

TWMS J. App. Eng. Math. V.3, N.2, 2013, pp. 147-159.

**SOLVABILITY OF ITERATIVE SYSTEMS OF THREE-POINT  
BOUNDARY VALUE PROBLEMS**K. R. PRASAD<sup>1</sup>, N. SREEDHAR<sup>2</sup>, K. R. KUMAR<sup>3</sup> §

ABSTRACT. We establish a criterion for the existence of at least one positive solution for the iterative system of three-point boundary value problems by determining the eigenvalues  $\lambda_i$ ,  $1 \leq i \leq n$ , using Guo–Krasnosel'skii fixed point theorem.

Keywords: Iterative system, boundary value problem, eigenvalue, positive solution, cone.

AMS Subject Classification: 34B18, 34A34

## 1. INTRODUCTION

In this paper, we are concerned with determining the eigenvalues  $\lambda_i$ ,  $1 \leq i \leq n$ , for which there exist positive solutions of the iterative system of second order differential equations,

$$\left. \begin{aligned} y_i''(t) + \lambda_i p_i(t) f_i(y_{i+1}(t)) &= 0, \quad 1 \leq i \leq n, \quad t \in [t_1, t_3], \\ y_{n+1}(t) &= y_1(t), \quad t \in [t_1, t_3], \end{aligned} \right\} \quad (1)$$

satisfying the different three-point boundary conditions,

$$\alpha_i y_i(t_1) - \beta_i y_i'(t_1) = 0 \text{ and } \gamma_i y_i(t_3) + \delta_i y_i'(t_3) = y_i'(t_2), \quad 1 \leq i \leq n, \quad (2)$$

where  $0 \leq t_1 < t_2 < t_3$ ,  $\alpha_i > 0$ ,  $\beta_i > 0$ ,  $\gamma_i > 0$  and  $\delta_i > 1$  are real numbers, for  $1 \leq i \leq n$ . We assume the following conditions hold throughout the paper:

- (A1)  $f_i : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  is continuous, for  $1 \leq i \leq n$ ,  
 (A2)  $p_i : [t_1, t_3] \rightarrow \mathbb{R}^+$  is continuous and  $p_i$  does not vanish identically on any closed subinterval of  $[t_1, t_3]$ , for  $1 \leq i \leq n$ ,  
 (A3)  $\alpha_i > 0$ ,  $\beta_i > 0$ ,  $\gamma_i > 0$ ,  $\delta_i > 1$  and  $\gamma_i > \frac{\alpha_i \delta_i}{\alpha_i(t_2 - t_1) + \beta_i}$ , for  $1 \leq i \leq n$ ,  
 (A4) each of

$$f_{i0} = \lim_{x \rightarrow 0^+} \frac{f_i(x)}{x} \text{ and } f_{i\infty} = \lim_{x \rightarrow \infty} \frac{f_i(x)}{x},$$

for  $1 \leq i \leq n$ , exists as positive real number.

<sup>1</sup> Department of Applied Mathematics, Andhra University, Visakhapatnam, 530 003, India,  
e-mail: rajendra92@rediffmail.com

<sup>2</sup> Department of Mathematics, GITAM University, Visakhapatnam, 530 045, India,  
e-mail: sreedharnamburi@rediffmail.com

<sup>3</sup> Department of Mathematics, VITAM College of Engineering, Visakhapatnam, 531 173, India,  
e-mail: rkkona72@rediffmail.com

§ Manuscript received August 01, 2013.

TWMS Journal of Applied and Engineering Mathematics, Vol.3, No.2; © Işık University, Department of Mathematics, 2013; all rights reserved.

Recently, the existence of positive solutions for the system of differential equations with multi-point boundary conditions have been studied by many authors due to their striking applications to almost all area of science, engineering and technology. In 2007, Henderson and Ntouyas [6] established the existence of positive solutions for the system of  $n^{\text{th}}$  order differential equations,

$$\begin{aligned}u^{(n)}(t) + \lambda a(t)f(v(t)) &= 0, \quad 0 < t < 1, \\v^{(n)}(t) + \lambda b(t)g(u(t)) &= 0, \quad 0 < t < 1,\end{aligned}$$

satisfying the three-point nonlocal boundary conditions,

$$\begin{aligned}u(0) = 0, u'(0) = 0, \dots, u^{(n-2)}(0) = 0, \quad u(1) = \alpha u(\eta), \\v(0) = 0, v'(0) = 0, \dots, v^{(n-2)}(0) = 0, \quad v(1) = \alpha v(\eta),\end{aligned}$$

where  $0 < \eta < 1$  and  $0 < \alpha \eta^{n-1} < 1$ .

In 2008, Henderson, Ntouyas and Purnaras [7] deals the existence of positive solutions for the system of second order differential equations,

$$\begin{aligned}u''(t) + \lambda a(t)f(v(t)) &= 0, \quad 0 < t < 1, \\v''(t) + \lambda b(t)g(u(t)) &= 0, \quad 0 < t < 1,\end{aligned}$$

satisfying the three-point boundary conditions,

$$\begin{aligned}u(0) = \beta u(\eta), \quad u(1) = \alpha u(\eta), \\v(0) = \beta v(\eta), \quad v(1) = \alpha v(\eta),\end{aligned}$$

where  $0 < \eta < 1$ ,  $0 < \alpha < \frac{1}{\eta}$ ,  $0 < \beta < \frac{1-\alpha\eta}{1-\eta}$ , and also in the same year, same authors, (see [8] in references) studied the existence of positive solutions for the system of second order differential equations,

$$\begin{aligned}u''(t) + \lambda a(t)f(v(t)) &= 0, \quad 0 < t < 1, \\v''(t) + \mu b(t)g(u(t)) &= 0, \quad 0 < t < 1,\end{aligned}$$

satisfying the four-point boundary conditions,

$$\begin{aligned}u(0) = \alpha u(\xi), \quad u(1) = \beta u(\eta), \\v(0) = \alpha v(\xi), \quad v(1) = \beta v(\eta),\end{aligned}$$

where  $0 < \xi < \eta < 1$ ,  $0 \leq \alpha, \beta < 1$ .

Till now in the literature, the authors established results for the existence of positive solutions for the system of two differential equations satisfying the same boundary conditions. We wish to extend these results to system of  $n$  differential equations satisfying the different boundary conditions.

The rest of the paper is organized as follows. In Section 2, we construct the Green's function for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the Green's function. In Section 3, we determine the eigenvalues for which there exist positive solutions of the boundary value problem (1)-(2) by using Guo-Krasnosel'skii fixed point theorem for operators on a cone in a Banach space. Finally as an application, we give an example to illustrate our result.

## 2. GREEN'S FUNCTION AND BOUNDS

In this section, we construct the Green's function for the homogeneous problem corresponding to (1)-(2) and estimate bounds for the Green's function.

Let  $G_i(t, s)$  be the Green's function for the homogeneous boundary value problem,

$$-y_i''(t) = 0, \quad t \in [t_1, t_3], \tag{3}$$

$$\alpha_i y_i(t_1) - \beta_i y_i'(t_1) = 0 \text{ and } \gamma_i y_i(t_3) + \delta_i y_i'(t_3) = y_i'(t_2), \tag{4}$$

for  $1 \leq i \leq n$ .

**Lemma 2.1.** *Let  $d_i = \alpha_i[\gamma_i(t_3 - t_1) + \delta_i - 1] + \beta_i\gamma_i \neq 0$ ,  $1 \leq i \leq n$ . Then, for  $1 \leq i \leq n$ , the Green's function  $G_i(t, s)$  for the homogeneous boundary value problem (3)-(4) is given by*

$$G_i(t, s) = \begin{cases} G_{i1}(t, s), & t_1 \leq s \leq t \leq t_2 < t_3, \\ G_{i2}(t, s), & t_1 \leq t \leq s \leq t_2 < t_3, \\ G_{i3}(t, s), & t_1 < t \leq t_2 \leq s \leq t_3, \\ \\ G_{i4}(t, s), & t_1 < t_2 \leq s \leq t \leq t_3, \\ G_{i5}(t, s), & t_1 < t_2 \leq t \leq s \leq t_3, \\ G_{i6}(t, s), & t_1 \leq s \leq t_2 \leq t < t_3, \end{cases} \tag{5}$$

where

$$\begin{aligned} G_{i1}(t, s) &= \frac{1}{d_i} [\alpha_i(s - t_1) + \beta_i][\gamma_i(t_3 - t) + \delta_i - 1], \\ G_{i2}(t, s) &= \frac{1}{d_i} [\alpha_i(t - t_1) + \beta_i][\gamma_i(t_3 - s) + \delta_i - 1], \\ G_{i3}(t, s) &= \frac{1}{d_i} [\alpha_i(t - t_1) + \beta_i][\gamma_i(t_3 - s) + \delta_i], \\ G_{i4}(t, s) &= \frac{1}{d_i} [(\alpha_i(s - t_1) + \beta_i)(\gamma_i(t_3 - t) + \delta_i) + \alpha_i(t - s)], \\ G_{i5}(t, s) &= \frac{1}{d_i} [\alpha_i(t - t_1) + \beta_i][\gamma_i(t_3 - s) + \delta_i], \\ G_{i6}(t, s) &= \frac{1}{d_i} [\alpha_i(s - t_1) + \beta_i][\gamma_i(t_3 - t) + \delta_i - 1]. \end{aligned}$$

**Lemma 2.2.** *Assume that the condition (A3) is satisfied. Then, for  $1 \leq i \leq n$ , the Green's function  $G_i(t, s)$  of (3)-(4) is positive, for all  $(t, s) \in (t_1, t_3) \times (t_1, t_3)$ .*

*Proof.* By simple algebraic calculations, we can easily establish the positivity of the Green's function. □

**Lemma 2.3.** *Assume that the condition (A3) is satisfied. Then, for  $1 \leq i \leq n$ , the Green's function  $G_i(t, s)$  in (5) satisfies the following inequality,*

$$k_i G_i(s, s) \leq G_i(t, s) \leq G_i(s, s), \text{ for all } (t, s) \in [t_1, t_3] \times [t_1, t_3], \tag{6}$$

where

$$k_i = \min \left\{ \frac{\delta_i - 1}{\gamma_i(t_3 - t_1) + \delta_i - 1}, \frac{\beta_i}{\alpha_i(t_3 - t_1) + \beta_i} \right\} < 1.$$

*Proof.* For  $1 \leq i \leq n$ , the Green's function  $G_i(t, s)$  is given in (5). In each case, we prove the inequality as in (6).

**Case 1.** For  $t_1 \leq s \leq t \leq t_2 < t_3$ ,

$$\frac{G_i(t, s)}{G_i(s, s)} = \frac{G_{i1}(t, s)}{G_{i1}(s, s)} = \frac{\gamma_i(t_3 - t) + \delta_i - 1}{\gamma_i(t_3 - s) + \delta_i - 1} \leq 1$$

and also

$$\frac{G_i(t, s)}{G_i(s, s)} = \frac{G_{i1}(t, s)}{G_{i1}(s, s)} = \frac{\gamma_i(t_3 - t) + \delta_i - 1}{\gamma_i(t_3 - s) + \delta_i - 1} \geq \frac{\delta_i - 1}{\gamma_i(t_3 - t_1) + \delta_i - 1}.$$

**Case 2.** For  $t_1 \leq t \leq s \leq t_2 < t_3$ ,

$$\frac{G_i(t, s)}{G_i(s, s)} = \frac{G_{i2}(t, s)}{G_{i2}(s, s)} = \frac{\alpha_i(t - t_1) + \beta_i}{\alpha_i(s - t_1) + \beta_i} \leq 1$$

and also

$$\frac{G_i(t, s)}{G_i(s, s)} = \frac{G_{i2}(t, s)}{G_{i2}(s, s)} = \frac{\alpha_i(t - t_1) + \beta_i}{\alpha_i(s - t_1) + \beta_i} \geq \frac{\beta_i}{\alpha_i(t_3 - t_1) + \beta_i}.$$

**Case 3.** For  $t_1 < t \leq t_2 \leq s \leq t_3$ ,

$$\frac{G_i(t, s)}{G_i(s, s)} = \frac{G_{i3}(t, s)}{G_{i3}(s, s)} = \frac{\alpha_i(t - t_1) + \beta_i}{\alpha_i(s - t_1) + \beta_i} \leq 1$$

and also

$$\frac{G_i(t, s)}{G_i(s, s)} = \frac{G_{i3}(t, s)}{G_{i3}(s, s)} = \frac{\alpha_i(t - t_1) + \beta_i}{\alpha_i(s - t_1) + \beta_i} \geq \frac{\beta_i}{\alpha_i(t_3 - t_1) + \beta_i}.$$

**Case 4.** For  $t_1 < t_2 \leq s \leq t \leq t_3$ ,

$$\begin{aligned} \frac{G_i(t, s)}{G_i(s, s)} &= \frac{G_{i4}(t, s)}{G_{i4}(s, s)} = \frac{[\alpha_i(s - t_1) + \beta_i][\gamma_i(t_3 - t) + \delta_i] + \alpha_i(t - s)}{[\alpha_i(s - t_1) + \beta_i][\gamma_i(t_3 - s) + \delta_i]} \\ &\leq \frac{\gamma_i(t_3 - s) + \delta_i}{\gamma_i(t_3 - s) + \delta_i} = 1 \end{aligned}$$

and also

$$\begin{aligned} \frac{G_i(t, s)}{G_i(s, s)} &= \frac{G_{i4}(t, s)}{G_{i4}(s, s)} = \frac{[\alpha_i(s - t_1) + \beta_i][\gamma_i(t_3 - t) + \delta_i] + \alpha_i(t - s)}{[\alpha_i(s - t_1) + \beta_i][\gamma_i(t_3 - s) + \delta_i]} \\ &\geq \frac{\delta_i - 1}{\gamma_i(t_3 - t_1) + \delta_i - 1}. \end{aligned}$$

Similarly, we can easily establish the inequality, when the Green's function  $G_i(t, s) = G_{i5}(t, s)$  and  $G_i(t, s) = G_{i6}(t, s)$  as in case 3 and case 1 respectively. Hence the inequality (6).  $\square$

**Lemma 2.4.** Assume that the condition (A3) is satisfied. Then, for  $1 \leq i \leq n$ , the Green's function  $G_i(t, s)$  in (5) satisfies the following inequality,

$$G_i(t, s) \geq kG_i(s, s), \text{ for all } (t, s) \in [t_1, t_3] \times [t_1, t_3],$$

where  $k = \min\{k_1, k_2, \dots, k_n\}$ .

We note that an  $n$ -tuple  $(y_1(t), y_2(t), \dots, y_n(t))$  is a solution of the boundary value problem (1)-(2) if and only if

$$\begin{aligned} y_1(t) &= \lambda_1 \int_{t_1}^{t_3} G_1(t, s_1) p_1(s_1) f_1 \left( \lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) \dots \right. \\ &\quad \left. f_{n-1} \left( \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n \right) \dots ds_2 \right) ds_1, \quad t \in [t_1, t_3], \\ y_i(t) &= \lambda_i \int_{t_1}^{t_3} G_i(t, s) p_i(s) f_i(y_{i+1}(s)) ds, \quad 2 \leq i \leq n, \quad t \in [t_1, t_3], \end{aligned}$$

where

$$y_{n+1}(t) = y_1(t), \quad t \in [t_1, t_3].$$

To determine the eigenvalues for which the boundary value problem (1)-(2) has at least one positive solution, we will employ the following Guo–Krasnosel’skii fixed point theorem.

**Theorem 2.1.** *Let  $X$  be a Banach Space,  $\kappa \subseteq X$  be a cone and suppose that  $\Omega_1, \Omega_2$  are open subsets of  $X$  with  $0 \in \Omega_1$  and  $\bar{\Omega}_1 \subset \Omega_2$ . Suppose further that  $T : \kappa \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow \kappa$  is completely continuous operator such that either*

- (i)  $\|Tu\| \leq \|u\|$ ,  $u \in \kappa \cap \partial\Omega_1$  and  $\|Tu\| \geq \|u\|$ ,  $u \in \kappa \cap \partial\Omega_2$ , or
- (ii)  $\|Tu\| \geq \|u\|$ ,  $u \in \kappa \cap \partial\Omega_1$  and  $\|Tu\| \leq \|u\|$ ,  $u \in \kappa \cap \partial\Omega_2$  holds.

Then  $T$  has a fixed point in  $\kappa \cap (\bar{\Omega}_2 \setminus \Omega_1)$ .

### 3. POSITIVE SOLUTIONS IN A CONE

In this section, we establish criteria to determine the eigenvalues for which the boundary value problem (1)-(2) has at least one positive solution in a cone.

For our construction, let  $B = \{x \mid x \in C[t_1, t_3]\}$  be a Banach space with the norm

$$\|x\| = \sup_{t \in [t_1, t_3]} |x(t)|.$$

Define a cone  $P \subset B$  by

$$P = \left\{ x \in B \mid x(t) \geq 0 \text{ on } [t_1, t_3] \text{ and } \min_{t \in [t_1, t_3]} x(t) \geq k\|x\| \right\},$$

where  $k$  is given in Lemma 2.4.

Now, we define an integral operator  $T : P \rightarrow B$ , for  $y_1 \in P$ , by

$$\begin{aligned} Ty_1(t) = & \lambda_1 \int_{t_1}^{t_3} G_1(t, s_1) p_1(s_1) f_1 \left( \lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) \cdots \right. \\ & \left. f_{n-1} \left( \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n \right) \cdots ds_2 \right) ds_1. \end{aligned} \tag{7}$$

Notice from (A1), (A2) and Lemma 2.2 that, for  $y_1 \in P$ ,  $Ty_1(t) \geq 0$  on  $[t_1, t_3]$ . Also, for  $y_1 \in P$ , we have from Lemma 2.3 that

$$\begin{aligned} Ty_1(t) \leq & \lambda_1 \int_{t_1}^{t_3} G_1(s_1, s_1) p_1(s_1) f_1 \left( \lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) \cdots \right. \\ & \left. f_{n-1} \left( \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n \right) \cdots ds_2 \right) ds_1 \end{aligned}$$

so that

$$\begin{aligned} \|Ty_1\| \leq & \lambda_1 \int_{t_1}^{t_3} G_1(s_1, s_1) p_1(s_1) f_1 \left( \lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) \cdots \right. \\ & \left. f_{n-1} \left( \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n \right) \cdots ds_2 \right) ds_1. \end{aligned} \tag{8}$$

Next, if  $y_1 \in P$ , we have from Lemma 2.4 and (8) that

$$\begin{aligned} \min_{t \in [t_1, t_3]} Ty_1(t) &= \min_{t \in [t_1, t_3]} \left\{ \lambda_1 \int_{t_1}^{t_3} G_1(t, s_1) p_1(s_1) f_1 \left( \lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) \cdots \right. \right. \\ &\quad \left. \left. f_{n-1} \left( \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n \right) \cdots ds_2 \right) ds_1 \right\} \\ &\geq \lambda_1 k \int_{t_1}^{t_3} G_1(s_1, s_1) p_1(s_1) f_1 \left( \lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) \cdots \right. \\ &\quad \left. f_{n-1} \left( \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n \right) \cdots ds_2 \right) ds_1 \\ &\geq k \|Ty_1\|. \end{aligned}$$

Hence,  $Ty_1 \in P$  and so  $T : P \rightarrow P$ . Further, the operator  $T$  is completely continuous operator by an application of the Ascoli-Arzelà Theorem.

Now, we seek suitable fixed points of  $T$  belonging to the cone  $P$ . For our first result, define positive numbers  $M_1$  and  $M_2$  by

$$M_1 = \max_{1 \leq i \leq n} \left\{ \left[ k^2 \int_{t_1}^{t_3} G_i(s, s) p_i(s) ds f_{i\infty} \right]^{-1} \right\}$$

and

$$M_2 = \min_{1 \leq i \leq n} \left\{ \left[ \int_{t_1}^{t_3} G_i(s, s) p_i(s) ds f_{i0} \right]^{-1} \right\}.$$

**Theorem 3.1.** *Assume that the conditions (A1)-(A4) are satisfied. Then, for each  $\lambda_1, \lambda_2, \dots, \lambda_n$  satisfying*

$$M_1 < \lambda_j < M_2, \quad 1 \leq j \leq n, \quad (9)$$

*there exists an  $n$ -tuple  $(y_1, y_2, \dots, y_n)$  satisfying (1)-(2) such that  $y_j(t) > 0$ ,  $1 \leq j \leq n$ , on  $(t_1, t_3)$ .*

*Proof.* Let  $\lambda_j$ ,  $1 \leq j \leq n$ , be given as in (9). Now, let  $\epsilon > 0$  be chosen such that

$$\max_{1 \leq i \leq n} \left\{ \left[ k^2 \int_{t_1}^{t_3} G_i(s, s) p_i(s) ds (f_{i\infty} - \epsilon) \right]^{-1} \right\} \leq \min_{1 \leq j \leq n} \lambda_j$$

and

$$\max_{1 \leq j \leq n} \lambda_j \leq \min_{1 \leq i \leq n} \left\{ \left[ \int_{t_1}^{t_3} G_i(s, s) p_i(s) ds (f_{i0} + \epsilon) \right]^{-1} \right\}.$$

We seek fixed points of the completely continuous operator  $T : P \rightarrow P$  defined by (7). Now, from the definitions of  $f_{i0}$ ,  $1 \leq i \leq n$ , there exists an  $H_1 > 0$  such that, for each  $1 \leq i \leq n$ ,

$$f_i(x) \leq (f_{i0} + \epsilon)x, \quad 0 < x \leq H_1.$$

Let  $y_1 \in P$  with  $\|y_1\| = H_1$ . We first have from Lemma 2.3 and the choice of  $\epsilon$ , for  $t_1 \leq s_{n-1} \leq t_3$ ,

$$\begin{aligned} \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n &\leq \lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n \\ &\leq \lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n) p_n(s_n) (f_{n0} + \epsilon) y_1(s_n) ds_n \\ &\leq \lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n) p_n(s_n) ds_n (f_{n0} + \epsilon) \|y_1\| \\ &\leq \|y_1\| = H_1. \end{aligned}$$

It follows in a similar manner from Lemma 2.3 and the choice of  $\epsilon$  that, for  $t_1 \leq s_{n-2} \leq t_3$ ,

$$\begin{aligned} \lambda_{n-1} \int_{t_1}^{t_3} G_{n-1}(s_{n-2}, s_{n-1}) p_{n-1}(s_{n-1}) \\ f_{n-1} \left( \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n \right) ds_{n-1} \\ \leq \lambda_{n-1} \int_{t_1}^{t_3} G_{n-1}(s_{n-1}, s_{n-1}) p_{n-1}(s_{n-1}) ds_{n-1} (f_{n-1,0} + \epsilon) H_1 \\ \leq H_1. \end{aligned}$$

Continuing with this bootstrapping argument, we have, for  $t_1 \leq t \leq t_3$ ,

$$\lambda_1 \int_{t_1}^{t_3} G_1(t, s_1) p_1(s_1) f_1 \left( \lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) \cdots f_n(y_1(s_n)) ds_n \cdots ds_2 \right) ds_1 \leq H_1,$$

so that, for  $t_1 \leq t \leq t_3$ ,

$$Ty_1(t) \leq H_1.$$

Hence,  $\|Ty_1\| \leq H_1 = \|y_1\|$ . If we set

$$\Omega_1 = \{x \in B \mid \|x\| < H_1\},$$

then

$$\|Ty_1\| \leq \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_1. \tag{10}$$

Next, from the definitions of  $f_{i\infty}$ ,  $1 \leq i \leq n$ , there exists  $\overline{H}_2 > 0$  such that, for each  $1 \leq i \leq n$ ,

$$f_i(x) \geq (f_{i\infty} - \epsilon)x, \quad x \geq \overline{H}_2.$$

Let

$$H_2 = \max \left\{ 2H_1, \frac{\overline{H}_2}{k} \right\}.$$

Choose  $y_1 \in P$  and  $\|y_1\| = H_2$ . Then,

$$\min_{t \in [t_1, t_3]} y_1(t) \geq k \|y_1\| \geq \overline{H}_2.$$

From Lemma 2.4 and choice of  $\epsilon$ , for  $t_1 \leq s_{n-1} \leq t_3$ , we have that

$$\begin{aligned} \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n &\geq k \lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n) p_n(s_n) (f_{n\infty} - \epsilon) y_1(s_n) ds_n \\ &\geq k^2 \lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n) p_n(s_n) ds_n (f_{n\infty} - \epsilon) \|y_1\| \\ &\geq \|y_1\| = H_2. \end{aligned}$$

It follows in a similar manner from Lemma 2.4 and choice of  $\epsilon$ , for  $t_1 \leq s_{n-2} \leq t_3$ ,

$$\begin{aligned} & \lambda_{n-1} \int_{t_1}^{t_3} G_{n-1}(s_{n-2}, s_{n-1}) p_{n-1}(s_{n-1}) \\ & f_{n-1} \left( \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n \right) ds_{n-1} \\ & \geq k \lambda_{n-1} \int_{t_1}^{t_3} G_{n-1}(s_{n-1}, s_{n-1}) p_{n-1}(s_{n-1}) ds_{n-1} (f_{n-1, \infty} - \epsilon) H_2 \\ & \geq k^2 \lambda_{n-1} \int_{t_1}^{t_3} G_{n-1}(s_{n-1}, s_{n-1}) p_{n-1}(s_{n-1}) ds_{n-1} (f_{n-1, \infty} - \epsilon) H_2 \\ & \geq H_2. \end{aligned}$$

Again, using a bootstrapping argument, we have

$$\lambda_1 \int_{t_1}^{t_3} G_1(t, s_1) p_1(s_1) f_1 \left( \lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) \cdots f_n(y_1(s_n)) ds_n \cdots ds_2 \right) ds_1 \geq H_2,$$

so that

$$Ty_1(t) \geq H_2 = \|y_1\|.$$

Hence,  $\|Ty_1\| \geq \|y_1\|$ . So, if we set

$$\Omega_2 = \{x \in B \mid \|x\| < H_2\},$$

then

$$\|Ty_1\| \geq \|y_1\|, \text{ for } y_1 \in P \cap \partial\Omega_2. \quad (11)$$

Applying Theorem 2.1 to (10) and (11), we obtain that  $T$  has a fixed point  $y_1 \in P \cap (\bar{\Omega}_2 \setminus \Omega_1)$ . As such, setting  $y_{n+1} = y_1$ , we obtain a positive solution  $(y_1, y_2, \dots, y_n)$  of (1)-(2) given iteratively by

$$y_j(t) = \lambda_j \int_{t_1}^{t_3} G_j(t, s) p_j(s) f_j(y_{j+1}(s)) ds, \quad j = n, n-1, \dots, 1.$$

The proof is completed.  $\square$

Prior to our next result, we define the positive numbers  $M_3$  and  $M_4$  by

$$M_3 = \max_{1 \leq i \leq n} \left\{ \left[ k^2 \int_{t_1}^{t_3} G_i(s, s) p_i(s) ds f_{i0} \right]^{-1} \right\}$$

and

$$M_4 = \min_{1 \leq i \leq n} \left\{ \left[ \int_{t_1}^{t_3} G_i(s, s) p_i(s) ds f_{i\infty} \right]^{-1} \right\}.$$

**Theorem 3.2.** *Assume that the conditions (A1)-(A4) are satisfied. Then, for each  $\lambda_1, \lambda_2, \dots, \lambda_n$  satisfying*

$$M_3 < \lambda_j < M_4, \quad 1 \leq j \leq n, \quad (12)$$

*there exists an  $n$ -tuple  $(y_1, y_2, \dots, y_n)$  satisfying (1)-(2) such that  $y_j(t) > 0$ ,  $1 \leq j \leq n$ , on  $(t_1, t_3)$ .*

*Proof.* Let  $\lambda_j$ ,  $1 \leq j \leq n$  be given as in (12). Now, let  $\epsilon > 0$  be chosen such that

$$\max_{1 \leq i \leq n} \left\{ \left[ k^2 \int_{t_1}^{t_3} G_i(s, s) p_i(s) ds (f_{i0} - \epsilon) \right]^{-1} \right\} \leq \min_{1 \leq j \leq n} \lambda_j$$



and

$$\max_{1 \leq j \leq n} \lambda_j \leq \min_{1 \leq i \leq n} \left\{ \left[ \int_{t_1}^{t_3} G_i(s, s) p_i(s) ds (f_{i\infty} + \epsilon) \right]^{-1} \right\}.$$

Let  $T$  be the cone preserving, completely continuous operator that was defined by (7). From the definitions of  $f_{i0}$ ,  $1 \leq i \leq n$ , there exists  $\bar{H}_3 > 0$  such that, for each  $1 \leq i \leq n$ ,

$$f_i(x) \geq (f_{i0} - \epsilon)x, \quad 0 < x \leq \bar{H}_3.$$

Also, from the definitions of  $f_{i0}$ , it follows that  $f_{i0}(0) = 0$ ,  $1 \leq i \leq n$ , and so there exist  $0 < K_n < K_{n-1} < \dots < K_2 < \bar{H}_3$  such that

$$\lambda_i f_i(t) \leq \frac{K_{i-1}}{\int_{t_1}^{t_3} G_i(s, s) p_i(s) ds}, \quad t \in [0, K_i], \quad 3 \leq i \leq n,$$

and

$$\lambda_2 f_2(t) \leq \frac{\bar{H}_3}{\int_{t_1}^{t_3} G_2(s, s) p_2(s) ds}, \quad t \in [0, K_2].$$

Choose  $y_1 \in P$  with  $\|y_1\| = K_n$ . Then, we have

$$\begin{aligned} \lambda_n \int_{t_1}^{t_3} G_n(s_{n-1}, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n &\leq \lambda_n \int_{t_1}^{t_3} G_n(s_n, s_n) p_n(s_n) f_n(y_1(s_n)) ds_n \\ &\leq \frac{\int_{t_1}^{t_3} G_n(s_n, s_n) p_n(s_n) K_{n-1} ds_n}{\int_{t_1}^{t_3} G_n(s_n, s_n) p_n(s_n) ds_n} \\ &= K_{n-1}. \end{aligned}$$

Continuing with this bootstrapping argument, it follows that

$$\lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) f_2 \left( \lambda_3 \int_{t_1}^{t_3} G_3(s_2, s_3) p_3(s_3) \dots f_n(y_1(s_n)) ds_n \dots ds_3 \right) ds_2 \leq \bar{H}_3.$$

Then,

$$\begin{aligned} Ty_1(t) &= \lambda_1 \int_{t_1}^{t_3} G_1(t, s_1) p_1(s_1) f_1 \left( \lambda_2 \int_{t_1}^{t_3} G_2(s_1, s_2) p_2(s_2) \dots f_n(y_1(s_n)) ds_n \dots ds_2 \right) ds_1 \\ &\geq k^2 \lambda_1 \int_{t_1}^{t_3} G_1(s_1, s_1) p_1(s_1) (f_{1,0} - \epsilon) \|y_1\| ds_1 \\ &\geq \|y_1\|. \end{aligned}$$

So,  $\|Ty_1\| \geq \|y_1\|$ . If we put

$$\Omega_3 = \{x \in B \mid \|x\| < K_n\},$$

then

$$\|Ty_1\| \geq \|y_1\|, \quad \text{for } y_1 \in P \cap \partial\Omega_3. \tag{13}$$

Since each  $f_{i\infty}$  is assumed to be a positive real number, it follows that  $f_i$ ,  $1 \leq i \leq n$ , is unbounded at  $\infty$ .

For each  $1 \leq i \leq n$ , set

$$f_i^*(x) = \sup_{0 \leq s \leq x} f_i(s).$$

Then, it is straightforward that, for each  $1 \leq i \leq n$ ,  $f_i^*$  is a nondecreasing real-valued function,  $f_i \leq f_i^*$  and

$$\lim_{x \rightarrow \infty} \frac{f_i^*(x)}{x} = f_{i\infty}.$$

Next, by definition of  $f_{i\infty}$ ,  $1 \leq i \leq n$ , there exists  $\bar{H}_4 > 0$  such that, for each  $1 \leq i \leq n$ ,

$$f_i^*(x) \leq (f_{i\infty} + \epsilon)x, \quad x \geq \bar{H}_4.$$

It follows that there exists  $H_4 > \max\{2\bar{H}_3, \bar{H}_4\}$  such that, for each  $1 \leq i \leq n$ ,

$$f_i^*(x) \leq f_i^*(H_4), \quad 0 < x \leq H_4.$$

Choose  $y_1 \in P$  with  $\|y_1\| = H_4$ . Then, using the usual bootstrapping argument, we have

$$\begin{aligned} Ty_1(t) &= \lambda_1 \int_{t_1}^{t_3} G_1(t, s_1) p_1(s_1) f_1(\lambda_2 \cdots) ds_1 \\ &\leq \lambda_1 \int_{t_1}^{t_3} G_1(t, s_1) p_1(s_1) f_1^*(\lambda_2 \cdots) ds_1 \\ &\leq \lambda_1 \int_{t_1}^{t_3} G_1(s_1, s_1) p_1(s_1) f_1^*(H_4) ds_1 \\ &\leq \lambda_1 \int_{t_1}^{t_3} G_1(s_1, s_1) p_1(s_1) ds_1 (f_{1\infty} + \epsilon) H_4 \\ &\leq H_4 = \|y_1\|. \end{aligned}$$

Hence,  $\|Ty_1\| \leq \|y_1\|$ . So, if we let

$$\Omega_4 = \{x \in B \mid \|x\| < H_4\},$$

then

$$\|Ty_1\| \leq \|y_1\|, \quad \text{for } y_1 \in P \cap \partial\Omega_4. \quad (14)$$

Applying Theorem 2.1 to (13) and (14), we obtain that  $T$  has a fixed point  $y_1 \in P \cap (\bar{\Omega}_4 \setminus \Omega_3)$ , which in turn with  $y_{n+1} = y_1$ , yields an  $n$ -tuple  $(y_1, y_2, \dots, y_n)$  satisfying (1)-(2) for the chosen values of  $\lambda_i$ ,  $1 \leq i \leq n$ . The proof is completed.  $\square$

**Example 3.1.** Let us consider an example to illustrate the above result. Take  $n = 3$ ,  $t_1 = 0$ ,  $t_2 = \frac{1}{2}$ ,  $t_3 = 1$ ,  $\alpha_1 = 2$ ,  $\beta_1 = 1$ ,  $\gamma_1 = 4$ ,  $\delta_1 = 3$ ,  $\alpha_2 = 1$ ,  $\beta_2 = 2$ ,  $\gamma_2 = 3$ ,  $\delta_2 = 4$ ,  $\alpha_3 = 3$ ,  $\beta_3 = 1$ ,  $\gamma_3 = 4$ ,  $\delta_3 = 2$ ,  $p_1(t) = p_2(t) = p_3(t) = 1$  and

$$f_1(y_2) = y_2(800 - 795.5e^{-y_2})(900 - 899.5e^{-2y_2}),$$

$$f_2(y_3) = y_3(1400 - 1398.5e^{-3y_3})(600 - 596.5e^{-y_3}),$$

$$f_3(y_1) = y_1(2000 - 1997.5e^{-2y_1})(260 - 258.5e^{-4y_1}).$$

The Green's functions  $G_i(t, s)$ , for  $i = 1, 2, 3$ , in Lemma 2.1 is

$$G_i(t, s) = \begin{cases} G_i(t, s) = \begin{cases} G_{i1}(t, s), & 0 \leq s \leq t \leq \frac{1}{2} < 1, \\ G_{i2}(t, s), & 0 \leq t \leq s \leq \frac{1}{2} < 1, \\ G_{i3}(t, s), & 0 < t \leq \frac{1}{2} \leq s \leq 1, \end{cases} \\ G_i(t, s) = \begin{cases} G_{i4}(t, s), & 0 < \frac{1}{2} \leq s \leq t \leq 1, \\ G_{i5}(t, s), & 0 < \frac{1}{2} \leq t \leq s \leq 1, \\ G_{i6}(t, s), & 0 \leq s \leq \frac{1}{2} \leq t < 1, \end{cases} \end{cases}$$

where

$$\begin{aligned} G_{11}(t, s) &= \frac{1}{16}[2s + 1][6 - 4t], G_{12}(t, s) = \frac{1}{16}[2t + 1][6 - 4s], \\ G_{13}(t, s) &= \frac{1}{16}[2t + 1][7 - 4s], G_{14}(t, s) = \frac{1}{16}[(2s + 1)(7 - 4t) + 2(t - s)], \\ G_{15}(t, s) &= \frac{1}{16}[2t + 1][7 - 4s], G_{16}(t, s) = \frac{1}{16}[2s + 1][6 - 4t], \\ G_{21}(t, s) &= \frac{1}{12}[s + 2][6 - 3t], G_{22}(t, s) = \frac{1}{12}[t + 2][6 - 3s], \\ G_{23}(t, s) &= \frac{1}{12}[t + 2][7 - 3s], G_{24}(t, s) = \frac{1}{12}[(s + 2)(7 - 3t) + t - s], \\ G_{25}(t, s) &= \frac{1}{12}[t + 2][7 - 3s], G_{26}(t, s) = \frac{1}{12}[s + 2][6 - 3t], \\ G_{31}(t, s) &= \frac{1}{19}[3s + 1][5 - 4t], G_{32}(t, s) = \frac{1}{19}[3t + 1][5 - 4s], \\ G_{33}(t, s) &= \frac{1}{19}[3t + 1][6 - 4s], G_{34}(t, s) = \frac{1}{19}[(3s + 1)(6 - 4t) + 3(t - s)], \\ G_{35}(t, s) &= \frac{1}{19}[3t + 1][6 - 4s], G_{36}(t, s) = \frac{1}{19}[3s + 1][5 - 4t]. \end{aligned}$$

From Lemma 2.4, we get  $k = \frac{1}{5}$ . We found that

$$f_{10} = 2.25, f_{20} = 5.25, f_{30} = 3.75, f_{1\infty} = 720000, f_{2\infty} = 840000, f_{3\infty} = 520000,$$

$$M_1 = \max\{0.00001129943503, 0.00002886002886, 0.00011242603\},$$

and

$$M_2 = \min\{0.8284789644, 0.184704184, 0.6235897436\}.$$

Employing Theorem 3.1, we get an eigenvalue interval  $0.00011242603 < \lambda_i < 0.184704184$ ,  $i = 1, 2, 3$ , for which the boundary value problem (1)-(2) has a positive solution.

#### 4. CONCLUSION

We derived sufficient conditions for the existence of positive solutions for the iterative system of second order differential equations satisfying the general three-point boundary conditions. We determine the eigenvalue intervals of the parameters for which the three-point boundary value problems possess a positive solution.

**Acknowledgement:** The authors thank the referees for their valuable suggestions.

#### REFERENCES

- [1] Agarwal, R. P., O'Regan, D. and Wong, P. J. Y., (1999), Positive solutions of Differential, Difference and Integral Equations, Kluwer, Dordrecht.
- [2] Erbe, L. H. and Wang, H., (1994), On the existence of positive solutions of ordinary differential equations, Proc. Amer. Math. Soc., 120, 743-748.
- [3] Fink, A. M. and Gatica, J. A., (1993), Positive solutions of second order systems of boundary value problem, J. Math. Anal. Appl., 180, 93-108.
- [4] Graef, J. R. and Yang, B., (2002), Boundary value problems for second order nonlinear ordinary differential equations, Comm. Appl. Anal., 6, 273-288.
- [5] Guo, D. and Lakshmikantham, V., (1988), Nonlinear Problems in Abstract Cones, Academic Press, Orlando.
- [6] Henderson, J. and Ntouyas, S. K., (2007), Positive solutions for systems of nth order three-point nonlocal boundary value problems, Elec. J. Qual. Theory Differ. Equ., 2007, No. 18, 1-12.

- [7] Henderson, J., Ntouyas, S. K. and Purnaras, I. K., (2008), Positive solutions for systems of generalized three-point nonlinear boundary value problems, *Comment. Math. Univ. Carolin.*, 49, 79-91.
- [8] Henderson, J., Ntouyas, S. K. and Purnaras, I. K., (2008), Positive solutions for systems of second order four-point nonlinear boundary value problems, *Comm. Appl. Anal.*, 12, No. 1, 29-40.
- [9] Henderson, J. and Wang, H., (1997), Positive solutions for nonlinear eigenvalue problems, *J. Math. Anal. Appl.*, 208, 1051-1060.
- [10] Henderson, J. and Wang, H., (2005), Nonlinear eigenvalue problems for quasilinear systems, *Comput. Math. Appl.*, 49, 1941-1949.
- [11] Henderson, J. and Wang, H., (2007), An eigenvalue problem for quasilinear systems, *Rockey Mountain J. Math.*, 37, 215-228.
- [12] Hu, L. and Wang, L. L., (2007), Multiple positive solutions of boundary value problems for systems of nonlinear second order differential equations, *J. Math. Anal. Appl.*, 335, 1052-1060.
- [13] Infante, G., (2003), Eigenvalues of some nonlocal boundary value problems, *Proc. Edinburgh Math. Soc.*, 46, 75-86.
- [14] Ma, R., (2000), Multiplicity of nonnegative solutions of second order systems of boundary value problems, *Nonlinear Anal.*, 42, 1003-1010.
- [15] Raffoul, Y., (2002), Positive solutions of three-point nonlinear second order boundary value problems, *Elec. J. Qual. Theory Differ. Equ.*, 2002, No. 15, 1-11.
- [16] Wang, H., (2003), On the number of positive solutions of nonlinear systems, *J. Math. Anal. Appl.*, 281, 287-306.
- [17] Webb, J. R. L., (2001), Positive solutions of some three-point boundary value problems via fixed point index theory, *Nonlinear Anal.*, 47 4319-4332.
- [18] Zhou, Y. and Xu, Y., (2006), Positive solutions of three-point boundary value problems for systems of nonlinear second order differential equations, *J. Math. Anal. Appl.*, 320, 578-590.



**Kapula Rajendra Prasad** received M.Sc. and Ph.D. degrees from Andhra University, Visakhapatnam, India. Dr. Prasad did his post doctoral work at Auburn University, Auburn, USA. Presently, Dr. Prasad is working as Professor and Head of the department of Applied Mathematics, Andhra University, Visakhapatnam, India. His major research interest includes ordinary differential equations, difference equations, dynamic equations on time scales, fractional order differential equations and boundary value problems. He published several research papers on the above topics in various national and international journals of high repute. He has been serving as referee for various national, international journals and reviewer for *Zentralblatt MATH*.



**Sreedhar Namburi** received M.Sc. and M.Phil. degrees from Andhra University, Visakhapatnam, India. Presently, Mr. Sreedhar is working as Assistant Professor in the department of Mathematics, GITAM University, Visakhapatnam, India. His major research interest includes ordinary differential equations, dynamic equations on time scales and boundary value problems. He published research papers on the above topics in various national and international journals.



**Kona Rajendra Kumar** received M.Sc. and M.Phil. degrees from Andhra University, Visakhapatnam, India. Presently, Mr. Kumar is working as Associate Professor in the department of Mathematics, VITAM College of Engineering, Visakhapatnam, India. His major research interest includes ordinary differential equations and boundary value problems.

---

---