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LINEAR RETRIAL INVENTORY SYSTEM WITH SECOND OPTIONAL SERVICE UNDER MIXED PRIORITY SERVICE

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Abstract. The present paper deals with a generalization of the homogeneous single server finite source retrial inventory system with two classes of customers - one with high priority customer and the other with low priority customer. The inventory is replenished according to an (s, Q) policy and the replenishing times are assumed to be exponentially distributed. The server provides two types of services - one with essential service and the other with a second optional service. The service times of the 1st (essential) and 2nd (optional) services are independent and exponentially distributed. The high priority customers have a mixed priority over the low priority customers. Retrial is introduced for low priority customers only. The joint probability distribution of the number of customers in the waiting hall, the number of customers in the orbit and the inventory level is obtained for the steady state case. Some important system performance measures in the steady state are derived and the long-run total expected cost rate is also derived.

Keywords: Markov process, continuous review, inventory with service time, priority customers, essential and optional service, finite populations.

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1. Introduction

Several researchers have studied the inventory systems in which demanded items are instantaneously distributed from stock (if available) to the customers. During stock out period, the demands of a customer are either not satisfied (lost sales case) or satisfied only after getting the receipt of the ordered items (backlog case). In the backlog case, either all demands (full backlog case) or only a limited number of demands (partial backlog case) are satisfied during stock out period. To know the review of these works see Çakanyildirim et al., [7], Durán et al. [9], Elango and Arivarignan[10], Goyal and Giri [12], Kalpakam and Arivarignan([16], [17]), Liu and Yang[20], Nahmias[21], Raafat[23] and Yadavalli et al.[25] and the references therein.

However, in the case of inventories maintained at service facilities, after some service is performed on the demanded items they are distributed to the customers. In such situations, the items are issued not on demanding rather it is done after a random time of service. It causes the formation of queues in front of service centres. As a result there is a need for study of both the inventory level and the queue length in the long run. Berman and Kim [2] analyzed a queueing - inventory system with Poisson arrivals, exponential

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service times and zero lead times. The authors proved that the optimal policy is never to order when the system is empty. Berman and Sapna [5] studied queueing - inventory systems with Poisson arrivals, arbitrary distribution service times and zero lead times.

Berman and Sapna [6] discussed a finite capacity system with Poisson arrivals, exponential distributed lead times and service times. The existence of a stationary optimal service policy has been proved. Berman and Kim [3] addressed an infinite capacity queueing - inventory system with Poisson arrivals, exponential distributed lead times and service times. The authors identified a replenishment policy which maximized the system profit. Berman and Kim [4] studied internet based supply chains with Poisson arrivals, exponential service times, the Erlang lead times and found that the optimal ordering policy has a monotonic threshold structure.

In all the above models, the authors assumed that after completion of the service (namely, regular service or main service or essential service), immediately the customers leave the system. But in many real life situation, all the arriving customers first require an essential service and some of them may require the secondary service provided by the same server. In queueing systems the latter type of service referred to as second optional service. The concept of the second optional service with queue has been studied by several researchers in the past. As a related work we refer [14, 15].

An important issue in the queueing-inventory system with two types of customers is the priority assignment problem. For example, in assembly manufacturing system customers with long-term supply contracts have been given high priority than the other ordinary customers. In multi-specialty hospitals patients with serious illness are given high priority than the other patients opting for routine check or else. These real life problems stimulate us to study the queueing-inventory systems with two types of customers.

Ning Zhao and Zhaotong Lian [22] analyzed a queueing-inventory system with two classes of customers. The authors have assumed the arrival of the two-types of customers form independent Poisson processes and exponential service times. It may be noted that recently Jeganathan et al. [13] studied a retrial inventory system with non-preemptive priority service. The authors have assumed the arrival of the two-types of customers form independent with Poisson arrival and exponential service times. Artalejo et al. [1] analyzed queue with retrial customers looked to have preemptive priority over the waiting room. In related bibliography [8, 11, 19], the high priority customers have either preemptive or non-preemptive priority over the low priority customers.

In this paper, we introduce the mixed priority concept in the context of a queueing - inventory models with repeated attempts, second optional service and a finite populations. Our research work on finite source inventory system is motivated by the service facility system with restricted customers for example military canteen providing service to soldiers or a company canteen serving the members of the specific working community in the company. Machine service problems within an industry is also a problem which motivated us to create the present stochastic model. The problem we consider is more relevant to the real life situation. The joint probability distribution of the number of customers in the waiting line, the number of customers in the orbit and the inventory level is obtained for the steady state case. Various measures of system performance are computed in the steady state case.

The rest of the paper is organized as follows. In the next section, the problem formulation and the notations used in this paper are described. Analysis of the model and the steady state solutions of the model are obtained in section 3. Some key system performance measures are derived in section 4. In the section 5, we derive the total expected cost rate in the steady state. The last section is meant for conclusion.

2. Mathematical modelling

In this paper, we considered the stochastic model of inventory system with service facility and the demands originated from a finite population of sources $K = N + M$, $0 < K < \infty$. Maximum inventory level is denoted by S and the inventory is replenished according to (s, Q) ordering policy. According to this policy, the reorder level is fixed as s and an order is placed when the inventory level reaches the reorder level. The ordering quantity is $Q(= S-s > s+1)$ items. The condition $S-s > s+1$ ensures that no perpetual shortage in the stock after replenishment. The lead time is assumed to be exponential with parameter $\beta(> 0)$. The arrival of high priority customer and low priority customer follow an independent Poisson processes with parameters λ_1 and λ_2 respectively. The customers (high/low) receive their service one by one and they demand single item. The items are issued to the demanding customers only after some random time due to some service (i.e.: 1st service or essential service or regular service) on it. The service time of the customer depends on the priority level of the customer. The waiting area is limited to accommodate a maximum number N of high priority customers including the one at service point. An arriving low priority customer finds the server is idle with the positive inventory level he immediately taken for service by the server. The retrial is introduced for low priority customers only. Whenever the server is busy (with high priority / low priority customer) or the inventory level is zero, an arriving primary low priority customer enters into an orbit of finite size M. The orbiting customers compete for service by sending out signals that are exponentially distributed.There are two types of retrial policy considered in the literature of queuing systems:

(1) The probability of a repeated attempt depends on the number of orbiting demands (classical retrial policy).
(2) The probability of a repeated attempt is independent of the number of orbiting demands (constant retrial policy).

In this article we consider both types of retrial policy (classical retrial policy and constant retrial policy). More explicitly, the probability of a repeated attempt during the interval $(t, t + dt)$, given that j customers are in the orbit at time t, is $\theta_j = \gamma(1 - \delta_{j0}) + j\nu$, where δ denotes the Kronecker delta. Note that the latter type of retrial policy is described as linear retrial policy. In the special case $\gamma = 0$ and $\nu > 0$ our model becomes a single server inventory model with classical retrial policy and priority service. Alternatively, when $\nu = 0$ and $\gamma > 0$ our model becomes a single server inventory model with constant retrial policy and priority service.

The 'Mixed Priority' (i.e., preemptive priority and non-preemptive priority) concept plays an important role here. The high priority customers have a mixed priority over the low priority customers. More precisely, if a high priority customer arrives during the service of a low priority customer, then with probability r interrupts this service period and force the server to start serving him, while the interrupted customer joins the orbit. It may be called as preemptive priority service. After the completion of the essential service of the high priority customer, he requires for second optional service with probability p or he leaves the system with probability $1 - p$. The additional optional service is offered only for high priority customers. During the time of low priority customer's service, with probability $1 - r$, any arriving high priority customer decides to wait in the waiting hall (i.e.: ordinary queue) and to get service after the completion of the low priority customer's service. This type of service may be called as non-preemptive priority service. The head of the customer in the waiting hall is immediately taken to service after the present low/high priority customer's service, irrespective of the number of customers in the waiting hall.

The service times of the high priority customer, low priority customer and the second optional service are independent and assumed to follow an exponential distribution with parameters μ_1, μ_2 and μ_3 respectively. Any arriving high priority customer or low priority customer, who finds his waiting line is full are considered to be lost. Various stochastic processes involved in the system are independent of each other.

2.1. **Notations:**

$$
\begin{aligned}
\mathbf{e} &: \text{ a column vector of appropriate dimension containing all ones}\\
\mathbf{0} &: \text{ Zero matrix}\\
[A]_{ij} &: \text{ entry at } (i,j)^{th} \text{ position of a matrix A}\\
\delta_{ij} &: \begin{cases} 1 & \text{if} \quad j = i \\ 0 & \text{otherwise} \end{cases}\\
\bar{\delta_{ij}} &: 1 - \delta_{ij}\\
k \in V_i^j &: k = i, i+1, \ldots j
\end{aligned}
$$

3. Analysis

Let $L(t)$, $X_1(t)$ and $X_2(t)$ respectively, denote the inventory level, the number of high priority customers in the waiting hall and the number of low priority customers in the orbit at time t.
Further, server status $Y(t)$ be defined as follows:

$$
Y(t) \quad : \begin{cases} \alpha_0, & \text{if the server is idle at time t} \\ \alpha_1, & \text{if the server is providing essential service to a high priority customer at time t} \\ \alpha_2, & \text{if the server is providing second optional service to a high priority customer at time t} \\ \alpha_3, & \text{if the server is providing essential service to a low priority customer at time t} \end{cases}
$$

From the assumptions made on the input and output processes, it can be shown that the stochastic process $I(t) = \{(L(t), Y(t), X_1(t), X_2(t)), t \geq 0\}$ is a continuous time Markov chain with state space given by $E = E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5$. where

$$
\begin{aligned}
E_1 &: \{(i_1, \alpha_0, 0, i_4) \mid i_1 = 0, 1, 2, \ldots, S, i_4 = 0, 1, 2, \ldots, M, \}\\
E_2 &: \{(0, \alpha_0, i_3, i_4) \mid i_3 = 1, 2, \ldots, N, \ i_4 = 0, 1, 2, \ldots, M, \}\\
E_3 &: \{(i_1, \alpha_1, i_3, i_4) \mid i_1 = 1, 2, \ldots, S, \ i_3 = 1, 2, \ldots, N, \ i_4 = 0, 1, 2, \ldots, M, \}\\
E_4 &: \{(i_1, \alpha_2, i_3, i_4) \mid i_1 = 0, 1, 2, \ldots, S, \ i_3 = 1, 2, \ldots, N, \ i_4 = 0, 1, 2, \ldots, M, \}\\
E_5 &: \{(i_1, \alpha_3, i_3, i_4) \mid i_1 = 1, 2, \ldots, S, \ i_3 = 0, 1, 2, \ldots, N, \ i_4 = 0, 1, 2, \ldots, M, \}
\end{aligned}
$$

Define the following ordered sets:

$$
< i_1, i_2, i_3 > \quad = \quad \begin{cases} ((i_1, \alpha_0, 0, 0), (i_1, \alpha_0, 0, 1), \ldots, (i_1, \alpha_0, 0, M)), & i_1 = 0, 1, \ldots, S; \\ ((i_1, \alpha_0, i_3, 0), (i_1, \alpha_0, i_3, 1), \ldots, (i_1, \alpha_0, i_3, M)), & i_1 = 0; i_3 = 1, \ldots, N; \\ ((i_1, \alpha_1, i_3, 0), (i_1, \alpha_1, i_3, 1), \ldots, (i_1, \alpha_1, i_3, M)), & i_1 = 1, 2, \ldots S; i_3 = 1, \ldots, N; \\ ((i_1, \alpha_2, i_3, 0), (i_1, \alpha_2, i_3, 1), \ldots, (i_1, \alpha_2, i_3, M)), & i_1 = 0, 1, \ldots, S; i_3 = 1, 2, \ldots, N; \\ ((i_1, \alpha_3, i_3, 0), (i_1, \alpha_3, i_3, 1), \ldots, (i_1, \alpha_3, i_3, M)), & i_1 = 1, 2, \ldots S; i_3 = 0, 1, \ldots, N; \end{cases}
$$

$$\ll i_1, i_2 \gg \;=\; \begin{cases} < i_1, \alpha_0, 0 >, & i_1 = 0, 1, \ldots, S; \\ < i_1, \alpha_0, 1 >, < i_1, \alpha_0, 2 >, \ldots, < i_1, \alpha_0, N >, & i_1 = 0; \\ < i_1, \alpha_1, 1 >, < i_1, \alpha_1, 2 >, \ldots, < i_1, \alpha_1, N >, & i_1 = 1, 2, \ldots S; \\ < i_1, \alpha_2, 1 >, < i_1, \alpha_2, 2 >, \ldots, < i_1, \alpha_2, N >, & i_1 = 0, 1, \ldots, S; \\ < i_1, \alpha_3, 0 >, < i_1, \alpha_, 1 >, \ldots, < i_1, \alpha_3, N >, & i_1 = 1, 2, \ldots S; \end{cases}$$

$$\lll i_1 \ggg \;=\; \begin{cases} \ll i_1, \alpha_0 \gg, & i_1 = 0, 1, \ldots, S; \\ \ll i_1, \alpha_0 \gg, & i_1 = 0; \\ \ll i_1, \alpha_1 \gg, & i_1 = 1, 2, \ldots S; \\ \ll i_1, \alpha_2 \gg,, & i_1 = 0, 1, \ldots, S; \\ \ll i_1, \alpha_3 \gg, & i_1 = 1, 2, \ldots S; \end{cases}$$

Then the state space can be ordered as $(\lll 0 \ggg, \lll 1 \ggg, \ldots, \lll S \ggg)$.

By ordering the state space $(\lll 0 \ggg, \lll 1 \ggg, \ldots, \lll S \ggg)$, the infinitesimal generator Θ can be conveniently written in a block partitioned matrix with entries

$$A = \begin{array}{c} \\ \lll 0 \ggg \\ \lll 1 \ggg \\ \lll 2 \ggg \\ \vdots \\ \lll S-1 \ggg \\ \lll S \ggg \end{array} \begin{array}{c} \begin{array}{cccccc} \lll 0 \ggg & \lll 1 \ggg & \lll 2 \ggg & \cdots & \lll S-1 \ggg & \lll S \ggg \end{array} \\ \begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} & \cdots & A_{0,S-1} & A_{0,S} \\ A_{1,0} & A_{1,1} & A_{1,2} & \cdots & A_{1,S-1} & A_{1,S} \\ A_{2,0} & A_{2,1} & A_{2,2} & \cdots & A_{2,S-1} & A_{2,S} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{S-1,0} & A_{S-1,1} & A_{S-1,2} & \cdots & A_{S-1,S-1} & A_{S-1,S} \\ A_{S,0} & A_{S,1} & A_{S,2} & \cdots & A_{S,S-1} & A_{S,S} \end{pmatrix} \end{array}$$

More explicitly,

Due to the assumptions made on the demand and replenishment processes, we note that

$$A_{i_1,j_1} = \mathbf{0}, \quad \text{for} \quad j_1 \neq i_1, i_1 - 1, i_1 + Q.$$

We first consider the case A_{i_1,i_1+Q}. This will occur only when the inventory level is replenished. First we consider the inventory level is zero, that is $A_{0,Q}$. For this

Case (i) : When there is no customer in both the waiting hall and server is idle, at the time of replenishment the state of the system changes from $(0, \alpha_0, 0, i_4)$ to $(Q, \alpha_0, 0, i_4)$, with intensity of transition β. The sub matrix of the transition rates from $\ll 0, \alpha_0 \gg$ to $\ll Q, \alpha_0 \gg$, is given by

$$[C_{00}^{(0)}]_{i_3 j_3} \;=\; \begin{cases} E_0, & j_3 = 0, \qquad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

where $[E_0]_{i_4,i_4} = \beta I_{(M+1)}$

Case (ii): When there is a customer in the waiting hall and server is idle, at the time of replenishment takes the system state from $(0, \alpha_0, i_3, i_4)$ to (Q, α_1, i_3, i_4), $i_3 = 1, 2, \ldots, N$, $i_4 = 0, 1, 2, \ldots, M$. The sub matrix of the transition rates from $\ll 0, \alpha_0 \gg$ to $\ll Q, \alpha_1 \gg$, is given by $C_{01}^{(0)}$

$$[C_{01}^{(0)}]_{i_3 j_3} \;=\; \begin{cases} E_0, & j_3 = i_3, \qquad i_3 = 1, 2, \ldots, N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

Case (iii): When the server is providing second optional service to a high priority customer, the replenishment takes the system state from $(0, \alpha_2, i_3, i_4)$ to

(Q, α_2, i_3, i_4), $i_3 = 1, 2, \dots, N$, $i_4 = 0, 1, 2, \dots, M$. The sub matrix of this transition rate $\ll 0, \alpha_2, \gg$ to $\ll Q, \alpha_2, \gg$, is given by

$$[C_{22}^{(0)}]_{i_3 j_3} = \begin{cases} E_0, & j_3 = i_3, \quad i_3 \in V_1^N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

Hence,

$$[A_{0,Q}]_{i_2 j_2} = \begin{cases} C_{00}^{(0)}, & j_2 = i_2, \quad i_2 = \alpha_0, \\ C_{01}^{(0)}, & j_2 = \alpha_1, \quad i_2 = \alpha_0, \\ C_{22}^{(0)}, & j_2 = i_2, \quad i_2 = \alpha_2, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

We denote $A_{0,Q}$ as C_0.

We now consider the case when the inventory level lies between one to s. We note that for this case, only the inventory level changes from i_1 to $i_1 + Q$. The other system states does not change. Hence $A_{i_1, i_1+Q} = \beta I_{(3N+2)(M+1)}$. More clearly,

$$[A_{i_1, i_1+Q}]_{i_2 j_2} = \begin{cases} C_{00}^{(3)}, & j_2 = i_2, \quad i_2 = \alpha_0, \\ C_{11}^{(3)}, & j_2 = i_2, \quad i_2 = \alpha_1, \\ C_{22}^{(3)}, & j_2 = i_2, \quad i_2 = \alpha_2, \\ C_{33}^{(3)}, & j_2 = i_2, \quad i_2 = \alpha_3, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

where

$$C_{00}^{(3)} = \beta I_{(M+1)}, C_{11}^{(3)} = \beta I_{N(M+1)}, C_{22}^{(3)} = \beta I_{N(M+1)}, C_{33}^{(3)} = \beta I_{(N+1)(M+1)},$$

We denote A_{i_1, i_1+Q} is denoted by C.

Next, we consider the case $A_{i_1, i_1-1}, i_1 = 1, 2, \dots, S$. This will occur only when either when the essential service completion of the high priority customer or the low priority customer. For this, we have the following cases occur:

Case(i): When the inventory level is one and the server is providing essential service to a high priority customer

- as soon as the essential service of a high priority customer is completed, then with probability p the customer may opt to leave the system, in which case both the inventory level and customer level in the waiting hall decrease by one and the server becomes idle. The state of the system change from $(1, \alpha_1, i_3, i_4)$ to $(0, \alpha_0, i_3 - 1, i_4)$, $i_3 = 1, 2, \dots, N$, $i_4 = 0, 1, 2, \dots, M$, with intensity of this transition $p\mu_1$, or with probability $(1 - p)$ he may ask for second optional service, in which case his optional service will immediately commence. The state of the system change from $(1, \alpha_1, i_3, i_4)$ to $(0, \alpha_2, i_3, i_4)$, $i_3 = 1, 2, \dots, N$, $i_4 = 0, 1, 2, \dots, M$, with intensity of this transition $(1 - p)\mu_1$. Form this argument, we have

$$[A_{10}^{(0)}]_{i_3 j_3} = \begin{cases} H_3, & j_3 = i_3 - 1, \quad i_3 \in V_1^N \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_{12}^{(0)}]_{i_3 j_3} = \begin{cases} H_4, & j_3 = i_3, \quad i_3 \in V_1^N, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where,

$$H_3 = p\mu_1 I_{(M+1)}, \; H_4 = (1-p)\mu_1 I_{(M+1)}$$

Case (ii): When the inventory level is one and the server is providing essential service to a low priority customer

- after the low priority customer served in the service station, the inventory level decreases by one and immediately the server becomes idle with intensity of transition is given by μ_3. The sub matrix of the transition rates from $(1, \alpha_3, i_3, i_4)$ to $(0, \alpha_0, i_3, i_4)$, $i_3 = 0, 1, 2, \ldots, N$, $i_4 = 0, 1, 2, \ldots, M$ is given by,

$$[A_{30}^{(0)}]_{i_3 j_3} = \begin{cases} C_3, & j_3 = i_3, \quad i_3 \in V_0^N, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

where,

$$C_3 = \mu_3 I_{(M+1)}.$$

We denote $A_{1,0}$ as A_0.

Case (i): When the inventory level is more than one and the server is providing essential service to a high priority customer

- if the number of customers in the waiting hall is one, at the time of service completion of a high priority customer both the inventory level and the customer level in the waiting hall decrease by one, and server becomes idle (i.e.: if the customer may opt to leave the system without getting addition service) with intensity of passage for this transition is given by $p\mu_1$. Suppose, the serviced customer wants to require second optional service, in which case his service will immediately commence with intensity of this transition $(1-p)\mu_1$. If there are more than one customer in the waiting hall, then after a service completion of a customer server remains busy (i.e.: if the customer may opt to leave the system without getting addition service) servicing (essential service) the high priority customer.

Using this argument, we have constructed the following matrices

$$[A_{10}^{(1)}]_{i_3 j_3} = \begin{cases} H_3, & j_3 = i_3 - 1, \quad i_3 = 1, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_{11}^{(1)}]_{i_3 j_3} = \begin{cases} H_3, & j_3 = i_3 - 1, \quad i_3 \in V_2^N, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

$$[A_{12}^{(1)}]_{i_3 j_3} = \begin{cases} H_4, & j_3 = i_3, \quad i_3 \in V_1^N, \\ \mathbf{0}, & \text{otherwise,} \end{cases}$$

Case (ii): When the inventory level is more than one and the server is providing essential service to a low priority customer

- if there is no customer in the waiting hall, then at the time of service completion of a low priority customer the inventory level decreases by one and server becomes idle with intensity of transition is given by μ_3. The sub matrix of the transition rates from $(i_1, \alpha_3, 0, i_4)$ to $(i_1 - 1, \alpha_0, 0, i_4)$, $i_1 = 2, \dots, S$, $i_4 = 0, 1, 2, \dots, M$ is given by, $B_{30}^{(1)}$.
- If there is a customer (high priority) in the waiting hall, then at the time of service completion of a low priority customer the inventory level decreases by one, in which case the customer (high priority) at the head of the waiting hall is taken up for his first essential service. The system state from $(i_1, \alpha_3, i_3, i_4)$ to $(i_1 - 1, \alpha_1, i_3, i_4)$, $i_1 = 2, \dots, S$, $i_3 = 1, 2, \dots, N$, $i_4 = 0, 1, 2, \dots, M$ is given by, $B_{31}^{(1)}$. Using this argument, we have constructed the following matrices

$$[A_{30}^{(1)}]_{i_3 j_3} = \begin{cases} C_3, & j_3 = 0, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[A_{31}^{(1)}]_{i_3 j_3} = \begin{cases} C_3, & j_3 = i_3, \quad i_3 \in V_1^N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

Hence the matrix $A_{i_1, i_1 - 1} (i_1 = 2, 3, \dots, S)$ is given by

$$[A_{i_1, i_1-1}]_{i_2 j_2} = \begin{cases} A_{10}^{(1)}, & j_2 = \alpha_0, \quad i_2 = \alpha_1, \\ A_{11}^{(1)}, & j_2 = i_2, \quad i_2 = \alpha_1, \\ A_{12}^{(1)}, & j_2 = \alpha_2, \quad i_2 = \alpha_1, \\ A_{30}^{(1)}, & j_2 = \alpha_0, \quad i_2 = \alpha_3, \\ A_{31}^{(1)}, & j_2 = \alpha_1, \quad i_2 = \alpha_3, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

We will denote $A_{i_1, i_1 - 1} (i_1 = 2, 3, \dots, S)$, as A_1.
Finally, we consider the case $A_{i_1, i_1}, i_1 = 0, 1, \dots, S$. Here due to each one of the following mutually exclusive cases, a transition results:

- an arrival of a high priority customer may occur
- an arrival of a low priority customer may occur
- a retrial customer (low priority customer) enter into the service station may occur
- a second optional service of the high customer may be completed

When the inventory level is zero and server is idle, we have the following three state changes may arise:

Case(i): • an arrival of a high priority customer increases the number of customer waiting in the waiting hall increases by one and the state of the arrival process moves from $(0, \alpha_0, i_3, i_4)$ to $(0, \alpha_0, i_3 + 1, i_4), i_3 = 0, 1, \dots, N - 1; i_4 = 0, 1, \dots, M$, with intensity of transition $(K - (i_3 + i_4))\lambda_1$. The sub matrix of this transition rate is given by $\ll 0, \alpha_0 \gg$ to $\ll 0, \alpha_0 \gg$ is $(K - (i_3 + i_4))\lambda_1$ and is denoted by C_1.

Case(ii): • an arrival of a primary low priority customer increases the number of customer in the orbit increases by one and the state of the arrival process moves from $(0, \alpha_0, i_3, i_4)$ to $(0, \alpha_0, i_3, i_4 + 1), i_3 = 0, 1, \dots, N; i_4 = 0, 1, \dots, M - 1$, with intensity of transition $(K - (i_3 + i_4))\lambda_2$.

When the inventory level is zero and the server is providing second optional service to a high priority customer, we have the following state changes may arise:

Case(i): • an arrival of a high priority customer increases the number of customer waiting in the waiting hall increases by one and the state of the arrival process moves from $(0, \alpha_2, i_3, i_4)$ to $(0, \alpha_2, i_3+1, i_4), i_3 = 1, \dots, N-1; i_4 = 0, 1, \dots, M$, with intensity of transition $(K - (i_3 + i_4))\lambda_1$.

Case(ii): • an arrival of a primary low priority customer increases the number of customer in the orbit increases by one and the state of the arrival process moves from $(0, \alpha_2, i_3, i_4)$ to $(0, \alpha_2, i_3, i_4+1), i_3 = 0, 1, \dots, N; i_4 = 0, 1, \dots, M-1$, with intensity of transition $(K - (i_3 + i_4))\lambda_2$.

Case(iv): • the completion of second optional service for a high priority customer makes a transition from $(0, \alpha_2, i_3, i_4)$ to $(0, \alpha_0, i_3 - 1, i_4), i_3 = 1, \dots, N; i_4 = 0, 1, \dots, M$, with intensity of transition μ_2.

The transition rates for any other transitions not considered above, when the inventory level is zero, are zero. The intensity of passage in the state $(0, i_2, i_3, i_4)$ is given by

$$-\sum_{(0,i_2,i_3,i_4)\neq(0,j_2,j_3,j_4)} a\left((0, i_2, i_3, i_4); (0, j_2, j_3, j_4)\right).$$

Using the above arguments, we have constructed the following matrices: $C_3 = \mu_2 I_{(M+1)}$

$$[D_0]_{i_4j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1 + \lambda_2\bar{\delta}_{i_4M}) + \beta), & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[D]_{i_4j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))\lambda_2\bar{\delta}_{i_4M} + \beta), & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[D_1]_{i_4j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1 + \lambda_2\bar{\delta}_{i_4M}) + \beta + \mu_2), & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[D_2]_{i_4j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -(\lambda_2\bar{\delta}_{i_4M} + \mu_3 + \beta), & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & \text{otherwise}, \end{cases}$$

Combining theses matrices in suitable form, we get

$$[B_{00}^{(0)}]_{i_3j_3} = \begin{cases} C_1, & j_3 = i_3+1, \quad i_3 \in V_0^{N-1}, \\ D_0, & j_3 = i_3, \quad i_3 \in V_0^{N-1}, \\ D, & j_3 = i_3, \quad i_3 = N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{20}^{(0)}]_{i_3j_3} = \begin{cases} C_2, & j_3 = i_3-1, \quad i_3 \in V_1^N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{22}^{(0)}]_{i_3j_3} = \begin{cases} C_1, & j_3 = i_3+1, \quad i_3 \in V_1^N, \\ D_1, & j_3 = i_3, \quad i_3 \in V_1^{N-1}, \\ D_2, & j_3 = i_3, \quad i_3 = N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

Hence the matrix A_{00} is given by

$$[A_{00}]_{i_2 j_2} = \begin{cases} B_{00}^{(0)}, & j_2 = 0, \quad i_2 = \alpha_0, \\ B_{20}^{(0)}, & j_2 = \alpha_0, \quad i_2 = \alpha_2, \\ B_{22}^{(0)}, & j_2 = \alpha_2, \quad i_2 = \alpha_2, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

and A_{00} is denoted by B_0. Arguments similar to above yields: For $i_1 = 1, 2, \ldots, s$,

$$[A_{i_1,i_1}]_{i_2 j_2} = \begin{cases} B_{00}^{(1)}, & j_2 = \alpha_0, \quad i_2 = \alpha_0, \\ B_{01}^{(1)}, & j_2 = \alpha_1, \quad i_2 = \alpha_0, \\ B_{11}^{(1)}, & j_2 = \alpha_1, \quad i_2 = \alpha_1, \\ B_{03}^{(1)}, & j_2 = \alpha_3, \quad i_2 = \alpha_0, \\ B_{20}^{(1)}, & j_2 = \alpha_0, \quad i_2 = \alpha_2, \\ B_{21}^{(1)}, & j_2 = \alpha_1, \quad i_2 = \alpha_2, \\ B_{22}^{(1)}, & j_2 = \alpha_2, \quad i_2 = \alpha_2, \\ B_{31}^{(1)}, & j_2 = \alpha_1, \quad i_2 = \alpha_3, \\ B_{33}^{(1)}, & j_2 = \alpha_3, \quad i_2 = \alpha_3, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

with

$$[B_{00}^{(1)}]_{i_3 j_3} = \begin{cases} G, & j_3 = 0, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[G]_{i_4 j_4} = \begin{cases} -((K - (i_3 + i_4))(\lambda_1 + \lambda_2) + \beta + \theta_{i_4}), & j_4 = 0, \quad i_4 \in V_0^M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[B_{01}^{(1)}]_{i_3 j_3} = \begin{cases} C_1, & j_3 = 1, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{03}^{(1)}]_{i_3 j_3} = \begin{cases} F, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[F]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4, \quad i_4 \in V_0^M, \\ \theta_{i_4}, & j_4 = i_4 - 1, \quad i_4 \in V_1^M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[B_{11}^{(1)}]_{i_3 j_3} = \begin{cases} C_1, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\ G_1, & j_3 = i_3, \quad i_3 \in V_1^{N-1}, \\ G_2, & j_3 = i_3, \quad i_3 = N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[G_1]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1 + \lambda_2 \bar{\delta}_{i_4 M}) + \mu_1 + \beta), & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[G_2]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))\lambda_2 \bar{\delta}_{i_4 M} + \mu_1 + \beta), & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[B_{20}^{(1)}]_{i_3 j_3} = \begin{cases} C_2, & j_3 = 0, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{21}^{(1)}]_{i_3 j_3} = \begin{cases} C_2 & j_3 = i_3 - 1, \quad i_3 \in V_2^N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{22}^{(1)}]_{i_3 j_3} = \begin{cases} C_1, & j_3 = i_3 + 1, \quad i_3 \in V_1^{N-1}, \\ G_3, & j_3 = i_3, \quad i_3 \in V_1^{N-1}, \\ G_4, & j_3 = i_3, \quad i_3 = N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[G_3]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1 + \lambda_2 \bar{\delta}_{i_4 M}) + \mu_2 + \beta), & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[G_4]_{i_4 j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4 + 1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))\lambda_2 \bar{\delta}_{i_4 M} + \mu_2 + \beta), & j_4 = i_4, \quad i_4 \in V_0^M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[B_{31}^{(1)}]_{i_3 j_3} = \begin{cases} F_1, & j_3 = 1, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[F_1]_{i_4 j_4} = \begin{cases} r(K-(i_3+i_4))\lambda_1, & j_4 = i_4 + 1, \quad i_4 \in V_0^{M-2}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[B_{33}^{(1)}]_{i_3j_3} = \begin{cases} F_0, & j_3 = 1, \quad i_3 = 0, \\ C_1, & j_3 = i_3, \quad i_3 \in V_1^{N-1}, \\ H, & j_3 = i_3, \quad i_3 = 0, \\ H_1, & j_3 = i_3, \quad i_3 \in V_1^{N-1}, \\ H_2, & j_3 = i_3, \quad i_3 = N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[F_0]_{i_4j_4} = \begin{cases} (1-r)(K-(i_3+i_4))\lambda_1, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[H]_{i_4j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1+\lambda_2)+\mu_3+\beta), & j_4 = i_4, \quad i_4 \in V_0^{M-1}, \\ -((1-r)(K-(i_3+i_4))\lambda_1+\mu_3+\beta), & j_4 = i_4, \quad i_4 = M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[H_1]_{i_4j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1+\lambda_2\bar{\delta}_{i_4M})+\mu_3+\beta), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[H_2]_{i_4j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))\lambda_2\bar{\delta}_{i_4M}+\mu_3+\beta), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

For $i_1 = s+1, s+2, \dots, S$,

$$[A_{i_1,i_1}]_{i_2j_2} = \begin{cases} B_{00}^{(2)}, & j_2 = \alpha_0, \quad i_2 = \alpha_0, \\ B_{01}^{(2)}, & j_2 = \alpha_1, \quad i_2 = \alpha_0, \\ B_{11}^{(2)}, & j_2 = \alpha_1, \quad i_2 = \alpha_1, \\ B_{03}^{(2)}, & j_2 = \alpha_3, \quad i_2 = \alpha_0, \\ B_{20}^{(2)}, & j_2 = \alpha_0, \quad i_2 = \alpha_2, \\ B_{21}^{(2)}, & j_2 = \alpha_1, \quad i_2 = \alpha_2, \\ B_{22}^{(2)}, & j_2 = \alpha_2, \quad i_2 = \alpha_2, \\ B_{31}^{(2)}, & j_2 = \alpha_1, \quad i_2 = \alpha_3, \\ B_{33}^{(2)}, & j_2 = \alpha_3, \quad i_2 = \alpha_3, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{00}^{(2)}]_{i_3j_3} = \begin{cases} L, & j_3 = i_3, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[L]_{i_4j_4} = \begin{cases} -((K-(i_3+i_4))(\lambda_1+\lambda_2)+\theta_{i_4}, & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[B_{01}^{(2)}]_{i_3 j_3} = \begin{cases} C_1, & j_3 = 1, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{03}^{(2)}]_{i_3 j_3} = \begin{cases} F, & j_3 = 0, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{11}^{(2)}]_{i_3 j_3} = \begin{cases} C_1, & j_3 = i_3 + 1, & i_3 \in V_1^{N-1}, \\ L_1, & j_3 = i_3, & i_3 \in V_1^{N-1}, \\ L_2, & j_3 = i_3, & i_3 = N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[L_1]_{i_4 j_4} = \begin{cases} (K - (i_3 + i_4))\lambda_2, & j_4 = i_4 + 1, & i_4 \in V_0^{M-2}, \\ -((K - (i_3 + i_4))(\lambda_1 + \lambda_2 \bar{\delta}_{i_4 M}) + \mu_1), & j_4 = i_4, & i_4 \in V_0^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[L_2]_{i_4 j_4} = \begin{cases} (K - (i_3 + i_4))\lambda_2, & j_4 = i_4 + 1, & i_4 \in V_0^{M-1}, \\ -((K - (i_3 + i_4))\lambda_2 \bar{\delta}_{i_4 M} + \mu_1), & j_4 = i_4, & i_4 \in V_0^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[B_{20}^{(2)}]_{i_3 j_3} = \begin{cases} C_2, & j_3 = 0, \quad i_3 = 1, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{21}^{(2)}]_{i_3 j_3} = \begin{cases} C_2, & j_3 = i_3 - 1, \quad i_3 \in V_2^{N}, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{22}^{(2)}]_{i_3 j_3} = \begin{cases} C_1, & j_3 = i_3 + 1, & i_3 \in V_1^{N-1}, \\ K_1, & j_3 = i_3, & i_3 \in V_1^{N-1}, \\ K_2, & j_3 = i_3, & i_3 = N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[K_1]_{i_4 j_4} = \begin{cases} (K - (i_3 + i_4))\lambda_2, & j_4 = i_4 + 1, & i_4 \in V_0^{M-1}, \\ -((K - (i_3 + i_4))(\lambda_1 + \lambda_2 \bar{\delta}_{i_4 M}) + \mu_2), & j_4 = i_4, & i_4 \in V_0^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[K_2]_{i_4 j_4} = \begin{cases} (K - (i_3 + i_4))\lambda_2, & j_4 = i_4 + 1, & i_4 \in V_0^{M-1}, \\ -((K - (i_3 + i_4))\lambda_2 \bar{\delta}_{i_4 M} + \mu_2), & j_4 = i_4, & i_4 \in V_0^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[B_{31}^{(2)}]_{i_3 j_3} = \begin{cases} F_1, & j_3 = 1, \quad i_3 = 0, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[B_{33}^{(2)}]_{i_3 j_3} = \begin{cases} F_0, & j_3 = 1, & i_3 = 0, \\ C_1, & j_3 = i_3 + 1, & i_3 \in V_1^{N-1}, \\ U, & j_3 = i_3, & i_3 = 0, \\ U_1, & j_3 = i_3, & i_3 \in V_1^{N-1}, \\ V, & j_3 = i_3, & i_3 = N, \\ \mathbf{0}, & \text{otherwise}, \end{cases}$$

$$[U]_{i_4j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_1^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1+\lambda_2)+\mu_3), & j_4 = i_4, \quad i_4 \in V_0^{M-1}, \\ -((1-r)(K-(i_3+i_4))\lambda_1+\mu_3), & j_4 = i_4, \quad i_4 = M, \\ 0, & \text{otherwise}, \end{cases}$$

$$[U_1]_{i_4j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_1^{M-1}, \\ -((K-(i_3+i_4))(\lambda_1+\lambda_2\bar{\delta}_{i_4M})+\mu_3), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

$$[V]_{i_4j_4} = \begin{cases} (K-(i_3+i_4))\lambda_2, & j_4 = i_4+1, \quad i_4 \in V_0^{M-1}, \\ -((K-(i_3+i_4))\lambda_2\bar{\delta}_{i_4M}+\mu_3), & j_4 = i_4, \quad i_4 \in V_0^{M}, \\ 0, & \text{otherwise}, \end{cases}$$

We denote $A_{i_1,i_1}, i_1 = 1,2,\ldots,s$ as B_1 and $A_{i_1,i_1}, i_1 = s+1, s+2, \ldots, S$ as B_2. Hence the matrix A can be written in the following form

$$[A]_{i_1j_1} = \begin{cases} A_0, & j_1 = i_1-1, \quad i_1 = 1, \\ A_1, & j_1 = i_1-1, \quad i_1 \in V_2^S, \\ C, & j_1 = i_1+Q, \quad i_1 \in V_1^s, \\ C_0, & j_1 = i_1+Q, \quad i_1 = 0, \\ B_0, & j_1 = i_1, \quad i_1 = 0, \\ B_1, & j_1 = i_1, \quad i_1 \in V_1^s, \\ B_2, & j_1 = i_1, \quad i_1 \in V_{s+1}^S, \\ \mathbf{0}, & \text{otherwise}. \end{cases}$$

It can be noted that the matrices B_1, B_2, A_1 and C are square matrices of size $(3N+2)(M+1)$. A_0 and C_0 are matrices of size $(3N+2)(M+2)\times(2N+1)(M+1)$ and $(2N+1)(M+1)\times(3N+2)(M+1)$ respectively. B_0 is the square matrix of order $(2N+1)(M+1)$. $C_{00}^{(0)}$, $C_{00}^{(3)}$, $B_{00}^{(1)}$, $B_{00}^{(2)}$, E_0, H_3, H_4, C_3, C_1, D_0, D, C_2, D_1, D_2, G, G_1, G_2, G_3, G_4, F, F_1, F_0, H_1, H, H_2, L, L_1, L_2, K_1, K_2, U, U_1 and V are square matrices of size $(M{+}1)$. $C_{01}^{(0)}$, $C_{22}^{(0)}$, $C_{11}^{(3)}$, $C_{22}^{(3)}$, $A_{11}^{(1)}$, $A_{12}^{(1)}$, $A_{12}^{(0)}$, $B_{22}^{(0)}$,$B_{11}^{(1)}$, $B_{21}^{(1)}$, $B_{22}^{(1)}$, $B_{11}^{(2)}$, $B_{21}^{(2)}$ and $B_{22}^{(2)}$ are square matrices of size $N(M+1)$. $C_{33}^{(3)}$, $A_{30}^{(0)}$, $B_{00}^{(0)}$, $B_{33}^{(1)}$ and $B_{33}^{(2)}$ are square matrices of size $(N+1)(M+1)$. $A_{10}^{(1)}$, $B_{20}^{(1)}$ and $B_{20}^{(2)}$ are matrices of size $N(M+1)\times(M+1)$. $A_{31}^{(1)}$, $B_{31}^{(1)}$ and $B_{31}^{(2)}$ are matrices of size $(N+1)(M+1)\times N(M+1)$. $A_{10}^{(0)}$ and $B_{20}^{(0)}$ are matrices of size $N(M+1)\times(N+1)(M+1)$. $B_{01}^{(1)}$ and $B_{01}^{(2)}$ are matrices of size $(M+1)\times N(M+1)$. $B_{03}^{(1)}$ and $B_{03}^{(2)}$ are matrices of size $(M+1)\times(N+1)(M+1)$. $A_{30}^{(1)}$ is the matrix of size $(N+1)(M+1)\times(M+1)$.

3.1. **Steady state analysis.** It can be seen from the structure of A that the homogeneous Markov process $\{(L(t), Y(t), X_1(t), X_2(t)) : t \geq 0\}$ on the finite space E is irreducible, aperiodic and persistent non-null. Hence the limiting distribution

$$\phi^{(i_1,i_2,i_3,i_4)} = \lim_{t\to\infty} Pr[L(t)=i_1, Y(t)=i_2, X_1(t)=i_3, X_2(t)=i_4 | L(0), Y(0), X_1(0), X_2(0)] \text{ exists.}$$

Let $\mathbf{\Phi} = (\mathbf{\Phi}^{(0)}, \mathbf{\Phi^{(1)}}, \ldots, \mathbf{\Phi^{(S)}})$ and partitioning the vector, $\mathbf{\Phi^{(i_1)}}$ as follows:

$$\begin{aligned}
\mathbf{\Phi^{(0)}} &= (\mathbf{\Phi^{(0,\alpha_0)}}, \\
\mathbf{\Phi^{(i_1)}} &= (\mathbf{\Phi^{(i_1,\alpha_0)}}, \quad i_1 = 0,1,2,3,\ldots,S; \\
\mathbf{\Phi^{(i_1)}} &= (\mathbf{\Phi^{(i_1,\alpha_1)}}, \quad i_1 = 1,2,3,\ldots,S; \\
\mathbf{\Phi^{(i_1)}} &= (\mathbf{\Phi^{(i_1,\alpha_2)}}, \quad i_1 = 0,1,2,3,\ldots,S; \\
\mathbf{\Phi^{(i_1)}} &= (\mathbf{\Phi^{(i_1,\alpha_3)}}, \quad i_1 = 1,2,3,\ldots,S;
\end{aligned}$$

which is partitioned as follows:

$$\begin{aligned}
\mathbf{\Phi^{(i_1,\alpha_0)}} &= (\Phi^{(i_1,\alpha_0,0)}), & i_1 = 0,1,2,3,\ldots,S; \\
\mathbf{\Phi^{(i_1,\alpha_0)}} &= (\Phi^{(i_1,\alpha_0,1)}, \Phi^{(i_1,\alpha_0,2)}, \ldots, \Phi^{(i_1,\alpha_0,N)}), & i_1 = 0; \\
\mathbf{\Phi^{(i_1,\alpha_1)}} &= (\Phi^{(i_1,\alpha_1,1)}, \Phi^{(i_1,\alpha_1,2)}, \ldots, \Phi^{(i_1,\alpha_1,N)}), & i_1 = 1,2,3,\ldots,S; \\
\mathbf{\Phi^{(i_1,\alpha_2)}} &= (\Phi^{(i_1,\alpha_2,1)}, \Phi^{(i_1,\alpha_2,2)}, \ldots, \Phi^{(i_1,\alpha_2,N)}), & i_1 = 0,1,2,3,\ldots,S; \\
\mathbf{\Phi^{(i_1,\alpha_3)}} &= (\Phi^{(i_1,\alpha_3,0)}, \Phi^{(i_1,\alpha_3,1)}, \ldots, \Phi^{(i_1,\alpha_3,N)}), & i_1 = 1,2,3,\ldots,S;
\end{aligned}$$

Further the above vectors also partitioned as follows:

$$\begin{aligned}
\mathbf{\Phi^{(i_1,\alpha_0,0)}} &= (\phi^{(i_1,\alpha_0,0,0)}, \phi^{(i_1,\alpha_0,0,1)}, \ldots, \phi^{(i_1,\alpha_0,0,M)}), & i_1 = 0,1,2,3,\ldots,S; \\
\mathbf{\Phi^{(0,\alpha_0,i_3)}} &= (\phi^{(0,\alpha_0,i_3,0)}, \phi^{(0,\alpha_0,i_3,1)}, \ldots, \phi^{(0,\alpha_0,i_3,M)}), & i_3 = 1,2,3,\ldots,N; \\
\mathbf{\Phi^{(i_1,\alpha_1,i_3)}} &= (\phi^{(i_1,\alpha_1,i_3,0)}, \phi^{(i_1,\alpha_1,i_3,1)}, \ldots, \phi^{(i_1,\alpha_1,i_3,M)}), & i_1 = 1,2,3,\ldots,S;\ i_3 = 1,2,3,\ldots,N; \\
\phi^{(\mathbf{i_1,\alpha_2,i_3})} &= (\phi^{(i_1,\alpha_2,i_3,0)}, \phi^{(i_1,\alpha_2,i_3,1)}, \ldots, \phi^{(i_1,\alpha_2,i_3,M)}), & i_1 = 0,1,2,3,\ldots,S;\ i_3 = 1,2,3,\ldots,N; \\
\phi^{(\mathbf{i_1,\alpha_3,i_3})} &= (\phi^{(i_1,\alpha_3,i_3,0)}, \phi^{(i_1,\alpha_3,i_3,1)}, \ldots, \phi^{(i_1,\alpha_3,i_3,M)}), & i_1 = 1,2,3,\ldots,S;\ i_3 = 0,1,2,3,\ldots,M;
\end{aligned}$$

Then the steady state probability $\mathbf{\Phi}$ satisfies

$$\mathbf{\Phi}\Theta = \mathbf{0} \quad \text{and} \tag{1}$$

$$\sum\sum\sum_{(i_1,i_2,i_3,i_4)} \phi^{(i_1,i_2,i_3,i_4)} = 1. \tag{2}$$

The equation (1) yields the following set of equations:

$$\begin{aligned}
\mathbf{\Phi^{i_1}} B_0 + \mathbf{\Phi^{i_1+1}} A_0 &= \mathbf{0}, & i_1 = 0, \\
\mathbf{\Phi^{i_1}} B_1 + \mathbf{\Phi^{i_1+1}} A_1 &= \mathbf{0}, & i_1 = 1,2,\ldots,s, \\
\mathbf{\Phi^{i_1}} B_2 + \mathbf{\Phi^{i_1+1}} A_1 + \mathbf{\Phi^{(i_1-1-Q)}} C_0 &= \mathbf{0}, & i_1 = s+1,\ldots,Q-1, \\
\mathbf{\Phi^{0}} C_0 + \mathbf{\Phi^{i_1}} B_2 + \mathbf{\Phi^{(i_1+1)}} A_1 &= \mathbf{0}, & i_1 = Q \qquad (*) \\
\mathbf{\Phi^{i_1-Q}} C + \mathbf{\Phi^{i_1}} B_2 + \mathbf{\Phi^{(i_1+1)}} A_1 &= \mathbf{0}, & i_1 = Q+1,\ldots,S-1 \\
\mathbf{\Phi^{i_1-Q}} C + \mathbf{\Phi^{i_1}} B_2 &= \mathbf{0}, & i_1 = S.
\end{aligned}$$

The steady state probability distribution $\mathbf{\Phi^{i_1}}, i_1 = 0,1,2,\ldots,S$, can be obtained using the following algorithm.

1. : Solve the following system of equations to find the value of $\mathbf{\Phi^{Q}}$

$$\mathbf{\Phi^{(Q)}} \Big[(-1)^Q (A_1 {B_2}^{-1})^{(Q-(s+1))} (A_1 {B_1}^{-1})^s (A_0 B_0^{-1}) C_0 + B_2 + (-1)^Q \sum_{j=0}^{s-1} (A_1 B_2^{-1})^{(2(s-1)-j)} (A_1 B_1^{-1})^{(j+1)} (C B_2^{-1}) A_1 \Big] = \mathbf{0}$$

and

$$\mathbf{\Phi^{(Q)}}\Big[(-1)^{Q}(A_1B_2^{-1})^{(Q-(s+1))}(A_1B_1^{-1})^{s}(A_0B_0^{-1})+$$
$$\sum_{i_1=1}^{s}(-1)^{Q-i_1}(A_1B_2^{-1})^{(Q-(s+1))}(A_1B_1^{-1})^{(s+1)-j}+\sum_{i_1=s+1}^{Q-1}(-1)^{Q-i_1}(A_1B_2^{-1})^{(Q-i_1)}+I$$
$$+\sum_{i_1=Q+1}^{S}(-1)^{2Q+1-i_1}\sum_{j=0}^{S-i_1}(A_1B_2^{-1})^{(S+s-(i_1+j)-1)}(A_1B_1^{-1})^{(j+1)}(CB_2^{-1})\Big]\mathbf{e}=1$$

2. : Compute the values of

$$\Omega_{i_1}=\begin{cases}(-1)^{Q-i_1}(A_1B_2^{-1})^{(Q-(s+1))}(A_1B_1^{-1})^{s}(A_0B_0^{-1}), & i_1=0,\\ (-1)^{Q-i_1}(A_1B_2^{-1})^{(Q-(s+1))}(A_1B_1^{-1})^{((s+1)-i_1)}, & i_1=1,2,\dots,s,\\ (-1)^{Q-i_1}(A_1B_2^{-1})^{(Q-i_1)}, & i_1=s+1,\dots,Q-1,\\ I, & i_1=Q\\ (-1)^{2Q+1-i_1}\sum\limits_{j=0}^{S-i_1}(A_1B_2^{-1})^{(S+s-(i_1+1)-j)}(A_1B_1^{-1})^{(j+1)}(CB_2^{-1}), & i_1=Q+1,Q+2,\dots,S.\end{cases}$$

3. : Using $\mathbf{\Phi^{(Q)}}$ and $\Omega_{i_1}, i_1=0,1,\dots,S$, calculate the value of $\mathbf{\Phi^{(i_1)}}, i_1=0,1,\dots,S$. That is,

$$\mathbf{\Phi^{(i_1)}} = \mathbf{\Phi^{(Q)}}\Omega_{i_1}, \quad i_1=0,1,\dots,S.$$

4. System performance measures

In this section, we derive some measures of system performance in the steady state. Using this, we calculate the total expected cost rate.

4.1. **Expected Inventory Level.** Let η_I denote the excepted inventory level in the steady state. Since $\Phi^{(i_1)}$ is the steady state probability vector that there are i_1 items in the inventory with each component represents a particular combination of the number of customers in the waiting hall, number of customers in the orbit and the status of the server, $\Phi^{(i_1)}\mathbf{e}$ gives the probability of i_1 item in the inventory in the steady state. Hence η_I is given by

$$\eta_I=\sum_{i_1=1}^{S} i_1\Phi^{(i_1)}\mathbf{e}$$

4.2. **Expected Reorder Rate.** Let η_R denote the expected reorder rate in the steady state. A reorder is placed when the inventory level drops from $s+1$ to s. This may occur in the following two cases:

- the server completes the essential service for a high priority customer
- the server completes a service for a low priority customer

Hence we get

$$\eta_R=\mu_1\sum_{i_3=1}^{N}\sum_{i_4=0}^{M}\phi^{(s+1,\alpha_1,i_3,i_4)}+\mu_3\sum_{i_3=0}^{N}\sum_{i_4=1}^{M}\phi^{(s+1,\alpha_3,i_3,i_4)}.$$

4.3. **Expected Loss Rate for High Priority Customers.** Let η_{LH} denote the expected loss rate for a high priority customer in the steady state. Any arriving high priority customer finds the waiting hall is full and leaves the system without getting service. These customers are considered to be lost. Thus we obtain

$$\begin{aligned}\eta_{LH} &= \sum_{i_4=0}^{M}(K-(N+i_4))\lambda_1\phi^{(0,\alpha_0,N,i_4)} + \sum_{i_1=1}^{S}\sum_{i_4=0}^{M}(K-(N+i_4)\lambda_1\phi^{(i_1,\alpha_1,N,i_4)} \\ &\quad + \sum_{i_1=1}^{S}\sum_{i_4=0}^{M}(K-(N+i_4)\lambda_1\phi^{(i_1,\alpha_3,N,i_4)} + \sum_{i_1=0}^{S}\sum_{i_4=0}^{M}(K-(N+i_4)\lambda_1\phi^{(i_1,\alpha_2,N,i_4)}\end{aligned}$$

4.4. **Expected Loss Rate for Low Priority Customers.** Let η_{LL} denote the expected loss rate for a low priority customer (i.e.: expected number of low priority customers lost before entering the orbit per unit time) in the steady state. Any arriving low priority customer finds either the server busy or inventory level is zero and orbit size is full, leaves the system. These customers are considered to be lost. Hence η_{LL} is given by

$$\begin{aligned}\eta_{LL} &= \sum_{i_3=1}^{N}(K-(i_3+M)\lambda_2\phi^{(0,\alpha_0,i_3,M)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}(K-(i_3+M)\lambda_2\phi^{(i_1,\alpha_1,i_3,M)} \\ &\quad + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N}(K-(i_3+M)\lambda_2\phi^{(i_1,\alpha_2,i_3,M)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}(K-(i_3+M)\lambda_2\phi^{(i_1,\alpha_3,i_3,M)} \\ &\quad + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}r(K-(i_3+M)\lambda_1\phi^{(i_1,\alpha_3,i_3,M)}\end{aligned}$$

4.5. **Expected Waiting Time for High Priority Customers.** Let η_{WH} denote the expected waiting time for high priority customers in the waiting hall. We get

$$\eta_{WH} = \frac{\eta_{QH}}{\eta_{AH}}$$

where η_{QH} is the expected queue length for high priority customers in the waiting hall. It is given by

$$\begin{aligned}\eta_{QH} &= \sum_{i_3=1}^{N}\sum_{i_4=0}^{M} i_3\phi^{(0,\alpha_0,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M} i_3\phi^{(i_1,\alpha_1,i_3,i_4)} \\ &\quad + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M} i_3\phi^{(i_1,\alpha_2,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M} i_3\phi^{(i_1,\alpha_3,i_3,i_4)}\end{aligned}$$

and η_{AH} is the effective arrival rate for high priority customers (Ross [24]) in the waiting hall. Then

$$\begin{aligned}\eta_{AH} &= \sum_{i_1=0}^{S}\sum_{i_4=0}^{M}(K-(i_3+i_4)\lambda_1\phi^{(i_1,\alpha_0,0,i_4)} + \sum_{i_3=1}^{N-1}\sum_{i_4=0}^{M}(K-(i_3+i_4)\lambda_1\phi^{(0,\alpha_0,i_3,i_4)}\\ &+ \sum_{i_1=1}^{S}\sum_{i_3=1}^{N-1}\sum_{i_4=0}^{M}(K-(i_3+i_4)\lambda_1\phi^{(i_1,\alpha_1,i_3,i_4)} + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N-1}\sum_{i_4=0}^{M}(K-(i_3+i_4)\lambda_1\phi^{(i_1,\alpha_2,i_3,i_4)}\\ &+ \sum_{i_1=1}^{S}\sum_{i_3=0}^{N-1}\sum_{i_4=0}^{M}(K-(i_3+i_4)\lambda_1\phi^{(i_1,\alpha_3,i_3,i_4)}\end{aligned}$$

4.6. **Expected Number of Low Priority Customers in the Orbit.** Let η_{WL} denote the expected number of low priority customers in the orbit. We get

$$\begin{aligned}\eta_{WL} &= \sum_{i_3=1}^{N}\sum_{i_4=1}^{M} i_4\phi^{(0,\alpha_0,i_3,i_4)} + \sum_{i_1=0}^{S}\sum_{i_4=1}^{M} i_4\phi^{(i_1,\alpha_0,0,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=1}^{M} i_4\phi^{(i_1,\alpha_1,i_3,i_4)}\\ &+ \sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\sum_{i_4=1}^{M} i_4\phi^{(i_1,\alpha_2,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}\sum_{i_4=1}^{M} i_4\phi^{(i_1,\alpha_3,i_3,i_4)}\end{aligned}$$

4.7. **Probability that Server is Busy.** Let η_{SB} denote the probability that server is busy is given by

$$\eta_{SB} = \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M}\phi^{(i_1,\alpha_1,i_3,i_4)} + \sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M}\phi^{(i_1,\alpha_2,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}\sum_{i_4=0}^{M}\phi^{(i_1,\alpha_3,i_3,i_4)}$$

4.8. **Probability that Server is Idle.** Let η_{SI} denote the probability that server is idle is given by

$$\eta_{SI} = \sum_{i_1=0}^{S}\sum_{i_4=0}^{M}\phi^{(i_1,\alpha_0,0,i_4)} + \sum_{i_3=1}^{N}\sum_{i_4=0}^{M}\phi^{(0,\alpha_0,i_3,i_4)}$$

4.9. **The Overall Rate of Retrials.** Let η_{OR} denote the overall rate of retrials in the steady state. Then

$$\begin{aligned}\eta_{OR} &= \sum_{i_3=1}^{N}\sum_{i_4=0}^{M}\theta_{i_4}\phi^{(0,\alpha_0,i_3,i_4)} + \sum_{i_1=0}^{S}\sum_{i_4=0}^{M}\theta_{i_4}\phi^{(i_1,\alpha_0,0,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M}\theta_{i_4}\phi^{(i_1,\alpha_1,i_3,i_4)}\\ &+ \sum_{i_1=0}^{S}\sum_{i_3=1}^{N}\sum_{i_4=0}^{M}\theta_{i_4}\phi^{(i_1,\alpha_2,i_3,i_4)} + \sum_{i_1=1}^{S}\sum_{i_3=0}^{N}\sum_{i_4=0}^{M}\theta_{i_4}\phi^{(i_1,\alpha_3,i_3,i_4)}\end{aligned}$$

4.10. **The Successful Retrial Rate.** Let η_{SR} denote the successful retrial rate in the steady state. Then

$$\eta_{SR} = \sum_{i_1=1}^{S}\sum_{i_4=1}^{M} \theta_{i_4}\pi^{(i_1,\alpha_0,0,i_4)}$$

4.11. **The Fraction of Successful Rate of Retrial.** Let η_{FR} denote the fraction of successful retrial rate in the steady state. Then

$$\eta_{FR} = \frac{\eta_{SR}}{\eta_{OR}}$$

5. Expected Total Cost Rate

We assume various cost elements associated with different system performance measures are given as follows:

c_h – Inventory carrying cost per unit per unit time
c_s – Setup cost per order
c_{lh} – Cost per high priority customer lost
c_{ll} – Cost per low priority customer lost
c_{wh} – Waiting cost of a high priority customer per unit time
c_{wl} – Waiting cost of a low priority customer per unit time

We construct the function for the expected total cost per unit time as follows:

$$TC(S, s, N, M) = c_h\eta_I + c_s\eta_R + c_{lh}\eta_{LH} + c_{ll}(\eta_{LL} + \eta_{RR}) + c_{wh}\eta_{WH} + c_{wl}\eta_{WL}$$

where η's are as given in the above measures of system performance.

6. Conclusion

In this article, we have presented a continuous review stochastic retrial queueing-inventory system with finite source, (s, Q) policy and Mixed-priority discipline. The lead times of reorder, service times and the retrial demand time points form independent exponential distributions. The model is analyzed within the framework of Markov processes. Joint probability distribution of the number of customers in the waiting area, the number of customers in the orbit and the inventory level are obtained in the steady state. Various system performance measures and the long-run total expected cost rate are derived.

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