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A DISCUSSION ON SOME RECENT COUPLED FIXED POINT RESULTS VIA NEW GENERALIZED NONLINEAR CONTRACTIVE CONDITIONS

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Abstract. Recently, Samet et al. [34], by using the equivalence of the three basic metrics showed that certain coupled fixed point results can be obtained immediately from the well-known fixed point theorems. In the setting of partially ordered metric spaces, we establish a generalization of the recent coupled fixed/ coincidence point results under new nonlinear contractive conditions. The significant feature of the presented work is that, our obtained results are not the immediate consequence of the already existing results in the literature. Presented work generalizes some of the results of Bhaskar and Lakshmikantham [6], Berinde [7], Choudhury et al. [10], Harjani et al. [17], Jain et al. [21] , Karapinar et al. [22], Luong and Thuan [25], and Rasouli and Bahrampour [30].

Keywords: ordered metric spaces, mixed monotone mappings, coupled fixed point, w^*-compatible mappings

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1. Introduction and preliminaries

Fixed point theory is an important tool to study the phenomenon of nonlinear analysis and is a bridge between pure and applied mathematics. The theory has its wide applications in engineering, computer science, physical and life sciences, economics and other fields. Banach [4] introduced the well known classical and valuable theorem in nonlinear analysis, which after his name, is known as the Banach contraction principle. This celebrated principle has been extended and improved by various authors in many ways over the years (see, for instance [5, 8, 26, 28, 31]). Nowadays, fixed point theory has been receiving much attention in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. By extending the Banach contraction principle to partially ordered sets, Turinici [36] laid the foundation for a new trend in fixed point theory. Ran and Reurings [31] developed some applications of Turinici's theorem to matrix equations and they were the first to establish the results in this direction. Their results were then extended by Nieto and Rodríguez-López [28] for non-decreasing mappings. Works noted in the references [12, 18, 19, 29] are some examples in this direction. The existence of the fixed points for

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weak and generalized contractions was extended to partially ordered metric spaces using the altering distance functions by many authors (see [12, 18, 19]). Such functions were introduced by Khan et al. [23], where they presented some fixed point theorems with the help of such functions.

Definition 1.1 ([23]). *An altering distance function is a function $\psi : [0, +\infty) \to [0, +\infty)$ which satisfies*

(ψ_i) *ψ is continuous and nondecreasing;*

(ψ_{ii}) *$\psi(t) = 0$ if and only if $t = 0$.*

Lemma 1.1 ([32]). *If ψ is an altering distance function and $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous function with the condition $\psi(t) > \phi(t)$ for all $t > 0$, then $\phi(0) = 0$.*

The notion of coupled fixed points was introduced by Guo and Lakshmikantham [14]. Since then, the concept has been of interest to many researchers in metrical fixed point theory. The work of Bhaskar and Lakshmikantham [6] is worth mentioning, as they introduced the mixed monotone property, and thereby proved some coupled fixed point theorems for mappings satisfying this property in ordered metric spaces. Lakshmikantham and Ćirić [24] extended the notion of the mixed monotone property to the mixed g-monotone property and generalized the results of Bhaskar and Lakshmikantham [6] by establishing the existence of coupled coincidence points, using a pair of commutative maps. This proved to be a milestone in the development of fixed point theory with applications to partially ordered sets. Since then much work has been done in this direction by different authors. For more details the reader may consult the works in the cited references ([1-3, 7, 9-11, 13, 16, 20-22, 25, 27, 30, 33-35]).

Definition 1.2 ([6]). *Let $(X, \preccurlyeq)$ be a partially ordered set. The mapping $F : X \times X \to X$ is said to have the mixed monotone property if $F(x, y)$ is monotone non-decreasing in x and monotone non-increasing in y; that is, for any $x, y \in X$, we have*

$$x_1, x_2 \in X, \quad x_1 \preccurlyeq x_2 \text{ implies } F(x_1, y) \preccurlyeq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad y_1 \preccurlyeq y_2 \text{ implies } F(x, y_1) \succcurlyeq F(x, y_2).$$

Definition 1.3 ([14, 6]). *An element $(x, y) \in X \times X$, is called a coupled fixed point of the mapping $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.*

Definition 1.4 ([24]). *Let $(X, \preccurlyeq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$ be the mappings. The mapping F is said to have the mixed g-monotone property if $F(x, y)$ is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument; that is, for any $x, y \in X$, we have*

$$x_1, x_2 \in X, \quad gx_1 \preccurlyeq gx_2 \text{ implies } F(x_1, y) \preccurlyeq F(x_2, y)$$

and

$$y_1, y_2 \in X, \quad gy_1 \preccurlyeq gy_2 \text{ implies } F(x, y_1) \succcurlyeq F(x, y_2).$$

Definition 1.5 ([24]). *An element $(x, y) \in X \times X$, is called a coupled coincidence point of the mappings $F : X \times X \to X$and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.*

In this case, we call (gx, gy) the coupled point of coincidence of the mappings F and g. Also, if (gx, gy) is the coupled point of coincidence of the mappings F and g, then (gy, gx) is also the coupled point of coincidence of F and g.

Definition 1.6 ([24]). *An element $(x, y) \in X \times X$, is called a coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.*

Definition 1.7 ([24]). *Let X be a non-empty set. The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be commutative if $gF(x,y) = F(gx, gy)$ for all $x, y \in X$.*

Later, Choudhury and Kundu [9] introduced the notion of compatibility in the context of coupled coincidence point problems and used this notion to improve the results noted in [24].

Definition 1.8 ([9]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if $\lim_{n\to\infty} d(g(F(x_n, y_n), F(gx_n, gy_n)) = 0$ and $\lim_{n\to\infty} d(g(F(y_n, x_n), F(gy_n, gx_n)) = 0$, whenever x_n and y_n are sequences in X such that $\lim_{n\to\infty} F(x_(n,)y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_(n,)x_n) = \lim_{n\to\infty} gy_n = y$ for some $x, y \in X$.*

Abbas et al. [2] introduced the new concept of w-compatible mappings to obtain coupled coincidence point and coupled common fixed point for nonlinear contractive mappings in cone metric space.

Definition 1.9 ([2]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are called w-compatible if $gF(x,y) = F(gx, gy)$ whenever $g(x) = F(x,y)$ and $g(y) = F(y,x)$ for $x, y \in X$.*

Remark 1.1. *We note that compatible mappings are w-compatible but converse need not be true.*

The following example illustrates that w-compatible mappings need not be compatible.

Example 1.1. *Let $X = [3, 20]$. Define $d(x,y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a metric space. Consider the mappings $F : X \times X \to X, g : X \to X$ defined by*

$$F(x,y) = \begin{cases} 3, & \text{if } x = 3 \text{ or } x > 5, y \in X \\ 6, & \text{if } 3 < x \leq 5,\ y \in X, \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 3, & \text{if } x = 3 \\ 18, & \text{if } 3 < x \leq 5 \\ x - 2, & \text{if } x > 5. \end{cases}$$

The only coupled coincidence point of the pair (F, g) is $(3, 3)$. The mappings F and g are non-compatible, since for the sequences $\{x_n\} = \{y_n\} = \{5 + (1/n) : n \geq 1\}$ we have $F(x_n, y_n) \to 3$, $g(x_n) \to 3$, $F(y_n, x_n) \to 3$, $g(y_n) \to 3$, $d(F(gx_n, gy_n), gF(x_n, y_n)) \not\to 0$ as $n \to \infty$. But they are w-compatible since they commute at their (only) coupled coincidence point $(3, 3)$.

On the other hand, by assigning $y = x$ in the Definition 1.9, the concept of w^*-compatible mappings came into existence which was enjoyed by various authors [3, 27, 35] to obtain coupled common fixed points.

Definition 1.10 ([2, 27]). *The mappings $F : X \times X \to X$ and $g : X \to X$ are called w^*-compatible if $gF(x,x) = F(gx, gx)$ whenever $g(x) = F(x,x)$ for $x \in X$.*

Remark 1.2. *Mappings that are w^*-compatible need not be w-compatible, as shown in the following example.*

Example 1.2. *Let $X = [0, \infty)$ and $F : X \times X \to X$ and $g : X \to X$ be defined by*

$$F(x,y) = \begin{cases} 3, & (x,y) = (0,1), \\ 5, & (x,y) = (1,0), \\ 10, & \text{otherwise}, \end{cases} \quad g(x) = \begin{cases} 3, & x = 0, \\ 5, & x = 1, \\ 10, & x \in \{6, 8, 10, \ldots\}, \\ 9, & \text{otherwise}. \end{cases}$$

We note that $g(0) = 3 = F(0,1)$ *and* $g(1) = 5 = F(1,0)$*, but* $gF(0,1) = 9 \neq 10 = F(g0, g1)$*. Hence,* F *and* g *are not* w*-compatible.*

Also, $F(x,x) = gx$ *is possible only if* $x \in \{6, 8, 10, \ldots\}$ *and for all points in this case, we get* $gF(x,x) = 10 = F(gx, gx)$*. Therefore,* F *and* g *are* w^**-compatible.*

Remark 1.3. *It also follows that* w^**-compatible mappings need not be compatible.*

Very recently, Đorić et al. [13] showed that the mixed monotone property in coupled fixed point results can be replaced by another property which is automatically satisfied in the case of a totally ordered space, the case which is the most important in applications. Hence, these results can be applied in a much wider class of problems. Following the work of Đorić et al. [13], different authors generalized the previously presented work in the literature of coupled fixed points (see [3, 11]).

If elements x, y of a partially ordered set $(X, \preccurlyeq)$ are comparable (that is, $x \preccurlyeq y$ or $y \preccurlyeq x$ holds), we will write $x \asymp y$. Let $F : X \times X \to X$ and $g : X \to X$ be two mappings. We shall consider the following condition:

$$\text{if } x, y, u \in X \text{ are such that } gx \asymp F(x,y) = gu, \text{ then } F(x,y) \asymp F(u,v) \text{ for } v \in X. \quad (1.1)$$

In particular, when $g = I_X$ (the identity mapping on X), (1.1) reduces to

$$\text{for all } x, y \in X, if x \asymp F(x,y), \text{ then } F(x,y) \asymp F(F(x,y), v) \text{ for } v \in X. \quad (1.2)$$

Various authors [3, 13] discussed some examples that the condition (1.1) ((1.2), respectively) may be satisfied when F does not have the g-mixed monotone property (monotone property, respectively).

In 2010, As an application of axiom of choice, Haghi et al. [15] proved a lemma and showed that some coincidence point or common fixed point generalizations in fixed point theory are not real generalizations as they could easily be obtained from the corresponding fixed point theorems. The lemma (below) provides an interesting criterion for categorization of generalized common fixed point theorem into classes of well known fixed point theorem that imply each other in the setting of metric spaces or more general metric spaces (for more details see [15]).

Lemma 1.2 ([15])**.** *Let* X *be a nonempty set and* $f : X \to X$ *a function. Then there exists a subset* $E \subseteq X$ *such that* $f(E) = f(X)$ *and* $f : E \to X$ *is one-to-one.*

The technique of Haghi et al. [15] was further extended by Sintunavarat et al. [33] to obtain coupled coincidence points of mappings satisfying contractive conditions without the need of commutative condition in intuitionistic fuzzy normed spaces, which was then used by Hussain et al. [16] to generalize the results noted in the references [1, 9, 24]. In their remarkable work, Jain et al. [20] improved this technique and yields a direct method to compute coupled coincidence points for a pair of mappings without first proving the result for a single mapping.

In this paper, using the technique of Jain et al. [20] we establish some coupled coincidence point results for the pair of non-compatible mappings lacking the mixed monotone property under a new generalized nonlinear contractive condition. By using the concept of w^*-compatible mappings, the presented results are also extended to ensure the existence and uniqueness of coupled fixed points. The main result is equipped with a suitable example. Contractive conditions presented in this paper extend, complement, and unify the contractions given by Bhaskar and Lakshmikantham [6], Berinde [7], Choudhury et al. [10], Harjani et al. [17], Jain et al. [21], Karapinar et al. [22], Luong and Thuan [25], Rasouli and Bahrampour [30] as well as several other contractions as in relevant items from the reference section of this paper and in the literature in general. Very recently, Samet et al. [34], by using the equivalence of the three basic metrics showed that certain coupled fixed point results follows immediately from the well-known fixed point theorems. The importance of our presented work is that, there exists no equivalence of the obtained coupled coincidence and coupled fixed point results with the already existing results in the literature.

2. Coupled coincidence point theorems lacking the mixed g-monotone property

In this section, we give the existence of coupled coincidence point theorems in ordered metric spaces lacking the mixed g-monotone property. Our first main result is the following theorem.

Theorem 2.1. *Let $(X, \preccurlyeq, d)$ be a partially ordered metric space. Suppose that $F : X\times X \to X$ and $g : X \to X$ be the mappings. Assume that the following conditions hold:*

(i) *$g(X)$ is complete;*
(ii) *$F(X \times X) \subseteq g(X)$;*
(iii) *g and F satisfy property (1.1);*
(iv) *there exist $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$;*
(v) *there exists a non-negative real number L such that*

$$\begin{aligned}\psi(d(F(x, y), F(u, v))) \leq \phi(&\max\{d(gx, gu), d(gy, gv)\}) \\ &+ L\min\{(d(F(x, y), gu), d(F(u, v), gx), \\ &\quad d(F(x, y), gx), d(F(u, v), gu))\} \qquad (2.1)\end{aligned}$$

for all $x, y, u, v \in X$ with $gx \asymp gu$ and $gy \asymp gv$, where ψ is an altering distance function and $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous function with the condition that $\psi(t) > \phi(t)$ for all $t > 0$;

(vi) (a) *F and g both are continuous, or*
(b) *$x_n \to x$, when $n \to \infty$ in X, then $x_n \asymp x$ for sufficiently large n.*

Then there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$; that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. By (iv), there exist $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$. Using (ii), we can construct the sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying $g(x_{n+1}) = F(x_n, y_n)$ and $g(y_{n+1}) = F(y_n, x_n)$ for $n = 1, 2, \dots$.

Again by (iv), $gx_0 \asymp F(x_0, y_0) = gx_1$ and $gy_0 \asymp F(y_0, x_0) = gy_1$, then using condition (iii), we obtain that $gx_1 = F(x_0, y_0) \asymp F(x_1, y_1) = gx_2$ and $gy_1 = F(y_0, x_0) \asymp F(y_1, x_1) = gy_2$. Applying induction, we obtain that $gx_{n-1} \asymp gx_n$ and $gy_{n-1} \asymp gy_n$ for all $n \in \mathbb{N}$.

Now by the contractive condition (2.1), we have

$$\begin{aligned}\psi(d(gx_{n+1}, gx_n)) &= \psi(d(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \\ &\leq \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}) \\ &\quad + L\min\{(d(F(x_n, y_n), gx_{n-1}), d(F(x_{n-1}, y_{n-1}), gx_n), \\ &\qquad d(F(x_n, y_n), gx_n), d(F(x_{n-1}, y_{n-1}), gx_{n-1}))\}, \qquad (2.2)\end{aligned}$$

which implies that

$$\psi(d(gx_{n+1}, gx_n)) \leq \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}). \qquad (2.3)$$

Similarly, we obtain that

$$\psi(d(gy_{n+1}, gy_n)) \leq \phi(\max\{d(gy_n, gy_{n-1}), d(gx_n, gx_{n-1})\}). \qquad (2.4)$$

Since $\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}$ is either $d(gx_{n+1}, gx_n)$ or $d(gy_{n+1}, gy_n)$, in both the cases, from (2.3) and (2.4) we obtain that

$$\psi(\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}) \le \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}). \quad (2.5)$$

Using the condition of Theorem 2.1, we obtain that

$$\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} \le \max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}.$$

Let $\delta_n := \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}$, then $\{\delta_n\}$ is a non-increasing sequence of positive real numbers. Therefore, there exists some $\delta \ge 0$, such that $\lim_{n\to\infty} \delta_n = \delta$.

Suppose that $\delta > 0$, on letting $n \to \infty$ on both the sides of (2.5) and the properties of ψ and ϕ, we obtain that

$$\begin{aligned}\psi(\delta) &= \lim_{n\to\infty} \psi(\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\})\\ &\le \lim_{n\to\infty} \phi(\max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}) = \phi(\delta) < \psi(\delta), \quad (2.6)\end{aligned}$$

which is a contradiction. Therefore $\delta = 0$, that is,

$$\lim_{n\to\infty} \delta_n = \lim_{n\to\infty} \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} = 0. \quad (2.7)$$

In what follows, we shall show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not a Cauchy sequence. Then, there exist an $\varepsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that for all positive integers $k, n(k) > m(k) > k$,

$$r_k = \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \ge \varepsilon. \quad (2.8)$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) > k$ and it satisfies (2.8). Then

$$\max\{d(gx_{n(k)-1}, gx_{m(k)}), d(gy_{n(k)-1}, gy_{m(k)})\} < \varepsilon. \quad (2.9)$$

Using the triangle inequality and (2.9), we obtain that

$$\begin{aligned}d(gx_{n(k)}, gx_{m(k)}) &\le d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)})\\ &< d(gx_{n(k)}, gx_{n(k)-1}) + \varepsilon, \quad (2.10)\end{aligned}$$

and

$$\begin{aligned}d(gy_{n(k)}, gy_{m(k)}) &\le d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)})\\ &< d(gy_{n(k)}, gy_{n(k)-1}) + \varepsilon. \quad (2.11)\end{aligned}$$

By (2.8), (2.10) and (2.11), we obtain that

$$\begin{aligned}\varepsilon \le r_k &= \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\}\\ &< \max\{d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\} + \varepsilon. \quad (2.12)\end{aligned}$$

Letting $k \to \infty$ in (2.12) and using (2.7), we obtain that

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty} \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} = \varepsilon. \quad (2.13)$$

Using the triangle inequality,

$$d(gx_{n(k)}, gx_{m(k)}) \le d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)-1}) + d(gx_{m(k)-1}, gx_{m(k)}), \quad (2.14)$$

and

$$d(gy_{n(k)}, gy_{m(k)}) \le d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1}) + d(gy_{m(k)-1}, gy_{m(k)}). \quad (2.15)$$

From (2.14) and (2.15), we obtain that

$$\begin{aligned}\varepsilon \leq r_k &= \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \\ &\leq \max\{d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\} \\ &\quad + \max\{d(gx_{m(k)-1}, gx_{m(k)}), d(gy_{m(k)-1}, gy_{m(k)})\} \\ &\quad + \max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\} \\ &= \delta_{n(k)-1} + \delta_{m(k)-1} + \max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}. \end{aligned} \tag{2.16}$$

Using triangle inequality and (2.9), we obtain that

$$\begin{aligned} d(gx_{n(k)-1}, gx_{m(k)-1}) &\leq d(gx_{n(k)-1}, gx_{m(k)}) + d(gx_{m(k)}, gx_{m(k)-1}) \\ &< d(gx_{m(k)}, gx_{m(k)-1}) + \varepsilon, \end{aligned}$$

and

$$\begin{aligned} d(gy_{n(k)-1}, gy_{m(k)-1}) &\leq d(gy_{n(k)-1}, gy_{m(k)}) + d(gy_{m(k)}, gy_{m(k)-1}) \\ &< d(gy_{m(k)}, gy_{m(k)-1}) + \varepsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} &\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\} \\ &\quad < \max\{d(gx_{m(k)}, gx_{m(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\} + \varepsilon. \end{aligned} \tag{2.17}$$

Taking the limit as $k \to \infty$ in (2.16) and (2.17), and using (2.7) and (2.13), we obtain that

$$\lim_{k\to\infty} \max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\} = \varepsilon. \tag{2.18}$$

Since $gx_{n(k)-1} \asymp gx_{m(k)-1}$ and $gy_{n(k)-1} \asymp gy_{m(k)-1}$, then by (2.1), we obtain that

$$\begin{aligned} \psi(d(gx_{n(k)}, gx_{m(k)})) &= \psi(d(F(x_{n(k)-1}, y_{n(k)-1}), F(x_{m(k)-1}, y_{m(k)-1}))) \\ &\leq \phi(\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}) \\ &\qquad + L\min\{(d(F(x_{n(k)-1}, y_{n(k)-1}), gx_{m(k)-1}), \\ &\qquad d(F(x_{m(k)-1}, y_{m(k)-1}), gx_{n(k)-1}), d(F(x_{n(k)-1}, y_{n(k)-1}), \\ &\qquad gx_{n(k)-1}), d(F(x_{m(k)-1}, y_{m(k)-1}), gx_{m(k)-1}))\} \\ &\leq \phi(\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}) \\ &\quad + L\min\{d(gx_{n(k)}, gx_{n(k)-1}), d(gx_{m(k)}, gx_{m(k)-1})\}. \end{aligned} \tag{2.19}$$

Similarly,

$$\begin{aligned} \psi(d(gy_{n(k)}, gy_{m(k)})) &\leq \phi(\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}) \\ &\quad + L\min\{d(gy_{n(k)}, gy_{n(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\}. \end{aligned} \tag{2.20}$$

Since $\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\}$ is either $d(gx_{n(k)}, gx_{m(k)})$ or $d(gy_{n(k)}, gy_{m(k)})$ using (2.19) and (2.20), we obtain that

$$\begin{aligned} &\psi(\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\}) \\ &\leq \phi(\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}) \\ &\quad + L\min\{d(gx_{n(k)}, gx_{n(k)-1}), d(gx_{m(k)}, gx_{m(k)-1})\} \\ &\quad + L\min\{d(gy_{n(k)}, gy_{n(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\}. \end{aligned} \tag{2.21}$$

Letting $n \to \infty$ in (2.21) and using (2.7), (2.13), (2.18) and the properties of ψ and ϕ, we obtain that

$$\psi(\varepsilon) \leq \phi(\varepsilon) + 2L\min\{0, 0\} < \psi(\varepsilon),$$

which is a contradiction. Therefore, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences and hence by completeness of $g(X)$, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = gx, \tag{2.22}$$

$$\lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = gy. \tag{2.23}$$

Suppose that the condition (a) of hypothesis (vi) holds.

As in [15], let us define a multifunction $G : g(X) \to 2^X$ by $G(y) = \{x \in X : g(x) = y\}$. Using Axiom of choice, we can construct a function $h : g(X) \to X$ so that $h(y) \in G(y)$ for all $y \in g(X)$. Clearly, $g(h(y)) = y$ for all $y \in g(X)$. Define the set $E = \{h(y) : y \in g(X)\} \subseteq X$. Then, the mapping $g : E \to X$ is one-one and $g(E) = g(X)$.

Define another mapping $H : g(E) \times g(E) \to X$ by

$$H(ga, gb) = F(a, b) \quad \text{for all } g(a), g(b) \in g(E)(= g(X)). \tag{2.24}$$

Since $g : E \to X$ is one-one, the mapping H is well-defined. By (2.22), (2.23) and (2.24), we obtain that

$$\begin{aligned} \lim_{n\to\infty} H(gx_n, gy_n) &= \lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} g(x_n) = gx, \\ \lim_{n\to\infty} H(gy_n, gx_n) &= \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} g(y_n) = gy. \end{aligned} \tag{2.25}$$

Also, the continuity of the mappings F and g implies the continuity of the mapping H, then by (2.25) we obtain that

$$H(gx, gy) = gx \quad \text{and} \quad H(gy, gx) = gy. \tag{2.26}$$

By (2.26) and using the definition of H, we obtain that $F(x, y) = gx$ and $F(y, x) = gy$. Next, we suppose that the condition (b) of hypotheses (vi) holds. Then by (2.22) and (2.23), we obtain that $gx_n \asymp gx$ and $gy_n \asymp gy$ for sufficiently large n. For such large n, using the triangle inequality and the monotone property of ψ, we have $\psi(d(F(x, y), gx)) \leq \psi(d(F(x, y), F(x_n, y_n)) + d(F(x_n, y_n), gx))$.

Letting $n \to +\infty$, using the continuity of ψ and (2.22), we have

$$\begin{aligned} \psi(d(F(x, y), gx)) &\leq \psi(\lim_{n\to+\infty} (d(F(x, y), F(x_n, y_n)) + d(F(x_n, y_n), gx))) \\ &= \psi(\lim_{n\to+\infty} (d(F(x, y), F(x_n, y_n)))) \\ &= \lim_{n\to+\infty} \psi(d(F(x, y), F(x_n, y_n))). \end{aligned} \tag{2.27}$$

Also, by (2.1), we have

$$\begin{aligned} \psi(d(F(x, y), F(x_n, y_n))) \leq \phi(&\max\{d(gx, gx_n), d(gy, gy_n)\}) \\ &+ L \min\{(d(F(x, y), gx_n), d(F(x_n, y_n), gx), \\ &\quad d(F(x, y), gx), d(F(x_n, y_n), gx_n))\}. \end{aligned} \tag{2.28}$$

Using (2.27), (2.28), the properties of ϕ and Lemma 1.1, we have

$$\begin{aligned} \psi(d(F(x, y), gx)) &\leq \phi(\lim_{n\to+\infty} \max\{d(gx, gx_n), d(gy, gy_n)\}) \\ &\quad + \lim_{n\to+\infty} L \min\{(d(F(x, y), gx_n), \\ &\quad\quad d(F(x_n, y_n), gx), d(F(x, y), gx), d(F(x_n, y_n), gx_n))\} \\ &= \phi(\max\{0, 0\}) + 0 = 0. \end{aligned}$$

Hence $d(F(x, y), gx) \leq 0$, which implies that $F(x, y) = gx$. Similarly, we obtain that $F(y, x) = gy$. This completes the proof of our main result. □

Remark 2.1. *In Theorem 2.1, condition (iii) (that is, property* (1.1)*) is a substitution for the mixed monotone property that has been used in most of the coupled fixed point results so far. This condition is trivially satisfied if the order $\preccurlyeq$ on X is total, which is the case in most of the examples in articles mentioned in the reference section. Also, in the obtained result, the mappings are neither commuting nor compatible. Further, the completeness of the space (X, d) has also been replaced by the completeness of the range subspace.*

Next, we give an example to support Theorem 2.1.

Example 2.1. *Let $(X, \preccurlyeq)$ be the partially ordered set with $X = (-1, 1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preccurlyeq$. Define $d(x, y) = |x - y|$ for all $x, y \in X$, then (X, d) is a metric space which is not complete. Let $g : X \to X$ and $F : X \times X \to X$ be respectively defined by $gx = \frac{x^2+1}{2}$ and $F(x, y) = \frac{x^2+y^2+4}{8}$. Clearly, the mappings F and g are not compatible. Consider $y_1 = \frac{-1}{4}$ and $y_2 = \frac{-1}{2}$, then we have $gy_1 = g(\frac{-1}{4}) = \frac{17}{32} \leq \frac{5}{8} = g(\frac{-1}{2}) = gy_2$, but for $x = 0$, we have $F(x, y_1) = F(0, \frac{-1}{4}) = \frac{65}{128} \leq \frac{17}{32} = F(0, \frac{-1}{2}) = F(x, y_2)$. So the mapping F does not satisfy the mixed g-monotone property. Clearly, $g(X) = [\frac{1}{2}, 1]$ is complete and $F(X \times X) \subseteq g(X)$. Let the mappings $\psi, \phi : [0, +\infty) \to [0, +\infty)$ be respectively defined by $\psi(t) = \frac{t}{2}$ and $\phi(t) = \frac{t}{4}$ for $t \in [0, +\infty)$. Then ψ is an altering distance function and ϕ is continuous such that $\psi(t) > \phi(t)$ for all $t > 0$. Next, we verify the inequality* (2.1)*. Let $L \geq 0$.*

For $x, y, u, v \in X$ satisfying $gx \asymp gu$ and $gy \asymp gv$, we have

$$\begin{aligned}
\psi(d(F(x,y), F(u,v))) &= \frac{1}{2}\left(\left|\frac{x^2+y^2+4}{8} - \frac{u^2+v^2+4}{8}\right|\right) \\
&= \frac{1}{8}\left(\left|\frac{x^2+y^2}{2} - \frac{u^2+v^2}{2}\right|\right) \\
&\leq \frac{1}{8}\left(\left|\frac{x^2-u^2}{2}\right| + \left|\frac{y^2-v^2}{2}\right|\right) \\
&= \frac{1}{8}\left(\left|\frac{x^2+1}{2} - \frac{u^2+1}{2}\right| + \left|\frac{y^2+1}{2} - \frac{v^2+1}{2}\right|\right) \\
&= \frac{1}{8}(d(gx, gu) + d(gy, gv)) \leq \frac{1}{4}(\max\{d(gx, gu), d(gy, gv)\}) \\
&\leq \phi(\max\{d(gx, gu), d(gy, gv)\}) \\
&\qquad + L \min\{(d(F(x,y), gu), d(F(u,v), gx), \\
&\qquad\quad d(F(x,y), gx), d(F(u,v), gu))\}.
\end{aligned}$$

Further, the other conditions in Theorem 2.1 are satisfied. Now, we can apply Theorem 2.1 to conclude the existence of coupled coincidence point of F and g that is a point $(0, 0)$.

Corollary 2.1. *Let $(X, \preccurlyeq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X \times X \to X$ be the mapping such that the following conditions hold:*

(i) *F satisfy property* (1.2)*;*
(ii) *there exist $x_0, y_0 \in X$ such that $x_0 \asymp F(x_0, y_0)$ and $y_0 \asymp F(y_0, x_0)$;*
(iii) *there exists a non-negative real number L such that*

$$\begin{aligned}
\psi(d(F(x,y), F(u,v))) \leq\ & \phi(\max\{d(x,u), d(y,v)\}) \\
& + L \min\{(d(F(x,y), u), d(F(u,v), x), \\
& \quad d(F(x,y), x), d(F(u,v), u))\} \qquad (2.29)
\end{aligned}$$

for all $x, y, u, v \in X$ with $x \asymp u$ and $y \asymp v$, where ψ is an altering distance function and $\phi : [0, +\infty) \to [0, +\infty)$ is a continuous function with the condition that $\psi(t) > \phi(t)$ for all $t > 0$;

(iv) (a) *F is continuous, or*

(b) *$x_n \to x$, when $n \to \infty$ in X, then $x_n \asymp x$ for sufficiently large n.*

Then there exist $x, y \in X$ such that $F(x, y) = x$ and $F(y, x) = y$; that is, F has a coupled fixed point $(x, y) \in X \times X$.

Proof. Taking $g = I_X$ (the identity mapping on X) in Theorem 2.1, the result follows immediately. □

Remark 2.2. *Corollary 2.1 improves the results due to Harjani et al.* [17, Theorems 2, 3]. *On setting $L = 0$ and substituting $\psi(x) - \phi(x)$ for $\phi(x)$ in Corollary 2.1, we can see that the contractive condition* (2.29) *becomes*

$$\psi(d(F(x, y), F(u, v))) \leq \psi(\max\{d(x, u), d(y, v)\}) - \phi(\max\{d(x, u), d(y, v)\}), \qquad (2.30)$$

which is the same contractive condition as discussed by Harjani et al. [17] *but our obtained result will be more general than the results of Harjani et al.* [17] *since in our results, we do not require mixed monotone operators, that is, we do not require the functions satisfying mixed monotone property. We illustrate this fact using the following example:*

Example 2.2. *Let $(X, \preccurlyeq)$ be the partially ordered set with $X = [-1, 1]$ and the natural ordering $\leq$ of the real numbers as the partial ordering $\preccurlyeq$. Define $d(x, y) = |x - y|$ for all $x, y \in X$, then (X, d) is a complete metric space. Let the mapping $F : X \times X \to X$ be defined by $F(x, y) = \frac{x^2+y^2}{8}$. Next, we consider $y_1 = 1$ and $y_2 = \frac{1}{2}$, then we have $y_1 > y_2$ but for $x = 0$, we have $F(x, y_1) = F(0, 1) = \frac{1}{8} > \frac{1}{32} = F(0, \frac{1}{2}) = F(x, y_2)$. Clearly, the mapping F does not satisfy the mixed monotone property. Therefore, Theorems 2, 3 of Harjani et al.* [17] *cannot be used to reach the conclusion. Let the mappings $\psi, \phi : [0, +\infty) \to [0, +\infty)$ be defined by $\psi(t) = \frac{t}{2}$ and $\phi(t) = \frac{t}{4}$ for $t \in [0, +\infty)$. Now, we verify the inequality* (2.30).

For $x, y, u, v \in X$ satisfying $x \asymp u$ and $y \asymp v$, we have

$$\begin{aligned}
\psi(d(F(x, y), F(u, v))) &= \frac{1}{2}\left(\left|\frac{x^2+y^2}{8} - \frac{u^2+v^2}{8}\right|\right) \\
&= \frac{1}{16}(|(x^2 - u^2) - (y^2 - v^2)|) \\
&\leq \frac{1}{16}(|x^2 - u^2| + |y^2 - v^2|) \\
&= \frac{1}{16}(|x - u||x + u| + |y - v|\ |y + v|) \\
&\leq \frac{1}{16}(|x - u|(|x| + |u|) + |y - v|(|y| + |v|)) \\
&\leq \frac{1}{16}(|x - u|(1 + 1) + |y - v|(1 + 1)) \quad (\textit{since } x, y, u, v \in [-1, 1]) \\
&= \frac{1}{8}(|x - u| + |y - v|) \\
&= \frac{1}{8}(d(x, u) + d(y, v)) \\
&\leq \frac{1}{4}(\max\{d(x, u), d(y, v)\}) \\
&= \psi(\max\{d(x, u), d(y, v)\}) - \phi(\max\{d(x, u), d(y, v)\}).
\end{aligned}$$

Hence, the inequality (2.30) *holds. Applying Corollary 2.1 with Remark 2.2, we can conclude that* $(0, 0)$ *is the coupled fixed point of* F.

Corollary 2.2. *Let* $(X, \preccurlyeq, d)$ *be a partially ordered metric space. Suppose that* $F : X \times X \to X$ *and* $g : X \to X$ *be the mappings such that the following conditions hold:*

(i) $g(X)$ *is complete;*
(ii) $F(X \times X) \subseteq g(X)$*;*
(iii) g *and* F *satisfy property* (1.1)*;*
(iv) *there exist* $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0, y_0)$ *and* $gy_0 \asymp F(y_0, x_0)$*;*
(v) *there exist non-negative real numbers* α, β *with* $\alpha + \beta < 1$ *such that*

$$\begin{aligned}\psi(d(F(x,y),F(u,v))) \leq\ & \alpha d(gx,gu) + \beta d(gy,gv)\\ & + L\min\{(d(F(x,y),gu),d(F(u,v),gx),\\ & \quad d(F(x,y),gx),d(F(u,v),gu))\}\end{aligned} \tag{2.31}$$

for all $x, y, u, v \in X$ *with* $gx \asymp gu$ *and* $gy \asymp gv$*, where* ψ *is an altering distance function with the condition that* $\psi(t) > (\alpha + \beta)(t)$ *for all* $t > 0$*;*

(vi) (a) F *and* g *both are continuous, or*
(b) $x_n \to x$*, when* $n \to \infty$ *in* X*, then* $x_n \asymp x$ *for sufficiently large* n*.*

Then there exist $x, y \in X$ *such that* $F(x, y) = g(x)$ *and* $F(y, x) = g(y)$*; that is,* F *and* g *have a coupled coincidence point* $(x, y) \in X \times X$*.*

Proof. For all $x, y, u, v \in X$, $\alpha, \beta \geq 0$ with $\alpha + \beta < 1$, we have

$$\begin{aligned}\alpha d(gx,gu) + \beta d(gy,gv) &\leq (\alpha+\beta)\max\{d(gx,gu),d(gy,gv)\}\\ &= \phi(\max\{d(gx,gu),d(gy,gv)\}),\end{aligned}$$

where $\phi(t) = (\alpha + \beta)(t)$ for all $t \geq 0$. It is easy to observe that ψ and ϕ satisfy all the properties as in Theorem 2.1 and hence the result follows immediately by applying Theorem 2.1. □

Remark 2.3. *Corollary 2.2 improves and extends the result due to Luong and Thuan* [25, Theorem 2.1]. *On setting* $\psi(x) = t$ *for all* $t \geq 0$ *and taking* $g = I_X$ *(the identity mapping on* X*), we can see that the contractive condition* (2.31) *becomes*

$$\begin{aligned}d(F(x,y),F(u,v)) \leq\ & \alpha d(x,u) + \beta d(y,v)\\ & + L\min\{(d(F(x,y),u),d(F(u,v),x),\\ & \quad d(F(x,y),x),d(F(u,v),u))\},\end{aligned} \tag{2.32}$$

which is the same contractive condition as produced by Luong and Thuan [25] *but our obtained result will be more general than the Theorem 2.1 of Luong and Thuan* [25]*, which can be illustrated using the following example:*

Example 2.3. *Let* $(X, \preccurlyeq)$ *be the partially ordered set with* $X = [-1, 1]$ *and the natural ordering* $\leq$ *of the real numbers as the partial ordering* $\preccurlyeq$*. Define* $d(x, y) = |x - y|$ *for all* $x, y \in X$*, then* (X, d) *is a complete metric space. Let the mapping* $F : X \times X \to X$ *be defined by* $F(x, y) = \frac{x^2+y^2}{16}$*. Clearly, the mapping* F *does not satisfy the mixed monotone property. Therefore, Theorem 2.1 of Luong and Thuan* [25] *is not applicable here. Take* $\alpha = \frac{1}{4}$ *and* $\beta = \frac{1}{3}$ *and* $L \geq 0$*. Now, we verify the inequality* (2.32)*.*

For $x, y, u, v \in X$ *satisfying* $x \asymp u$ *and* $y \asymp v$*, we have*

$$d(F(x,y),F(u,v)) = \left(\left|\frac{x^2+y^2}{16} - \frac{u^2+v^2}{16}\right|\right)$$

$$
\begin{aligned}
&= \frac{1}{16}(|(x^2-u^2)-(y^2-v^2)|) \\
&\leq \frac{1}{16}(|x^2-u^2|+|y^2-v^2|) \\
&= \frac{1}{16}(|x-u|\ |x+u|+|y-v|\,|y+v|) \\
&\leq \frac{1}{16}(|x-u|(|x|+|u|)+|y-v|(|y|+|v|)) \\
&\leq \frac{1}{16}(|x-u|(1+1)+|y-v|(1+1)) \quad (\textit{since } x,y,u,v \in [-1,1]) \\
&= \frac{1}{8}(|x-u|+|y-v|) \\
&= \frac{1}{8}d(x,u)+\frac{1}{8}d(y,v) \\
&\leq \alpha d(x,u)+\beta d(y,v) \\
&\quad + L\min\{(d(F(x,y),u),d(F(u,v),x),d(F(x,y),x),d(F(u,v),u))\}.
\end{aligned}
$$

Hence, the inequality (2.32) *holds. Applying Corollary 2.2 with Remark 2.3, we obtain that* $(0,0)$ *is the coupled fixed point of* F*.*

Corollary 2.3. *Let* $(X,\preccurlyeq,d)$ *be a partially ordered metric space. Suppose that* $F : X \times X \to X$ *and* $g : X \to X$ *be the mappings such that the following conditions hold:*

(i) $g(X)$ *is complete;*
(ii) $F(X \times X) \subseteq g(X)$*;*
(iii) g *and* F *satisfy property* (1.1)*;*
(iv) *there exist* $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0,y_0)$ *and* $gy_0 \asymp F(y_0,x_0)$*;*
(v) *there exists a non-negative real number* L *such that*

$$
\begin{aligned}
d(F(x,y),F(u,v)) \leq{}& \phi(\max\{d(gx,gu),d(gy,gv)\}) \\
&+ L\min\{(d(F(x,y),gu),d(F(u,v),gx), \\
&\quad d(F(x,y),gx),d(F(u,v),gu))\} \qquad (2.33)
\end{aligned}
$$

for all $x,y,u,v \in X$ *with* $gx \asymp gu$ *and* $gy \asymp gv$*, where* $\phi : [0,+\infty) \to [0,+\infty)$ *is a continuous function with the condition that* $\phi(t) < t$ *for all* $t > 0$*;*

(vi) (a) F *and* g *both are continuous, or*
(b) $x_n \to x$*, when* $n \to \infty$ *in* X*, then* $x_n \asymp x$ *for sufficiently large* n*.*

Then there exist $x,y \in X$ *such that* $F(x,y) = g(x)$ *and* $F(y,x) = g(y)$*; that is,* F *and* g *have a coupled coincidence point* $(x,y) \in X \times X$*.*

Proof. Taking $\psi(t) = t$ in Theorem 2.1, the result follows immediately. □

Remark 2.4. (i) *Corollary 2.3 improves the recent results due to Karapinar et al.* [22, Theorems 2.1, 2.3]. *Contractive condition* (2.33) *is the same as contractive condition discussed by Karapinar et al.* [22] *but Corollary 2.3 is more general than the results of Karapinar et al.* [22]*, this can be easily justified using Remark 2.1.*

(ii) *Also in Example 2.1, by redefining the mappings* $\psi,\phi : [0,+\infty) \to [0,+\infty)$ *respectively as* $\psi(t) = t$ *and* $\phi(t) = \frac{t}{2}$ *for* $t \in [0,+\infty)$ *and then following the similar steps as in Example 2.1, it is easy to see that by applying Corollary 2.3, we can obtain the existence of coupled coincidence point of the mappings* F *and* g*, that is a point* $(0,0)$ *but here we cannot apply Theorems 2.1 and 2.3 of Karapinar et al.* [22] *since the mappings* F *and* g *are not compatible, the mapping* F *does not satisfy the mixed* g*-monotone property and the space* (X,d) *is not complete.*

Corollary 2.4. *Let $(X, \preccurlyeq, d)$ be a partially ordered metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ be the mappings such that the following conditions hold:*

(i) *$g(X)$ is complete;*
(ii) *$F(X \times X) \subseteq g(X)$;*
(iii) *g and F satisfy property (1.1);*
(iv) *there exist $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$;*
(v) *there exists $k \in [0, 1)$ such that*

$$\begin{aligned}\psi(d(F(x,y), F(u,v))) \le \frac{k}{2}&[d(gx, gu) + d(gy, gv)]\\ &+ L\min\{(d(F(x,y), gu), d(F(u,v), gx),\\ &\quad d(F(x,y), gx), d(F(u,v), gu))\}\end{aligned} \tag{2.34}$$

for all $x, y, u, v \in X$ with $gx \asymp gu$ and $gy \asymp gv$, where ψ is an altering distance function with the condition that $\psi(t) > kt$ for all $t > 0$;
(vi) (a) *F and g both are continuous, or*
(b) *$x_n \to x$, when $n \to \infty$ in X, then $x_n \asymp x$ for sufficiently large n.*

Then there exist $x, y \in X$ such that $F(x, y) = g(x)$ and $F(y, x) = g(y)$; that is, F and g have a coupled coincidence point $(x, y) \in X \times X$.

Proof. Taking $\alpha = \beta = \frac{k}{2}$ with $k \in [0, 1)$ in Corollary 2.2, the result follows immediately. □

Remark 2.5. *Corollary 2.4 improves and extends the results due to Bhaskar and Lakshmikantham* [6, Theorems 2.1, 2.2]. *On setting $L = 0$, $\psi(t) = t$ for all $t \ge 0$ and $g = I_X$ (the identity mapping on X), we can see that the contractive condition* (2.34) *becomes*

$$d(F(x,y), F(u,v)) \le \frac{k}{2}[d(x,u) + d(y,v)], \tag{2.35}$$

which is the same contractive condition as discussed by Bhaskar and Lakshmikantham [6]. *Interestingly, we again note that our obtained result will be more general than the results of Bhaskar and Lakshmikantham* [6]. *In order to support this fact, we have the following illustration:*

Example 2.4. *Let $(X, \preccurlyeq)$ be the partially ordered set with $X = [-1, 1]$ and the natural ordering $\le$ of the real numbers as the partial ordering $\preccurlyeq$. Define $d(x, y) = |x - y|$ for all $x, y \in X$, then (X, d) is a complete metric space. Let the mapping $F : X \times X \to X$ be defined by $F(x, y) = \frac{x+y+3}{8}$. Clearly, the mapping F does not satisfy the mixed monotone property. Therefore, Theorems 2.1, 2.2 of Bhaskar and Lakshmikantham* [6] *cannot be used to reach the conclusion. Take $k = \frac{1}{4} \in [0, 1)$.*

Now, we verify the inequality (2.35).

For $x, y, u, v \in X$ satisfying $x \asymp u$ and $y \asymp v$, we have

$$\begin{aligned}d(F(x,y), F(u,v)) &= \left(\left|\frac{x+y+3}{8} - \frac{u+v+3}{8}\right|\right)\\ &= \frac{1}{8}(|(x-u) - (y-v)|)\\ &\le \frac{1}{8}(|x-u| + |y-v|)\\ &= \frac{k}{2}[d(x,u) + d(y,v)].\end{aligned}$$

Hence, the inequality (2.35) *holds. Applying Corollary 2.4 with Remark 2.5, we conclude that* $(\frac{1}{2}, \frac{1}{2})$ *is the coupled fixed point of* F.

Corollary 2.5. *Let* $(X, \preccurlyeq, d)$ *be a partially ordered metric space. Suppose that* $F : X \times X \to X$ *and* $g : X \to X$ *be the mappings such that the following conditions hold:*

(i) $g(X)$ *is complete;*
(ii) $F(X \times X) \subseteq g(X)$*;*
(iii) g *and* F *satisfy property* (1.1)*;*
(iv) *there exist* $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0, y_0)$ *and* $gy_0 \asymp F(y_0, x_0)$*;*
(v) *there exists* $k \in [0, 1)$ *such that*

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k[d(gx, gu) + d(gy, gv)] \tag{2.36}$$

for all $x, y, u, v \in X$ *with* $gx \asymp gu$ *and* $gy \asymp gv$*;*

(vi) (a) F *and* g *both are continuous, or*
(b) $x_n \to x$*, when* $n \to \infty$ *in* X*, then* $x_n \asymp x$ *for sufficiently large* n.

Then there exist $x, y \in X$ *such that* $F(x, y) = g(x)$ *and* $F(y, x) = g(y)$*; that is,* F *and* g *have a coupled coincidence point* $(x, y) \in X \times X$.

Proof. On setting $\psi(t) = t$ and $L = 0$ in the inequality (2.34), for all $x, y, u, v \in X$ with $gx \asymp gu$ and $gy \asymp gv$, we have

$$d(F(x, y), F(u, v)) \leq \frac{k}{2}[d(gx, gu) + d(gy, gv)]$$

and

$$d(F(y, x), F(v, u)) \leq \frac{k}{2}[d(gy, gv) + d(gx, gu)].$$

This implies that for all $x, y, u, v \in X$ with $gx \asymp gu$ and $gy \asymp gv$, we have

$$d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \leq k[d(gx, gu) + d(gy, gv)].$$

Hence the result follows immediately. □

Remark 2.6. *Corollary 2.5 improves the recent result of Jain et al* [21, Corollary 2.3].

Example 2.5. *Let* $(X, \preccurlyeq)$ *be the partially ordered set with* $X = [-1, 1)$ *and the natural ordering* $\leq$ *of the real numbers as the partial ordering* $\preccurlyeq$*. Define* $d(x, y) = |x - y|$ *for all* $x, y \in X$*, then* (X, d) *is a metric space which is not complete. Let* $g : X \to X$ *and* $F : X \times X \to X$ *be respectively defined by* $gx = \frac{2x^2+1}{4}$ *and* $F(x, y) = \frac{x^2+y^2+2}{8}$*. Clearly, the mappings* F *and* g *are not compatible. Consider* $y_1 = \frac{1}{4}$ *and* $y_2 = \frac{1}{2}$*, then we have* $gy_1 = g(\frac{1}{4}) = \frac{9}{32} < \frac{12}{32} = g(\frac{1}{2}) = gy_2$ *but for* $x = 0$*, we have* $F(x, y_1) = F(0, \frac{1}{4}) = \frac{33}{128} < \frac{36}{128} = F(0, \frac{1}{2}) = F(x, y_2)$*. So the mapping* F *does not satisfy the mixed* g*-monotone property. Clearly,* $g(X) = [\frac{1}{4}, \frac{3}{4}]$ *is complete and* $F(X \times X) \subseteq g(X)$*. Next, we verify the inequality* (2.36).

For $x, y, u, v \in X$ *satisfying* $gx \asymp gu$ *and* $gy \asymp gv$*, we have*

$$\begin{aligned}
&d(F(x, y), F(u, v)) + d(F(y, x), F(v, u)) \\
&= \left(\left|\frac{x^2 + y^2 + 2}{8} - \frac{u^2 + v^2 + 2}{8}\right|\right) + \left(\left|\frac{y^2 + x^2 + 2}{8} - \frac{v^2 + u^2 + 2}{8}\right|\right) \\
&= \left|\frac{x^2 + y^2}{4} - \frac{u^2 + v^2}{4}\right| \\
&= \frac{1}{2}\left(\left|\frac{2x^2 + 2y^2}{4} - \frac{2u^2 + 2v^2}{4}\right|\right)
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{2}\Big(\Big|\frac{2x^2-2u^2}{4}\Big|+\Big|\frac{2y^2-2v^2}{4}\Big|\Big)\\
&=\frac{1}{2}\Big(\Big|\frac{2x^2+1}{4}-\frac{2u^2+1}{4}\Big|+\Big|\frac{2y^2+1}{4}-\frac{2v^2+1}{4}\Big|\Big)\\
&=k[d(gx,gu)+d(gy,gv)].
\end{aligned}
$$

Further, the other conditions in Corollary 2.5 are satisfied. Now, we can apply Corollary 2.5 to conclude the existence of coupled coincidence point of F and g that is a point $(0,0)$. Note that, here Corollary 2.3 of Jain et al. [21] *cannot be applied to conclude the result since in the presented illustration F does not satisfy the mixed g-monotone property, the space X is not complete and the pair (F,g) is not compatible.*

Remark 2.7. *Setting $g = I_X$ (the identity mapping on X) in Corollary 2.5, we can see that the obtained result yields the same contractive condition as produced by Berinde* [7, Theorem 3].

Corollary 2.6. *Let $(X, \preccurlyeq, d)$ be a partially ordered metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ be the mappings such that the following conditions hold:*

(i) *$g(X)$ is complete;*
(ii) *$F(X \times X) \subseteq g(X)$;*
(iii) *g and F satisfy property* (1.1)*;*
(iv) *there exist $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0, y_0)$ and $gy_0 \asymp F(y_0, x_0)$;*
(v) *there exists a non-negative real number L such that*

$$
\begin{aligned}
\psi(d(F(x,y),F(u,v))) \leq\ & \psi(\max\{d(gx,gu),d(gy,gv)\}) - \phi(\max\{d(gx,gu),d(gy,gv)\})\\
&+ L\min\{(d(F(x,y),gu),d(F(u,v),gx),\\
&\quad d(F(x,y),gx),d(F(u,v),gu))\} \qquad (2.37)
\end{aligned}
$$

for all $x, y, u, v \in X$ with $gx \asymp gu$ and $gy \asymp gv$, where ψ is an altering distance function and $\phi : [0,+\infty) \to [0,+\infty)$ is a continuous function with the condition that $\psi(t) > \phi(t)$ for all > 0;

(vi) (a) *F and g both are continuous, or*
(b) *$x_n \to x$, when $n \to \infty$ in X, then $x_n \asymp x$ for sufficiently large n.*

Then there exist $x, y \in X$ such that $F(x,y) = g(x)$ and $F(y,x) = g(y)$; that is, F and g have a coupled coincidence point $(x,y) \in X \times X$.

Proof. Substituting $\psi(x)-\phi(x) for \phi(x)$ in Theorem 2.1, the result follows immediately. □

Remark 2.8. *Corollary 2.6 improves the result of Choudhury et al.* [10, Theorem 3.1]. *On setting $L = 0$, we can see that the contractive condition* (2.37) *becomes*

$$
\psi(d(F(x,y),F(u,v))) \leq \psi(\max\{d(gx,gu),d(gy,gv)\}) - \phi(\max\{d(gx,gu),d(gy,gv)\}), \qquad (2.38)
$$

which is the same contractive condition as discussed by Choudhury et al. [10] *but in view of Remark 2.1, our obtained result will be more general than the result of Choudhury et al.* [10]. *Also, Example 2.1 can be used in support of this fact.*

Corollary 2.7. *Let $(X, \preccurlyeq, d)$ be a partially ordered metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ be the mappings such that the following conditions hold:*

(i) *$g(X)$ is complete;*
(ii) *$F(X \times X) \subseteq g(X)$;*
(iii) *g and F satisfy property* (1.1)*;*

(iv) *there exist* $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0, y_0)$ *and* $gy_0 \asymp F(y_0, x_0)$*;*
(v) *there exists a non-negative real number* L *such that*

$$d(F(x,y),F(u,v)) \leq \beta(\max\{d(gx,gu),d(gy,gv)\})\max\{d(gx,gu),d(gy,gv)\} + L\min\{(d(F(x,y),gu),d(F(u,v),gx),d(F(x,y),gx),d(F(u,v),gu))\} \tag{2.39}$$

for all $x, y, u, v \in X$ *with* $gx \asymp gu$ *and* $gy \asymp gv$*, where the function* $\beta : [0, +\infty) \to [0, 1)$ *satisfies the condition that* $\beta(t_n) \to 1 \Rightarrow t_n \to 0$*.*
(vi) (a) F *and* g *both are continuous, or*
(b) $x_n \to x$*, when* $n \to \infty$ *in* X*, then* $x_n \asymp x$ *for sufficiently large* n*.*

Then there exist $x, y \in X$ *such that* $F(x, y) = g(x)$ *and* $F(y, x) = g(y)$*; that is,* F *and* g *have a coupled coincidence point* $(x, y) \in X \times X$*.*

Proof. On setting ψ to be the identity function and $\phi(x) = \beta(x)x$ (where, the function $\beta : [0, +\infty) \to [0, 1)$ satisfies the condition that $\beta(t_n) \to 1 \Rightarrow t_n \to 0$) in Theorem 2.1, the result follows immediately. □

Remark 2.9. *Corollary 2.7 improves and extends the results of Rasouli and Bahrampour* [30, Theorem 3]*. Setting* $L = 0$ *and* $g = I_X$ *(the identity mapping on* X*), we can see that the contractive condition* (2.39) *becomes*

$$d(F(x,y),F(u,v)) \leq \beta(\max\{d(x,u),d(y,v)\})\max\{d(x,u),d(y,v)\}, \tag{2.40}$$

which is the same contractive condition as produced by Rasouli and Bahrampour [30]*. The following illustration supports the usability of our obtained result:*

Example 2.6. *Let* $(X, \preccurlyeq)$ *be the partially ordered set with* $X = [0, 1]$ *and the natural ordering* $\leq$ *of the real numbers as the partial ordering* $\preccurlyeq$*. Define* $d(x, y) = |x - y|$ *for all* $x, y \in X$*, then* (X, d) *is a complete metric space. Define the mapping* $F : X \times X \to X$ *by* $F(x, y) = \frac{x+y}{24}$*. Clearly, the mapping* F *does not satisfy the mixed monotone property. Therefore, Theorem 3 of Rasouli and Bahrampour* [30] *cannot be applied to reach the conclusion. Define the function* $\beta : [0, +\infty) \to [0, 1)$ *by* $\beta(t) = \frac{e^{-t}}{t+1}$ *for* $t > 0$ *and* $\beta(0) \in [0, 1)$*, then* $\beta(t_n) \to 1 \Rightarrow t_n \to 0$*. Then, for all* $x, y, u, v \in X$ *with* $x \asymp u$ *and* $y \asymp v$*, we have*

$$\begin{aligned} d(F(x,y),F(u,v)) &= d\left(\frac{x+y}{24}, \frac{u+v}{24}\right) \\ &= \left|\frac{x+y}{24} - \frac{u+v}{24}\right| \\ &\leq \frac{1}{24}\{|x-u| + |v-y|\} \\ &= \frac{1}{24}\{d(x,u) + d(y,v)\} \\ &\leq \frac{1}{12}\max\{d(x,u),d(y,v)\}. \end{aligned}$$

Clearly, $\max\{d(x, u), d(y, v)\} \in [0, 1]$ *for all* $x, y, u, v \in [0, 1]$*. We consider* $\max\{d(x, u), d(y, v)\} \neq 0$*, since otherwise condition* (2.40) *is obvious.*

Also,

$$\begin{aligned} &\beta(\max\{d(x,u),d(y,v)\})\max\{d(x,u),d(y,v)\} \\ &= \frac{e^{-(\max\{d(x,u),d(y,v)\})}}{(\max\{d(x,u),d(y,v)\}) + 1} \cdot (\max\{d(x,u),d(y,v)\}). \end{aligned}$$

Since $\frac{e^{-t}}{t+1} \geq \frac{1}{2e} > \frac{1}{12}$ *for* $t \in [0,1]$*, thus it follows that*

$$d(F(x,y), F(u,v)) \leq \beta(\max\{d(x,u), d(y,v)\}) \max\{d(x,u), d(y,v)\}.$$

Hence contractive condition(2.40) *holds. Applying Corollary 2.7 together with Remark 2.9, we obtain that* $(0,0)$ *is the coupled fixed point of the mapping* F.

Remark 2.10. *Contraction* (2.1) *becomes the contraction as discussed in* [11] *for* ψ *to be the identity function.*

Remark 2.11. *Very recently, using the equivalence of the three basic metrics, Samet et al.* [34] *showed that many of the coupled fixed point theorems are immediate consequences of well-known fixed point theorems in the literature. In our obtained results, it is easy to see that if* $L > 0$*, there is no such equivalence and hence, the obtained results are not the consequences of the known fixed point theorems.*

3. Common coupled fixed point theorems lacking the mixed monotone property

In this section we prove the existence and uniqueness of coupled fixed points. Before we proceed, we need to consider the following notion:

For a partially ordered set $(X, \preccurlyeq)$, we will denote also by $\preccurlyeq$ the order on $X \times X$ given by

$$(u,v) \preccurlyeq (x,y) \Rightarrow u \preccurlyeq x,\ y \preccurlyeq v \text{ for all } (x,y),(u,v) \in X \times X. \tag{3.1}$$

In this case, we say that (u,v) and (x,y) are comparable if either $(u,v) \preccurlyeq (x,y)$ or $(x,y) \preccurlyeq (u,v)$ and we will also denote this fact by $(u,v) \asymp (x,y)$.

Theorem 3.1. *In addition to the hypotheses of Theorem 2.1, suppose that for every* $(x,y),(x^*,y^*) \in X \times X$ *there exists a* $(u,v) \in X \times X$ *such that* $(F(u,v), F(v,u)) \asymp (F(x,y), F(y,x))$ *and* $(F(u,v), F(v,u)) \asymp (F(x^*,y^*), F(y^*,x^*))$*. If the pair of the mappings* (F,g) *is* w^**-compatible, then* F *and* g *have a unique coupled common fixed point, that is, there exists a unique* $(\bar{x},\bar{y}) \in X \times X$ *such that* $\bar{x} = g(\bar{x}) = F(\bar{x},\bar{y})$ *and* $\bar{y} = g(\bar{y}) = F(\bar{y},\bar{x})$.

Proof. From Theorem 2.1, the set of coupled coincidences of the mappings F and g is non-empty. In order to prove the theorem, we shall first show that if (x,y) and (x^*,y^*) are coupled coincidence points, that is, if $g(x) = F(x,y), g(y) = F(y,x)$ and $g(x^*) = F(x^*,y^*), g(y^*) = F(y^*,x^*)$, then

$$g(x) = g(x^*) \quad \text{and} \quad g(y) = g(y^*). \tag{3.2}$$

By assumption, there exists $(u,v) \in X \times X$ such that $(F(u,v), F(v,u)) \asymp (F(x,y), F(y,x))$ and $(F(u,v), F(v,u)) \asymp (F(x^*,y^*), F(y^*,x^*))$. Put $u_0 = u, v_0 = v$ and choose $u_1, v_1 \in X$ so that $gu_1 = F(u_0,v_0), gv_1 = F(v_0,u_0)$.

Then, as in the proof of Theorem 2.1, we can inductively define sequences $\{gu_n\}$ and $\{gv_n\}$ such that $gu_{n+1} = F(u_n,v_n)$ and $gv_{n+1} = F(v_n,u_n)$.

Further, set $x_0 = x$, $y_0 = y$, $x_0^* = x^*$, $y_0^* = y^*$ and on the same way define the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gx_n^*\}$, $\{gy_n^*\}$. Then, it is easy to show that

$$gx_{n+1} = F(x_n,y_n), \quad gy_{n+1} = F(y_n,x_n)$$

and

$$gx_{n+1}^* = F(x_n^*,y_n^*),\ gy_{n+1}^* = F(y_n^*,x_n^*) \quad \text{for all } n \geq 0.$$

Since $(F(u,v),F(v,u)) = (gu_1,gv_1) \asymp (gx,gy) = (F(x,y),\ F(y,x)) = (gx_1,gy_1)$ are comparable, then $gu_1 \asymp gx$ and $gv_1 \asymp gy$. It is easy to show that (gu_n,gv_n) and (gx,gy) are comparable; that is, $gu_n \asymp gx$ and $gv_n \asymp gy$ for $n \in N$. Thus from(2.1), we have

$$\begin{aligned}\psi(d(gu_{n+1},gx)) = \psi(d(F(u_n,v_n),F(x,y))) \\ \leq \phi(\max\{d(gu_n,gx),d(gv_n,gy)\}) \\ + L\min\{(d(F(u_n,v_n),gx),d(F(x,y),gu_n), \\ d(F(u_n,v_n),gu_n),d(F(x,y),gx))\},\end{aligned}$$

which implies that

$$\psi(d(gu_{n+1},gx)) \leq \phi(\max\{d(gu_n,gx),d(gv_n,gy)\}). \tag{3.3}$$

Similarly, it follows that

$$\psi(d(gv_{n+1},gy)) \leq \phi(\max\{d(gv_n,gy),d(gu_n,gx)\}). \tag{3.4}$$

Now, $\max\{d(gu_{n+1},gx),d(gv_{n+1},gy)\}$ is either $d(gu_{n+1},gx)$ or $d(gv_{n+1},gy)$, in both the cases, from(3.3) and (3.4) we obtain that

$$\psi(\max\{d(gu_{n+1},gx),d(gv_{n+1},gy)\}) \leq \phi(\max\{d(gu_n,gx),d(gv_n,gy)\}). \tag{3.5}$$

Let us define $\sigma_n := \max\{d(gu_{n+1},gx),d(gv_{n+1},gy)\}$, then by (3.5) we have that $\psi(\sigma_n) \leq \phi(\sigma_{n-1})$. Using the conditions on ψ and ϕ, we obtain that $\psi(\sigma_n) \leq \phi(\sigma_{n-1}) < \psi(\sigma_{n-1})$, then by monotone property of ψ, it follows that $\{\sigma_n\}$ is a non-negative and decreasing sequence and hence there exists some $\sigma \geq 0$, such that $\lim_{n\to\infty} \sigma_n = \sigma$.

We claim that $\sigma = 0$. Assume that $\sigma > 0$, taking the limit as $n \to \infty$ on both the sides of (3.5) and using the properties of ψ and ϕ, we obtain that

$$\psi(\sigma) \leq \phi(\sigma) < \psi(\sigma),$$

which is a contradiction. Hence $\sigma = 0$, that is, $\lim_{n\to\infty} \max\{d(gu_{n+1},gx),d(gv_{n+1},gy)\} = 0$, hence

$$\lim_{n\to\infty} d(gu_{n+1},gx) = 0 = \lim_{n\to\infty} d(gv_{n+1},gy). \tag{3.6}$$

Similarly, we can show that

$$\lim_{n\to\infty} d(gu_{n+1},gx^*) = 0 = \lim_{n\to\infty} d(gv_{n+1},gy^*). \tag{3.7}$$

By uniqueness of limit it follows that $gx = gx^*$ and $gy = gy^*$. Hence (3.2) is proved. Therefore, (gx,gy) is the unique coupled point of coincidence of F and g.

Also, if (gx,gy) is a coupled point of coincidence of F and g, then (gy,gx) is also a coupled point of coincidence of F and g. Then $gx = gy$ and therefore (gx,gx) is the unique coupled point of coincidence of F and g.

Next, we show that F and g have a common coupled fixed point. Let $\bar{x} := gx$. Then we have $\bar{x} = gx = F(x,x)$. Since F and g are w^*-compatible, we have

$$g\bar{x} = ggx = gF(x,x) = F(gx,gx) = F(\bar{x},\bar{x}).$$

Thus, $(g\bar{x},g\bar{x})$ is a coupled point of coincidence of F and g. By the uniqueness of a coupled point of coincidence of F and g, we obtain that $g\bar{x} = gx$. Therefore, $\bar{x} = g\bar{x} = F(\bar{x},\bar{x})$, that is, $(\bar{x},\bar{x})$ is a common coupled fixed point of F and g.

Finally, we show the uniqueness of a common coupled fixed point of F and g. Let $(\bar{y},\bar{y}) \in X \times X$ be any common coupled fixed point of F and g. So,

$$\bar{y} = g\bar{y} = F(\bar{y},\bar{y}).$$

Then $(g\bar{x}, g\bar{x})$ and $(g\bar{y}, g\bar{y})$ are two common coupled points of coincidence of F and g and, as was proved previously, it must be $g\bar{x} = g\bar{y}$, and so $\bar{x} = g\bar{x} = g\bar{y} = \bar{y}$. This completes the proof. □

Remark 3.1. (i) *In addition to the hypotheses of any of the Corollaries 2.2-2.7, suppose that for every* $(x, y), (x^*, y^*) \in X \times X$ *there exists a* $(u, v) \in X \times X$ *such that* $(F(u, v), F(v, u)) \asymp (F(x, y), F(y, x))$ *and* $(F(u, v), F(v, u)) \asymp (F(x^*, y^*), F(y^*, x^*))$. *If the pair of the mappings* (F, g) *is* w^**-compatible, then* F *and* g *have a unique coupled common fixed point, that is, there exists a unique* $(\bar{x}, \bar{y}) \in X \times X$ *such that* $\bar{x} = g(\bar{x}) = F(\bar{x}, \bar{y})$ *and* $\bar{y} = g(\bar{y}) = F(\bar{y}, \bar{x})$.

(ii) *In addition to the hypotheses of Corollary 2.1, suppose that for every* $(x, y), (x^*, y^*) \in X \times X$ *there exists a* $(u, v) \in X \times X$ *such that* $(F(u, v), F(v, u)) \asymp (F(x, y), F(y, x))$ *and* $(F(u, v), F(v, u)) \asymp (F(x^*, y^*),\ F(y^*, x^*))$, *then* F *has a unique coupled fixed point, that is, there exists a unique* $(\bar{x}, \bar{y}) \in X \times X$ *such that* $\bar{x} = F(\bar{x}, \bar{y})$ *and* $\bar{y} = F(\bar{y}, \bar{x})$.

References

[1] Alotaibi,A. and Alsulami,S.M., (2011), Coupled coincidence points for monotone operators in partially ordered metric spaces, Fixed Point Theory Appl. 2011, 2011: 44.

[2] Abbas,M., Khan,M.A., and Radenović,S., (2010), Common coupled fixed point theorem in cone metric space for w-compatible mappings, Appl. Math. Comput., 217, pp.195-202.

[3] Agarwal,R.P., Sintunavarat,W., and Kumam,P., (2013), Coupled coincidence point and common coupled fixed point theorems lacking the mixed monotone property, Fixed Point Theory Appl. 2013, 2013:22.

[4] Banach,S., (1922), Surles operations dans les ensembles et leur application aux equation sitegrales, Fund. Math., 3, pp.133-181.

[5] Boyd,D.W. and Wong,J.S.W., (1969), On nonlinear contractions, Proc. Amer. Math. Soc., 20, pp.458-464.

[6] Bhaskar,T.G. and Lakshmikantham,V., (2006), Fixed point theorems in partially ordered metric spaces and applications, Nonlinear Anal., 65(7), pp.1379-1393.

[7] Berinde,V., (2011), Generalized coupled fixed point theorems for mixed monotone mappings partially ordered metric spaces, Nonlinear Anal., 74, pp.7347-7355.

[8] Ćirić,L.B., (1974), A generalization of Banachs contraction principle, Proc. Amer. Math. Soc., 45, pp.267-273.

[9] Choudhury,B.S. and Kundu,A., (2010), A coupled coincidence point result in partially ordered metric spaces for compatible mappings, Nonlinear Anal., 73, pp.2524-2531.

[10] Choudhury,B.S., Metiya,N., and Kundu,A., (2011), Coupled coincidence point theorems in ordered metric spaces, Ann. Univ. Ferrara, 57, pp.1-16.

[11] Chandok,S. and Tas,K., (2014), An original coupled coincidence point result for a pair of mappings without MMP, J. Inequal. Appl., 2014, 2014:61.

[12] Dutta,P. and Choudhury,B.S., (2008), A generalization of contractions in partially ordered metric spaces, Appl. Anal., 87, pp.109-116.

[13] Đorić,D., Kadelburg,Z., and Radenović,S., (2012), Coupled fixed point results for mappings without mixed monotone property, Appl. Math. Lett., DOI 10.1016/j.aml.2012.02.022.

[14] Guo,D. and Lakshmikantham,V., (1987), Coupled fixed points of nonlinear operators with applications, Nonlinear Anal., 11, pp.623-632.

[15] Haghi, R.H., Rezapour,Sh., and Shahzad,N., (2011), Some fixed point generalizations are not real generalizations, Nonlinear Anal., 74, pp.1799-1803.

[16] Hussain,N., Latif,A., and Shah,M.H., (2012), Coupled and tripled coincidence point results without compatibility, Fixed Point Theory Appl., 2012, 2012: 77.

[17] Harjani,J., Lopez,B. and Sadarangani K., (2011), Fixed point theorems for mixed monotone operators and applications to integral equations, Nonlinear Anal., 74, pp.1749-1760.

[18] Harjani,J. and Sadarangni,K., (2009), Fixed point theorems for weakly contraction mappings in partially ordered sets, Nonlinear Anal., 71, pp.3403-3410.

[19] Harjani,J. and Sadarangani,K., (2010), Generalized contractions in partially ordered metric spaces and applications to ordinary differential equations, Nonlinear Anal., 72, pp.1188-1197.

[20] Jain,M., Gupta,N., and Kumar,S., (2014), A New Technique to Compute Coupled Coincidence Points, Chinese Journal of Mathematics, Vol.2014, Article ID 652107, 6 pages.

[21] Jain,M., Tas,K., Kumar,S., and Gupta,N., (2012), Coupled common fixed points involving a (φ, ψ)-contractive condition for mixed g-monotone operators in partially ordered metric spaces, J. Inequal. Appl., 2012, 2012:285.

[22] Karapinar,E., Luong,N.V., and Thuan N.X., (2012), Coupled coincidence points for mixed monotone operators in partially ordered metric spaces, Arab J Math, 1, pp.329-339. DOI 10.1007/s40065-012-0027-0

[23] Khan,M.S., Swaleh,M., and Sessa,S., (1984), Fixed point theorems by altering distances between the points, Bull. Austral. Math. Soc., 30(1), pp.1-9.

[24] Lakshmikantham,V. and Ćirić,L.B., (2009), Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Anal., 70, pp.4341-4349.

[25] Luong,N.V. and Thuan,N.X., (2010), Coupled fixed point theorems in partially ordered metric spaces, Bull. Math. Anal. Appl., 2, pp.16-24.

[26] Meir A. and Keeler E., (1969), A theorem on contraction mappings, J. Math. Anal. Appl., 28, pp.326-329.

[27] Nashine,H.K., Kadelburg,Z., and Radenović,S., (2012), Coupled common fixed point theorems for w^*-compatible mappings in ordered cone metric spaces, Appl. Math. Comput., 218, pp.5422-5432.

[28] Nieto,J.J. and López,R.R., (2005), Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations, Order, 22(3), pp.223-239.

[29] Nieto,J.J. and López,R.R., (2007), Existence and uniqueness of fixed point in partially ordered sets and applications to ordinary dierential equations, Acta Math. Sin. (Engl. Ser.), 23(12), pp.2205-2212.

[30] Rasouli,S.H. and Bahrampour,M., (2011), A remark on the coupled fixed point theorems for mixed monotone operators in partially ordered metric spaces, J. Math. Comput. Sci., 3(2), pp.246-261.

[31] Ran,A.C.M. and Reurings,M.C.B., (2004), A fixed point theorem in partially ordered sets and some applications to matrix equations, Proc. Amer. Math. Soc., 132(5), pp.1435-1443.

[32] Su,Y., Feng,Q., Zhang,J., Cheng,Q., and Yan,F., (2012), A new contraction mapping principle in partially ordered metric spaces and applications to ordinary differential equations, Fixed Point Theory Appl., 2012, 2012:152.

[33] Sintunavarat,W., Cho,Y.J., and Kumam,P., (2011), Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces, Fixed Point Theory Appl., 2011, 2011: 81.

[34] Samet,B., Karapinar,E., Aydi,H., and Rajić,V.Ć., (2013), Discussion on some coupled fixed point theorems, Fixed Point Theory Appl., 2013 2013:50.

[35] Sintunavarat,W., Petruel,A. and Kumam,P., (2012) Common coupled fixed point theorems for w^*-compatible mappings without mixed monotone property, Rend. Circ. Mat. Palermo, 61, pp.361-383. DOI 10.1007/s12215-012-0096-0.

[36] Turinici,M., (1986), Abstract comparison principles and multivariable Gronwall-Bellman inequalities, J. Math. Anal. Appl., 117(1), pp.100-127.

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