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CERTAIN CLASS OF HARMONIC MAPPINGS RELATED TO STARLIKE FUNCTIONS

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Abstract. Let S^* be the class of starlike functions and let S_H be the class of harmonic mappings in the plane. In this paper we investigate harmonic mapping related to the starlike functions.

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1. Introduction

Let Ω be the family of functions $\phi(z)$ regular in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} | |z| < 1\}$ and satisfying the conditions $\phi(0) = 0$, $\phi(z) < 1$ for every $z \in \mathbb{D}$.
Next, for arbitrary fixed numbers A, B, $-1 \leq B < A \leq 1$, we denote by $P(A, B)$, the family of functions $p(z) = 1 + p_1 z + p_2 z^2 + ...$ regular in $\mathbb{D}$ and such that $p(z)$ is in $P(A, B)$ if and only if

$$p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, \tag{1}$$

for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. (Janowski) [5].
Moreover, let $S^*(A, B)$ denote the family of functions $h(z) = z + a_2 z^2 + ...$ regular in $\mathbb{D}$ and such that $h(z)$ is in $S^*(A, B)$ if and only if

$$z\frac{h'(z)}{h(z)} = \frac{1 + A\phi(z)}{1 + B\phi(z)} \tag{2}$$

for some $\phi(z) \in \Omega$ and all $z \in \mathbb{D}$.
Let $F(z) = z + \alpha_2 z^2 + \alpha_3 z^3 + ...$ and $G(z) = z + \beta_2 z^2 + \beta_3 z^3 + ...$ be analytic functions in the open unit disc. If there exists a function $\phi(z) \in \Omega$ such that $F(z) = G(\phi(z))$ for all $z \in \mathbb{D}$, then we say that $F(z)$ is subordinated to $G(z)$ and we write $F(z) \prec G(z)$. Specially if $F(z)$ is univalent in $\mathbb{D}$, then $F(z) \prec G(z)$ if and only if $F(\mathbb{D}) \subset G(\mathbb{D})$, implies $F(\mathbb{D}_r) \subset G(\mathbb{D}_r)$, where $\mathbb{D}_r = \{z | |z| < r, 0 < r < 1\}$ (Subordination and Lindelof Principle [3]) Finally, a planar harmonic mapping in the open unit disc $\mathbb{D}$ is a complex valued harmonic function f, which maps $\mathbb{D}$ onto the some planar domain $f(\mathbb{D})$. Since $\mathbb{D}$ is a simply connected

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domain, the mapping f has a canonical decomposition $f = h(z) + \overline{g(z)}$, where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ and have the following power series expansions,

$$h(z) = \sum_{n=0}^{\infty} a_n z^n,$$

$$g(z) = \sum_{n=0}^{\infty} b_n z^n, z \in \mathbb{D},$$

where a_n,$b_n \in \mathbb{C}$, $n = 0, 1, 2, ...$

We call $h(z)$ the analytic part of f and $\overline{g(z)}$ co-analytic part of f, an elegant and complete treatment theory of harmonic mapping in given Duren's monography [2]. Lewy proved in 1936 that the harmonic mapping f is locally univalent in $\mathbb{D}$ if and only if its jacobian $J_{f(z)} = |h'(z)|^2 - |g'(z)|^2 \neq 0$ in $\mathbb{D}$. When $J_f > 0$ in $\mathbb{D}$, the harmonic function f is called sense-preserving. In view of this result, locally univalent harmonic mappings in the open unit disc are either sense-preserving if $|g'(z)| < |h'(z)|$ in $\mathbb{D}$ or sense-reversing if $|h'(z)| < |g'(z)|$. Throughout this paper we will restrict ourselves to the study of sense-preserving if and only if $h'(z)$ does not vanish in the unit disc $\mathbb{D}$, and the second dilatation $w(z) = (\frac{g'(z)}{h'(z)})$ has the property $|w(z)| < 1$ in $\mathbb{D}$.

The class of all sense-preserving harmonic mappings in the open unit disc $\mathbb{D}$ with $a_0 = b_0 = 0$ and $a_1 = 1$ will be denoted by S_H. Thus S_H contains the standart class S of univalent functions.

The family of all mappings $f \in S_H$ with the additional property $g'(0) = 0$, i. e. , $b_1 = 0$ is denoted by S_H^0. Thus it is clear that $S \subset S_H^0 \subset S_H$ [Duren].

Now we consider the following class of harmonic mappings

$$S_{HS^*}(A, B) = \left\{ f = h(z) + \overline{g(z)} \in S_H \mid \frac{g'(z)}{h'(z)} \prec b_1 \frac{1 + A\phi(z)}{1 + B\phi(z)}, h(z) \in S^* \right\} \tag{3}$$

The main purpose of this paper is to investigate the class $S_{HS^*}(A, B)$. For this aim, we will need the following lemma and theorem.

Lemma 1.1. *([4]) Let $\phi(z)$ be regular in the open unit disc $\mathbb{D}$ with $\phi(0) = 0$ and $|\phi(z)| < 1$. If $|\phi(z)|$ attains its maximum value on the circle $|z| = r$ at the point z_1, then we have $z_1.\phi'(z) = k\phi(z_1)$ for some $k \geq 1$.*

Theorem 1.1. *([3]) Let $h(z)$ be an element of S^*, then*

$$\frac{r}{(1+r)^2} \leq |h(z)| \leq \frac{r}{(1-r)^2}$$

$$\frac{1-r}{(1+r)^3} \leq \left|h'(z)\right| \leq \frac{1+r}{(1-r)^3}$$

2. Main Results

Theorem 2.1. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HS^*}(A, B)$. If $h(z) \in S^*(A, B)$ and $\frac{g'(z)}{b_1 h'(z)} \in P(A, B)$, then $\frac{g(z)}{b_1 h(z)} \in P(A, B)$.*

Proof. Since the linear transformation $\frac{1+Az}{1+Bz}$ maps $|z| = r$ onto the disc with the centre

$$C(r) = (\frac{1 - ABr^2}{1 - B^2r^2}, 0)$$

and the radius $\rho(r)=\frac{(A-B)r}{1-B^2r^2}$ and using the subordination principle we can write

$$\frac{g'(z)}{h'(z)}\prec b_1\frac{1+Az}{1+Bz} \tag{4}$$

$$\frac{1}{b_1}\frac{g'(z)}{h'(z)}\prec\frac{1+Az}{1+Bz}\Rightarrow\left|\frac{1}{b_1}\frac{g'(z)}{h'(z)}-\frac{1-ABr^2}{1-B^2r^2}\right|\leq\frac{(A-B)r}{1-B^2r^2}$$

thus

$$\left|\frac{g'(z)}{h'(z)}-\frac{b_1(1-ABr^2)}{1-B^2r^2}\right|\leq\frac{|b_1|\,(A-B)r}{1-B^2r^2} \tag{5}$$

Therefore the inequality (4) shows that the values of $\frac{g'(z)}{h'(z)}$ are in the disc

$$D_r(b_1)=\begin{cases}\frac{g'(z)}{h'(z)}\mid\left|\frac{g'(z)}{h'(z)}-\frac{b_1(1-AB)r^2}{1-B^2r^2}\right|\leq\frac{|b_1|(A-B)r}{1-B^2r^2}, & B\neq 0;\\ \\ \frac{g'(z)}{h'(z)}\mid\left|\frac{g'(z)}{h'(z)}-b_1\right|\leq|b_1|\,Ar, & B=0.\end{cases} \tag{6}$$

Now we define a function $\phi(z)$ by

$$\frac{g(z)}{h(z)}=b_1\frac{1+A\phi(z)}{1+B\phi(z)} \tag{7}$$

then $\phi(z)$ is analytic in $\mathbb{D}$ and $\phi(0)=0$, on the other hand since $h(z)\in S^*(A,B)$ then

$$D_r=\begin{cases}\left|z\frac{h'(z)}{h(z)}-\frac{1-ABr^2}{1-B^2r^2}\right|\leq\frac{(A-B)r}{1-B^2r^2}, & B\neq 0;\\ \\ \left|z\frac{h'(z)}{h(z)}-1\right|\leq Ar, & B=0.\end{cases} \tag{8}$$

for all $|z|=r<1$. Thus for a point z_1 on the bound of this disc we have

$$z_1\frac{h'(z)}{h(z)}-\frac{1-ABr^2}{1-B^2r^2}=\frac{(A-B)r}{1-B^2r^2}e^{i\theta},B\neq 0$$

$$z_1\frac{h'(z)}{h(z)}-1=Are^{i\theta},B=0$$

$$\frac{h(z_1)}{z_1h'(z_1)}=\frac{1-B^2r^2}{(1-ABr^2)+(A-B)re^{i\theta}}\in\partial\mathbb{D}_r,B\neq 0$$

$$\frac{h(z_1)}{z_1h'(z_1)}=\frac{1}{1+Are^{i\theta}}\in\partial\mathbb{D}_r,B=0$$

where $\partial\mathbb{D}_r$ is the boundary of the disc $\mathbb{D}_r$. Therefore by Jack's Lemma $z_1\phi'(z_1)=k\phi(z_1)=k\phi(z_1)$ and $k\geq 1$, we have that

$$w(z_1)=\frac{g'(z_1)}{b_1h'(z_1)}=\begin{cases}\frac{1+A\phi(z_1)}{1+B\phi(z_1)}+\frac{(A-B)k\phi(z_1)}{(1+B\phi(z_1))^2}\frac{1-B^2r^2}{(1-ABr^2)+(A-B)re^{i\theta}}\notin w(\mathbb{D}_r(b_1)), & B\neq 0;\\ \\ 1+A\phi(z_1)+Ak\phi(z_1)\frac{1}{1+Are^{i\theta}}\notin w(\mathbb{D}_r(b_1)), & B=0.\end{cases} \tag{9}$$

because $|\phi(z_1)|=1$ and $k\geq 1$. But this is a contradiction to the condition

$$\frac{g'(z)}{h'(z)}\prec b_1\frac{1+Az}{1+Bz}$$

and so we have $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. □

Lemma 2.1. *Let $f = h(z) + \overline{g(z)}$ be an element of S_H, then for a function defined by*

$$w(z) = \frac{g'(z)}{h'(z)}$$

we have

$$\frac{|b_1| - r}{1 - |b_1| r} \leq |w(z)| \leq \frac{|b_1| + r}{1 + |b_1| r} \tag{10}$$

$$\frac{(1 - r^2)(1 - |b_1|^2)}{(1 + |b_1| r)^2} \leq (1 - |w(z)|^2) \leq \frac{(1 - r^2)(1 - |b_1|^2)}{(1 - |b_1| r)^2} \tag{11}$$

and

$$\frac{(1 - r)(1 + |b_1|)}{(1 - |b_1| r)} \leq (1 + |w(z)|) \leq \frac{(1 + r)(1 + |b_1|)}{(1 + |b_1| r)} \tag{12}$$

$$\frac{(1 - r)(1 - |b_1|)}{(1 + |b_1| r)} \leq (1 - |w(z)|) \leq \frac{(1 + r)(1 - |b_1|)}{(1 - |b_1| r)} \tag{13}$$

for all $|z| = r < 1$

Proof. Since $f = h(z) + \overline{g(z)}$ be an element of S_H, it follows that
$w(z) = \frac{g'(z)}{h'(z)} = \frac{b_1 + 2b_2 z + 3b_3 z^2 + \ldots}{1 + 2a_2 z + 3a_3 z^2 + \ldots} \Rightarrow w(0) = b_1, |w(z)| < 1,$
so the function

$$\phi(z) = \frac{w(z) - w(0)}{1 - \overline{w(0)} w(z)} = \frac{w(z) - b_1}{1 - \overline{b_1} w(z)}$$

satisfies the conditions of Schwarz lemma. Therefore we have
$w(z) = \frac{b_1 + \phi(z)}{1 + \overline{b_1} \phi(z)}$ if and only if $w(z) \prec \frac{b_1 + z}{1 + \overline{b_1} z}$.
On the other hand, the linear transformation $(\frac{b_1 + z}{1 + \overline{b_1} z})$ maps $|z| = r$ onto the disc with the center

$$C(r) = (\frac{(1 - r^2) Re(b_1)}{1 - |b_1|^2 r^2}, \frac{(1 - r^2) Im(b_1)}{1 - |b_1|^2 r^2})$$

with the radius

$$\rho(r) = \frac{(1 - |b_1|^2) r}{1 - |b_1|^2 r^2}$$

$$\left| w(z) - \frac{b_1 (1 - r^2)}{1 - |b_1|^2} \right| \leq \frac{(1 - |b_1|^2) r}{1 - |b_1|^2 r^2}$$

which gives (11), (12) and (13). □

Theorem 2.2. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HS^*}(A, B)$, then*

$$|b_1| \frac{1 - Ar}{1 - Br} \frac{r}{(1 + r)^2} \leq |g(z)| \leq |b_1| \frac{1 + Ar}{1 + Br} \frac{r}{(1 - r)^2} \tag{14}$$

$$|b_1| \frac{1 - Ar}{1 - Br} \frac{(1 - r)}{(1 + r)^3} \leq |g'(z)| \leq |b_1| \frac{1 + Ar}{1 + Br} \frac{(1 + r)}{(1 - r)^3} \tag{15}$$

Proof. Since $f = (h(z) + \overline{g(z)})$ be an element of $S_{HS^*}(A, B)$ then we have

$$\left| \frac{g'(z)}{h'(z)} - b_1 \frac{1 - ABr^2}{1 - B^2 r^2} \right| \leq \frac{|b_1| (A - B) r}{1 - B^2 r^2} \tag{16}$$

and using Theorem 2.1, then we write

$$\left|\frac{g(z)}{h(z)} - b_1\frac{1-ABr^2}{1-B^2r^2}\right| \leq \frac{|b_1|\,(A-B)r}{1-B^2r^2} \tag{17}$$

After the simple calculations from (16) and (17) we get

$$\frac{|b_1|\,(1-Ar)}{(1-Br)} \leq \left|\frac{g'(z)}{h'(z)}\right| \leq \frac{|b_1|\,(1+Ar)}{(1+Br)} \tag{18}$$

$$\frac{|b_1|\,(1-Ar)}{(1-Br)} \leq \left|\frac{g(z)}{h(z)}\right| \leq \frac{|b_1|\,(1+Ar)}{(1+Br)} \tag{19}$$

Considering (18), (19) and Theorem 1.1 together, we obtain (14) and (15). □

Corollary 2.1. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HS^*}(A,B)$, then*

$$\frac{(1-r)^2}{(1+r)^6}\frac{(1+Br)^2 - |b_1|^2\,(1+Ar)^2}{(1+Br)^2} \leq J_f \leq \frac{(1+r)^2}{(1-r)^6}\frac{(1+Br)^2 - |b_1|^2\,(1-Ar)^2}{(1-Br)^2} \tag{20}$$

Proof. Since

$$J_f = \left|h'(z)\right|^2 - \left|g'(z)\right|^2 = \left|h'(z)\right|^2 - \left|h'(z)\right|^2|w(z)|^2 = \left|h'(z)\right|^2(1-|w(z)|^2),$$

$$|w(z)| = \left|\frac{g'(z)}{h'(z)}\right|,$$

using Theorem 2.2 and Lemma 2.1 we get (20). □

Corollary 2.2. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HS^*}(A,B)$, then*

$$\frac{1}{(B-1)^3}\Big[\frac{(B-1)(-B(A\,|b_1| + |b_1| + 2) + (3A-1)\,|b_1| + B^2 + 1)}{1+r}$$

$$-\frac{(B-1)^2(-A\,|b_1| + B + |b_1| - 1)}{(1+r)^2} - (B+1)\,|b_1|\,(A-B)\log(r+1) + (B+1)\,|b_1|\,(A-B)\log(Br+1)\Big]$$

$$\leq |f| \leq$$

$$\frac{1}{(B-1)^3}\Big[\frac{(B-1)(B(A\,|b_1| + |b_1| - 2) - 3A\,|b_1| + B^2 + |b_1| + 1)}{r-1}$$

$$+\frac{(B-1)^2((A-1)\,|b_1| + B - 1)}{(r-1)^2} - (B+1)\,|b_1|\,(A-B)\log(1-r) + (B+1)\,|b_1|\,(A-B)\log(1-Br)\Big]$$

Proof. Since

$$(\left|h'(z)\right| - \left|g'(z)\right|)\,|dz| \leq |df| \leq (\left|h'(z)\right| + \left|g'(z)\right|)\,|dz| \Rightarrow$$

$$\left|h'(z)\right|(1-|w(z)|)\,|dz| \leq |df| \leq \left|h'(z)\right|(1+|w(z)|)\,|dz|$$

Using Theorem 1.1 and Lemma 2.1 we get

$$\frac{1-r}{(1+r)^3}(1-|b_1|\frac{1+Ar}{1+Br})dr \leq |df| \leq \frac{1+r}{(1-r)^3}(1+|b_1|\frac{1-Ar}{1-Br})dr \tag{21}$$

After integrating from (21), we get the result. □

Theorem 2.3. *Let $f = (h(z) + \overline{g(z)})$ be an element of $S_{HS^*}(A,B)$, then*

$$\sum_{k=2}^{n}|b_k - b_1a_k|^2 \leq |b_1|^2\,(A-B)^2 + \sum_{k=2}^{n}|b_1Aa_k - Bb_k|^2 \tag{22}$$

Proof. Using Theorem 2.1, then we write

$$\frac{g(z)}{h(z)} \prec b_1 \frac{1+Az}{1+Bz} \Rightarrow \frac{g(z)}{h(z)} = b_1 \frac{1+A\phi(z)}{1+B\phi(z)} \Rightarrow$$

$$(g(z) - b_1 h(z)) = [b_1 A h(z) - B g(z)]\phi(z) \Rightarrow$$

$$\sum_{k=2}^{n} (b_k - b_1 a_k) z^k + \sum_{k=n+1}^{\infty} d_k z^k = [b_1(A-B)z + \sum_{k=2}^{n} (b_1 A a_k - B b_k) z^k]\phi(z) \qquad (23)$$

The equality (23) can be written in the following form

$$F(z) = G(z)\phi(z), |\phi(z)| < 1.$$

Therefore we have

$$|F(z)|^2 \le |G(z)|^2$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} \left|F(re^{i\theta})\right|^2 d\theta \le \frac{1}{2\pi}\int_0^{2\pi} \left|G(re^{i\theta})\right|^2 d\theta \qquad (24)$$

for each r $(0 < r < 1)$. Expressing (2.21) in terms of the coefficients in (2.20) we obtain the inequality

$$\sum_{k=2}^{n} |b_k - b_1 a_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} \le |b_1|^2 (A-B)^2 + \sum_{k=2}^{n} |b_1 A a_k - B b_k|^2 r^{2k} \qquad (25)$$

From letting $r \to 1$ in (25) we conclude that

$$\sum_{k=2}^{n} |b_k - b_1 a_k|^2 \le |b_1|^2 (A-B)^2 + \sum_{k=2}^{n} |b_1 A a_k - B b_k|^2$$

We note that the proof of this theorem is based on Clunie method [1].

□

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