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THE PERTURBATION TO NON-MARKOVIAN EQUATION OF MOTION CORRESPONDING TO COHERENT AND QUADRATURE NON-MARKOVIAN SSES

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Abstract. In this paper, we derive the perturbation and post Markovian perturbation to non-Markovian equation of motion(NMEM) that correspond to coherent and quadrature non-Markovian stochastic Schrödinger equations (SSE). In that case, we derive two perturbation approaches for zero and first orders to the coherent and quadrature NMEM. In order to explain both approaches, we apply two examples of non-Markovian.

Keywords: perturbation method, post Markovian perturbation method, non-Markovian equation of motion, probability operator, functional operator.

AMS Subject Classification: 81V80, 60G15.

1. Introduction

Master equations have served as an essential tool in the dynamic systems which is introduced in quantum mechanical setting [14]. The probability operators are defined as ,

$$\tilde{P}(t) = |\tilde{\psi}_z(t)\rangle\langle\tilde{\psi}_z(t)|, \tag{1}$$

where $|\tilde{\psi}_z(t)\rangle$ is the solution for linear coherent and quadrature non Markovian stochastic Schrödinger equations(SSE). The general linear coherent and quadrature non Markovian stochastic equations of motion (SEM) that correspond to linear coherent and quadrature non Markovian SSEs are derived as,

$$\begin{aligned}\dot{\tilde{P}}(t) &= -i[\hat{H},\tilde{P}(t)] + (Z^*(t)\hat{L}\tilde{P}(t) + \tilde{P}(t)Z(t)\hat{L}^\dagger) - \int_0^t \alpha(t-s)\hat{L}^\dagger\hat{d}_0(z,t,s)\tilde{P}(t)ds \\ &\quad - \int_0^t \alpha^*(t-s)\tilde{P}(t)\hat{d}_0^\dagger(z,t,s)\hat{L}ds, \end{aligned}\tag{2}$$

and,

$$\begin{aligned}\dot{\tilde{P}}(t) &= -i[\hat{H},\tilde{P}(t)] + (Z^*(t)\hat{L}^\dagger\tilde{P}(t) + \tilde{P}(t)Z(t)\hat{L}) - \int_0^t \beta(t-s)[\hat{L}_x\hat{m}_0(z,t,s)\tilde{P}(t) \\ &\quad + \tilde{P}(t)\hat{m}_0^\dagger(z,t,s)\hat{L}_x]ds, \end{aligned}\tag{3}$$

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where,

$$D_0(z,t)=\int_0^t \alpha(t-s)\hat{d}_0(z,t,s)ds, \quad (4)$$

and

$$M_0(z,t)=\int_0^t \beta(t-s)\hat{m}_0(z,t,s)ds, \quad (5)$$

are called functional operators, such that

$$\hat{d}_0(z,t,s)=\frac{\delta}{\delta z_t^*}, \quad (6)$$

and

$$\hat{m}_0(z,t,s)=\frac{\delta}{\delta z_t}, \quad (7)$$

are called functional derivative operators (anzats). Some methods are suggested to study the non Markovian open quantum systems which are given by Ref. [9, 12, 6, 2, 8]. The Lindblad master equation [5] is solved specifically by the stochastic Schrödinger theory [3, 10, 4, 7, 15]. The density matrix perturbation theory (PT)is developed in [1], that is applicable to all eigenstates of the operator $\hat{L}$ including the steady state. The post Markovian perturbation theory is derived to non Markovian stochastic Schrödinger equation(SSE)[13]. This derivation is based on the following consistency condition,

$$\partial_t \frac{\delta}{\delta z_t^*}|\tilde{\psi}_z(t)\rangle=\frac{\delta}{\delta z_t^*}\partial_t|\tilde{\psi}_z(t)\rangle, \quad (8)$$

and Taylor expansion for functional derivative operator and it is applied on the non Markovian master equation for density operator. A perturbation approach is introduced to non Markovian coherent and quadrature SSEs [11] which is based on the expansion of the functional operators and above consistency condition and the following consistency condition,

$$\partial_t \frac{\delta}{\delta z_t}|\tilde{\psi}_z(t)\rangle=\frac{\delta}{\delta z_t}\partial_t|\tilde{\psi}_z(t)\rangle. \quad (9)$$

In this paper, we determine the functional operator which it depend on time and noise, by deriving the perturbation theory and the post Markovian perturbation theory to coherent and quadrature(NMEM). In this side, we take the combined functional derivative for probability operator. And we use in derivation, the expansion for functional operator and Taylor expansion for functional derivative operator in [11] and [13] respectively. We apply above approaches on two examples. This article can be divided into five sections. In sec.II , we derive the zero and first orders stochastic perturbation equations for functional operator $\hat{D}_0(z,t)$ and $\hat{M}_0(z,t)$ in coherent and quadrature (NMEM) respectively. In quadrature unravelling we find two zero order stochastic perturbation equations and four first order stochastic perturbation equations for $\hat{M}_0(z,t)$. In sec.III, we derive the post Markovian perturbation to coherent and quadrature (NMEM). We find the zero and first orders stochastic perturbation for functional derivative operators $\hat{d}_0(z,t,s)$ and $\hat{m}_0(z,t,s)$. Here we can calculate the functional operators $\hat{D}_0(z,t)$ and $\hat{M}_0(z,t)$ also we find first order coherent and quadrature (NMEM). In sec.IV, we apply our approaches on two examples. Finally the conclusion and suggestion for future work is given in sec.V.

2. The perturbation to coherent and quadrature (NMEM)

In this section, we derive zero and first orders stochastic perturbation equations to equation(2) and equation(3). For equation(2), the expansion for memory function and functional operator are defined as,

$$\alpha(t-s) = \sum_{j=1} \alpha^j(t-s), \tag{10}$$

where

$$\alpha^j(t-s) = |G_j|^2 \exp^{\frac{-k_j}{2}|t-s|} \exp^{-i\Omega_j(t-s)}, \tag{11}$$

and

$$D_0(z,t) = \sum_j D_0^j(z,t), \tag{12}$$

such that functional operator$D_0^j(z,t)$ is defined as,

$$D_0^j(z,t) = \int_0^t \alpha^j(t-s)\hat{d}_0(z,t,s)ds. \tag{13}$$

The time derivative of equation (13) is,

$$\partial_t D_0^j(z,t) = \alpha^j(0)\hat{d}_0(z,t,t) - [\frac{k_j}{2} + i\Omega_j]D_0^j(z,t) + \int_0^t \alpha^j(t-s)\partial_t\hat{d}_0(z,t,s)ds. \tag{14}$$

To find $\partial_t\hat{d}_0(z,t,s)$, one can obtain the combined functional derivative for probability operator $\tilde{P}(t)$ as,

$$\begin{aligned}\frac{\delta}{\delta z^*(s)}\tilde{P}(t) &= \frac{\delta}{\delta z^*(s)}|\tilde{\psi}_z(t)\rangle\langle\tilde{\psi}_z(t)| \\ &= (\frac{\delta}{\delta z^*(s)}|\tilde{\psi}_z(t)\rangle)\langle\tilde{\psi}_z(t)| + |\tilde{\psi}_z(t)\rangle(\frac{\delta}{\delta z^*(s)}\langle\tilde{\psi}_z(t)|).\end{aligned} \tag{15}$$

Since linear coherent non-Markovian SSE [13] we get,

$$\begin{aligned}\partial_t D_0^j(z,t) &= |G_j|^2\hat{L} - [\frac{k_j}{2} + i\Omega_j]D_0^j(z,t) + (Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)D_0^j(z,t) - D_0^\dagger(z,t)\hat{L}D_0^j(z,t) \\ &\quad -\hat{L}^\dagger\sum_k D_1^{j,k}(z,t).\end{aligned} \tag{16}$$

Equation (16) is zero order stochastic perturbation equation for functional operator $D_0(z,t)$, where $D_1^{j,k}(z,t)$ is first order functional operator which can be defined as

$$D_1^{j,k}(z,t) = \int_0^t \alpha^j(t-s)\hat{d}_1^k(z,t,s)ds. \tag{17}$$

Such that

$$\frac{\delta}{\delta z_t^*}D_0^k(z,t) = \hat{d}_1^k(z,t,s). \tag{18}$$

The time derivative of equation (17) is,

$$\partial_t D_1^{j,k}(z,t) = |G_j|^2\hat{d}_1^k(z,t,t) + \int_0^t \partial_t\alpha^j(t-s)\hat{d}_1^k(z,t,s)ds + \int_0^t \alpha^j(t-s)\partial_t\hat{d}_1^k(z,t,s)ds. \tag{19}$$

The first order is obtained,

$$\begin{aligned}\partial_t D_1^{j,k}(z,t) &= |G_j|^2\hat{d}_1^k(z,t,t) - [\frac{k_j}{2} + i\Omega_j]D_1^{j,k}(z,t) + |G_j|^2 D_0^j(z,t)\hat{L} \\ &-[\frac{k_k}{2} + i\Omega_k]D_0^j(z,t)D_0^k(z,t) + D_0^j(z,t)(Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)D_0^k(z,t) \\ &-D_0^j(z,t)D^\dagger(z,t)\hat{L}D_0^k(z,t) - D_0^k(z,t)\sum_l D_2^{j,k,l}(z,t). \end{aligned} \tag{20}$$

Equation (20) is first order stochastic perturbation equation for functional operator $D_0(z,t)$, where here the operators $\hat{L}^\dagger$ and $\hat{d}_0(z,t,s)$ are commutator. For equation (3), we find the zero and first orders stochastic perturbation equations in four cases; In case A, we will use the expansion of the memory function and functional operator which are defined by,

$$\beta(t-s) = \sum_j \beta^{j,\cos}(t-s), \tag{21}$$

and

$$M_0(z,t) = \sum_j M_0^{j,\cos}(z,t), \tag{22}$$

such that

$$\beta^{j,\cos}(t-s) = 2|G_j|^2 \exp^{-\frac{k_j}{2}|t-s|}\cos\Omega_j(t-s). \tag{23}$$

The functional operator $M_0^{j,\cos}(z,t)$ is defined as,

$$M_0^{j,\cos}(z,t) = \int_0^t \beta^{j,\cos}(t-s)\hat{m}_0(z,t,s)ds. \tag{24}$$

The time derivative of the equation (24) is,

$$\begin{aligned}\partial_t M_0^{j,\cos}(z,t) &= 2|G_j|^2\hat{L} - \frac{k_j}{2}M_0^{j,\cos}(z,t) - \Omega_j M_0^{j,\sin}(z,t) \\ &+ \int_0^t \beta^{j,\cos}(t-s)\partial_t\hat{m}_0(z,t,s)ds, \end{aligned} \tag{25}$$

where $M_0^{j,\sin}(z,t) = \int_0^t \beta^{j,\sin}(t-s)\hat{m}_0(z,t,s)ds$. In order to find $\partial_t\hat{m}_0(z,t,s)$, we use the combined functional derivative of the probability operator as done in coherent case, we have,

$$\begin{aligned}\partial_t M_0^{j,\cos}(z,t) &= 2|G_j|^2\hat{L} - \frac{k_j}{2}M_0^{j,\cos}(z,t) - \Omega_j M_0^{j,\sin}(z,t) \\ &+(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)M_0^{j,\cos}(z,t) - M_0^\dagger(z,t)\hat{L}_x M_0^{j,\cos}(z,t) \\ &-\hat{L}_x\sum_k M_1^{j,k,\cos,\cos}(z,t). \end{aligned} \tag{26}$$

Equation (26) is zero order stochastic perturbation equation. The first order perturbation $M_1^{j,k,\cos,\cos}(z,t)$ is defined by,

$$M_1^{j,k,\cos,\cos}(z,t) = \int_0^t \beta^{j,\cos}(t-s)\hat{m}_1^k(z,t,s)ds, \tag{27}$$

where

$$\frac{\delta}{\delta z_s}M_0^{k,\cos}(z,t) = \hat{m}_1^k(z,t,s). \tag{28}$$

Then the functional operator $M_1^{j,k,\cos,\cos}(z,t)$ is,

$$\begin{aligned}\partial_t M_1^{j,k,\cos,\cos}(z,t) &= 2|G_j|^2\hat{m}_1^k(z,t,t) - \frac{k_j}{2}M_1^{j,k,\cos,\cos}(z,t)\\ &-\Omega_j M_1^{j,k,\sin,\cos}(z,t) + 2|G_j|^2 M_0^{j,\cos}(z,t)\hat{L} - \frac{k_j}{2}M_0^{j,\cos}(z,t)M_0^{k,\cos}(z,t)\\ &-\Omega_k M_0^{j,\cos}(z,t)M_0^{k,\cos}(z,t) + (Z_t\hat{L} + Z_t^*\hat{L}^\dagger)M_0^{j,\cos}(z,t)M_0^{k,\cos}(z,t)\\ &-M_0^{j,\cos}(z,t)M_0^\dagger(z,t)\hat{L}_x M_0^{k,\cos}(z,t) - \hat{L}_x\sum_l M_2^{j,k,l,\cos,\cos,\cos}(z,t), \quad (29)\end{aligned}$$

where $\hat{m}_0(z,t,s)$ and $\hat{L}_x$ are commutators and equation (29) is first order stochastic perturbation equation. Now, we back to case B, we will take the expansion of the memory function and functional operator which are defined by,

$$\beta(t-s) = \sum_j \beta^{j,\sin}(t-s), \quad (30)$$

and

$$M_0(z,t) = \sum_j M_0^{j,\sin}(z,t), \quad (31)$$

where

$$M_0^{j,\sin}(z,t) = \int_0^t \beta^{j,\sin}(t-s)\hat{m}_0(z,t,s)ds, \quad (32)$$

and

$$\beta^{j,\sin}(t-s) = 2|G_j|^2 \exp^{-\frac{k_j}{2}|t-s|}\sin\Omega_j(t-s). \quad (33)$$

The zero order is,

$$\begin{aligned}\partial_t M_0^{j,\sin}(z,t) &= 2|G_j|^2\hat{L} - \frac{k_j}{2}M_0^{j,\sin}(z,t) + \Omega_j M_0^{j,\cos}(z,t)\\ &+(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)M_0^{j,\sin}(z,t) - (Z_t\hat{L} + Z_t^*\hat{L}^\dagger)M_0^{j,\sin}(z,t)\\ &-M_0^\dagger(z,t)\hat{L}_x M_0^{j,\sin}(z,t) - \hat{L}_x\sum_k M_1^{j,,k,\sin,\sin}(z,t), \quad (34)\end{aligned}$$

Equation (34) is zero order stochastic perturbation equation. The first order perturbation can be defined as,

$$M_1^{j,,k,\sin,\sin}(z,t) = \int_0^t \beta^{j,\sin}(t-s)\hat{m}_1^k(z,t,s)ds, \quad (35)$$

where

$$\hat{m}_1^k(z,t,s) = \hat{m}_0(z,t,s)M_0^{k,\sin}(z,t). \quad (36)$$

Taking time derivative of equation(35), one can obtain

$$\begin{aligned}\partial_t M_1^{j,,k,\sin,\sin}(z,t) &= 2|G_j|^2\hat{m}_1^k(z,t,t) - \frac{k_j}{2}M_1^{j,,k,\sin,\sin}(z,t) + \Omega_j M_1^{j,k,\cos,\sin}(z,t)\\ &+ \int_0^t \beta^{j,\sin}(t-s)\partial_t\hat{m}_1^k(z,t,s)ds. \quad (37)\end{aligned}$$

The first order is,

$$\begin{aligned}\partial_t M_1^{j,,k,\sin,\sin}(z,t) &= 2|G_j|^2\hat{m}_1^k(z,t,t) - \frac{k_j}{2}M_1^{j,,k,\sin,\sin}(z,t)\\ &+\Omega_j M_1^{j,k,\cos,\sin}(z,t) + 2|G_j|^2\hat{L}M_0^{j,\sin}(z,t) - \frac{k_k}{2}M_0^{j,\sin}(z,t)M_0^{k,\sin}(z,t)\\ &+\Omega_k M_0^{j,\sin}(z,t)M_0^{k,\cos}(z,t) + (Z_t\hat{L} + Z_t^*\hat{L}^\dagger)M_0^{j,\sin}(z,t)M_0^{k,\sin}(z,t)\\ &-M_0^{j,\sin}(z,t)M_0^\dagger(z,t)\hat{L}_x M_0^{k,\sin}(z,t) - \hat{L}_x\sum_l M_2^{j,,k,l,\sin,\sin,\sin}(z,t), \quad (38)\end{aligned}$$

where the functional operator $M_0(z,t)$ is defined by equation(31). Here, we are going to study the third case. In this case, we use equation (21) then we get,

$$\begin{aligned}\partial_t M_0^{j,\cos}(z,t) &= 2|G_j|^2\hat{L} - \frac{k_j}{2}M_0^{j,\cos}(z,t) - \Omega_j M_0^{j,\sin}(z,t)\\ &+(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)M_0^{j,\cos}(z,t) - M_0^\dagger(z,t)\hat{L}_x M_0^{j,\cos}(z,t)\\ &-\hat{L}_x\sum_k M_1^{j,k,\cos,\sin}(z,t), \quad (39)\end{aligned}$$

Equation (39) is zero order stochastic perturbation equation. Using equation (36), one can find following equation,

$$\begin{aligned}\partial_t M_1^{j,k,\cos,\sin}(z,t) &= 2|G_j|^2\hat{m}_1^k(z,t,t) - \frac{k_j}{2}M_1^{j,k,\cos,\sin}(z,t)\\ &-\Omega_j M_1^{j,k,\sin,\sin}(z,t) + 2|G_k|^2\hat{L}M_0^{j,\cos}(z,t) - \frac{k_k}{2}M_0^{j,\cos}(z,t)M_0^{k,\sin}(z,t)\\ &+\Omega_k M_0^{j,\cos}(z,t)M_0^{k,\cos}(z,t) + (Z_t\hat{L} + Z_t^*\hat{L}^\dagger)M_0^{j,\cos}(z,t)M_0^{k,\sin}(z,t)\\ &-M_0^{j,\cos}(z,t)M_0^\dagger(z,t)\hat{L}_x M_0^{k,\sin}(z,t) - \hat{L}_x\sum_l M_2^{j,k,l,\cos,\sin,\sin}(z,t), \quad (40)\end{aligned}$$

Equation (40) is first order stochastic perturbation equation. Now we back to last case. In this case we use equation (30) and rewrite equation (34), we have,

$$\begin{aligned}\partial_t M_0^{j,\sin}(z,t) &= 2|G_j|^2\hat{L} - \frac{k_j}{2}M_0^{j,\sin}(z,t) + \Omega_j M_0^{j,\cos}(z,t)\\ &+(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)M_0^{j,\sin}(z,t) - (Z_t\hat{L} + Z_t^*\hat{L}^\dagger)M_0^{j,\sin}(z,t)\\ &-M_0^\dagger(z,t)\hat{L}_x M_0^{j,\sin}(z,t) - \hat{L}_x\sum_k M_1^{j,k\,\sin,\cos}(z,t), \quad (41)\end{aligned}$$

Equation (41)is zero order stochastic perturbation equation. The first order perturbation $M_1^{j,k\,\sin,\cos}(z,t)$ can be defined as,

$$M_1^{j,k\,\sin,\cos}(z,t) = \int_0^t \beta^{j,\sin}(t-s)\hat{m}_1^k(z,t,s)ds. \quad (42)$$

Then the stochastic equation for the $M_1^{j,k\,\sin,\cos}(z,t)$ is,

$$\begin{aligned}\partial_t M_1^{j,k\,\sin,\cos}(z,t) &= 2|G_j|^2\hat{m}_1^k(z,t,t) - \frac{k_j}{2}M_1^{j,k\,\sin,\cos}(z,t)\\ &-\Omega_j M_1^{j,k,\cos,\cos}(z,t) + 2|G_k|^2\hat{L}M_0^{j,\sin}(z,t) - \frac{k_k}{2}M_0^{j,\sin}(z,t)M_0^{k,\cos}(z,t)\\ &-M_0^{j,\sin}(z,t)M_0^\dagger(z,t)\hat{L}_x M_0^{k,\cos}(z,t) - \hat{L}_x\sum_l M_2^{j,k,l,\sin,\cos,\cos}(z,t), \quad (43)\end{aligned}$$

Equation (43) is first order stochastic perturbation equation.

3. Post Markovian perturbation to coherent and quadrature (NMEM)

In this section, we derive the first order coherent and quadrature (NMEM) by deriving post Markovian perturbation to equation (2) and equation(3). For equation (2), we use combined functional derivative for the probability operator $\tilde{P}(t)$ and Taylor expansion for functional derivative operator $\hat{d}_0(z,t,s)$ in power of $(t-s)$, which is defined as,

$$\hat{d}_0(z,t,s) = \hat{L} + [\partial_t \hat{d}_0(z,t,s)|_{t=s}](t-s) + \frac{1}{2}[\partial_t^2 \hat{d}_0(z,t,s)|_{t=s}](t-s)^2. \tag{44}$$

The first order coherent NMEM is,

$$\begin{aligned}
\dot{\tilde{P}}(t) \;=\; & -i[\hat{H},\tilde{P}(t)] + (Z^*(t)\hat{L}\tilde{P}(t) + \tilde{P}(t)Z(t)\hat{L}^\dagger) - \hat{L}^\dagger[\hat{L}g_0(t) + [(Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)\hat{L}g_1(t) \\
& -\hat{L}^\dagger\hat{L}^2 g_2(t) - \hat{L}^\dagger\hat{L}^2 g_3(t)] + \frac{1}{2}[[(Z^*(t)\hat{L})^2 + |Z(t)|^2(\hat{L}\hat{L}^\dagger + \hat{L}^\dagger\hat{L}) + (Z(t)\hat{L}^\dagger)^2]\hat{L}g_4(t) \\
& -(Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)\hat{L}^\dagger\hat{L}^2 g_5(t) - (Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)\hat{L}^\dagger\hat{L}^2 g_6(t)] \\
& +\frac{1}{2}[-\hat{L}^\dagger\hat{L}(Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)\hat{L}g_5(t) + \hat{L}^\dagger\hat{L}\hat{L}^\dagger\hat{L}^2 0g_7(t) + \hat{L}^\dagger\hat{L}\hat{L}^\dagger\hat{L}^2 g_8(t)] \\
& +\frac{1}{2}[-\hat{L}^2\hat{L}^\dagger(Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)g_5(t) + \hat{L}^2(\hat{L}^\dagger)^2\hat{L}g_8(t) + \hat{L}^2(\hat{L}^\dagger)^2\hat{L}g_7(t)] \\
& +\frac{1}{2}[-2\hat{L}^\dagger\hat{L}(Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)\hat{L}g_6(t) + 2\hat{L}^\dagger\hat{L}\hat{L}^\dagger\hat{L}^2 g_9(t) \\
& +2\hat{L}^\dagger\hat{L}\hat{L}^\dagger\hat{L}^2 g_{10}(t)]]\tilde{P}(t) - \tilde{P}(t)[\hat{L}g_0(t) + \frac{1}{2}[(Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)\hat{L}g_1(t) \\
& -\hat{L}^\dagger\hat{L}^2 g_2(t) - \hat{L}^\dagger\hat{L}^2 g_3(t)] + \frac{1}{2}[[(Z^*(t)\hat{L})^2 + |Z(t)|^2(\hat{L}\hat{L}^\dagger + \hat{L}^\dagger\hat{L}) + (Z(t)\hat{L}^\dagger)^2]\hat{L}g_4(t) \\
& -(Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)\hat{L}^\dagger\hat{L}^2 g_5(t) - (Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)\hat{L}^\dagger\hat{L}^2 g_6(t)] + \frac{1}{2}[-\hat{L}^\dagger\hat{L}(Z^*(t)\hat{L} \\
& +Z(t)\hat{L}^\dagger)\hat{L}g_5(t) + \hat{L}^\dagger\hat{L}\hat{L}^\dagger\hat{L}^2 g_7(t) + \hat{L}^\dagger\hat{L}\hat{L}^\dagger\hat{L}^2 g_8(t)] + \frac{1}{2}[-\hat{L}^2\hat{L}^\dagger(Z^*(t)\hat{L} \\
& +Z(t)\hat{L}^\dagger)g_5(t) + \hat{L}^2(\hat{L}^\dagger)^2\hat{L}g_8(t) + \hat{L}^2(\hat{L}^\dagger)^2\hat{L}g_7(t)] + \frac{1}{2}[-2\hat{L}^\dagger\hat{L}(Z^*(t)\hat{L} + Z(t)\hat{L}^\dagger)\hat{L}g_6(t) \\
& +2\hat{L}^\dagger\hat{L}\hat{L}^\dagger\hat{L}^2 g_9(t) + 2\hat{L}^\dagger\hat{L}\hat{L}^\dagger\hat{L}^2 g_{10}(t)]]^\dagger\hat{L},
\end{aligned} \tag{45}$$

Equation (45) is first order coherent (NMEM). where,

$$g_0(t) = \int_0^t \alpha(t-s)ds, \tag{46}$$

$$g_1(t) = \int_0^t \alpha(t-s)(t-s)ds, \tag{47}$$

$$g_2(t) = \int_0^t \int_0^u \alpha(t-s)\alpha^*(s-u)(t-s)duds, \tag{48}$$

$$g_3(t) = \int_0^t \int_0^u \alpha(t-s)\alpha(s-u)(t-s)duds, \tag{49}$$

$$g_4(t) = \int_0^t \alpha(t-s)(t-s)^2 ds, \tag{50}$$

$$g_5(t) = \int_0^t \int_0^u \alpha(t-s)\alpha^*(s-u)(t-s)^2 duds, \tag{51}$$

$$g_6(t) = \int_0^t \int_0^u \alpha(t-s)\alpha(s-u)(t-s)^2 duds, \tag{52}$$

$$g_7(t) = \int_0^t \int_0^u \int_0^v \alpha(t-s)\alpha^*(s-u)\alpha^*(u-v)(t-s)^2 dvduds, \tag{53}$$

$$g_8(t) = \int_0^t \int_0^u \int_0^v \alpha(t-s)\alpha^*(s-u)\alpha(u-v)(t-s)^2 dvduds, \tag{54}$$

$$g_9(t) = \int_0^t \int_0^u \int_0^v \alpha(t-s)\alpha(s-u)\alpha^*(u-v)(t-s)^2 dvduds, \tag{55}$$

$$g_{10}(t) = \int_0^t \int_0^u \int_0^v \alpha(t-s)\alpha(s-u)\alpha(u-v)(t-s)^2 dvduds. \tag{56}$$

Now, for equation (3), also we use functional derivative for probability operator $\tilde{P}(t)$ and Taylor expansion for functional derivative operator $\hat{m}_0(z,t,s)$ in power of $(t-s)$, which is defined as

$$\hat{m}_0(z,t,s) = \hat{L} + [\partial_t \hat{m}_0(z,t,s)|_{t=s}](t-s) + \frac{1}{2}[\partial_t^2 \hat{m}_0(z,t,s)|_{t=s}](t-s)^2. \tag{57}$$

The first order quadrature NMEM is,

$$\begin{aligned}
\dot{\tilde{P}}(t) \; = \; & -i[\hat{H}, \tilde{P}(t)] + (Z^*(t)\hat{L}^\dagger \tilde{P}(t) + \tilde{P}(t)Z(t)\hat{L}) - \hat{L}_x[\hat{L}r_0(t) + [(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)\hat{L}r_1(t) - (\hat{L}^\dagger\hat{L}_x\hat{L} \\
& \times(\hat{L}^\dagger\hat{L}_x\hat{L} + \hat{L}_x\hat{L}^2)r_2(t)] + \frac{1}{2}[[(Z^*(t)\hat{L}^\dagger)^2 + |Z(t)|^2(\hat{L}^\dagger\hat{L} + \hat{L}\hat{L}^\dagger) + (Z(t)\hat{L})^2]\hat{L}r_3(t) \\
& -(Z_t\hat{L} + Z_t^*\hat{L}^\dagger) + \hat{L}_x\hat{L}^2)r_4(t)] + \frac{1}{2}[-\hat{L}^\dagger\hat{L}_x(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)\hat{L}r_4(t) \\
& +\hat{L}^\dagger\hat{L}_x(\hat{L}^\dagger\hat{L}_x\hat{L} + \hat{L}_x\hat{L}^2)r_5(t)] + \frac{1}{2}[-\hat{L}_x\hat{L}\hat{L}^\dagger(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)r_4(t) \\
& +\hat{L}_x\hat{L}(\hat{L}^\dagger\hat{L}_x\hat{L} + \hat{L}_x\hat{L}^2)r_5(t)] + \frac{1}{2}[-\hat{L}_x(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)\hat{L}r_3(t) \\
& +\hat{L}_x(\hat{L}^\dagger\hat{L}_x\hat{L} + \hat{L}_x\hat{L}^2)r_5(t)]]\tilde{P}(t) + \tilde{P}(t)[\hat{L}r_0(t) + [(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)\hat{L}r_1(t) - (\hat{L}^\dagger\hat{L}_x\hat{L} \\
& +\hat{L}_x\hat{L}^2)r_2(t)] + \frac{1}{2}[[(Z^*(t)\hat{L}^\dagger)^2 + |Z(t)|^2(\hat{L}^\dagger\hat{L} + \hat{L}\hat{L}^\dagger) + (Z(t)\hat{L})^2]\hat{L}r_3(t) - (Z_t\hat{L} + Z_t^*\hat{L}^\dagger) \\
& \times(\hat{L}^\dagger\hat{L}_x\hat{L} + \hat{L}_x\hat{L}^2)r_4(t)] + \frac{1}{2}[-\hat{L}^\dagger\hat{L}_x(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)\hat{L}r_4(t) + \hat{L}^\dagger\hat{L}_x(\hat{L}^\dagger\hat{L}_x\hat{L} + \hat{L}_x\hat{L}^2)r_5(t)] \\
& +\frac{1}{2}[-\hat{L}_x\hat{L}\hat{L}^\dagger(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)r_4(t) + \hat{L}_x\hat{L}(\hat{L}^\dagger\hat{L}_x\hat{L} + \hat{L}_x\hat{L}^2)r_5(t)] + \frac{1}{2}[-\hat{L}_x(Z_t\hat{L} + Z_t^*\hat{L}^\dagger)\hat{L}r_3(t) \\
& +\hat{L}_x(\hat{L}^\dagger\hat{L}_x\hat{L} + \hat{L}_x\hat{L}^2)r_5(t)]]^\dagger\hat{L}_x),
\end{aligned} \tag{58}$$

Equation (58) is first order quadrature (NMEM). where,

$$r_0(t) = \int_0^t \beta(t-s)ds, \tag{59}$$

$$r_1(t) = \int_0^t \beta(t-s)(t-s)ds, \tag{60}$$

$$r_2(t) = \int_0^t \int_0^u \beta(t-s)\beta(s-u)(t-s)duds, \tag{61}$$

$$r_3(t) = \int_0^t \int_0^u \beta(t-s)(t-s)^2 duds, \tag{62}$$

$$r_4(t) = \int_0^t \int_0^u \beta(t-s)\beta(s-u)(t-s)^2 duds, \tag{63}$$

$$r_5(t) = \int_0^t \int_0^u \int_0^v \beta(t-s)\beta(s-u)\beta(u-v)(t-s)^2 dvduds, \qquad (64)$$

4. Examples

In order to have some application for the corresponding approach we need to consider two systems. One is two level atom and the other is dissipative two level systems. We describe these models by coherent and quadrature (NMEM) and first order coherent and quadrature post Markovian (EM) respectively.

4.1. **Two-level atom system.** In this subsection, the corresponding system is given by,

$$\hat{L} = \lambda\sigma_z. \qquad (65)$$

The coherent NMEM for this model is defined as,

$$\dot{\tilde{P}}(t) = -i[\frac{\omega}{2}\sigma_x, \tilde{P}(t)] + (Z^*(t)\lambda\sigma_z\tilde{P}(t) + \tilde{P}(t)Z(t)\lambda\sigma_z) - \lambda\sigma_z D_0(z,t)\tilde{P}(t) - \tilde{P}(t)D_0^\dagger(z,t)\lambda\sigma_z. \qquad (66)$$

Assume that J=1, then $D_0^j(z,t) = D_0(z,t)$, we can find the zero order stochastic perturbation equation to this system by using equation (16)

$$\begin{aligned}\partial_t D_0(z,t) &= |G|^2\lambda\sigma_z - \frac{k}{2}D_0(z,t) + (Z^*(t)\lambda\sigma_z + Z(t)\lambda\sigma_z)D_0(z,t) - D_0^\dagger(z,t)\lambda\sigma_z D_0(z,t) \\ &-\lambda\sigma_z D_0^2(z,t). \end{aligned} \qquad (67)$$

Also $D_1^{j,k}(z,t) = D_1(z,t)$, one can obtain, the first order stochastic perturbation equation by using equation (20)

$$\begin{aligned}\partial_t D_1(z,t) &= |G|^2\lambda\sigma_z D_0(t,t) - \frac{k}{2}D_1(z,t) + |G|^2 D_0(z,t)\lambda\sigma_z \\ &-\frac{k}{2}D_0^2(z,t) + D_0(z,t)(Z^*(t)\lambda\sigma_z + Z(t)\lambda\sigma_z)D_0(z,t) \\ &-D_0(z,t)D^\dagger(z,t)\lambda\sigma_z D_0(z,t) - D_1^2(z,t) \end{aligned} \qquad (68)$$

The quadrature NMEM for this model is defined as,

$$\begin{aligned}\dot{\tilde{P}}(t) &= -i[\frac{\omega}{2}\sigma_x, \tilde{P}(t)] + (Z^*(t)\lambda\sigma_z\tilde{P}(t) + \tilde{P}(t)Z(t)\lambda\sigma_z) \\ &-(\lambda\sigma_z + \langle\langle\lambda\sigma_z\rangle)M_0(z,t)\tilde{P}(t) + \tilde{P}(t)M_0^\dagger(z,t)(\lambda\sigma_z \\ &+\langle\langle\lambda\sigma_z\rangle) \end{aligned} \qquad (69)$$

Assume J=1,then $M_0^{j,\cos}(z,t) = M_0^{j,\sin}(z,t) = M_0(z,t)$, we have, zero order by equation(26)

$$\begin{aligned}\partial_t M_0(z,t) &= 2|G|^2\lambda\sigma_z - \frac{k}{2}M_0^2(z,t) + (Z_t\lambda\sigma_z + Z_t^*\lambda\sigma_z)M_0(z,t) \\ &-M_0^\dagger(z,t)(\lambda\sigma_z + \langle\langle\lambda\sigma_z\rangle)M_0(z,t) - (\lambda\sigma_z + \langle\langle\lambda\sigma_z\rangle)M_0^2(z,t) \end{aligned} \qquad (70)$$

Also $M_1^{j,k,\sin,\sin}(z,t) = M_1^{j,k,\sin,\sin}(z,t) = M_1^{j,k,\cos,\sin}(z,t) = M_1(z,t)$ The first order by using equation(29)

$$\begin{aligned}\partial_t M_1(z,t) &= 2|G|^2\lambda\sigma_z M_0(t,t) - \frac{k}{2}M_1(z,t) + 2|G|^2 M_0(z,t)\lambda\sigma_z - \frac{k}{2}M_0(z,t)M_0(z,t) \\ &+(Z_t\lambda\sigma_z + Z_t^*\lambda\sigma_z)M_0^2(z,t) - M_0(z,t)M_0^\dagger(z,t)(Z_t\lambda\sigma_z \\ &+Z_t^*\lambda\sigma_z)M_0(z,t) - (\lambda\sigma_z + \langle\langle\lambda\sigma_z\rangle)M_1^2(z,t) \end{aligned} \qquad (71)$$

4.2. **Dissipative two level atom model.** In this subsection, we calculate the first order coherent and quadrature (NMEM) for this model. The Hamiltonian and system operators are defined as

$$\hat{L} = \lambda\sigma_-. \tag{72}$$

The first order coherent NMEM for two level atom is

$$\begin{aligned}\dot{\tilde{P}}(t) &= -i[\frac{\omega}{2}\sigma_z, \tilde{P}(t)] + \lambda(Z^*(t)\sigma_-\tilde{P}(t) + \tilde{P}(t)Z(t)\sigma_+) - \lambda^2\sigma_+\sigma_-g_0(t)\tilde{P}(t) \\ &+\frac{\lambda^4}{2}|Z(t)|^2\sigma_+\sigma_-\sigma_+\sigma_-g_4(t)\tilde{P}(t) + \frac{\lambda^2}{2}\sigma_+Z(t)\sigma_-g_5(t)\tilde{P}(t) - \lambda^2\tilde{P}(t)g_0^*(t)\sigma_+\sigma_- \\ &+\frac{\lambda^4}{2}|Z(t)|^2\tilde{P}(t)g_4^*(t)\sigma_+\sigma_-\sigma_+\sigma_- \end{aligned} \tag{73}$$

The first order quadrature NMEM for this model,

$$\begin{aligned}\dot{\tilde{P}}(t) &= -i[\frac{\omega}{2}\sigma_z, \tilde{P}(t)] + \lambda(Z^*(t)\sigma_+\tilde{P}(t) + \tilde{P}(t)Z(t)\sigma_-) - \lambda^2\sigma_+\sigma_-r_0(t)\tilde{P}(t) - \lambda^3(\sigma_-)_xZ^*(t) \\ &\times\sigma_+\sigma_-r_1(t)\tilde{P}(t) - \frac{\lambda^4}{2}|Z(t)|^2\sigma_+\sigma_-\sigma_+\sigma_-r_3(t)\tilde{P}(t) + \frac{\lambda^5}{2}\sigma_-\sigma_+\sigma_-Z^*(t)\sigma_+\sigma_-r_4(t) \\ &+\frac{\lambda^5}{2}\sigma_-\sigma_+\sigma_-\sigma_+Z^*(t)\sigma_-r_4(t)\tilde{P}(t) + \frac{\lambda^4}{2}\sigma_+\sigma_-Z^*(t)\sigma_+\sigma_-r_3(t)\tilde{P}(t) \\ &-\lambda\tilde{P}(t)r_0^*\sigma_+\sigma_-\sigma_+ + \frac{\lambda^4}{2}|Z(t)|^2\tilde{P}(t)r_3(t)\sigma_+\sigma_-\sigma_+\sigma_- \\ &-\frac{\lambda^5}{2}r_4^*(t)\tilde{P}(t)\sigma_+Z^*(t)\sigma_-\sigma_+\sigma_-\sigma_+. \end{aligned} \tag{74}$$

References

[1] Li,A.C.Y., Petruccione,F., and Koch,J., (2014), Scientific Reports 4, 4887.
[2] Garraway,B.M., (1997), Phys. Rev. A55,2290.
[3] Gardiner,C.W., Parkins,A.S., and Zoller,P., (1992), Phy. Rev. A46, 4363.
[4] Gardiner,C.W. and Zoller,P., Quantum noise, (2000), Springer - Verlag, Berlin.
[5] Intravia,F., Maniscalco,S. and Messina,A., (2003), Phys. Rev. A67,042108.
[6] Shibata,F., Takahashi,Y., and Hashitsume,N.,J., (1977), Stat. Phys.
[7] Carmichael,H.J., (1993), An open systems approach to Quantum optics, Springer - Verlag, Berlin.
[8] Kappler,H.P.B. and Petruccione,F., (1999), Phys. Rev. A59, 1633.
[9] Breuerand,H.P. and Petruccioue,F., (2002), The Theory of open Quantum systems, Oxford University Press.
[10] Dalibard,J., Castin,Y., and Molmer,K., (1992), Phys. Rev. Lett. , 68, 580.
[11] Gambetta,J. and Wiseman,H M., (2002), Physical Review A 66(5), 052105, APS.
[12] Nakajima,S., (1958), Progr. Theor. Phys. 20, 948.
[13] Yu,T., Diosi,L., Gisin,N., and Strunz,W.T., (1999), Phys. Rev. A , V.60.
[14] Pauli,W., in: Debye(Ed),problems dar modernen physik, Arnold sommerfeld zum60.Geburtsatge, in:Gewidmet von Seinen Schulern, Hirzel, S. Leipzig, 1928, pp.30-45.
[15] Strunz,W.T., (1996), Phys. Lett. A224, 25.

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