}$ be a frame of subspaces with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ for H with frame operator $S_{H,v}$ and let L be as in (1). Let $u_0 = 0$ and for $k \geq 1$,*

$$u_k = u_{k-1} + \frac{2}{c_1^2 A + c_2^2 B} LS'_{H,v}(f - Lu_{k-1}),$$

where $S'_{H,v}$ is the frame operator for the frame of subspaces $\{L(H_\lambda)\}_{\lambda\in\Lambda}$ with respect to $\{v_\lambda\}_{\lambda\in\Lambda}$ with bounds A, B, and c_1, c_2 as in (6). Then

$$\|u - u_k\|_H \leq \left(\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}\right)^k \|u\|_H.$$

In particular the vectors u_k *converges to* u *as* $k \to \infty$.

Proof. By definition of u_k we obtain

$$\begin{aligned} u - u_k &= u - u_{k-1} + \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v}(f - L u_{k-1}) \\ &= (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)(u - u_{k-1}) \\ &= (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)^2 (u - u_{k-2}) \\ &= \ldots = (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L)^k (u - u_0), \end{aligned}$$

therefore

$$\|u - u_k\|_H \leq \|I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L\|^k \|u\|_H. \tag{7}$$

But for every $v \in H$ we have

$$\begin{aligned} \langle (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L) v, v \rangle &= \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \langle S'_{H,v} L v, L v \rangle \\ &= \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \langle \sum_{\lambda \in \Lambda} v_\lambda^2 \pi_{L H_\lambda}(L v), L v \rangle \\ &= \|v\|_H^2 - \frac{2}{c_1^2 A + c_2^2 B} \sum_{\lambda \in \Lambda} v_\lambda^2 \|\pi_{L H_\lambda}(L v)\|_H^2 \\ &\leq \|v\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} \|L v\|_H^2 \\ &\leq \|v\|_H^2 - \frac{2A}{c_1^2 A + c_2^2 B} c_1^2 \|v\|_H^2 \\ &= (\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}) \|v\|_H^2, \end{aligned}$$

where in the first inequality we used the property of the lower bound of the frame of subspaces and in the second inequality we used the property of c_1 in (6). Similarly we have

$$-(\frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}) \|v\|_H^2 \leq \langle (I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L) v, v \rangle,$$

and so we conclude that

$$\|I - \frac{2}{c_1^2 A + c_2^2 B} L S'_{H,v} L\| \leq \frac{c_2^2 B - c_1^2 A}{c_1^2 A + c_2^2 B}. \tag{8}$$

Combining this inequality with (7) gives the result. □

Now, let u be the solution of the equation(1) and $T_{H,v}$ be the synthesis operator of the frame of subspaces $\{H_\lambda\}_{\lambda \in \Lambda}$ for H. Since $T_{H,v}$ is onto then there exists $U \in \sum_{\lambda \in \Lambda} \oplus H_\lambda$ such that $u = T_{H,v} U$, so the equation (1) is equivalent to $L T_{H,v} U = f$, or $T_{H,v}^* L T_{H,v} U = T_{H,v}^* f$, where $T_{H,v}^*$ is the analysis operator of the frame of subspaces. Therefore finding the solution u of the equation (1) is equivalent to finding the solution U of the equation

$$MU = F, \tag{9}$$

where $M := T_{H,v}^* L T_{H,v}$ *and* $F := T_{H,v}^* f$.

Note that we can consider the equation (9) as a matrix equation from $(\sum_{\lambda \in \Lambda} \oplus H_\lambda)_{\ell_2}$ to

itself , where the entries of M are the operators of the form $m_{\lambda,\lambda'} = v_\lambda v_{\lambda'} \pi_{H_{\lambda'}} L$.
We note that $(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2} = RanT^*_{H,v} \oplus KerT_{H,v}$, and the following lemma holds.

Lemma 2.1. *The orthogonal projector onto $RanT^*_{H,v}$ is $Q = T^*_{H,v} S^{-1}_{H,v} T_{H,v}$.*

Proof. For $x \in T^*_{H,v} f$,

$$\begin{aligned}
Qx &= Q(\{v_\lambda \pi_{H_\lambda} f\}_{\lambda\in\Lambda}) \\
&= T^*_{H,v} S^{-1}_{H,v} T_{H,v}(\{v_\lambda \pi_{H_\lambda} f\}_{\lambda\in\Lambda}) \\
&= T^*_{H,v} S^{-1}_{H,v}(\sum_{\lambda\in\Lambda} v_\lambda^2 \pi_{H_\lambda} f) \\
&= T^*_{H,v}(\sum_{\lambda\in\Lambda} S^{-1}_{H,v} v_\lambda^2 \pi_{H_\lambda} f) \\
&= \{v_\lambda \pi_{H_\lambda}(\sum_{\lambda\in\Lambda} S^{-1}_{H,v} v_\lambda^2 \pi_{H_\lambda} f)\}_{\lambda\in\Lambda} \\
&= \{v_\lambda \pi_{H_\lambda}(f)\}_{\lambda\in\Lambda}.
\end{aligned}$$

That is $Q = id$ on $RanT^*_{H,v}$ and $Q = 0$ on $KerT_{H,v}$.

□

Therefore, $M|_{RanT^*} : RanT^*_{H,v} \longrightarrow RanT^*_{H,v}$ is boundedly invertible and we have $\|M\| \leq B\|L\|$ and $\|M|^{-1}_{RanT^*_{H,v}}\| \leq A^{-1}\|L^{-1}\|$.

3. An adaptive algorithm based on a frame of subspaces

In this section, we construct an adaptive algorithm in order to give an approximate solution to the exact solution U of the equation (9). In order to analyze adaptive methods, we compare them with the best N-term approximation. The aim is to balance between the accuracy and computational complexity at the same time.
For $N \in \mathbb{N}$, define

$$\sum_N := \{V \in (\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2} : \#supp \ \ V \leq N\},$$

and the corresponding error

$$\rho_N(V) := \inf_{V_N \in \sum_N} \|V - V_N\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}}, \quad V \in (\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2},$$

where $\#supp \ \ V$ is the number of nonzero entries of V.
A best approximation to V from $\sum_N$ (called the best N-term approximation to V) is obtained by taking a set $\Lambda_N \subset \Lambda$ with $\#\Lambda_N \leq N$ on which $\|v_\lambda\|$ takes its N largest values. Note that the set Λ_N is not unique.
Given a sequence $V = (v_\lambda)_\lambda \in (\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}$, for each $n \geq 1$ let v_n^* be the n-th largest of the values $\|v_\lambda\|$ and define the decreasing rearrangement V^* of V by $V^* := (v_n^*)_{n=1}^\infty$. For each $0 < \tau < 2$ we let $\ell^\omega_\tau(\Lambda)$ denote the collection of all vectors $V \in (\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}$ for which $|V|_{\ell^\omega_\tau(\Lambda)} := sup_{n\geq 1} n^{\frac{1}{\tau}} v_n^*$ is finite. This expression defines a quasi norm for $\ell^\omega_\tau(\Lambda)$. A corresponding norm is defined by $\|V\|_{\ell^\omega_\tau(\Lambda)} := |V|_{\ell^\omega_\tau(\Lambda)} + \|V\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}}$. Also there exists a constant C_τ such that

$$|V + W|_{\ell^\omega_\tau(\Lambda)} \preceq (|V|_{\ell^\omega_\tau(\Lambda)} + |W|_{\ell^\omega_\tau(\Lambda)}), \tag{10}$$

where $a \preceq b$ means that, there is a constant c such that $a \leq cb$. Now let V_N be the best N-term approximation of V such that $\|V - V_N\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}} \leq \epsilon$. If for some $s > 0$

$$\|V - V_N\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}} \preceq N^{-s}, \tag{11}$$

then $N \preceq \epsilon^{\frac{-1}{s}}$. For $\tau = (\frac{1}{2} + s)^{-1}$, (11) means that $V \in \ell^\omega_\tau(\Lambda)$ and for $0 < \tau < 2$,

$$sup_{N\in\mathbb{N}}\{N^s\|V - V_N\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}}\} \simeq |V|_{\ell^\omega_\tau(\Lambda)}. \tag{12}$$

One can see [4, 9] for further details on the quasi-Banach spaces $\ell^\omega_\tau(\Lambda)$.

Proposition 3.1. *Let $s > 0$ and $\tau = (s + \frac{1}{2})^{-1}$. If $V \in \ell^\omega_\tau(\Lambda)$, then*

$$\rho_N(V) \preceq N^{-s}\|V\|_{\ell^\omega_\tau(\Lambda)}, \tag{13}$$

with a constant only depending on τ for $\tau \searrow 0$.

Proof. See [4]. □

Assumption. We assume that the matrix M is s^*-compressible, in the sense that for $0 < s < s^*$ there exists a sequence $\alpha = (\alpha_j)_j \in \ell_1(\Lambda)$, and a matrix M_j having at most $\alpha_j 2^j$ nonzero entries per row and column, and a positive constant C_M such that

$$\|M - M_j\| \leq C_M \alpha_j 2^{-js}, \tag{14}$$

for all $j \in \mathbb{N}$. ($\|M - M_j\|$ is the spectral norm of $(M - M_j)$.) Such a matrix maps $\ell^\omega_\tau(\Lambda)$ boundedly into itself for $\tau = (\frac{1}{2} + s)^{-1}$. [4].

Remark 3.1. *If M is s^* compressible then M maps $\ell^\omega_\tau(\Lambda)$ boundedly into itself for every $\tau = (\frac{1}{2} + s)^{-1}$.* [4].

For a finite support vector V, $N := (\#supp(V)) < \infty$, we denote the best 2^j-term approximation to V in $(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}$ by $V_{[j]}$, for $j = 1, 2, ..., \lfloor \log N \rfloor$, and let $V_{[j]} = V$, for $j > \log N$. For a given $K \in \mathbb{N}$ define

$$W_K := M_K V_{[0]} + \sum_{j=0}^{K-1} M_j (V_{[K-j]} - V_{[K-j-1]}),$$

where M_j is as (14). In this case

$$\begin{aligned} MV - W_K &= MV - M_K V_{[0]} - \sum_{j=0}^{K-1} M_j (V_{[K-j]} - V_{[K-j-1]}) \\ &= MV - M_K V_{[0]} - M_{K-1}(V_{[1]} - V_{[0]}) - ... - M_0(V_{[K]} - V_{[K-1]}) \\ &= M(V - V_{[K]}) + (M - M_0)(V_{[K]} - V_{[K-1]}) + ... + (M - M_K)V_{[0]}, \end{aligned}$$

and since M is s^*-compressible then

$$\begin{gathered} \|MV - W_K\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}} \leq \\ \|M\|\|V - V_{[K]}\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}} + \|M - M_0\|\|V_{[K]} - V_{[K-1]}\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}} \\ +... + \|M - M_K\|\|V_{[0]}\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}} \leq \|M\|\|V - V_{[K]}\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}} + \\ C_M(\alpha_0\|V_{[K]} - V_{[K-1]}\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}} + ... + \alpha_K 2^{-Ks}\|V_{[0]}\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}}). \end{gathered}$$

In the other words there exists a constant C_3 such that

$$\begin{gathered} \|MV - W_K\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}} \leq \\ C_3(\|V - V_{[K]}\|_{(\sum_{\lambda\in\Lambda} \oplus H_\lambda)_{\ell_2}} + \end{gathered} \tag{15}$$

$$\alpha_0\|V_{[K]}-V_{[K-1]}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}+...+\alpha_K 2^{-Ks}\|V_{[0]}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}).$$

Now by proposition 3.1 and the definition of $V_{[j]}$ there exists a constant C_4 such that

$$\|V-V_{[j]}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}=\rho_{2^j}(V)\le C_4 2^{-js}\|V\|_{\ell_\tau^\omega(\Lambda)},$$

hence

$$\begin{aligned}\|V_{[j]}-V_{[j-1]}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} &\le \|V-V_{[j]}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}+\|V-V_{[j-1]}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}\\ &=\rho_{2^j}(V)+\rho_{2^{j-1}}(V)\\ &\le C_4(2^{-s}+1)2^{-(j-1)s}\|V\|_{\ell_\tau^\omega(\Lambda)}.\end{aligned}$$

Applying the above inequalities and the fact that $\|V_{[0]}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}\le\|V_{[0]}\|_{\ell_\tau^\omega(\Lambda)}$, the inequality (15) induces a constant C such that

$$\|MV-W_K\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}\le C2^{-Ks}\|V\|_{\ell_\tau^\omega(\Lambda)}. \tag{16}$$

Now we are ready to design our algorithm. First, following [4], we introduce the following routine.

APPLY $[M,\ \ V,\ \ \epsilon]$:
i) Compute $V_{[0]},\ \ V_{[j]}-V_{[j-1]},\ \ j=1,...,\lfloor\log N\rfloor$ and define $V_{[j]}:=V$ for $j>logN$.
ii) Compute K as the smallest integer such that $2^K\ge C^{\frac{1}{s}}\epsilon^{-\frac{1}{s}}\|V\|^{\frac{1}{s}}_{\ell_\tau^\omega(\Lambda)}$.
iii) For $k=1$ to K compute
$R_k:=\|M\|\|V-V_{[k]}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}+C_A(\alpha_0\|V_{[k]}-V_{[k-1]}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}+...+\alpha_k 2^{-ks}\|V_{[0]}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}})$.
iv) If $R_k\le\epsilon$ then exit.
v) $W_K:=M_KV_{[0]}+\sum_{j=0}^{K-1}M_j(V_{[k-j]}-V_{[k-j-1]})$.

Remark 3.2. *Since K is the smallest integer such that $2^K\ge C^{\frac{1}{s}}\epsilon^{-\frac{1}{s}}\|V\|^{\frac{1}{s}}_{\ell_\tau^\omega(\Lambda)}$, then $2^{K-1}<C^{\frac{1}{s}}\epsilon^{-\frac{1}{s}}\|V\|^{\frac{1}{s}}_{\ell_\tau^\omega(\Lambda)}$. Thus $2^K<2C^{\frac{1}{s}}\epsilon^{-\frac{1}{s}}\|V\|^{\frac{1}{s}}_{\ell_\tau^\omega(\Lambda)}$, which $2^K\preceq\epsilon^{-\frac{1}{s}}\|V\|^{\frac{1}{s}}_{\ell_\tau^\omega(\Lambda)}$.*

Lemma 3.1. *Let $V\in\ell_\tau^\omega(\Lambda)$ with $\tau=(s+\frac{1}{2})^{-1}$. For a given accuracy $\epsilon>0$, the output W_K of **APPLY** $[M,\ \ V,\ \ \epsilon]$ satisfies*

$$\|MV-W_K\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}\le\epsilon, \tag{17}$$

and

$$\#supp(W_K)\preceq\epsilon^{-\frac{1}{s}}\|V\|^{\frac{1}{s}}_{\ell_\tau^\omega(\Lambda)}. \tag{18}$$

Also the number of arithmetic operations to compute W_K is at most a multiple of $\epsilon^{-\frac{1}{s}}\|V\|^{\frac{1}{s}}_{\ell_\tau^\omega(\Lambda)}$.

Proof. The inequality (17) comes from the inequality (16) and the definition of K in **APPLY**.
Recalling the number of nonzero entries in $V_{[j]}$, and definition of M_j we conclude

$$\begin{aligned}\#supp(W_K)&\le\#rows(M_K)+\#rows(M_{K-1})+...+\#rows(M_0)\\ &\le\alpha_K2^K+\alpha_{K-1}2^{K-1}+...+\alpha_0\le(|\alpha_K|+|\alpha_{K-1}|+...+|\alpha_0|)2^K\preceq 2^K,\end{aligned}$$

where the last inequality is induced by $(\alpha_j)_j\in\ell_1(\Lambda)$. Now by using remark 3.2, we obtain the inequality (18). Also if N_K denotes the number of arithmetic operation needed to compute W_K we have

$$\begin{aligned}N_K\le\#rows(M_K)\#supp(V_{[0]})+\#rows(M_{K-1})\#supp(V_{[1]}-V_{[0]})\\ +...+\#rows(M_0)\#supp(V_{[K]}-V_{[K-1]})\end{aligned}$$

$$\leq \alpha_K 2^K + \alpha_{K-1}2^{K-1} + ... + \alpha_0 2^{K-1} \preceq 2^K \preceq \epsilon^{-\frac{1}{s}} \|V\|^{\frac{1}{s}}_{\ell^{\omega}_{\tau}(\Lambda)}.$$

□

Also, as in [4], for an accuracy $\epsilon > 0$ and a finitely supported vector W with $N = \#(spport(W))$, we introduce the following basic numerical ingredient that we will use in our algorithm.

COARSE $[W, \epsilon] \to (\Lambda, \bar{W})$
(i) Sort the nonzero entries of W into decreasing order in modulus and obtain the vector $\lambda^* := (\lambda_1, ..., \lambda_N)$ of indices which gives the decreasing rearrangement $W^* = (\|W_{\lambda_1}\|_H, ..., \|W_{\lambda_N}\|_H)$ of nonzero entries of W; then compute $\|W\|^2_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} = \sum_{i=1}^N \|W_{\lambda_i}\|^2_H$.

(ii) Find the smallest $K \in \mathbb{N}$ such that $\sum_{i=1}^K \|W_{\lambda_i}\|^2$ exceeds $\|W\|^2_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} - \epsilon^2$. For this K define $\Lambda := \{\lambda_i : i = 1, ..., K\}$ and $\bar{W}$ by $\bar{W}_\lambda = W_\lambda$ for $\lambda \in \Lambda$ and $\bar{W}_\lambda = 0$ for $\lambda \notin \Lambda$.

Now, let $0 < \epsilon < \|V\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}$ and W be a finitely supported approximation to V such that $\|V - W\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq d\epsilon$ for some $d < 1$, then it is obvious that the **COARSE** $[W, (1-d)\epsilon]$ produces $\bar{W}$ supported on Λ which $\|V - \bar{W}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq \epsilon$. (Note that the output $\bar{W}$ of **COARSE**, by construction, satisfies $\|W - \bar{W}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq \epsilon$). Moreover, we have the following lemma [4].

Lemma 3.2. *If* $V \in \ell^{\omega}_{\tau}(\Lambda), \tau = (s + \frac{1}{2})^{-1}$, *for some* $s > 0$ *then the outputs* $\bar{W}$, Λ *of* ***COARSE*** $[W, (1-d)\epsilon]$ *requires at most* $2N$ *arithmetic operations and* $NlogN$ *sorts,where* $N = \#supp(W)$. *Moreover,*

$$|\bar{W}|_{\ell^{\omega}_{\tau}(\Lambda)} \preceq |V|_{\ell^{\omega}_{\tau}(\Lambda)}, \tag{19}$$

and $\#\Lambda$ *(the cardinality of* $supp(\bar{W})$ *) satisfies*

$$\#(\Lambda) \preceq |V|^{\frac{1}{s}}_{\ell^{\omega}_{\tau}(\Lambda)} \epsilon^{\frac{1}{s}}. \tag{20}$$

Also for $F = T^*_{H,v} f$ we assume that the routine:
RHS $[\varepsilon, F] \to F_\epsilon$
determines a finitely supported vector $F_\epsilon \in (\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}$ satisfying

$$\|F - F_\epsilon\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq \epsilon.$$

Assuming Q is bounded on $\ell^{\omega}_{\tau'}(\Lambda)$ for $\tau' = (\frac{1}{2} + s')^{-1}$, $0 < s' < s$ (hence Q is bounded on ℓ^{ω}_{τ}, [9])we construct our algorithm for the target accuracy $\epsilon > 0$. At first, for some $0 < d < \frac{1}{3}$ and $\rho := \|I - \alpha M\| < 1$ (since M is a positive definite matrix this real number α exists) set $K := min\{k \in \mathbb{N} :\ 3\rho^k < d\ \ min\{1, [C_1 C_2 |I - Q|_{\ell^{\omega}_{\tau} \to \ell^{\omega}_{\tau}}]^{\frac{s}{s'-s}}\}\}$, where C_1, C_2 are two constants induced from (10) and (19) for τ'.

SOLVE $[\epsilon, M, F] \to (U_\epsilon, \Lambda_\epsilon)$
(i) Set $i = 0, U^{(0)} = 0, \Lambda_0 = \emptyset, \epsilon_0 := \|M|^{-1}_{Ran(T^*)}\|\ \ \|F\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}$.

(ii) If $\epsilon_i \leq \epsilon$ stop and set $U_\epsilon := U^i$, otherwise

(ii.1) $i := i + 1$, $\epsilon_i := 3\rho^K \frac{\epsilon_{i-1}}{d}$.
(ii.2) $F^i :=$ **RHS** $[F, \frac{d\epsilon_i}{6\alpha K}]$.
(ii.3) $V^{(i,0)} := U^{i-1}$

(ii.4) For $j = 1, ..., K$ compute

(1) $W^{j-1} :=$ **APPLY** $[M, V^{(i,j-1)}, \frac{d\epsilon_i}{6\alpha K}]$.
(2) $V^{(i,j)} := V^{(i,j-1)} + \alpha(F^i - W^{j-1})$.

(iii) $U^i :=$ **COARSE** $[V^{(i,K)}, (1-d)\epsilon_i]$ and go to **(ii)**.

Remark 3.3. *By definition of $V^{(i,j)}$, F^i in **SOLVE** and lemma (3.1) and since $MQU = F$,*

$$\begin{aligned}
&\|QU - V^{(i,1)} - (I - \alpha M)(QU - U^{i-1})\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \\
&= \|QU - (V^{(i,0)} - \alpha(\boldsymbol{APPLY}\ [M, V^{(i,0)}, \frac{d\epsilon_i}{6\alpha}] - F^i)) \\
&\quad -(I - \alpha M)(QU - U^{i-1})\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \\
&= \|QU - U^{i-1} + \alpha(\boldsymbol{APPLY}\ [M, V^{(i,0)}, \frac{d\epsilon_i}{6\alpha}] - \alpha F^i) \\
&\quad -QU + U^{i-1} + \alpha MQU - \alpha MU^{i-1}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \\
&= \|\alpha\boldsymbol{APPLY}\ [M, V^{(i,0)}, \frac{d\epsilon_i}{6\alpha}] - \alpha F^i + \alpha F - \alpha MU^{i-1}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \\
&= \|\alpha(\boldsymbol{APPLY}\ [M, V^{(i,0)}, \frac{d\epsilon_i}{6\alpha}] - MU^{i-1}) + \alpha(F - F^i)\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \\
&\leq \alpha\|\boldsymbol{APPLY}\ [M, V^{(i,0)}, \frac{d\epsilon_i}{6\alpha}] - MU^{i-1}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} + \alpha\|F - F^i\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \\
&\leq \alpha\frac{d\epsilon_i}{6\alpha} + \alpha\frac{d\epsilon_i}{6\alpha} = \frac{d\epsilon_i}{3}.
\end{aligned}$$

Similarly we can prove

$$\|QU - V^{(i,K)} - (I - \alpha M)^K(QU - U^{i-1})\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq \frac{d\epsilon_i}{3}. \tag{21}$$

Theorem 3.1. *If U is a solution for (9) then the following inequalities hold for the algorithm **SOLVE**:*

$$\begin{aligned}
&\|Q(U - U_\epsilon)\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq \epsilon \\
&\|QU + (I - Q)U^{i-1} - V^{(i,K)}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq \frac{2}{3}d\epsilon_i, \qquad (i \geq 1)
\end{aligned} \tag{22}$$

Proof. In order to prove the first inequality, it is enough to prove $\|Q(U - U^i)\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq \epsilon_i$ for each $i \geq 0$. For $i = 0$, since $QU = M|^{-1}_{RanT^*_{H,v}} F$ then

$$\begin{aligned}
\|Q(U - U^0)\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} &= \|QU\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} = \|M|^{-1}_{RanT^*}F\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \\
&\leq \|M|^{-1}_{RanT^*}\|\|F\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} = \epsilon_0.
\end{aligned}$$

Now for an $i \geq 1$, let $\|Q(U - U^{i-1})\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq \epsilon_{i-1}$. Since $MQU = F$ and

$$(I - \alpha M)^K(QU - U^{i-1}) = (I - \alpha M)^K Q(U - U^{i-1}) - (I - Q)U^{i-1},$$

by using the inequality (21)

$$\begin{aligned}
&\|QU + (I - Q)U^{i-1} - V^{(i,K)}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \\
&= \|QU + (I - Q)^K Q(U - U^{i-1}) - (I - Q)^K(QU - U^{i-1}) - V^{(i,K)}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}
\end{aligned}$$

$$\leq \|(I-\alpha M)^K Q(U-U^{i-1})\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}+$$
$$\|QU-V^{(i,K)}-(I-\alpha M)^K(QU-U^{i-1})\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}$$
$$\leq \rho^K\|Q(U-U^{i-1})\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}+\frac{d\epsilon_i}{3}$$
$$\leq \rho^K\epsilon_{i-1}+\frac{d\epsilon_i}{3}=\frac{2}{3}d\epsilon_i,$$

as we desired in (22).
Now by using (22) and the definition of U^i in **SOLVE** we obtain,

$$\|QU+(I-Q)U^{i-1}-U^i\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}$$
$$=\|QU+(I-Q)U^{i-1}-V^{(i,K)}+V^{(i,K)}-U^i\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}$$
$$\leq \|QU+(I-Q)U^{i-1}-V^{(i,K)}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}+\|V^{(i,K)}-U^i\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}$$
$$\leq (\frac{2d}{3}+(1-d))\epsilon_i=(1-\frac{d}{3})\epsilon_i.$$

Therefore

$$\|Q(U-U^i)\|^2_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}\leq \|Q(U-U^i)\|^2_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}+$$
$$\|(I-Q)(U^{i-1}-U^i)\|^2_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}$$
$$=\|Q(U-U^i)+(I-Q)(U^{i-1}-U^i)\|^2_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}$$
$$=\|QU+(I-Q)U^{i-1}-U^i\|^2_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}$$
$$\leq (1-\frac{d}{3})^2\epsilon_i^2\leq\epsilon_i^2,$$

and so

$$\|Q(U-U^i)\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}\leq\epsilon.$$

□

In the following theorems, we investigate the optimal computational complexity of the algorithm **SOLVE** as it recovers an approximate solution with desired accuracy at a computational expense that stays proportional to the number of terms in a corresponding wavelet-best N-term approximation.

Theorem 3.2. *Assume that the solution U of (9) belongs to ℓ^ω_τ. Then*

$$\#(supp(U_\epsilon))\preceq \epsilon^{\frac{1}{s}}|U|^{-\frac{1}{s}}_{\ell^\omega_\tau}.$$

Proof. Let $(QU)_{N_i}$ be the best N_i-term approximation for QU such that

$$\|QU-(QU)_{N_i}\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}}\leq\frac{d\epsilon_i}{3}, \tag{23}$$

where $0<d<\frac{1}{3}$. Since $U\in\ell^\omega_\tau$, by (11) we have

$$N_i\preceq\epsilon_i^{-\frac{1}{s}}|QU|^{\frac{1}{s}}_{\ell^\omega_\tau}\preceq\epsilon_i^{-\frac{1}{s}}|U|^{\frac{1}{s}}_{\ell^\omega_\tau}. \tag{24}$$

Since $s'<s$, then for a vector V with $\#supp(V)=N$

$$|V|_{\ell^\omega_{\tau'}}\leq N^{s-s'}|V|_{\ell^\omega_\tau}, \tag{25}$$

combination (25) and (24) gives

$$\epsilon_i^{1-\frac{s'}{s}}|(QU)_{N_i}|_{\ell^\omega_{\tau'}}\preceq\epsilon_i^{1-\frac{s'}{s}}N_i^{(s-s')}|(QU)_{N_i}|_{\ell^\omega_\tau}\preceq$$

$$\epsilon_i^{1-\frac{s'}{s}}(\epsilon_i^{-\frac{1}{s}})^{(s-s')}(|U|_{\ell^\omega_\tau}^{\frac{1}{s}})^{(s-s')}|(QU)_{N_i}|_{\ell^\omega_\tau} \preceq |U|_{\ell^\omega_\tau}^{1-\frac{s'}{s}}|QU|_{\ell^\omega_\tau} \preceq |U|_{\ell^\omega_\tau}^{-\frac{s'}{s}},$$

therefore

$$\epsilon_i^{1-\frac{s'}{s}}|(QU)_{N_i}|_{\ell^\omega_{\tau'}} \preceq |U|_{\ell^\omega_\tau}^{\frac{s'}{s}}. \tag{26}$$

Using (22) and (23),

$$\|(QU)_{N_i} + (I-Q)U^{i-1} - V^K\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq$$

$$\|(QU)_{N_i} - QU\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} + \|QU + (I-Q)U^{i-1} - V^K\|_{(\sum_{\lambda\in\Lambda}\oplus H_\lambda)_{\ell_2}} \leq$$

$$\frac{d\epsilon_i}{3} + \frac{2d\epsilon_i}{3} = d\epsilon_i,$$

then by lemma 3.2 and 10 we have

$$|U^i|_{\ell^\omega_{\tau'}} \leq C_2|(QU)_{N_i} + (I-Q)U^{i-1}|_{\ell^\omega_{\tau'}}$$

$$\leq C_1C_2|(QU)_{N_i}|_{\ell^\omega_{\tau'}} + C_1C_2|I-Q|_{\ell^\omega_{\tau'}\to\ell^\omega_{\tau'}}|U^{i-1}|_{\ell^\omega_{\tau'}},$$

by (26) and since $\epsilon_i = \frac{3\rho^k\epsilon_{i-1}}{d}$

$$\epsilon_i^{1-\frac{s'}{s}}|U^i|_{\ell^\omega_{\tau'}} \leq C|U|_{\ell^\omega_\tau}^{-\frac{s'}{s}} + C_1C_2|I-Q|_{\ell^\omega_{\tau'}\to\ell^\omega_{\tau'}}(\frac{3\rho^K}{d})^{1-\frac{s'}{s}}(\epsilon_{i-1}^{1-\frac{s'}{s}}|U^{i-1}|_{\ell^\omega_{\tau'}}), \tag{27}$$

for some constant C. Now by the properties of K we obtain

$$\epsilon_i^{1-\frac{s'}{s}}|U^i|_{\ell^\omega_{\tau'}} \preceq |U|_{\ell^\omega_\tau}^{-\frac{s'}{s}}. \tag{28}$$

Finally by (28), (26) and lemma 3.2 we conclude

$$\#(supp(U^i)) \preceq \epsilon_i^{-\frac{1}{s'}}|(QU)_{N_i} + (I-Q)U^{i-1}|_{\ell^\omega_{\tau'}}^{\frac{1}{s'}}$$

$$\preceq \epsilon_i^{-\frac{1}{s}}(\epsilon_i^{1-\frac{s'}{s}}[|(QU)_{N_i}|_{\ell^\omega_{\tau'}} + |I-Q|_{\ell^\omega_{\tau'}\to\ell^\omega_{\tau'}}|U^{i-1}|_{\ell^\omega_{\tau'}}])^{\frac{1}{s'}}$$

$$\preceq \epsilon_i^{\frac{1}{s}}|U|_{\ell^\omega_\tau}^{-\frac{1}{s}},$$

and it is obvious that this proves our request. □

Theorem 3.3. *Assume that the solution U of (9) belongs to ℓ^ω_τ. Then the number of arithmetic operations needed to compute U_ϵ is bounded by a multiple of $\epsilon^{\frac{1}{s}}|U|_{\ell^\omega_\tau}^{-\frac{1}{s}}$.*

Proof. Since $MU = F$ and M is bounded on ℓ^ω_τ then $|F|_{\ell^\omega_\tau} \preceq |U|_{\ell^\omega_\tau}$, and therefore by lemma 3.2

$$\#supp(F^i) \preceq \epsilon_i^{-\frac{1}{s}}|U|_{\ell^\omega_\tau}^{\frac{1}{s}}, \tag{29}$$

and

$$|F^i|_{\ell^\omega_\tau} \preceq |U|_{\ell^\omega_\tau}. \tag{30}$$

Now by (25), (29) and (30) we obtain

$$|F^i|_{\ell^\omega_{\tau'}} \preceq (\#supp(F^i))^{s-s'}|F^i|_{\ell^\omega_\tau}$$

$$\preceq (\epsilon_i^{-\frac{1}{s}}|U|_{\ell^\omega_\tau}^{\frac{1}{s}})^{s-s'}|U|_{\ell^\omega_\tau} \preceq \epsilon_i^{\frac{s'}{s}-1}|U|_{\ell^\omega_\tau}^{-\frac{s'}{s}}.$$

Thus

$$(\epsilon_i)^{1-\frac{s'}{s}}|F^i|_{\ell^\omega_{\tau'}} \preceq |U|_{\ell^\omega_\tau}^{-\frac{s'}{s}}. \tag{31}$$

Lemma 3.1 together with (28) and (31) give $(\epsilon_i)^{1-\frac{s'}{s}}|V^{(i,j)}|_{\ell^{\omega}_{\tau'}} \preceq |U|_{\ell^{\omega}_{\tau}}^{-\frac{s'}{s}}$, $0 \leq j \leq K$, therefore by lemma 3.1 (for s), $\#supp(W^{j-1}) \preceq \epsilon_i^{-\frac{1}{s'}}|V^{(i,j-1)}|^{\frac{1}{s'}}_{\ell^{\omega}_{\tau'}} \preceq \epsilon_i^{\frac{1}{s}}|U|^{-\frac{1}{s}}_{\ell^{\omega}_{\tau}}$. Also by the previous theorem $\#supp(U^{i-1}) = \#supp(V^{i,0}) \preceq \epsilon_{i-1}^{\frac{1}{s}}|U|^{-\frac{1}{s}}_{\ell^{\omega}_{\tau}}$, while by step (2) in **SOLVE**,

$$\#supp(V^{(i,j)}) \leq \#supp(V^{(i,j-1)}) + \#supp(F^i) + \#supp(W^{j-1}).$$

Therefore we conclude

$$\#supp(V^{i,j}) \preceq \epsilon_i^{\frac{1}{s}}|U|^{-\frac{1}{s}}_{\ell^{\omega}_{\tau}}. \tag{32}$$

Now by lemmas 3.2 and 3.1 together with (32), the number of arithmetic operations needed from i to i+1 is at most a multiple of $\epsilon^{\frac{1}{s}}|U|^{-\frac{1}{s}}_{\ell^{\omega}_{\tau}}$, which is the desired result.

□

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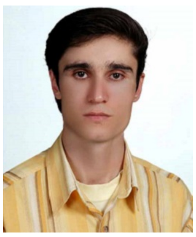

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